ABSTRACT

Title of dissertation: ASYMPTOTICS OF THE YANG-MILLS FLOW FOR HOLOMORPHIC VECTOR BUNDLES OVER KÄHLER MANIFOLDS

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In this thesis we study the limiting properties of the Yang-Mills flow associated to a holomorphic vector bundle E over an arbitrary Kähler manifold (X, ω). In particular we show that the flow is determined at infinity by the holomorphic structure of E. Namely, if we fix an integrable unitary reference connection A_0 defining the holomorphic structure, then the Yang-Mills flow with initial condition A_0, converges (away from an appropriately defined singular set) in the sense of the Uhlenbeck compactness theorem to a holomorphic vector bundle E_∞, which is isomorphic to the associated graded object of the Harder-Narasimhan-Seshadri filtration of (E, A_0). Moreover, E_∞ extends as a reflexive sheaf over the singular set as the double dual of the associated graded object. This is an extension of previous work in the cases of 1 and 2 complex dimensions and proves the general case of a conjecture of Bando and Siu. Chapter 1 is an introduction and a review of the background material. Chapter 2 gives the proof of several critical intermediate results, including the existence of an approximate critical hermitian structure.

Chapter 3 concludes the proof of the main theorem.

ASYMPTOTICS OF THE YANG-MILLS FLOW FOR HOLOMORPHIC VECTOR BUNDLES OVER KÄHLER MANIFOLDS

by

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Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2013

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Acknowledgments

I would like to use this space to thank several people. Mostly, I would like to thank my thesis advisor Richard Wentworth, who has been giving me valuable advice since the Summer of 2010. He introduced me to the topic of this thesis, very patiently explained the details of his papers on the subject, and gave me considerable help in solving the problem. He has also very helpfully arranged to have me periodically relieved of my teaching duties, which has been a great aid in completing the present manuscript. In this regard I also thank the National Science Foundation, and in particular would like to mention NSF Grant Number 1037094.

I am also indebted to the other members of my thesis committee: William Goldman, Jonathan Rosenberg, and Yanir Rubinstein for agreeing to read this thesis and give advice. I particularly thank Professor Rubinstein for his questions and corrections during a talk on the subject of this thesis in the Geometric Anlaysis Seminar. I am also particularly grateful to Professor Raman Sundrum for agreeing to be the Dean's Representative on the committee.

The anonymous referee of the article [S] also deserves a great deal of gratitude for a particularly helpful referee report which provided knowledgeable comments at a microscopic level of detail. The referee has greatly improved the expostion and hopefully made the manuscript more readable.

My former flatmate Xuwen Chen, who lived with me for five years has offered his friendship and occasional mathematical advice. He has also given me a great deal of LaTex advice, and in particular was extremely helpful in showing me the ropes of the somewhat confusing University of Maryland thesis format.

Finally, I would like to thank my family: Dad, Mum and Laura. I am grateful to them for giving me all the opportunities they have, and for enduring endless discussion about my extremely esoteric and impractical subject.

Table of Contents

Chapter 1

Introduction and Background

1.1 Introduction

This thesis is a study of the Yang-Mills flow, the L^2-gradient flow of the Yang-Mills functional; and in particular its convergence properties at infinity. The flow is (after imposing the Coulomb gauge condition) a parabolic equation for a connection on a holomorphic vector bundle. Very soon after the introduction of the flow equations, Donaldson proved that in the case of a stable bundle, the gradient flow converges smoothly at infinity. In the unstable case the behaviour of the flow is more ambiguous. Nevertheless, even in the general case there is an appropriate notion of convergence (a version of Uhlenbeck's compactness theorem) that is always satisfied. The goal of this thesis is to prove that this notion depends only on the holomorphic structure of the original bundle.

We follow up on work whose origin lies in two principal directions, both related to stability properties of holomorphic vector bundles over compact Kähler manifolds. The first strain is the seminal work of Atiyah and Bott [AB], in which the authors study the moduli space of stable holomorphic bundles over Riemann surfaces. In particular, they compute the $\mathcal{G}^{\mathbb{C}}$-equivariant Betti numbers of this space in certain cases, where $\mathcal{G}^{\mathbb{C}}$ is the complex gauge group of a holomorphic vector bundle E (over a

Riemann surface X) acting on the space $\mathcal{A}_{\text{hol}}$ of holomorphic structures of E. Their approach was to stratify $\mathcal{A}_{\text{hol}}$ by Harder-Narasimhan type. The type is a tuple of rational numbers $\mu = (\mu_1, ..., \mu_R)$ associated to a holomorphic structure $(E, \bar{\partial}_E)$, defined using a filtration of E by analytic subsheaves whose successive quotients are semi-stable, called the Harder-Narasimhan filtration. One of the resulting strata of $\mathcal{A}_{\text{hol}}$ consists of the semi-stable bundles. Furthermore the action of $\mathcal{G}^{\mathbb{C}}$ preserves the stratification, and the main result that yields the computation of the equivariant Betti numbers is that the stratification by Harder-Narasimhan type is equivariantly perfect under this action.

Atiyah and Bott also noticed that the problem might be amenable to a more analytic approach. Specifically they considered the Yang-Mills functional YM on the space $\mathcal{A}_h$ of integrable, unitary connections with respect to a fixed hermitan metric on E. The space $\mathcal{A}_h$ may be identified with $\mathcal{A}_{\text{hol}}$ by sending a connection ∇_A to its $(0,1)$ part $\bar{\partial}_A$. The Yang-Mills functional is defined by taking the L^2 norm of ∇_A, and is a Morse function on $\mathcal{A}_h$. Therefore this functional induces the usual stable-unstable manifold stratification on $\mathcal{A}_h$ (or equivalently $\mathcal{A}_{\text{hol}}$) familiar from Morse theory. It is natural to conjecture that this analytic stratification is in fact the same as the algebraic stratification given by the Harder-Narasimhan type. The authors of [AB] stopped short of proving this statement, instead leaving it at the conjectural level, and working directly with the algebraic stratification. They noted however that a key technical point in proving the equivalence was to show the convergence of the gradient flow of the Yang-Mills functional at infinity. This was proven in [D] by Daskalopoulos (see also [R]). Specifically, in the case of Riemann surfaces, Daskalopoulos showed the asymptotic convergence of the Yang-Mills Flow, that there is indeed a well-defined stratification in the sense of Morse theory in this case, and that it coincides with the algebraic stratification (which makes sense in all dimensions).

When (X, ω) is a higher dimensional Kähler manifold, the Yang-Mills flow fails to converge in the usual sense. This brings us to the second strain of ideas of which the present paper is a continuation: the so called "Kobayashi-Hitchin correspondences". These are statements (in various levels of generality) relating the existence of Hermitian-Einstein metrics on a holomorphic bundle E, to the stability of E. Namely, E admits an Hermitian-Einstein metric if and only if E is polystable. This was originally proven in [DO1] by Donaldson, for algebraic surfaces. The idea of the proof was to reformulate the flow as an equivalent parabolic PDE, show long-time existence of the equation, and then prove that for a stable bundle, this modified flow indeed converges, the solution being the desired Hermitan-Einstein metric. This was generalised by Donaldson to higher dimensions in the algebraic case in [DO2] and by Uhlenbeck and Yau in [UY] in the case of a compact Kähler manifold. Finally, in [BS], Bando and Siu extended the correspondence to coherent analytic sheaves on Kähler manifolds by considering what they called "admissible" hermitian metrics, which are metrics on the locally free part of the sheaf having controlled curvature. They also conjectured that there should also be a correspondence (albeit far less detailed) between the Yang-Mills flow and the Harder-Narasimhan filtration in higher dimensions despite the absence of a Morse theory for the Yang-Mills functional.

There are two main features that distinguish the higher dimensional case from the case of Riemann surfaces. As previously mentioned, the flow does not converge in general. However, the only obstruction to convergence is bubbling phenomena. Specifically, one of Uhlenbeck's compactness results applies to the flow, which means that there are always subsequences that converge (in a certain Sobolev norm) away from a singular set of Hausdorff codimension 4 inside X (which we will denote by $Z_{\rm an}$), to a connection on a possibly different vector bundle E_∞. A priori, this pair of a limiting connection and bundle depends on the subsequence. In the case of two complex dimensions, the singular set is a locally finite set of points (finite in the compact case) and by Uhlenbeck's removable singularities theorem E_∞ extends over the singular set as a vector bundle with a Yang-Mills connection. In higher dimensions, again due to a result of Bando and Siu, E_∞ extends over the singular set, but only as a reflexive sheaf. Although we will not use their result, Hong and Tian have proven in [HT] that in fact the convergence is in C^∞ and that $Z_{\rm an}$ is a holomorphic subvariety.

A separate, but intimately related issue is the Harder-Narasimhan filtration. In the case of a Riemann surface the filtration is given by subbundles. In higher dimensions, it is only a filtration by subsheaves. Again however, away from a singular set $Z_{\rm alg}$, which is a complex analytic subset of X of complex codimension 2, the filtration is indeed given by subbundles. Once more, in the case of a Kahler surface this is a locally finite set of points (finite in the compact case).

The main result of this thesis (the conjecture of Bando and Siu), describes the relationship between the analytic and algebraic sides of the above picture. To state it, we recall that there is a refinement of the Harder-Narasimhan filtration called the Harder-Narasimhan-Seshadri filtration, which is a double filtration whose successive quotients are stable rather than merely semi-stable. Then if $(E, \bar{\partial}_E)$ is a holomorphic vector bundle where the operator $\bar{\partial}_E$ denotes the holomorphic structure, write $Gr_\omega^{HNS}(E, \bar{\partial}_E)$ for the associated graded object (the direct sum of the stable quotients) of the Harder-Narasimhan-Seshadri filtration. Notice that by the Kobayashi-Hitchin correspondence, $Gr_\omega^{HNS}(E, \bar{\partial}_E)$ also carries a natural Yang-Mills connection on its locally free part, given by the direct sum of the Hermitian-Einstein connections on each of the stable factors. The main theorem says in particular that the limiting bundle along the flow is in fact independent of the the subsequence chosen in order to employ Uhlenbeck compactness, and is determined entirely by the holomorphic structure $\bar{\partial}_E$ of E. Furthermore, the limiting connection is precisely the connection on $Gr_\omega^{HNS}(E, \bar{\partial}_E)$.

Theorem 1 *Let (X, ω) be a compact Kähler manifold, and $E \to X$ an hermitian vector bundle. Let A_0 denote an integrable, unitary connection endowing E with a holomorphic structure $\bar{\partial}_E = \bar{\partial}_{A_0}$. Let A_∞ denote the Yang-Mills connection on $Gr_\omega^{HNS}(E, \bar{\partial}_E)$ restricted to $X - Z_{\mathrm{alg}}$ induced from the Kobayashi-Hitchin correspondence. Let A_t be the time t solution of the flow with initial condition A_0. Then as $t \to \infty$, $A_t \to A_\infty$ in the sense of Uhlenbeck, and on $X - Z_{\mathrm{alg}} \cup Z_{\mathrm{an}}$, the vector bundles $Gr_\omega^{HNS}(E, \bar{\partial}_E)$ and the limiting bundle E_∞ are holomorpically isomorphic. Moreover, E_∞ extends over Z_{an} as a reflexive sheaf to $\left(Gr_\omega^{HNS}(E, \bar{\partial}_E)\right)^{**}$.*

This theorem was proven in [DW1] by Daskalopoulos and Wentworth in the case when $\dim X = 2$. In this case, the filtration consists of vector bundles, whose successive quotients may have point singularities. As stated earlier, this means E_∞ extends as a vector bundle and [DW1] proves that this bundle is isomorphic to the vector bundle $\left(Gr_\omega^{HNS}(E, \bar{\partial}_E)\right)^{**}$.

We now give an overview of the thesis and in particular our proof of Theorem 1, pointing out what goes through directly from [DW1] and where we require new arguments. The remainder of Chapter 1 consists of all the various background topics we will need to employ in the proof of Theorem 1 and consists of no original material. We begin by giving basic definitions in complex geometry, describing the space of integrable unitary connections on a holomorphic vector bundle, and give the equivalence of this space with the space of holomorphic structures. Then we discuss the Yang-Mills functional, the Hermitian-Yang-Mills functional and the Yang-Mills flow and their basic properties. In particular, we prove short-time existence via a standard gauge-fixing trick, showing the equivalence of the Yang-Mills flow with a certain flow of metrics and sketch the proof of long-time existence on a Kahler manifold due to Donaldson and Simpson. We then state Simpson's version of the fact that for a stable bundle the heat flow converges to an Hermitian-Einstein metric.

Next we give basic definitions from sheaf theory, including the Harder-Narasimhan and Harder-Narasimhan-Seshadri filtrations and their associated graded objects, as well as the corresponding types for future use. We also prove a few basic results about these filtrations for later use. We also introduce the weakly holomorphic projection operators for a saturated subsheaf due to Uhlenbeck and Yau, and recall the proof of a lemma on the boundedness of second fundamental forms from [DW1].

We finish Chapter 1 by tying up some loose ends. We state two versions of the Uhlenbeck compactness result that we will need. Although we will primarily be concerned with the flow, the proof of Theorem 1 is set up to work for slightly more general sequences of connections, so we state the compactness theorem in this generality first, and then specialise to the flow. We state the removable singularities theorem of Bando and Siu [BS] and discuss Kahler metrics on a resolution of singularities, a topic that will be central to the proofs of the original results in this thesis. We also give a discussion of the proof of the main result of [BS]. We recall the proof of one of the main theorems of [DW1], that the Harder-Narasimhan-Seshadri type of an Uhlenbeck limit is bounded from below by the type of the initial bundle with respect to the partial ordering on types. Chapter 1 ends with a discussion of Yang-Mills type functionals associated to ad-invariant convex functions on the lie algebra of the unitary group.

Chapter 2 is the technical heart of the proof. It begins by detailing the main results we will need about resolution of singularities. This is the first place in which our presentation differs fundamentally from that of [DW1]. The main strategy of the proof is to eliminate the singular set of the Harder-Narasimhan-Seshadri filtration by blowing up, and doing all the necessary analysis on the blowup. In the two-dimensional case, since the singularities consist only of points, this can be done directly by hand as in [DW1] see also [BU1]. In the general case we must appeal to the resolution of singularities theorem of Hironaka see [H1] and [H2]. We consider the filtration as a rational section of a flag bundle, and apply the resolution of indeterminacy theorem for rational maps. If we write $\pi : \tilde{X} \to X$

for the composition of the blowups involved in resolution, the result of is that the pullback bundle $\pi^* E \to \tilde{X}$ has a filtration by subbundles, which away from the exceptional divisor $\mathbf{E}$ is precisely the filtration on X.

We will need to consider a natural family of Kähler metrics ω_ε on $\tilde{X}$, which are perturbations of the pullback form $\pi^*\omega$ by the irreducible components of the exceptional divisor, and which are introduced in order to compensate for the fact that $\pi^*\omega$ fails to be a metric on $\mathbf{E}$. The filtration of π^*E by subbundles is not quite the Harder-Narasimhan-Seshadri filtration with respect to ω_ε but is closely related. In particular, the main result of this section is that the Harder-Narasimhan-Seshadri type of π^*E with respect to ω_ε converges to the type of E with respect to ω. This was proven in the surface case in [DW1] using an argument of Buchdahl from [BU1]. The proof contained in [DW1] seems to be insufficient in the higher dimensional case, so we give a rather different proof of this result. The main ingredient is a bound on the ω_ε degree of a subsheaf of π^*E with torsion-free quotient in terms of its pushforward sheaf that is uniform as $\varepsilon \to 0$. To prove this we use standard algebro-geometric facts together with a modification of an argument of Kobayashi [KOB] first used to prove the uniform boundedness of the degree of subsheaves of a vector bundle with respect to a fixed Kähler metric. In particular we prove the following theorem:

Theorem 2 *Let (X, ω) be a compact Kähler manifold and $\tilde{S}$ be a subsheaf (with torsion free quotient $\tilde{Q}$) of a holomorphic vector bundle $\tilde{E}$ on $\tilde{X}$, where $\pi : \tilde{X} \to X$ is given by a sequence of blowups along complex submanifolds of* codim ≥ 2. *Then then there is a uniform constant M such that the degrees of $\tilde{S}$ and $\tilde{Q}$ with respect to*

ω_ε satisfy: $\deg(\tilde{S}, \omega_\varepsilon) \leq \deg(\pi_* \tilde{S}) + \varepsilon M$*, and* $\deg(\tilde{Q}, \omega_\varepsilon) \geq \deg(\pi_* \tilde{Q}) - \varepsilon M$*.*

Similar statements are proven in the case of a surface by Buchdahl [BU1] and for projective manifolds by Daskalopoulos and Wentworth see [DW3].

An essential fact needed to complete the proof of Theorem 1 is that the Harder-Narasimhan-Seshadri type of the limiting sheaf is in fact equal to the type of the initial bundle. This fact seems to be closely related to the existence of what is called an L^p-approximate critical hermitian structure. In rough terms this is an hermitian metric on a holomorphic vector bundle whose Hermitian-Einstein tensor is L^p-close to that of a Yang-Mills connection (a critical value) determined by the Harder-Narasimhan-Seshadri type of the bundle (see Definition 6). Since any connection on E has Hermitian-Yang-Mills energy bounded below by the type of E, and we have a monotonicity property along the flow, the result of section 3 implies that the existence of an approximate structure then ensures that the flow starting from this initial condition realises the correct type in the limit. Then one shows that *any* initial condition flows to the correct type, essentially by proving that the set of such metrics is open and closed (and non-empty by the existence of an approximate structure) in the space of smooth metrics, and applying the connectivity of the latter space. This last argument appears in detail in [DW1], but we repeat the argument here for completeness. The main theorem of Chapter 2 is the following:

Theorem 3 *Let $E \to X$ be a holomorphic vector bundle of over a Kähler manifold with Kähler form ω. Then given $\delta > 0$ and any $1 \leq p < \infty$, E has an L^p δ-approximate critical hermitian structure.*

The method does not extend to $p = \infty$. This is straightforward in the case when the filtration is given by subbundles (even for $p = \infty$). Given an exact sequence of holomorphic vector bundles:

$$0 \longrightarrow S \longrightarrow E \longrightarrow Q \longrightarrow 0$$

and hermitian metrics on S and Q, one can scale the second fundamental form $\beta \mapsto t\beta$ to obtain an isomorphic bundle whose Hermitian-Einstein tensor is close to the direct sum of those of S and Q. In general it seems difficult to do this directly. The problem here is that the filtration is not in general given by subbundles, and so the vast majority chapter is an argument needed to address this point. This is precisely where we need the resolution of the filtration obtained earlier. We first take the direct sum of the Hermitian Einstein metrics on the stable quotients in the resolution by subbundles, which sits inside the pullback π^*E under the blowup map $\pi : \tilde{X} \to X$. Then the argument above shows that after modifying this metric by a gauge transformation, its Hermitian-Einstein tensor becomes close to the type in the L^p norm. We complete the proof by pushing this metric down to $E \to X$ using a cutoff argument.

In broad outline our discussion follows the ideas in [DW1] but we point out two things. First of all, since we are varying the Kähler metric on $\tilde{X}$ by a parameter ε, one has to fix a value ε_1 and consider stable quotients with respect to this metric. Therefore in order to show that the metric on the blowup is L^p-close, one also needs some sort of uniform control over the Hermitian-Einstein tensor as $\varepsilon \to 0$. The author has noticed an error in [DW1] on this point. In particular, *Lemma*

3.14 is slightly incorrect. Instead, the right hand side should have an additional term involving the L^2 norm of the full curvature. This does not essentially disrupt the proof, because the Yang-Mills and Hermitian-Yang-Mills functionals differ only by a topological term, but it has the effect of changing the logic of the argument somewhat, as well as increasing the technical complexity.

Secondly, the authors of [DW1] were able to rely on the fact that the singular set was given by points when applying the cutoff argument, in particular they knew that there were uniform bounds on the derivatives of the cutoff function. We must allow for the fact that the singular set is higher dimensional, and therefore need to replace their arguments involving coverings of the singular set by disjoint balls of arbitrarily small radius by calculations in a tubular neighbourhood. We first assume Z_{alg} is smooth and that blowing up once along Z_{alg} resolves the singularities. The essential point is that the Hausdorff codimension of Z_{alg} is large enough to allow the arguments of [DW1] to go through in this case. We then reduce the general theorem to this case by applying an inductive argument on the number of blowups required to resolve the filtration. It is here that we crucially use the convergence of the Harder-Narasimhan-Seshadri type.

In Chapter 3, following Bando and Siu, we introduce a degenerate Yang-Mills flow on the composition of blowups $\tilde{X}$ with respect to the degenerate metric $\pi^*\omega$. We review some basic properties of this flow that are necessary for the proof of Theorem 1. In particular we show that a solution of this degenerate flow is in fact an hermitian metric, and solves the ordinary flow equations with respect to the metric $\pi^*\omega$ away from the exceptional divisor $\mathbf{E}$.

The remainder of Chapter 3 completes the proof of the Theorem 1 by showing the isomorphism of the limit E_∞ with $\left(Gr_\omega^{HNS}(E,\bar{\partial}_E)\right)^{**}$. The basic idea follows that of [DW1] which in turn is a generalisation of the argument of Donaldson in [DO1]. His idea is to construct a non-zero holomorphic map to the limiting bundle as the limit of the sequence of gauge transformations defined by the flow. In the case that the initial bundle is stable and has stable image, one may apply the basic fact that such a map is always an isomorphism. In general, the idea in [DW1] is simply to apply this argument to the first factor of the associated graded object (which is stable) and then perform an induction. The image of the first factor will be stable because of the result in *Chapter* 2 about the type of the limiting sheaf. The difficulty with this method is in proving that the limiting map is in fact non-zero. This follows directly from Donaldson's proof in the case of a single subsheaf, but it is more complicated to construct such a map on the entire filtration. The authors of [DW1] avoid applying Donaldson's method directly by appealing to a complex analytic argument involving analytic extension see also [BU2]. Arguing in this fashion makes the induction rather easier. However, this requires the complement of the singular set to have strictly pseudo-concave boundary, which is true in the case of surfaces, but is not guaranteed in higher dimensions.

Therefore we give a proof of a slightly more differential geometric character. Namely, in the case that the filtration is given by subbundles, we follow the argument of Donaldson, which goes through with modest corrections in higher dimensions, and does indeed suffice to complete the induction alluded to. In the general case, we must again appeal to a resolution of singularities of the filtration and apply the previous strategy to the pullback bundle over the composition of blowups $\tilde{X}$. The problem one encounters with this approach is that the induction breaks down due to the appearance of second fundamental forms of each piece of the filtration, which are not bounded in L^∞ with respect to the degenerate metric $\pi^*\omega$. To rectify this, we apply the degenerate flow for some fixed non-zero time t to each element of the sequence of connections, and this new sequence does have the desired bound. This is due to the key observation of Bando and Siu that the Sobolev constant of $\tilde{X}$ with respect to the metrics ω_ε is bounded away from zero. A theorem of Cheng and Li then implies uniform control over subsolutions to the heat equation, which is sufficient to understand the degenerate flow. One then has to show that the limit obtained from this new sequence of connections is independent of t and is the correct one. This is an expanded and slightly modified account of an argument contained in the unpublished preprint [DW3].

We conclude the introduction with some general comments. First of all, as pointed out in [DW1], the proof of Theorem 1 is essentially independent of the flow, and one obtains a similar theorem by restricting to sequences of connections which are minimising with respect to certain Hermitian-Yang-Mills type functionals. Indeed, the statement appears explicitly as Theorem 15. Secondly, one expects that there should be a relationship between the two singular sets Z_{alg} and Z_{an}. Namely, in the best case Z_{alg} should be exactly the set of points where bubbling occurs. One always has containment $Z_{\mathrm{alg}} \subset Z_{\mathrm{an}}$, and in the separate article [DW2] Daskalopoulos and Wentworth have shown that in the surface case equality does in fact hold. We hope to be able to clarify this issue in higher dimensions in a future paper.

Finally, the author is aware of a recent series of preprints [J1],[J2],[J3] by Adam Jacob which collectively give a proof of Theorem 1 using different methods.

1.2 Kähler Manifolds, The Space of Holomorphic Structures, Hermitian Einstein Metrics and Connections, and the Yang-Mills Functional

1.2.1 Kahler Manifolds

The setting for this thesis will be a compact Kähler manifold (X, ω). That is, a complex manifold X, equipped with a Kähler form ω. We briefly explain the terminology. We assume that the real tangent bundle of X is equipped with an Hermitian metric g (i.e. a Riemannian metric such that $g(JX, JY) = g(X, Y)$ for every pair of tangent vectors X, Y where J is the almost complex structure on X, also called a compatible Riemannian metric) and ω is defined to be the two form given by $\omega(X, Y) = g(JX, Y)$. Note that the Riemannian metric extends $\mathbb{C}$-linearly to the complexification $TM \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X$, where the two direct summands are the the $\pm i$ eigenspaces of J.

By compatibility g restricts to be zero on each summand, so the only relevant data is the Hermitian matrix given by:

$$g_{i\bar{j}} = g(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}).$$

Then one checks that:

$$g = g_{i\bar{j}} dz^i \otimes d\bar{z}^j + g_{\bar{i}j} d\bar{z}^i \otimes dz^j$$

and if we consider the term:

$$g_{herm} = g_{i\bar{j}} dz^i \otimes d\bar{z}^j$$

then this gives a metric which is Hermitian on the fibres of the holomorphic tangent bundle $T^{1,0}X$ (more correctly we see that $\langle X, Y \rangle = g_{herm}(X, \bar{Y})$ is Hermitian on the fibres of $T^{1,0}$) and one computes that $\operatorname{Re} g_{herm} = \frac{1}{2} g$, where now g is the original Riemannian metric. In local coordinates the form ω can be written:

$$\omega = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

and the identity:

$$\operatorname{Im} g_{herm} = -\omega$$

may be shown to hold, so that we have:

$$g_{herm} = \frac{1}{2} g - \sqrt{-1}\omega.$$

Sometimes g_{herm} (which from here on out we will simply write as g) is called the Hermitian metric on X. This is consistent with the terminology (to be introduced below) for Hermitian metrics on vector bundles. If the two form ω happens to be closed, then we say that ω is a **Kähler form** and we say that the metric g is a **Kähler metric**.

1.2.2 Holomorphic Vector Bundles, Hermitian-Einstein Metrics and Connections, and the Yang-Mills Functional

Many arguments in this thesis will rely on the interplay between two different types of structure on a C^∞ $\mathbb{C}$-vector bundle $E \longrightarrow X$. The first is that of a holomorphic structure on E. One standard definition of this is a choice of local trivialisations such the transition maps are holomorphic. However, more useful for us will be the following. A **holomorphic structure** on E is a map $\bar{\partial}_E : \Gamma(E) \longrightarrow \Omega^{0,1}(E)$ that satisfies the Liebniz rule:

$$\bar{\partial}_E(f\sigma) = \bar{\partial} f \otimes \sigma + f\bar{\partial}_E\sigma$$

and the integrability condition $\bar{\partial}_E^2 = 0$. It can be shown (see [KOB] Chapter 1) that every such operator defines a unique holomorphic structure on E such that $\bar{\partial}_E = \bar{\partial}$. We will write $\mathcal{A}_{hol}$ for the set of holomorphic structures (suppressing the notation for E).

We now consider the group $\mathcal{G}^{\mathbb{C}}$ of smooth automorphisms of E that are complex linear on the fibres of E (sometimes this is written $GL(E)$). Then the group acts on $\mathcal{A}_{hol}$ by conjugation:

$$\bar{\partial}_E \longrightarrow g^{-1} \circ \bar{\partial}_E \circ g.$$

If we act on a section σ :

$$\begin{aligned} g^{-1} \circ \bar{\partial}_E \circ g(\sigma) &= g^{-1}(\bar{\partial}_E(g(\sigma)) \\ &= g^{-1}(\bar{\partial}_{\mathrm{End}\,E}(g)(\sigma) + g(\bar{\partial}_E\sigma) \end{aligned}$$

where we have used the expression:

$$\bar{\partial}_{\mathrm{End}\,E}(g)(\sigma) = \bar{\partial}_E(g(\sigma)) - g(\bar{\partial}_E\sigma).$$

This explains the notation:

$$\bar{\partial}_E \longrightarrow g^{-1} \circ \bar{\partial}_E \circ g = \bar{\partial}_E + g^{-1}\bar{\partial} g.$$

The quotient $\mathcal{M}_{hol} = \mathcal{A}_{hol}/\mathcal{G}^{\mathbb{C}}$ is the moduli space of holomorphic structures on E. Two holomorphic structures are considered to be equivalent if they lie in the same $\mathcal{G}^{\mathbb{C}}$ orbit.

The second type of structure is an **Hermitian metric** h on E, which is simply a smoothly varying choice of a positive definite Hermitian form on the fibres of E. Then if h is an Hermitian metric on E, we will write $\mathcal{G}$ for the subgroup of $\mathcal{G}^{\mathbb{C}}$ consisting of unitary automorphisms of (E, h), that is, elements for which $g^* g = id$ (g^* will denote the conjugate transpose). We will write $\mathcal{A}_h$ for the set of connections on E preserving the Hermitian metric, i.e. connections ∇ for which:

$$d(h(s,t)) = h(\nabla s, t) + h(s, \nabla t).$$

Here we extend the metric h to 1-forms with values in E simply by ignoring the 1-form component, so that the right hand side is indeed a 1-form on X. Write $\mathcal{A}_{\bar{\partial}}$ for the space of $\bar{\partial}_E$ operators (not necessarily integrable). Then note that the map:

$$\mathcal{A}_h \longrightarrow \mathcal{A}_{\bar{\partial}}, \ \nabla_A \longrightarrow \bar{\partial}_A$$

gives a bijection. In fact, given $\bar{\partial}_E$, the $(1,0)$ part ∂_A of the connection is determined by the relation:

$$\bar{\partial}(h(s,t)) = h(\bar{\partial}_E s, t) + h(s, \partial_A t).$$

Therefore $\nabla_A = \partial_A + \bar{\partial}_E$ is in $\mathcal{A}_h$. Now consider the set of integrable unitary connections, i.e. those with $\bar{\partial}_A^2 = 0$ or equivalently those with $(1,1)$ curvature (i.e. their curvature satisfies $F_A^{0,2} = 0$). We will write $\mathcal{A}_h^{1,1}$ for this set. If we use an element $\nabla_A \in \mathcal{A}_h^{1,1}$ to define a holomorphic structure $\bar{\partial}_A = \bar{\partial}$ on E, then ∇_A is the unique integrable, unitary connection for this holomorphic structure. In other words ∇_A is the **Chern connection** for $\bar{\partial}_A$. The connection 1-form and curvature for ∇_A may be written in a local holomorphic frame (with respect to $\bar{\partial}_A$) as:

$$A = \bar{h}^{-1}\partial\bar{h}, \;\; F_A = \bar{\partial}(\bar{h}^{-1}\partial\bar{h}).$$

Conversely, if we fix a holomorphic structure $\bar{\partial}_E \in \mathcal{A}_{hol}$, then the corresponding Chern connection defines an element of $\mathcal{A}_h^{1,1}$ so we obtain a further bijection:

$$\mathcal{A}_{hol} \longleftrightarrow \mathcal{A}_h^{1,1}.$$

Throughtout this thesis, for a fixed holomorphic structure and Hermitian metric we will denote by $(\bar{\partial}_E, h)$ the Chern connection associated to the pair of structures on E.

Note that $\mathcal{G}$ acts on $\mathcal{A}_h$ by conjugation:

$$\nabla \longrightarrow g^{-1} \circ \nabla \circ g = \nabla + g^{-1}\nabla g.$$

The corresponding action on the curvature is given by:

$$g \cdot F_\nabla = g^{-1} \circ F_\nabla \circ g$$

and so the subspace $\mathcal{A}_h^{1,1}$ is preserved under the action of $\mathcal{G}$. By the correspondence above, the action of $\mathcal{G}^{\mathbb{C}}$ on $\mathcal{A}_{hol}$ induces an action of $\mathcal{G}^{\mathbb{C}}$ on $\mathcal{A}_h^{1,1}$. To write this action down explicitly, we put $g \cdot \nabla_A = \partial_{A'} + \bar{\partial}_{A'}$. Since $\mathcal{A}_{hol} \longleftrightarrow \mathcal{A}_h^{1,1}$ is given by $\nabla_A \longrightarrow \bar{\partial}_A$, we have $\bar{\partial}_{A'} = g^{-1} \circ \bar{\partial}_A \circ g$. Now, writing g^* for the adjoint of g we compute:

$$
\begin{aligned}
h(s, \partial_{A'} g^* t) &= \bar{\partial}(h(s, g^* t)) - h(\bar{\partial}_{A'} s, g^* t) \\
&= \bar{\partial}(h(gs, t)) - h(\bar{\partial}_A(gs), t) \\
&= h(gs, \partial_A t) = h(s, g^* \partial_A t).
\end{aligned}
$$

It now follows that $\partial_{A'} = g^* \circ \partial_A \circ g^{*-1}$ so that the action of $\mathcal{G}^{\mathbb{C}}$ on $\mathcal{A}_h^{1,1}$ is given by:

$$
\nabla_A \longrightarrow g^* \circ \partial_A \circ g^{*-1} + g^{-1} \circ \bar{\partial}_A \circ g.
$$

Note that in case $g \in \mathcal{G}$, then $g^* = g^{-1}$ and this action agrees with the action of $\mathcal{G}$ on $\mathcal{A}_h^{1,1}$ previously mentioned. We will write:

$$
\mathcal{B}_h = \mathcal{A}_h / \mathcal{G} \ , \ \mathcal{B}_h^{1,1} = \mathcal{A}_h^{1,1} / \mathcal{G}, \mathcal{M}_h = \mathcal{A}_h / \mathcal{G}^{\mathbb{C}}, \mathcal{M}_h^{1,1} = \mathcal{A}_h^{1,1} / \mathcal{G}^{\mathbb{C}}
$$

for the respective quotient spaces. Note that we have a bijection:

$$
\mathcal{M}_h^{1,1} \simeq \mathcal{M}_{hol}.
$$

Moreover, $\mathcal{G}^{\mathbb{C}}$ also acts on the space of Hermitian metrics via $h \mapsto g \cdot h$ where $g \cdot h(s_1, s_2) = h(g(s_1), g(s_2))$. In matrix form this reads $g \cdot h = g^* h g$ where g^* is the conjugate transpose. Note that the action of $\mathcal{G}^{\mathbb{C}}$ on the space $Herm^+(E)$ extends to an action on the space $Herm(E)$ of all Hermitian forms on E, and is transitive on $Herm^+(E)$. Furthermore, the isotropy subgroup at the identity is clearly the unitary gauge group $\mathcal{G}$. Therefore we have the identification:

$$
Herm^+(E) \simeq \mathcal{G}^{\mathbb{C}} / \mathcal{G}.
$$

Now, starting from a holomorphic bundle E with Hermitian metric h and Chern connection $(\bar{\partial}_E, h)$, we may use a complex gauge transformation to perturb this connection in two different ways. We may either let g act on $\bar{\partial}_E$ or on h. A calculation relates the curvatures of the corresponding connections:

$$F_{(g\cdot\bar{\partial}_E,h)} = g^{-1} \circ F_{(\bar{\partial}_E,g\cdot h)} \circ g.$$

If we denote by $\mathfrak{u}(E) \subset End(E)$ the subbundle of skew-hermitian endomorphisms (i.e. the Lie algebra of $\mathcal{G}$), then for a section σ of $\mathfrak{u}(E)$, we will write $|\sigma|$ for its pointwise norm. This is defined as usual by

$$|\sigma| = \left(\sum_{i=1}^{R} |\lambda_i|^2 \right)^{\frac{1}{2}}$$

where the λ_i are the eigenvalues of σ at a given point and R is the rank of E. Combining this with the usual pointwise norm on 2-forms, we obtain a pointwise (Hilbert-Schmidt) norm on the curvature F_A of a connection. Now we may define the **Yang-Mills functional** (YM functional) by:

$$YM(\nabla_A) = \int_X |F_A|^2 \, dvol.$$

If we assume that X is Kähler, we have:

$$YM(\nabla_A) = \int_X |F_A|^2 \frac{\omega^n}{n!}.$$

This functional is invariant under the action of $\mathcal{G}$ and so defines a map $YM : \mathcal{B}_h \rightarrow \mathbb{R}$. Its critical points are the so called **Yang-Mills connections** and by computing the first variation of YM one sees that they satisfy the Euler-Lagrange equations: $d_A^* F_A = 0$, where d_A is the covariant derivative induced on $End(E)$ valued 2 forms by ∇_A. If $\nabla_A \in \mathcal{A}_h^{1,1}$ then we may also define the **Hermitian-Yang-Mills functional**:

$$HYM(\nabla_A) = \int_X |\Lambda_\omega F_A|^2 \frac{\omega^n}{n!},$$

where Λ_ω denotes contraction with the Kähler form. This is the formal adjoint of the Lefshetz operator $\wedge\omega$ obtained by wedging with ω. In local coordinates one sees that for a $(1,1)$ form $G = G_{i,\bar{j}} dz^i \wedge d\bar{z}^j$ we have:

$$\Lambda_\omega G = g^{i,\bar{j}} G_{i,\bar{j}}.$$

The quantity $\Lambda_\omega F_A$ is called the **Hermitian-Einstein tensor** of A.

Again, HYM is invariant under the action of $\mathcal{G}$ and so defines a functional $HYM : \mathcal{B}_h^{1,1} \to \mathbb{R}$. Critical points of the functional satisfy the Euler-Lagrange equations: $d_A \Lambda_\omega F_A = 0$. On the other hand, just as in the preceding discussion, we may regard the holomorphic stucture as being fixed and consider the space of $(1,1)$ connections as being the set of pairs $(\bar{\partial}_E, h)$ where h varies over all Hermitian metrics. We may therefore think of HYM as a functional $HYM(h) = HYM(\bar{\partial}_E, h)$ on the space of Hermitian metrics on E. A critical metric of HYM is referred to a **critical Hermitian structure** on $(E, \bar{\partial})$.

An important fact that we will use is that when X is compact, there is a relation between the two functionals YM and HYM. Explicitly:

$$YM(\nabla_A) = HYM(\nabla_A) + \frac{4\pi^2}{(n-2)!} \int_X \big(2c_2(E) - c_1^2(E)\big) \wedge \omega^{n-2}$$

for any $A \in \mathcal{A}_h^{1,1}$. The second term depends only on the topology of E and the form ω, so YM and HYM have the same critical points on $\mathcal{A}_h^{1,1}$. Furthermore, ∇_A is a critical point of YM and HYM, iff and only if h is a critical hermitian structure for the holomorphic stucture on E given by A.

On Kahler manifolds, Yang-Mills connections have a very special property.

The Kahler identities together with the Yang-Mills condition give:

$$\begin{aligned} 0 &= d_A^* F_A = -i(\bar{\partial}_A - \partial_A)\Lambda_\omega F_A = 0 \\ &\iff d_A \Lambda_\omega F_A = 0 \end{aligned}$$

(this is another way of seeing that YM and HYM have the same critical points). Therefore the eigenspaces of the Hermitian-Einstein tensor of a Yang-Mills connection are constant, so we have the following proposition.

Proposition 1 *Let $\nabla_A \in \mathcal{A}_h^{1,1}$ be a Yang-Mills connection on an hermitian vector bundle (E, h) over a Kähler manifold X. Then $\nabla_A = \oplus_{i=1}^l \nabla_{A_i}$ where $E = \oplus_{i=1}^l Q_i$ is an orthogonal splitting of E, and where $\sqrt{-1}\Lambda_\omega F_{A_i} = \lambda_i \mathbf{Id}_{Q_i}$, where λ_i are constant. If X is compact, then $\lambda_i = \mu(Q_i)$.*

Proof. Let $\nabla_A \in \mathcal{A}_h^{1,1}$ be a YM connection. By the YM equations and the Kahler identities:

$$\sqrt{-1}\left(\partial_A - \bar{\partial}_A\right)\Lambda_\omega F_A = d_A^* F_A = 0.$$

Therefore, decomposing into types, we have $\partial_A \Lambda_\omega F_A = \bar{\partial}_A \Lambda_\omega F_A = d_A \Lambda_\omega F_A = 0$. Then this implies that the eigenvalues are constant, and so we may decompose E into its eigenbundles Q_i. By construction, if we let ∇_{A_i} be the restriction of ∇_A to Q_i, then ∇_{A_i} is Hermitian-Einstein. In the case where X is compact, Chern-Weil theory computes the Hermitian-Einstein constants explicitly in terms of the slopes to be:

$$\lambda_i = 2\pi n \mu(Q_i) / \int_X \omega^n,$$

and since we have normalised the volume to be $2\pi/(n-1)!$ the result follows. ■

The following definition is now natural.

Definition 1 *Let $E \to (X, \omega)$ be a holomorphic bundle. Then a connection ∇_A such that there exists a constant λ with:*

$$\sqrt{-1}\Lambda_\omega F_A = \lambda \mathbf{Id}_E$$

*is called an **Hermitian-Einstein connection**. If A is the Chern connection of $(\bar{\partial}_E, h)$ for some hermitian metric h, then h is called an **Hermitian-Einstein metric**.*

There is a topological lower bound for the functional HYM depending only on the first Chern class of E and the cohomology class of ω. This bound is realised precisely for connections (metrics) that are Hermitian-Einstein. In other words, Hermitian-Einstein connections (metrics) are the absolute minima of the functional HYM.

1.3 The Yang-Mills Flow and Basic Properties

1.3.1 Yang-Mills and Hermitian Yang-Mills Flow Equations: Equivalence up to Gauge

Throughout this section, we follow the reference [WIL]. As stated in the introduction, although many of our arguments are valid for minimising sequences of unitary connections, our primary interest will be in sequences obtained from the **Yang-Mills flow**. This is a one parameter family of integrable unitary connections

A_t obtained as solutions of the L^2-gradient flow equations for the YM functional. Explicitly:

$$\frac{\partial A_t}{\partial t} = -d^*_{A_t} F_{A_t}, \quad A_0 \in \mathcal{A}^{1,1}_h.$$

We will eventually sketch a proof of the fact, following the references [DO1] and [SI] that the above equations have a unique solution in $\mathcal{A}^{1,1}_h \times [0, \infty)$. Moreover, the flow preserves complex gauge orbits, that is, A_t lies in the orbit $\mathcal{G}^{\mathbb{C}} \cdot A_0$. This may be seen as follows. Instead of solving for the connection, fix A_0 so that $\bar{\partial}_{A_0} = \bar{\partial}_E$, and consider instead the family of hermitian metrics h_t satisfying the **Hermitian-Yang-Mills flow equations**:

$$h_t^{-1} \frac{\partial h_t}{\partial t} = -2 \left(\sqrt{-1} \Lambda_\omega F_{h_t} - \mu_\omega(E) Id_E \right).$$

In the above, F_{h_t} is the curvature of $(\bar{\partial}_E, h_t)$ and $\mu_\omega(E)$ is a real number called the slope of E with respect to ω (to be defined later). We will now show that these two equations are equivalent in a very precise sense. Namely, given a solution to the Hermitian-Yang-Mills flow, we produce a solution to Yang-Mills flow and vice-versa.

First, we assume that the Hermitian-Yang-Mills flow has a solution. To construct a solution to the Yang-Mills flow, we will need to first consider the following **equivalent Yang-Mills flow equations**:

$$\frac{\partial \tilde{A}_t}{\partial t} = -d^*_{\tilde{A}_t} F_{\tilde{A}_t} + d_{\tilde{A}_t} \alpha(t), \quad \tilde{A}_0 \in \mathcal{A}^{1,1}_h, \alpha(t) \in \Omega^0(\mathfrak{u}(E)).$$

Here, the one-parameter family $d_{\tilde{A}_t} \alpha(t)$ of endomorphism valued 1-forms are elements of the tangent space to a $\mathcal{G}$ orbit, which is the space $\Omega^1(\mathfrak{u}(E))$. Therefore, up to the action of $\mathcal{G}$ (i.e. in the quotient space $\mathcal{B}^{1,1}_h$), one expects this equation to have the same solutions as the Yang-Mills flow. We will show that a solution of this equivalent flow can be obtained from a solution to the HYM flow equations.

If we consider two different Hermitian metrics on h_1 and h_2 on the fixed holomorphic vector bundle E, then we may define a positive definite, self-adjoint element of $\mathcal{G}^{\mathbb{C}}$ by $k = h_2^{-1}h_1$, where

$$h_1(\sigma, \tau) = h_2(k\sigma, \tau).$$

Then the corresponding Chern connections $\nabla_1 = \partial_{h_1} + \bar{\partial}_{h_1}$ and $\nabla_2 = \partial_{h_2} + \bar{\partial}_{h_2}$ can be seen by a simple computation to satisfy

$$\begin{aligned} \bar{\partial}_{h_2} &= \bar{\partial}_{h_1} \\ \partial_{h_2} &= k^{-1} \circ \partial_{h_1} \circ k = \partial_{h_1} + k^{-1}\partial_{h_1} k \end{aligned}$$

so that also

$$F_{h_2} - F_{h_1} = \bar{\partial}_{h_1}(k^{-1}\partial_{h_1} k).$$

These relations hold for any two metrics.

Now for a fixed Hermitian metric $h_0 = h(0)$, write $h_0^{-1}h(t) = k(t)$. Note that $h^{-1}\frac{\partial h_t}{\partial t} = k^{-1}\frac{\partial k_t}{\partial t}$. Let A_0 be the Chern connection for the metric h_0. Since by the above relation:

$$F_{h(t)} = F_{h_0} + \bar{\partial}_{h_0}(k(t)^{-1}\partial_{h_0} k(t))$$

we may consider (in place of the Hermitian-Yang-Mills flow) the equation:

$$k(t)^{-1}\frac{\partial k(t)}{\partial t} = -2(\sqrt{-1}\Lambda_\omega(F_{h_0} + \bar{\partial}_{h_0}(k(t)^{-1}\partial_{h_0} k(t))) - \mu_\omega(E)Id_E).$$

In other words, the existence of a solution to the HYM equations implies existence for the above system. Since $k(0) = id$ and $k(t)$ is positive definite, there is a complex gauge transformation $g(t) \in \mathcal{G}^{\mathbb{C}}$ such that $g(t)g(t)^* = k(t)^{-1}$. A priori this choice is not unique.

Lemma 1 *Let $k(t)$ be a solution to the above equation. Let $g(t) \in \mathcal{G}^{\mathbb{C}}$ such that $g(t)g(t)^* = k(t)^{-1}$, and let $\tilde{A}(t) = g(t) \cdot A_0$. Then $\tilde{A}(t)$ is a solution to*

$$\frac{\partial \tilde{A}_t}{\partial t} = -d^*_{\tilde{A}_t} F_{\tilde{A}_t} + d_{\tilde{A}_t}\alpha(t)$$

with $\alpha(t) = \frac{1}{2}(g^{-1}\frac{\partial g}{\partial t} - \frac{\partial g^*}{\partial t}(g^*)^{-1})$.

Proof. Let $\tilde{A}(t) = g(t) \cdot A_0$. Then we have the identity:

$$gF_{\tilde{A}(t)}g^{-1} = F_{A_0} + \bar{\partial}_{A_0}(k^{-1}(\partial_{A_0}k)).$$

Differentiating at $\tilde{A}(t)$ gives:

$$\begin{aligned} \frac{\partial \tilde{A}_t}{\partial t} &= \frac{\partial}{\partial \varepsilon}|_{\varepsilon=0}\ d_{(g+\varepsilon\frac{\partial g}{\partial t})\cdot A_0} = (\bar{\partial}_{\tilde{A}(t)}(g^{-1}\frac{\partial g}{\partial t}) - \partial_{\tilde{A}(t)}(\frac{\partial g^*}{\partial t})(g^*)^{-1}) \\ &= \frac{1}{2}(\bar{\partial}_{\tilde{A}(t)} - \partial_{\tilde{A}(t)})(g^{-1}\frac{\partial g}{\partial t} + (\frac{\partial g^*}{\partial t})(g^*)^{-1}) \\ &\quad + \frac{1}{2}(\bar{\partial}_{\tilde{A}(t)} + \partial_{\tilde{A}(t)})((g^{-1})\frac{\partial g}{\partial t} - (\frac{\partial g^*}{\partial t})(g^*)^{-1}). \end{aligned}$$

Then let $\alpha(t) = \frac{1}{2}(g^{-1}\frac{\partial g}{\partial t} - \frac{\partial g^*}{\partial t}(g^*)^{-1})$. Since $g(t)g(t)^* = k(t)^{-1}$, we have:

$$\frac{\partial k}{\partial t} = -(g^*)^{-1}(\frac{\partial g^*}{\partial t}(g^*)^{-1} + g^{-1}\frac{\partial g}{\partial t})g^{-1}.$$

Since we are assuming that $k(t)$ satisfies:

$$k(t)^{-1}\frac{\partial k(t)}{\partial t} = -2(\sqrt{-1}\Lambda_\omega(F_{h_0} + \bar{\partial}_{h_0}(k(t)^{-1}\partial_{h_0}k(t))) - \mu_\omega(E)Id_E)$$

we have that:

$$\begin{aligned} & -(g^*)^{-1}(\frac{\partial g^*}{\partial t}(g^*)^{-1} + g^{-1}\frac{\partial g}{\partial t})g^{-1} \\ = \ & -2(g^*)^{-1}g^{-1}(\sqrt{-1}g\Lambda_\omega F_{\tilde{A}(t)}g^{-1} - \mu_\omega(E)Id_E) \end{aligned}$$

so that

$$\frac{1}{2}(\frac{\partial g^*}{\partial t}(g^*)^{-1} + g^{-1}\frac{\partial g}{\partial t}) = \sqrt{-1}\Lambda_\omega F_{\tilde{A}(t)} - \mu_\omega(E)Id_E.$$

Together with the equation for $\frac{\partial \tilde{A}_t}{\partial t}$ this gives:

$$\frac{\partial \tilde{A}_t}{\partial t} = \sqrt{-1}(\bar{\partial}_{\tilde{A}(t)} - \partial_{\tilde{A}(t)})\Lambda_\omega F_{\tilde{A}(t)} + d_{\tilde{A}(t)}\alpha(t)$$

and by the Kähler identities this is the same as

$$\frac{\partial \tilde{A}_t}{\partial t} = -d^*_{\tilde{A}(t)}F_{\tilde{A}(t)} + d_{\tilde{A}(t)}\alpha(t).$$

■

Proposition 2 *The existence and uniqueness of a long time solution for all time to the Hermitian-Yang-Mills flow implies the existence and uniqueness of a long time solution to the Yang-Mills flow with a fixed initial condition.*

Proof. We have already seen that a solution to the HYM flow equations gives a solution to the equivalent Yang-Mills flow equations. Therefore we construct a unique solution to the Yang-Mills flow equations from a solution to the equivalent Yang-Mills flow equations. Consider the ODE for a one-parameter subgroup $S(t) \in \mathcal{G}^{\mathbb{C}}$ given by:

$$\frac{\partial S}{\partial t} = S(t)(\alpha(t) - \mu_\omega(E)Id)$$

where $\alpha : \mathbb{R} \longrightarrow \mathfrak{u}(E)$ is defined as in the previous lemma. Since $S(0) = Id$ and $\frac{\partial S}{\partial t} \in S(t) \cdot \mathfrak{u}(E)$ we have that $S(t) \in \mathcal{G}$ for all t. The previous lemma shows that $\alpha(t)$ is defined for all t. Therefore there is a solution to the ODE for all time by the theory of linear ODEs. As in the previous lemma, write $\tilde{A}(t)$ for a solution to the equivalent flow equations. Write $\tilde{\alpha} = \alpha - \sqrt{-1}\mu_{\omega}(E)Id$ and define $A(t) = S^{-1}(t) \cdot \tilde{A}(t)$. Then $A(t)$ exists for all t. We show that this is a solution to the Yang-Mills flow. Differentiating with respect to t gives:

$$\begin{aligned}
\frac{\partial A_t}{\partial t} &= \frac{\partial}{\partial t}(S \circ d_{\tilde{A}} \circ S^{-1}) \\
&= S\tilde{\alpha} d_{\tilde{A}} S^{-1} - S d^*_{\tilde{A}} F_{\tilde{A}} S^{-1} + S(d_{\tilde{A}}\tilde{\alpha})S^{-1} - d_A \frac{\partial S}{\partial t} S^{-1} \\
&= S\tilde{\alpha} d_{\tilde{A}} S^{-1} - d^*_A F_A + S(d_{\tilde{A}}\tilde{\alpha})S^{-1} - S d_A \tilde{\alpha} S^{-1} \\
&= -d^*_A F_A.
\end{aligned}$$

As for uniqueness, we will see in subsequent sections that by arguments of Donaldson and Simpson, a solution to the HYM flow equations is unique. Therefore, in the above construction, the only place where we might have introduced non-uniqueness is in the selection of $g(t) \in \mathcal{G}^{\mathbb{C}}$ such that $g(t)g(t)^* = k(t)^{-1}$. We show that in fact, any two such choices yield the same solution of the YM flow.

Let $g_1(t)$, $g_2(t) \in \mathcal{G}^{\mathbb{C}}$ where $g_1(t)g_1(t)^* = h(t)^{-1} = g_2(t)g_2(t)^*$. Let $S_1(t)$ and $S_2(t)$ be the solutions of the corresponding ODEs as defined above. Then define also:

$$\begin{aligned}
A_1(t) &= S_1(t)^{-1} \cdot g_1(t) \cdot A_0 = (g_1(t)S_1(t)^{-1}) \cdot A_0 \\
A_2(t) &= S_2(t)^{-1} \cdot g_2(t) \cdot A_0 = (g_2(t)S_2(t)^{-1}) \cdot A_0.
\end{aligned}$$

We claim that $A_1(t) = A_2(t)$. Note that $g_1^{-1}g_2g_2^*(g_1^*)^{-1} = id$, so if we set $u(t) = g_1^{-1}g_2$, then $u(t) \in \mathcal{G}$. Now, once again, define the gauge-fixing terms $\alpha_1(t)$ and $\alpha_2(t)$

by:

$$
\begin{aligned}
\alpha_1(t) &= \frac{1}{2}(g_1^{-1}\frac{\partial g_1}{\partial t} - \frac{\partial g_1^*}{\partial t}(g_1^*)^{-1}) \\
\alpha_2(t) &= \frac{1}{2}(g_2^{-1}\frac{\partial g_2}{\partial t} - \frac{\partial g_2^*}{\partial t}(g_2^*)^{-1}) \\
&= u(t)^{-1}\alpha_1(t)u(t) + u(t)^{-1}\frac{\partial u(t)}{\partial t}.
\end{aligned}
$$

Then the corresponding ODEs are:

$$
\begin{aligned}
S_1(t)^{-1}\frac{\partial S_1}{\partial t} &= \alpha_1(t) - \mu_\omega(E)Id \\
S_2(t)^{-1}\frac{\partial S_2}{\partial t} &= \alpha_2(t) - \mu_\omega(E)Id \\
&= u(t)^{-1}\alpha_1(t)u(t) + u(t)^{-1}\frac{\partial u}{\partial t}.
\end{aligned}
$$

Now note that $S_2(t) = S_1(t)u(t)$ is a solution of the second equation, and solutions of this equation are unique by the theory of linear ODEs. Therefore we have:

$$
g_2(t)S_2(t)^{-1} = g_1(t)u(t)u(t)^{-1}S_1(t)^{-1} = g_1(t)S_1(t)^{-1}
$$

which implies that $A_1(t) = A_2(t)$.

One can also show that given a solution $A(t) = g(t) \cdot A_0$ of the Yang-Mills flow, the metric $h(t) = g(t)g(t)^* h_0$ is a solution of the Hermitian-Yang-Mills flow.

■

1.3.2 Short-Time Existence of the Flow

We have shown in the last section that a solution of

$$
h_t^{-1}\frac{\partial h_t}{\partial t} = -2\left(\sqrt{-1}\Lambda_\omega F_{h_t} - \mu_\omega(E)Id_E\right)
$$

for some finite time implies the existence of a solution to the YM flow. Therefore, to understand existence and uniqueness questions of the YM flow, it suffices to study the above equation. We have seen that this equation is in turn equivalent to the equation:

$$k(t)^{-1}\frac{\partial k(t)}{\partial t} = -2(\sqrt{-1}(\Lambda_\omega((F_{h_0} + \bar{\partial}_{h_0}(k(t)^{-1}\partial_{h_0}k(t))) - \mu_\omega(E)Id_E)$$

for some positive definite self-adjoint endomorphism $k(t)$. The term

$$-2\sqrt{-1}\Lambda_\omega(\bar{\partial}_{h_0}(k(t)^{-1}\partial_{h_0}k(t))$$

may be written as:

$$\begin{aligned} & 2\sqrt{-1}k^{-1}(\Lambda_\omega(\bar{\partial}_{h_0}k)k^{-1}(\partial_{h_0}k) - 2\sqrt{-1}k^{-1}\bar{\partial}_{h_0}\partial_{h_0}k \\ = \; & 2\sqrt{-1}k^{-1}(\Lambda_\omega(\bar{\partial}_{h_0}k)k^{-1}(\partial_{h_0}k) - \sqrt{-1}\Delta_{\partial_{h_0}}k) \end{aligned}$$

Now writing A_0 for the Chern connection associated to $(h_0, \bar{\partial}_E)$, the Kahler identities imply that:

$$\begin{aligned} \Delta_{A_0} &= d^*_{A_0}d_{A_0} \\ &= \sqrt{-1}\Lambda_\omega(\bar{\partial}_{A_0}\partial_{A_0} - \partial_{A_0}\bar{\partial}_{A_0}) \\ &= \Delta_{\partial_{A_0}} + \Delta_{\bar{\partial}_{A_0}} \end{aligned}$$

and

$$\Delta_{\partial_{A_0}} - \Delta_{\bar{\partial}_{A_0}} = \sqrt{-1}\Lambda_\omega(\bar{\partial}_{A_0}\partial_{A_0} + \partial_{A_0}\bar{\partial}_{A_0}) = \sqrt{-1}\Lambda_\omega F_{A_0}.$$

Therefore, adding these two together:

$$\frac{1}{2}(\Delta_{A_0} + \sqrt{-1}\Lambda_\omega F_{A_0}) = \Delta_{\partial_{A_0}}$$

and therefore:

$$
\begin{aligned}
& -2\sqrt{-1}\Lambda_\omega(\bar{\partial}_{h_0}(k(t)^{-1}\partial_{h_0}k(t)) \\
= \; & 2\sqrt{-1}k^{-1}(\Lambda_\omega(\bar{\partial}_{h_0}k)k^{-1}(\partial_{h_0}k) - \sqrt{-1}\frac{1}{2}(\Delta_{A_0} + \sqrt{-1}\Lambda_\omega F_{A_0})k) \\
= \; & 2\sqrt{-1}k^{-1}(\Lambda_\omega(\bar{\partial}_{h_0}k)k^{-1}(\partial_{h_0}k)) + \sqrt{-1}k^{-1}\Lambda_\omega F_{A_0}k + \Delta_{A_0}k.
\end{aligned}
$$

Finally, this implies:

$$
\frac{\partial k}{\partial t} = -\{\Delta_{A_0}k + \sqrt{-1}(\Lambda_\omega F_{A_0}k + k\Lambda_\omega F_{A_0} - 2\mu_\omega(E)k) + 2\sqrt{-1}(\Lambda_\omega((\bar{\partial}_{h_0}k)k^{-1}(\partial_{h_0}k)))\}.
$$

Now if we set $k = Id + K$, for small K, the linearisation of this equation is:

$$
-\Delta_{A_0}K - \sqrt{-1}(\Lambda_\omega F_{A_0}K + K\Lambda_\omega F_{A_0} - 2\mu_\omega(E)K) - 2\sqrt{-1}(\Lambda_\omega F_{A_0} - \mu_\omega(E)Id)
$$

and this equation is parabolic. In particular, the fact that this equation has short-time solutions is an application of [HAM] Part IV, Section 11, p.122. Therefore we have:

Proposition 3 *For sufficiently small $\epsilon > 0$ (possibly depending on the initial condition) the Hermitian-Yang-Mills flow, and hence the Yang-Mills flow, has a solution defined for $0 \leq t < \varepsilon$.*

1.3.3 Uniqueness and Long-Time Existence of the Flow, Convergence for Stable bundles

First we take care of the much easier problem of uniqueness. To do this we will first define (following [DO1]) a distance function on the space of Hermitian metrics.

Let:

$$\begin{aligned}\tau(h_1, h_2) &= Tr(h_1^{-1}h_2) \\ \sigma(h_1, h_2) &= \tau(h_1, h_2) + \tau(h_2, h_1) - 2\,\mathrm{rk}\,E.\end{aligned}$$

These are both C^∞ functions on X. Note that it follows from the inequality:

$$\lambda + \frac{1}{\lambda} \geq 2, \text{ for all } \lambda \geq 0$$

that $\sigma(h_1, h_2) \geq 0$ with equality if and only if $h_1 = h_2$.

Proposition 4 *If $h_1(t)$ and $h_2(t)$ are two solutions of the equation:*

$$h_t^{-1}\frac{\partial h_t}{\partial t} = -2\left(\sqrt{-1}\Lambda_\omega F_{h_t} - \mu_\omega(E)Id_E\right),$$

then if we write $\sigma = \sigma(h_1(t), h_2(t))$ then we have

$$\frac{\partial \sigma}{\partial t} + \Delta\sigma \leq 0.$$

Proof. Clearly it suffices to show:

$$\frac{\partial \tau}{\partial t} + \Delta\tau \leq 0.$$

Note that

$$\frac{\partial \tau}{\partial t} = \mathrm{Tr}\left(h_1^{-1}\frac{\partial h_2}{\partial t} - h_1^{-1}\frac{\partial h_1}{\partial t}h_1^{-1}h_2\right).$$

By assumption:

$$h_1^{-1}(t)\frac{\partial h_1(t)}{\partial t} = -2\left(\sqrt{-1}\Lambda_\omega F_{h_1(t)} - \mu_\omega(E)Id_E\right)$$

and

$$h_2^{-1}(t)\frac{\partial h_2(t)}{\partial t} = -2\left(\sqrt{-1}\Lambda_\omega F_{h_2(t)} - \mu_\omega(E)Id_E\right),$$

so letting $k = h_1^{-1} h_2$ we have

$$\begin{aligned}
\frac{\partial \tau}{\partial t} &= -2\sqrt{-1}\,\mathrm{Tr}\left(k\left(\Lambda_\omega F_{h_1(t)} - \Lambda_\omega F_{h_2(t)}\right)\right) \\
&= -2\sqrt{-1}\,\mathrm{Tr}\left(k\left(\Lambda_\omega \bar{\partial}_{h_1}\left(k^{-1}\partial_{h_1} k\right)\right)\right) \\
&= -2\sqrt{-1}\,\mathrm{Tr}\left(\Lambda_\omega\left(\bar{\partial}_{h_1}\partial_{h_1} k - \left(\bar{\partial}_{h_1} k\right) k^{-1}\left(\partial_{h_1} k\right)\right)\right) \\
&= -\,\mathrm{Tr}\left(\triangle_{h_1} k\right) + 2\sqrt{-1}\Lambda_\omega\,\mathrm{Tr}\left(\left(\bar{\partial}_{h_1} k\right) k^{-1}\left(\partial_{h_1} k\right)\right).
\end{aligned}$$

Note that the second term is negative and

$$\mathrm{Tr}\left(\triangle_{h_1} k\right) = \triangle\,\mathrm{Tr}\, k = \Delta\tau.$$

Therefore we have:

$$\frac{\partial \tau}{\partial t} + \Delta\tau \leq 0.$$

■

Corollary 1 *If $h_1(t)$ and $h_2(t)$ are solution of the HYM flow for $0 \leq t < \varepsilon$ and have the same initial condition $h_1(0) = h_2(0)$, then $h_1(t)$ and $h_2(t)$ agree on $X \times [0, \varepsilon)$.*

Proof. Apply the parabolic maximum principle to $\sigma(h_1(t), h_2(t))$ using the previous proposition. ■

Long-time existence is rather more difficult. The strategy is a common one in parabolic theory. First one starts with a short time solution, defined on an interval say $[0, T)$. Then one shows that h_t converges in C^∞ to a metric h_T. Then we may use this metric as an initial condition, and apply short-time existence to extend to a solution on an interval $[0, T + \varepsilon)$.

The difficult part of this of course is to prove C^∞ convergence. It is a straightforward corollary of the previous propostion and the maximum principle that there

h_t converges to such a metric h_T in C^0. C^∞ convergence can be proven using the following a priori estimates on the curvature and Hermitian-Einstein tensor. These estimates will be generally useful.

Lemma 2 *Let A_t be a path of connections that is formally gauge equivalent to a solution of the YM flow on some unspecified (possibly infinite) interval. Then:*

(1)

$$\frac{\partial F_{A_t}}{\partial t} = -\triangle_{A_t} F_{A_t}$$

and therefore,

$$\left(\frac{\partial}{\partial t} + \triangle_{A_t}\right) \operatorname{Tr} F_{A_t} = 0,$$

$\|\operatorname{Tr} F_{A_t}\|_{L^\infty}$ *is decreasing in t and so is bounded, and* $\operatorname{Tr} F_{A_t}$ *converges to a harmonic 2-form. Also:*

$$\frac{d}{dt}\|F_{A_t}\|_{L^2}^2 = -2\left\|d_{A_t}^* F_{A_t}\right\|_{L^2}^2 \le 0.$$

Hence, $t \longrightarrow YM(A_t)$, and $t \longrightarrow HYM(A_t)$ are non-increasing.

(2) $|\Lambda_\omega F_{A_t}|^2$ *satisfies*

$$\frac{\partial |\Lambda_\omega F_{A_t}|^2}{\partial t} + \triangle |\Lambda_\omega F_{A_t}|^2 = -2\left|d_{A_t}^* F_{A_t}\right|^2 \le 0,$$

so by the maximum principle for the heat operator $\frac{\partial}{\partial t} + \triangle$, $\sup |\Lambda_\omega F_{A_t}|^2$ is decreasing in t, and therefore $\Lambda_\omega F_{A_t}$ is bounded in L^∞.

(3)

$$\left(\frac{\partial}{\partial t} + \triangle_{A_t}\right) |F_{A_t}|^2 \le C\left(|F_{A_t}|^3 + |F_{A_t}|^2\right)$$

(4)

$$\left(\frac{\partial}{\partial t}+\triangle_{A_t}\right)\left|\nabla^k_{A_t}F_{A_t}\right|^2$$
$$\leq \; C_k\left|\nabla^k_{A_t}F_{A_t}\right|\left(\sum_{i+j=k}\left|\nabla^i_{A_t}F_{A_t}\right|\left(\left|\nabla^j_{A_t}F_{A_t}\right|+1\right)\right).$$

Proof. From the flow equations, we have:

$$\frac{\partial A_t}{\partial t}=-d^*_{A_t}F_{A_t}.$$

Taking d_{A_t} on both sides we get:

$$\frac{\partial F_{A_t}}{\partial t}=-d_{A_t}d^*_{A_t}F_{A_t}=-\triangle_{A_t}F_{A_t}$$

by the Bianchi identity. Taking the bundle trace gives the equation in (1), and the statement about convergence of the trace now follows from a standard result in parabolic theory.

Now by taking Λ_ω of both sides we get

$$\frac{\partial \Lambda_\omega F_{A_t}}{\partial t}=-\triangle_{A_t}\Lambda_\omega F_{A_t}=-d_{A_t}d^*_{A_t}\Lambda_\omega F_{A_t}.$$

Since the pointwise norm is defined by:

$$\left|\Lambda_\omega F_{A_t}\right|^2=\mathrm{Tr}(\Lambda_\omega F_{A_t}\circ\Lambda_\omega F_{A_t})=\sum\left(\Lambda_\omega F_{A_t}\right)^i_j\left(\Lambda_\omega F_{A_t}\right)^j_i$$

we have:

$$\begin{aligned}\frac{\partial}{\partial t}\left|\Lambda_\omega F_{A_t}\right|^2 &= 2\left\langle\frac{\partial}{\partial t}\Lambda_\omega F_{A_t},\Lambda_\omega F_{A_t}\right\rangle=-2\left\langle\triangle_{A_t}\Lambda_\omega F_{A_t},\Lambda_\omega F_{A_t}\right\rangle,\\ \triangle_{A_t}\left|\Lambda_\omega F_{A_t}\right|^2 &= 2\left\langle d_{A_t}d^*_{A_t}\Lambda_\omega F_{A_t},\Lambda_\omega F_{A_t}\right\rangle-2\left|d_{A_t}\Lambda_\omega F_{A_t}\right|\end{aligned}$$

and adding these two we get:

$$\frac{\partial}{\partial t}\left|\Lambda_\omega F_{A_t}\right|^2+\triangle_{A_t}\left|\Lambda_\omega F_{A_t}\right|^2=-2\left|d_{A_t}\Lambda_\omega F_{A_t}\right|\leq 0.$$

The equality in (2) follows from the Kähler identities.

Parts (3) and (4) are more labour intensive. Proofs can be found for example in [DO1] or in [KOB] Chapter 6, Section 8. ■

The import of these estimates is the following lemma.

Lemma 3 *Let h_t be a smooth solution of the HYM flow for $0 \leq t < T$. Then if there is a uniform bound on the curvature $|F_{h_t}| \leq B$, on $X \times [0,T)$. Then all the covariant derviatives are also bounded uniformly: $\left|\nabla^k F_{h_t}\right| \leq B_k$ on $X \times [0,T)$.*

Proof. The proof is by induction on k. The case $k = 0$ is the hypothesis. For the inductive step suppose that $\left|\nabla^j F_{h_t}\right|$ are bounded for all $j < k$. By (4) of the previous lemma we have:

$$\left(\frac{\partial}{\partial t} + \triangle_{h_t}\right) \left|\nabla^k_{h_t} F_{h_t}\right|^2 \leq C\left(1 + \left|\nabla^k_{h_t} F_{h_t}\right|^2\right).$$

The linear equation of the form:

$$\left(\frac{\partial}{\partial t} + \triangle\right) u = \left(\frac{\partial}{\partial t} + \triangle\right)(1+u) = C\,(1+u)\,,\ u(0) = \left|\nabla^k_{h_t} F_{h_t}\right|^2(0)$$

is linear in $(1+u)$, and so has a unique solution u for all t. Then computing:

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + \triangle\right)\left(\left(\left|\nabla^k_{h_t} F_{h_t}\right|^2\right) - u\right) e^{-Ct} \\ = \quad & e^{-Ct}\left(\left(\frac{\partial}{\partial t} + \triangle\right)\left|\nabla^k_{h_t} F_{h_t}\right|^2 - C\left(\left|\nabla^k_{h_t} F_{h_t}\right|^2 + 1\right)\right) \leq 0 \end{aligned}$$

and so by the maximum principle we have $\left|\nabla^k_{h_t} F_{h_t}\right|^2 \leq u$. ■

From the facts that a one parameter family h_t of metrics along the flow has a C^0 limit as $t \longrightarrow T$, and has uniformly bounded Hermitian-Einstein tensor, it is fairly straightforward to prove that h_t is bounded uniformly in C^1 and L^p_2 and F_{h_t} is uniformly bounded in L^p for any $p < \infty$. In turn, the asymptotic expansion for the heat kernel of $\frac{\partial}{\partial t} + \triangle$ can be used to show that an L^p bound on F_{h_t} in fact implies an L^∞ bound. Now the previous lemma implies that all the derivatives $\nabla^k F_{h_t}$ are bounded on $X \times [0,T)$. This furthermore imples that the Hermitian-Einstein tensor is also bounded in C^k for all k. Then, using the local expression for the curvature gives:

$$\Lambda_\omega F_{h_t} = h_t^{-1} \triangle h_t - \sqrt{-1} \Lambda_\omega \left(\partial h_t \right) h_t^{-1} \left(\bar{\partial} h_t \right).$$

Assuming inductively that h_t is bounded in C^l for all $l < k$, this means that $\triangle h_t$ is bounded uniformly in C^{k-2}, so by elliptic regularity, h_t is bounded in C^k. Now long-time existence of the equation follows.

Long-time existence of the Yang-Mills flow, as sketched above was originally proven in [DO1], for a compact, Kähler X. The main acheivement of [DO1] and [DO2] was to prove that in the case that the bundle is stable and X is projective, the flow converges to an Hermitian-Einstein metric. This requires the introduction of of an alternative functional on the space of metrics, which is defined using Bott-Chern classes. The projectivity assumption was necessary because Donaldson used the theorem of Mehta-Ramanathan that says that the for some positive m the restriction of a semi-stable bundle to a generic smooth hypersurface in the linear system $|\mathcal{O}(m)|$ remains semi-stable. This result requires projectivity.

Finally, we note that in [SI], Simpson was able to drop the compactness restriction on X and instead impose the following assumptions:

- X has finite volume.

- There exists an exhaustion function ψ with $\Delta\psi$ bounded. Take $\psi \geq 0$.

- There is an increasing function $a : [0, \infty) \longrightarrow [0, \infty)$ with $a(0) = 0$ and $a(x) = x$ for $x > 1$, such that if f is a bounded, positive function on X with $\Delta f \leq B$ then
$$\sup_X |f| \leq C(B) a \left(\int_X |f| \right).$$
Furthermore, if $\Delta f \leq 0$ then $\Delta f = 0$.

These assumptions are satisfied if X is compact, and more generally if X is the complement of a holomorphic subvariety in a compact Kähler manifold $\bar{X}$ such that the ω for X extends to a Kähler form on $\bar{X}$. This latter condition is the one we will actually need to use.

The proof of longtime existence and convergence of the flow in [SI] for X satisfying these somewhat more general assumptions is based on an adaptation of Donaldson's work, coupled with the use of techniques of Uhlenbeck and Yau, whose proof of the Kobayashi-Hitchin correspondence in [UY] works for arbitary compact Kähler manifolds and does not use the flow. In particular, [SI] uses the existence of weakly holomorphic projection operators proved in [UY] (and to be discussed in the next section).

The basic strategy is to solve the equation on a compact manifold X_c satisfying certain boundary conditions, and then takes the limit as $c \longrightarrow \infty$. More explicitly, fix c and let X_c be the compact space with $\psi(x) \leq c$, and denote the boundary by Y_c. Let H be a metric on $E \longrightarrow X$, and $\frac{\partial}{\partial \nu}$ denote differentiation of sections of E in the direction perpendicular to the boundary using the Chern connection associated to H. We will consider metrics h that either satisfy:

$$\frac{\partial}{\partial \nu} h_{|Y_c} = 0$$

or

$$h_{|Y_c} = H_{|Y_c}.$$

These are the **Neumann** and **Dirichlet** boundary conditions respectively.

For completeness we state Simpson's result.

Theorem 4 *Let (X, ω) be a Kähler manifold satisfy the conditions stated above. Let S be a stable bundle on X with an hermitian metric h_0. Then the equation:*

$$h_t^{-1} \frac{\partial h_t}{\partial t} = -2 \left(\sqrt{-1} \Lambda_\omega F_{h_t} - \mu_\omega(E) Id_E \right), \ h(0) = h_0$$

has a solution for all time and converges at infinity to an Hermitian-Einstein metric on S.

1.4 Properties of Sheaves, the HNS filtration, Weakly Holomorphic Projections, and Second Fundamental Forms

1.4.1 Subsheaves of Holomorphic Bundles and the HNS Filtration

As stated in the introduction, the main obstacle we will face is that we must consider arbitrary subsheaves of a holomorphic vector bundle. Throughout, X will be a compact Kahler manifold (unless otherwise stated) with Kahler form ω, E a holomorphic vector bundle, and $S \subset E$ a subsheaf.

Recall that an analytic sheaf $\mathcal{F}$ on X is called torsion free if the natural map $\mathcal{F} \longrightarrow \mathcal{F}^{**}$ is injective. We call $\mathcal{F}$ reflexive if this map is an isomorphism. Of vital importance is the fact that a torsion free sheaf is "almost a vector bundle" in the following sense. For $\mathcal{F}$ a sheaf on X we define the singular set:

$$\operatorname{Sing}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \; is \; not \; free\}.$$

Here $\mathcal{F}_x$ is the stalk of $\mathcal{F}$ over x. In other words $\operatorname{Sing}(\mathcal{F})$ is the set of points where $\mathcal{F}$ fails to be locally free, i.e. a vector bundle. The set $\operatorname{Sing}(\mathcal{F})$ is closed, and furthermore is a complex analytic subvariety of X. We have the following result.

Proposition 5 *If $\mathcal{F}$ is torsion free, then* $\operatorname{codim} \operatorname{Sing}(\mathcal{F}) \geq 2$. *If $\mathcal{F}$ is reflexive then* $\operatorname{codim} \operatorname{Sing}(\mathcal{F}) \geq 3$.

For the proof see [KOB].

Now in our case, a vector bundle E is clearly torsion free, so any subsheaf S is also. Therefore the above result applies to S. On the other hand, the quotient $Q = E/S$ may not be torsion free. We define the torsion $\operatorname{Tor}(Q)$ to be the kernel of the sheaf map $Q \longrightarrow Q^{**}$. To obtain a sheaf which does have torsion-free quotient, define the saturation of S in E by $\operatorname{Sat}_E(S) = \ker(E \longrightarrow Q/\operatorname{Tor}(Q))$. Note that S is a subsheaf of $\operatorname{Sat}_E(S)$ with torsion quotient, and the quotient $E/\operatorname{Sat}_E(S)$ is torsion free. The same holds true of course for subsheaves of an arbitrary torsion free sheaf $\mathcal{F}$. We also have the following lemma.

Lemma 4 *Let $\mathcal{F}$ be torsion free. Suppose $S_1 \subset S_2 \subset \mathcal{F}$ are subsheaves with S_2/S_1 torsion. Then* $\operatorname{Sat}_E(S_1) = \operatorname{Sat}_E(S_2)$.

Proof. We claim $\operatorname{Sat}_E(S_1) \subset \operatorname{Sat}_E(S_2)$. The natural map $\operatorname{Sat}_E(S_1) \longrightarrow \mathcal{F}/\operatorname{Sat}_E(S_2)$ given by inclusion followed by projection factors through a map $\operatorname{Sat}_E(S_1)/S_1 \longrightarrow \mathcal{F}/\operatorname{Sat}_E(S_2)$ since $S_1 \subset \operatorname{Sat}_E(S_2)$. But on the other hand $\operatorname{Sat}_E(S_1)/S_1$ is torsion and so has torsion image, but then its image must be zero since $\mathcal{F}/\operatorname{Sat}_E(S_2)$ is torsion free. Thus we have the first inclusion. We therefore have a map $\operatorname{Sat}_E(S_2)/\operatorname{Sat}_E(S_1) \longrightarrow \mathcal{F}/\operatorname{Sat}_E(S_1)$. By assumption $\operatorname{Sat}_E(S_2)/\operatorname{Sat}_E(S_1)$ is torsion, and so has torsion (and hence zero) image. Then $\operatorname{Sat}_E(S_2) \subset \operatorname{Sat}_E(S_1)$. ■

The ω-**slope** of a torsion free sheaf $\mathcal{F}$ on X is defined by:

$$\mu_\omega(\mathcal{F}) = \deg_\omega(\mathcal{F})/\operatorname{rk}(\mathcal{F}) = \frac{1}{\operatorname{rk}(\mathcal{F})}\int_X c_1(\mathcal{F}) \wedge \omega^{n-1}.$$

Note that the right hand side is well defined independently of the representative for $c_1(\mathcal{F})$ since ω is closed. Throughout we will assume that the volume of X with respect to ω is normalised to be $2\pi/(n-1)!$, where $n = \dim_{\mathbb{C}} X$.

Definition 2 *We say that a torsion free sheaf $\mathcal{F}$ is ω-**stable** (ω-**semistable**) if for all proper subsheaves $S \subset \mathcal{F}$, $\mu_\omega(S) < \mu_\omega(\mathcal{F})$ ($\mu_\omega(S) \leq \mu_\omega(\mathcal{F})$). Equivalently $\mu_\omega(Q) > \mu_\omega(\mathcal{F})$ ($\mu_\omega(Q) \geq \mu_\omega(\mathcal{F})$) for every torsion free quotient Q.*

We have the following important proposition.

Proposition 6 *There is an upper bound on the set of slopes $\mu_\omega(S)$ of subsheaves of a torsion free sheaf $\mathcal{F}$, and more over this upper bound is realised by some subsheaf $\mathcal{F}_1 \subset \mathcal{F}$. Moreover, we can choose $\mathcal{F}_1$ so that for any $S \subset \mathcal{F}$, if $\mu_\omega(S) = \mu_\omega(\mathcal{F}_1)$ then $rk(S) \leq rk(\mathcal{F}_1)$.*

For the proof see Kobayashi. The sheaf $\mathcal{F}_1$ is called the **maximal destabilising subsheaf** of $\mathcal{F}$. This sheaf is also clearly semistable.

Remark 1 *If $S \subset \mathcal{F}$ is a subsheaf with torsion free quotient $Q = \mathcal{F}/S$, then $Q^* \hookrightarrow \mathcal{F}^*$ is a subsheaf and $\deg(Q^*) = -\deg(Q)$. By the above proposition $\mu_\omega(Q^*)$ is bounded from above, so $\mu_\omega(Q)$ is bounded from below.*

Remark 2 *Note also that the saturation of a sheaf has slope at least as large as the slope of the original sheaf. Therefore the maximal destabilising subsheaf is saturated by definition.*

Definition 3 *We will write $\mu^{\max}(\mathcal{F})$ for the maximal slope of a subsheaf, and $\mu^{\min}(\mathcal{F})$ for the minimal slope of a torsion free quotient. Clearly we have the equality $\mu^{\min}(\mathcal{F}) = -\mu^{\max}(\mathcal{F}^*)$.*

We now specialise to the case of a holomorphic vector bundle E, although the following all holds also for an arbitrary torsion-free sheaf.

Proposition 7 *There is a filtration:*

$$0 = E_0 \subset E_1 \subset ... \subset E_l = E$$

such that the quotients $Q_i = E_i/E_{i-1}$ are torsion free and semistable, and $\mu_\omega(Q_{i+1}) < \mu_\omega(Q_i)$. Furthermore, the associated graded object:

$$Gr^{HN}_\omega(E) = \oplus_i Q_i$$

is uniquely determined by the isomorphism class of E and is called the ***Harder-Narasimhan filtration****.*

In the sequel we will usually abbreviate this as the HN filtration, and we will write $\mathbb{F}_i^{HN}(E)$ for the i^{th} piece of the filtration. The previous proposition follows from Proposition 2. The maximal destabilising subsheaf is $\mathbb{F}_1^{HN}(E)$. Then consider the quotient $E/\mathbb{F}_1^{HN}(E)$ and its maximal destabilising subshseaf. Define $\mathbb{F}_2^{HN}(E)$ to be the pre-image of this subsheaf under the natural projection. Iterating this process gives the stated filtration, and one easily checks that it has the desired properties.

Another invariant of the isomorphism class of E is the collection of all slopes of the quotient Q_i.

Definition 4 *Let E have rank R. Then we form an R-tuple*

$$\mu(E) = (\mu(Q_1), ...\mu(Q_1), ..., \mu(Q_i), ..., \mu(Q_i), ...\mu(Q_l), ...\mu(Q_l))$$

where $\mu(Q_i)$ is repeated $\mathrm{rk}(Q_i)$ times. Then $\mu(E)$ is called the ***Harder-Narasimhan*** *(or HN)* ***type*** *of E.*

The set of all HN types of holomorphic bundles on X has a partial ordering due to Shatz. For a pair of R-tuples μ and λ with $\mu_1 \geq \mu_2 \geq ... \geq \mu_R$ and $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_R$ and $\sum_i \mu_i = \sum_i \lambda_i$, we write

$$\mu \leq \lambda \Longleftrightarrow \sum_{j \leq k} \mu_j \leq \sum_{j \leq k} \lambda_j \; for \; all \; k = 1, ..., R.$$

This partial ordering was originally used by Atiyah and Bott to stratify the space of holomorphic structures on a complex vector bundle over a Riemann surface.

We have the following fact.

Lemma 5 *Let $\mu = (\mu_1, ..., \mu_R)$ and $\lambda = (\lambda_1, ..., \lambda_R)$ be R-tuples with non-increasing entries as above. Suppose there is a partition $0 = R_0 < R_1 < ... < R_l = R$ such* *that* $\mu_i = \mu_j$ *for all pairs* i, j *satisfying:* $R_{k-1} + 1 \leq i, j \leq R_k$, $k = 1, ..., l$. *If* $\sum_{j \leq R_k} \mu_j \leq \sum_{j \leq R_k} \lambda_j$, *for all* $k = 1, ..., l$, *then* $\mu \leq \lambda$.

For the proof see Atiyah-Bott 7.

We will also need a result describing the HN filtration of E in terms of then HN filtration of a subsheaf S and its quotient Q.

Lemma 6 *Let* E *be a holomorphic vector bundle. Consider the subsheaf* $\mathbb{F}_1^{HN}(E) \subset E$ *and set* $Q = E/\mathbb{F}_1^{HN}(E)$. *Then*

$$\mathbb{F}_{i+1}^{HN}(E) = \ker\left(E \longrightarrow Q/\mathbb{F}_i^{HN}(Q)\right).$$

Therefore in particular,$\mathbb{F}_{i+1}^{HN}(E)/\mathbb{F}_1^{HN}(E) = \mathbb{F}_i^{HN}(Q)$.

Proof. If $i = 0$ this is true by definition of the objects involved. If $i = 1$, then $\mathbb{F}_2^{HN}(E)$ is the pre-image of $\mathbb{F}_1^{HN}(Q)$ under the quotient map $E \longrightarrow Q$, in other words, exactly the statement of the lemma. Now we proceed by induction. Assume that we have:

$$\mathbb{F}_i^{HN}(E) = \ker\left(E \longrightarrow Q/\mathbb{F}_{i-1}^{HN}(Q)\right).$$

Then by definition of $\mathbb{F}_{i+1}^{HN}(E)$:

$$\begin{aligned} \mathbb{F}_{i+1}^{HN}(E) &= \ker\left(E \longrightarrow \frac{E/\mathbb{F}_i^{HN}(E)}{\mathbb{F}_1^{HN}\left(E/\mathbb{F}_i^{HN}(E)\right)}\right) = \ker\left(E \longrightarrow \frac{Q/\mathbb{F}_{i-1}^{HN}(Q)}{\mathbb{F}_1^{HN}\left(Q/\mathbb{F}_{i-1}^{HN}(Q)\right)}\right) \\ &= \ker\left(E \longrightarrow \frac{Q/\mathbb{F}_{i-1}^{HN}(Q)}{\mathbb{F}_i^{HN}(Q)/\mathbb{F}_{i-1}^{HN}(Q)}\right) = \ker\left(E \longrightarrow Q/\mathbb{F}_i^{HN}(Q)\right) \end{aligned}$$

■

Proposition 8 *Let*

$$0 \longrightarrow S \longrightarrow E \longrightarrow Q \longrightarrow 0$$

be an exact sequence of torsion free sheaves with E a holomorphic vector bundle such that $\mu^{\min}(S) > \mu^{\max}(Q)$. Then the HN filtration of E is given by:

$$0 \subset \mathbb{F}_1^{HN}(S) \subset ... \subset \mathbb{F}_k^{HN}(S) = S \subset \mathbb{F}_{k+1}^{HN}(E) \subset ... \subset \mathbb{F}_l^{HN}(E) = E,$$

where

$$\mathbb{F}_{k+i}^{HN}(E) = \ker\left(E \longrightarrow Q/\mathbb{F}_i^{HN}(Q)\right) \;\; for \;\; i = 0, 1, ..., l-k.$$

In particular, this means that $Q_i = \mathbb{F}_{k+i}^{HN}(E)/\mathbb{F}_{k+i-1}^{HN}(E) = \mathbb{F}_i^{HN}(Q)$ and therefore

$$Gr^{HN}(E) = Gr^{HN}(S) \oplus Gr^{HN}(Q).$$

Proof. Let E_1 be the maximal destabilising subsheaf of E. Then by assumption we have:

$$\mu_\omega(E_1) \geq \mu_\omega^{\max}(S) \geq \mu_\omega^{\min}(S) > \mu_\omega^{\max}(Q).$$

If the projection map $E_1 \longrightarrow Q$ were non-zero, by semi-stability of E_1we would have:

$$\mu_\omega\left(\mathrm{im}\left(E_1 \longrightarrow Q\right)\right) \geq \mu_\omega(E_1) > \mu_\omega^{\max}(Q),$$

which condradicts the definition of $\mu_\omega^{\max}(Q)$. Then necessarily $E_1 \subset S$, and if $E_1 \neq S$, then E_1 must be the maximal destabilising subsheaf of S.

We proceed by induction on the length of the HN filtration of S. If S is semi-stable then the above argument implies that $S = E_1 = \mathbb{F}_1^{HN}(E)$. In the statement of the proposition is exactly the same as that of the preceding lemma. Now let S be arbitrary and suppose that the statement has been proven for all such exact sequences such that the HN filtration of the subsheaf in the sequence is strictly shorter than that of S. Now we have an induced exact sequence:

$$0 \longrightarrow S/E_1 \longrightarrow E/E_1 \longrightarrow Q \longrightarrow 0$$

and furthermore this sequence still satisfies $\mu_\omega^{\min}(S/E_1) = \mu_\omega^{\min}(S) > \mu_\omega^{\max}(Q)$. By the inductive hypothesis we have:

$$\begin{aligned} 0 &\subset \mathbb{F}_1^{HN}(S/E_1) \subset \ldots \subset \mathbb{F}_{k-1}^{HN}(S/E_1) \\ &= S/E_1 \subset \mathbb{F}_k^{HN}(E/E_1) \subset \ldots \subset \mathbb{F}_{l-1}^{HN}(E/E_1) = E/E_1, \end{aligned}$$

where

$$\mathbb{F}_{k+i-1}^{HN}(E/E_1) = \ker\left(E/E_1 \longrightarrow Q/\mathbb{F}_i^{HN}(Q)\right).$$

Now by the previous lemma we have:

$$\mathbb{F}_{i+1}^{HN}(E) = \ker\left(E \longrightarrow \frac{E/E_1}{\mathbb{F}_i^{HN}(E/E_1)}\right).$$

Combining these two equalities gives:

$$\begin{aligned} \mathbb{F}_{k+i}^{HN}(E/E_1) &= \ker\left(E \longrightarrow \frac{E/E_1}{\ker\left(E/E_1 \longrightarrow Q/\mathbb{F}_i^{HN}(Q)\right)}\right) \\ &= \ker\left(E \longrightarrow Q/\mathbb{F}_i^{HN}(Q)\right). \end{aligned}$$

Now for $i \leq k-1$, by induction and the previous lemma we have:

$$\mathbb{F}_{i+1}^{HN}(E)/E_1 = \mathbb{F}_i^{HN}(E/E_1) = \mathbb{F}_i^{HN}(S/E_1).$$

Therefore:

$$\begin{aligned} \mathbb{F}_{i+1}^{HN}(E) &= \ker\left(E \longrightarrow \frac{E/E_1}{\mathbb{F}_i^{HN}(E/E_1)}\right) = \ker\left(E \longrightarrow \frac{E/E_1}{\mathbb{F}_i^{HN}(S/E_1)}\right) \\ &= \ker\left(E \longrightarrow \frac{E/E_1}{\mathbb{F}_{i+1}^{HN}(S)/E_1}\right) = \ker\left(E \longrightarrow E/\mathbb{F}_{i+1}^{HN}(S)\right) = \mathbb{F}_{i+1}^{HN}(S). \end{aligned}$$

■

Corollary 2 *Suppose that*

$$0 \subset E_1 \subset ... \subset E_{l-1} \subset E_l = E$$

is a filtration of E *by subbundles, and suppose that for each* i $\mu^{\min}(E_i) > \mu^{\max}(E/E_i)$*. Then the Harder-Narasimhan filtration of* E *is given by:*

$$\begin{aligned} 0 &\subset F_1^{HN}(E_1) \subset ... \subset F_{k_1}^{HN}(E_1) = E_1 \subset ... \subset F_{k_1+...+k_{l-1}}^{HN}(E_{l-1}) = E_{l-1} \\ &\subset F_{k_1+...+k_{l-1}+1}^{HN}(E) \subset ... \subset F_{k_1+...+k_l}^{HN}(E) = E. \end{aligned}$$

Proof. This is immediate from the previous proposition. ∎

Now we will define the double filtration that appears in the statement of the Main Theorem. Its existence follows from the existence of the HN filtration and the following proposition.

Proposition 9 *Let* Q *be a semi-stable torsion free sheaf on* X*. Then there is a filtration:*

$$0 \subset F_1 \subset ... \subset F_l = Q$$

such that F_i/F_{i-1} *is stable and torsion-free. Also, for each* i *we have* $\mu(F_i/F_{i-1}) = \mu(Q)$*. The associated graded object:*

$$Gr_{\omega}^{S}(Q) = \oplus_i F_i/F_{i-1}$$

is uniquely determined by the isomorphism class of Q*. This filtration is called the* ***Seshadri filtration*** *of* Q*.*

For the proof see Kobayashi. An immediate corollary is the following.

Proposition 10 *Let E be a holomorphic vector bundle on X. Then there is a double filtration $\{E_{i,j}\}$ with the following properties. If the HN filtration is given by:*

$$0 \subset E_1 \subset ... \subset E_{l-1} \subset E_l = E,$$

then

$$E_{i-1} = E_{i,0} \subset E_{i,1} \subset ... \subset E_{i,l_i} = E_i$$

where the successive quotients

$$Q_{i,j} = E_{i,j}/E_{i,j-1}$$

are stable and torsion-free. Furthermore:

$$\begin{aligned} \mu_\omega(Q_{i,j}) &= \mu_\omega(Q_{i,j+1}) \\ \mu_\omega(Q_{i,j}) &> \mu_\omega(Q_{i+1,j}). \end{aligned}$$

The associated graded object

$$Gr_\omega^{HNS}(E) = \oplus_i \oplus_j Q_{i,j}$$

is uniquely determined by the isomorphism class of E. This double filtration is called the ***Harder-Narasimhan-Seshadri filtration*** *(or HNS filtration) of E.*

Similarly, we define the corresponding type of E as the R-tuple:

$$\mu = (\mu(Q_{1,1}), ..., \mu(Q_{i,j}), ..., \mu(Q_{l,k_l}))$$

where each $\mu(Q_{i,j})$ is repeated according to $\mathrm{rk}(Q_{i,j})$. Note that this is exactly the same vector as the HN type. Since each of the quotients $Q_{i,j}$ is torsion-free,

$\mathrm{Sing}(Q_{ij})$ lies in codimension 2. We will write:

$$Z_{\mathrm{alg}} = \cup_{i,j} \mathrm{Sing}(E_{i,j}) \cup \mathrm{Sing}(Q_{i,j}).$$

This is a complex analytic subset (again, we ignore multiplicities) of codimension at least two, and corresponds exactly to the set of points at which the HNS filtration fails to be given by subbundles. We will refer to it as the algebraic singular set of the filtration.

1.4.2 Weakly Holomorphic Projections/Second Fundamental Forms

Let $S \subset E$ be a subsheaf with quotient Q. Then away from $\mathrm{Sing}(S) \cup \mathrm{Sing}(Q)$, S is a subbundle. If we fix an hermitian metric h on E, then we may think of S as a direct summand away from the singular set, and there is a corresponding smooth projection operator $\pi : E \to S$ depending on h. The condition of being a holomorphic subbundle almost everywhere can be shown to be equivalent to the condition: $(\mathbf{Id}_E - \pi)\,\bar{\partial}_E \pi = 0$. Since π is a projection operator we also have $\pi^2 = \pi = \pi^*$. Furthermore it can be shown that π extends to an L^2_1 section of $\mathrm{End}\, E$. Conversely it turns out that an operator with these properties determines a subsheaf.

Definition 5 *An element $\pi \in L^2_1(\mathrm{End}\, E)$ is called a weakly holomorphic projection operator if the conditions*

$$(\mathbf{Id}_E - \pi)\,\bar{\partial}_E \pi = 0 \textit{ and } \pi_j^2 = \pi_j = \pi_j^* \qquad *$$

hold almost everywhere.

Theorem 5 *(Uhlenbeck-Yau) A weakly holomorphic projection operator π of a holomorphic vector bundle (E, h) with a smooth hermitian metric over a compact Kähler manifold (X, ω) determines a coherent subsheaf of E. That is, there exists a coherent subsheaf S of E together with a singular set $V \subset X$ with the following properties:*

$\cdot$ Codim $V \geq 2$,

$\cdot \pi_{|X-V}$ *is C^∞ and satisfies* $*$,

$\cdot S_{|X-V} = \pi_{|X-V}(E_{|X-V}) \hookrightarrow E_{|X-V}$ *is a holomorphic subbundle.*

The proof of this theorem is contained in [UY]. From here on out we will identify a subsheaf with its weakly holomorphic holomorphic projection.

If $S \subset E$ is a subsheaf, then away from $\mathrm{Sing}(S) \cup \mathrm{Sing}(Q)$ there is an orthogonal splitting $E = S \oplus Q$. In general we may write the Chern connection $\nabla_{(\bar{\partial}_E, h)}$ connection on E as:

$$\nabla_{(\bar{\partial}_E, h)} = \begin{pmatrix} \nabla_{(\bar{\partial}_S, h_S)} & \beta \\ -\beta^* & \nabla_{(\bar{\partial}_Q, h_Q)} \end{pmatrix}$$

where $\nabla_{(\bar{\partial}_S, h_S)}$ and $\nabla_{(\bar{\partial}_Q, h_Q)}$ are the induced Chern connections on S and Q respectively, and β is the second fundamental form. Recall that $\beta \in \Omega^{0,1}(Hom(Q, S))$. More specifically, in terms of the projection operator, we have $\bar{\partial}_E \pi = \beta$ and $\partial_E \pi = -\beta^*$. Also β extends to an L^2 section of $\Omega^{0,1}(Hom(Q, S))$ everywhere as $\bar{\partial}_E \pi$ since π is L^2_1. We also have the following well-known formula for the degree of a subsheaf in terms of its weakly holomorphic projection.

Theorem 6 *(Chern-Weil Formula) Let $S \subset E$ be a saturated subsheaf of a holomorphic vector bundle with hermitian metric h, and π the associated weakly holomorphic* *projection. Let $\bar{\partial}_E$ denote the holomorphic structure on E. Then we have:*

$$\deg S = \frac{1}{2\pi n}\int_X \operatorname{Tr}\left(\sqrt{-1}\Lambda_\omega F_{(\bar{\partial}_E,h)}\pi\right)\omega^n - \frac{1}{2\pi n}\int_X |\beta|^2\,\omega^n$$

The statement of this theorem as well as a sketch of the proof may be found in [SI]. This formula will also follow as a special case of our discussion in Section 4.

Clearly any sequence π_j of such projection operators is uniformly bounded in $L^\infty(X)$. As an immediate corollary of the Chern-Weil formula we have the following.

Corollary 3 *A sequence π_j of weakly holomorphic projection operators such that $\deg \pi_j$ is bounded from below is uniformly bounded in L^2_1. In particular, if $\deg \pi_j$ is constant then π_j is bounded in L^2_1.*

Now suppose ∇_{A_0} is a reference connection, $g_j \in \mathcal{G}^{\mathbb{C}}$ is a sequence of complex gauge transformations, and ∇_{A_j} is the sequence of integrable unitary connections on an hermitian vector bundle (E, h) given by $\nabla_{A_j} = g_j \cdot \nabla_{A_0}$. Let $S \subset E$ be a subbundle with quotient Q. We have a sequence of projection operators π_j given by orthogonal projection onto $g_j(S)$ (with respect to the metric h) from E to holomorphic subbundles S_j (whose holomorphic structures are induced by the connections ∇_{A_j}) smoothly isomorphic to S. We will denote by Q_j the corresponding quotients. Each of these holomorphic subbundles has a second fundamental form which we will write as β_j. Assume that the β_j are also uniformly bounded in L^2 (this will later be a consequence of our hypotheses). Then with all of the above understood, we have the following result.

Lemma 7 *For any $1 \leq p < \infty$, the β_j are bounded in $L^p_{1,loc}(X - Z_{\mathrm{an}})$, uniformly for all j. In particular the β_j are uniformly bounded on compact subsets of $X - Z_{\mathrm{an}}$.*

Proof. To simplify notation, in the following proof we will continue to write ∇_{A_j} and $\bar{\partial}_{A_j}$ for the induced operators on $\operatorname{End} E$. By weak convergence of the sequence ∇_{A_j} in $L^p_{1,loc}$ for $p > n$ (see the next section), if we write $\Omega_j = \nabla_{A_j} - \nabla_{A_0}$, we may assume Ω_j is uniformly bounded in $L^p_{1,loc}$ for any p, and so in particular the Ω_j are bounded in C^0_{loc} since we have the imbedding $L^p_{1,loc} \hookrightarrow C^0_{loc}$. We will write $\Omega_j^{1,0}$ and $\Omega_j^{0,1}$ for the $(1,0)$ and $(0,1)$ parts of Ω_j. Now:

$$\bar{\partial}_{A_0}\pi_j = \bar{\partial}_{A_j}\pi_j + \Omega_j^{0,1}\pi_j = \beta_j + \Omega_j^{0,1}\pi_j$$

and the β_j are bounded in L^2. Recall also that π_j is bounded in L^2_1 and L^∞. On $\Omega^{1,0}(\operatorname{End}(E))$ and $\Omega^{0,1}(\operatorname{End}(E))$ the Kahler identities are: $\bar{\partial}^* = \sqrt{-1}\Lambda_\omega \partial$ and $\partial^* = -\sqrt{-1}\Lambda_\omega \bar{\partial}$.

We compute:

$$\begin{aligned}
\triangle_{\bar{\partial}_{A_0}}\pi_j &= \bar{\partial}^*_{A_0}\bar{\partial}_{A_0}\pi_j = \sqrt{-1}\Lambda_\omega \partial_{A_0}\bar{\partial}_{A_0}\pi_j \\
&= \sqrt{-1}\Lambda_\omega\left(\left(\partial_{Aj} - \Omega_j^{1,0}\right)\left(\bar{\partial}_{A_j} - \Omega_j^{0,1}\right)\pi_j\right) \\
&= \sqrt{-1}\Lambda_\omega \partial_{Aj}\bar{\partial}_{A_j}\pi_j - \sqrt{-1}\Lambda_\omega \partial_{Aj}\left(\Omega_j^{0,1}\pi_j\right) \\
&\quad -\sqrt{-1}\Lambda_\omega\left(\Omega_j^{1,0}\bar{\partial}_{A_j}\pi_j\right) + \sqrt{-1}\Lambda_\omega\left(\left(\Omega_j^{1,0}\wedge\Omega_j^{0,1}\right)\pi_j\right) \\
&= \triangle_{\bar{\partial}_{A_j}}\pi_j - \sqrt{-1}\Lambda_\omega\left(\left(\partial_{Aj}\Omega_j^{0,1}\right)\pi_j\right) - \Lambda_\omega\left(\Omega_j^{1,0}\wedge\bar{\partial}_{A_j}\pi_j\right) \\
&\quad -\sqrt{-1}\Lambda_\omega\left(\Omega_j^{0,1}\wedge\partial_{Aj}\pi_j\right) + \sqrt{-1}\Lambda_\omega\left(\left(\Omega_j^{1,0}\wedge\Omega_j^{0,1}\right)\pi_j\right) \\
&= \triangle_{\bar{\partial}_{A_j}}\pi_j - \sqrt{-1}\Lambda_\omega\left(\left(\partial_{A_0}\Omega_j^{0,1}\right)\pi_j\right) - \Lambda_\omega\left(\Omega_j^{1,0}\bar{\partial}_{A_0}\pi_j\right) \\
&\quad -\sqrt{-1}\Lambda_\omega\left(\Omega_j^{0,1}\wedge\partial_{A_0}\pi_j\right) - \sqrt{-1}\Lambda_\omega\left(\left(\Omega_j^{1,0}\wedge\Omega_j^{0,1}\right)\pi_j\right) \\
&\quad -\sqrt{-1}\Lambda_\omega\left(\left(\Omega_j^{0,1}\wedge\Omega_j^{1,0}\right)\pi_j\right).
\end{aligned}$$

Now since $\bar{\partial}_{Aj}\pi_j = \beta_j$, we have the expression:

$$\triangle_{\bar{\partial}_{A_j}}\pi_j = \sqrt{-1}\Lambda_\omega \partial_{Aj}\bar{\partial}_{Aj}\pi_j = \sqrt{-1}\Lambda_\omega \partial_{Aj}\beta_j.$$

On the other hand:

$$F_{(\bar{\partial}_{A_j},h)} = \begin{pmatrix} F_{S_j} - \beta_j \wedge \beta_j^* & \partial_{Aj}\beta_j \\ -\overline{\partial}\beta_j^* & F_{Q_j} - \beta_j^* \wedge \beta_j \end{pmatrix}$$

and since we assume $\Lambda_\omega F_{A_j}$ is uniformly bounded, this implies $\triangle_{\bar{\partial}_{A_j}}\pi_j$ is bounded uniformly. By the preceeding discussion, we therefore know that the right hand side of the expression for $\triangle_{\bar{\partial}_{A_0}}\pi_j$ is bounded in L^2_{loc}. Recall also the Weitzenbock formula:

$$\triangle_{\bar{\partial}_{A_0}} = \frac{1}{2}\nabla^*_{A_0}\nabla_{A_0} + \sqrt{-1}\Lambda_\omega F_{A_0}.$$

Again, the second term is bounded, so we may replace $\triangle_{\bar{\partial}_{A_0}}$ by $\nabla^*_{A_0}\nabla_{A_0}$ at the cost of adding a bounded term to the right hand side. Therefore $\nabla^*_{A_0}\nabla_{A_0}\pi_j$ is bounded uniformly in L^2_{loc}.

We now bootstrap this expression. Since $\nabla^*_{A_0}\nabla_{A_0}$ is elliptic, by the usual elliptic estimate:

$$\|\pi_j\|_{L^2_{2,loc}} \leq C\left(\left\|\nabla^*_{A_0}\nabla_{A_0}\pi_j\right\|_{L^2_{loc}} + \|\pi_j\|_{L^2_1}\right),$$

so π_j is bounded in $L^2_{2,loc}$ and hence in $L^p_{1,loc}$ for $1 \leq p \leq \frac{2n}{n-1}$ by Sobolev imbedding. Therefore, if we consider again the expression for $\triangle_{\bar{\partial}_{A_0}}\pi_j$ above, it follows that $\nabla^*_{A_0}\nabla_{A_0}\pi_j$ is in fact bounded in L^p_{loc} for p in this range. Applying the L^p elliptic estimate:

$$\|\pi_j\|_{L^p_{2,loc}} \leq C\left(\left\|\nabla^*_{A_0}\nabla_{A_0}\pi_j\right\|_{L^p_{loc}} + \|\pi_j\|_{L^2_1}\right),$$

and so π_j is uniformly bounded in $L^p_{2,loc}$ for $1 \leq p \leq \frac{2n}{n-1}$. Therefore in particular $\beta_j = \bar{\partial}_{A_j}\pi_j$ is bounded in L^p_{loc} for all p. Applying the L^p elliptic estimate again implies that β_j is bounded in $L^p_{1,loc}$ for all p, and so by Sobolev imbedding β_j is locally bounded. ■

1.5 Uhlenbeck Compactness, Results of Bando and Siu, Hermitan-Yang-Mills Type Functionals, and a Theorem About the HN type

1.5.1 Uhlenbeck Compactness and Removable Singularities

We now give the statement of the general Uhlenbeck compactness theorem. Although we will be primarily concerned with theorem as it applies to the Yang-Mills flow of the next section, the proof of the main theorem in Section 7 will also rely on this more general statement.

Theorem 7 *Let X be a Kahler manifold (not necessarily compact) and $E \longrightarrow X$ a hermitian vector bundle with metric h. Fix any $p > n$. Let ∇_{A_j} be a sequence of integrable, unitary connections on E, on E such that $\|F_{A_j}\|_{L^2(X)}$ and $\|\Lambda_\omega F_{A_j}\|_{L^\infty(X)}$ are uniformly bounded. Then there is a subsequence (still denoted A_j), a closed subset $Z_{\mathrm{an}} \subset X$ with Hausdorff codimension 4, and a smooth hermitian vector bundle (E_∞, h_∞) defined on the complement $X - Z_{\mathrm{an}}$ with a finite action Yang-Mills connection ∇_{A_∞} on E_∞, such that $\nabla_{A_j|X-Z_{\mathrm{an}}}$ is gauge equivalent to a sequence of connections that converges to ∇_{A_∞} weakly in $L^p_{1,loc}(X - Z_{\mathrm{an}})$.*

The statement of this version of Uhlenbeck compactness may be found for example in Uhlenbeck-Yau (Theorem 5.2). The proof is essentially contained in [U2] and the statement about the singular set follows from the arguments in [NA]. We will call such a limit ∇_{A_∞} an **Uhlenbeck limit**. Furthermore, we have the following crucial extension of this theorem due essentially to Bando and Siu.

Corollary 4 *If in addition to the assumptions in the previous theorem, we also require that:*

$$\left\|d_{A_j}\Lambda_\omega F_{Aj}\right\|_{L^2(X)} \longrightarrow 0,$$

then any Uhlenbeck limit ∇_{A_∞} is Yang-Mills. On $X - Z_{\mathrm{an}}$ we therefore have a holomorphic, orthogonal, splitting:

$$(E_\infty, h_\infty, \nabla_{A_\infty}) = \oplus_{i=1}^{l}(Q_{\infty,i}, h_{\infty,i}, \nabla_{A_{\infty,i}})$$

Moreover E_∞ extends to a reflexive sheaf (still denoted E_∞) on all of X.

Proof. Most of the content of this theorem resides in the last statement, and this is due to Bando-Siu ([BS]) Corollary 2. The statement about the splitting follows directly from the fact that an Uhlenbeck limit is Yang-Mills and Proposition 1. Therefore it only remains to prove that the stated condition implies the first statement. Since $A_j \longrightarrow A_\infty$ weakly in in $L^p_{1,loc}(X - Z_{\mathrm{an}})$, and by the Rellich compactness theorem there is a compact imbedding $L^p_1(X) \hookrightarrow C^0(X)$, we can assume:

$$A_j \longrightarrow A_\infty \textit{ in } C^0_{loc} \textit{ and } \Lambda_\omega F_{A_j} \longrightarrow \Lambda_\omega F_{A_\infty} \textit{ weakly in } L^p_{loc}$$

since we also have a uniform bound on $\left\|\Lambda_\omega F_{A_j}\right\|_{L^\infty(X)}$ by assumption. On the other hand, writing $\nabla_{A_\infty} = \nabla_{A_j} + (\nabla_{A_\infty} - \nabla_{A_j})$, and using the expression for a connection on an associated bundle, we have:

$$d_{A_\infty}\Lambda_\omega F_{A_j} = d_{A_j}\Lambda_\omega F_{A_j} + \left[\nabla_{A_\infty} - \nabla_{A_j}, \Lambda_\omega F_{A_j}\right],$$

where the $[,]$ notation is a combination of wedge-product and composition of endomorphisms. By the previous argument and our additional assumption, this implies $d_{A_\infty}\Lambda_\omega F_{A_j} \longrightarrow 0$ in L^2_{loc}. We claim that also

$$d_{A_\infty}\Lambda_\omega F_{A_j} \longrightarrow d_{A_\infty}\Lambda_\omega F_{A_\infty} \textit{ weakly in } L^2_{loc}.$$

If we locally write $\nabla_{A_\infty} = d + A_\infty$ for some smooth connection d (here we are thinking of A_∞ as the connection 1-form, which is continuous), then again we may write locally:

$$d_{A_\infty}\Lambda_\omega F_{A_j} = d\Lambda_\omega F_{A_j} + \left[A_\infty, \Lambda_\omega F_{A_j}\right].$$

Then for a neighbourhood $W \subset\subset X - Z_{\mathrm{an}}$ and u an L^2 test section of $\mathfrak{u}(E)$, we have:

$$\begin{aligned}
\int_W \left\langle d\Lambda_\omega F_{A_j}, u\right\rangle &= \int_W \left\langle \Lambda_\omega F_{A_j}, d^*u\right\rangle \\
&\longrightarrow \int_W \left\langle \Lambda_\omega F_{A_\infty}, d^*u\right\rangle = \int_W \left\langle d\Lambda_\omega F_{A\infty}, u\right\rangle,
\end{aligned}$$

so $d\Lambda_\omega F_{A_j} \rightharpoonup d\Lambda_\omega F_{A\infty}$ in L^2_{loc}. Similarly the pointwise $U(n)$ invariant inner product $\langle , \rangle$ on $\mathfrak{u}(n)$ enjoys the property $\langle [u,v], w\rangle = \langle u, [v,w]\rangle$ with respect to the bracket. Then for an L^2 test section of $\Omega^1(\mathfrak{u}(E))$:

$$\begin{aligned}
\int_W \left\langle \left[A_\infty, \Lambda_\omega F_{A_j}\right], u\right\rangle &= \int_W \left\langle [A_\infty, u], \Lambda_\omega F_{A_j}\right\rangle \\
&\longrightarrow \int_W \left\langle [A_\infty, u], \Lambda_\omega F_{A_\infty}\right\rangle = \int_W \left\langle [A_\infty, \Lambda_\omega F_{A_\infty}], u\right\rangle.
\end{aligned}$$

Therefore $\left[A_\infty, \Lambda_\omega F_{A_j}\right] \longrightarrow [A_\infty, \Lambda_\omega F_{A_\infty}]$, and so since $d_{A_\infty}\Lambda_\omega F_{A_\infty} = d\Lambda_\omega F_{A\infty} + [A_\infty, \Lambda_\omega F_{A_\infty}]$ the claim follows. Therefore $d_{A_\infty}\Lambda_\omega F_{A_\infty} = 0$, and these are exactly the HYM equations, so A_∞ is HYM and therefore Yang-Mills. ■

Corollary 5 *With the same assumptions as in Theorem 7, $\Lambda_\omega F_{A_j} \longrightarrow \Lambda_\omega F_{A_\infty}$ in $L^p(X - Z_{\rm an})$ for all $1 \leq p < \infty$.*

Proof. Let $\psi_k = \Lambda_\omega F_{A_j} - \Lambda_\omega F_{A_\infty}$. As in the proof of the preceeding theorem, $\psi_k \longrightarrow 0$ weakly in L^p_{loc} and $d_{A_\infty}\psi_k \longrightarrow 0$ strongly in L^2 since ∇_{A_∞} is Yang-Mills. By Kato's inequality we have $|d\,|\psi_k|| \leq |d_{A_\infty}\psi_k|$, so $|\psi_k|$ is bounded on $L^2_{1,loc}$ and therefore $|\psi_k| \longrightarrow 0$ strongly in L^2_{loc}. Since ψ_k is also bounded in L^∞ this implies $|\psi_k| \longrightarrow 0$ in L^p for all p. ■

We may also apply the Uhlenbeck compactness theorem to the sequence of connections given by the flow.

Proposition 11 *Let X be a compact Kahler manifold. Let A_0 be any fixed connection, and A_t denote its evolution along the flow. For any sequence $t_j \longrightarrow \infty$ there is a subsequence (still denoted t_j), a closed subset $Z_{\rm an} \subset X$ with Hausdorff codimension 4, and a smooth hermitian vector bundle (E_∞, h_∞) defined on the complement $X - Z_{\rm an}$ with a finite action Yang-Mills connection A_∞ on E_∞, such that $A_{t_j|X-Z_{\rm an}}$ is gauge equivalent to a sequence of connections that converges to A_∞ in $L^p_{1,loc}(X - Z_{\rm an})$. Away from $Z_{\rm an}$ there is a smooth splitting:*

$$(E_\infty, A_\infty, h_\infty) = \oplus_{i=l}^{l}\,(Q_{\infty,i}, A_{\infty,i}, h_{\infty,i})$$

where $A_{\infty,i}$ is the induced connection on Q_i, and $h_{\infty,,i}$ is an Hermitian-Einstein metric. Furthermore, E_∞ extends over $Z_{\rm an}$ as a reflexive sheaf (still denoted E_∞).

Proof. The functions $\|F_{A_t}\|_{L^2}$ and $\|\Lambda_\omega F_{A_t}\|_{L^\infty}$ are uniformly bounded by parts (1) and (2) of *Lemma* 2 respectively. By [DOKR] Proposition 6.2.14,

$$\lim_{t\longrightarrow\infty} \|\nabla_{A_t}\Lambda_\omega F_{A_t}\|_{L^2} = 0.$$

The remaining statements follow from Corollary 4. ■

Just as before we call A_∞ an Uhlenbeck limit of the flow.

1.5.2 The Kobayashi-Hitchin Correspondence for Reflexive Sheaves

In general, if $\mathcal{E}$ is only a reflexive sheaf, Bando and Siu ([BS]) defined the notion of an **admissable hermitian metric**. This is an hermitian metric h on the locally free part of $\mathcal{E}$ such that:

· $\Lambda_\omega F_h \in L^\infty(X,\omega)$

· $F_h \in L^2(X,\omega)$.

Corollary 4 says that the limiting metric is an admissable hermitian metric on the reflexive sheaf E_∞ that is a direct sum of admissable Hermitian-Einstein metrics. We also point out the version of the Kobayashi-Hitchin correspondence for reflexive sheaves, due to Bando and Siu [BS].

Theorem 8 *(Bando-Siu) A reflexive sheaf $\mathcal{E}$ on a compact Kähler manifold (X,ω) admits an admissible Hermitian-Einstein metric if and only if it is polystable. Such a metric is unique up to a positive constant.*

Note that this theorem says the $\left(Gr_\omega^{HNS}(E)\right)^{**}$ carries an admissible Yang-Mills connection (where admissible has the same meaning for connections), which is unique up to gauge. We sketch a proof of this result in this section..

We will need the following two propositions from [BS]., which we will also use in Chapter 3.

Proposition 12 *Let (X, ω) be an n-dimensional compact Kähler manifold and $\pi : \tilde{X} \longrightarrow X$ a blowup along a compact complex submanifold. Let η be a Kähler metric on $\tilde{X}$ and consider the family of Kähler metrics $\omega_\varepsilon = \pi^*\omega + \varepsilon\eta$ with $0 < \varepsilon \leq 1$. Let K_ε be the heat kernel with respect to the metric ω_ε, then we have a uniform estimate $0 \leq K_\varepsilon \leq C\,(t^{-n} + 1)$.*

In the above proposition we use the general fact that the blowup along a compact complex submanifold of a Kähler manifold is Kähler. We will sketch a proof of this fact in the next chapter. We will use the family ω_ε throughout Chapters 2 and 3.

We will construct the admissible Hermitian-Einstein metric on $\mathcal{E}$, we will patch together metrics on a local resolution by vector bundles. More explicitly, let $\mathcal{E}^*$ be the dual, and recall that locally, there is a resolution of the dual by holomorphic vector bundles. Let U_α be an open subset on which such a presentation exists and let $E^*_{i,\alpha}$ be the bundles in the resolution:

$$E^*_{1,\alpha} \xrightarrow{\phi^*_{0,\alpha}} E^*_{0,\alpha} \xrightarrow{\phi^*_\alpha} \mathcal{E}^*_{|U_\alpha} \longrightarrow 0.$$

Then taking duals we have an inclusion:

$$0 \longrightarrow \mathcal{E}_{|U_\alpha} \xrightarrow{\phi_\alpha} E_{0,\alpha} \xrightarrow{\phi_{0,\alpha}} E_{1,\alpha}.$$

In other words, we may view $\mathcal{E}$ as a subsheaf of locally defined holomorphic vector bundles $E_{0,\alpha}$. Away from $\operatorname{Sing}\mathcal{E}$, this inclusion realises $\mathcal{E}$ as a holomorphic subbundle. The key point is that we can make an actual holomorphic subbundle if we are willing to go to a blowup. Namely, there is a finite sequence of blowups:

$$\tilde{X} = X_k \xrightarrow{\pi_k} X_{k-1} \longrightarrow \dots \longrightarrow X_1 \xrightarrow{\pi_1} X_0 = X$$

along compact, complex submanifolds, such that if we denote by π the composition of all the π_i, then $\pi^*\mathcal{E}/\operatorname{Tor}(\pi^*\mathcal{E})$ is locally free. This is a consequence of Hironaka's flattening theorem, which says that there is such a sequence of blowups such that $\pi^*\mathcal{E}/\operatorname{Tor}(\pi^*\mathcal{E}) = \tilde{\mathcal{E}}$ is flat, together with the fact that a flat module over a local ring is free. We will discuss resolution of singularities in more detail in Chapter 2.

Then on $\tilde{U}_\alpha = \pi^{-1}(U_\alpha)$ there is an inclusion of vector bundles $\tilde{\mathcal{E}} \hookrightarrow E_{0,\alpha} = E_\alpha$ where we continue to denote by $E_{i,\alpha}$ the pullbacks of these bundles to $\tilde{X}$. Now covering $\tilde{X}$ by such neighbourhoods, we can fix hermitian metrics h_α on each E_α and let ρ_α be a partition of unity with respect to $\tilde{U}_\alpha$ then we may write $h = \sum \rho_\alpha \phi_\alpha^* h_\alpha$. This defines an hermitian metric on $\tilde{\mathcal{E}}$ that restricts to an hermitian metric, still denoted h, on $\mathcal{E}_{|X - \operatorname{Sing}\mathcal{E}}$.

Now we would like to deform this metric using the HYM flow on $\tilde{X}$ to an admissable Hermitian-Einstein metric. For the rest of this section we will denote objects on the blowups and on the base by the same symbols without reference to the pullback. Fix arbitrary Kähler metrics η_i on X_i ane write $\omega_{i,\varepsilon} = \omega + \varepsilon_1\eta_1 + \dots + \varepsilon_i\eta_i$. Now consider the HYM flow equations on $\tilde{X}$:

$$h_t^{-1}\frac{dh_t}{dt} = -\left(\sqrt{-1}\Lambda_{\omega_{k,\varepsilon}}F_{h_t} - \lambda(\mathcal{E})Id\right),\ h(0) = h,$$

where

$$\lambda(\mathcal{E}) = 2\pi n\mu(\mathcal{E})/\int_X \omega^n.$$

By Donaldson's work this equation has a long-time solution. Now the curvature enjoys the following properties:

· $\mathbf{P}_1$

$$\frac{\partial \left|\Lambda_{\omega_{k,\varepsilon}} F_{h_t}\right|^2}{\partial t} = -\triangle \left|\Lambda_{\omega_{k,\varepsilon}} F_{h_t}\right|^2 - 2\left|d^*_{A_t} F_{h_t}\right|^2$$

· $\mathbf{P}_2$

$$\frac{\partial \left|\Lambda_{\omega_{k,\varepsilon}} F_{h_t}\right|}{\partial t} \leq -\triangle \left|\Lambda_{\omega_{k,\varepsilon}} F_{h_t}\right|$$

· $\mathbf{P}_3$

$$\frac{d}{dt}\int_{X_k} \left|\Lambda_{\omega_{k,\varepsilon}} F_{h_t}\right|^2 = -\int_{X_k} \left|\nabla \Lambda_{\omega_{k,\varepsilon}} F_{h_t}\right|^2$$

· $\mathbf{P}_4$

$$\int_{X_k} \left|\Lambda_{\omega_{k,\varepsilon}} F_{h_t}\right|(y) \leq \int_{X_k} \left|\Lambda_{\omega_{k,\varepsilon}} F_h\right|(y)$$

· $\mathbf{P}_5$

$$\left|\Lambda_{\omega_{k,\varepsilon}} F_{h_t}\right|(x) \leq \int_{X_k} K^t_{\omega_{k,\varepsilon}}(x,y)\left|\Lambda_{\omega_{k,\varepsilon}} F_h\right|(y)$$

where $K^t_{\omega_{k,\varepsilon}}(x,y)$ is the heat kernel with respect to $\omega_{k,\varepsilon}$ and h is the metric constructed above on $\tilde{\mathcal{E}}$. Then for a fixed ε_1 we have:

$$\omega^n_{i,\varepsilon} = \frac{\det g_{i,\varepsilon}}{\det g_{i,\varepsilon_1}} \omega^n_{i,\varepsilon_1}$$

and

$$\left|\Lambda_{\omega_{i,\varepsilon}} F_h\right| = \left|\frac{F_h \wedge \omega^{n-1}_{i,\varepsilon}}{\omega^n_{i,\varepsilon}}\right|$$

so

$$\left|\Lambda_{\omega_{i,\varepsilon}} F_h\right| \omega^n_{i,\varepsilon} = \left|\frac{F_h \wedge \omega^{n-1}_{i,\varepsilon}}{\omega^n_{i,\varepsilon_1}}\right| \omega^n_{i,\varepsilon_1}$$

and clearly this is uniformly bounded since as $\varepsilon \longrightarrow 0$ we have $\omega_{i,\varepsilon} \longrightarrow \omega$. Therefore $\Lambda_{\omega_{i,\varepsilon}} F_h$ is uniformly integrable.

Now Proposition 12 says that if we fix a $k-1$ tuple $\varepsilon^{'} = (\varepsilon_1, ..., \varepsilon_{k-1})$, then the heat kernel $K^t_{\omega_{k,\varepsilon}}(x,y)$ has a uniform bound. Furthermore, outside the exceptional divisor, $K^t_{\omega_{k,\varepsilon}}(x,y)$ converges to $K^t_{\omega_{k,\varepsilon'}}(x,y)$ as $\varepsilon_k \longrightarrow 0$. $\mathbf{P}_4$ and $\mathbf{P}_5$ together with the above discussion imply that $\left|\Lambda_{\omega_{k,\varepsilon}} F_{h_t}\right|$ has a uniform L^1 bound for $t \geq 0$ and a uniform L^∞ bound for $t \geq t_0 > 0$, or on a compact set disjoint from the exceptional divisor. The usual relationship between the full curvature and the Hermitian-Einstein tensor now give a uniform L^2 bound on F_{h_t}.

This means that for any fixed $t > 0$, as $\varepsilon_k \longrightarrow 0$ the limit $h_{t,\varepsilon'} = \lim_{\varepsilon_k \longrightarrow 0} h_{t,\varepsilon}$ solves the HYM equations on X_{k-1} and is an admissible metric. Continuing by induction, for each $t > 0$ we obtain an admissible hermitian metric h_t on $\mathcal{E}$ solving the HYM equations on $X - \mathrm{Sing}(\mathcal{E})$.

Now if $\mathcal{E}$ is stable, then Theorem 4 implies that there is a sequence of times t_i such that h_{t_i} converges to an admissible Hermitian-Einstein metric. More generally, in the polystable case we have obtained an admissible Hermitian-Einstein metric.

If $\mathcal{E}$ is a general reflexive sheaf, $\mathbf{P}_3$ still holds for the family of metrics h_t, so integrating gives:

$$\int_{t_0}^{\infty} \int_X |\nabla \Lambda_\omega F_{h_t}|^2 \leq \int_X \left|\Lambda_\omega F_{h_{t_0}}\right|^2.$$

In other words, there is a subsequence of times t_i such that

$$\int_X \left|\nabla \Lambda_\omega F_{h_{t_i}}\right|^2 \longrightarrow 0.$$

Now by Corollary 4 there is a subset $S \subset X - \mathrm{Sing}(\mathcal{E})$ or Hausdorff codimension 4, and a further subsequence of times t_i such that h_{t_i} converges to a weak solution h_∞ of the equation $\nabla \Lambda_\omega F_{h_\infty} = 0$. Then, as usual, the limiting bundle E_∞ defined on $X - (\mathrm{Sing}(\mathcal{E}) \cup S)$ breaks up into a direct sum of the eigenspaces of F_{h_∞}. Furthermore these bundles extend to reflexive sheaves over $\mathrm{Sing}(\mathcal{E}) \cup S$. This shows that the limiting sheaf we have $\mathcal{E}_\infty$ breaks up into a direct sum of reflexive sheaves admitting admissible Hermitian-Einstein metrics, which is exactly what we claimed in Proposition 11.

Furthermore, the following can be used to show uniqueness up to a positive constant.

Proposition 13 *Let $(\mathcal{E}, h)$ be a reflexive sheaf with an admissible Einstein-Hermitian metric on a compact Kahler manifold (X, ω). If $\mu(\mathcal{E}) < 0$ $(= 0)$ then $\mathcal{E}$ admits only the zero section (every section is parallel).*

Proof. If s is a global section of $\mathcal{E}$, then [BS] Theorem 2 b) gives a bound on $|s|$ on all of X. It satisfies:

$$\triangle |s|^2 = |\nabla s|^2 - \langle (\Lambda_\omega F_h)\, s, s\rangle = |\nabla s|^2 - \lambda(\mathcal{E})\, |s|^2 \geq 0.$$

Since subharmonic functions satisfy the maximum principle $|s|$ is constant, which implies $|\nabla s|^2 = \lambda(\mathcal{E})\, |s|^2$ and the result follows. ■

1.5.3 A Remark About the the HN Type of the Limit

Lemma 8 *Let A_{t_j} be a sequence of connections along the YM flow with Uhlenbeck limit A_∞. Then For $t_j \geq t_0 \geq 0$,*

$$\|\Lambda_\omega F_{A_\infty}\|_{L^\infty} \leq \left\|\Lambda_\omega F_{A_{t_j}}\right\|_{L^\infty} \leq \left\|\Lambda_\omega F_{A_{t_0}}\right\|_{L^\infty}.$$

Proof. Again, $|\Lambda_\omega F_{A_t}|^2$ is decreasing in t. Fix $t \geq 0$. Then for any $1 \leq p < \infty$ and j sufficiently large we have:

$$\left\|\Lambda_\omega F_{A_{t_j}}\right\|_{L^p} \leq (2\pi)^{\frac{1}{p}} \left\|\Lambda_\omega F_{A_{t_j}}\right\|_{L^\infty} \leq (2\pi)^{\frac{1}{p}} \left\|\Lambda_\omega F_{A_t}\right\|_{L^\infty}$$

(recall $vol(X) = 2\pi$). By *Corollary* 4,

$$\lim_{j \cdot \infty} \left\|\Lambda_\omega F_{A_{t_j}}\right\|_{L^p} = \left\|\Lambda_\omega F_{A_\infty}\right\|_{L^p},$$

for all p. So

$$\left\|\Lambda_\omega F_{A_\infty}\right\|_{L^p} \leq (2\pi)^{\frac{1}{p}} \left\|\Lambda_\omega F_{A_t}\right\|_{L^\infty}$$

for all p. Therefore letting $p \longrightarrow \infty$,

$$\left\|\Lambda_\omega F_{A_\infty}\right\|_{L^\infty} \leq \left\|\Lambda_\omega F_{A_t}\right\|_{L^\infty}.$$

■

Lemma 9 *If A_∞ is an Uhlenbeck limit of A_{t_j}, then $\Lambda_\omega F_{A_j} \longrightarrow \Lambda_\omega F_{A_\infty}$ in L^p for all $1 \leq p < \infty$. Moreover, $\lim_{t \longrightarrow \infty} HYM(A_t) = HYM(A_\infty)$.*

Proof. The first part is immediate from *Corollary* 5. The second statement is immediate from the facts that $t \longrightarrow HYM(A_t)$ is non-increasing, and

$$HYM(A_{t_j}) \longrightarrow HYM(A_\infty).$$

■

We will need the following lemma from linear algebra:

Lemma 10 *Let V be a finite dimensional hermitian vector space of complex dimension R, and $L \in End(V)$ an hermitan operator with eigenvalues $\lambda_1 \geq ... \geq \lambda_R$*

(counted with multiplicities). Let $\pi = \pi^2 = \pi^*$ *denote the orthogonal projection onto a subspace of dimension* r*. Then* $Tr(L\pi) \leq \sum_{i \leq r} \lambda_i$.

For a sketch of the proof see [DW1] *Section* 2.3. Now we discuss the HNS type of an Uhlenbeck limit.

Lemma 11 *Let* $A_j = g_j(A_0)$ *be a sequence of complex gauge equivalent integrable connections in a complex vector bundle of rank* R *with hermitian metric* h_0*. Let* S *be a coherent subsheaf of* $(E, \bar{\partial}_{A_0})$ *of rank* r*. Suppose that* $\sqrt{-1}\Lambda_\omega F_{A_j} \longrightarrow \mathbf{v}$ *in* L^1*, where* $\mathbf{v} \in L^1(\sqrt{-1}\mathfrak{u}(E))$*, and that the eigenvalues* $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_R$ *of* $\mathbf{v}$ *counted with multiplicities are constant almost everywhere. Then:* $\deg(S) \leq \sum_{i \leq r} \lambda_i$.

Proof. As stated earlier, $\deg(S) \leq \deg(\mathrm{Sat}_E(S))$, so we may assume S is saturated. Let π_j denote the weakly holomorphic projection to $g_j(S)$ with respect to h_0. Then by the Chern-Weil formula:

$$\begin{aligned}
\deg(S) &= \frac{1}{2\pi n}\int_X \left(\mathrm{Tr}(\sqrt{-1}\Lambda_\omega F_{A_j} - \left|\bar{\partial}_{A_j}\pi_j\right|^2\right)\omega^n \\
&\leq \frac{1}{2\pi n}\int_X \mathrm{Tr}(\sqrt{-1}\Lambda_\omega F_{A_j\pi_j})\omega^n \\
&= \frac{1}{2\pi n}\int_X \mathrm{Tr}(\mathbf{v}\pi_j)\omega^n - \frac{1}{2\pi n}\int_X \mathrm{Tr}\left((\sqrt{-1}\Lambda_\omega F_{A_j} - \mathbf{v})\pi_j)\right)\omega^n.
\end{aligned}$$

Therefore since $\|\pi_j\|_{L^\infty} \leq 1$, $vol(X) = \frac{2\pi}{(n-1)!}$, by the previous lemma we have:

$$\deg(S) \leq \sum_{i \leq r} \lambda_i + \frac{1}{2\pi n}\left\|\sqrt{-1}\Lambda_\omega F_{A_j} - \mathbf{v}\right\|_{L^1}.$$

Letting $j \longrightarrow \infty$ we have the result. ■

The following simple fact will be crucial in section 5.

Proposition 14 *Let A_j be a sequence of connections along the YM flow on a holomorphic vector bundle of rank R, with Uhlenbeck limit A_∞. Let μ_0 be the HNS type of E with holomorphic structure $\bar{\partial}_{A_0}$. Let λ_∞ be the HNS type of $\bar{\partial}_{A_\infty}$. Then $\mu_0 \leq \lambda_\infty$.*

Proof. Let $0 = E_0 \subset E_1 \subset ... \subset E_l = E_{\bar{\partial}_{A_0}}$ be the HNS filtration of $\bar{\partial}_{A_0}$., and let $\mu_0 = (\mu_1, ..., \mu_R)$ (here we are ignoring the notation indicating the fact that it is a double filtration). Then

$$\deg(E_i) = \sum_{j \leq r\,\mathrm{rk}(E_i)} \mu_j.$$

By *Lemma* 9, $\Lambda_\omega F_{A_j} \longrightarrow \Lambda_\omega F_{A_\infty}$ in L^1. The type $\lambda_\infty = (\lambda_1, ..., \lambda_R)$ corresponds to the (constant) eigenvalues of $\Lambda_\omega F_{A_\infty}$. By the previous lemma applied to $S = E_i$, we have

$$\deg(E_i) \leq \sum_{j \leq rk(E_i)} \lambda_j.$$

Therefore

$$\sum_{j \leq \mathrm{rk}(E_i)} \mu_j \leq \sum_{j \leq rk(E_i)} \lambda_j,$$

and the result follows from *Lemma* 5. ■

Corollary 6 *Let $\mu = (\mu_1, ..., \mu_R)$ be the HNS type of a rank R holomorphic vector bundle $(E, \bar{\partial}_E)$ on X. Then*

$$\sum_{i=1}^{R} \mu_i^2 \leq \frac{1}{2\pi n} \int_X |\Lambda_\omega F_A|^2 \, \omega^n$$

and

$$\left(\sum_{i=1}^{R} \mu_i^2 \right)^{\frac{1}{2}} \leq \frac{1}{2\pi n} \int_X |\Lambda_\omega F_A| \, \omega^n$$

for all unitary connections ∇_A in the $\mathcal{G}^{\mathbb{C}}$ orbit of $(E, \bar{\partial}_E)$.

Proof. Let A_t denote the YM flow with initial condition A. By 3.1(2) we know:

$$\int_X |\Lambda_\omega F_{A_t}|^2 \omega^n \leq \int_X |\Lambda_\omega F_A|^2 \omega^n,$$

for every $t \geq 0$. Let A_∞ be an Uhlenbeck limit along a subsequence $t_j \longrightarrow \infty$. By *Lemma* 9,

$$\int_X |\Lambda_\omega F_{A_\infty}|^2 \omega^n = \lim_{j\longrightarrow\infty} \int_X \left|\Lambda_\omega F_{A_{t_j}}\right|^2 \omega^n.$$

As before, $\sqrt{-1}\Lambda_\omega F_{A_\infty}$ has constant eigenvalues λ_∞ and by the previous proposition $\mu \leq \lambda_\infty$. It follows from Atiyah-Bott, 12.8 that

$$\sum_{i=1}^R \mu_i^2 \leq \sum_{i=1}^R \lambda_i^2.$$

This, together with the previous two inequalities and the normalisation $vol(X) = \frac{2\pi}{(n-1)!}$ gives the first result. The second follows in exactly the same way. ■

1.5.4 Hermitian-Yang-Mills Type Functionals

The YM and HYM functionals are not sufficient to distinguish different HNS types in general. In other words there may be multiple connections with the same YM number, but which induce holomorphic structures with different HNS types. In this subsection we introduce generalisations of the HYM functional that can be used to distinguish different types. This is only a technical device, but will be used essentially in *Section* 5.

Write $\mathfrak{u}(R)$ for the Lie algebra of the unitary group $U(R)$. Fix a real number $\alpha \geq 1$. The for $\mathbf{v} \in \mathfrak{u}(R)$, a skew hermitian matrix with eigenvalues $\sqrt{-1}\lambda_1, ..., \sqrt{-1}\lambda_R$, let $\varphi_\alpha(\mathbf{v}) = \sum_{i=1}^R |\lambda_i|^\alpha$. It can be seen that there is a family $\varphi_{\alpha,\rho}$, $0 < \rho \leq 1$, of smooth convex Ad-invariant functions such that $\varphi_{\alpha,\rho} \longrightarrow \varphi_\alpha$ uniformly on compact subsets of $\mathfrak{u}(R)$. By Atiyah-Bott, Proposition 12.16, φ_α is a convex function on $\mathfrak{u}(R)$. We may consider a section $\sigma \in \Gamma(X, \mathfrak{u}(E))$ as collection of local sections $\{\sigma_\beta\}$ such that $\sigma_\beta = \mathrm{Ad}(g_{\beta\rho})\sigma_\rho$ where $g_{\beta\rho}$ are the transition functions for E. By the Ad-invariance of φ_α, $\varphi_\alpha(\sigma_\beta) = \varphi_\alpha(\sigma_\rho)$, so φ_α induces a well-defined function Φ_α on $\mathfrak{u}(E)$. Then for a fixed real number N, define:

$$HYM_{\alpha,N}(A) = \int_X \Phi_\alpha(\Lambda_\omega F_A + \sqrt{-1} N \mathbf{Id}_E) dvol_\omega$$

and $HYM_\alpha(A) = HYM_{\alpha,0}(A)$. Note that $HYM = HYM_2$ is the usual HYM functional. In the sequel we will write:

$$\begin{aligned} HYM_{\alpha,N}(\mu) &= HYM_\alpha(\mu + N) = \frac{2\pi}{(n-1)!}\Phi_\alpha(\sqrt{-1}\,(\mu+N)), \\ where\ \mu + N &= (\mu_1 + N, ..., \mu_R + N) \end{aligned}$$

is identified with the matrix $\mathrm{diag}\,(\mu_1 + N, ..., \mu_R + N)$. Therefore:

$$HYM(\mu) = \frac{2\pi}{(n-1)!}\sum_{i=1}^{R} \mu_i^2.$$

We have the following elementary lemma.

Lemma 12 *The functional* $\mathbf{v} \longrightarrow \left(\int_X \Phi_\alpha(\mathbf{v})\right)^{\frac{1}{\alpha}}$ *is equivalent to the* $L^\alpha(\mathfrak{u}(E))$ *norm.*

Proof. There are universal constants C_1 and C_2 (depending on R) such that for any real numbers $\lambda_1, ..., \lambda_R$, and $\alpha \geq 1$:

$$\begin{aligned} \frac{1}{C_1}\left(\sum_{i=1}^{R} |\lambda_i|^2\right)^{\frac{\alpha}{2}} &\leq \frac{1}{C_1}\left(\sum_{i=1}^{R} |\lambda_i|\right)^{\alpha} \leq \sum_{i=1}^{R} |\lambda_i|^\alpha \\ &\leq C_1\left(\sum_{i=1}^{R} |\lambda_i|\right)^{\alpha} \leq C_2\left(\sum_{i=1}^{R} |\lambda_i|^2\right)^{\frac{\alpha}{2}}. \end{aligned}$$

Applied to the eigenvalues of $\mathbf{v}$, this gives:

$$\frac{1}{C_1}\int_X (\mathrm{Tr}\,\mathbf{v}\mathbf{v}^*)^{\frac{\alpha}{2}}\,dvol_\omega \leq \int_X \Phi_\alpha(\mathbf{v})dvol_\omega \leq C_2\int_X (\mathrm{Tr}\,\mathbf{v}\mathbf{v}^*)^{\frac{\alpha}{2}}\,dvol_\omega.$$

■

The following three propositions will be crucial in *Section* 5.

Proposition 15 (1) *If* $\mu \leq \lambda$, *then* $\Phi_\alpha(\sqrt{-1}\mu) \leq \Phi_\alpha(\sqrt{-1}\lambda)$ *for all* $\alpha \geq 1$.

(2) *Assume* $\mu_R \geq 0$ *and* $\lambda_R \geq 0$. *If* $\Phi_\alpha(\sqrt{-1}\mu) = \Phi_\alpha(\sqrt{-1}\lambda)$ *for all* α *in some set*

$A \subset [1,\infty)$ *possessing a limit point, then* $\mu = \lambda$.

Proof. (1) follows from Atiyah-Bott 12.8. For (2), consider $f(\alpha) = \Phi_\alpha(\sqrt{-1}\lambda)$ and $g(\alpha) = \Phi_\alpha(\sqrt{-1}\mu)$ as functions of α. These functions have complex anlalytic extensions to $\mathbb{C} - \{\alpha \leq 0\}$. If $f(\alpha) = g(\alpha)$ for all $\alpha \in A$, then by the uniqueness principal for analytic functions, $f = g$ on $\mathbb{C} - \{\alpha \leq 0\}$. If $\mu \neq \lambda$, then there is some $1 \leq k \leq R$, such that $\mu_i = \lambda_i$ for $i < k$,and $\mu_k \neq \lambda_k$. Without loss of generality assume $\mu_k > \lambda_k$. Then for any $\alpha > 0$:

$$\left(\frac{\mu_k}{\lambda_k}\right)^\alpha \leq \sum_{i=k}^{R}\left(\frac{\mu_i}{\lambda_k}\right)^\alpha = \sum_{i=k}^{R}\left(\frac{\lambda_i}{\lambda_k}\right)^\alpha = R.$$

Letting $\alpha \longrightarrow \infty$ therefore gives a contradiction. ■

Proposition 16 *Let* A_t *be a solution of the* YM *flow. Then for any* $\alpha \geq 1$ *and any* N, $t \longrightarrow HYM_{\alpha,N}(A_t)$ *is non-increasing.*

Proof. Since φ_α can be approximated by smooth, convex, ad-invariant functions $\varphi_{\alpha,\rho} \longrightarrow \varphi_\alpha$, it is enough to show that

$$t \longrightarrow \int_X \Phi_{\alpha,\rho}(\Lambda_\omega F_{A_t} + \sqrt{-1}N\mathbf{Id}_E)dvol_\omega$$

is non-increasing along the flow for each ρ. This follows from the fact that $\Phi_{\alpha,\rho}(\Lambda_\omega F_A + \sqrt{-1}N\mathbf{Id}_E)$ is a subsolutions of the heat equation, which we now show. Let $\sigma = \Lambda_\omega F_A + \sqrt{-1}N\mathbf{Id}_E$ and $\Phi = \Phi_{\alpha,\rho}$. We claim:

$$\triangle\left(\Phi\circ\sigma\right)(x) = -*\varphi^{''}_{\sigma(x)}\left\langle *\nabla_{A_t}\sigma, \nabla_{A_t}\sigma\right\rangle + \varphi^{'}_{\sigma(x)}(\triangle_{A_t}\sigma)(x).$$

We will explain our notation as we derive this formula. We have:

$$\begin{aligned}\triangle\left(\Phi\circ\sigma\right)(x) &= -*d*d(\Phi\circ\sigma)(x)\\ &= -*d*d_{\sigma(x)}\Phi(d\sigma)_x.\end{aligned}$$

Now note that if we fix any connection A on E, we may think of this as a horizontal splitting $\mathcal{H}$ of the associated principal bundle P. Thinking of σ as a map $\hat{\sigma} : P \longrightarrow \mathfrak{u}(n)$, we have $\Phi\circ\sigma(x) = \varphi\circ\hat{\sigma}(p)$ for any $p \in P_x$ and so $d_{\sigma(x)}\Phi(d\sigma)_x = d_{\hat{\sigma}(p)}\varphi(d\hat{\sigma})_p$. The derivative $d\hat{\sigma}$ splits as $d\hat{\sigma}_{|\mathcal{H}} \oplus d\hat{\sigma}_{|\mathcal{H}^\perp}$ where $\mathcal{H}^\perp$ consists of the tangent directions to the fibres P_x. Since $\varphi\circ\hat{\sigma}$ is constant on the fibres, $d\varphi\circ d\hat{\sigma}_{|\mathcal{H}^\perp} = 0$. Thus $d\Phi d\sigma = d\varphi d\hat{\sigma}_{|\mathcal{H}}$, but $d\hat{\sigma}_{|\mathcal{H}}$ is precisely the induced covariant derivative $\nabla_A\hat{\sigma}$. Therefore, appying this argument to a connection A_t along the flow, we may write:

$$\triangle\left(\Phi\circ\sigma\right) = -*d*d\varphi(\sigma)(\nabla_{A_t}\hat{\sigma}).$$

Now since $d\varphi \in \Omega^1(\mathfrak{su}(n))$ and $T^*\mathfrak{su}(n) = \mathfrak{su}(n)\times\mathfrak{su}(n)^*$ we may think of $d\varphi$ as a map $\varphi^{'} : \mathfrak{su}(n) \longrightarrow \mathfrak{su}(n)^*$ so that $\varphi^{'}(\sigma) : P \longrightarrow \mathfrak{su}(n)^*$. The expression

$$\varphi^{'}(\hat{\sigma})(\nabla_{A_t}\hat{\sigma})$$

may therefore be thought of as an element of $\Omega^1(P)$, and we interpret this expression as evaluation in the lie algebra component and multiplication in the form component.

Therefore we have

$$\begin{aligned} -*d*\varphi^{'}(\hat{\sigma})(\nabla_{A_t}\hat{\sigma}) &= -*d(\varphi^{'}(\hat{\sigma})(*\nabla_{A_t}\hat{\sigma})) \\ & -*d\left(\varphi^{'}(\hat{\sigma})\right)(*\nabla_{A_t}\sigma) - *\varphi^{'}(\tilde{\sigma})(d(*\nabla_{A_t}\hat{\sigma})) \end{aligned}$$

Differentiating again, for each $p \in P$ we may also think of $\varphi^{''}$ as a map $\varphi^{''}_{\hat{\sigma}(p)}$: $\mathfrak{su}(n) \longrightarrow \mathfrak{su}(n)^*$ or alternatively as a pairing $\varphi^{''}_{\hat{\sigma}(p)}(-,-)$ on $\mathfrak{su}(n)$. Then with this notation, since the maps $\varphi^{'}$ and $\varphi^{''}$ are also Ad-invariant, we have $d\left(\varphi^{'}(\hat{\sigma})\right) = \varphi^{''}(-,\nabla_{A_t}\hat{\sigma})$, and $\varphi^{'}(\tilde{\sigma})(d(*\nabla_{A_t}\sigma))(p) = \varphi^{'}_{\hat{\sigma}(p)}(\nabla_{A_t} * \nabla_{A_t}\hat{\sigma})$ so

$$\begin{aligned} \triangle(\Phi\circ\sigma)(x) &= -*\varphi^{''}_{\hat{\sigma}(p)} * (\nabla_{A_t}\hat{\sigma},\nabla_{A_t}\hat{\sigma}) - *\varphi^{'}_{\hat{\sigma}(p)}(\nabla_{A_t} * \nabla_{A_t}\hat{\sigma}) \\ &= -*\varphi^{''}_{\hat{\sigma}(p)}(*\nabla_{A_t}\hat{\sigma},\nabla_{A_t}\hat{\sigma}) + d\Phi_{\sigma(x)}(\triangle_{A_t}\sigma) \\ &\leq d\Phi_{\sigma(x)}(\triangle_{A_t}\sigma). \end{aligned}$$

In the last line we have used the fact that $\varphi^{''}$ is positive definite (φ is convex) and that $\varphi^{''}$only acts on the lie algebra component. This proves the claim. this implies:

$$\begin{aligned} \triangle\Phi_{\alpha,\rho}\left(\Lambda_\omega F_{A_t} + \sqrt{-1}N\mathbf{Id}_E\right) &\leq d\Phi_{\alpha,\rho}\left(\triangle_{A_t}\left(\Lambda_\omega F_{A_t} + \sqrt{-1}N\mathbf{Id}_E\right)\right) \\ &= -d\Phi_{\alpha,\rho}\left(\frac{\partial\Lambda_\omega F_{A_t}}{\partial t}\right) \\ &= -\frac{\partial}{\partial t}\Phi_{\alpha,\rho}\left(\left(\Lambda_\omega F_{A_t} + \sqrt{-1}N\mathbf{Id}_E\right)\right). \end{aligned}$$

■

Proposition 17 *Let A_∞ be a subsequential Uhlenbeck limit of A_t where A_t is a solution of the YM flow. Then for all $\alpha \geq 1$,*

$$\lim_{t\longrightarrow\infty} HYM_{\alpha,N}(A_t) = HYM_{\alpha,N}(A_\infty).$$

Proof. If we write t_j for the subsequence, then by Lemma 9 we have $\Lambda_\omega F_{A_{t_j}} \xrightarrow{L^p} \Lambda_\omega F_{A_\infty}$, so by Lemma 12 it follows that $\lim_{t \longrightarrow \infty} HYM_{\alpha,N}(A_{t_j}) = HYM_{\alpha,N}(A_\infty)$. The statement now follows from Proposition 16. ■

Chapter 2

Resolution of Singularities and Approximate Critical Structures

2.1 Properties of Blowups and Resolution of the HNS Filtration

In this section we discuss the properties of blowups of complex manifolds along complex submanifolds that will be used in the subsequent discussion. Essentially all of this material is standard, but we review it carefully now because we will need to employ these facts often in the proofs of the main results.

2.1.1 Resolution of Singularities Type Theorems

The HNS filtration is in general only given by subsheaves, making it difficult to do analysis. We will therefore need some way of obtaining a filtration by subbundles, that is, a way of resolving the singularities. In two dimensions, when the singular set consists of point singularities this can be done by hand (see [BU1]), but in higher dimensions the only available tool seems to be the general resolution of singularities theorem of Hironaka. Specifically:

Theorem 9 *(Resolution of Singularities) Let X be a compact, complex space (or $\mathbb{C}$-scheme). Then there exists a finite sequence of of blowups with smooth centres:*

$$\tilde{X} = X_m \xrightarrow{\pi_m} X_{m-1} \longrightarrow \ldots \longrightarrow X_1 \xrightarrow{\pi_1} X_0 = X$$

such that $\tilde{X}$ is compact and non-singular (a complex manifold) and the centre Y_{j-1}

of each blowup π_j is contained in the singular locus of X_{j-1}.

For the proof see [H1] and [H2]. What we will actually use is the following corollary:

Corollary 7 *(Resolution of the Locus of Indeterminacy) Let X and Y be compact, complex spaces and let $\varphi : X \dashrightarrow Y$ be a rational (meromorphic) map. Then there exists a compact, complex space $\tilde{X} \xrightarrow{\pi} X$ obtained from X by a sequence of blowups with smooth centres and a holomorphic map $\psi : \tilde{X} \to Y$ such that the following diagram commutes:*

$$\begin{array}{ccc} \widetilde{X} & & \\ \downarrow & \searrow & \\ X & \underset{\varphi}{\dashrightarrow} & Y \end{array} .$$

In our case both X and Y (and hence also $\tilde{X}$) will be complex manifolds. Note that in this case a blowup with "smooth centre" is the same as the blowup along a complex submanifold. We will apply the Corollary in the following way.

The HNS filtration of a bundle E, which in the sequel we will abbreviate for simplicity as:

$$0 = E_0 \subset E_1 \subset ... \subset E_{l-1} \subset E_l = E$$

(i.e. we ignore the notation indicating that it is a double filtration), as stated previously, is in general a filtration only by subsheaves of E. We may think of a subbundle $S \subset E$ of rank k as a holomorphic section of the Grassmann bundle $Gr(k, E)$, the bundle whose fibre at each point is the set of k-dimensional complex subspaces of the fibre of E. Similarly a filtration by subbundles corresponds to a holomorphic section of the partial flag bundle $\mathbb{FL}(d_1, ..., d_l, E)$, the bundle whose fibre at each point is the set of l flags of type $(d_1, ..., d_l)$ where $d_i = rk(E_i)$. On the other hand a filtration by subsheaves corresponds to a rational section $X \overset{\sigma}{\dashrightarrow} \mathbb{FL}(d_1, ..., d_l, E)$. The corollary says that by blowing up finitely many times along complex submanifolds, we obtain an honest section $\tilde{X} \rightarrow \mathbb{FL}(d_1, ..., d_l, \pi^* E)$. More explicity, we have a diagram:

$$\begin{array}{ccc} \tilde{X} & \overset{\tilde{\sigma}}{\underset{}{\overset{\longrightarrow}{\longleftarrow}}} & \mathbb{FL}(\pi^* E) \\ \downarrow & \searrow & \downarrow \\ X & \overset{\sigma}{\underset{p}{\overset{\dashrightarrow}{\longleftarrow}}} & \mathbb{FL}(E) \end{array}$$

where $\tilde{\sigma}$ will be constructed below. The outer square is just the pullback diagram for the map $\tilde{X} \xrightarrow{\pi} X$. First we claim that the triangle:

$$\begin{array}{ccc} \tilde{X} & & \\ \downarrow & \searrow & \\ X & \longleftarrow & \mathbb{FL}(E) \end{array}$$

commutes. If we write ψ for the desingularised map $\tilde{X} \longrightarrow \mathbb{FL}(E)$, then note that for a point $\tilde{x} \in \tilde{X} - \mathbf{E}$, we have $\psi(\tilde{x}) = \psi(\pi^{-1}(x))$ for $x \in Z_{\mathrm{alg}}$. Then we have: $p(\psi(\tilde{x})) = p(\sigma(\pi(\tilde{x}))) = x = \pi(\tilde{x})$ since σ is well-defined and a section away from Z_{alg} and we know the diagram:

$$\begin{array}{ccc} \tilde{X} & & \\ \downarrow & \searrow & \\ X & \overset{\sigma}{\dashrightarrow} & \mathbb{FL}(E) \end{array}$$

commutes. In other words on $\tilde{X} - \mathbf{E}$ we have $p \circ \psi = \pi$. But since both of these are holomorphic maps $\tilde{X} \longrightarrow X$, $p \circ \psi = \pi$ on $\tilde{X}$ by the uniqueness principle for holomorphic maps, since they agree on a non-empty open subset. Now $\mathbb{FL}(\pi^*E) = \pi^*\mathbb{FL}(E) = \{(\tilde{x}, \nu) \in \tilde{X} \times \mathbb{FL}(E) \mid \pi(\tilde{x}) = p(\nu)\}$. Now define $\tilde{\sigma} : \tilde{X} \longrightarrow \mathbb{FL}(\pi^*E)$ by $\tilde{\sigma}(\tilde{x}) = (\tilde{x}, \psi(\tilde{x}))$. Since $p \circ \psi = \pi$ this is indeed a map into $\mathbb{FL}(\pi^*E)$, and it is manifestly a section.

In other words there is a filtration of π^*E:

$$0 = \tilde{E}_0 \subset \tilde{E}_1 \subset \ldots \subset \tilde{E}_{l-1} \subset \tilde{E}_l = \pi^*E$$

where the $\tilde{E}_i$ are subbundles.

Now note that we have the following diagram:

$$\begin{array}{ccc} & & \tilde{Q}_i^E \\ & & \uparrow \\ & & \pi^*E \\ & \nearrow & \uparrow \\ \pi^*E_i & \dashrightarrow & \tilde{E}_i \end{array}$$

where the dashed line is the rational map corresponding to the equality of π^*E_i and $\tilde{E}_i$ away from $\mathbf{E}$ (both are equal to E_i), and $\tilde{Q}_i^E$ is the quotient of π^*E by $\tilde{E}_i$. Then $\tilde{Q}_i^E$ is a vector bundle and in particular torsion free. On the other hand the image of π^*E_i under the composition $\pi^*E_i \to \pi^*E \to \tilde{Q}_i^E$ is torsion since it is supported on the divisor $\mathbf{E}$, and hence must be zero. If we write $\operatorname{Im} \pi^*E_i$ for the image of $\pi^*E_i \longrightarrow \pi^*E$, this means there is an actual inclusion of sheaves $\operatorname{Im} \pi^*E_i \hookrightarrow \tilde{E}_i$. The quotient sheaf $\tilde{E}_i / \operatorname{Im} \pi^*E_i$ is supported on $\mathbf{E}$, hence torsion and so it follows from Lemma 4 that $\tilde{E}_i = \operatorname{Sat}_{\pi^*E}(\operatorname{Im} \pi^*E_i)$.

Since $\pi_* \tilde{E}_i$ is equal to E_i away from $\operatorname{Sing} E_i$ there is a birational map $E_i \dashrightarrow \pi_* \tilde{E}_i$. Since $\tilde{E}_i$ is a bundle, it is in particular reflexive, so $\pi_* \tilde{E}_i$ is also reflexive. Because E_i is saturated by construction, it is also reflexive. Therefore both of these sheaves are normal, and since $\operatorname{Sing} E_i$ has singular set of codimension at least 3, this map extends to an isomorphism $E_i \cong \pi_* \tilde{E}_i$.

Similarly, if $\tilde{Q}_i = \tilde{E}_i / \tilde{E}_{i-1}$, then $\pi_* \tilde{Q}_i$ is equal to Q_i away from $\operatorname{Sing} Q_i$ so again we have a birational map $(Q_i)^{**} \dashrightarrow \left(\pi_* \tilde{Q}_i\right)^{**}$. Since the double dual is always reflexive, these sheaves are normal, so the map extends to an isomorphism. To summarise:

Proposition 18 *Let*

$$0 = E_0 \subset E_1 \subset ... \subset E_{l-1} \subset E_l = E$$

be a filtration of a holomorphic vector bundle $E \to X$ by saturated subsheaves and let $Q_i = E_i / E_{i-1}$. Then there is a finite sequence of blowups along complex submanifolds whose composition $\pi : \tilde{X} \to X$ enjoys the following properties. There is a filtration

$$0 = \tilde{E}_0 \subset \tilde{E}_1 \subset ... \subset \tilde{E}_{l-1} \subset \tilde{E}_l = \tilde{E} = \pi^* E$$

by subbundles. If we write $\operatorname{Im} \pi^* E_i$ for the image of $\pi^* E_i \hookrightarrow \pi^* E_i$, then $\tilde{E}_i = \operatorname{Sat}_{\pi^* E} (\operatorname{Im} \pi^* E_i)$. If $\tilde{Q}_i = \tilde{E}_i / \tilde{E}_{i-1}$ then we have $\pi_* \tilde{E}_i = E_i$ and $Q_i^{**} = (\pi_* \tilde{Q}_i)^{**}$.

We will also have occasion to consider ideal sheaves $\mathcal{I} \subset \mathcal{O}_X$ whose vanishing set is a closed complex subspace $Y \subset X$. If Y is smooth for example then we may blowup along Y to obtain a smooth manifold $\pi : \tilde{X} \longrightarrow X$. Denote by $\pi^* \mathcal{I}$ the ideal sheaf generated by pulling back local sections of $\mathcal{I}$, in other words the ideal sheaf in $\mathcal{O}_{\tilde{X}}$ generated by the image of $\pi^{-1}\mathcal{I}$ under the map $\pi^{-1}\mathcal{O}_X \longrightarrow \mathcal{O}_{\tilde{X}}$ where $\pi^{-1}\mathcal{I}$ and $\pi^{-1}\mathcal{O}_X$ are the inverse image sheaves. Note that this is not necessarily equal to the usual sheaf theoretic pullback of $\mathcal{I}$ which is given by $\pi^{-1}\mathcal{I}\otimes_{\pi^{-1}\mathcal{O}_X}\mathcal{O}_{\tilde{X}}$ and may for example have torsion. The sheaf $\pi^*\mathcal{I}$ is sometimes called the "inverse image ideal sheaf". If the order of vanishing of $\mathcal{I}$ along Y is m, then $\pi^*\mathcal{I} \subset \mathcal{O}_{\tilde{X}}(-m\mathbf{E})$, that is, every element of $\pi^*\mathcal{I}$ vanishes to order at least m along the smooth divisor $\mathbf{E}$. In this situation we will use this notation without further comment. In general Y is not smooth, so we appeal to the following resolution of singularities theorem, which is sometimes referred to as "principalisation of $\mathcal{I}$" or more specifically "monomialisation of $\mathcal{I}$" , and results of this type are usually used to prove resolution of singularities.

Theorem 10 *Let X be a complex manifold and Y a closed complex subspace. Then there is a finite sequence of blowups along smooth centres whose composition yields a map $\pi : \tilde{X} \to X$ such that $\pi : \tilde{X} - \mathbf{E} \to X - W$ is biholomorphic, $\mathbf{E} = \boldsymbol{\pi}^{-1}(W)$ is a normal crossings divisor, and $\pi^*\mathcal{I} = \mathcal{O}_{\tilde{X}}(-\sum_i m_i\mathbf{E}_i)$ where the $\mathbf{E}_i$ are the irreducible components of $\mathbf{E}$. Moreover, $\pi^*\mathcal{I}$ is locally principal (monomial) in the following sense: for any $x \in X$ there is a local coordinate neighbourhood $U \subset X$ containing x and a local section f_0 of $\mathcal{O}_{\tilde{X}}(-\sum_i m_i\mathbf{E}_i)$ over $\boldsymbol{\pi}^{-1}(U)$, such that if f_j is any local section of $\mathcal{I}$ over U, then $\pi^* f_j = f_0 f_j'$ where f_j' is a non-vanishing holomorphic function on $\boldsymbol{\pi}^{-1}(U)$. Furthermore, if ξ_k are local normal crossings coordinates for $\mathbf{E}$, then there is a factorisation:*

$$f_0 = \prod_k \xi_k^{m_k}$$

so that we may write:

$$\pi^* f_j = \prod_k \xi_k^{m_k} \cdot f_j'.$$

For the proof, see for example Kollar [KO].

2.1.2 Metrics on Blowups and Uniform Bounds on the Degree

Now we consider the case that the original manifold is Kähler. The following proposition is standard in Kähler geometry. It says that the property of being Kähler is preserved under blowing up.

Proposition 19 *Let (X, ω) be a Kähler manifold, and Y a compact, complex submanifold. Then the blowup $\tilde{X} = Bl_Y X$ along Y is also Kähler. Moreover $\tilde{X}$ possesses a one parameter family of Kähler metrics given by $\omega_\varepsilon = \pi^*\omega + \varepsilon\eta$ where $\varepsilon > 0$, $\pi : \tilde{X} \to X$ is the blowup map and η is itself a Kähler form on $\tilde{X}$.*

For the proof see for example [VO].

We will need a bound on the ω_ε degree of an arbitrary subsheaf of a holomorphic vector bundle E that depends on ε in such a way that as $\varepsilon \to 0$ the degree converges to the degree of a subsheaf on the base (namely the pushforward). This will be a consequence of the following lemma.

Lemma 13 *Let X be a compact complex manifold and let τ and η be closed $(1,1)$ forms with τ semi-positve and η a Kähler form. Let $E \to X$ a holomorphic vector bundle. Then there is a constant M depending on the L^2 form of F_E such that for* *any subsheaf* $S \subset E$ *with torsion free quotient and any* $0 < k \leq n-1$*:*

$$\deg_k(S, \tau, \eta) \equiv \int_X c_1(S) \wedge \tau^{n-k-1} \wedge \eta^k \leq M.$$

Proof. Note that when $k = n-1$, $\deg_k(S, \tau, \eta)$ is the ordinary η degree of S. We follow Kobayashi's proof that the degree of an arbitrary subsheaf is bounded. Fix an hermitian metric h on E. The general case will follow from the case when S is a line subbundle L. In this case we can use the formula: $F_L = \pi F_E \pi + \beta \wedge \beta^*$, where π is the orthogonal projection to L and β is the second fundamental form. Since $c_1(L) = \frac{i}{2\pi} F_L$ we have that:

$$\deg_k(L, \tau, \eta) = \frac{i}{2\pi} \int_X \pi F_E \pi \wedge \tau^{n-k-1} \wedge \eta^k + \frac{i}{2\pi} \int_X \beta \wedge \beta^* \wedge \tau^{n-k-1} \wedge \eta^k.$$

Since $\|\pi\|_{L^\infty(X)} \leq 1$, the first term is clearly bounded from above. Therefore we only need to check that the second term is non-positive. This is the case since β is a $(0,1)$ form, and therefore $i\beta \wedge \beta^* \leq 0$. Therefore $\deg_k(L, \tau, \eta) \leq M$, for a constant independent of L. To extend the result to all subbundles $F \subset E$, simply find such an M as above for each exterior power $\Lambda^p E$ for $p = 1, ..., \operatorname{rk} E$, and take the maximum. Then apply the above argument to the line bundle $L = \det F \hookrightarrow \Lambda^p E$.

In general $S \overset{\imath}{\hookrightarrow} E$ is not a subbundle but there is an inclusion of sheaves $\det S \hookrightarrow \Lambda^p E$ where p is the rank of S. If V is the singular set of S, then moreover S is a subbundle away from V, and so the inclusion $\det S \overset{\imath}{\hookrightarrow} \Lambda^p E$ is a line subbundle away from V. Let σ be any local holomorphic frame for $\det S$. Now consider the set: $W = \{x \in X \mid \imath(\sigma)(x) = 0\}$. Since $\det S$ is a line bundle this is clearly independent of σ. Furthermore because $\imath$ is an injective bundle map away from V, any $x \in W$ must be in V, that is, $W \subset V$. Now write $H = \imath^* (\Lambda^p h)$. This is an Hermitian metric on $\det S$ over $X - W$. On the other hand there is some Hermitian metric G on $\det S$ over all of X. We would like to show that:

$$\deg_k(S, \tau, \eta) = \int_X c_1(\det S, G) \wedge \tau^{n-k-1} \wedge \eta^k = \int_{X-W} c_1(\det S, H) \wedge \tau^{n-k-1} \wedge \eta^k$$

Then applying the above reasoning, the last integral is bounded since just as before

$$\begin{aligned}\int_{X-W} c_1(\det S, H) \wedge \tau^{n-k-1} \wedge \eta^k &= \int_{X-V} c_1(S, h_S) \wedge \tau^{n-k-1} \wedge \eta^k \\ &\leq \frac{i}{2\pi} \int_{X-V} \pi F_E \pi \wedge \tau^{n-k-1} \wedge \eta^k\end{aligned}$$

where h_S is the metric on $S_{|X-V}$ induced by h. Again this is bounded independently of π.

We will construct a C^∞ function f on X such that $H = fG$ on $X - W$. Then the usual formula for the curvature of the associated Chern connections implies:

$$\begin{aligned} c_1(\det S, H) &= \frac{i}{2\pi}\bar{\partial}\partial \log H = \frac{i}{2\pi}\bar{\partial}\partial \log f + c_1(\det S, G) \\ &\implies c_1(\det S, G) = c_1(\det S, H) - \frac{i}{2\pi}\bar{\partial}\partial \log f \ \textit{on}\ X - W.\end{aligned}$$

Finally we will show:

$$\int_{X-W} \frac{i}{2\pi}\bar{\partial}\partial \log f \wedge \tau^{n-k-1} \wedge \eta^k = 0.$$

To construct f, let σ be any local holomorphic frame for $\det S$. If $(e_{1,}....., e_r)$ is a local holomorphic frame for E, then define: $\imath(\sigma) = \sum_I \sigma^I e_I$, where $e_I = e_{i_1} \wedge ... \wedge e_{i_p}$, with $i_1 < ... < i_p$. Then let

$$f = H(\sigma, \sigma)/G(\sigma, \sigma) = \sum_{I,J} H_{IJ} \sigma^I \bar{\sigma}^J$$

where $H_{IJ} = \Lambda^p h(e_I, e_J)/G(\sigma, \sigma)$. Then one may check that f is well-defined independently of σ. It is a smooth non-negative function vanishing exactly on W. Since the matrix (H_{IJ}) is positive definite, f vanishes exactly where all the σ_I vanish. It is also clear that we have the equality $H = fG$.

To complete the argument we will show that $\frac{i}{2\pi}\bar{\partial}\partial \log f$ integrates to zero. Let $\mathcal{I}$ be the sheaf of ideals in $\mathcal{O}_X$ generated by $\{\sigma_I\}$. By Theorem 10 there is a sequence of smooth blowups $\pi : \tilde{X} \to X$ such that $\pi^*\mathcal{I}$ the inverse image ideal sheaf of $\mathcal{I}$, is the ideal sheaf of a divisor $\mathbf{E} = \sum_i m_i \mathbf{E}_i$ where the $\mathbf{E}_i$ are the irreducible components of the support of the exceptional divisor $\operatorname{supp} \mathbf{E} = \cup_i\ \mathbf{E}_i$. In other words $\pi^*\mathcal{I} = \mathcal{O}_{\tilde{X}}(-\sum_i m_i \mathbf{E}_i)$ for some natural numbers m_i. Furthermore, we have: $\pi^*\sigma^I = \rho^I \cdot \xi_{i_1}^{m_{i_1}} ... \xi_{i_s}^{m_{i_s}}$, where $\{\xi_{i_j}\}$ are normal crossings coordinates for $\mathbf{E}$ on an open set where $\pi^*\sigma^I$ is defined, and ρ^I is a non-vanishing holomorphic function. Therefore we may locally write: $\pi^* f = \chi \cdot \left|\xi_{i_1}\right|^{2m_{i_1}} ... \left|\xi_{i_s}\right|^{2m_{i_s}}$, where χ is a strictly positive C^∞ function defined on $\tilde{X}$. If we write $\Phi = \frac{i}{2\pi}\partial \log \chi$, and $T_{d\Phi}$ for the current defined by $d\Phi = \frac{i}{2\pi}\bar{\partial}\partial \log \chi$, then since by definition:

$$
\begin{aligned}
T_{d\Phi}\left(\pi^*(\tau^{n-k-1} \wedge \eta^k)\right) &= -dT_\Phi(\pi^*(\tau^{n-k-1} \wedge \eta^k)) \\
T_\Phi(d(\pi^*(\tau^{n-k-1} \wedge \eta^k)) &= 0
\end{aligned}
$$

since $\pi^*(\tau^{n-k-1} \wedge \eta^k)$ is closed. Away from the exceptional set we may write locally:

$$
\begin{aligned}
\frac{i}{2\pi}\partial \log \pi^* f &= \frac{i}{2\pi}\left(\partial \log \chi + 2m_{i_1}\partial \log \left|\xi_{i_1}\right| + ... + 2m_{i_s}\partial \log \left|\xi_{i_s}\right|\right) \\
&= \Phi + \frac{i}{2\pi}\left(\frac{m_{i_1} d\xi_{i_1}}{\xi_{i_1}} + ... + \frac{m_{i_s} d\xi_{i_s}}{\xi_{i_s}}\right).
\end{aligned}
$$

The second term is integrable on its domain of definition and so $\frac{i}{2\pi}\bar{\partial}\partial \log \pi^* f$ is a $(1,1)$ form with $L^1_{loc}(\tilde{X})$ coefficients, and so defines a current. On the other hand by the Poincaré-Lelong formula, $\bar\partial$ applied to the second term is equal to $\sum_{i_j} m_{i_j} T_{\mathbf{E}_{i_j}}$, in the sense of currents, where $T_{\mathbf{E}_{i_j}}$ is the current defined by the smooth hypersurface $\mathbf{E}_{i_j}$. Finally then:

$$\int_{X-W} \frac{i}{2\pi}\bar\partial\partial \log f \wedge \pi^*\tau^{n-k-1} \wedge \pi^*\eta^k = \int_{\tilde X-\mathbf{E}} \frac{i}{2\pi}\bar\partial\partial \log \pi^* f \wedge \pi^*\tau^{n-k-1} \wedge \pi^*\eta^k$$

$$= T_{\frac{i}{2\pi}\bar\partial\partial\log \pi^* f}(\pi^*\tau^{n-k-1} \wedge \pi^*\eta^k) = \left(\sum_i m_i T_{\mathbf{E}_i}\right)(\pi^*\tau^{n-k-1} \wedge \pi^*\eta^k)$$

$$= \sum_i m_i \int_{\mathbf{E}_i} \pi^*\tau^{n-k-1} \wedge \pi^*\eta^k = 0$$

since the image of $\mathbf{E}_i$ under π has codimension at least two. This completes the proof. ■

Remark 3 *If $0 \to S \to E \to Q \to 0$ is an exact sequence, where E is a vector bundle and Q is torsion free, then the dualised sequence $0 \to Q^* \to E^* \to S^*$ is exact, and so as in the above lemma there is a constant M associated to E independent of Q so that*

$$-\int_X c_1(Q) \wedge \tau^{n-k-1} \wedge \eta^k = \int_X c_1(Q^*) \wedge \tau^{n-k-1} \wedge \eta^k \leq M.$$

In other words there is a uniform constant M so that: $-M \leq \int_X c_1(Q) \wedge \tau^{n-k-1} \wedge \eta^k$, where Q is any torsion-free quotient of E.

Remark 4 *In the case that $k = n-1$, $\deg_k(S,\tau,\eta) = \deg(S,\eta)$ and the above constitutes a proof of Simpson's degree formula.*

We note that if $\tilde X \to X$ is a composition of finitely many blowups then we also have a family of Kähler metrics on $\tilde X$ by interatively applying Proposition 19.

We would now like to compute the degree of an arbitrary torsion-free sheaf $\tilde{S}$ on $\tilde{X}$ with respect to each metric ω_ε on $\tilde{X}$.

Theorem 11 *Let $\tilde{S}$ be a subsheaf (with torsion free quotient $\tilde{Q}$) of a holomorphic vector bundle $\tilde{E}$ on $\tilde{X}$, where $\pi : \tilde{X} \to X$ is given by a sequence of blowups along complex submanifolds of* codim ≥ 2*. Then then there is a uniform constant M independent of $\tilde{S}$ such that the degrees of $\tilde{S}$ and $\tilde{Q}$ with respect to ω_ε satisfy:* $\deg(\tilde{S}, \omega_\varepsilon) \leq \deg(\pi_* \tilde{S}) + \varepsilon M$*, and* $\deg(\tilde{Q}, \omega_\varepsilon) \geq \deg(\pi_* \tilde{Q}) - \varepsilon M$*.*

Proof. The general case will follow from the case when $\tilde{S}$ is a line bundle $\tilde{L}$ (perhaps not a line subbundle). Recall that the Picard group of the blowup $Pic(\tilde{X}) = Pic(X) \oplus \mathbb{Z}\mathcal{O}(\mathbf{E}_1) \oplus ... \oplus \mathbb{Z}\mathcal{O}(\mathbf{E}_m)$ where the $\mathbf{E}_i$ are the irreducible components of the exceptional divisor. That is, we may write an arbitrary line bundle as $\tilde{L} = \pi^* L \otimes \mathcal{O}_{\tilde{X}}(\sum_i m_i \mathbf{E}_i)$ where L is a line bundle on X. Then by definition:

$$\deg(\tilde{L}, \omega_\varepsilon) = \int_{\tilde{X}} c_1(\tilde{L}) \wedge \omega_\varepsilon^{n-1} = \int_{\tilde{X}} c_1(\tilde{L}) \wedge (\pi^*\omega + \varepsilon\eta)^{n-1}.$$

Then we have an expansion:

$$(\pi^*\omega + \varepsilon\eta)^{n-1} = (\pi^*\omega)^{n-1} + \varepsilon (\pi^*\omega)^{n-2} \wedge \eta + ... + \varepsilon^{n-2}\pi^*\omega \wedge \eta^{n-2} + \varepsilon^{n-1}\eta^{n-1}.$$

Note that $\int_{\tilde{X}} c_1\mathcal{O}_{\tilde{X}}(\mathbf{E}_i) \wedge (\pi^*\omega)^{n-1} = \int_{\mathbf{E}_i} (\pi^*\omega)^{n-1} = 0$, since the image in X of each $\mathbf{E}_i$ lives in codimension 2. Therefore we are left with

$$\begin{aligned}
\deg(\tilde{L}, \omega_\varepsilon) &= \int_{\tilde{X}} \left(c_1(\tilde{L}\right) \wedge (\pi^*\omega)^{n-1} + \sum_k \varepsilon^k \left(\int_{\tilde{X}} c_1(\tilde{L}) \wedge (\pi^*\omega)^{n-k-1} \wedge \eta^k\right) \\
&= \int_{\tilde{X}} (\pi^* c_1(L) \wedge (\pi^*\omega)^{n-1} + \sum_i m_i \int_{\tilde{X}} (c_1(\mathcal{O}_{\tilde{X}}(\mathbf{E}_i)) \wedge (\pi^*\omega)^{n-1} \\
&\quad + \sum_k \varepsilon^k \left(\int_{\tilde{X}} c_1(\tilde{L}) \wedge (\pi^*\omega)^{n-k-1} \wedge \eta^k\right) \\
&= \deg(L, \omega) + \sum_k \varepsilon^k \left(\int_{\tilde{X}} c_1(\tilde{L}) \wedge (\pi^*\omega)^{n-k-1} \wedge \eta^k\right)
\end{aligned}$$

By the previous lemma the terms $\int_{\tilde{X}} c_1(\tilde{L}) \wedge (\pi^*\omega)^{n-k-1} \wedge \eta^k$, are all bounded uniformly independently of ε since $\pi^*\omega$ is semi-positive and η is a Kähler form. Therefore we have: $\deg(\tilde{L}, \omega_\varepsilon) \leq \deg(L, \omega) + \varepsilon M$.

Now note that if $\tilde{X} = Bl_Y X$ then $\pi_*\mathcal{O}(m\mathbf{E}) = \mathcal{O}_X$ if $m \geq 0$ and $\pi_*\mathcal{O}(m\mathbf{E}) = I_Y^{\otimes m}$ if $m < 0$, where I_Y is the ideal sheaf of holomorphic functions on X vanishing on Y. The determinant of an ideal sheaf is trivial if Y has codimension at least 2, so we have $\det(\pi_*\tilde{L}) = \det(L)$ so finally: $\deg(\tilde{L}, \omega_\varepsilon) \leq \deg(\pi_*\tilde{L}) + \varepsilon M$.

Now for an arbitrary subsheaf $\tilde{S} \subset \tilde{E}$, by definition $\deg(\tilde{S}, \omega_\varepsilon) = \deg(\det(\tilde{S}), \omega_\varepsilon)$. When $\pi_*\tilde{S}$ is a vector bundle, that is, away from its algebraic singular set, we have an isomorphism $\det(\pi_*\tilde{S}) = \pi_* \det \tilde{S}$. Their determinants are therefore isomorphic away from this set, and so by Hartogs' theorem there is an isomorphism of line bundles: $\det(\pi_*\tilde{S}) = \det(\pi_* \det \tilde{S})$ on X. Therefore by the previous argument:

$$\deg(\tilde{S}, \omega_\varepsilon) = \deg(\det(\tilde{S}), \omega_\varepsilon) \leq \deg(\pi_* \det \tilde{S}) + \varepsilon M = \deg(\pi_*\tilde{S}) + \varepsilon M \ .$$

The exact same argument together with the previous remark proves the second inequality as well. ■

2.1.3 Stability on Blowups and Convergence of the HN Type

Proposition 20 *Let $\tilde{E} \to \tilde{X}$ a holomorphic vector bundle where $\tilde{X} \to X$ is a sequence of blowups. If $\pi_*\tilde{E}$ is ω-stable, then there is an ε_2 such that $\tilde{E}$ is ω_ε stable for all $0 < \varepsilon \leq \varepsilon_2$.*

Proof. Suppose there is a destabilising subsheaf $\tilde{S}_\varepsilon \subset \tilde{E}$, i.e. $\mu_{\omega_\varepsilon}(\tilde{S}_\varepsilon) \geq \mu_{\omega_\varepsilon}(\tilde{E})$ for each ε. Now among all proper subsheaves of $\pi_*\tilde{E}$, the maximal slope is realised by some subsheaf $\mathcal{F}$. Then by the previous theorem we have:

$$\mu_\omega(\pi_*\tilde{E}) - \varepsilon M \leq \mu_{\omega_\varepsilon}(\tilde{E}) \leq \mu_\omega(\pi_*\tilde{S}_\varepsilon) + \varepsilon M \leq \mu_\omega(\mathcal{F}) + \varepsilon M < \mu_\omega(\pi_*\tilde{E}) + \varepsilon M$$

where we have used that $\mathcal{F}$ is proper and $\pi_*\tilde{E}$ is ω-stable. Now letting $\varepsilon \to 0$ we have $\mu_\omega(\pi_*\tilde{E}) < \mu_\omega(\pi_*\tilde{E})$ and the proposition follows. ■

Remark 5 *This shows in particular that for any resolution of a HNS filtration, the quotients $\tilde{Q}_i = \tilde{E}_i/\tilde{E}_{i-1}$ are stable with respect to ω_ε for ε sufficiently small, since the double dual of the pushforward is the double dual of Q_i which is stable by construction. This fact will be important in Section 5.*

For each of the metrics ω_ε there is also an HNS filtration of the pullback π^*E. We will need information about what happens to the corresponding HN types as $\varepsilon \to 0$. Namely we have:

Proposition 21 *Let $E \to X$ a holomorphic vector bundle and $\pi : \tilde{X} \to X$ be a finite sequence of blowups resolving the HNS filtration. Then the HN type $(\mu_1^\varepsilon, ..., \mu_K^\varepsilon)$ of π^*E with respect to ω_ε converges to the HN type $(\mu_1, ..., \mu_K)$ of E with respect to ω as $\varepsilon \longrightarrow 0$.*

Proof. Let

$$0 = \tilde{E}_0 \subset \tilde{E}_1 \subset \tilde{E}_2 \subset ... \subset \tilde{E}_{n-1} \subset \tilde{E}_l = \pi^* E$$

be a resolution of the HNS filtration. Since all the information about the HN type is contained in the HN filtration

$$0 = \mathbb{F}_0^{HN} \subset \mathbb{F}_1^{HN}(E) \subset \mathbb{F}_2^{HN}(E) \subset ... \subset \mathbb{F}_l^{HN}(E) = E,$$

we will just regard this as a resolution of singularities of the HN filtration and forget about Seshadri filtrations for the rest of this proof.

We would like to relate the resolution of the HN filtration of (E, ω), to the HN filtration of $(\pi^* E, \omega_\varepsilon)$ for small ε. We claim that for all ε in a sufficient range we may arrange that $\mu_{\omega_\varepsilon}^{\min}(\tilde{E}_i) > \mu_{\omega_\varepsilon}^{\max}(\pi^* E/\tilde{E}_i)$. Let $\mathcal{F}_1 \subset \tilde{E}_i \subset \mathcal{F}_2 \subset \pi^* E$ be any subsheaves such that $\tilde{E}_i/\mathcal{F}_1$ is torsion free. Note that for $\tilde{x} \in \tilde{X}$ with $\pi(\tilde{x}) = x$, we always have maps on the stalks $(\pi_* \mathcal{F}_i)_x \to (\mathcal{F}_i)_{\tilde{x}}$. Since π is in particular a biholomorphism away from $\mathbf{E}$, when $\tilde{x} \in \tilde{X} - \mathbf{E}$ these maps are isomorphisms. In other words the sequences:

$$0 \longrightarrow \pi_* \mathcal{F}_1 \longrightarrow E_i \longrightarrow \pi_* \left(\tilde{E}_i/\mathcal{F}_1 \right) \longrightarrow 0$$

and

$$0 \longrightarrow E_i \longrightarrow \pi_* \mathcal{F}_2 \longrightarrow \pi_* \left(\mathcal{F}_2/\tilde{E}_i \right) \longrightarrow 0$$

are exact away from the singular set Z_{alg}. In particular this means $E_i/\pi_* \mathcal{F}_1 \hookrightarrow \pi_*(\tilde{E}_i/\mathcal{F}_1)$ and $\pi_* \mathcal{F}_2/E_i \hookrightarrow \pi_*(\mathcal{F}_2/\tilde{E}_i)$ with torsion quotients, which implies $(E_i/\pi_* \mathcal{F}_1)^{**} = (\pi_*(\tilde{E}_i/\mathcal{F}_1))^{**}$ and $(\pi_* \mathcal{F}_2/E_i)^{**} = (\pi_*(\mathcal{F}_2/\tilde{E}_i))^{**}$. Then finally we have $\mu_\omega(E_i/\pi_* \mathcal{F}_1) = \mu_\omega(\pi_*(\tilde{E}_i/\mathcal{F}_1))$ and $\mu_\omega(\pi_* \mathcal{F}_2/E_i) = \mu_\omega(\pi_*(\mathcal{F}_2/\tilde{E}_i))$.

The above argument together with $Theorem$ 11 now implies that $\mu_{\omega_\varepsilon}(\tilde{E}_i/\mathcal{F}_1) \geq \mu_\omega(E_i/\pi_*\mathcal{F}_1) - \varepsilon M$ and $\mu_{\omega_\varepsilon}(\mathcal{F}_2/\tilde{E}_i) \leq \mu_\omega(\pi_*\mathcal{F}_2/E_i) + \varepsilon M$. On the other hand: $\mu_\omega(E_i/\pi_*\mathcal{F}_1) \geq \mu_\omega(Q_i) > \mu_\omega(Q_{i+1}) \geq \mu_\omega(\pi_*\mathcal{F}_2/E_i)$, where we have used the facts that $\mu_\omega(Q_i) = \mu_\omega^{\min}(E_i)$ and $\mu_\omega(Q_{i+1}) = \mu_\omega^{\max}(E/E_i)$. Therefore we have:

$$\mu_{\omega_\varepsilon}(\tilde{E}_i/\mathcal{F}_1) - \mu_{\omega_\varepsilon}(\mathcal{F}_2/\tilde{E}_i) \geq (\mu_\omega(E_i/\pi_*\mathcal{F}_1) - \mu_\omega(\pi_*\mathcal{F}_2/E_i)) - 2\varepsilon M.$$

As we have shown, the first term on the right hand side is strictly positive, so when ε is sufficiently small the entire right hand side is strictly positive. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ were arbitrary, for ε small $\mu_{\omega_\varepsilon}^{\min}(\tilde{E}_i)$ must be strictly bigger than $\mu_{\omega_\varepsilon}^{\max}(\pi^*E/\tilde{E}_i)$.

Now it follows from Proposition 8 that the HN filtration of $(\pi^*E, \omega_\varepsilon)$ is:

$$\begin{aligned} 0 \ &\subset \ \mathbb{F}_1^{HN,\varepsilon}(\tilde{E}_1) \subset ... \subset \mathbb{F}_{k_1}^{HN,\varepsilon}(\tilde{E}_1) = \tilde{E}_1 \subset ... \subset \mathbb{F}_{k_1+...+k_{l-1}}^{HN,\varepsilon}(\tilde{E}_{l-1}) = \tilde{E}_{l-1} \\ &\subset \ \mathbb{F}_{k_1+...+k_{l-1}+1}^{HN,\varepsilon}(\tilde{E}_l) \subset ... \subset \mathbb{F}_{k_1+...+k_l}^{HN,\varepsilon}(\tilde{E}_l) = \pi^*E. \end{aligned}$$

That is, the resolution appears within the HN filtration with respect to ω_ε, and two successive subbundles in the resolution are separated by the HN filtration of the larger bundle. Then for any i we consider the following part of the above filtration:

$$\begin{aligned} \tilde{E}_{i-1} \ &= \ \mathbb{F}_{k_1+...+k_{i-1}}^{HN,\varepsilon}(\tilde{E}_{i-1}) \subset \mathbb{F}_{k_1+...+k_{i-1}+1}^{HN,\varepsilon}(\tilde{E}_i) \subset \\ ... \ &\subset \ \mathbb{F}_{k_1+...+k_i-1}^{HN,\varepsilon}(\tilde{E}_i) \subset \mathbb{F}_{k_1+...+k_i}^{HN,\varepsilon}(\tilde{E}_i) = \tilde{E}_i. \end{aligned}$$

We claim that:

$$\mu_{\omega_\varepsilon}\left(\mathbb{F}_{k_1+...+k_{i-1}+j}^{HN,\varepsilon}(\tilde{E}_i)/\mathbb{F}_{k_1+...+k_{i-1}+j-1}^{HN,\varepsilon}(\tilde{E}_i)\right) \longrightarrow \mu_\omega(E_i/E_{i-1}) = \mu_\omega(Q_i)$$

for each $1 \leq j \leq k_i$. Then the proposition will follow immediately. The slopes of the quotients in the HN filtration are strictly decreasing so we have:

$$\begin{aligned} \mu_{\omega_\varepsilon}\left(\tilde{E}_i/\mathbb{F}^{HN,\varepsilon}_{k_1+\ldots+k_i-1}(\tilde{E}_i)\right) &< \mu_{\omega_\varepsilon}\left(\mathbb{F}^{HN,\varepsilon}_{k_1+\ldots+k_{i-1}+j}(\tilde{E}_i)/\mathbb{F}^{HN,\varepsilon}_{k_1+\ldots+k_{i-1}+j-1}(\tilde{E}_i)\right) \\ &< \mu_{\omega_\varepsilon}\left(\mathbb{F}^{HN,\varepsilon}_{k_1+\ldots+k_{i-1}+1}(\tilde{E}_{i-1})/\tilde{E}_{i-1}\right). \end{aligned}$$

Therefore it suffices to prove convergence of

$$\mu_{\omega_\varepsilon}\left(\tilde{E}_i/\mathbb{F}^{HN,\varepsilon}_{k_1+\ldots+k_i-1}(\tilde{E}_i)\right) \text{ and } \mu_{\omega_\varepsilon}\left(\mathbb{F}^{HN,\varepsilon}_{k_1+\ldots+k_{i-1}+1}(\tilde{E}_{i-1})/\tilde{E}_{i-1}\right)$$

to $\mu_\omega(Q_i)$ as $\varepsilon \to 0$. Note that just as before we may argue that

$$\mu_\omega\left(\pi_*\left(\tilde{E}_i/\mathbb{F}^{HN,\varepsilon}_{k_1+\ldots+k_i-1}(\tilde{E}_i)\right)\right) = \mu_\omega\left(E_i/\pi_*\mathbb{F}^{HN,\varepsilon}_{k_1+\ldots+k_i-1}(\tilde{E}_i)\right)$$

and

$$\mu_\omega\left(\pi_*\left(\mathbb{F}^{HN,\varepsilon}_{k_1+\ldots+k_{i-1}+1}(\tilde{E}_{i-1})/\tilde{E}_{i-1}\right)\right) = \mu_\omega\left(\pi_*\mathbb{F}^{HN,\varepsilon}_{k_1+\ldots+k_{i-1}+1}(\tilde{E}_{i-1})/E_{i-1}\right).$$

By *Theorem* 11 we have:

$$\begin{aligned} \mu_\omega(Q_i) - \varepsilon M &= \mu_\omega(\pi_*\tilde{Q}_i) - \varepsilon M \leq \mu_{\omega_\varepsilon}(\tilde{Q}_i) \leq \mu_{\omega_\varepsilon}\left(\mathbb{F}^{HN,\varepsilon}_{k_1+\ldots+k_{i-1}+1}(\tilde{E}_{i-1})/\tilde{E}_{i-1}\right) \\ &\leq \mu_\omega\left(\pi_*\mathbb{F}^{HN,\varepsilon}_{k_1+\ldots+k_{i-1}+1}(\tilde{E}_{i-1})/E_{i-1}\right) + \varepsilon M \leq \mu_\omega\left(E_i/E_{i-1}\right) + \varepsilon M \\ &= \mu_\omega(Q_i) + \varepsilon M \end{aligned}$$

where we have used that $F^{HN,\varepsilon}_{k_1+\ldots+k_{i-1}+1}(\tilde{E}_{i-1})$ is maximally destabilising in $\pi^*E/\tilde{E}_{i-1}$ and E_i/E_{i-1} is maximally destabilising in E/E_{i-1}. So

$$\mu_{\omega_\varepsilon}\left(\mathbb{F}^{HN,\varepsilon}_{k_1+\ldots+k_{i-1}+1}(\tilde{E}_{i-1})/\tilde{E}_{i-1}\right) \longrightarrow \mu_\omega(Q_i).$$

Similiarly we have:

$$\begin{aligned}\mu_\omega(Q_i) - \varepsilon M &= \mu_\omega(E_i/E_{i-1}) - \varepsilon M \leq \mu_\omega\left(E_i/\pi_*\mathbb{F}^{HN,\varepsilon}_{k_1+...+k_i-1}(\tilde{E}_i)\right) - \varepsilon M \\ &\leq \mu_{\omega_\varepsilon}\left(\tilde{E}_i/\mathbb{F}^{HN,\varepsilon}_{k_1+...+k_i-1}(\tilde{E}_i)\right) \leq \mu_{\omega_\varepsilon}(\tilde{Q}_i) \leq \mu_\omega\left(\pi_*\tilde{Q}_i\right) + \varepsilon M \\ &= \mu_\omega(Q_i) + \varepsilon M\end{aligned}$$

where we have used that $\mu_\omega(E_i/E_{i-1}) = \mu_\omega^{\min}(E_i)$ and $\mu_{\omega_\varepsilon}\left(\tilde{E}_i/\mathbb{F}^{HN,\varepsilon}_{k_1+...+k_i-1}(\tilde{E}_i)\right) = \mu^{\min}_{\omega_\varepsilon}(\tilde{E}_i)$. Then taking limits implies $\mu_{\omega_\varepsilon}\left(\tilde{E}_i/\mathbb{F}^{HN,\varepsilon}_{k_1+...+k_i-1}(\tilde{E}_i)\right) \to \mu_\omega(Q_i)$. This completes the proof. ■

Remark 6 *Note that the argument of the above proof also shows that we have convergence:*

$$\left(\mu_{\omega_\varepsilon}(\tilde{Q}_1), ..., \mu_{\omega_\varepsilon}(\tilde{Q}_l)\right) \longrightarrow (\mu_\omega(Q_1), ..., \mu_\omega(Q_l)),$$

where as usual $\mu_{\omega_\varepsilon}(\tilde{Q}_i)$ is repeated $\mathrm{rk}(\tilde{Q}_i)$ times. We will use this fact in the following section.

2.2 Approximate Critical Hermitian Structures/HN Type of the Limit

In this section we accomplish two important aims. One is the construction of a certain canonical type of metric on a holomorphic vector bundle over a Kähler manifold called an L^p-approximate critical hermitian structure. The other is identifying the Harder-Narasimhan type of the limiting vector bundle E_∞ along the flow, namely we prove that this is the same as the type of the original bundle E. This latter fact will be a crucial element in the proof of the main theorem, whereas the former will play no role in the remainder of the proof. However we remark that these two theorems are, due to certain technical considerations, very much intertwined.

Our argument is as follows: first we construct an L^p-approximate critical hermitian structure for p very close to 1 in the special case that the analytic singular set is a complex submanifold, and a single blowup along Z_{an} suffices to resolve the singularities of the HNS filtration. In this case (small p), note that in fact the metric produced will be independent of p. We obtain the result about the HN type in the same special case as a corollary. This in turn may be used to prove, again in the special case, the existence of an L^p-approximate critical hermitian structure for all p. We then use this to prove the existence of such a structure in the general case by blowing up finitely many times and applying an inductive argument. Finally, we point out that along the way we have proven the theorem (in general) that the HN type of the limit is the correct one.

We will need to work with the varying family of Kähler metrics on $\tilde{X}$ given by $\pi^*\omega + \varepsilon\eta$ in Section 4. As we will see, the construction of an L^p-approximate critical hermitian structure requires us to fix a value ε_1 and consider stable quotients with respect to this metric. We will therefore need some sort of uniform control over the Hermitian-Einstein tensor as $\varepsilon \to 0$. The author has noticed an error in [DW1] on this point. In particular, *Lemma* 3.14 is slightly incorrect. Instead, the right hand side should have an additional term involving the L^2 norm of the full curvature. This does not essentially disrupt the proof, because the Yang-Mills and Hermitian-Yang-Mills functionals differ only by a topological term, but it has the effect of changing the logic of the argument somewhat, as well as increasing the technical complexity.

If we fix a holomorphic structure on E, then a critical point of the HYM functional thought of as a map $h \mapsto HYM(\bar{\partial}_E, h)$ on the space of metrics is called (see Kobayashi [KOB]) a *critical hermitian structure.* The Kähler identities imply that this happens exactly when the corresponding connection $(\bar{\partial}_E, h)$ is Yang-Mills, and hence in this case the Hermitian-Einstein tensor splits: $i\Lambda_\omega F_{(\bar{\partial}_E,h)} = \mu_1 Id_{Q_i} \oplus \ldots \oplus \mu_K Id_{Q_K}$. Here the holomorphic structure $\bar{\partial}_E$ splits into the direct sum $\oplus_i Q_i$ and the metric induced on each summand is Hermitian-Einstein with constant factor μ_i.

In general, the holomorphic structure on E is not split, and of course the Q_i may not be subbundles as at all, so it is not the case that we always have a critical hermitian structure. We therefore need to define a correct approximate notion of a critical point. In the subsequent discussion we follow Daskalopoulos-Wentworth [DW1].

Let h be a smooth metric on E and $\mathcal{F} = \{F_i\}_{i=0}^K$ a filtration of E by saturated subsheaves. For every F_i we have the corresponding weakly holomorphic projection π_i^h. These are bounded, L_1^2 hermitian endomorphisms of E. Here $F_0 = 0$, and so $\pi_0^h = 0$. Given real numbers $\mu_1, \ldots, \mu_K$, define the following L_1^2 hermitian endomorphism of E :

$$\Psi(\mathcal{F}, (\mu_1, \ldots, \mu_K), h) = \sum_{i=1}^{K} \mu_i \left(\pi_i^h - \pi_{i-1}^h\right).$$

Notice that away from the singular set of the filtration (points where it is given by sub-bundles), the bundle E splits smoothly as $\oplus Q_i = \oplus_i E_i/E_{i-1}$ and with respect to the splitting the endomorphism $\Psi(\mathcal{F}, (\mu_1, \ldots, \mu_K), h)$ is just diagonal map $\mu_1 Id_{Q_i} \oplus \ldots \oplus \mu_K Id_{Q_K}$.

In the special case where E is a holomorphic vector bundle over a Kähler manifold (X, ω), we will write $\Psi_\omega^{HNS}(\bar{\partial}_E, h)$ when the filtration of E is the HNS filtration $F_i = \mathbb{F}_i^{HNS}(E)$ and $(\mu_1, ..., \mu_K)$ is the HN type.

Definition 6 *Fix $\delta > 0$ and $1 \leq p \leq \infty$. An L^p δ-approximate critical hermitian structure on a holomorphic bundle E is a smooth metric h such that:*

$$\left\| i\Lambda_\omega F_{(\bar{\partial}_E, h)} - \Psi_\omega^{HNS}(\bar{\partial}_E, h) \right\|_{L^p(\omega)} \leq \delta.$$

For the proof of the following theorem, see [DW1]:

Theorem 12 *If the HNS filtration of E is given by subbundles, then for any $\delta > 0$, E has an L^∞ approximate critical hermitian structure.*

In general, we will not obtain an L^∞ approximate structure. In the following we show that for an arbitrary holomorphic bundle we have such a metric for $1 \leq p < \infty$. We begin with two preliminary technical lemmas.

Lemma 14 *Let X be a compact Kähler manifold of dimension n, and let $\pi : \tilde{X} \to X$ be a of blowup along a complex submanifold Y of complex codimension k where $k \geq 2$. Consider the natural family $\omega_\varepsilon = \pi^*\omega + \varepsilon\eta$ where $0 < \varepsilon \leq \varepsilon_1$ and η is a Kähler form on $\tilde{X}$. Then given any α and $\tilde{\alpha}$ such that $1 < \alpha < 1 + \frac{1}{2(k-1)}$, and $\frac{\alpha}{1-2(k-1)(\alpha-1)} < \tilde{\alpha} < \infty$, and if we let $s = \frac{\tilde{\alpha}}{\tilde{\alpha}-\alpha}$ then for the Kähler metric g^ε, we have: $\det g^\varepsilon / \det \varpi \in L^{2(1-\alpha)s}(\tilde{X}, \varpi)$, for any hermitian metric ϖ on $\tilde{X}$, and the value of the $L^{2(1-\alpha)s}$ norm is uniformly bounded in ε.*

Proof. Since g^ε converges to the Kähler metric $\pi^*\omega$ away from the exceptional divisor $\mathbf{E}$, on the complement of a neighbourhood of $\mathbf{E}$ there is always such a uniform bound (and on this set $(\det g^{\varepsilon}/\det g^{\varepsilon_1})^{2(1-\alpha)s}$ is clearly integrable). It therefore suffices to prove the result in a neighbourhood of the exceptional divisor. Let $y \in Y$ and U be a local coordinate chart containing y consisting of coordinates $(z, ..., z_n)$. Now Y has codimension k so that locally Y is given by the slice coordinates $\{z_1 = z_2 = ... = z_k = 0\}$. Recall that on the blow-up $\tilde{X}$ we have explicit coordinate charts $\tilde{U}_m \subset \tilde{U} = \pi^{-1}(U)$ where $\tilde{U}_m = \{z \in U - Y \mid z_m \neq 0\} \cup \{(z, [\nu]) \in \mathbb{P}(\boldsymbol{\zeta})_{|Y \cap U} \mid \nu_m \neq 0\}$, where $\mathbb{P}(\boldsymbol{\zeta})$ is the projectivisation of the normal bundle of Y. Let $(\xi_{1,...,}\xi_n)$ denote local coordinates on $\tilde{U}_m$. In these coordinates the map $\pi : \tilde{X} \to X$ is given by:

$$(\xi_{1,...,}\xi_n) \longrightarrow (\xi_1\xi_m, ..., \xi_{s-1}\xi_m, \xi_m, \xi_{m+1}\xi_m, ..., \xi_k\xi_m, \xi_{k+1}, ..., \xi_n).$$

Now locally we have: $\omega^n = (i/2)^n \det g_{ij}\, dz_1 \wedge d\bar{z}_1 \wedge ... \wedge dz_n \wedge d\bar{z}_n$, and using the above coordinate description we may compute: $\pi^*\omega^n = (i/2)^n\,(\pi^* \det g_{ij})\, |\xi_m|^{2k-2}\ \ d\xi_1 \wedge d\bar{\xi}_1 \wedge ... \wedge d\xi_n \wedge d\bar{\xi}_n$.

Note that $\pi^* \det g_{ij}$ is non-vanishing since $\det g_{ij}$, and so degeneracy of the pullback occurs only along the hypersurface defined by $\xi_m = 0$. In other words, $(\xi_{1,...,}\xi_n)$ are normal crossings coordinates on the blow-up for the exceptional divisor $\mathbf{E}$, and locally $\mathbf{E}$ takes the form $\{\xi_m = 0\}$.

The top power of the Kähler form ω_ε is:

$$\omega_\varepsilon^n = \pi^*\omega^n + \varepsilon\pi^*\omega^{n-1} \wedge \eta + .. + \varepsilon^k\pi^*\omega^{n-l} \wedge \eta^l + ...\varepsilon^{n-1}\pi^*\omega \wedge \eta^{n-1} + \varepsilon^n\eta^n.$$

In the local coordinates $(\xi_1, ..., \xi_n)$ we have: $\omega_\varepsilon^n = (i/2)^n \det g_{ij}^{\varepsilon} d\xi_1 \wedge d\bar{\xi}_1 \wedge ... \wedge d\xi_1 \wedge d\bar{\xi}_1$. We may therefore obtain a lower bound (not depending on ε) on $\det g_{ij}^{\varepsilon}$ as follows. Since η is a metric $\eta > 0$. On the other hand, the only degeneracy of $\pi^*\omega$ is only on vectors tangent to the exceptional divisor, where it vanishes, so $\pi^*\omega \geq 0$. Therefore $\pi^*\omega^l \wedge \eta^{n-l}$ is non-negative for every l.

Then comparing the two expressions for ω_ε^n, this implies that we have the lower bound: $\det g_{ij}^\varepsilon \geq C\,|\xi_m|^{2k-2}$, where $C = \inf \pi^* \det g_{ij}$ on $\tilde{U}_m$ for each $0 < \varepsilon \leq \varepsilon_1$. Taking the $2(1-\alpha)s$ power of both sides we see that

$$\int_{\tilde{U}_m} \left(\det g^\varepsilon / \det \varpi\right)^{2(1-\alpha)s} \varpi^n \leq C \int_{\tilde{U}_m} (\det g_{ij}^\varepsilon)^{2(1-\alpha)s} \leq C \int_{\tilde{U}_m} |\xi_m|^{4(1-\alpha)(k-1)s},$$

where the last two integrals are with respect to the standard Euclidean measure. Using the condition on $\tilde{\alpha}$ one computes that $4(1-\alpha)(k-1)s > -2$ and so the functions $|\xi_m|^{4(1-\alpha)s(k-1)}$, are integrable (as can be seen by computing the integral in polar coordinates), and the result follows. ■

Lemma 15 *Let $\pi : \tilde{X} \to X$, the codimension k, and the family of metrics ω_ε be the same as in the previous lemma. Let $\tilde{B}$ be a holomorphic vector bundle on $\tilde{X}$ and F a $(1,1)$ form with values in the auxiliary vector bundle $End(\tilde{B})$. Let $1 < \alpha < 1 + \frac{1}{4k(k-1)}$ and $\frac{\alpha}{1-2(k-1)(\alpha-1)} < \tilde{\alpha} < 1 + \frac{1}{2(k-1)}$. Then there is a number κ_0 such that for any $0 < \kappa \leq \kappa_0$, there exists a constant C independent of ε, ε_1, and κ, and a constant $C(\kappa)$ such that:*

$$\|\Lambda_{\omega_\varepsilon} F\|_{L^\alpha(\tilde{X},\omega_\varepsilon)} \leq C\left(\left\|\Lambda_{\omega_{\varepsilon_1}} F\right\|_{L^{\tilde{\alpha}}(\tilde{X},\omega_{\varepsilon_1})} + \kappa \|F\|_{L^2(\tilde{X},\omega_{\varepsilon_1})}\right) + \varepsilon_1 C(\kappa) \|F\|_{L^2(\tilde{X},\omega_{\varepsilon_1})}$$

Proof. Recall that $(\Lambda_{\omega_\varepsilon} F)\,\omega_\varepsilon^n = F \wedge \omega_\varepsilon^{n-1}$ and $\left(\Lambda_{\omega_{\varepsilon_1}} F\right)\omega_{\varepsilon_1}^n = F \wedge \omega_{\varepsilon_1}^{n-1}$ so that:

$$\Lambda_{\omega_\varepsilon} F = \frac{F \wedge \omega_\varepsilon^{n-1}}{\omega_\varepsilon^n}, \Lambda_{\omega_{\varepsilon_1}} F = \frac{F \wedge \omega_{\varepsilon_1}^{n-1}}{\omega_{\varepsilon_1}^n}.$$

Note also that

$$\omega_\varepsilon^n = \frac{\det g^\varepsilon}{\det g^{\varepsilon_1}} \omega_{\varepsilon_1}^n$$

Now we write:

$$\begin{aligned}\Lambda_{\omega_\varepsilon} F &= \frac{F\wedge \omega_\varepsilon^{n-1}}{\omega_\varepsilon^n} = \frac{F\wedge (\omega_{\varepsilon_1}^{n-1}+\omega_\varepsilon^{n-1}-\omega_{\varepsilon_1}^{n-1})}{\omega_\varepsilon^n} \\ &= \left(\frac{F\wedge \omega_{\varepsilon_1}^{n-1}+\sum_{l=1}^{n-1}(\varepsilon^l-\varepsilon_1^l)\binom{n-l}{l}F\wedge\pi^*\omega^{(n-1)-l}\wedge\eta^l}{\omega_\varepsilon^n}\right). \\ &= \frac{\det g^{\varepsilon_1}}{\det g^{\varepsilon}}\left(\Lambda_{\omega_{\varepsilon_1}}F+\frac{\sum_{l=1}^{n-1}(\varepsilon^l-\varepsilon_1^l)\binom{n-l}{l}F\wedge\pi^*\omega^{(n-1)-l}\wedge\eta^l}{\omega_{\varepsilon_1}^n}\right).\end{aligned}$$

Therefore:

$$|\Lambda_{\omega_\varepsilon}F|^\alpha \le C\left|\frac{\det g^{\varepsilon_1}}{\det g^\varepsilon}\right|^\alpha\left(\left|\Lambda_{\omega_{\varepsilon_1}}F\right|^\alpha+\sum_{l=1}^{n-1}\left|\varepsilon^l-\varepsilon_1^l\right|^\alpha\left|\frac{F\wedge\pi^*\omega^{(n-1)-l}\wedge\eta^l}{\omega_{\varepsilon_1}^n}\right|^\alpha\right)$$

(by convexity of the function $|\cdot|^\alpha$ when $\alpha > 1$). Again, we set $s=\frac{\tilde\alpha}{\tilde\alpha-\alpha}$. By the above expression and Hölder's inequality with respect to the metric ω_{ε_1}:

$$\begin{aligned}\|\Lambda_{\omega_\varepsilon}F\|_{L^\alpha(\tilde X,\omega_\varepsilon)} &= \left(\int_{\tilde X}|\Lambda_{\omega_\varepsilon}F|^\alpha\,\omega_\varepsilon^n\right)^{\frac{1}{\alpha}} \le \\ C\left(\int_{\tilde X}\left(\frac{\det g^\varepsilon}{\det g^{\varepsilon_1}}\right)^{(1-\alpha)s}\omega_{\varepsilon_1}^n\right)^{\frac{1}{\alpha s}} \\ &\times\left(\left(\int_{\tilde X}\left|\Lambda_{\omega_{\varepsilon_1}}F\right|^{\tilde\alpha}\omega_{\varepsilon_1}^n\right)^{\frac{1}{\tilde\alpha}}+\left(\int_{\tilde X}\sum_{l=1}^{n-1}\left(\varepsilon_1^l\right)^{\tilde\alpha}\left|\frac{F\wedge\pi^*\omega^{(n-1)-l}\wedge\eta^l}{\omega_{\varepsilon_1}^n}\right|^{\tilde\alpha}\omega_{\varepsilon_1}^n\right)^{\frac{1}{\tilde\alpha}}\right)\end{aligned}$$

By the previous lemma the factor

$$\left(\int_{\tilde X}\left(\frac{\det g^\varepsilon}{\det g^{\varepsilon_1}}\right)^{(1-\alpha)s}\omega_{\varepsilon_1}^n\right)^{\frac{1}{\alpha s}}$$

is uniformly bounded in ε.

Now we need to control the second term of the second factor above. We divide $\tilde X$ into two pieces: an arbitrarily small neighbourhood V_κ with $\mathrm{Vol}(V_\kappa,\omega_{\varepsilon_1})=\kappa^{\frac{2}{2-\tilde\alpha}}$ of the exceptional divisor $\mathbf{E}$ and its complement. We will perform two separate estimates, one for each piece. Write the components of F in a local basis as $F_{\rho i\bar j}^{\gamma}$.

At any point we may choose an orthonormal basis for the tangent space so that η is standard and $\pi^*\omega$ is diagonal. Then if we call this basis $\{e_i\}$, we have

$$\begin{aligned}
&\left|\frac{F\wedge\pi^*\omega^{(n-1)-l}\wedge\eta^l}{\omega^n_{\varepsilon_1}}\right|^{2\tilde{\alpha}} \\
= &\left|\frac{\left(\sum_{i,j}F_{i\bar{j}}e_i\wedge\bar{e}_j\right)\wedge\left(\sum_i\pi^*g_{ii}e^i\wedge\bar{e}^i\right)^{(n-1)-l}\wedge\left(\sum_i e^i\wedge\bar{e}^i\right)^l}{\omega^n_{\varepsilon_1}}\right|^{2\tilde{\alpha}} \\
\leq &\frac{C}{\left|\omega^n_{\varepsilon_1}\right|^{2\tilde{\alpha}}}\left(\sum_{i.j,\gamma,\rho}\left|F^{\gamma}_{\rho i\bar{j}}\right|^2\right)^{\tilde{\alpha}} = C\frac{|F|^{2\tilde{\alpha}}_{\eta}}{\left|\omega^n_{\varepsilon_1}\right|^{2\tilde{\alpha}}}.
\end{aligned}$$

Now note that on $\tilde{X}-V_\kappa$ the pullback $\pi^*\omega$ determines a metric, in other words $(\pi^*\omega)^n$ is non-vanishing, so since $\omega^n_{\varepsilon_1}\longrightarrow(\pi^*\omega)^n$, the quantity $\left|\omega^n_{\varepsilon_1}\right|^{2\tilde{\alpha}}$ is uniformly bounded away from 0. Therefore

$$\left|\frac{F\wedge\pi^*\omega^{(n-1)-l}\wedge\eta^l}{\omega^n_{\varepsilon_1}}\right|^{\tilde{\alpha}}\leq C\,|F|^{\tilde{\alpha}}_{\eta}.$$

On the other hand, if we again choose a basis for which η is standard and such that ω_{ε_1} is diagonal, we have:

$$|F|^{2\tilde{\alpha}}_{\eta}=\left|\left(\sum_{i.j,\gamma,\rho}|F^{\gamma}_{\rho ij}|^2\right)\right|^{\tilde{\alpha}}\leq C\left|\left(\sum_{i.j,\gamma,\rho}\frac{1}{g^{\varepsilon_1}_{ii}g^{\varepsilon_1}_{jj}}|F^{\gamma}_{\rho ij}|^2\right)\right|^{\tilde{\alpha}}=C\,|F|^{2\tilde{\alpha}}_{\omega_{\varepsilon_1}}$$

since the product of the eigenvalues $g^{\varepsilon_1}_{ii}g^{\varepsilon_1}_{jj}$ is again uniformly bounded ($g^{\varepsilon_1}_{ii}g^{\varepsilon_1}_{jj}\to\pi^*g_{ii}\pi^*g_{jj}$ as $\varepsilon_1\to0$). Thus, on $\tilde{X}-V_\kappa$ we have the further pointwise bound: $|F|^{\tilde{\alpha}}_{\eta}\leq C\,|F|^{\tilde{\alpha}}_{\omega_{\varepsilon_1}}$.

$$\begin{aligned}
\left(\int_{\tilde{X}-V_\kappa}\left(\varepsilon_1^l\right)^{\tilde{\alpha}}\left|\frac{F\wedge\pi^*\omega^{(n-1)-l}\wedge\eta^l}{\omega^n_{\varepsilon_1}}\right|^{\tilde{\alpha}}\omega^n_{\varepsilon_1}\right)^{\frac{1}{\tilde{\alpha}}} &\leq C\varepsilon_1\left(\int_{\tilde{X}-V_\kappa}|F|^{\tilde{\alpha}}_{\omega_{\varepsilon_1}}\omega^n_{\varepsilon_1}\right)^{\frac{1}{\tilde{\alpha}}} \\
&\leq C\varepsilon_1\,\|F\|_{L^{\tilde{\alpha}}(\omega_{\varepsilon_1})}\leq C(\kappa)\varepsilon_1\,\|F\|_{L^2(\omega_{\varepsilon_1})}
\end{aligned}$$

since by assumption $\tilde{\alpha}<2$.

Now we estimate this term on V_κ. Choose an orthonormal basis for the tangent space at a point in V_κ such that ω_{ε_1} is standard and η is diagonal. Then we have

$g^{\varepsilon_1}_{ij} = \pi^* g_{ij} + \varepsilon_1 \eta_{ij}$, so if $i \neq j$ $\pi^* g_{ij} = 0$, and if $i = j$ $\eta_{ii} = \frac{1-\tilde{g}_{ii}}{\varepsilon_1}$. Note also that $0 \leq \tilde{g}_{ii} < 1$ since $0 < \eta_{ii}$. If we write Ω for the standard Euclidean volume form then:

$$\begin{aligned}
&\sum_{l=1}^{n-1} \left(\varepsilon_1^l\right)^{\tilde{\alpha}} \left| \frac{F \wedge \pi^* \omega^{(n-1)-l} \wedge \eta^l}{\omega^n_{\varepsilon_1}} \right|^{\tilde{\alpha}} \\
= &\sum_{l=1}^{n-1} \left| \frac{\left(\sum_{i,j} F_{i\bar{j}} e_i \wedge \bar{e}_j\right) \wedge \left(\sum_i \pi^* g_{i\bar{i}} e^i \wedge \bar{e}^i\right)^{(n-1)-l} \wedge \left(\sum_i \left(1 - \pi^* g_{ii}\right) e^i \wedge \bar{e}^i\right)^l}{\Omega} \right|^{\tilde{\alpha}} \\
\leq &\; C \left(\sum_{i.j,\gamma,\rho} \left| F^{\gamma}_{\rho i \bar{j}} \right| \right)^{\tilde{\alpha}} \leq C \left|F\right|^{\tilde{\alpha}}_{\omega_{\varepsilon_1}}.
\end{aligned}$$

Therefore:

$$\begin{aligned}
&\left(\int_{V_\kappa} \sum_{l=1}^{n-1} \left(\varepsilon_1^l\right)^{\tilde{\alpha}} \left| \frac{F \wedge \pi^* \omega^{(n-1)-l} \wedge \eta^l}{\omega^n_{\varepsilon_1}} \right|^{\tilde{\alpha}} \omega^n_{\varepsilon_1} \right)^{\frac{1}{\tilde{\alpha}}} \\
\leq &\; C \left(\int_{V_\kappa} \left|F\right|^{\tilde{\alpha}}_{\omega_{\varepsilon_1}} \omega^n_{\varepsilon_1} \right)^{\frac{1}{\tilde{\alpha}}} \leq C \,\mathrm{Vol}(V_\kappa, \omega_{\varepsilon_1})^{1-\frac{\tilde{\alpha}}{2}} \left\|F\right\|_{L^2(V_\kappa, \omega_{\varepsilon_1})} \leq C\kappa \left\|F\right\|_{L^2(\tilde{X}, \omega_{\varepsilon_1})} \;(\textit{Hölder}).
\end{aligned}$$

Now we obtain the desired estimate:

$$\left\|\Lambda_{\omega_\varepsilon} F\right\|_{L^\alpha(\tilde{X}, \omega_\varepsilon)} \leq C \left(\left\|\Lambda_{\omega_{\varepsilon_1}} F\right\|_{L^{\tilde{\alpha}}(\tilde{X}, \omega_{\varepsilon_1})} + \kappa \left\|F\right\|_{L^2(\tilde{X}, \omega_{\varepsilon_1})} \right) + \varepsilon_1 C(\kappa) \left\|F\right\|_{L^2(\tilde{X}, \omega_{\varepsilon_1})}.$$

■

Proposition 22 *Let $E \to X$ be a holomorphic vector bundle of rank K over a Kähler manifold with Kähler form ω. Assume that E has Harder-Narasimhan type $\mu = (\mu_1, ..., \mu_K)$ that the singular set Z_{alg} of the HNS filtration is smooth, and furthermore that blowing up along the singular set resolves the singularities of the HNS filtration. There is an $\alpha_0 > 1$ such that the following holds: given any $\delta > 0$ and any N, there is an hermitian metric h on E such that $HYM^{\omega}_{\alpha,N}(\bar{\partial}_E, h) \leq HYM_{\alpha,N}(\mu) + \delta$, for all $1 \leq \alpha < \alpha_0$.*

Proof. As before, let $\pi : \tilde{X} \to X$ be a blow-up along a smooth, complex submanifold Y, and we assume that this resolves the singularities of the HNS filtration. In other words there is a filtration of $\tilde{E} = \pi^* E$ on $\tilde{X}$ that is given by sub-bundles and is equal to the HNS filtration of E away from the divisor $\mathbf{E}$. Let ω_ε denote the aforementioned family of Kähler metrics on $\tilde{X}$ given by $\omega_\varepsilon = \pi^*\omega + \varepsilon\eta$ where $0 < \varepsilon \leq 1$ and η is a fixed Kähler metric on $\tilde{X}$. We will construct the metric on h on E from an hermitian metric $\tilde{h}$ on $\pi^* E$ to be specified later.

Since Z_{alg} is a complex submanifold, we consider its normal bundle $\boldsymbol{\zeta}$, or more particularly the open subset: $\boldsymbol{\zeta}_R = \{(x, \nu) \in \boldsymbol{\zeta} \mid |\nu| < R\}$. By the tubular neighbourhood theorem, for R sufficiently small this set is diffeomorphic to an open neighbourhood U_R of Z_{alg}. We choose a background metric H on this open set.

Let ψ be a smooth cut-off function supported in U_1 and and identically 1 on $U_{1/2}$ and such that $0 \leq \psi \leq 1$ everywhere. Then if we define $\psi_R(x, \nu) = \psi(x, \frac{\nu}{R})$, ψ_R is identically 1 on $U_{R/2}$ and supported in U_R with $0 \leq \psi_R \leq 1$ and furthermore there are bounds on the derivatives:

$$\left| \frac{\partial \psi_R}{\partial z_i} \right| \leq \frac{C}{R} \qquad , \qquad \left| \frac{\partial}{\partial \bar{z}_i} \frac{\partial \psi_R}{\partial z_i} \right| \leq \frac{C}{R^2}$$

where the constant C does not depend on R. Suppose for the moment that we have constructed an hermitian metric $\tilde{h}$ on $\pi^* E$. If we continue to denote by H and ψ_R their pullbacks to $\tilde{X}$, then we may define the following metric on $\pi^* E$:

$$h_{\psi_R} := \psi_R H + (1 - \psi_R)\tilde{h}$$

Observe that on $X - U_R$ we have $h_{\psi_R} = \tilde{h}$ and on $U_{R/2}$, $h_{\psi_R} = H$.

Now we need to estimate the difference:

$$
\begin{aligned}
& \left|HYM^{\omega_\varepsilon}_{\alpha,N}(\bar{\partial}_{\tilde{E}}, h_{\psi_R}) - HYM_{\alpha,N}(\mu)\right| \\
= & \left|\int_{\tilde{X}} \Phi_\alpha(\Lambda_{\omega_\varepsilon} F_{h_{\psi_R}} + \sqrt{-1}N\mathbf{I}_{\tilde{E}}) - \Phi_\alpha\left(i(\mu_1+N), ..., \sqrt{-1}(\mu_K+N)\right)\right|
\end{aligned}
$$

where Φ_α is the convex functional on $\mathfrak{u}(\tilde{E})$ given as in *Section* 3.2 by $\Phi_\alpha(a) = \sum_{j=1}^{k} |\lambda_j|^\alpha$, where the $i\lambda_j$ are the eigenvalues of a. From here on out we will write $i(\mu+N)$ for the matrix in the above expression. Therefore we have:

$$
\begin{aligned}
& \left|HYM^{\omega_\varepsilon}_{\alpha,N}(\bar{\partial}_{\tilde{E}}, h_{\psi_R}) - HYM_{\alpha,N}(\mu)\right| \\
\leq & \left|\int_{\tilde{X}-\pi^{-1}(U_{R/2})} \Phi_\alpha(\Lambda_{\omega_\varepsilon} F_{h_{\psi_R}} + \sqrt{-1}N\mathbf{I}_{\tilde{E}}) - \Phi_\alpha(i(\mu+N))\right| \\
+ & \left|\int_{\pi^{-1}(U_{R/2})} \Phi_\alpha(\Lambda_{\omega_\varepsilon} F_{h_{\psi_R}} + \sqrt{-1}N\mathbf{I}_{\tilde{E}}) - \Phi_\alpha(\sqrt{-1}(\mu+N))\right| \\
= & \left|\int_{\tilde{X}-\pi^{-1}(U_{R/2})} \Phi_\alpha(\Lambda_{\omega_\varepsilon} F_{h_{\psi_R}} + \sqrt{-1}N\mathbf{I}_{\tilde{E}}) - \Phi_\alpha(\sqrt{-1}(\mu+N))\right| \\
+ & \left|\int_{\pi^{-1}(U_{R/2})} \Phi_\alpha(\Lambda_{\omega_\varepsilon} F_H + \sqrt{-1}N\mathbf{I}_{\tilde{E}}) - \Phi_\alpha(\sqrt{-1}(\mu+N))\right|
\end{aligned}
$$

where the last equality comes from the fact that h_{ψ_R} is equal to H on $U_{R/2}$. Dividing the first integral further we have:

$$
\begin{aligned}
& \left|HYM^{\omega_\varepsilon}_{\alpha,N}(\bar{\partial}_E, h_{\psi_R}) - HYM_{\alpha,N}(\mu)\right| \\
\leq & \left|\int_{\pi^{-1}(U_R-U_{R/2})} \Phi_\alpha(\Lambda_{\omega_\varepsilon} F_{h_{\psi_R}} + \sqrt{-1}N\mathbf{I}_{\tilde{E}}) - \Phi_\alpha(\Lambda_{\omega_\varepsilon} F_{\tilde{h}} + \sqrt{-1}N\mathbf{I}_{\tilde{E}})\right| \\
& + \left|\int_{\tilde{X}-\pi^{-1}(U_{R/2})} \Phi_\alpha(\Lambda_{\omega_\varepsilon} F_{\tilde{h}} + \sqrt{-1}N\mathbf{I}_{\tilde{E}}) - \Phi_\alpha(\sqrt{-1}(\mu_{\omega_{\varepsilon_1}}+N))\right| \\
& + \left|\int_{\tilde{X}-\pi^{-1}(U_{R/2})} \Phi_\alpha(\sqrt{-1}(\mu_{\omega_{\varepsilon_1}}+N)) - \Phi_\alpha(\sqrt{-1}(\mu+N))\right| \\
& + \left|\int_{\pi^{-1}(U_{R/2})} \Phi_\alpha(\Lambda_{\omega_\varepsilon} F_H + \sqrt{-1}N\mathbf{I}_{\tilde{E}}) - \Phi_\alpha(\sqrt{-1}(\mu+N))\right|
\end{aligned}
$$

where in the first integral on the right hand side we have used the fact that outside of U_R the metrics h_{ψ_R} and $\tilde{h}$ agree. Here, $\mu_{\omega_{\varepsilon_1}}$ denotes the usual K-tuple of rational numbers made from the ω_{ε_1} slopes of the quotients of the resolution.

Recall that the norm on $L^\alpha(\mathfrak{u}(\tilde{E}))$ $a \mapsto \left(\int_M \Phi_\alpha(a)\right)^{1/\alpha}$ is equivalent to the L^α norm and so there is a universal constant C independent of R and ε such that:

$$\begin{aligned}
&\left|\int_{\pi^{-1}(U_R-U_{R/2})} \Phi_\alpha(\Lambda_{\omega_\varepsilon}F_{h_{\psi_R}} + \sqrt{-1}N\mathbf{I}_{\tilde{E}}) - \Phi_\alpha(\Lambda_{\omega_\varepsilon}F_{\tilde{h}} + \sqrt{-1}N\mathbf{I}_E)\right| \\
&+ \left|\int_{\tilde{X}-\pi^{-1}(U_{R/2})} \Phi_\alpha(\Lambda_{\omega_\varepsilon}F_{\tilde{h}} + \sqrt{-1}N\mathbf{I}_E) - \Phi_\alpha(\sqrt{-1}(\mu_{\omega_{\varepsilon_1}} + N))\right| \\
\leq\ & C\left(\left\|\Lambda_{\omega_\varepsilon}F_{h_{\psi_R}} - \Lambda_{\omega_\varepsilon}F_{\tilde{h}}\right\|^\alpha_{L^\alpha(\pi^{-1}(U_R-U_{R/2}),\omega_\varepsilon)} + \left\|\Lambda_{\omega_\varepsilon}F_{\tilde{h}} - \sqrt{-1}\mu_{\omega_{\varepsilon_1}}\right\|^\alpha_{L^\alpha(\tilde{X}-\pi^{-1}(U_{R/2}),\omega_\varepsilon)}\right).
\end{aligned}$$

First we dispose of

$$\left|\int_{\tilde{X}-\pi^{-1}(U_{R/2})} \Phi_\alpha(\sqrt{-1}(\mu_{\omega_{\varepsilon_1}} + N)) - \Phi_\alpha(\sqrt{-1}(\mu + N))\right|$$

by choosing ε_1 close to zero and using Remark 6. That is, we may choose ε_1 small enough so that

$$\left|\int_{\tilde{X}-\pi^{-1}(U_{R/2})} \Phi_\alpha(\sqrt{-1}(\mu_{\omega_{\varepsilon_1}} + N)) - \Phi_\alpha(\sqrt{-1}(\mu + N))\right| < \frac{\delta}{2}$$

Next will will bound:

$$\left\|\Lambda_{\omega_\varepsilon}F_{\tilde{h}} - \sqrt{-1}\mu_{\omega_{\varepsilon_1}} Id_{\tilde{E}}\right\|^\alpha_{L^\alpha(\tilde{X}-\pi^{-1}(U_{R/2}),\omega_\varepsilon)}.$$

. Note that at this point we have not specified the metric $\tilde{h}$ on π^*E. We will do so now. Each of the ω-stable quotients Q_i of the Harder-Narisimhan-Seshadri filtration remains stable on the blowup with respect to the metrics ω_ε with ε sufficiently small (see Remark 5), so that the quotients $\tilde{Q}_i$ are also ω_{ε_1}-stable and admit a unique

Hermitian-Einstein metric $\tilde{G}_i^{\varepsilon_1}$. The prototype for our metric $\tilde{h}$ will be the metric $\tilde{G}_{\varepsilon_1} = \oplus_i \tilde{G}_i^{\varepsilon_1}$. However we need to modify $\tilde{G}$ by a gauge transformation in order to obtain the appropriate bound on the second term. More precisely, since holomorphic structures on the bundle $\tilde{E}$ are equivalent to integrable unitary connections, this is the same as showing that if we fix the metric $\tilde{G}_{\varepsilon_1}$, there is a gauge transformation $\tilde{g}$ of $\tilde{E}$ such that $\|\Lambda_{\omega_\varepsilon} F_{(\tilde{g}(\bar{\partial}_{\tilde{E}}),\tilde{G}_{\varepsilon_1})} - i\mu_{\omega_{\varepsilon_1}} Id_{\tilde{E}}\|_{L^\alpha(\tilde{X}-\pi^{-1}(U_{R/2}),\omega_\varepsilon)}$ is small. When we take the direct sum, the second fundamental form enters into the curvature and so we ask that there is a gauge transformation making this contribution small. We can write the holomorphic structure $\bar{\partial}_{\tilde{E}}$ on $\tilde{E}$ as an upper triangular matrix with $\bar{\partial}_{\tilde{Q}_i}$ on the diagonal and β_i above the diagonal, where the β_i are the second fundamental forms for the splitting. Then define the complex gauge transformations $\tilde{g}_t = t^{1-l} Id_{Q_1} \oplus ... \oplus t^{-1} Id_{Q_{l-1}} \oplus Id_{Q_l}$. The action of $\tilde{g}_t$ on $\bar{\partial}_{\tilde{E}}$ is

$$\tilde{g}_t(\bar{\partial}_{\tilde{E}}) = \begin{pmatrix} \bar{\partial}_{\tilde{Q}_1} & t\beta_1 & \cdot & \cdot & t^{l-1}\beta_l \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \bar{\partial}_{\tilde{Q}_l} \end{pmatrix}.$$

and so we see that

$$\begin{aligned} &\|\Lambda_{\omega_\varepsilon} F_{(\tilde{g}(\bar{\partial}_{\tilde{E}}),\tilde{G}_{\varepsilon_1})} - \sqrt{-1}\mu_{\omega_{\varepsilon_1}} Id_{\tilde{E}}\|_{L^\alpha(\tilde{X}-\pi^{-1}(U_{R/2}),\omega_\varepsilon)} \\ &\qquad \le \|\Lambda_{\omega_\varepsilon} F_{\tilde{G}_1^{\varepsilon_1}} - \sqrt{-1}\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_1) Id_{\tilde{Q}_1}\|_{L^\alpha(\tilde{X}-\pi^{-1}(U_{R/2}),\omega_\varepsilon)} \\ &\qquad + ... + \|\Lambda_{\omega_\varepsilon} F_{\tilde{G}_K^{\varepsilon_1}} - \sqrt{-1}\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_l) Id_{\tilde{Q}_l}\|_{L^\alpha(\tilde{X}-\pi^{-1}(U_{R/2}),\omega_\varepsilon)} \\ &\qquad + \Theta(t\beta_1, ..t\beta_l) \end{aligned}$$

where $\Theta(t\beta_1, ..t\beta_l) \to 0$ as $t \to 0$. Therefore we have reduced this estimate to an estimate on each of the terms:

$$\left\| \Lambda_{\omega_\varepsilon} F_{\tilde{G}_i^{\varepsilon_1}} - \sqrt{-1}\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i} \right\|_{L^\alpha(\tilde{X}-\pi^{-1}(U_{R/2}),\omega_\varepsilon)}.$$

On the other hand we have:

$$\begin{aligned}
&\left\| \Lambda_{\omega_\varepsilon} F_{\tilde{G}_i^{\varepsilon_1}} - \sqrt{-1}\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i} \right\|_{L^\alpha(\tilde{X}-\pi^{-1}(U_{R/2}),\omega_\varepsilon)} \\
&\leq \left\| \Lambda_{\omega_\varepsilon} \left(F_{\tilde{G}_i^{\varepsilon_1}} - \frac{\sqrt{-1}}{n}\omega_{\varepsilon_1}\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i} \right) \right\|_{L^\alpha(\tilde{X}-\pi^{-1}(U_{R/2}),\omega_\varepsilon)} \\
&+ \left\| \frac{\sqrt{-1}}{n}\Lambda_{\omega_\varepsilon}(\omega_{\varepsilon_1} - \omega_\varepsilon)\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i} \right\|_{L^\alpha(\tilde{X}-\pi^{-1}(U_{R/2}),\omega_\varepsilon)}
\end{aligned}$$

where we have used the fact that $\Lambda_{\omega_\varepsilon}\omega_\varepsilon = n$. Now by Lemma 15 we have:

$$\begin{aligned}
&\left\| \Lambda_{\omega_\varepsilon} \left(F_{\tilde{G}_i^{\varepsilon_1}} - \frac{\sqrt{-1}}{n}\omega_{\varepsilon_1}\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i} \right) \right\|_{L^\alpha(\tilde{X}-\pi^{-1}(U_{R/2}),\omega_\varepsilon)} \\
&\qquad \leq C\left(\left\| \Lambda_{\omega_{\varepsilon_1}} F_{\tilde{G}_i^{\varepsilon_1}} - \sqrt{-1}\mu_{\omega_{\varepsilon_1}}(\tilde{Q}) Id_{\tilde{Q}_i} \right\|_{L^{\tilde{\alpha}}(\tilde{X},\omega_{\varepsilon_1})} \right) \\
&\qquad + \kappa C\left(\left\| F_{\tilde{G}_i^{\varepsilon_1}} \right\|_{L^2(\tilde{X},\omega_{\varepsilon_1})} + \frac{1}{n}\left\| \omega_{\varepsilon_1}\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i} \right\|_{L^2(\tilde{X},\omega_{\varepsilon_1})} \right) \\
&\qquad + \varepsilon_1 C(\kappa)\left\| F_{\tilde{G}_i^{\varepsilon_1}} \right\|_{L^2(\tilde{X},\omega_{\varepsilon_1})} \\
&\qquad + \varepsilon_1 C(\kappa)\frac{1}{n}\left\| \omega_{\varepsilon_1}\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i} \right\|_{L^2(\tilde{X},\omega_{\varepsilon_1})}
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \frac{\sqrt{-1}}{n}\Lambda_{\omega_\varepsilon}(\omega_{\varepsilon_1} - \omega_\varepsilon)\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i} \right\|_{L^\alpha(\tilde{X}-\pi^{-1}(U_{R/2}),\omega_\varepsilon)} \\
\leq\ & \frac{\varepsilon_1}{n}C\left(\left\| \left(\Lambda_{\omega_{\varepsilon_1}}\eta\right)\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i} \right\|_{L^{\tilde{\alpha}}(\tilde{X},\omega_{\varepsilon_1})} + \kappa\left\| \eta\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i} \right\|_{L^2(\tilde{X},\omega_{\varepsilon_1})} \right) \\
&+ \frac{\varepsilon_1^2}{n}C(\kappa)\left\| \eta\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i} \right\|_{L^2(\tilde{X},\omega_{\varepsilon_1})}
\end{aligned}$$

again using Lemma 15. Here we have used the fact that $\omega_{\varepsilon_1} - \omega_\varepsilon = (\varepsilon_1 - \varepsilon)\eta$ in the second inequality. Of course, $\left\| \Lambda_{\omega_{\varepsilon_1}} F_{\tilde{G}_i^{\varepsilon_1}} - \sqrt{-1}\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) id_{\tilde{Q}_i} \right\|_{L^{\tilde{\alpha}}(\tilde{X},\omega_{\varepsilon_1})} = 0$, by the construction of $G_i^{\varepsilon_1}$. On the other hand:

$$\begin{aligned}\left\|F_{\tilde{G}_i^{\varepsilon_1}}\right\|_{L^2(\tilde{X},\omega_{\varepsilon_1})} &= \left\|\Lambda_{\omega_{\varepsilon_1}} F_{\tilde{G}_i^{\varepsilon_1}}\right\|_{L^2(\tilde{X},\omega_{\varepsilon_1})} + \pi^2 n(n-1)\int_{\tilde{X}} \left(2c_2(\tilde{Q}_i) - c_1^2(\tilde{Q}_i)\right)) \wedge \omega_{\varepsilon_1}^{n-2} \\ &= \left\|\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i}\right\|_{L^2(\tilde{X},\omega_{\varepsilon_1})} + \pi^2 n(n-1)\int_{\tilde{X}} \left(2c_2(\tilde{Q}_i) - c_1^2(\tilde{Q}_i)\right) \wedge \omega_{\varepsilon_1}^{n-2}\end{aligned}$$

which is bounded. Likewise the terms

$$\left\|\omega_{\varepsilon_1}\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i}\right\|_{L^2(\tilde{X},\omega_{\varepsilon_1})}$$

and

$$\left\|\eta\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i}\right\|_{L^2(\tilde{X},\omega_{\varepsilon_1})}$$

are bounded. The only remaining issue is: $\left\|\left(\Lambda_{\omega_{\varepsilon_1}}\eta\right)\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i}\right\|_{L^{\tilde{\alpha}}(\tilde{X},\omega_{\varepsilon_1})}$. But writing

$$\left|\Lambda_{\omega_{\varepsilon_1}}\eta\right|^{\tilde{\alpha}} = \left|\frac{\eta\wedge\omega_{\varepsilon_1}^{n-1}}{\omega_{\varepsilon_1}^n}\right|^{\tilde{\alpha}} = \left|\frac{\eta\wedge\omega_{\varepsilon_1}^{n-1}}{\eta^n}\right|^{\tilde{\alpha}}\left|\frac{\det\eta}{\det\omega_{\varepsilon_1}}\right|^{\tilde{\alpha}}$$

and

$$\omega_{\varepsilon_1}^n = \left|\frac{\det\omega_{\varepsilon_1}}{\det\eta}\right|\eta^n$$

$$\begin{aligned}&\left\|\left(\Lambda_{\omega_{\varepsilon_1}}\eta\right)\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i}\right\|_{L^{\tilde{\alpha}}(\tilde{X},\omega_{\varepsilon_1})} \\ \leq\ & C\left(\int_{\tilde{X}}\left|\frac{\det\omega_{\varepsilon_1}}{\det\eta}\right|^{(1-\tilde{\alpha})\tilde{s}}\eta^n\right)^{\frac{1}{\tilde{\alpha}\tilde{s}}}\left(\int_{\tilde{X}}\left|\frac{\eta\wedge\omega_{\varepsilon_1}^{n-1}}{\eta^n}\right|^{\beta}\left|\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i}\right|^{\beta}\eta^n\right)^{\frac{1}{\beta}}\end{aligned}$$

by Hölder's inequality with respect to the metric η. Here again $\tilde{\alpha}$ is as in *Lemma 15* $\tilde{s} = \frac{\beta}{\beta-\tilde{\alpha}}$ where $\frac{\tilde{\alpha}}{1-2(k-1)(\tilde{\alpha}-1)} < \beta < \infty$. By Lemma 14 this is uniformly bounded in ε_1 since we also have $\omega_{\varepsilon_1}^{n-1} \longrightarrow \pi^*\omega^{n-1}$.

The consequence of the above argument is that we may choose t first, then κ, and then ε_1, all sufficiently small so that

$$\left\|\Lambda_{\omega_\varepsilon} F_{\tilde{G}_{\varepsilon_1}} - \sqrt{-1}\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{G}_{\varepsilon_1}}\right\|_{L^{\alpha}(\tilde{X}-\pi^{-1}(U_{R/2}),\omega_\varepsilon)} < \frac{\delta}{4}$$

for all ε and all α sufficiently close to 1. We will now fix this value of ε_1, so that all remaining quantities depending on ε_1 may be thought of as constant.

The term

$$\left| \int_{\pi^{-1}(U_{R/2})} \Phi_\alpha(\Lambda_{\omega_\varepsilon} F_H + \sqrt{-1} N \mathbf{I}_E) - \Phi_\alpha(\sqrt{-1}(\mu + N)) \right|$$

is bounded by:

$$C \left\| (\Lambda_{\omega_\varepsilon} F_H - \sqrt{-1}\mu \right\|_{L^\alpha(\pi^{-1}(U_{R/2}), \omega_\varepsilon)}.$$

Now write

$$|\Lambda_{\omega_\varepsilon} F_H|^\alpha = \left| \frac{F_H \wedge \omega_\varepsilon^{n-1}}{\omega_\varepsilon^{n-1}} \right|^\alpha = \left| \frac{F_H \wedge \omega_\varepsilon^{n-1}}{\eta^n} \right|^{\tilde{\alpha}} \left| \frac{\det \eta}{\det \omega_\varepsilon} \right|^{\tilde{\alpha}}$$

and

$$\omega_\varepsilon^n = \left| \frac{\det \omega_\varepsilon}{\det \eta} \right| \eta^n,$$

we have

$$\begin{aligned} \left\| (\Lambda_{\omega_\varepsilon} F_H - \sqrt{-1}\mu \right\|_{L^\alpha(\pi^{-1}(U_{R/2}), \omega_\varepsilon)} &\leq C_1 \left\| \Lambda_{\omega_\varepsilon} F_H \right\|_{L^\alpha(\pi^{-1}(U_{R/2}), \omega_\varepsilon)} + C_2 \operatorname{Vol}(U_{R/2}, \omega) \leq \\ & C_1 \left(\int_{\pi^{-1}(U_{R/2})} \left| \frac{\det \omega_\varepsilon}{\det \eta} \right|^{(1-\alpha)s} \eta^n \right)^{\frac{1}{s\tilde{\alpha}}} \left(\int_{\pi^{-1}(U_{R/2})} \left| \frac{F_H \wedge \omega_\varepsilon^{n-1}}{\eta^n} \right|^{\tilde{\alpha}} \eta^n \right)^{\frac{1}{\tilde{\alpha}}} \\ & + C_2 \operatorname{Vol}(U_{R/2}, \omega) \end{aligned}$$

where α and s are as in Lemma 14. By the lemma, the factor

$$\left(\int_{\pi^{-1}(U_{R/2})} \left| \frac{\det \omega_\varepsilon}{\det \eta} \right|^{(1-\alpha)s} \eta^n \right)$$

is uniformly bounded, and so the result is that there is an R such that

$$\left| \int_{\pi^{-1}(U_{R/2})} \Phi_\alpha(\Lambda_{\omega_\varepsilon} F_H + \sqrt{-1} N \mathbf{I}_E) - \Phi_\alpha(\sqrt{-1}(\mu + N)) \right| < \frac{\delta}{8}.$$

Therefore the only remaining estimates required are on:

$$\left\|\Lambda_{\omega_\varepsilon} F_{\tilde{h}_{\psi_R}} - \Lambda_{\omega_\varepsilon} F_{\tilde{h}}\right\|_{L^\alpha(\pi^{-1}(U_R - U_{R/2}),\omega_\varepsilon)}.$$

If we let k_{ψ_R} be an endomorphism such that $\tilde{h} = k_{\psi_R} h_{\psi_R}$. Then

$$F_{h_{\psi_R}} - F_{\tilde{h}} = \bar{\partial}_{\tilde{E}}(k_{\psi_R}^{-1}\partial_{\tilde{h}} k_{\psi_R})$$

where $\partial_{\tilde{h}}$ is the $(1,0)$ part of the Chern connection for $\tilde{h}$. The expression on the right hand side involves only two derivatives of ψ_R, and so, using the bound on the derivatives of ψ_R, there is a bound of the form:

$$\left|F_{h_{\psi_R}} - F_{\tilde{h}}\right| \leq C_1 + \frac{C_2}{R^2}.$$

where C_1 and C_2 are independent of both ε and R. Now as usual we have:

$$\begin{aligned}
\left|\Lambda_{\omega_\varepsilon}\left(F_{\tilde{h}_{\psi_R}} - F_{\tilde{h}}\right)\right|^\alpha &= \left|\frac{\left(F_{\tilde{h}_{\psi_R}} - F_{\tilde{h}}\right)\wedge\omega_\varepsilon^{n-1}}{\omega_\varepsilon^n}\right|^\alpha = \left|\frac{\left(F_{\tilde{h}_{\psi_R}} - F_{\tilde{h}}\right)\wedge\omega_\varepsilon^{n-1}}{\eta^n}\right|^\alpha \left|\frac{\det\eta}{\det\omega_\varepsilon}\right|^\alpha \\
\textit{and } \omega_\varepsilon^n &= \frac{\det\omega_\varepsilon}{\det\eta}\eta^n.
\end{aligned}$$

Then we compute:

$$\begin{aligned}
&\left\|\Lambda_{\omega_\varepsilon} F_{h_{\psi_R}} - \Lambda_{\omega_\varepsilon} F_{\tilde{h}}\right\|_{L^\alpha(\pi^{-1}(U_R - U_{R/2}),\omega_\varepsilon)} \\
= &\left(\int_{\pi^{-1}(U_R - U_{R/2})}\left|\frac{\left(F_{\tilde{h}_{\psi_R}} - F_{\tilde{h}}\right)\wedge\omega_\varepsilon^{n-1}}{\eta^n}\right|^\alpha\left|\frac{\det\eta}{\det\omega_\varepsilon}\right|^\alpha\frac{\det\omega_\varepsilon}{\det\eta}\eta^n\right)^{\frac{1}{\alpha}} \\
\leq &\left(\int_{\pi^{-1}(U_R - U_{R/2})}\left(\frac{\det\omega_\varepsilon}{\det\eta}\right)^{(1-\alpha)s}\eta^n\right)^{\frac{1}{\alpha s}}\left(\int_{\pi^{-1}(U_R - U_{R/2})}\left(C_1 + \frac{C_2}{R^{2\tilde{\alpha}}}\right)\eta^n\right)^{\frac{1}{\tilde{\alpha}}}
\end{aligned}$$

Here s and $\tilde{\alpha}$ are as in Lemma 14 and we have applied Hölder's inequality to the conjugate pair s and $\frac{\tilde{\alpha}}{\alpha}$. By the Lemma, the first factor is uniformly bounded in ε. We must therefore show that as $R \to 0$, the first factor can be made arbitrarily small. To do this we note that the open set U_R may be covered by a union of balls $\cup_j B_r^j$. Therefore:

$$\int_{\pi^{-1}(U_R - U_{R/2})} C_1 + C_2 R^{-2\tilde{\alpha}} \leq \sum_j (C_1 + C_2 R^{-2\tilde{\alpha}}) vol(B_r^j)$$

and up to a constant $vol(B_r^j) = r^{2n}$ where n is the complex dimension of X.

The key observation is now that the singular set Z_{alg} is a complex submanifold of X and has complex codimension at least 2, in other words it is of real dimension at most $2n-4$. This implies that Z_{alg} has Hausdorff dimension at most $2n-4$, i.e. it has zero d-dimensional Hausdorff measure for $d < 2n-4$. In other words, for each $0 \leq d < 4$, and a given $\delta > 0$, there is a cover of Z_{alg} and an $r > 0$ such that $\sum_j r^{2n-d} < \delta$. Now assume that we have chosen $R = r$. Then then the cover described above is also a cover for U_R so

$$\int_{\pi^{-1}(U_R - U_{R/2})} C_1 + C_2 R^{-2\tilde{\alpha}} \leq \sum_j (C_1 r^{2n} + C_2 r^{2n-2\tilde{\alpha}}).$$

Note that by assumption $\tilde{\alpha} < 2$. In other words, we may select R so that:

$$\left\| \Lambda_{\omega_\varepsilon} F_{\tilde{h}_{\psi_R}} - \Lambda_{\omega_\varepsilon} F_{\tilde{h}} \right\|_{L^\alpha(\pi^{-1}(U_R - U_{R/2}), \omega_\varepsilon)} < \frac{\delta}{16}.$$

Thus choosing ε_1 and R in the manner specified above gives us for each ε a bound on the difference of the HYM functionals: $\left| HYM_{\alpha,N}^{\omega_\varepsilon}(\bar{\partial}_E, \tilde{h}_{\psi_R}) - HYM_{\alpha,N}(\mu) \right| \leq \delta$. Now sending $\varepsilon \to 0$ we finally see that there exists a metric h with

$$\left| HYM_{\alpha,N}^{\omega}(\bar{\partial}_E, h) - HYM_{\alpha,N}(\mu) \right| < \delta$$

for all N and all α sufficiently close to 1. ■

Lemma 16 *Let $E \to X$ and α_0 be the same as in the proposition. Let h be any smooth Hermitian metric on E and A_t a solution of the Yang-Mills flow whose initial condition is $(\bar\partial_E, h)$. Let μ_0 denote the Harder-Narasimhan type of E. Then* $\lim_{t\to\infty} HYM_{\alpha,N}(A_t) = HYM_{\alpha,N}(\mu_0)$, *for all* $1 \le \alpha \le \alpha_0$ *and all* N.

As a consequence, if A_∞ is an Uhlenbeck limit along the flow: $HYM_{\alpha,N}(A_\infty) = HYM_{\alpha,N}(\mu_0)$, since $HYM_{\alpha,N}(A_\infty) = \lim_{t\to\infty} HYM_{\alpha,N}(A_t)$.

Proof. Define the number $\delta_0 > 0$ by the condition:

$$2\delta_0 + HYM_{\alpha,N}(\mu) = \min\{HYM_{\alpha,N}(\mu) \mid HYM_{\alpha,N}(\mu) > HYM_{\alpha,N}(\mu_0)\}$$

where μ runs over all possible HNS types of holomorphic vector bundles on X with the same rank as E.

Given a metric h and a corresponding initial condition $A_0^h = (\bar\partial_E, h)$ for the flow, we write A_t^h the solution at time t. Let $\mathcal{H}_\delta$ denote the set of all metrics h on E such that for any $\delta > 0$ there is a $T \ge 0$ such that for all $t \ge T$:

$$HYM_{\alpha,N}(A_t^h) < HYM_{\alpha,N}(\mu_0) + \delta.$$

We will show that every Hermitian metric is in $\mathcal{H}_\delta$ by showing that it is open and closed in the space of metrics with the C^∞ topology. Notice first that any metric h satisfying:

$$HYM_{\alpha,N}(A_0^h) \le HYM_{\alpha,N}(\mu_0) + \delta$$

is in $\mathcal{H}_\delta$ since

$$HYM_{\alpha,N}(A_t^h) \le HYM_{\alpha,N}(A_0^h)$$

for all t (monotonicity along the flow). Such a metric always exists by the above proposition. Therefore the set $\mathcal{H}_\delta$ is non-empty. Since the flow depends continuously on the initial condition $\mathcal{H}_\delta$ is also open.

Now assume without loss of generality that $0 \leq \delta \leq \frac{\delta_0}{2}$. To show $\mathcal{H}_\delta$ is closed we will show that it contains all of its limit points. So let h_j be a sequence of Hermitian metrics on E contained in the set $\mathcal{H}_\delta$ and suppose $h_j \longrightarrow H$ in the C^∞ topology, where H is an Hermitian metric. For each h_j let T_j be the corresponding time such that for all $t \geq T_j$ we have:

$$HYM_{\alpha,N}(A_t^{h_j}) \leq HYM_{\alpha,N}(\mu_0) + \delta.$$

By Uhlenbeck compactness, we may find a sequence of times $t_j \geq T_j$, Yang-Mills connections $A_\infty^{(1)}$ and $A_\infty^{(2)}$, and bubbling sets $Z_{\text{an}}^{(1)}$ and $Z_{\text{an}}^{(2)}$ such that $A_t^{h_j} \longrightarrow A_\infty^{(1)}$ in $L^p_{1,loc}(X - Z_{\text{an}}^{(1)})$ and $A_{t_j}^H \longrightarrow A_\infty^{(2)}$ in $L^p_{1,loc}(X - Z_{\text{an}}^{(2)})$. We also have $\Lambda_\omega F_{A_{t_j}^{h_j}} \longrightarrow \Lambda_\omega F_{A_\infty^{(1)}}$ and $\Lambda_\omega F_{A_{t_j}^H} \longrightarrow \Lambda_\omega F_{A_\infty^{(2)}}$ strongly in L^p for all $1 \leq p < \infty$.

We claim that $A_\infty^{(1)} = A_\infty^{(2)}$.

Proof of the Claim.

Define the automorphisms $k_j^{t_j}$ of E by $h_j^{t_j} = k_j^{t_j} H_{t_j}$, in other words $k_j^{t_j}$ is the gauge transformation taking the connection $A_{t_j}^H$ to $A_{t_j}^{h_j}$ by the action of conjugation. It follows from [DO1] Proposition 13 that

$$\sup \sigma(h_t^j, H_t) \longrightarrow 0$$

as $j \longrightarrow \infty$, uniformly in t, where:

$$\sigma(h, H) = Tr(h^{-1}H) + Tr(H^{-1}h) - 2rk(E)$$

is the C^0-distance function on the space of Hermitian metrics. In particular we have that:

$$\sup \left| k_j^{t_j} - I_E \right| \longrightarrow 0$$

as $j \longrightarrow \infty$. Let $Z_{\mathrm{an}} = Z_{\mathrm{an}}^{(1)} \cup Z_{\mathrm{an}}^{(1)}$ and choose a smooth test form $\phi \in \Omega^{1,0}(\mathrm{End}\, E)$ compactly supported on $X - Z_{\mathrm{an}}$.

Denote by ∂_{h_j,t_j},∂_{H,t_j} and $\partial_{\infty,(2)}$ the $(1,0)$ parts of the covariant derivatives corresponding to the connections $A_{t_j}^{h_j}$,$A_{t_j}^{H}$ and $A_\infty^{(2)}$. Then for any section s of E one computes:

$$\begin{aligned}
\left(\left(k_j^{t_j} \right)^{-1} \partial_{H,t_j}(k_j^{t_j}) \right)(s) &= \left(\left(k_j^{t_j} \right)^{-1} \partial_{H,t_j}\left(k_j^{t_j} s \right) \right) - \left(\nabla_{t_j}^{H} \right)^{1,0}(s) \\
&= \left(k_j^{t_j} \cdot \partial_{H,t_j} - \left(\nabla_{t_j}^{H} \right)^{1,0} \right) s \\
&= \left(\partial_{h_j,t_j} - \partial_{H,t_j} \right) s.
\end{aligned}$$

In other words:

$$\partial_{h_j,t_j} - \partial_{H,t_j} = \left(k_j^{t_j} \right)^{-1} \partial_{H,t_j}(k_j^{t_j}).$$

Now, there is a constant C such that:

$$\begin{aligned}
&\left| \langle \partial_{h_j,t_j} - \partial_{H,t_j}, \phi \rangle_{L^2} \right| \\
= &\left| \left\langle \left(k_j^{t_j} \right)^{-1} \partial_{H,t_j}(k_j^{t_j}), \phi \right\rangle_{L^2} \right| \leq C \left| \left\langle k_j^{t_j}, \partial_{H,t_j}^* \phi \right\rangle_{L^2} \right| \\
\leq &\ C \left\{ \left| \left\langle k_j^{t_j}, \left(\partial_{H,t_j} - \partial_{\infty,(2)}^* \right) \phi \right\rangle_{L^2} \right| + \left| \left\langle k_j^{t_j}, \partial_{\infty,(2)}^* \phi \right\rangle_{L^2} \right| \right\}.
\end{aligned}$$

Note that since $\partial_{H,t_j} \longrightarrow \partial_{\infty,(2)}$ in C^∞ and $k_j^{t_j}$ is bounded uniformly in L^∞ the first term goes to zero. On the other hand $k_j^{t_j} \longrightarrow I_E$ in C^0 so that:

$$\begin{aligned}\left\langle k_j^{t_j}, \partial^*_{\infty,(2)}\phi \right\rangle_{L^2} &\longrightarrow \left\langle I_E, \partial^*_{\infty,(2)}\phi \right\rangle_{L^2} = \int_X Tr\left(\partial^*_{\infty,(2)}\phi\right)\, dvol_\omega \\ &= \int_X \partial^* Tr\phi \, dvol_\omega = 0\end{aligned}$$

by Stokes' theorem. Therefore $\partial_{h_j,t_j} - \partial_{H,t_j} \longrightarrow 0$ in $L^2_{\text{loc}}(X - Z_{\text{an}})$ and so the two limits are equal.

Set $A_\infty = A_\infty^{(1)} = A_\infty^{(2)}$. Because $\Lambda_\omega F_{A_t^{h_j}} \longrightarrow \Lambda_\omega F_{A_\infty}$ and $\Lambda_\omega F_{A_t^H} \longrightarrow \Lambda_\omega F_{A_\infty}$ in L^p:

$$\lim_{j\longrightarrow\infty} HYM_{\alpha,N}\left(A_{t_j}^{h_j}\right) = \lim_{j\longrightarrow\infty} HYM_{\alpha,N}\left(A_{t_j}^{H}\right) = HYM_{\alpha,N}\left(A_\infty\right).$$

For large j we have:

$$\begin{aligned}HYM_{\alpha,N}\left(A_{t_j}^H\right) &\leq HYM_{\alpha,N}\left(A_\infty\right) + \delta = \lim_{j\longrightarrow\infty} HYM_{\alpha,N}\left(A_{t_j}^{h_j}\right) + \delta \\ &\leq HYM_{\alpha,N}(\mu_0) + 2\delta \leq HYM_{\alpha,N}(\mu_0) + \delta_0\end{aligned}$$

where we have used that $h_j \in \mathcal{H}_\delta$ and $\delta \leq \frac{\delta_0}{2}$. By the definition of δ_0,

$$\lim_{t\longrightarrow\infty} HYM_{\alpha,N}\left(A_{t_j+t}^H\right) = HYM_{\alpha,N}(\mu_0)$$

and so there is a $T \geq 0$ such that for all sufficiently large j, when $t \geq T$, $HYM_{\alpha,N}\left(A_{t_j+t}^H\right) < HYM_{\alpha,N}(\mu_0) + \delta$, in other words $H \in \mathcal{H}_\delta$, and so $\mathcal{H}_\delta$ is closed.

Then every Hermitian metric h is in $\mathcal{H}_\delta$ for all δ. In particular we may choose $\delta \leq \delta_0$ so that:

$$\begin{aligned}HYM_{\alpha,N}(\mu_0) &\leq HYM_{\alpha,N}(\mu_\infty) = HYM_{\alpha,N}(A_\infty) \\ &\leq HYM_{\alpha,N}\left(A_t^h\right) \leq HYM_{\alpha,N}\left(\mu_0\right) + \delta_0\end{aligned}$$

and again by the definition of δ_0 we have $HYM_{\alpha,N}(\mu_\infty) = HYM_{\alpha,N}(\mu_0)$ so the result follows. ■

We can now identify the Harder-Narasimhan type of the limit.

Proposition 23 *Let $E \to X$ have the same properties as before. Let A_t be a solution to the YM flow with initial condition A_0 whose limit along the flow is A_∞. Let E_∞ be the corresponding holomorphic vector bundle defined away from Z_{an}. Then the HN type of (E_∞, A_∞) is the same as (E_0, A_0).*

Proof. Let $\mu_0 = (\mu_1, ..., \mu_K)$ and $\mu_\infty = (\mu_1^\infty, ..., \mu_K^\infty)$ be the HN types of (E_0, A_0) and (E_∞, A_∞). A restatement of the above lemma is that $\Phi_\alpha(\mu_0+N) = \Phi_\alpha(\mu_\infty+N)$ for all $1 \leq \alpha \leq \alpha_0$ and all N. Choose N to be large enough so that $\mu_K + N \geq 0$. Then we also have $\mu_K^\infty + N \geq 0$ by Proposition 14, and therefore $\mu_K + N = \mu_K^\infty + N$ by Proposition 15, so $\mu_K = \mu_K^\infty$. ■

Let $(E, \bar{\partial}_{A_0})$ be a holomorphic bundle, and A_0 an initial connection, and A_{t_j} its evolution along the flow for a sequence of times t_j. Then we have the following.

Lemma 17 (1) *Let $\{\pi_j^{(i)}\}$ be the HN filtration of $(E,\bar{\partial}_{A_{t_j}})$ and $\{\pi_\infty^{(i)}\}$ the HN filtration of $(E_\infty,\partial_{A_\infty})$. Then after passing to a subsequence, $\pi_j^{(i)} \to \pi_\infty^{(i)}$ strongly $L^p \cap L^2_{1,loc}$ for all $1 \leq p < \infty$ and all i.*

(2) *Assume the original bundle $(E, \bar{\partial}_{A_0})$ is semi-stable and $\{\pi_{ss,j}^{(i)}\}$ are Seshadri filtrations of $(E, \bar{\partial}_{A_{t_j j}})$. Without loss of generality assume the ranks of the subsheaves $\pi_{ss,j}^{(i)}$ are constant in j. Then there is a filtration $\{\pi_{ss,\infty}^{(i)}\}$ of $(E,\bar{\partial}_{A_\infty})$ such that after passing to a subsequence $\{\pi_{ss,j}^{(i)}\} \to \{\pi_{ss,\infty}^{(i)}\}$ strongly in $L^p \cap L^2_{1,loc}$*

for all $1 \leq p < \infty$ and all i. The rank and degree of $\pi^{(i)}_{ss,\infty}$ is equal to the rank and degree of $\pi^{(i)}_{ss,j}$ for all i and j.

Proof. We will write $E^{(i)} = \mathbb{F}^{HN}_i(E, \bar{\partial}_{A_0})$ and $E^{(i)}_\infty = \mathbb{F}^{HN}_i(E, \bar{\partial}_{A_\infty})$ and $\pi^{(i)}_j$ the orthogonal projection onto the subsheaf $g_j(E_i)$. From the standard Chern-Weil formula (Simpson) we again have:

$$\deg(E_i) + \frac{1}{2\pi}\int_X \left\| \bar{\partial}_{A_j}\pi^{(i)}_j \right\|^2 dvol_\omega \leq \sum_{k \leq rk(E_i)} \mu^\infty_k + \frac{1}{2\pi}\left\| \Lambda_\omega F_{A_j} - \Lambda_\omega F_{A_\infty} \right\|_{L^1(X)}.$$

By the second assumption, $\mu = \mu_\infty$ so $\deg(E_i) = \sum_{k \leq rk(E_i)} \mu^\infty_k$, and so by the third assumption:

$$\bar{\partial}_{A_j}\pi^{(i)}_j \xrightarrow{L^2} 0.$$

Since $L^2_{1,loc}$ is weakly compact and $\left\{\pi^{(i)}_j\right\}$ is uniformly bounded in $L^2_{1,loc}$, after passing to a subsequence if necessary, $\pi^{(i)}_j \longrightarrow \tilde{\pi}^{(i)}_\infty$ weakly in $L^2_{1,loc}$ for an L^2_1 projection $\tilde{\pi}^{(i)}_\infty$. We claim that $\bar{\partial}_{A_\infty}\tilde{\pi}^{(i)}_j = 0$. For any compactly supported test form $\phi \in \Omega^{0,1}(X - Z_{\text{an}}, \mathfrak{u}(E))$:

$$\begin{aligned} \int_{X - Z_{\text{an}}} \langle \phi, \bar{\partial}_{A_\infty}\tilde{\pi}_\infty \rangle &= \int_{X - Z_{\text{an}}} \langle (\bar{\partial}_{A_\infty})^* \phi, \tilde{\pi}^{(i)}_\infty \rangle \\ &= \lim_{j \longrightarrow \infty} \int_{X - Z_{\text{an}}} \left\langle (\bar{\partial}_{A_\infty})^* \phi, \pi^{(i)}_j \right\rangle \\ &= \lim_{j \longrightarrow \infty} \int_{X - Z_{\text{an}}} \left\langle \phi, \bar{\partial}_{A_\infty}\pi^{(i)}_j \right\rangle \\ &= \lim_{j \longrightarrow \infty} \int_{X - Z_{\text{an}}} \left\langle \phi, (\bar{\partial}_{A_\infty} - \bar{\partial}_{A_j})\,\pi^{(i)}_j \right\rangle + \left\langle \phi, \bar{\partial}_{A_j}\pi^{(i)}_j \right\rangle = 0 \end{aligned}$$

where we have used that $A_j \longrightarrow A_\infty$ in $C^\infty(X - Z_{\text{an}})$, $\left\| \pi^{(i)}_j \right\|_{L^\infty} \leq 1$. In particular this means $\tilde{\pi}^{(i)}_\infty$ defines a saturated subsheaf which we will denote by $\tilde{E}^{(i)}_\infty$. Clearly $rk(\tilde{E}^{(i)}_\infty) = rk(E^{(i)}_\infty)$. We claim that $\deg(\tilde{E}^{(i)}_\infty) = \deg(E^{(i)}_\infty)$. Since $\bar{\partial}_{A_\infty}\tilde{\pi}^{(i)}_j = 0$ and

$\Lambda_\omega F_{A_j} \longrightarrow \Lambda_\omega F_{A_\infty}$ and $\pi_j^{(i)} \longrightarrow \tilde{\pi}_\infty^{(i)}$ in L^2:

$$\begin{aligned} \deg(\tilde{E}_\infty^{(i)}) &= \frac{1}{2\pi}\int_X Tr(i\Lambda_\omega F_{A_\infty}\tilde{\pi}_\infty^{(i)})dvol_\omega = \lim_{j\longrightarrow\infty}\frac{1}{2\pi}\int_X Tr(i\Lambda_\omega F_{A_\infty}\tilde{\pi}_j^{(i)})dvol_\omega \\ &= \deg(E_\infty^{(i)}) + \frac{1}{2\pi}\lim_{j\longrightarrow\infty}\int_X \left\|\bar{\partial}_{A_j}\pi_j^{(i)}\right\|^2 dvol_\omega = \deg(E_\infty^{(i)}). \end{aligned}$$

The maximal destabilising subsheaf $\mathbb{F}_1^{HN}(E_\infty)$ of E_∞ is the unique saturated subsheaf of E_∞ of the given rank and degree, so that $\tilde{\pi}_\infty^{(1)} = \pi_\infty^{(1)}$. We proceed by induction. Let $1 \leq k < l$ and assume $\tilde{\pi}_\infty^{(i)} = \pi_\infty^{(i)}$ for all $i \leq k$. Then $\frac{\tilde{E}_\infty^{(k+1)}}{E_\infty^{(k)}}$ has the same rank and slope as the maximal destabilising subsheaf of $\frac{E_\infty}{E_\infty^{(k)}}$ and so $\tilde{E}_\infty^{(k+1)} = E_\infty^{(k+1)}$. Continuing until $k = l$ completes the proof of part 1. For part 2 just notice that the same proof applies to a Seshadri filtration, but since these are not unique we can only conclude that the sheaves in the limiting filtration have the same rank and degree. ■

Proposition 24 *Assume as before that $E \to X$ is a holomorphic vector bundle such that Z_{an} is smooth and that blowing up once resolves the singularities of the HNS filtration. Then given $\delta > 0$ and any $1 \leq p < \infty$, E has an L^p δ-approximate critical hermitian structure.*

Proof. Let A_t be a solution to the YM flow with initial condition $A_0 = (\bar{\partial}_E, h)$, and let A_∞ be the limit along the flow for some sequence A_{t_j}. Then we may apply the previous lemma to conclude that $\Psi_\omega^{HNS}(\bar{\partial}_{A_{t_j}}, h) \stackrel{L^p}{\to} \Psi_\omega^{HNS}(\bar{\partial}_{A_\infty}, h_\infty)$ after passing to another subsequence if necessary. Since A_∞ is a Yang-Mills connection, $i\Lambda_\omega F_{A_\infty} =$

$\Psi^{HN}_{\omega}(\bar{\partial}_{A_\infty}, h_\infty)$. Therefore:

$$\left\| i\Lambda_\omega F_{A_{t_j}} - \Psi^{HNS}_{\omega}(\bar{\partial}_{A_{t_j}}, h) \right\|_{L^p(\omega)} \leq$$

$$\left\| \Lambda_\omega F_{A_{t_j}} - \Lambda_\omega F_{A_\infty} \right\|_{L^p(\omega)} + \left\| \Psi^{HNS}_{\omega}(\bar{\partial}_{A_{t_j}}, h) - \Psi^{HNS}_{\omega}(\bar{\partial}_{A_\infty}, h_\infty) \right\|_{L^p(\omega)} \longrightarrow 0$$

where we have also used *Lemma* 9. ■

Now we would like to eliminate the assumptions that Z_{an} is smooth and that blowing up once resolves the singularities of the HNS filtration.

Theorem 13 *Let $E \to X$ be a holomorphic vector bundle over a Kähler manifold with Kähler form ω. Then given $\delta > 0$ and any $1 \leq p < \infty$, E has an L^p δ-approximate critical hermitian structure.*

Proof. By 18, we know that we can resolve the singularities of the HNS filtration by blowing up finitely many times. Moreover, the i^{th} blowup is obtained by blowing up along a complex submanifold contained in the singular set associated to the pullback bundle over the manifold produced at the $(i-1)st$ stage of the process. In other words there is a tower of blow-ups:

$$\tilde{X} = X_m \xrightarrow{\pi_m} X_{m-1} \xrightarrow{\pi_{m-1}} ... \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = X$$

such that if $E = E_0$ is the original bundle, and $E_i = \pi_i^*(E_{i-1})$, then there is a filtration of $\tilde{E} = \pi_m^*(E_{m-1})$ that is given by sub-bundles and isomorphic to the HNS filtration of E away from $\mathbf{E}$. Note that on each blowup X_i we have a family of Kähler metrics defined iteratively by $\omega_{\varepsilon_1,\ldots,\varepsilon_i} = \pi^*\omega_{\varepsilon_1,\ldots,\varepsilon_{i-1}} + \varepsilon_i \eta_i$, where η_i is any Kähler form on X_i. Then consider $\omega_{\varepsilon_1,\ldots,\varepsilon_m}$ on $\tilde{X}$ to be a fixed metric for specified values of $\varepsilon_1, ..., \varepsilon_m$, and fix $\delta > 0$. Fix δ_0 to be a number that is very small with respect to δ. By the previous proposition, for every p there is a δ_0-approximate critical hermitian structure on E_{n-1}. In particular there is such a metric for $p = 2$. In other words there is a metric h_{m-1} so that:

$$\left\| \sqrt{-1}\Lambda_{\omega_{\varepsilon_1,\ldots\varepsilon_{m-1}}} F_{(\bar{\partial}_{E_{m-1}}, h_{m-1})} - \Psi^{HNS}_{\omega_{\varepsilon_1,\ldots\varepsilon_{m-1}}}(\bar{\partial}_{E_{m-1}}, h_{m-1}) \right\|_{L^2(\omega_{\varepsilon_1,\ldots\varepsilon_{m-1}})} < \delta_0.$$

By construction this metric depends on the values of $\varepsilon_1, ..., \varepsilon_m$, since it is constructed from a metric on the blowup which itself is constructed using the notion of stability with respect to $\omega_{\varepsilon_1,...,\varepsilon_m}$.

We prove the result by induction on the number of blowups. Assume that we have an L^2 δ_0-approximate critical hermitian structure for each of the bundles $E_i \to X_i$ for $1 \leq i \leq m - 2$. Then in particular, with respect to the metric ω_{ε_1} on X_1, we have a metric h_1 on $E_1 \to X_1$ such that:

$$\left\| \sqrt{-1}\Lambda_{\omega_{\varepsilon_1}} F_{(\bar{\partial}_{E_1}, h_1)} - \Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar{\partial}_{E_1}, h_1) \right\|_{L^2(\omega_{\varepsilon_1})} < \delta_0.$$

Since X_1 is obtained from X by blowing up along a smooth, complex submanifold, we may use the exact same cut-off argument, choosing a cutoff function with respect to a neighbourhood U_R as in Proposition 22 to construct a metric h_R on the bundle $E \to X$ which depends on the value of ε_1. In the following we will continue to denote by h_R its pullback to X_1. As in the proof of Proposition 22 we have $h_R = h_1$ outside of the set $\pi_1^{-1}(U_R)$. We divide the proof into two steps.

(Step 1) There is an L^p δ-approximate critical hermitian structure for p close to 1

First let us assume that p satisfies the hypotheses of Lemma 15. In other words, substitute p for α in the statement. Similarly, substitute $\tilde{p}$ for $\tilde{\alpha}$. We will show that a single metric, namely h_R, gives an L^p δ-approximate critical hermitian structure for all p within this range. We need to estimate the difference

$$\left\|\sqrt{-1}\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{E_1},h_R)} - \Psi^{HNS}_{\omega}(\bar{\partial}_E, h_R)\right\|_{L^p(\omega_\varepsilon)}$$

where $\tilde{h} = \pi_1^* h$. Now:

$$\begin{aligned}\left\|\sqrt{-1}\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{E_1},h_R)} - \Psi^{HNS}_{\omega}(\bar{\partial}_E, h_R)\right\|_{L^p(\omega_\varepsilon)} \leq \\ \left\|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{E_1},h_R)} - \Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{E_1},h_1)}\right\|_{L^p(\omega_\varepsilon)} \\ + \left\|\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar{\partial}_E, h_1) - \Psi^{HNS}_{\omega}(\bar{\partial}_E, h_R)\right\|_{L^p(\omega_\varepsilon)} \\ + \left\|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{E_1},h_1)} - \Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar{\partial}_E, h_1)\right\|_{L^p(\omega_\varepsilon)}.\end{aligned}$$

We can make the second term smaller than $\frac{\delta}{3}$ by choosing ε_1 small and using the convergence of the HN types. The third term is bounded by two applications of Lemma 15 as follows:

$$\begin{aligned}&\left\|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{E_1},h_1)} - \Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar{\partial}_E, h_1)\right\|_{L^p(\omega_\varepsilon)} \leq \\ &\left\|\Lambda_{\omega_\varepsilon}\left(F_{(\bar{\partial}_{E_1},h_1)} - \frac{1}{n}\omega_{\varepsilon_1}\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar{\partial}_E, h_1)\right)\right\|_{L^p(\omega_\varepsilon)} + \left\|\frac{1}{n}\Lambda_{\omega_\varepsilon}\left(\omega_{\varepsilon_1} - \omega_\varepsilon\right)\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar{\partial}_E, h_1)\right\|_{L^p(\omega_\varepsilon)} \\ \leq\ & C\left\|\Lambda_{\omega_{\varepsilon_1}} F_{(\bar{\partial}_{E_1},h_1)} - \Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar{\partial}_E, h_1)\right\|_{L^{\tilde{p}}(\omega_{\varepsilon_1})} \\ &+\kappa C\left(\left\|F_{(\bar{\partial}_{E_1},h_1)}\right\|_{L^2(\tilde{X},\omega_{\varepsilon_1})} + \frac{1}{n}\left\|\omega_{\varepsilon_1}\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar{\partial}_E, h_1)\right\|_{L^2(\tilde{X},\omega_{\varepsilon_1})}\right) \\ &+\varepsilon_1 C(\kappa)\left(\left\|F_{(\bar{\partial}_{E_1},h_1)}\right\|_{L^2(\tilde{X},\omega_{\varepsilon_1})} + \frac{1}{n}\left\|\omega_{\varepsilon_1}\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar{\partial}_E, h_1)\right\|_{L^2(\tilde{X},\omega_{\varepsilon_1})}\right) \\ &+\frac{\varepsilon_1^2}{n}C(\kappa)\left\|\eta\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar{\partial}_E, h_1)\right\|_{L^2(\tilde{X},\omega_{\varepsilon_1})} \\ &+\frac{\varepsilon_1}{n}C\left(\left\|\Lambda_{\omega_{\varepsilon_1}}\eta\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar{\partial}_E, h_1)\right\|_{L^{\tilde{p}}(\tilde{X},\omega_{\varepsilon_1})} + \kappa\left\|\eta\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar{\partial}_E, h_1)\right\|_{L^2(\tilde{X},\omega_{\varepsilon_1})}\right).\end{aligned}$$

Recall from the statement of Lemma 15 that none of the above constants depend on ε_1. All terms with a κ in front and no $C(\kappa)$ can be made small by choosing κ small, so these terms can be ignored. Clearly the terms

$$\left\|\omega_{\varepsilon_1}\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar{\partial}_E, h_1)\right\|_{L^2(\tilde{X},\omega_{\varepsilon_1})}, \left\|\eta\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar{\partial}_E, h_1)\right\|_{L^2(\tilde{X},\omega_{\varepsilon_1})}$$

are bounded independently of ε_1 since the HN type converges. Therefore we need only show that

$$\left\|\Lambda_{\omega_{\varepsilon_1}}F_{(\bar{\partial}_{E_1},h_1)} - \Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar{\partial}_E, h_1)\right\|_{L^{\tilde{p}}(\omega_{\varepsilon_1})}, \left\|F_{(\bar{\partial}_{E_1},h_1)}\right\|_{L^2(\tilde{X},\omega_{\varepsilon_1})},$$
$$\left\|\Lambda_{\omega_{\varepsilon_1}}\eta\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar{\partial}_E, h_1)\right\|_{L^{\tilde{p}}(\tilde{X},\omega_{\varepsilon_1})}$$

are uniformly bounded in ε_1. Then we can choose κ first and then ε_1 so that:

$$\left\|\Lambda_{\omega_\varepsilon}F_{(\bar{\partial}_{E_1},h_1)} - \Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar{\partial}_E, h_1)\right\|_{L^p(\omega_\varepsilon)} < \frac{\delta}{3}.$$

Firstly we have:

$$\begin{aligned}\left\|\Lambda_{\omega_{\varepsilon_1}}F_{(\bar{\partial}_{E_1},h_1)} - \Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar{\partial}_E, h_1)\right\|_{L^{\tilde{p}}(\omega_{\varepsilon_1})} &\leq \\ C\left\|\Lambda_{\omega_{\varepsilon_1}}F_{(\bar{\partial}_{E_1},h_1)} - \Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar{\partial}_E, h_1)\right\|_{L^2(\omega_{\varepsilon_1})} &< \delta_0\end{aligned}$$

by Hölder's inequality (since $\tilde{p} < 2$), and the induction hypothesis. Note that the constant above is independent of ε_1 since the ω_{ε_1} volume is bounded. Also, the following bound:

$$\begin{aligned}\left\|F_{(\bar{\partial}_{E_1},h_1)}\right\|_{L^2(\omega_{\varepsilon_1})} &= \left\|\Lambda_{\omega_{\varepsilon_1}}F_{(\bar{\partial}_{E_1},h_1)}\right\|_{L^2(\omega_{\varepsilon_1})} + \\ &\quad \pi^2(n)(n-1)\int_{\tilde{X}}\left(2c_2(E_1) - c_1^2(E_1)\right)\wedge\omega_{\varepsilon_1}^{n-2} \\ &\leqslant \left\|\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar{\partial}_E, h_1)\right\|_{L^2(\omega_{\varepsilon_1})} + \\ &\quad \delta_0 + \pi^2(n)(n-1)\int_{\tilde{X}}\left(2c_2(E_1) - c_1^2(E_1)\right)\wedge\omega_{\varepsilon_1}^{n-2}\end{aligned}$$

obtained from the usual relationship between the Hermitian-Einstein tensor and the full curvature in L^2, together with the induction hypothesis, shows that this term is bounded in ε_1 as well. Finally, writing

$$\begin{aligned}\Lambda_{\omega_{\varepsilon_1}}\eta &= \frac{\eta\wedge\omega_{\varepsilon_1}^{n-1}}{\omega_{\varepsilon_1}^n} = \frac{\eta\wedge\omega_{\varepsilon_1}^{n-1}}{\eta^n}\frac{\det\eta}{\det\omega_{\varepsilon_1}} \\ \omega_{\varepsilon_1}^n &= \frac{\det\omega_{\varepsilon_1}}{\det\eta}\eta^n\end{aligned}$$

then by Hölder's inequality we have:

$$\left\|\Lambda_{\omega_{\varepsilon_1}}\eta\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar\partial_E,h_1)\right\|_{L^{\tilde p}(\tilde X,\omega_{\varepsilon_1})} \leq$$
$$\left(\int_{\tilde X}\left|\frac{\det\omega_{\varepsilon_1}}{\det\eta}\right|^{(1-\tilde p)(\tilde s)}\eta^n\right)^{\frac{1}{\tilde p\tilde s}}\left(\int_{\tilde X}\left|\frac{\eta\wedge\omega_{\varepsilon_1}^{n-1}}{\eta^n}\right|^w\left|\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar\partial_E,h_1)\right|^w\eta^n\right)^{\frac{1}{w}}$$

where $\tilde s = \frac{w}{w-\tilde p}$ and $\frac{\tilde p}{1-2(k-1)(\tilde p-1)} < w < \infty$. By Lemma 14 this is bounded in ε_1.

We have already seen that

$$\left\|\Lambda_{\omega_\varepsilon}F_{(\bar\partial_{E_1},h_R)} - \Lambda_{\omega_\varepsilon}F_{(\bar\partial_{E_1},h_1)}\right\|_{L^p(\omega_\varepsilon)}$$

can be estimated, since it is 0 outside of U_R and the same argument as in the proof of Proposition 22, shows that by making R sufficiently small, we can make the contribution from this term over U_R less than $\frac{\delta}{3}$. Therefore the estimate on $\left\|i\Lambda_\omega F_{(\bar\partial_E,h)} - \Psi^{HNS}_\omega(\bar\partial_E,h)\right\|_{L^p(\omega)}$ for these values of p follows by sending $\varepsilon\to 0$.

Step 2 (Extending to all p)

Repeating the arguments of Lemma 16, Proposition 23, Lemma 17, and Proposition 24, now gives the existence of an L^p δ-approximate critical hermitian structure on E for each p. This metric will depend on p. ■

Notice that during the course of the above proof we have also proven the following:

Theorem 14 *Let $E \to X$ be a holomorphic vector bundle over a Kähler manifold. Let A_t be a solution to the YM flow with initial condition A_0 whose limit along the flow is A_∞. Let E_∞ be the corresponding holomorphic vector bundle defined away from Z_{an}. Then the HN type of (E_∞, A_∞) is the same as (E_0, A_0).*

Chapter 3

Proof of the Main Theorem

3.1 The Degenerate Yang-Mills Flow

In this section we introduce a version of the Yang-Mills flow on a sequence with respect to the degenerate metric $\omega_0 = \pi^*\omega$ on a sequence of blowups $\pi : \tilde{X} \to X$ along complex submanifolds. This flow will correspond exactly to the usual Yang-Mills flow on $\tilde{X} - \mathbf{E}$ with respect the metric ω. It will be useful in the proof of the main theorem, because we will again need to desingularise the HNS filtration, and consider a sequence of blowups. The argument will rely on having a flow with the correct properties that is well-defined on all of $\tilde{X}$ rather than just on the complement of $\mathbf{E}$. The idea here is due to Bando and Siu (see [BS]).

Let $\pi : \tilde{X} \to X$ be a sequence of smooth blowups, and let ω_ε be the usual family of Kähler metrics on $\tilde{X}$. We will write $L^p_k(\tilde{X}, \omega_\varepsilon)$ for the corresponding Sobolev spaces. The following lemma is clear.

Lemma 18 *Fix a compact subset $W \subset\subset \tilde{X} - \mathbf{E}$. Let $\tilde{E}$ be a vector bundle. Then there exists a family of constants $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that for any r-form $F \in \Omega^r(\tilde{X} - \mathbf{E}, \tilde{E})$*

$$(1 - C(\varepsilon)) \|F\|_{L^p_k(W,\omega_0)} \leq \|F\|_{L^p_k(W,\omega_\varepsilon)} \leq (1 + C(\varepsilon)) \|F\|_{L^p_k(W,\omega_0)} .$$

Throughout this section $\tilde{E} \to \tilde{X}$ will be a holomorphic vector bundle of rank

K, equipped with a smooth hermitian metric $\tilde{h}_0$. Note that $\left\|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}},\tilde{h}_0)}\right\|_{L^1(\omega_\varepsilon)}$ is uniformly bounded in ε, since for any fixed Kähler form (metric) ϖ on $\tilde{X}$ we have:

$$\begin{aligned} \left|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}},\tilde{h}_0)}\right| &= \left|\frac{F_{(\bar{\partial}_{\tilde{E}},\tilde{h}_0)} \wedge \omega_\varepsilon^{n-1}}{\omega_\varepsilon^n}\right| = \left|\frac{F_{(\bar{\partial}_{\tilde{E}},\tilde{h}_0)} \wedge \omega_\varepsilon^{n-1}}{\varpi^n}\right| \left|\frac{\det \varpi}{\det g^\varepsilon}\right|, \\ \omega_\varepsilon^n &= \frac{\det g^\varepsilon}{\det \varpi} \varpi^n \end{aligned}$$

so

$$\left\|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}},\tilde{h}_0)}\right\|_{L^1(\omega_\varepsilon)} = \int_{\tilde{X}} \left|\frac{F_{(\bar{\partial}_{\tilde{E}},\tilde{h}_0)} \wedge \omega_\varepsilon^{n-1}}{\varpi^n}\right| \varpi^n$$

which is clearly bounded uniformly in ε. Write $\tilde{h}_{\varepsilon,t}$ for the evolution of $\tilde{h}_0$ under the HYM flow with respect to the metric ω_ε.

Lemma 19 (1) *Let $t_0 > 0$. Then $\left|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}},\tilde{h}_{\varepsilon,t})}\right|$ is uniformly bounded for all $t \geq t_0 > 0$ and all $\varepsilon > 0$. The bound depends only on t_0 and the uniform bound on $\left\|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}},\tilde{h}_0)}\right\|_{L^1(\omega_\varepsilon)}$.*

(2) *$\left|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}},\tilde{h}_{\varepsilon,t})}\right|$ is bounded uniformly on compact subsets of $\tilde{X} - \mathbf{E}$ for all $t \geq 0$ and all $\varepsilon > 0$. The bound depends only on the local bound on $\left|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}},\tilde{h}_0)}\right|$ and the uniform bound on $\left\|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}},\tilde{h}_0)}\right\|_{L^1(\omega_\varepsilon)}$.*

Proof. By Lemma 2 (2), the pointwise norm $\left|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}},\tilde{h}_{\varepsilon,t})}\right|$ is a subsolution of the heat equation on $(\tilde{X}, \omega_\varepsilon)$ (see also [BS] equation 3.3). If $K_t^\varepsilon(x,y)$ is the heat kernel for the ω_ε Laplacian on $\tilde{X}$ then

$$\int_{\tilde{X}} K_t^\varepsilon(x,y) \left|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}},\tilde{h}_0)}\right| (y) dvol_{\omega_\varepsilon}(y)$$

is a solution of the heat equation and therefore:

$$\left|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}},\tilde{h}_{\varepsilon,t})}\right| (x) - \int_{\tilde{X}} K_t^\varepsilon(x,y) \left|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}},\tilde{h}_0)}\right| (y) dvol_{\omega_\varepsilon}(y)$$

is also a subsolution. Because

$$\int_{\tilde{X}} K_0^\varepsilon(x,y) \left|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_0)}\right|(y) dvol_{\omega_\varepsilon}(y) = \left|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_0)}\right|(x),$$

the maximum principle for the heat equation now implies that

$$\left|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}} \tilde{h}_{\varepsilon,t})}\right|(x) \leq \int_{\tilde{X}} K_t^\varepsilon(x,y) \left|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_0)}\right|(y) dvol_{\omega_\varepsilon}(y).$$

By [BS] *Lemma* 4, there is a bound: $K_t^\varepsilon(x,y) \leq C\left(1 + 1/t^n\right)$ for some constant C independent of ε. Part (1) now follows.

For part (2), let $\Omega_1 \subset\subset \Omega \subset\subset \tilde{X} - \mathbf{E}$, and let ψ be a smooth cut-off function supported in Ω and identically 1 in a neighbourhood of $\bar{\Omega}_1$. Then just as in part (1) we have:

$$\begin{aligned} \left|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}} \tilde{h}_{\varepsilon,t})}\right|(x) &\leq \int_{\tilde{X}} K_t^\varepsilon(x,y) \left|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_0)}\right|(y) dvol_{\omega_\varepsilon}(y) \\ &= \int_{\tilde{X}} \psi K_t^\varepsilon(x,y) \left|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_0)}\right|(y) dvol_{\omega_\varepsilon}(y) \\ &\quad + \int_{\tilde{X}} (1-\psi) K_t^\varepsilon(x,y) \left|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_0)}\right|(y) dvol_{\omega_\varepsilon}(y). \end{aligned}$$

By the maximum principle, the first term on the right hand side is bounded from above by:
$\sup\left\{\left|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_0)}\right|(y) \mid y \in \Omega\right\}$. Since $\Omega \subset\subset \tilde{X} - \mathbf{E}$, the function $1/\det g_{ij}^\varepsilon$ is uniformly bounded in ε, so this sup and hence the first integral above are uniformly bounded in ε. By [GR] Theorem 3.1, there are positive constants δ, C_1, C_2, independent of t and ε, such that for $x \neq y$,

$$K_t^\varepsilon(x,y) \leq C_1 \left(1 + \frac{1}{\delta^2 t^2}\right) \exp\left(-\frac{\left(d_{\omega_\varepsilon}(x,y)\right)^2}{C_2 t}\right).$$

where d_{ω_ε} is the distance function on $\tilde{X}$ with respect to the Riemannian metric induced by ω_ε. Of course $d_{\omega_\varepsilon}(x,y)$ is bounded from below for $x \in \Omega_1$ and $y \in$

supp$(1-\psi)$ uniformly in ε. Therefore, $K_t^\varepsilon(x,y)$ is uniformly bounded in ε and t, for these values of x and y. Then the second term on the right is uniformly bounded in terms of $\left\|\Lambda_\omega F_{(\bar\partial_{\tilde E}\tilde h_0)}\right\|_{L^1(\omega_\varepsilon)}$, so $\left|\Lambda_{\omega_\varepsilon} F_{(\bar\partial_{\tilde E}\tilde h_{\varepsilon,t})}\right|$ is uniformly bounded on Ω_1. ■

If we write $\tilde h_{\varepsilon,t} = \tilde k_{\varepsilon,t}\tilde h_0$, then it follows from the HYM flow equations and the second part of the previous lemma that both $\tilde k_{\varepsilon,t}$ and $\tilde k_{\varepsilon,t}^{-1}$ are uniformly bounded on compact subsets of $\tilde X - \mathbf{E}$ for $0 \le t \le t_0$. The statement that $\left|\Lambda_{\omega_\varepsilon} F_{(\bar\partial_{\tilde E}\tilde h_{\varepsilon,t})}\right|$ is uniformly bounded on compact subsets of $\tilde X - \mathbf{E}$ translates to the statement that there is a section $f_{\varepsilon,t} \in \mathfrak{u}(E)$, uniformly bounded on compact subsets of $\tilde X - \mathbf{E}$, such that:

$$\sqrt{-1}\Lambda_{\omega_\varepsilon}\bar\partial_{A_0}\left(\tilde k_{\varepsilon,t}^{-1}\partial_{A_0}\tilde k_{\varepsilon,t}\right) = f_{\varepsilon,t},$$

where A_0 is the connection $(\bar\partial_E, \tilde h_0)$. It therefore follows from [BS] Proposition 1, that $\tilde k_{\varepsilon,t}$ has a uniform $C^{1,\alpha}$ bound on compact subsets of $\left(\tilde X - \mathbf{E}\right) \times [0,\infty)$. Furthermore, we may write:

$$\begin{aligned}\sqrt{-1}\Lambda_{\omega_\varepsilon}\bar\partial_{A_0}\left(\tilde k_{\varepsilon,t}^{-1}\partial_{A_0}\tilde k_{\varepsilon,t}\right) &= \tilde k_{\varepsilon,t}^{-1}\sqrt{-1}\Lambda_{\omega_\varepsilon}\left(\bar\partial_{A_0}\partial_{A_0}\tilde k_{\varepsilon,t}\right) + \sqrt{-1}\Lambda_{\omega_\varepsilon}\left(\bar\partial_{A_0}\tilde k_{\varepsilon,t}^{-1}\right)\left(\partial_{A_0}\tilde k_{\varepsilon,t}\right)\\ &= \tilde k_{\varepsilon,t}^{-1}\triangle_{(\bar\partial_{A_0},\omega_\varepsilon)}\tilde k_{\varepsilon,t} + \tilde k_{\varepsilon,t}^{-1}\sqrt{-1}\Lambda_{\omega_\varepsilon}\left(\bar\partial_{A_0}\tilde k_{\varepsilon,t}\right)\tilde k_{\varepsilon,t}^{-1}\left(\partial_{A_0}\tilde k_{\varepsilon,t}\right),\end{aligned}$$

where in the last equality we have used the Kähler identities and the expression for $\bar\partial_{A_0}\tilde k_{\varepsilon,t}^{-1}$. Therefore we have:

$$\triangle_{(\bar\partial_{A_0},\omega_\varepsilon)}\tilde k_{\varepsilon,t} + \sqrt{-1}\Lambda_{\omega_\varepsilon}\left(\bar\partial_{A_0}\tilde k_{\varepsilon,t}\right)\tilde k_{\varepsilon,t}^{-1}\left(\partial_{A_0}\tilde k_{\varepsilon,t}\right) = \tilde k_{\varepsilon,t} f_{\varepsilon,t}.$$

By elliptic regularity, this yields a uniform L_2^p bound on $\tilde k_{\varepsilon,t}$ on compact subsets of $\left(\tilde X - \mathbf{E}\right) \times [0,\infty)$. It now follows from the HYM the flow equations, that $\frac{\partial \tilde h_{\varepsilon,t}}{\partial t}$ has a uniform L^p bound on compact subsets of $\left(\tilde X - \mathbf{E}\right) \times [0,\infty)$, and so for any

$W \subset\subset \left(\tilde{X} - \mathbf{E}\right)$ and $T \geq 0$, there is a uniform $L^p_{2/1}(W \times [0, T))$ bound on $\tilde{h}_{\varepsilon,t}$, where the 2/1 in the previous notation refers to the fact that there is 1 derivative in the time variable and 2 derivatives in the space variables. By weak compactness, there is a subsequence $\varepsilon_j \to 0$, so that $\tilde{h}_{\varepsilon,t} \to \tilde{h}_t$ in $L^p_{2/1}$ on compact subsets. By the Sobolev imbedding theorem, $\tilde{h}_{\varepsilon,t} \to \tilde{h}_t$ in $C^{1/0}$ on compact subsets. By a further diagonalisation as $T \to \infty$, $\tilde{h}_{\varepsilon,t} \to \tilde{h}_t$ for all $t \geq 0$.

Definition 7 *We will refer to the resulting limit $\tilde{h}_t$ corresponding to the initial metric $\tilde{h}_0$ and the degenerate metric ω_0 as the* ***degenerate Hermitian-Yang-Mills flow****.*

Of course a priori $\tilde{h}_t$ may depend on the subsequence ε_j. We will show that in fact $\tilde{h}_t$ solves the HYM equations on $\tilde{X} - \mathbf{E}$ with respect to the metric ω_0.

Lemma 20 *Let $\tilde{h}_t$ be defined as above. Then $\tilde{h}_t$ is an hermitan metric on $\tilde{E} \to \tilde{X} - \mathbf{E}$ for all $t \geq 0$, and solves the HYM equations on $\tilde{X} - \mathbf{E}$:*

$$\tilde{h}_t^{-1} \frac{\partial \tilde{h}_t}{\partial t} = -2 \left(\Lambda_{\omega_0} F_{\tilde{h}_t} - \mu_{\omega_0}(E) \mathbf{Id}_E \right).$$

Proof. Clearly $\tilde{h}_t$ is positive semi-definite since it is a limit of metrics. Therefore we only need to check that $\det \tilde{h}_t$ is positive. Taking the trace of both sides of the HYM equations for the metric ω_ε, we get:

$$\frac{\partial}{\partial t} \left(\log \det \tilde{h}_{\varepsilon,t} \right) = -2 \operatorname{Tr} \left(\Lambda_{\omega_\varepsilon} F_{\tilde{h}_{\varepsilon,t}} - \mu_{\omega_\varepsilon}(E) \mathbf{Id}_E \right)$$

integrating both sides:

$$\left| \log \left(\frac{\det \tilde{h}_{\varepsilon,T}}{\det \tilde{h}_0} \right) \right| = 2 \left| \int_0^T \operatorname{Tr} \left(\Lambda_{\omega_\varepsilon} F_{\tilde{h}_{\varepsilon,t}} - \mu_{\omega_\varepsilon}(E) \mathbf{Id}_E \right) \right|.$$

By the previous lemma, the right hand side is bounded uniformly in ε, so $\det \tilde{h}_T = \lim_{\varepsilon_j \to 0} \det \tilde{h}_{\varepsilon_j,T}$ must be positive. Since $\tilde{h}_{\varepsilon_j,t} \to \tilde{h}_t$ weakly in $L^p_{2/1}$ and $C^{1/0}$ it follows that $\tilde{h}_t$ solves the HYM equations on $\tilde{X} - \mathbf{E}$. ■

Lemma 21 $\left\|F_{\tilde{h}_t}\right\|_{L^2(\tilde{X},\omega_0)}$ *and* $\left\|\Lambda_{\omega_0} F_{\tilde{h}_t}\right\|_{L^\infty(\tilde{X},\omega_0)}$ *are uniformly bounded for all* $t \geq t_0 > 0$. *The bound depends only on* t_0 *and the uniform bound on* $\left\|\Lambda_{\omega_\varepsilon} F_{\tilde{h}_0}\right\|_{L^1(\omega_\varepsilon)}$.

Proof. Let $W \subset\subset \tilde{X} - \mathbf{E}$ be a compact subset. By construction $F_{\tilde{h}_{\varepsilon_j},t} \to F_{\tilde{h}_t}$ weakly in $L^2(W,\omega_0)$. Applying Lemma 18 and the relation between $F_{\tilde{h}_{\varepsilon,t}}$ and $\Lambda_{\omega_\varepsilon} F_{\tilde{h}_{\varepsilon,t}}$ in L^2 we have:

$$
\begin{aligned}
\left\|F_{\tilde{h}_t}\right\|_{L^2(W,\omega_0)} &\leq \liminf_{\varepsilon \longrightarrow 0} \left\|F_{\tilde{h}_{\varepsilon,t}}\right\|_{L^2(W,\omega_0)} \leq C_1 \liminf_{\varepsilon \longrightarrow 0} \left\|F_{\tilde{h}_{\varepsilon,t}}\right\|_{L^2(W,\omega_\varepsilon)} \\
&\leq C_1 \liminf_{\varepsilon \longrightarrow 0} \left\|F_{\tilde{h}_{\varepsilon,t}}\right\|_{L^2(\tilde{X},\omega_\varepsilon)} \leq C_1 \liminf_{\varepsilon \longrightarrow 0} \left\|\Lambda_{\omega_\varepsilon} F_{\tilde{h}_{\varepsilon,t}}\right\|_{L^2(\tilde{X},\omega_\varepsilon)} + C_2 \\
&\leq C_3 \liminf_{\varepsilon \longrightarrow 0} \left\|\Lambda_{\omega_\varepsilon} F_{\tilde{h}_{\varepsilon,t}}\right\|_{L^\infty(\tilde{X})} + C_2,
\end{aligned}
$$

where C_3 is independent of W, and C_2 is the product of C_1 with a topological constant. The bound in L^2 now follows from Lemma 19 (1).

For the second part again fix $W \subset\subset \tilde{X} - \mathbf{E}$. We claim that for a fixed t and W, as $\varepsilon \to 0$ there is a uniform bound

$$
\left\|\Lambda_{\omega_0} F_{\tilde{h}_{\varepsilon,t}}\right\|_{L^p(W,\omega_0)} \leq \left\|\Lambda_{\omega_\varepsilon} F_{\tilde{h}_{\varepsilon,t}}\right\|_{L^p(W,\omega_0)} + 1.
$$

Otherwise, there is a sequence ε_j such that:

$$
\left\|\Lambda_{\omega_0} F_{h_{\varepsilon_j,t}}\right\|_{L^p(W,\omega_0)} \geq \left\|\Lambda_{\omega_{\varepsilon_j}} F_{\tilde{h}_{\varepsilon_j,t}}\right\|_{L^p(W,\omega_0)} + 1.
$$

Then

$$
\left|\Lambda_{\omega_0} - \Lambda_{\omega_{\varepsilon_j}}\right| \left\|F_{h_{\varepsilon_j,t}}\right\|_{L^p(W,\omega_0)} \geq \left\|\left(\Lambda_{\omega_0} - \Lambda_{\omega_{\varepsilon_j}}\right)\left(F_{\tilde{h}_{\varepsilon_j,t}}\right)\right\|_{L^p(W,\omega_0)} \geq 1.
$$

where $\left|\Lambda_{\omega_0} - \Lambda_{\omega_{\varepsilon_j}}\right|$ denotes the operator norm. Since $\left\|\Lambda_{\omega_0} F_{\tilde{h}_{\varepsilon_j,t}}\right\|_{L^p(W,\omega_0)}$ is uniformly bounded in ε_j and $\Lambda_{\omega_{\varepsilon_j}} \to \Lambda_{\omega_0}$ on W, this is a contradiction, and so we have proved the claim. Now we have $\tilde{h}_{\varepsilon_j,t} \to \tilde{h}_t$ weakly in $L_2^p(\omega_0, W)$, so $\left\|F_{\tilde{h}_{\varepsilon_j,t}}\right\|_{L^p(W,\omega_0)}$ is uniformly bounded. Therefore:

$$\begin{aligned}
\left\|\Lambda_{\omega_0} F_{\tilde{h}_t}\right\|_{L^p(W,\omega_0)} &\leq \lim \inf_{\varepsilon \longrightarrow 0} \left\|\Lambda_{\omega_0} F_{\tilde{h}_{\varepsilon,t}}\right\|_{L^p(W,\omega_0)} \leq \lim \inf_{\varepsilon \longrightarrow 0} \left\|\Lambda_{\omega_\varepsilon} F_{\tilde{h}_{\varepsilon,t}}\right\|_{L^p(W,\omega_0)} + 1 \\
&\leq C \lim \inf_{\varepsilon \longrightarrow 0} \left\|\Lambda_{\omega_\varepsilon} F_{\tilde{h}_{\varepsilon,t}}\right\|_{L^\infty(\tilde{X})} + 1.
\end{aligned}$$

Taking $p \to \infty$, the lemma now follows from Lemma 19. ■

Proposition 25 *For almost all $t \geq t_0 > 0$, we have:*

$$\left\|\nabla_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_t)} \Lambda_{\omega_0} F_{\tilde{h}_t}\right\|_{L^2(\tilde{X},\omega_0)} \leq \lim \inf_{\varepsilon \longrightarrow 0} \left\|\nabla_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_{\varepsilon,t})} \Lambda_{\omega_\varepsilon} F_{\tilde{h}_{\varepsilon,t}}\right\|_{L^2(\tilde{X},\omega_\varepsilon)} < \infty.$$

In particular, $\displaystyle\int_{t_0}^{\infty} \left\|\nabla_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_t)} \Lambda_{\omega_0} F_{\tilde{h}_t}\right\|_{L^2(\omega_0)} < \infty.$

Proof. By Lemma 2 (1) we have:

$$\frac{d}{dt} \left\|F_{\tilde{h}_{\varepsilon,t}}\right\|^2_{L^2(\tilde{X},\omega_\varepsilon)} = -2 \left\|d^*_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_t)} F_{\tilde{h}_{\varepsilon,t}}\right\|^2_{L^2(\tilde{X},\omega_\varepsilon)} = -2 \left\|\nabla_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_t)} \Lambda_{\omega_\varepsilon} F_{\tilde{h}_{\varepsilon,t}}\right\|^2_{L^2(\omega_\varepsilon)}.$$

Then:

$$2 \int_{t_0}^{\infty} \left\|\nabla_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_t)} \Lambda_{\omega_\varepsilon} F_{\tilde{h}_{\varepsilon,t}}\right\|^2_{L^2(\tilde{X},\omega_\varepsilon)} \leq \left\|F_{\tilde{h}_{\varepsilon,t}}\right\|^2_{L^2(\tilde{X},\omega_\varepsilon)} \leq \left\|\Lambda_{\omega_\varepsilon} F_{\tilde{h}_{\varepsilon,t}}\right\|^2_{L^2(\tilde{X},\omega_\varepsilon)} + C.$$

By *Lemma* 19 the right hand side is uniformly bounded as $\varepsilon \to 0$. Then by Fatou's lemma:

$$\lim \inf_{\varepsilon \longrightarrow 0} \left\|\nabla_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_t)} \Lambda_{\omega_\varepsilon} F_{\tilde{h}_{\varepsilon,t}}\right\|^2_{L^2(\tilde{X},\omega_\varepsilon)} < \infty$$

for almost all $t \geq t_0$. Then if the first inequality in the statement of the proposition is true, we have:

$$\begin{aligned}
\int_{t_0}^{\infty} \left\| \nabla_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_t)} \Lambda_{\omega_0} F_{\tilde{h}_{\varepsilon,t}} \right\|^2_{L^2(\tilde{X},\omega_0)} &\leq \int_{t_0}^{\infty} \lim_{\varepsilon \longrightarrow 0} \inf \left\| \nabla_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_{\varepsilon,t})} \Lambda_{\omega_\varepsilon} F_{\tilde{h}_{\varepsilon,t}} \right\|_{L^2(\tilde{X},\omega_\varepsilon)} \\
(Fatou) &\leq \lim_{\varepsilon \longrightarrow 0} \inf \int_{t_0}^{\infty} \left\| \nabla_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_{\varepsilon,t})} \Lambda_{\omega_\varepsilon} F_{\tilde{h}_{\varepsilon,t}} \right\|_{L^2(\tilde{X},\omega_\varepsilon)} \\
&\leq \lim_{\varepsilon \longrightarrow 0} \inf \left\| \Lambda_{\omega_\varepsilon} F_{\tilde{h}_{\varepsilon,t}} \right\|^2_{L^2(\tilde{X},\omega_\varepsilon)} + C < \infty.
\end{aligned}$$

Therefore it suffices to prove the first inequality:

$$\left\| \nabla_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_t)} \Lambda_{\omega_0} F_{\tilde{h}_t} \right\|_{L^2(\tilde{X},\omega_0)} \leq \lim_{\varepsilon \longrightarrow 0} \inf \left\| \nabla_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_{\varepsilon,t})} \Lambda_{\omega_\varepsilon} F_{\tilde{h}_{\varepsilon,t}} \right\|_{L^2(\tilde{X},\omega_\varepsilon)}.$$

It is enough to show this for an arbitrary compact subset $W \subset\subset \tilde{X} - \mathbf{E}$. For almost all $t \geq t_0$, we may choose a sequence $\varepsilon_j \to 0$ such that

$$\lim_{j \longrightarrow \infty} \left\| \nabla_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_{\varepsilon_j,t})} \Lambda_{\omega_{\varepsilon_j}} F_{\tilde{h}_{\varepsilon_j,t}} \right\|^2_{L^2(W,\omega_{\varepsilon_j})} = \lim_{\varepsilon \longrightarrow 0} \inf \left\| \nabla_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_{\varepsilon,t})} \Lambda_{\omega_\varepsilon} F_{\tilde{h}_{\varepsilon,t}} \right\|^2_{L^2(W,\omega_\varepsilon)} = b < \infty.$$

Since $\tilde{h}_{\varepsilon_j,t} \to \tilde{h}_t$ weakly in $L^p_2(\tilde{W})$, we have $\Lambda_{\omega_0} F_{\tilde{h}_{\varepsilon_j,t}} \to \Lambda_{\omega_0} F_{\tilde{h}_t}$ weakly in $L^p(\tilde{W})$, and $\nabla_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_{\varepsilon_j,t})} \to \nabla_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_t)}$ in $C^0(W)$. It follows by the triangle inequality and Lemma 18, that

$$\left\| \nabla_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_t)} \Lambda_{\omega_0} F_{\tilde{h}_{\varepsilon_j,t}} \right\|_{L^2(W,\omega_0)} \leq (1 + C_j) \left\| \nabla_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_{\varepsilon_j,t})} \Lambda_{\omega_{\varepsilon_j}} F_{\tilde{h}_{\varepsilon_j,t}} \right\|_{L^2(W,\omega_{\varepsilon_j})} + c_j$$

where C_j and $c_j \to 0$. Then, $\left\| \Lambda_{\omega_0} F_{\tilde{h}_{\varepsilon_j,t}} \right\|_{L^2_1(W,\omega_0)}$ is uniformly bounded as $j \to \infty$. Choose a subsequence (still written j) such that $\Lambda_{\omega_0} F_{\tilde{h}_{\varepsilon_j,t}}$ converges weakly in $L^2_1(W, \omega_0)$. By Rellich compactness we also have strong convergence $\Lambda_{\omega_0} F_{\tilde{h}_{\varepsilon_j,t}} \to \Lambda_{\omega_0} F_{\tilde{h}_t}$ in $L^2(W)$. By the choice of ε_j and the previous inequality, we have

$$\left\| \nabla_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_t)} \Lambda_{\omega_0} F_{\tilde{h}_{\varepsilon_j,t}} \right\|^2_{L^2(W,\omega_0)} \to b$$

. Then finally:

$$\begin{aligned}\left\|\Lambda_{\omega_0}F_{\tilde{h}_t}\right\|^2_{L^2_1(W,\omega_0)} &\leq \lim_{j\longrightarrow\infty}\inf \left\|\Lambda_{\omega_0}F_{\tilde{h}_{\varepsilon_j,t}}\right\|^2_{L^2_1(W,\omega_0)} \\ &\leq \lim_{j\longrightarrow\infty}\inf\left(\left\|\Lambda_{\omega_0}F_{\tilde{h}_{\varepsilon_j,t}}\right\|^2_{L^2(W,\omega_0)} + \left\|\nabla_{\left(\bar{\partial}_{\tilde{E}},\tilde{h}_t\right)}\Lambda_{\omega_0}F_{\tilde{h}_{\varepsilon_j,t}}\right\|^2_{L^2(W,\omega_0)}\right) \\ &\leq \left\|\Lambda_{\omega_0}F_{\tilde{h}_t}\right\|^2_{L^2(W,\omega_0)} + b.\end{aligned}$$

Since $\left\|\Lambda_{\omega_0}F_{\tilde{h}_t}\right\|^2_{L^2_1(W,\omega_0)} = \left\|\Lambda_{\omega_0}F_{\tilde{h}_t}\right\|^2_{L^2(W,\omega_0)} + \left\|\nabla_{\left(\bar{\partial}_{\tilde{E}},\tilde{h}_t\right)}\Lambda_{\omega_0}F_{\tilde{h}_t}\right\|^2_{L^2(W,\omega_0)}$, we have

$$\left\|\nabla_{\left(\bar{\partial}_{\tilde{E}},\tilde{h}_t\right)}\Lambda_{\omega_0}F_{\tilde{h}_t}\right\|^2_{L^2(W,\omega_0)} \leq b = \lim_{\varepsilon\longrightarrow 0}\inf\left\|\nabla_{\left(\bar{\partial}_{\tilde{E}},\tilde{h}_{\varepsilon,t}\right)}\Lambda_{\omega_\varepsilon}F_{\tilde{h}_{\varepsilon,t}}\right\|^2_{L^2(W,\omega_\varepsilon)},$$

which proves the proposition. ■

The following is an immediate consequence.

Corollary 8 *There is a sequence* $t_j \to \infty$ *such that* $\left\|\nabla_{\left(\bar{\partial}_{\tilde{E}},\tilde{h}_{t_j}\right)}\Lambda_{\omega_0}F_{\tilde{h}_t}\right\|_{L^2(\tilde{X},\omega_0)} \to 0$.

Proposition 26 *For almost all* $t > 0$, *there is a sequence* $\varepsilon_j(t) \to 0$ *such that* $\Lambda_{\omega_{\varepsilon_j}}F_{\tilde{h}_{\varepsilon_{j,t}}} \to \Lambda_{\omega_0}F_{\tilde{h}_t}$ *for all* $1 \leq p \leq \infty$. *In particular:* $HYM^{\omega_{\varepsilon_j}}_\alpha\left(\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_{\varepsilon_j,t})}\right) \to HYM^{\omega_0}_\alpha\left(\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_t)}\right)$ *for all* α.

Proof. Fix $\delta > 0$. Let $\tilde{U}$ be an open set containing $\mathbf{E}$ with $\mathrm{vol}(\tilde{U}) < \frac{\delta}{3C}$ where C is an upper bound on $\left|\Lambda_{\omega_\varepsilon}F_{\tilde{h}_{\varepsilon,t}}\right|$ which exists by Lemma 20. Now let t, ε_j be such that

$$\lim_{j\longrightarrow\infty}\left\|\nabla_{\left(\bar{\partial}_{\tilde{E}},\tilde{h}_{\varepsilon_j,t}\right)}\Lambda_{\omega_{\varepsilon_j}}F_{\tilde{h}_{\varepsilon_j,t}}\right\|^2_{L^2(W,\omega_{\varepsilon_j})} = \lim_{\varepsilon\longrightarrow 0}\inf\left\|\nabla_{\left(\bar{\partial}_{\tilde{E}},\tilde{h}_{\varepsilon,t}\right)}\Lambda_{\omega_\varepsilon}F_{\tilde{h}_{\varepsilon,t}}\right\|^2_{L^2(W,\omega_\varepsilon)} < \infty$$

as in the proof of the previous proposition, where $W = \tilde{X} - \tilde{U}$. Therefore, by the same argument as in the above proof we have strong convergence $\Lambda_{\omega_0}F_{\tilde{h}_{\varepsilon_j,t}} \to \Lambda_{\omega_0}F_{\tilde{h}_t}$

in $L^2(W,\omega_0)$. Therefore the same is true for $\Lambda_{\omega_{\varepsilon_j}} F_{\tilde{h}_{\varepsilon_j,t}}$. In particular there exists a J such that for $j,k \geq J$, we have:

$$\left\| \Lambda_{\omega_{\varepsilon_j}} F_{\tilde{h}_{\varepsilon_j,t}} - \Lambda_{\omega_{\varepsilon_k}} F_{\tilde{h}_{\varepsilon_k,t}} \right\|_{L^2(W,\omega_0)} \leq \frac{\delta}{3}.$$

By the choice of $\tilde{U}$, it follows that for $j,k \geq J$:

$$\left\| \Lambda_{\omega_{\varepsilon_j}} F_{\tilde{h}_{\varepsilon_j,t}} - \Lambda_{\omega_{\varepsilon_k}} F_{\tilde{h}_{\varepsilon_k,t}} \right\|_{L^2(\tilde{X},\omega_0)} \leq \delta.$$

Since $\Lambda_{\omega_{\varepsilon_j}} F_{\tilde{h}_{\varepsilon_j,t}}$ is a Cauchy sequence it converges strongly in $L^2(\tilde{X},\omega_0)$. Since $\Lambda_{\omega_{\varepsilon_j}} F_{\tilde{h}_{\varepsilon_j,t}} \to \Lambda_{\omega_0} F_{\tilde{h}_t}$ weakly in $L^2_{loc}(\tilde{X},\omega_0)$, it follows that $\Lambda_{\omega_{\varepsilon_j}} F_{\tilde{h}_{\varepsilon_j,t}} \to \Lambda_{\omega_0} F_{\tilde{h}_t}$ strongly in $L^2(\tilde{X},\omega_0)$. Since both $\Lambda_{\omega_{\varepsilon_j}} F_{\tilde{h}_{\varepsilon_j,t}}$ and $\Lambda_{\omega_0} F_{\tilde{h}_t}$ are bounded in L^∞ (see Lemma 20 and Lemma 21) it follows that $\Lambda_{\omega_{\varepsilon_j}} F_{\tilde{h}_{\varepsilon_j,t}} \to \Lambda_{\omega_0} F_{\tilde{h}_t}$ strongly in $L^p(\tilde{X},\omega_0)$ for all p. By Lemma 19 and Lemma 12 we have:

$$HYM_\alpha^{\omega_{\varepsilon_j}}\left(\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_{\varepsilon_j,t})}\right) \longrightarrow HYM_\alpha^{\omega_0}\left(\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_t)}\right).$$

■

3.2 Proof of the Main Theorem

In this section we complete the proof of the main theorem. The result is a direct corollary of the following theorem.

Theorem 15 *Let A_0 be an integrable, unitary connection on a holomorphic vector bundle E , μ_0 the Harder-Narasimhan type of $(E,\bar{\partial}_{A_0})$, and $A \subset [1,\infty)$ be any set containing an accumulation point. Let A_j be a sequence of integrable, unitary connections on E such that:*

- *$(E, \bar{\partial}_{A_j})$ is holomorphically isomorphic to $(E, \bar{\partial}_{A_0})$ for all i;*
- *$HYM_{\alpha,N}(A_j) \longrightarrow HYM_{\alpha,N}(\mu_0)$ for all $\alpha \in A \cup \{2\}$ and all $N > 0$.*

Then there is a Yang-Mills connection A_∞ on a bundle E_∞ defined outside a a closed subset of Hausdorff codimension 4 such that:

(1) *$(E_\infty, \bar{\partial}_{A_\infty})$ is isomorphic to $Gr^{HNS}(E, \bar{\partial}_{A_0})$ as a holomorphic bundle on $X - Z_{\mathrm{an}}$;*

(2) *After passing to a subsequence, $A_j \to A_\infty$ in $L^2_{loc}(X - Z_{an})$;*

(3) *There is an extension of the bundle E_∞ to a reflexive sheaf (still denoted E_∞) such that $E_\infty \approxeq Gr^{HNS}(E, \bar{\partial}_{A_0})^{**}$.*

The proof will be a modification of Donaldson's argument from [DO1] that there is a non-zero holomorphic map $(E, \bar{\partial}_{A_0}) \to (E_\infty, \bar{\partial}_{A_\infty})$ in the case that $(E, \bar{\partial}_{A_0})$ is semi-stable. If the bundles in question are actually stable, we may then apply the elementary fact that a non-zero holomorphic map between stable bundles with the same slope is necessarily an isomorphism. Of course in our case $(E, \bar{\partial}_{A_0})$ is not necessarily semi-stable so the argument must be modified. We first construct such a map on the maximal destabilising subsheaf $S \subset E$ (which is semi-stable). If we assume that S is stable (in other words if we construct the map on the first piece of the HNS filtration) this identifies S with a subsheaf of the limiting bundle E_∞. We then use an inductive argument to identify each of the successive quotients with a direct summand of E_∞. This is relatively straightforward in the case that the HNS filtration is given by subbundles, but in the general case technical complications arise. Therefore, to clearly illustrate our technique, we will first present an exposition of the simpler case where there are no singularities, and then explain the modifications necessary to complete the argument.

3.2.1 The Subbundles Case

We begin with the following proposition.

Proposition 27 *Let E be a holomorphic vector bundle and $A_j = g_j(A_0)$ be a sequence of integrable, unitary connections on E. Let $A \subset [1, \infty)$ be any set containing an accumulation point. Assume that $HYM_{\alpha,N}(A_j) \to HYM_{\alpha,N}(\mu_0)$ for all $N > 0$ and all $\alpha \in A \cup \{2\}$. Let $S \subset (E, \bar{\partial}_{A_0})$ be a holomorphic subbundle. Then there is closed subset Z_{an} of Hausdorff codimension 4, a reflexive sheaf E_∞ which is an Hermitian vector bundle away from Z_{an} and a Yang-Mills connection A_∞ on E_∞ such that:*

(1) After passing to a subsequence $A_j \to A_\infty$ in $L^2_{loc}(X - Z_{an})$;

(2) The Harder-Narasimhan type of $(E_\infty, \bar{\partial}_{A_\infty})$ is the same as that of $(E, \bar{\partial}_{A_0})$;

(3) There is a non-zero holomorphic map $g^S_\infty : S \longrightarrow (E_\infty, \bar{\partial}_{A_\infty})$.

Proof. We first reduce to the case where the Hermitian-Einstein tensors $\Lambda_\omega F_{A_j}$ are uniformly bounded. Write $A_{j,t}$ for the time t solution to the YM flow equations with initial condition A_j. By Lemma 2, $\left|\Lambda_\omega F_{A_{j,t}}\right|^2$ is a sub-solution of the heat equation. Then for each $t > 0$ and each $x \in X$:

$$\left|\Lambda_\omega F_{A_{j,t}}\right|^2 (x) \leq \int_X K_t(x, y) \left|\Lambda_\omega F_{A_{j,t}}\right|^2 (y) dvol_\omega(y).$$

Here $K_t(x,y)$ is the heat kernel on X. By a theorem of Cheng and Li (see [CHLI]) there is a bound:

$$0 < K_t(x,y) \leq C\left(1 + \frac{1}{t^n}\right),$$

and so for any fixed $t_0 > 0$ $\left\|\Lambda_\omega F_{A_{j,t_0}}\right\|_{L^\infty(X,\omega)}$ is uniformly bounded in terms of $\left\|\Lambda_\omega F_{A_j}\right\|_{L^2(X,\omega)}$. Since we assume in particular that $HYM(A_j) \to HYM(\mu_0)$ we know that $\left\|\Lambda_\omega F_{A_j}\right\|_{L^2(X,\omega)}$ is uniformly bounded independently of j, and therefore $\left\|\Lambda_\omega F_{A_{j,t_0}}\right\|_{L^\infty(X,\omega)}$ is uniformly bounded.

For the remainder of the argument we would like to replace A_j with A_{j,t_0}, so that we may assume in the sequel that we have the above bound. In order to do this we must know that the Uhlenbeck limit of the new sequence A_{j,t_0} is the same as that of A_j. We argue as follows:

$$\begin{aligned}\|A_{j,t_o} - A_j\|_{L^2} &\overset{Minkowski}{\leq} \int_0^{t_0} \left\|\frac{\partial A_{j,s}}{\partial s}\right\|_{L^2} \leq \sqrt{t_0}\left(\int_0^{t_0} \left\|d^*_{A_{j,s}} F_{A_{j,s}}\right\|^2_{L^2}\right)^{\frac{1}{2}} \\ &= \sqrt{t_0}\left(\int_0^{t_0} \frac{d}{ds}\left\|F_{A_{j,s}}\right\|^2_{L^2} ds\right)^{\frac{1}{2}} = \sqrt{t_0}\left(YM(A_j) - YM(A_{j,t_0})\right) \longrightarrow 0\end{aligned}$$

because D_j is minimising for the YM functional and YM is non-increasing along the flow. This shows that the two limits are equal, and moreover the proof also shows that $\left\|d^*_{A_{j,s}} F_{D_{j,s}}\right\|_{L^2} \to 0$ for almost all s, so we may arrange that this limit is a Yang-Mills connection. Since we have assumed additionally that $HYM_{\alpha,N}(A_j)$ (and hence $HYM_{\alpha,N}(A_{j,t_0})$) is minimising for $\alpha \in A$, it follows from Propositions 15 (2) and 17 that the HN type of (E_∞, A_∞) is the same as that of (E_0, A_0).

We may therefore assume from here on out that the Hermitian-Einstein tensors $\Lambda_\omega F_{A_j}$ are uniformly bounded independently of j. Note that we have already proven both (1) and (2) above. It remains to construct the non-zero holomorphic map.

Observe that for any holomorphic section σ of a holomorphic vector bundle $V \longrightarrow (X, \omega)$ equipped with an hermitian metric $\langle -, - \rangle$, and whose Chern connection is A, we have that

$$\begin{aligned} \sqrt{-1}\bar{\partial}\partial |\sigma|^2 &= \sqrt{-1}\bar{\partial}\partial \langle \sigma, \sigma \rangle = \sqrt{-1} \left(\langle \partial_A \sigma, \partial_A \sigma \rangle + \langle \sigma, \bar{\partial}_A \partial_A \sigma \rangle \right) \\ &= \sqrt{-1} \left(\langle \partial_A \sigma, \partial_A \sigma \rangle + \langle \sigma, F_A \sigma \rangle \right) \end{aligned}$$

since σ is holomorphic. Applying Λ_ω and using the Kähler identities, we have:

$$\triangle_\partial |\sigma|^2 = \sqrt{-1}\Lambda_\omega \bar{\partial}\partial |\sigma|^2 = -|\partial_A \sigma|^2 + \left\langle \sigma, \sqrt{-1} \left(\Lambda_\omega F_A \right) \sigma \right\rangle .$$

Now let $g_j^S : S \to (E, \bar{\partial}_{A_j})$ be given by the restriction of g_j to S. By definition, this is a holomorphic section of $\mathrm{Hom}(S, E)$, whose Chern connection is $A_0^* \otimes A_j$. Then applying the above formula to g_j^S and writing $k_j^S = (g_j^S)^* ({g_j}^S)$, and h^S and h_j for the metrics corresponding to $A_{0|S}$ and A_j, we have

$$\triangle_\partial \operatorname{Tr} k_j^S + \left| \partial_{A_0^* \otimes A_j} g_j^S \right|^2 = \left\langle g_j^S, \sqrt{-1} \left(\Lambda_\omega F_{h_j} g_j^S - g_j^S \Lambda_\omega F_{h^S} \right) \right\rangle ,$$

and so

$$\triangle_\partial (\operatorname{Tr} k_j^S) \leq (\operatorname{Tr} k_j^S) \left(\left| \Lambda_\omega F_{h_j} \right| + \left| \Lambda_\omega F_{h^S} \right| \right) .$$

Now we use the bound on $\left| \Lambda_\omega F_{h_j} \right|$. Let $C_1 = \sup_j \left\| \Lambda_\omega F_{h_j} \right\|_{L^\infty(X,\omega)}$ and $C_2 = \left\| \Lambda_\omega F_{h^S} \right\|_{L^\infty(X)}$. Multiplying both sides of the above inequality by Trk_j^S and integrating by parts shows:

$$\int_X \left| \nabla \operatorname{Tr} k_j^S \right|^2 dvol_\omega \leq (C_1 + C_2) \int_X \left| \operatorname{Tr} k_j^S \right|^2 dvol_\omega .$$

By the Sobolev imbedding $L_1^2 \hookrightarrow L^{\frac{2n}{n-1}}$ the previous inequality gives a bound

$$\left\| \operatorname{Tr} k_j^S \right\|_{L^{\frac{2n}{n-1}}(X,\omega)} \leq C \left\| \operatorname{Tr} k_j^S \right\|_{L^2(X,\omega)}$$

where C depends only on C_1,C_2 and the Sobolev constant of (X, ω). A standard Moser iteration gives a bound: $\left\|\operatorname{Tr} k_j^S\right\|_{L^\infty(X,\omega)} \leq C \left\|\operatorname{Tr} k_j^S\right\|_{L^2(X,\omega)}$.

At this point we may repeat Donaldson's argument (appropriately modified for higher dimensions). For the reader's convenience we reproduce it here. By definition $Tr(k_j^S) = \left|g_j^S\right|^2$. Since non-zero constants act trivially on $\mathcal{A}^{1,1}$ we may normalise the g_j^S so that $\left\|g_j^S\right\|_{L^4(X)} = \left\|Tr(k_j^S)\right\|_{L^2(X)} = 1$. The above bound implies that there is a subsequence of the g_j^S that converges to a limiting gauge transformation g_∞^S weakly in every L_2^p for example. Since Z_{an} has Hausdorff codimension 4, we may of course find a covering of Z_{an} by balls $\{B_i^r\}_i$ of radius r such that: $C\left(\sum_i Vol(B_i^r)\right) < 1/2$. If we write $K_r = X - \cup_i B_i \cup \mathrm{Sing}(E_\infty)$, then our L^∞ bound implies that: $\left\|g_j^S\right\|_{L^4(K_r)} \geq 1/2$ for all j. This implies that g_∞^S is non-zero. We now show g_∞^S is holomorphic.

If we denote by $\bar{\partial}_{A_0 \otimes A_\infty}$ the $(0,1)$ part of the connection on $E^* \otimes E_\infty = Hom(E, E_\infty)$ induced by the connections A_0 and A_∞. We will identify E and E_∞ on K_r. Then by definition we have:

$$\bar{\partial}_{A_0^* \otimes A_\infty} g_j^S = \left(g_j^S A_0 - A_\infty g_j^S\right) = (g_j^S A_0 (g_j^S)^{-1} - A_\infty) g_j^S = (A_j - A_\infty) g_j^S.$$

Since $A_0 \to A_\infty$ in $L^2(K_r)$ this implies $\bar{\partial}_{A_0 \otimes A_\infty} g_\infty^S = 0$, in other words g_∞^S is holomorphic on K_r. Since this argument works for any choice of r, and the K_r give an exhaustion of $X - Z_{\mathrm{an}} \cup \mathrm{Sing}(E_\infty)$, g_∞^S is holomorphic on $X - Z_{\mathrm{an}} \cup \mathrm{Sing}(E_\infty)$. By a version of Hartogs theorem (see [SHI] Lemma 3) there is an extension of g_∞^S to $X - \mathrm{Sing}(E_\infty)$. Finally, by normality of these sheaves (both are reflexive) there is an extension to a non-zero map $g_\infty^S : S \to E_\infty$. ■

We are now ready to perform the induction, and therefore prove the main theorem in the case when the HNS filtration is given by sub bundles. We first assume the quotients $Q_i = E_i/E_{i-1}$ in the Harder-Narasimhan filtration $0 = E_0 \subset E_1 \subset ... \subset E_l = (E, \bar{\partial}_{A_0})$ are stable (so the HN and HNS filtrations are the same). From Proposition 1 E_∞ has a holomorphic splitting $E_\infty = \oplus_{i=1}^{l'} Q_{\infty,i}$. By Theorem 14 the HN types of E and E_∞ are the same, so $l = l'$ and $\mu(E_1) = \mu(Q_{\infty,1}) > \mu(Q_{\infty,i})$ for $i = 2, ..., l$. By the above proposition there is a non-zero holomorphic map $g_\infty : E_1 \to E_\infty$. Since we are assuming E_1 is stable, and the $Q_{\infty,i}$ $(i > 1)$ have slope strictly smaller than E_1, the induced map onto these summands is 0 and hence $g_\infty : E_1 \to Q_{\infty,1}$. Again by stability of E_1 and $Q_{\infty,1}$ and the fact that E_1 and $Q_{\infty,1}$ have the same rank and degree, this map is an isomorphism. This is the first step in the induction.

The inductive hypothesis will be that the connections A_j restricted to E_{i-1} converge to connections on the bundle $Gr(E_{i-1})$, in other words $Gr(E_{i-1}) \subset E_\infty$. Let $E_{\infty,i} = \oplus_{j \leq i} Q_{\infty,j}$ and set: $E_\infty = Gr(E_{i-1}) \oplus R$, and consider the short exact sequence of bundles: $0 \to E_{i-1} \to E_i \to Q_i \to 0$. Since $Gr(E_i) = Gr(E_{i-1}) \oplus Q_i$, to complete the induction we need only show that Q_i is a direct summand of R. The sequence of connections on E_i^* induced by A_j satisfy the hypotheses of the proposition, so we may apply this result to the dual exact sequence: $0 \to Q_i^* \to E_i^* \to E_{i-1}^* \to 0$, and therefore obtain a holomorphic map $Q_i^* \to (E_{\infty,i})^*$. Because Q_i^* is the maximal destabilising subsheaf of $(E_{\infty,i})^*$ this implies that Q_i^* is isomorphic to a summand of R^*. This completes the proof under the assumption that the quotients are stable.

To extend this to the general case, it suffices to consider the case that the original bundle $(E, \bar{\partial}_{A_0})$ is semi-stable. In other words the filtration is a Seshadri filtration of E. Then as in the above argument we may conclude that E_1 is isomorphic to a factor of E_∞ we also again obtain a non-zero holomorphic map $g_\infty : Q_i^* \to (E_{\infty,i})^*$. However, the Seshadri quotients all have the same slope, so we do not know via slope considerations that Q_i^* maps into R^*. On the other hand we know that the weakly holomorphic projections converge. If $\pi_j^{(i-1)}$ denotes the sequence of projections to $g_j(E_{i-1})$ and $\pi_\infty^{(i-1)}$ the projection onto $E_{\infty,i-1}$, then $\pi_j^{(i-1)} \to \pi_\infty^{(i-1)}$ by the proof of Lemma 4.5 of [DW1]. If we denote by $\check{\pi}_j^{(i-1)}$ the dual projection, then for each j, the image of Q_i^* is in the kernel of $\check{\pi}_j^{(i-1)}$. In other words the image $g_\infty(Q_i^*)$ lies in the kernel of $\check{\pi}_j^{(i-1)}$. Therefore since we have convergence, the image of $g_\infty(Q_i^*)$ lies in the kernel of $\check{\pi}_\infty^{(i-1)}$ which is in R^*. Therefore Q_i^* is isomorphic with a factor of R^* and this completes the proof.

3.2.2 The General Case

In general the HNS filtration is not given by subbundles. The argument we have given in Proposition 27 for the construction of the holomorphic map $S \to E_\infty$ remains valid if S is an arbitrary torsion free subsheaf since the connections in question are all defined a priori on the ambient bundle E, and since the second fundamental form β of S drops out of the estimates, there is no problem obtaining a uniform bound on the Hermitian-Einstein tensors. On the other hand, when we try to run the inductive argument, the restrictions of the connections A_j to the pieces E_i

of the HNS filtration only make sense on the locally free part of these subsheaves. This prevents us from applying the argument of Proposition 27 in the inductive step because to do so requires global L^∞ bounds on the appropriate Hermitian-Einstein tensors, which we do not have, since the restrictions of the A_j do not extend over the singular set Z_{alg}.

The strategy for proving the main theorem in the general case mirrors our method in section 4. Roughly speaking we proceed as follows. Let $A_j = g_j(A_0)$ be a sequence of connections. First we pass to an arbitrary resolution $\pi : \tilde{X} \to X$ of singularities of the HNS filtration. Then we construct an isomorphism from the associated graded object of the filtration for the pullback bundle $\pi^* E$ (away from the exceptional set $\mathbf{E}$) to the Uhlenbeck limit of the sequence $\pi^* A_j$ on the Kähler manifold $(\tilde{X} - \mathbf{E}, \omega_0)$ where $\omega_0 = \pi^*\omega$. Then we will use the fact that these bundles extend as reflexive sheaves over the exceptional divisor to the double dual of the associated graded object of E and the Uhlenbeck limit of A_j respectively, and hence by normality of these sheaves, the isomorphism extends as well.

The outline of the proof given above has to be modified somewhat for technical reasons which we will now explain. Just as for the case of subbundles, by first running the YM flow for finite time we may assume there is a uniform bound $\left\|\Lambda_\omega F_{A_j}\right\|_{L^\infty(X)}$ or equivalently on $\left\|\Lambda_{\omega_0} F_{\tilde{A}_j}\right\|_{L^\infty(\tilde{X}-\mathbf{E})}$ where $\tilde{A}_j = \pi^* A_j$. As usual we will denote by A_∞ the Uhlenbeck limit of A_j on (X, ω) and we have $A_j \to A_\infty$ in $L^p_{1,loc}(X - Z_{\text{an}})$ for $p > 2n$. The proof of the proposition proves all but (3) of Theorem 15. Let $E_i \subset E$ be a factor of the HNS filtration and $A_j^{(i)} = \pi_j^{(i)} A_j$ be the connections on $g_j(E_i)$ induced from A_j, and $A_\infty^{(i)} = \pi_\infty^{(i)} A_\infty$. By Lemma 17 it follows that $A_j^{(i)} \to A_\infty^{(i)}$ weakly in $L^p_{1,loc}(X - Z_{\rm an} \cup Z_{\rm alg})$.

If $\pi : \tilde{X} \to X$ is the aforementioned resolution of singularities then the filtration of $\pi^* E = \tilde{E}$ is given by subbundles $\tilde{E}_i \subset \tilde{E}$, isomorphic to E_i away from the exceptional divisor $\mathbf{E}$. Write $\tilde{g}_j = g_j \circ \pi$ and let $\tilde{A}_j^{(i)}$ be the connection induced by $\tilde{A}_j = \pi^* A_j$ on $\tilde{g}_j(\tilde{E}_i)$. We will write $\tilde{\pi}_j$ for the projection to $\tilde{g}_j(\tilde{E}_i)$ and $\tilde{\beta}_j$ for the second fundamental forms for the connections $\tilde{A}_j$ with respect to the subbundles $\tilde{E}_i$; in other words these are sections of the bundle $\Omega^{0,1}\left(\tilde{X}, Hom(\tilde{Q}_i, \tilde{E}_i)\right)$ for an auxiliary bundle $\tilde{Q}_i$. Then this sequence of connections satisfies the following:

(1) There is a closed subset $\tilde{Z}_{\rm an} \subset \tilde{X} - \mathbf{E}$ of Hausdorff codimension 4 and a Yang-Mills connection $\tilde{A}_\infty^{(i)}$ defined on a bundle $\tilde{E}_{\infty,i} \to \tilde{X} - \mathbf{E}$, such that $\tilde{A}_j^{(i)} \to \tilde{A}_\infty^{(i)}$ weakly in $L^p_{1,loc}\left(\tilde{X} - (\tilde{Z}_{an} \cup \mathbf{E})\right)$.

(2) We have the standard formula for the curvature:

$$\sqrt{-1}\Lambda_{\omega_0} F_{\tilde{A}_j^{(i)}} = \sqrt{-1}\Lambda_{\omega_0}\left(\tilde{\pi}_j F_{\tilde{A}_j} \tilde{\pi}_j\right) + \sqrt{-1}\Lambda_{\omega_0}\left(\tilde{\beta}_j \wedge \tilde{\beta}_j^*\right).$$

Also:

- The $\tilde{\beta}_j$ are locally bounded on $\tilde{X} - (\tilde{Z}_{an} \cup \mathbf{E})$ uniformly in j (*Lemma* 7)

- The $\tilde{\beta}_j \to 0$ in $L^2(\omega_0)$. In particular, they are uniformly bounded in $L^2(\omega_0)$ (see the proof of [DW1] Lemma 4.5).

Note that the term

$$\sqrt{-1}\Lambda_{\omega_0}\left(\tilde{\pi}_j F_{\tilde{A}_j}\tilde{\pi}_j\right)$$

is bounded in $L^\infty(\tilde{X}-\mathbf{E},\omega_0)$ since $\tilde{A}_j=\pi^*A_j$. The key point here is that term

$$\sqrt{-1}\Lambda_{\omega_0}\left(\tilde{\beta}_j\wedge\tilde{\beta}_j^*\right)$$

is not bounded in $L^\infty(\tilde{X}-\mathbf{E},\omega_0)$ since it may be written as

$$\sqrt{-1}\frac{\left(\tilde{\beta}_j\wedge\tilde{\beta}_j^*\right)\wedge\omega_0^{n-1}}{\omega_0^n}$$

which blows up near $\mathbf{E}$. This is a problem because in order to carry out the induction in the preceeding sub-section we had to consider exact sequences of the form:

$$0\longrightarrow\tilde{Q}_i^*\longrightarrow\tilde{E}_i^*\longrightarrow\tilde{E}_{i-1}^*\longrightarrow 0$$

(here $\tilde{Q}_i=\tilde{E}_i/\tilde{E}_{i-1}$) and apply Proposition 27 to construct a non-zero holomorphic map $\tilde{Q}_i^*\to\tilde{E}_{\infty,i}^*$. This involved knowing that there was a uniform L^∞ bound on the Hermitian-Einstein tensors of the induced connections $(\tilde{A}_j^{(i)})^*$ and $(\tilde{A}_{j,Q}^{(i)})^*$ on $\tilde{E}_i^*$ and $\tilde{Q}_i^*$. Since this is not the case we cannot apply this argument directly. On the other hand we do know that for all positive times $t>0$, the degenerate Yang-Mills flow of Section 6 gives connections $\tilde{A}_{j,t}^{(i)}$ such that $\Lambda_{\omega_0}F_{\tilde{A}_{j,t}^{(i)}}$ is uniformly bounded (see Lemma 21). For each t the deformed sequence of connections has an Uhlenbeck limit $\tilde{A}_{\infty,t}^{(i)}$ on a bundle $\tilde{E}_{\infty,i}^t$ which a priori depends on t.

There are now two points to address. In parallel to Proposition 27 we will show that after resolving the singularities of the maximal destabilising subsheaf S to a bundle $\tilde{S}$ there is a non-zero holomorphic map $\tilde{S}\to\tilde{E}_\infty^t$ (where $\tilde{E}_\infty^t$ is an Uhlenbeck limit of $\tilde{A}_{j,t}$) away from $\mathbf{E}$. This is not automatic from the proof of Proposition 27 because the connections $\tilde{A}_{j,t}$ do not extend smoothly across $\mathbf{E}$, so the integration by parts involved in the proof is not valid. We will instead derive this map as a limit of the maps produced from the corresponding argument for the family of Kähler manifolds $(\tilde{X}, \omega_\varepsilon)$. Secondly we need to know that the Uhlenbeck limits $(\tilde{E}^t_\infty, \tilde{A}_{\infty,t})$ are independent of t and are all equal to $(\tilde{E}_\infty, \tilde{A}_\infty)$. Again, this does not follow from our previous argument since, as we have noted, the second fundamental forms of the restricted connections are only bounded in L^2 and therefore the curvatures are only bounded in L^1. In particular we do not have that $\tilde{A}^{(i)}_j$ is minimising for the functional YM. Establishing these two facts will complete the proof of the main theorem, since then we may use induction just as for the case when the HNS filtration is given by subbundles.

We begin with the first point.

Proposition 28 *Let $\tilde{E} \to \tilde{X}$ be a vector bundle with an hermitian metric $\tilde{h}$. Let $\tilde{A}_j = \tilde{g}_j(\tilde{A}_0)$ be a sequence of unitary connections on $\tilde{E}$, and assume $\Lambda_{\omega_0} F_{\tilde{A}_j}$ is bounded uniformly in j in $L^1(\tilde{X}, \omega_0)$. Let $\tilde{A}_{j,t}$ be the solution of the degenerate YM flow at time t with initial condition $\tilde{A}_j$, and suppose that this sequence has an Uhlenbeck limit $(\tilde{E}^t_\infty, \tilde{A}_{\infty,t})$. Finally let $\tilde{S} \subset \tilde{E}$ be a subbundle of $(\tilde{E}, \tilde{A}_0)$. Then there is a non-zero holomorphic map $\tilde{g}_\infty : \tilde{S} \to \tilde{E}^t_\infty$ on $\tilde{X} - \mathbf{E}$. Furthermore, let $(E^t_\infty, A_{\infty,t})$ be the extension of $(\tilde{E}^t_\infty, \tilde{A}_{\infty,t})$ over $\mathbf{E}$ to X, assume $\tilde{S}$ extends to a reflexive sheaf S on X. Then $\tilde{g}_\infty$ induces a non-zero holomorphic map $g_\infty : S \to E^t_\infty$.*

Proof. Let ω_ε be the standard family of Kähler metrics on $\tilde{X}$ and fix $t > 0$. Let

$\varepsilon_i \to 0$ be a sequence as in section 5, i.e. if $\tilde{A}^{\varepsilon_i}_{j,t}$ is the time t YM flow on $(\tilde{X}, \omega_{\varepsilon_i})$, then $\tilde{A}^{\varepsilon_i}_{j,t} \to \tilde{A}_{j,t}$ continuously on compact subsets of $\tilde{X} - \mathbf{E}$. Choose a family of metrics $\tilde{h}^{\tilde{S}}_{\varepsilon_i}$ on $\tilde{S}$ converging uniformly on compact subsets of $\tilde{X} - \mathbf{E}$ to a metric $\tilde{h}^{\tilde{S}}_0$ defined away from $\mathbf{E}$, and such that $\sup \left|\Lambda_{\omega_{\varepsilon i}} F_{\tilde{h}^{\tilde{S}}_{\varepsilon_i}}\right|$ is uniformly bounded as $\varepsilon_i \to 0$ (take for example the time 1 HYM flow of $\tilde{h}$ with respect ω_ε). For each j and each $\varepsilon_i > 0$, we have a non-zero holomorphic map $\tilde{g}^{\tilde{S}}_{\varepsilon_i,j} : \tilde{S} \to (\tilde{E}, \bar{\partial}_{\tilde{A}^{\varepsilon_i}_{j,t}})$. Just as in Section 7.1, we set $k^{\tilde{S}}_{\varepsilon_i,j} = \left(\tilde{g}^{\tilde{S}}_{\varepsilon_i,j}\right)^* \tilde{g}^{\tilde{S}}_{\varepsilon_i,j}$. As in Proposition 27 we have the inequality:

$$\Delta_{(\partial,\omega_\varepsilon)}(\operatorname{Tr} \tilde{k}^{\tilde{S}}_{\varepsilon_i,j}) \le (\operatorname{Tr} \tilde{k}^{\tilde{S}}_{\varepsilon_i,j}) \left(\left|\Lambda_{\omega_{\varepsilon_i}} F_{\tilde{A}^{\varepsilon_i}_{j,t}}\right| + \left|\Lambda_{\omega_{\varepsilon i}} F_{\tilde{h}^{\tilde{S}}_{\varepsilon_i}}\right| \right).$$

Both factors on the right are uniformly bounded as $\varepsilon_i \to 0$ by assumption. It follows that we have the inequality: $\left\|\operatorname{Tr} \tilde{k}^{\tilde{S}}_{\varepsilon_i,j}\right\|_{L^\infty(\tilde{X})} \le C \left\|\operatorname{Tr} \tilde{k}^{\tilde{S}}_{\varepsilon_i,j}\right\|_{L^2(\tilde{X},\omega_\varepsilon)}$, where the constant C depends only on these uniform bounds and the Sobolev constant of $(\tilde{X}, \omega_{\varepsilon_i})$ is also uniformly bounded away from zero by [BS] Lemma 3. As in the proof of Proposition 27 we rescale $\tilde{g}^{\tilde{S}}_{\varepsilon_i,j}$ so that $\left\|\tilde{g}^{\tilde{S}}_{\varepsilon_i,j}\right\|_{L^4(\tilde{X},\omega_\varepsilon)} = 1$. A diagonalisation argument for an exhaustion of $\tilde{X} - \mathbf{E}$ together with the sup bound gives a sequence of non-zero holomorphic maps $\tilde{g}^{\tilde{S}}_j : \tilde{S} \to (\tilde{E}, \bar{\partial}_{\tilde{A}_{j,t}})$ defined on $\tilde{X} - \mathbf{E}$ with $\tilde{g}^{\tilde{S}}_{\varepsilon_i,j} \to \tilde{g}^{\tilde{S}}_j$ uniformly on compact subsets as $\varepsilon_i \to 0$ such that: $\left\|\tilde{g}^{\tilde{S}}_j\right\|_{L^\infty} \le C$, and $\left\|\tilde{g}^{\tilde{S}}_j\right\|_{L^4(\omega_0)} = 1$. Repeating the proof of Proposition 27 yields a nonzero limit $\tilde{g}^{\tilde{S}}_\infty : \tilde{S} \to (\tilde{E}^t_\infty, \tilde{A}_{\infty,t})$. The last statement follows from the normality of the sheaves in question. ■

Secondly we have:

Proposition 29 *Let $\tilde{E} \to \tilde{X}$ be a Hermitian vector bundle with a unitary integrable connection $\tilde{A}_0$. We assume that the holomorphic bundle $(\tilde{E}, \bar{\partial}_{A_0})$ restricted to $\tilde{X} - \mathbf{E} = X - Z_{\text{alg}}$ extends to a holomorphic bundle $(E, \bar{\partial}_E)$ on X with Harder-*

Narasimhan type $\mu = (\mu_1, ..., \mu_R)$. *Let* $\tilde{A}_j = \tilde{g}_j(\tilde{A}_0)$ *be a sequence of unitary connections on* $\tilde{E}$, *and assume there is a subset* $\tilde{Z}_{\mathrm{an}} \subset \tilde{X} - \mathbf{E}$ *of Hausdorff codimension at least* 2, *and a* YM *connection* $\tilde{A}_\infty$ *on a bundle* $\tilde{E}_\infty \to \tilde{X} - \mathbf{E}$ *such that* $\tilde{A}_j \to \tilde{A}_\infty$ *weakly in* $L^p_{1,loc}$ *(where* $p > 2n$*) on compact subsets of* $\tilde{X} - \left(\tilde{Z}_{\mathrm{an}} \cup \mathbf{E}\right)$. *We assume that the constant eigenvalues of* $\sqrt{-1}\Lambda_{\omega_0} F_{\tilde{A}_\infty}$ *are given by the vector* μ. *Finally assume* $\Lambda_{\omega_0} F_{\tilde{A}_j} \to \Lambda_{\omega_0} F_{\tilde{A}_\infty}$ *in* $L^1(\omega_0)$. *Then there is a subsequence such that for almost all* $t > 0$ $\tilde{A}_{j,t} \to \tilde{A}_\infty$ *in* $L^p_{1,loc}$ *away from* $\tilde{Z}_{\mathrm{an}} \cup \mathbf{E}$ *where* $\tilde{A}_{j,t}$ *is the time* t *degenerate* YM *flow with initial condition* $\tilde{A}_j$.

This will follow from a sequence of lemmas.

Lemma 22 *For any* $t > 0$, $\left\|\Lambda_{\omega_0} F_{\tilde{A}_{j,t}}\right\|_{L^\infty(\tilde{X}-\mathbf{E})}$ *is uniformly bounded in* j. *Moreover, for almost all* $t > 0$, $\lim_{j\to\infty} HYM^{\omega_0}\left(\tilde{A}_{j,t}\right) = HYM(\mu)$.

Proof. The first statement follows from *Lemma* 21. By assumption, we have $\Lambda_{\omega_0} F_{\tilde{A}_j} \to \Lambda_{\omega_0} F_{\tilde{A}_\infty}$ in L^1, and $\Lambda_{\omega_0} F_{\tilde{A}_\infty}$ has constant eigenvalues $\mu_1, ..., \mu_R$. Set $M^2 = \sum_{i=1}^R \mu_i^2 = \frac{HYM(\mu)}{2\pi}$. Also let $\mu_{1,\varepsilon}, ..., \mu_{R,\varepsilon}$ be the HN type of $(E, \bar{\partial}_{\tilde{A}_0})$ with respect to ω_ε, and set $\tilde{M}_\varepsilon^2 = \sum_{i=1}^R \mu_{i,\varepsilon}^2$. By *Corollary* 6 we know:

$$\tilde{M}_\varepsilon \leq \frac{1}{2\pi} \int_{\tilde{X}} \left|\Lambda_{\omega_\varepsilon} F_{\tilde{A}^\varepsilon_{j,t}}\right| dvol_{\omega_\varepsilon}.$$

By Proposition 26 , for almost all t, we can find a sequence $\varepsilon_i = \varepsilon_i(t) \to 0$ such that $\Lambda_{\omega_{\varepsilon_i}} F_{\tilde{A}^{\varepsilon_i}_{j,t}} \to \Lambda_{\omega_0} F_{\tilde{A}_{j,t}}$ in any $L^p(\omega_0)$. Let $\varepsilon_i \to 0$ and using the convergence of the HN type:

$$M \leq \frac{1}{2\pi} \int_{\tilde{X}} \left|\Lambda_{\omega_0} F_{\tilde{A}_{j,t}}\right| dvol_{\omega_0}$$

for all j and almost all $t \geq 0$. We also have:

$$\begin{aligned} \left|\Lambda_{\omega_\varepsilon} F_{\tilde{A}^\varepsilon_{j,t}}\right|(x) &\leq \int_{\tilde{X}} K^\varepsilon_t(x,y)\left|\Lambda_{\omega_\varepsilon} F_{\tilde{A}_j}\right|(y) dvol_{\omega_\varepsilon}(y) \\ &= M + \int_{\tilde{X}} K^\varepsilon_t(x,y)\left(\left|\Lambda_{\omega_\varepsilon} F_{\tilde{A}_j}\right| - M\right) dvol_{\omega_\varepsilon} \end{aligned}$$

where $K^\varepsilon_t(x,y)$ is the heat kernel on $(\tilde{X}, \omega_\varepsilon)$ (since $K^\varepsilon_t(x,y)$ has integral equal to 1). Since we have the bound: $K^\varepsilon_t(x,y) \leq C(1 + 1/t^n)$, there is a constant $C(t)$ independent of ε such that:

$$\left|\Lambda_{\omega_\varepsilon} F_{\tilde{A}^\varepsilon_{j,t}}\right|(x) \leq M + C\left\|\left|\Lambda_{\omega_\varepsilon} F_{\tilde{A}_j}\right| - M\right\|_{L^1(\tilde{X},\omega_\varepsilon)}.$$

Then just as above we have:

$$\left|\Lambda_{\omega_0} F_{\tilde{A}_{j,t}}\right|(x) \leq M + C\left\|\left|\Lambda_{\omega_0} F_{\tilde{A}_j}\right| - M\right\|_{L^1(\tilde{X},\omega_0)}$$

for almost all $x \in \tilde{X} - \mathbf{E}$ and almost all $t > 0$. Since $\left|\Lambda_{\omega_0} F_{\tilde{A}_j}\right| \to \left|\Lambda_{\omega_0} F_{\tilde{A}_\infty}\right| = M$ in L^1, we have

$$\lim_{j\to\infty} \sup\left|\Lambda_{\omega_0} F_{\tilde{A}_{j,t}}\right|(x) \leq M$$

for almost all $x \in \tilde{X} - \mathbf{E}$ and almost all $t > 0$. On the other hand since $\Lambda_{\omega_0} F_{\tilde{A}_{j,t}}$ is uniformly bounded in j, we can use the lower bound for

$$\frac{1}{2\pi}\int_{\tilde{X}}\left|\Lambda_{\omega_0} F_{\tilde{A}_{j,t}}\right| dvol_{\omega_0}$$

and Fatou's Lemma to show:

$$M \leq \int_{\tilde{X}} \lim_{j\to\infty} \sup\left|\Lambda_{\omega_0} F_{\tilde{A}_{j,t}}\right| dvol_{\omega_0}.$$

It follows that $\lim_{j\to\infty} \sup\left|\Lambda_{\omega_0} F_{\tilde{A}_{j,t}}\right|^2 = M^2$ almost everywhere. By Fatou we therefore have:

$$
\begin{aligned}
HYM(\mu) &\leq \lim_{j\to\infty} \inf HYM^{\omega_0}(\tilde{A}_{j,t}) \leq \lim_{j\to\infty} \sup HYM^{\omega_0}(\tilde{A}_{j,t}) \\
&= \lim_{j\to\infty} \sup \int_{\tilde{X}} \left|\Lambda_{\omega_0} F_{\tilde{A}_{j,t}}\right| dvol_{\omega_0} \leq \int_{\tilde{X}} \lim_{j\to\infty} \sup \left|\Lambda_{\omega_0} F_{\tilde{A}_{j,t}}\right| dvol_{\omega_0} \\
&= 2\pi M^2 = HYM(\mu).
\end{aligned}
$$

■

Lemma 23 *For almost all* $t_0 > 0$, $\left\|\tilde{A}_{j,t} - \tilde{A}_{j,t_0}\right\|_{L^2(\tilde{X},\omega_0)} \to 0$, *uniformly for almost all* $t \geq t_0$.

Proof. As before let $\varepsilon_i \to 0$ be a sequence such that $\tilde{A}^{\varepsilon_i}_{j,t} \to \tilde{A}_{j,t}$ and $\tilde{A}^{\varepsilon_i}_{j,t_0} \to \tilde{A}_{j,t_0}$ in C^0_{loc}. Then we again have:

$$
\begin{aligned}
\left\|\tilde{A}^{\varepsilon_i}_{j,t} - \tilde{A}^{\varepsilon_i}_{j,t_0}\right\|_{L^2} &\overset{Minkowski}{\leq} \int_{t_0}^{t} \left\|\frac{\partial \tilde{A}^{\varepsilon_i}_{j,s}}{\partial s}\right\|_{L^2} \\
&\leq \sqrt{t}\left(\int_{t_0}^{t} \left\|d^*_{A_{j,s}} F_{\tilde{A}^{\varepsilon_i}_{j,s}}\right\|^2_{L^2}\right)^{\frac{1}{2}} = \sqrt{t}\left(\int_{t_0}^{t} \frac{d}{ds}\left\|F_{\tilde{A}^{\varepsilon_i}_{j,s}}\right\|^2_{L^2} ds\right)^{\frac{1}{2}} \\
&= \sqrt{t}\left(YM(\tilde{A}^{\varepsilon_i}_{j,t_0}) - YM(\tilde{A}^{\varepsilon_i}_{j,t})\right) = \sqrt{t}\left(HYM(\tilde{A}^{\varepsilon_i}_{j,t_0}) - HYM(\tilde{A}^{\varepsilon_i}_{j,t})\right) \\
&\leq \sqrt{t}\left(HYM(\tilde{A}^{\varepsilon_i}_{j,t_0}) - HYM(\mu_{\varepsilon_i})\right).
\end{aligned}
$$

Using Proposition 26 and Proposition 21 this yields:

$$
\left\|\tilde{A}_{j,t} - \tilde{A}_{j,t_0}\right\|_{L^2(\tilde{X},\omega_0)} \leq \sqrt{t}\left(HYM(\tilde{A}_{j,t_0}) - HYM(\mu)\right)
$$

The result follows by applying the previous lemma. ■

Lemma 24 *There is a* YM *connection* $\tilde{A}_{\infty,*}$ *on a bundle* $\tilde{E}_{\infty,*} \to \tilde{X} - \mathbf{E}$ *with the following property: for almost all* $t > 0$ *there is a subsequence and a closed subset* $\tilde{Z}^t_{\text{an}} \subset \tilde{X} - \mathbf{E}$, *possibly depending on* t *and the choice of subsequence, such that* $\tilde{A}_{j,t} \to \tilde{A}_{\infty,*}$ *in* $L^p_{1,loc}$ $(p > 2n)$ *away from* $\tilde{Z}^t_{an} \cup \mathbf{E}$.

Proof. As in Proposition 25 and using Proposition 26 we have:

$$HYM(\tilde{A}_{j,t_1}) - HYM(\tilde{A}_{j,t_2}) \geq 2\int_{t_1}^{t_2} \left\| d^*_{A_{j,s}} F_{\tilde{A}_{j,s}} \right\|_{L^2(\omega_0)} ds$$

for almost all $t_2 \geq t_1 > 0$. It follows from Lemma 22 and Fatou's lemma that:

$$\liminf_{j\to\infty} \left\| d^*_{A_{j,s}} F_{\tilde{A}_{j,s}} \right\|^2_{L^2(\omega_0)} = 0,$$

for almost all t. Choose a sequence t_k of such t with $t_k \to 0$. For each k there is a subsequence $j_m(t_k)$, a YM connection $\tilde{A}_{\infty,t_k}$, and a finite set of points $\tilde{Z}^{t_k}_{\mathrm{an}}$, depending on the choice of subsequence such that $\tilde{A}_{j_m,t_k} \to \tilde{A}_{\infty,t_k}$ in $L^p_{1,loc}$ away from $\tilde{Z}^{t_k}_{\mathrm{an}}$. By a diagonalisation argument, assume without loss of generality that the original sequence satisfies $\tilde{A}_{j,t_k} \to \tilde{A}_{\infty,t_k}$ for all t_k. On the other hand, by Lemma 23, $\tilde{A}_{\infty,t_k} = \tilde{A}_{\infty,*}$ is independent of t_k. For any t, there is a k with $t \geq t_k$, so Lemma 23 also implies $\tilde{A}_{j,t} \to \tilde{A}_{\infty,*}$ in L^2_{loc} for almost all $t > 0$. Hence, any Uhlenbeck limit of $\tilde{A}_{j,t}$ coincides with $\tilde{A}_{\infty,*}$. ■

The proof of Proposition 29 will be complete if we can show $\tilde{A}_\infty = \tilde{A}_{\infty,*}$. First we will need:

Lemma 25 *$\Lambda_{\omega_\varepsilon} F_{\tilde{A}_{j,t}}$ is bounded on compact subsets of $\tilde{X} - \mathbf{E}$, uniformly for all j, all $t \geq 0$, and all $\varepsilon > 0$.*

Proof. By our assumptions it follows that $\Lambda_{\omega_\varepsilon} F_{\tilde{A}_j}$ are uniformly bounded in L^1 and that they are uniformly locally bounded. The result now follows just as in the proof of Lemma 19(2). ■

Corollary 9 *$\left|\tilde{A}_{j,t}-\tilde{A}_\infty\right|$ is bounded in any $L^p_{1,loc}$ away from $\tilde{Z}_{\mathrm{an}}\cup\mathbf{E}$, uniformly for all j and all $0\le t\le t_0$. In particular, the singular set $\tilde{Z}^t_{\mathrm{an}}$ is independent of t and is equal to $\tilde{Z}_{\mathrm{an}}$.*

Proof. Since $\tilde{A}_j\to\tilde{A}_\infty$ in $L^p_{1,loc}$, it suffices to prove that $\left|\tilde{A}_{j,t}-\tilde{A}_j\right|$ is bounded in C^1_{loc}. Choose a sequence ε_i such that $\tilde{A}^{\varepsilon_i}_{j,t}\to\tilde{A}_{j,t}$ in C^1_{loc}. It suffices to prove $\left|\tilde{A}^{\varepsilon_i}_{j,t}-\tilde{A}_j\right|$ is bounded in C^1_{loc} uniformly in ε_i. Write $\tilde{A}^{\varepsilon_i}_{j,t}=\tilde{g}^{\varepsilon_i}_{j,t}(\tilde{A}_j)$ and $\tilde{k}^{\varepsilon_i}_{j,t}=(\tilde{g}^{\varepsilon_i}_{j,t})^*\tilde{g}^{\varepsilon_i}_{j,t}$. It suffices to show that $(\tilde{k}^{\varepsilon_i}_{j,t})^{-1}$ is bounded and $\tilde{k}^{\varepsilon_i}_{j,t}$ has bounded derivatives, locally with respect to a trivialisation of $\tilde{E}$. The local boundedness of $\tilde{k}^{\varepsilon_i}_{j,t}$ and $(\tilde{k}^{\varepsilon_i}_{j,t})^{-1}$ follows from the flow equations and the preceeding lemma. The boundedness of the derivatives follows from [BS] Proposition 1 applied to the equation

$$\triangle_{(\bar{\partial}_{A_0},\omega_\varepsilon)}\tilde{k}_{\varepsilon,t}+\sqrt{-1}\Lambda_{\omega_\varepsilon}\left(\bar{\partial}_{A_0}\tilde{k}_{\varepsilon,t}\right)\tilde{k}^{-1}_{\varepsilon,t}\left(\partial_{A_0}\tilde{k}_{\varepsilon,t}\right)=\tilde{k}_{\varepsilon,t}f_{\varepsilon,t}$$

. ■

Now we can complete the proof of Proposition 29. Fix a smooth test form $\phi\in\Omega^1(\tilde{X},\mathfrak{u}(E))$, compactly supported away from $\tilde{Z}_{\mathrm{an}}\cup\mathbf{E}$. Choose $0<\delta\le 1$. For $\varepsilon>0$ we have:

$$\begin{aligned}
\int_{\tilde{X}}\left\langle\phi,\tilde{A}^\varepsilon_{j,\delta}-A_j\right\rangle dvol_{\omega_\varepsilon} &= \int_0^\delta dt\int_{\tilde{X}}\left\langle\phi,\frac{\partial\tilde{A}^\varepsilon_{j,t}}{\partial t}\right\rangle dvol_{\omega_\varepsilon}\\
(\textit{flow equations}) &= -\int_0^\delta dt\int_{\tilde{X}}\left\langle\phi,\left(d_{\tilde{A}^\varepsilon_{j,t}}\right)^*F_{\tilde{A}^\varepsilon_{j,t}}\right\rangle dvol_{\omega_\varepsilon}\\
(\textit{Kähler identities}) &= \sqrt{-1}\int_0^\delta dt\int_{\tilde{X}}\left\langle\phi,\left(\partial_{\tilde{A}^\varepsilon_{j,t}}-\bar{\partial}_{\tilde{A}^\varepsilon_{j,t}}\right)\Lambda_{\omega_\varepsilon}F_{\tilde{A}^\varepsilon_{j,t}}\right\rangle dvol_{\omega_\varepsilon}\\
&= \sqrt{-1}\int_0^\delta dt\int_{\tilde{X}}\left\langle\left(\partial_{\tilde{A}^\varepsilon_{j,t}}-\bar{\partial}_{\tilde{A}^\varepsilon_{j,t}}\right)^*\phi,\Lambda_{\omega_\varepsilon}F_{\tilde{A}^\varepsilon_{j,t}}\right\rangle dvol_{\omega_\varepsilon}.
\end{aligned}$$

By *Lemma* 25 $\Lambda_{\omega_\varepsilon}F_{\tilde{A}^\varepsilon_{j,t}}$ is bounded on the support of ϕ for all j, all $\varepsilon>0$, and all

$0 \leq t \leq \delta$, and the bound may be taken to be independent of δ. Therefore:

$$\int_{\tilde{X}} \left\langle \phi, \tilde{A}^{\varepsilon}_{j,\delta} - A_j \right\rangle dvol_{\omega_\varepsilon} \leq C \int_0^\delta dt \left\| \left(\partial_{\tilde{A}^{\varepsilon}_{j,t}} - \bar{\partial}_{\tilde{A}^{\varepsilon}_{j,t}} \right)^* \phi \right\|_{L^1(\omega_0)}.$$

Applying this inequality to a sequence, $\tilde{A}^{\varepsilon_i}_{j,t} \to \tilde{A}_{j,t}$ in C^1_{loc},

$$\left| \int_{\tilde{X}} \left\langle \phi, \tilde{A}_{j,\delta} - A_j \right\rangle dvol_{\omega_0} \right| \leq C \int_0^\delta dt \left\| \left(\partial_{\tilde{A}_{j,t}} - \bar{\partial}_{\tilde{A}_{j,t}} \right)^* \phi \right\|_{L^1(\omega_0)}.$$

By the *Corollary* 9, $\left| \tilde{A}_{j,t} - \tilde{A}_\infty \right|$ is locally bounded in any L^p independently of j. Then

$$\left| \int_{\tilde{X}} \left\langle \phi, \tilde{A}_{j,\delta} - A_j \right\rangle dvol_{\omega_0} \right| \leq C\delta$$

where C depends only on the L^1 norm of $\partial_{\tilde{A}_\infty} \phi$, $\bar{\partial}_{\tilde{A}_\infty} \phi$ and the bounds on $\Lambda_{\omega_\varepsilon} F_{\tilde{A}^{\varepsilon}_{j,t}}$ and $\left| \tilde{A}_{j,t} - \tilde{A}_\infty \right|$. In particular C is independent of j. Taking limits as $j \to \infty$ we have:

$$\left| \int_{\tilde{X}} \left\langle \phi, \tilde{A}_{\infty,\delta} - A_\infty \right\rangle dvol_{\omega_0} \right| \leq C\delta$$

and since δ and was arbitrary and $\tilde{A}_{\infty,\delta} = \tilde{A}_{\infty,*}$ for almost all small δ, this implies $\tilde{A}_{\infty,*} = A_\infty$. This concludes the proof of Proposition 29 and hence the proof of the main theorem.

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