

# ABSTRACT

Title of dissertation: MAJORANA QUBITS IN NON-ABELIAN  
TOPOLOGICAL SUPERCONDUCTORS

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Non-Abelian superconductors are novel systems with exotic quasiparticle excitations, namely Majorana fermions, which obey non-Abelian quantum statistics. They are exploited as hardware platforms for fault-tolerant topological quantum computing. In this thesis, we primarily study the non-topological decoherence effects existing in realistic systems and how they affect the stability of topological qubits and gates built from the Majorana quasiparticles. The main decoherence effects are the tunneling splitting of the topological degeneracy, thermal excitations and superconducting fluctuations which are not treated in the usual BCS mean-field theory. We calculate the tunneling splitting between non-Abelian vortices in both chiral p-wave superconductors and the superconductor/topological insulator heterostructure, as a function of the inter-vortex distance, superconducting gap and the Fermi energy. It is shown that besides the well-known exponential suppression, the splitting also oscillates with the distance on the scale of Fermi wavelength. This implies that the fusion outcome of two non-Abelian particles depends strongly on microscopic details. We then investigate the robustness of topological qubits and

their braiding against thermal effects and non-adiabaticity, unavoidable in any realistic systems. We apply the formalism of density matrix and master equation and characterize the topological qubits in terms of physical observables. Based on this formulation, we show that the topological qubits are robust against both localized and extended fermionic excitations even when gapless bosonic modes are present. Finally, we explore the non-perturbative effect of strong fluctuations of superconducting order parameter, when the mean-field description in terms of Bogoliubov quasiparticles is invalidated. We consider a model of two-leg ladder of interacting fermions with only quasi-long-range superconducting order and derive the low-energy effective field theory using bosonization techniques. We find that although the whole spectrum is gapless, one can identify degeneracies of low-energy states resulting from Majorana edge modes. In the presence of certain impurity scatterings, we show that the splitting of the degeneracy has a power-law decay with the size of the system.

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SUPERCONDUCTORS

by

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## List of Abbreviations

IQHE	Integer Quantum Hall Effect
FQHE	Fractional Quantum Hall Effect
QSH	Quantum Spin Hall
BCS	Bardeen-Cooper-Schriffer
BdG	Bogoliubov-de Gennes
TI	Topological Insulator
TSC	Topological SuperConductor
TQC	Topological Quantum Computation

## Chapter 1

### Introduction

The subject of this thesis, the non-Abelian topological superconductors in two-dimensional space, is unique in many ways: it represents the simplest possible non-Abelian phase and the only one of the whole family of non-Abelian phases that can be understood completely in terms of non-interacting fermions, while all other cases require strong correlation. Although the BCS theory of superconductivity was born almost sixty years ago and superconductors are among the most well-studied condensed matter systems, it was only realized in 2000 that exotic non-Abelian phases can emerge in superconductors. The non-Abelian nature manifests itself in the unusual zero-energy excitations bound to topological defects such as vortices. Despite the theoretical interest, the study of non-Abelian superconductors is largely driven by a potential application to quantum computation, since non-Abelian excitations may be exploited as the fundamental building blocks of a fault-tolerant topological quantum computer.

The purpose of this chapter is to give a brief but self-contained account of the theory of non-Abelian topological superconductors. We will address the following questions:

1. What is a topological phase and how can a superconductor be topological?

2. What are non-Abelian statistics?
3. How is a non-Abelian superconductor related to quantum computation?

## 1.1 Topological Phases: An Overview

The 1980 discovery of Integer Quantum Hall Effect [1] opened the door to the fascinating world of topological phases in condensed matter systems. The remarkably precise quantization of Hall conductance, insensitive to many microscopic details such as disorder and geometry, is the first example of the “universal”, exact features common in topological phases, that are robust against any small perturbations to the system. The even more striking discovery of fractional quantum Hall effect [2] led to the conceptual formulation of the notion of topological order [3].

To understand its meaning, let us take a grand view of gapped quantum phases. In one sentence, topological order can be regarded as a “periodic table” of all gapped phases. Well-known examples of gapped phases are insulators: band insulators, Mott insulators, etc. All the excitations in these phases are gapped so correlation functions of any local observables decay exponentially in the limit of large space-time separation. However, it does not mean that they are all alike: IQH states, albeit gapped, have quantized Hall conductance while ordinary band insulators do not. Therefore a more refined notion of the equivalence classes between gapped quantum phases is needed, which is provided by the concept of adiabatic continuity [4]. Two gapped quantum phases are said to be adiabatically connected, if there exists a parameter path to connect their Hamiltonians such that the spectral gap is not

closed throughout the path. If one adiabatically follows the path of the Hamiltonian, the ground state smoothly evolves from one to the other. It is also clear that under such equivalence relations, there has to be a quantum phase transition if one wants to connect two different gapped phases. Interestingly, although it seems very natural to consider adiabatic continuity between gapped phases, historically the first application of adiabatic continuity in condensed matter physics was Landau's Fermi liquid theory [5, 6], where a Fermi liquid is in fact defined as a state adiabatically connected to a non-interacting Fermi gas, a gapless phase.

Topological phases are then defined as those gapped phases that can not be adiabatically connected to trivial phases. One may wonder what are trivial phases to be compared with. The canonical example of a trivial gapped phase is an atomic insulator, in which all electrons occupy localized atomic orbitals and the many-body wavefunction is just a Slater determinant of all real-space atomic orbitals. This definition of topological phases is quite general since we have not even invoked any physical characterizations. Theoretically, it is an extremely complicated problem to find all topological phases. Still, after thirty years of research, our understanding of topological phases has been greatly enriched [7, 8].

The above definition of topological phases gives no hint on how to characterize topological phases physically. It defines topological phases by what they are not. Conventionally, phases of matter are often associated with broken symmetries, characterized by local order parameters and correlation functions. This conventional approach has been very successful in describing many solid state systems such as magnets, superfluids and superconductors, but completely fails for topologi-

cal phases, since in general topological phases can exist without the presence of any symmetries. As we have mentioned, since all topological phases are gapped by definition, measurements of correlation functions can hardly tell us anything. So far we are not aware of any “universal” physical characterizations that apply to all topological phases. However, generally there are two types of physical characterization that are commonly found: first, topological phases often support gapless boundary modes when they are put on a manifold with open boundary, robust against any local perturbations. For example, the edge of IQHE supports a one-dimensional chiral Fermi liquid and FQHE has gapless edge modes described by a chiral Luttinger liquid [9, 10]. The existence of gapless edge modes is closely related to the quantization of Hall conductance. Second, topological phases can exhibit fractionalized quasiparticle excitations that carry fractional charges or anyonic statistics [11, 12]. We will discuss anyonic statistics in great detail in the next section. Each can be considered as a sufficient condition of topological phases, but not necessary.

## 1.2 Exchange Statistics and Anyons

Most of us have been familiar with the fact that nature only permits two kinds of exchanges statistics for indistinguishable particles: Bose-Einstein statistics and Fermi-Dirac statistics. The usual argument leading to this statement is the following: after exchanging a pair of identical particles, the many-body wavefunction can only acquire a phase factor  $e^{i\phi}$  because the configurations of the system before and after the exchange are the same. Since two exchanges must bring the system back to

its original configuration, we require  $e^{2i\phi} = 1$  and thus  $e^{i\phi} = \pm 1$ , corresponding to bosonic or fermionic statistics. Although intuitively the argument is sound, there is a deep subtlety which only became clarified in the late 1970's [13]. Namely, the fact that two exchanges equal an identity is a topological argument and only holds for spatial dimension  $d \geq 3$ . It is no longer true if  $d = 2$  ( $d = 1$  is another story since the notion of exchange statistics is not even defined).

Mathematically, exchange statistics is related to the homotopy classes of the world-line trajectories of identical particles starting and ending at the same spatial configurations. The configuration space of  $N$  identical hard-core particles living in  $d$ -dimensional space  $\mathbb{R}^d$  is

$$\mathcal{C}_N = \frac{R^{Nd} - \Delta_N}{S_N}. \quad (1.1)$$

Here  $\Delta_N = \{(\mathbf{x}_1, \dots, \mathbf{x}_N) \in \mathbb{R}^{Nd} \mid \mathbf{x}_i = \mathbf{x}_j \text{ for some } i, j\}$  is removed because of the hard-core condition.  $S_N$ , being the permutations group of  $N$  elements, is taken out to account for the indistinguishability of the particles. Trajectories that correspond to exchanges of particles correspond to “loops” in the many-particle configuration space  $\mathcal{C}_N$ , and therefore classified homotopically by the first fundamental group  $\pi_1(\mathcal{C}_N)$ . When quantizing the system using the path integral formalism [14] and “summing over all paths” to get the transition amplitude between different states, we clearly see that each path can be associated with an amplitude that is completely determined by the homotopy class of the path [15]. This “topological term” precisely represents the exchange statistics of identical particles. They must form unitary representations of  $\pi_1(\mathcal{C}_N)$  since quantum evolution is unitary. Although oftentimes only

one-dimensional representation is considered, there is no reason to exclude higher-dimensional representations. The whole subject of this thesis is about the physical realization of a two-dimensional irreducible representation of the fundamental group  $\pi_1(\mathcal{C}_N)$  for  $d = 2$ .

The physical universe has  $d = 3$  (as far as condensed matter system is concerned) and  $\pi_1(\mathcal{C}_N) = S_N$  [16]. It is known mathematically that  $S_N$  has two one-dimensional representations: the trivial one, corresponding to Bose-Einstein statistics and the “alternating” one corresponding to Fermi-Dirac statistics. Higher dimensional representations are possible but they are just disguised versions of bosonic and fermionic statistics with internal degrees of freedom [17].

If  $d = 2$ ,  $\pi_1(\mathcal{C}_N)$  is no longer isomorphic to  $S_N$ . Since the worldlines of particles are just curves in (2+1)-dimensional spacetime and exchanging particles “braids” the worldlines,  $\pi_1(\mathcal{C}_N)$  is called the braid group, denoted by  $\mathcal{B}_N$  and exchange statistics is often referred as braiding statistics. To represent the braid group, we need  $N - 1$  generators  $\sigma_i$  which are physically nothing but counterclockwise braiding two neighboring particles, subject to the following relations:

$$\begin{aligned}\sigma_i\sigma_j &= \sigma_j\sigma_i, \quad |i - j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}.\end{aligned}\tag{1.2}$$

Recall in the beginning of this section we gave the textbook argument why there are only bosonic and fermionic statistics when  $d = 3$ . The argument translates to the mathematical statement that if we supplement the definition of the braid group (1.2) with  $\sigma_i^2 = 1$ , the braid group reduces to the permutation group.

The study of unitary representations of the braid group is a rich subject of



mathematics. It is easy to check that  $D[\sigma_i] = e^{i\theta}$  gives one-dimensional unitary representation of  $\mathcal{B}_N$ . Since  $\theta$  can be any real number besides 0 and  $\pi$ , particles obeying this kind of statistics are called “anyons” [18]. Even more interesting are multi-dimensional unitary representations. Contrary to the  $d = 3$  case, here multi-dimensional representations can not be reduced to the one-dimensional ones by any means. Physically, a multi-dimensional representation requires the Hilbert space of several particles at fixed positions to be multi-dimensional. Particles with exchange statistics being multi-dimensional representation of  $\mathcal{B}_N$  are named “non-Abelian anyons” for obvious reasons [8]. Abelian and non-Abelian anyons arise in two-dimensional many-body systems as low-energy, point-like “quasiparticles” and in fact reflect the topological order of the underlying quantum phases. Topological phases with non-Abelian excitations are called non-Abelian phases.

### 1.3 Chiral Topological Superconductors

Among all theoretical models of non-Abelian topological phases, the  $p_x + ip_y$  superconductor is probably the simplest one and the only one that can be formulated completely in terms of non-interacting fermions, which not only allows a thorough theoretical understanding but also provides guidance for experimental searches. In this section we review the BCS mean-field description of chiral superconductors and their general topological classification.

The “prototype” of all chiral TSC is the chiral  $p_x + ip_y$  superconductor of

spinless fermions, given by the following BCS Hamiltonian [19, 20]:

$$H = \int d\mathbf{r}d\mathbf{r}' \psi^\dagger(\mathbf{r})h(\mathbf{r}, \mathbf{r}')\psi(\mathbf{r}') + \frac{1}{2} \int d\mathbf{r}d\mathbf{r}' \psi^\dagger(\mathbf{r})\Delta(\mathbf{r}, \mathbf{r}')\psi^\dagger(\mathbf{r}') + \text{h.c.} \quad (1.3)$$

$\psi$  is the fermionic field operator.  $h(\mathbf{r}, \mathbf{r}')$  contains the single-particle contribution, such as kinetic energy and potential energy.  $\Delta(\mathbf{r}, \mathbf{r}')$  is the superconducting order parameter. Due to the anti-commutation of the fermionic operators, the order parameter must be odd under the exchange of the two coordinates:  $\Delta(\mathbf{r}, \mathbf{r}') = -\Delta(\mathbf{r}', \mathbf{r})$ . Otherwise the pairing term would vanish identically. A typical expression for  $\Delta(\mathbf{r}, \mathbf{r}')$  with  $p_x + ip_y$  pairing symmetry is the following:

$$\Delta(\mathbf{r}, \mathbf{r}') = \Delta_0 \left( \frac{\mathbf{r} + \mathbf{r}'}{2} \right) (\partial_{x'} + i\partial_{y'}) \delta(\mathbf{r} - \mathbf{r}'). \quad (1.4)$$

To solve the Hamiltonian, we perform a Bogoliubov transformation [21, 22]

$$\psi(\mathbf{r}) = u(\mathbf{r})\gamma + v^*(\mathbf{r})\gamma^\dagger. \quad (1.5)$$

To diagonalize the Hamiltonian, we require that  $[H, \gamma] = -E\gamma$  where  $E$  is the corresponding eigenenergy, which implies that  $u$  and  $v$  should solve the following eigenvalue problem:

$$E \begin{pmatrix} u(\mathbf{r}) \\ v(\mathbf{r}) \end{pmatrix} = \int d\mathbf{r}' \begin{pmatrix} h(\mathbf{r}, \mathbf{r}') & \Delta(\mathbf{r}, \mathbf{r}') \\ \Delta^*(\mathbf{r}, \mathbf{r}') & -h^T(\mathbf{r}, \mathbf{r}') \end{pmatrix} \begin{pmatrix} u(\mathbf{r}') \\ v(\mathbf{r}') \end{pmatrix} \quad (1.6)$$

The  $2 \times 2$  matrix  $H_{\text{BdG}}$

$$H_{\text{BdG}} = \begin{pmatrix} h(\mathbf{r}, \mathbf{r}') & \Delta(\mathbf{r}, \mathbf{r}') \\ \Delta^*(\mathbf{r}, \mathbf{r}') & -h^T(\mathbf{r}, \mathbf{r}') \end{pmatrix} \quad (1.7)$$

is often called the BdG Hamiltonian. Due to the doubling of the degrees of freedom, the BdG Hamiltonian satisfies a particle-hole symmetry [23]:

$$\tau_x H_{\text{BdG}} \tau_x = -H_{\text{BdG}}^*. \quad (1.8)$$

Here  $\tau_x$  is the Pauli matrix acting on the particle-hole space. Notice that this is an anti-unitary symmetry for the BdG Hamiltonian matrix since the complex conjugation is involved. It implies that the solutions of BdG equation always come in pairs: for each solution  $\Psi_E = (u_E, v_E)^T$  with energy eigenvalue  $E$ , there is a corresponding solution  $\Psi_{-E} = \tau_x \Psi_E^*$  with energy  $-E$ . In terms of the Bogoliubov quasiparticles, we readily have  $\gamma_{-E} = \gamma_E^\dagger$ . This again confirms that the particle-hole symmetry reflects the doubling of the degrees of freedom: creating a hole excitation by  $\gamma_{-E}^\dagger$  is equivalent to annihilating a particle excitation  $\gamma_E$ . The Hamiltonian is now diagonalized using  $\gamma$  operators:

$$H = \sum_{E_n > 0} E_n \gamma_n^\dagger \gamma_n + E_0. \quad (1.9)$$

Here  $E_0 = -\frac{1}{2} \sum_{E_n > 0} E_n$  is a constant.

To reveal the topological nature of the  $p_x + ip_y$  superconductor, we first review the general framework of the topological classification of superconductors. Here by superconductor we mean fermionic systems described by BCS mean-field Hamiltonians. We do not assume any symmetries present in the system. Without loss of generality we consider lattice models of fermions with periodic boundary conditions, since any continuum model can be approached as a limiting case of a lattice model. A generic BCS Hamiltonian can be expressed in the momentum space as:

$$H = \sum_{\mathbf{k}} \Psi_{\mathbf{k}}^\dagger \mathcal{H}_{\mathbf{k}} \Psi_{\mathbf{k}}, \quad \Psi_{\mathbf{k}} = \begin{pmatrix} \psi_{\mathbf{k}} \\ \psi_{-\mathbf{k}}^\dagger \end{pmatrix}. \quad (1.10)$$

Here  $\mathbf{k}$  is the lattice momentum taking value in the first Brillouin zone.  $\psi_{\mathbf{k}}$  can have internal degrees of freedom, such as spin and orbital indices.  $\mathcal{H}_{\mathbf{k}}$  is the BdG

Hamiltonian:

$$\mathcal{H}_{\mathbf{k}} = \begin{pmatrix} h_{\mathbf{k}} & \Delta_{\mathbf{k}} \\ \Delta_{\mathbf{k}}^\dagger & -h_{-\mathbf{k}}^T \end{pmatrix} \quad (1.11)$$

$h_{\mathbf{k}}$  is the single-particle Hamiltonian and  $\Delta_{\mathbf{k}}$  represents BCS pairing. Notice that  $h_{\mathbf{k}}$  must be a Hermitian matrix  $h_{\mathbf{k}} = h_{\mathbf{k}}^\dagger$  and the order parameter matrix  $\Delta_{\mathbf{k}}$  always satisfies  $\Delta_{-\mathbf{k}} = -\Delta_{\mathbf{k}}^T$ . Let  $\tau_x$  be the Pauli matrix in Nambu (particle-hole) space, we have

$$\tau_x \mathcal{H}_{\mathbf{k}} \tau_x = - \begin{pmatrix} h_{-\mathbf{k}}^T & -\Delta_{\mathbf{k}}^\dagger \\ -\Delta_{\mathbf{k}} & h_{\mathbf{k}} \end{pmatrix} = -\mathcal{H}_{-\mathbf{k}}^*. \quad (1.12)$$

The last equality is derived from  $h_{\mathbf{k}}^\dagger = h_{\mathbf{k}}$ ,  $\Delta_{-\mathbf{k}}^* = -\Delta_{\mathbf{k}}^\dagger$ . This defines the particle-hole symmetry represented by  $\tau_x$ . A non-interacting fermionic Hamiltonian with such a symmetry is said to be in the class D [23]. The Hamiltonian can be diagonalized by Bogoliubov transformation, as we did for the spinless  $p_x + ip_y$  superconductor. The ground state is defined to be the state with no positive-energy Bogoliubov excitations.

At the level of topological classification, it is useful to consider the superconducting ground state as a state in which all “hole”-like states (i.e. with negative energy eigenvalue) are filled. It is in many ways like an insulator with the Hamiltonian matrix the same as the BdG Hamiltonian and the Fermi energy set to zero. Assume  $\mathcal{H}_{\mathbf{k}}$  is a  $2N \times 2N$  matrix and for each  $\mathbf{k}$  the negative energy eigenstates (the “occupied” states) are denoted by  $|u_{n\mathbf{k}}\rangle$  where  $n = 1, \dots, N$ . Mathematically, the ground state wavefunction defines a  $U(1)$  vector bundle on the Brillouin zone, which is a two-dimensional torus  $T^2$  [24, 25, 26]. This vector bundle can be specified

by constructing the ground state projection operator  $P_{\mathbf{k}} = \sum_n |u_{n\mathbf{k}}\rangle\langle u_{n\mathbf{k}}|$ . Topologically, we can characterize such a vector bundle by its first Chern number  $\mathcal{C}$  [27], which is the integral of the first Chern character:

$$\text{ch}_1(\mathcal{F}) = \text{Tr} \left( \frac{i\mathcal{F}}{2\pi} \right), \quad (1.13)$$

and

$$\mathcal{C} = \int_{T^2} \text{ch}_1(\mathcal{F}) = \frac{1}{2\pi} \int_{T^2} \mathcal{F}_{\mathbf{k}}. \quad (1.14)$$

Here  $\mathcal{F}_{\mathbf{k}}$  is the curvature form derived from the connection one-form:

$$\mathcal{A}_{\mathbf{k}} = i \sum_n \langle u_{n\mathbf{k}} | \nabla | u_{n\mathbf{k}} \rangle. \quad (1.15)$$

There are a couple of equivalent representations of the Chern number. It can be expressed solely in terms of the ground state projector [24]

$$\mathcal{C} = \frac{1}{2\pi i} \int \text{Tr}(P_{\mathbf{k}} dP_{\mathbf{k}} \wedge dP_{\mathbf{k}}) = \frac{1}{2\pi i} \int d\mathbf{k} \text{Tr} \left[ P_{\mathbf{k}} \left( \frac{\partial P_{\mathbf{k}}}{\partial k_x} \frac{\partial P_{\mathbf{k}}}{\partial k_y} - \frac{\partial P_{\mathbf{k}}}{\partial k_y} \frac{\partial P_{\mathbf{k}}}{\partial k_x} \right) \right]. \quad (1.16)$$

The first Chern number has very intuitive physical meaning. Any superconducting system with a non-zero Chern number supports chiral Majorana edge modes, the number of which is equal to the Chern number [19]. The Majorana edge mode carries energy, leading to quantized thermal Hall effect [19]. It also determines the existence of unpaired Majorana zero modes in topological defects, which will be discussed later.

We now apply the formula to the spinless  $p_x + ip_y$  superconductors with only one band, so the BdG Hamiltonian is a  $2 \times 2$  matrix. We write the Hamiltonian in terms of Pauli matrices in Nambu space:

$$H_{\mathbf{k}} = \mathbf{d}_{\mathbf{k}} \cdot \boldsymbol{\tau}. \quad (1.17)$$

The projector is then given by

$$P_{\mathbf{k}} = \frac{1}{2} \left( 1 + \frac{\mathbf{d}_{\mathbf{k}} \cdot \boldsymbol{\tau}}{|\mathbf{d}_{\mathbf{k}}|} \right). \quad (1.18)$$

Denote  $\hat{\mathbf{d}}_{\mathbf{k}} = \mathbf{d}_{\mathbf{k}}/|\mathbf{d}_{\mathbf{k}}|$  as the normalized  $\mathbf{d}$  vector, we find

$$\mathcal{C} = \frac{1}{4\pi} \int d^2\mathbf{k} \hat{\mathbf{d}}_{\mathbf{k}} \cdot \left( \frac{\partial \hat{\mathbf{d}}_{\mathbf{k}}}{\partial k_x} \times \frac{\partial \hat{\mathbf{d}}_{\mathbf{k}}}{\partial k_y} \right). \quad (1.19)$$

This formula has a simple geometrical interpretation. The normalized vector  $\mathbf{d}$  defines a mapping from the Brillouin zone  $T^2$  to the two-dimensional unit sphere  $S^2$  and the integral is nothing but the area of the image of  $T^2$ . Since the total area of the unit sphere is  $4\pi$ ,  $\mathcal{C}$  actually counts how many times the image of the torus is “wrapped” around the sphere. It is also known as the degree of the mapping in mathematics.

For  $p_x + ip_y$  superconductors, the  $\mathbf{d}$  vector is given by  $\mathbf{d}_{\mathbf{k}} = (\Delta k_x, -\Delta k_y, \frac{k^2}{2m} - \mu)$ . Direct evaluation of the integral yields

$$\mathcal{C} = \begin{cases} 1 & \mu > 0 \\ 0 & \mu < 0 \end{cases}. \quad (1.20)$$

Therefore  $p_x + ip_y$  superconductor is topological with Chern number 1 if the Fermi energy is above the band bottom. More generally, if the pairing order parameter  $\Delta(\mathbf{k}) \propto (k_x + ik_y)^n$  and  $\mu > 0$ , the Chern number is  $n$ .

Evaluating the Chern number analytically is a cumbersome task if the superconductor has multiple bands. Fortunately, there is a great simplification if we only want to know the parity of the Chern number which determines whether the superconductors have non-Abelian excitations or not [28, 29]. We present the formula

here and leaves its proof to Appendix A:

$$(-1)^{\mathcal{C}} = \prod_{\mathbf{K}} \text{Pf}[H_{\mathbf{K}}\Xi]. \quad (1.21)$$

Here the product is taken over all symmetric points  $\mathbf{K}$  satisfying  $\mathbf{K} \equiv -\mathbf{K}$  in the first Brillouin zone.

## 1.4 Majorana Zero Modes and TQC

As we have reviewed in the previous section, superconductors have particle-hole symmetry as a result of the redundant representation. The energy spectrum is symmetric with respect to zero and  $\gamma_{-E} = \gamma_E^\dagger$ . Thus the zero-energy states are very special because  $\gamma_{E=0} = \gamma_{E=0}^\dagger$ . Such a quasiparticle is self-conjugate, being identical to its “antiparticle”. Given two self-conjugate quasiparticles at  $E = 0$  denoted by  $\gamma_1$  and  $\gamma_2$ , it is straightforward to check that they still obey fermionic commutation relation:  $\{\gamma_i, \gamma_j\} = \delta_{ij}$ . It immediately follows that  $\gamma^2 = \frac{1}{2}$ . Such self-conjugate fermionic quasiparticles are called Majorana fermions(MF). Loosely speaking it can be regarded as half of an ordinary fermion, since an ordinary fermion can always be represented as two degenerate MFs: given a fermion annihilation operator  $c$  satisfying  $\{c, c^\dagger\} = 1$ , we can form two MFs  $\gamma_1 = \frac{c+c^\dagger}{\sqrt{2}}$ ,  $\gamma_2 = \frac{c-c^\dagger}{\sqrt{2}i}$  and  $c = \gamma_1 + i\gamma_2$ .

One may then wonder what is special about MF here in a topological superconductor since they are just the usual fermionic operators in disguise. It is important to clarify that what we are interested are unpaired (non-degenerate), localized Majorana fermionic excitations, which can not be obtained by naively rewriting a usual fermionic operator as two MF operators. The particle-hole symmetry ensures that

MF has zero excitation energy, implying a vanishing superconducting gap at where the MF emerges. Thus the zero-energy MFs can only exist at “defects” in the superconducting order parameter, such as domain walls or vortices. These defects can be spatially very well separated. Then one can form an ordinary fermion out of two Majorana fermions very far from each other, a highly non-local object. This non-local fermionic mode can be occupied or empty, yielding two degenerate many-body states. When there are  $2N$  Majorana fermions, we can group them pairwise and construct the Hilbert space from the  $N$  fermionic modes, leading to  $2^N$  degenerate states. If there are no other zero-energy modes in the system, there is a further superselection rule that the total fermion parity must be fixed. States with different global fermion parity belong to different superselection sectors and can not be connected by any physical matrix elements. Thus the actual degeneracy is reduced to  $2^{N-1}$ . These degenerate states have no difference if only local measurements are concerned. The only way to distinguish them is to measure the fermion parity stored in pairs of topological defects, which certainly requires non-local measurements. We therefore call such degeneracy “topological degeneracy”.

For spinless  $p_x + ip_y$  superconductors, a Majorana zero mode can be found in the core of an Abrikosov vortex [30, 22] around which the phase of the order parameter winds by  $2\pi$ :

$$\Delta_0(\mathbf{r}) = f(r)e^{i\varphi}. \quad (1.22)$$

Here  $(r, \varphi)$  is the polar coordinate of the two-dimensional plane.  $f(r)$  represents the profile of the order parameter. One can explicitly solve the BdG equation to find



the zero mode (see Chapter 3 for details). Besides the zero modes, there are other eigenstates with energy eigenvalues well below the bulk gap, the so-called midgap states [31, 32, 33, 34]. A semiclassical argument, treating the vortex core as a hole of size  $\xi \sim \frac{v_F}{\Delta_0}$ , gives an estimate of the energy of midgap states to be of the order  $\frac{1}{m\xi^2} \approx \frac{\Delta_0^2}{E_F}$  where  $E_F$  is the Fermi energy. As long as  $k_F\xi \gg 1$ , this energy scale is much smaller than the bulk gap.



Figure 1.1: Braiding of Majorana fermions bound to vortices.

We now demonstrate the very peculiar non-Abelian braiding statistics of MFs in superconducting vortices, first derived by Ivanov [35]. It is crucial to keep track of the branch cut where the superconducting phase jumps by  $2\pi$  to uniquely define the superconducting phase everywhere (except at the vortex cores). Pictorially we can attach a “string” to each vortex, which goes all the way to infinity (or the system boundary) to represent the branch cut. As the vortices are transported adiabatically, the branch cuts are also “dragged” along with the vortices and we have to make sure that the vortices do not cross the branch cuts, as depicted in Fig. 1.1. We denote the local phases seen by vortices 1 and 2 by  $\chi_1$  and  $\chi_2$  respectively. Before the exchange,  $\chi_1 = \pi + 0^+$ ,  $\chi_2 = 0^+$  and after the exchange  $\chi'_1 = 2\pi - 0^+$ ,  $\chi'_2 = \pi + 0^+$ . The gauge transformation then implies  $\gamma_1$  picks up a phase  $e^{i(\chi_1 - \chi'_1)/2} = -1$ . Consequently,  $\gamma_1$  is replaced by  $\gamma_2$  but  $\gamma_2$  is replaced by  $-\gamma_1$ . We therefore conclude that the

Majorana fermionic operators have the following transformation:

$$\gamma_1 \rightarrow \gamma_2, \gamma_2 \rightarrow -\gamma_1, \tag{1.23}$$

which can be realized by a unitary transformation  $U_{12} = \frac{e^{i\theta}}{\sqrt{2}}(1 + \gamma_2\gamma_1) = e^{i\theta}e^{\frac{\pi}{4}\gamma_2\gamma_1}$ :

$$U_{12}\gamma_1U_{12}^\dagger = \gamma_2, U_{12}\gamma_2U_{12}^\dagger = -\gamma_1. \tag{1.24}$$

The operator algebra determines  $U_{12}$  up to an Abelian phase. The Abelian phase is not arbitrary if the superconducting vortices are deconfined excitations (i.e. there are no long-range interactions between them except the topological ones). It is in fact related to the topological spin of the non-Abelian vortex [24] and the value is  $\theta = -\frac{\pi}{8}$ .

In summary, we have shown that  $2N$  superconducting vortices in a non-Abelian superconductor span a  $2^{N-1}$ -dimensional degenerate Hilbert space and braiding of vortices results in unitary transformations given by  $U_{12}$ . Both the degeneracy and the braiding operations are topologically protected, immune to arbitrary local perturbations. Such non-Abelian vortices are called “Ising anyons”, due to its connection with Ising conformal field theory.

Kitaev [36] and Freedman et. al. [37] had the great insight that non-Abelian anyons provide an ideal realization of a quantum computer: the ground state degeneracy due to multiple quasiparticles is exploited as qubits and quantum memory; the braiding operations generate quantum gates on the qubits. Both the quantum information stored in the qubits and the braiding operations are topologically protected, thus eliminating errors at the hardware level.

In the case of Ising anyons,  $2N$  vortices can realize  $N - 1$  qubits. A topological qubit thus requires at least 4 vortices. Let us label the four Majorana zero modes as  $\gamma_i, i = 1, 2, 3, 4$ . We can construct two complex fermionic modes  $c_1 = \frac{1}{\sqrt{2}}(\gamma_1 + i\gamma_2), c_2 = \frac{1}{\sqrt{2}}(\gamma_3 + i\gamma_4)$  and the degenerate states can be specified by the occupation numbers of  $c_1$  and  $c_2$ . Fixing the global fermion parity to be even, the two qubit states can be specified by the occupation numbers in the fermionic states  $|00\rangle$  and  $|11\rangle = c_1^\dagger c_2^\dagger |00\rangle$ . Braiding of the vortices generates  $\frac{\pi}{2}$  rotations  $e^{\pm i\frac{\pi}{4}\sigma_{x,y,z}} = \frac{1}{\sqrt{2}}(1 \pm i\sigma_{x,y,z})$ . The braids can be used to generate other single-qubit rotations, such as a Hadamard gate  $H$ :

$$H = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z) = \frac{1}{\sqrt{2}}(1 + i\sigma_y)\sigma_z. \quad (1.25)$$

So  $H$  can be implemented as a NOT gate (braiding twice) followed by another braid. We can go on and consider two qubits constructed from six vortices and braidings generate two-qubit gates in addition to single-qubit operations.

In addition, one also needs to find ways to read out the quantum information stored in the topological qubits. Let us still take the Ising anyons as an example. Since the qubit states are labeled by the fermion parity eigenvalues in pairs of vortices, to read out the qubit is amount to measure the fermion parity contained in a finite spatial region, which can be done typically by interferometry experiments [38, 39]. The basic idea is to exploit the fact that in a superconductor, when a fermion goes around a superconducting vortex, a  $\pi$  Berry phase is experienced by the fermion due to Aharonov-Bohm effect [40]. We have already made crucial use of this fact when deriving the non-Abelian statistics of vortices in  $p_x + ip_y$  superconductors. Now

to measure the fermion parity, we need the “dual” version, the so-called Aharonov-Casher effect [41], that when a superconducting vortex moves around a fermion it also acquires a  $\pi$  Berry phase, which is almost a trivially obvious statement if we put our reference frame on the fermion. Based on Aharonov-Casher effect one can design appropriate interferometers to measure the fermion parity. For a detailed exposition we refer the readers to Chapter 6.

However, it has been mathematically proved that the braiding operations can not realize all possible single-qubit rotations. In fact, Bravyi has established [42] that the Ising anyon computation model is an intersection of two classically simulatable models, quantum circuits with Clifford gates and fermionic linear optics. So the computational power of Ising anyon model with braidings and measurements is quite limited and is equivalent a classical computer. Thus, Ising anyons only offer topologically protected quantum memory and a limited set of protected gates. Generally speaking non-topological operations are needed in order to perform universal quantum computations [43]. There are also a number of interesting proposals to implement universal quantum computation using Ising anyons in a fully topologically protected way, e.g. by dynamically changing the topology [43], which we will not go into details in this introduction.

## 1.5 Physical Realizations of TSC

In this section we discuss possible realizations of TSC. Although  $p_x + ip_y$  superconductor is a rather simple model in theory, it turns out to be very challenging to

find a realistic material in nature. Most electronic superconductors in metals have  $s$ -wave pairing, which can be traced back to the electron-phonon mediated pairing mechanism. To have the required  $p$ -wave pairing symmetry one clearly needs unconventional pairing mechanisms. There are a number of candidates, though, including the  $^3\text{He}$  film in the superfluid  $A$  phase [44] and the oxide compound  $\text{Sr}_2\text{RuO}_4$  [45]. Although a lot of experimental efforts have been taken, progress in identifying the topological superconductivity/superfluidity in both systems is quite limited. Among the many obstacles we just mention that in both cases, due to the spin degeneracy, to observe a single Majorana zero mode requires creating a half-quantum vortex in the superfluid [46], in which the phase of the order parameter and the Cooper pair spin vector both wind by  $\pi$ . However this type of vortices are not thermodynamically stable: Its free energy diverges logarithmically with the system size. This apparently hinders the observation of Majorana excitations. In addition, the unconventional  $p$ -wave pairing symmetry, believed to be caused by ferromagnetic spin fluctuations, results in very low superconducting transition temperature, making the experimental setup very delicate.

Recent theoretical progress has revealed a completely new avenue towards realizing chiral  $p$ -wave superconductivity, which becomes by far the most promising direction in the search of non-Abelian superconductivity. The approach is to engineer chiral  $p$ -wave superconductor from conventional materials instead of trying one's luck in nature. In particular, the stringent requirement of the  $p$ -wave pairing is removed and all the proposals only involve ordinary  $s$ -wave superconductivity. In the following we discuss three independent different proposals for the practical realiza-

tions of the chiral p-wave superconductivity: i) topological insulator/superconductor heterostructure [47] ii) cold fermionic atoms with s-wave interactions [48] iii) ferromagnet/semiconductor/superconductor heterostructure [49].

We first introduce the TI/SC heterostructure proposal. The surface states of three-dimensional time-reversal-invariant topological insulators (TI) [50, 51, 52] are described by a two-component Dirac Hamiltonian at low-energy:

$$H_{\text{surface}} = v_F \boldsymbol{\sigma} \cdot \mathbf{p} - \mu, \quad (1.26)$$

protected by the nontrivial  $\mathbb{Z}_2$  topological invariant in the bulk of the TI. We notice that this Hamiltonian is formally nothing but a spin-orbit coupling term, resulting in helical spin textures on the Fermi surface. Namely, the (in-plane) spin direction is aligned to the momentum. Therefore the electronic states at opposite momenta on Fermi surface have opposite in-plane spin projections, which allows s-wave pairing to take place on a single Fermi surface, effectively generating *p*-wave pairing. Here it is crucial to exploit the surface states because they are fundamentally different from electronic dispersions arising from a true two-dimensional lattice model where the “fermion doubling” phenomena usually occurs.

S-wave pairing is induced by depositing a 3D superconductor on top of the TI surface. The full second-quantized Hamiltonian density is

$$\mathcal{H} = \psi^\dagger [v_F \boldsymbol{\sigma} \cdot (-i\nabla) - \mu] \psi + \Delta \psi_\uparrow^\dagger \psi_\downarrow^\dagger + \text{h.c.} \quad (1.27)$$

To make the connection to  $p_x + ip_y$  superconductor more explicit, we assume  $\mu > 0$  and  $\Delta \ll \mu$ . So to understand the low-energy physics we can project to the electronic

states near the Fermi circle. We first diagonalize the single-particle Hamiltonian:

$$E_{\pm}(\mathbf{p}) = \pm v_F |\mathbf{p}| - \mu. \quad (1.28)$$

And the eigenvectors are given by  $|\pm, \mathbf{p}\rangle = \frac{1}{\sqrt{2}}(1, \pm e^{i\theta_{\mathbf{p}}})^T$ . Projecting onto the + band, we find the effective Hamiltonian is given by

$$H_{\text{eff}} = \sum_{\mathbf{p}} f_{\mathbf{p}}^{\dagger} (v_F |\mathbf{p}| - \mu) f_{\mathbf{p}} - \frac{1}{2} \Delta e^{-i\theta_{\mathbf{p}}} f_{\mathbf{p}}^{\dagger} f_{-\mathbf{p}}^{\dagger} + \text{h.c.} \quad (1.29)$$

Here  $f_{\mathbf{p}} = \frac{1}{\sqrt{2}}(\psi_{\uparrow\mathbf{p}} + e^{-i\theta_{\mathbf{p}}}\psi_{\downarrow\mathbf{p}})$ . So in this basis, the chirality of the Dirac Hamiltonian results in the  $p_x + ip_y$  pairing symmetry. However, we would like to remark that the surface states do not break time-reversal symmetry while  $p_x + ip_y$  superconductors break time-reversal symmetry.

We can further solve the corresponding BdG equation with superconducting vortices and find a single Majorana zero-energy bound state in a  $\frac{hc}{2e}$  vortex. The explicit solution is displayed in Chapter 3.

We now turn to the second proposal, where effective  $p$ -wave pairing is realized in cold fermionic atoms [48]. The idea is that for spin-1/2 fermions, a single Fermi surface can be created by simply applying a Zeeman field to polarize the fermions and tune the Fermi energy in the Zeeman gap. Notice that the Zeeman splitting explicitly breaks the time-reversal symmetry for spin-1/2 fermions. S-wave interactions between the fermions lead to the formation of a BCS superfluid with s-wave singlet pairing. So Rashba spin-orbit coupling is needed to allow pairing on the same Fermi surface. Therefore we are led to the following theoretical model describing spin-orbit coupled fermions subject to a Zeeman field and s-wave superconducting

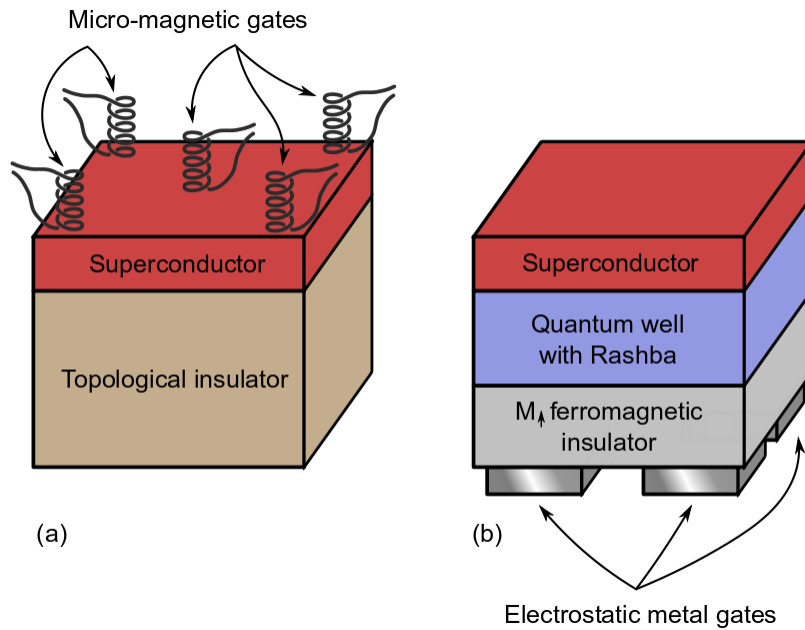


Figure 1.2: Illustrations of the two heterostructure that support Majorana excitations: (a) a superconductor-topological insulator heterostructure, (b) a superconductor-semiconductor-magnetic insulator heterostructure. Reproduced from Bonderson, Das Sarma, Freedman and Nayak, arXiv:1003.2856



pairing:

$$\mathcal{H} = \psi^\dagger \left( \frac{\mathbf{p}^2}{2m} - \mu + \alpha(\boldsymbol{\sigma} \times \mathbf{p}) \cdot \hat{\mathbf{z}} + V_z \sigma_z \right) \psi + g \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow. \quad (1.30)$$

Here  $m$  is the mass of the atoms,  $\alpha$  is the strength of spin-orbit coupling and  $V_z$  the Zeeman splitting. We have chosen the cold atomic gas to be confined in the  $xy$  plane. Experimentally, Rashba spin-orbit coupling can be engineered by a variety of ways [53, 54, 55, 56, 57].

The interaction can be treated by the standard BCS mean-field theory. A careful analysis of the pairing symmetry reveals that even in the presence of spin-orbit coupling, the short-range interaction we use in (1.30) only gives rise to s-wave singlet pairing [48, 58]. So we can proceed with the following mean-field Hamiltonian:

$$\mathcal{H}_{\text{MF}} = \psi^\dagger \left( \frac{\mathbf{p}^2}{2m^*} - \mu + \alpha(\boldsymbol{\sigma} \times \mathbf{p}) \cdot \hat{\mathbf{z}} + V_z \sigma_z \right) \psi + \Delta \psi_\uparrow^\dagger \psi_\downarrow^\dagger + \text{h.c.} \quad (1.31)$$

The value of the s-wave gap  $\Delta$  can be determined self-consistently from the gap equation.

Intuitively, the Zeeman field opens a “magnetic” gap  $2|V_z|$  at the band crossing point  $\mathbf{k} = 0$ . When the Fermi level lies within the Zeeman gap, there is only one Fermi surface, a “parent” state of chiral  $p$ -wave superconductor. To determine precisely the condition under which TSC exists, we need to calculate the parity of the Chern number using the Pfaffian formula (1.21), and the result is

$$(-1)^{\mathcal{C}} = \text{sgn}(\mu^2 + \Delta^2 - V_z^2). \quad (1.32)$$

Therefore if  $V_z^2 > \mu^2 + \Delta^2$ , the Chern number must be odd which ensures the existence of unpaired Majorana zero modes in superconducting vortices. A more

careful calculation shows that  $\mathcal{C} = \Theta(V_z^2 - \mu^2 - \Delta^2)$  where  $\Theta$  is the step function. Solving the BdG equation with a  $\frac{\hbar c}{2e}$  vortex confirms the existence of a Majorana zero-energy bound state in the parameter regime  $V_z^2 > \mu^2 + \Delta^2$ .

The last proposal can be regarded as a solid-state version of the previous one [49]. We notice that the key ingredients of the mean-field Hamiltonian (1.31), the spin-orbit coupling, Zeeman splitting and s-wave pairing, are known to occur in many solid state systems. Two-dimensional electron gas with strong spin-orbit coupling arise in semiconductor quantum wells, such as InAs. Zeeman splitting can be introduced by proximity to a ferromagnetic insulator (a perpendicular magnetic field can also work, but it brings in unwanted orbital effect). S-wave superconductivity can be simply induced by superconducting proximity effect. We therefore have a “sandwich”-like heterostructure consist of a ferromagnet, semiconductor and a s-wave superconductor, as depicted in Fig. 1.2. The Hamiltonian for the semiconductor is essentially the same as given in (1.31) and TSC occurs when  $V_z^2 > \mu^2 + \Delta^2$ . It is an appealing proposal because the materials involved are all conventional and well-studied in solid state physics.

So far we have focused on two-dimensional systems. By dimensional reduction (e.g. putting a confining potential along one of the dimensions), one can go smoothly from two-dimensional TSCs to their one-dimensional descendants. Following our previous discussions, it is quite natural to envision engineering heterostructures to realize one-dimensional  $p$ -wave superconductors. For example, if we pattern the superconductors on the surface of a three-dimensional topological insulator to form a SC/TI/SC line junction, it has been shown to lead to one-

dimensional non-chiral Majorana modes [47]. Just as the spinless  $p_x + ip_y$  superconductor is the prototype of most two-dimensional non-Abelian superconductors, its one-dimensional descendant, spinless fermions with  $p$ -wave pairing, can be considered as the prototype model of all one-dimensional topological superconductors, where Majorana zero-energy bound states are found at the ends of the topological regions [28]. Similarly, one can realize one-dimensional  $p$ -wave superconductor in semiconductor nanowire/superconductor heterostructures [59, 60]. An great advantage of the one-dimensional realization is that the Zeeman field can be applied along the wire, thus avoiding the destructive orbital effect without the need to introduce the second interface to a ferromagnet.

The proposals have stimulated a burst of theoretical and experimental efforts to design and engineer low-dimensional electronic systems that behaves like chiral  $p_x + ip_y$  superconductors [61, 62, 63, 64]. Quite recently, the proposal involving semiconductor nanowire/superconductor structure [59] has been claimed to be realized in experiments and possible signatures of the desired Majorana zero modes have been reported [65, 66, 67, 68].

## 1.6 Decoherence and Stability of Majorana Qubits

Ideally, a topological quantum computer based on non-Abelian anyons is free of any errors and decoherence. This is because only the topological degrees of freedom are used to build the quantum computer and non-topological degrees of freedom do not participate in the low-energy physics. However, in reality this idealized scheme

can only be regarded as an approximation, although a very good one. Understanding quantitatively the non-topological aspects of topological qubits is of fundamental importance to the practical implementation of topological quantum computation. A large part of this thesis will be devoted to the study of problems related to this topic. We briefly review the subject here in the context of Majorana qubits.

We first consider the stability of Majorana zero modes. Due to the particle-hole symmetry, the only way a single Majorana fermion can decohere is to couple to another Majorana fermion since it has no internal degrees of freedom. Given a Majorana fermion  $\gamma$ , one can write a mass term:

$$H_{\text{mass}} \propto i\gamma(u\psi + u^*\psi^\dagger). \quad (1.33)$$

Here  $u$  is a complex number and  $\psi$  is a fermion. However, since there is a superconducting gap everywhere in the bulk except at the defect where the Majorana fermion resides, any other single-particle excitation must have a gap. The above Hamiltonian (1.33) is thus irrelevant at energy below the bulk gap. We therefore conclude that a single Majorana fermion in a gapped superconductor is stable. On contrary, two or more Majorana fermions are generically not stable, since one can pairwise gap them out. They can be stabilized if there is a certain symmetry in the system that prohibits the mass term.

The existence of a bulk gap for single-particle excitations plays a crucial role for the stability of Majorana fermions. In fact, it implies an important conservation law in fermionic systems. In a superconductor, due to the condensation of Cooper pairs the total fermion number is not conserved: adding or removing a Cooper pair, which

is formed out of two fermions, does not cost any energy. However, the parity of the fermion number is conserved at low energy since exciting a single particle/hole costs a lot of energy. The conservation of the discrete physical observable, the fermion parity, plays an important role not only in protecting the Majorana fermions from developing a gap but also the measurement of the topological qubits. We give here a concise derivation of the non-Abelian statistics of Majorana fermions based on fermion parity conservation. Let us consider adiabatically exchanging two Majorana fermions  $\gamma_1$  and  $\gamma_2$ , the net effect of which is a unitary transformation  $U$ . Since before and after the exchange the configuration is exactly the same, we expect the Majorana nature of the excitations remains intact. Therefore,

$$U\gamma_1U^\dagger = s_1\gamma_2, U\gamma_2U^\dagger = s_2\gamma_1. \quad (1.34)$$

Here  $s_1^2 = s_2^2 = 1$ , required by the Majorana condition  $\gamma_{1,2}^2 = 1$ . It follows that

$$Ui\gamma_1\gamma_2U^\dagger = -s_1s_2 \cdot i\gamma_1\gamma_2. \quad (1.35)$$

Since  $i\gamma_1\gamma_2 = 1 - 2c^\dagger c$  measures the fermion parity, it should be invariant through the entire process of the adiabatic braiding. Therefore we must have  $s_1s_2 = -1$ , i.e.  $s_1$  and  $s_2$  must have opposite signs. We therefore reproduce the Ivanov's rule derived previously for Majorana fermions in superconducting vortices. The implication is that the non-Abelianness of the braiding is closely related, or even a consequence of, the conservation of fermion parity and the adiabaticity of the braiding process.

There are a number of ways that the fermion parity protection can be spoiled. We have mentioned that in any realistic superconductors Majorana fermions must come in pairs, but they can be very well separated from each other. The couplings

between these Majorana fermions, which is equivalent to the process of tunneling of a Majorana quasiparticle from one place to another, split the topological degeneracy. The energy splitting determines the fusion channel of two non-Abelian vortices. However, such tunneling process has to overcome the bulk gap, very similar to tunneling of a quantum-mechanical particle through a potential barrier that is higher than its kinetic energy. So the splitting is exponentially suppressed in the topological phase as  $e^{-R/\xi}$  where  $R$  is the separation between anyons and  $\xi$  is the correlation length (the coherence length in a superconductor).

Another possibility is thermal excitations of non-Majorana fermionic modes. The process (1.33) has non-vanishing probability to occur at finite temperature and as such, represents a thermal decoherence of Majorana qubits. This issue is particularly pronounced when there are low-energy (but not zero) bound states present together with the Majorana zero-energy states which is the case in superconducting vortices.

At a more fundamental level, the BCS theory which all our discussions are based on, is a mean-field theory neglecting all quantum and thermal superconducting fluctuations. In three-dimensional electronic superconductors the fluctuations are gapped due to the famous Anderson-Higgs mechanism [69] and the relevant energy scale is the plasmon frequency. However, since non-Abelian topological superconductors all exist in dimensions smaller than three, the fluctuation effect needs to be reconsidered. For example, in quasi-two-dimensional superconductors the London penetration length is inversely proportional to the thickness of the system. In the limit of vanishing thickness, the superconducting fluctuations are essentially

gapless and should not be neglected for low-energy physics.

## 1.7 Outline of the Thesis

In this thesis, we present a systematic study of non-Abelian topological superconductors, focusing on the interplay between non-topological aspects and the topological degrees of freedom, motivated by the question of how these non-universal effects affect topological quantum computation. The model system that is primarily concerned is the chiral  $p$ -wave superconductor.

In Chapter 2 we address the question of how chiral  $p$ -wave superconductivity arises from microscopic lattice models with four-fermion interaction. By analyzing the BCS energetics of a minimal “Hubbard” model of spinless fermions with nearest-neighbor interaction on a two-dimensional lattice, we show that the pairing symmetry selected by the energetics strongly depends on the spatial symmetry. We prove a general theorem specifying a set of necessary conditions that guarantees the existence of chiral  $p$ -wave superconductivity.

In Chapter 3, we review the Majorana bound states as zero-energy solutions of BdG equations when a superconducting vortex is present in the order parameter. These analytical expressions will be used frequently in the rest of the thesis.

From Chapter 4 we turn to the study of non-topological effects. In Chapter 4 we calculate the splitting of topological degeneracy due to quasiparticle tunneling when there are multiple non-Abelian vortices. We devise a WKB-like method to calculate the energy splitting for  $p_x + ip_y$  superconductors and also TI/Superconductor

heterostructure. The splitting shows interesting oscillations with inter-vortex separation besides the well-known exponential decay. The implication of such oscillation on the fusion of non-Abelian anyons is discussed.

In Chapter 5 we consider the thermal effects on the stability of Majorana-based topological qubits. Thermal excitations of subgap states and possibly the continuum states can change the quantum information stored in the topological qubits resulting in qubit decoherence. We first consider the effect of thermally excited subgap, localized states and by exploiting a density matrix formulation show that the topological qubits are robust against this type of thermal excitations. However, they do have a destructive effect on the read out of qubits which is demonstrated explicitly in a measurement scheme based on vortex interferometry. We then analyze the depolarization of qubits due to coupling to a fluctuating environment modeled by a collection of bosonic modes and derive the master equation for the reduced density matrix of the topological qubit. The decay rate is shown to be exponentially suppressed at temperatures much lower than the bulk gap.

In Chapter 6, we ask the question of what is the correction to the braiding operations due to non-adiabaticity. To answer this question, we develop the framework of time-dependent Bogoliubov equation to track the time evolution of various physical quantities in terms of Bogoliubov wavefunctions. We then derive analytical expressions for the time evolutions of the Majorana quasiparticles when non-adiabaticity is taken into account, so tunneling splitting and fermionic excitations can not be neglected.

In Chapter 7, we consider the effect of quantum fluctuations on Majorana



zero modes. We study a one-dimensional descendant of a chiral  $p$ -wave superconductor where the effect of quantum fluctuations is mostly pronounced. We found by bosonization technique that, in the absence of long-range superconducting order, there can still be degeneracies of low-energy eigenstates that should be related to Majorana zero modes on the edges. We show the explicit forms of the zero modes in a strongly interacting but still exactly-solvable case, the Luther-Emery liquid. We then discuss the stability of the zero modes under various perturbations.

In Chapter 8 we present our conclusions and discuss possible future research directions and open problems.

## Chapter 2

# Topological Superconductivity in Fermionic

## Lattice Models

Topological SCs, most notably  $p_x + ip_y$  models, have been considered in the theoretical literature in great detail. However, the starting point of all theoretical models has been a quadratic mean-field Hamiltonian, with a predetermined topological order parameter of interest, or equivalently a reduced BCS Hamiltonian with exotic interactions that are difficult to imagine being realized in the laboratory. Such models are capable of answering some key questions related to the properties of a given topological phase, but they do not provide much guidance in the search of Hamiltonians that would host those phases. In other words, these models are sufficient to produce nontrivial topological order by design, but do not shed light on the minimal necessary conditions for the emergence of topological order.

In this chapter we prove a general theorem that allows us to construct a large family of lattice models that give rise to topological superconducting states. We show that contrary to a common perception, the nontrivial topological phases do not necessarily arise from exotic Hamiltonians, but instead appear naturally within a range of simple models of spinless (or spin-polarized) fermions with physically reasonable interactions. Our theorem is based on examining the BCS free energy

of possible paired states which is known to be asymptotically exact for weak coupling since BCS instability is an infinitesimal instability and the use of the Jensen’s inequality, which ensures that topological phases are often selected naturally by energetics.

## 2.1 Spinless Fermion Superconductivity in Two dimensions

We start our general discussion with the following single-band Hamiltonian for spinless fermions

$$\hat{\mathcal{H}} = \int_{\mathbf{k} \in \text{BZ}} \xi_{\mathbf{k}} \hat{c}_{\mathbf{k}}^{\dagger} \hat{c}_{\mathbf{k}} + \frac{1}{2} \int_{\mathbf{q}/2, \mathbf{k}, \mathbf{k}' \in \text{BZ}} f_{\mathbf{k}\mathbf{k}', \mathbf{q}} \hat{c}_{\mathbf{k}+\mathbf{q}}^{\dagger} \hat{c}_{-\mathbf{k}}^{\dagger} \hat{c}_{-\mathbf{k}'} \hat{c}_{\mathbf{k}'+\mathbf{q}}, \quad (2.1)$$

where  $\hat{c}_{\mathbf{k}}^{\dagger}/\hat{c}_{\mathbf{k}}$  are the fermion creation/annihilation operators corresponding to momentum  $\mathbf{k}$ , “BZ” stands for “Brillouin zone,”  $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$  with  $\epsilon_{\mathbf{k}}$  being the dispersion relation of the fermions and  $\mu$  the chemical potential, and  $f_{\mathbf{k}, \mathbf{k}', \mathbf{q}}$  describes an interaction, which is assumed to have an attractive channel.

We assume that the Hamiltonian (2.1) arises from a real-space lattice or continuum model and is invariant with respect to the underlying spatial symmetry group, which we denote as  $\mathbb{G}$ , and the time-reversal group,  $\mathbb{T}$ . We note that in two dimensions (2D) the range of possible spatial groups,  $\mathbb{G}$ , is limited to the following dihedral point-symmetry groups:  $D_1, D_2, D_3, D_4$ , and  $D_6$  in the case of a lattice or orthogonal group of rotations  $O(2) = D_{\infty}$  in continuum. We recall that the group  $D_n$  includes  $\frac{360^\circ}{n}$ -rotations and in-plane reflections with respect to  $n$  axes. The superconducting order parameter is classified according to the irreducible representations of the full group  $\mathbb{T} \otimes \mathbb{G}$ . Since,  $\mathbb{T} = \mathbb{Z}_2, \mathbb{Z}_2 \otimes D_1 = D_2$  and  $\mathbb{Z}_2 \otimes D_3 = D_6$ , we

can confine ourselves to studying representations of  $D_2$ ,  $D_4$ ,  $D_6$ , and  $O(2)$ , which exhaust all physically relevant possibilities.

### 2.1.1 BCS Mean Field Theory

Now, we define the superconducting order parameter as

$$\Delta_{\mathbf{k}} = \int_{\mathbf{k}' \in \text{BZ}} \tilde{f}_{\mathbf{k},\mathbf{k}'} \langle \hat{c}_{-\mathbf{k}'} \hat{c}_{\mathbf{k}'} \rangle, \quad (2.2)$$

where  $\tilde{f}_{\mathbf{k},\mathbf{k}'} = (f_{\mathbf{k},\mathbf{k}',\mathbf{q}=0} - f_{\mathbf{k},-\mathbf{k}',\mathbf{q}=0})/2$  is the antisymmetrized BCS coupling strength. Using a Hubbard-Stratonovich decoupling in Eq. (2.1) with  $\mathbf{q} = \mathbf{0}$  and ignoring superconducting fluctuations, we arrive at

$$\hat{\mathcal{H}}^{\text{MF}} = \int_{\mathbf{k} \in \text{BZ}} \left( \xi_{\mathbf{k}} \hat{c}_{\mathbf{k}}^\dagger \hat{c}_{\mathbf{k}} + \frac{1}{2} \Delta_{\mathbf{k}} \hat{c}_{\mathbf{k}}^\dagger \hat{c}_{-\mathbf{k}}^\dagger + \frac{1}{2} \Delta_{\mathbf{k}}^* \hat{c}_{-\mathbf{k}} \hat{c}_{\mathbf{k}} \right) - \frac{1}{2} \int_{\mathbf{k},\mathbf{k}' \in \text{BZ}} \Delta_{\mathbf{k}}^* \tilde{f}_{\mathbf{k},\mathbf{k}'}^{-1} \Delta_{\mathbf{k}'}, \quad (2.3)$$

with  $\tilde{f}_{\mathbf{k},\mathbf{k}'}^{-1}$  being the matrix inverse of  $\tilde{f}_{\mathbf{k},\mathbf{k}'}$ . By integrating out the fermions we find the BCS free energy functional expressed in terms of  $\Delta$ . It contains two parts,  $\mathcal{F}[\Delta_{\mathbf{k}}] = \mathcal{F}_I + \mathcal{F}_{II}$ , with

$$\mathcal{F}_I[\Delta_{\mathbf{k}}] = -T \int_{\mathbf{k} \in \text{BZ}} \ln \left[ 2 \cosh \frac{1}{2T} \sqrt{\xi_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2} \right] \quad (2.4)$$

$$\mathcal{F}_{II}[\Delta_{\mathbf{k}}] = -\frac{1}{2} \int_{\mathbf{k},\mathbf{k}' \in \text{BZ}} \Delta_{\mathbf{k}}^* \tilde{f}_{\mathbf{k},\mathbf{k}'}^{-1} \Delta_{\mathbf{k}'}. \quad (2.5)$$

### 2.1.2 Topological Classification

As we have reviewed in Chapter 1, 2D superconductors in class D is characterized by the topological invariant (the Chern number) as follows [70]

$$C = \int_{\mathbf{k} \in \text{BZ}} \frac{d^2 \mathbf{k}}{4\pi} \mathbf{d} \cdot \partial_{k_x} \mathbf{d} \times \partial_{k_y} \mathbf{d}, \quad (2.6)$$

where  $\mathbf{d} \equiv (d_1, d_2, d_3) = (\Re\Delta_{\mathbf{k}}, -\Im\Delta_{\mathbf{k}}, \xi_{\mathbf{k}})/E_{\mathbf{k}}$  and  $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2}$ . In fact it has a geometric interpretation: this topological index classifies all maps from  $T^2$  to  $S^2$  representing the unit vector  $\mathbf{d}(\mathbf{k})$  into equivalent homotopy classes. We will call a SC state topological, if  $C \neq 0$ .

The Chern number is equal to the sum of the winding numbers,  $C = \sum_{\sigma} W_{\sigma}$ , which can be defined for each segment of the Fermi surface (FS),  $\mathcal{P}_{\sigma}$ , as follows:

$$2\pi W_{\sigma} = \oint_{\mathcal{P}_{\sigma}} \nabla_{\mathbf{k}} \varphi_{\mathbf{k}} \cdot d\mathbf{k} \quad (2.7)$$

where  $\varphi_{\mathbf{k}}$  is the complex phase of  $\Delta_{\mathbf{k}}$ . Note that even though we assume a single-band picture, a general situation is allowed where the FS is formed by one or more disconnected components,  $\text{FS} = \sum_{\sigma} \mathcal{P}_{\sigma}$  with  $\sigma = 1, 2, \dots, n$ .

To prove the relation between  $C$  and  $W_{\sigma}$ 's, we separate the closed Brillouin zone,  $\partial(\text{BZ}) = \partial(S^1 \times S^1) = 0$ , into an ‘‘electron’’ region,  $E_{\text{BZ}} = \{\mathbf{k} \in \text{BZ} : d_3(\mathbf{k}) > 0\}$  and a ‘‘hole’’ region,  $H_{\text{BZ}} = \{\mathbf{k} \in \text{BZ} : m_3(\mathbf{k}) < 0\}$ . The Fermi surface is a directed boundary of these regions,  $\text{FS} = \sum_{\sigma} \mathcal{P}_{\sigma} = \partial E_{\text{BZ}} = -\partial H_{\text{BZ}}$ . One can show that

$$C = \frac{1}{2} \left[ \int_{\mathbf{k} \in E_{\text{BZ}}} - \int_{\mathbf{k} \in H_{\text{BZ}}} \right] \nabla_{\mathbf{k}} \times \left[ \frac{d_1 \nabla_{\mathbf{k}} d_2 - d_2 \nabla_{\mathbf{k}} d_1}{1 + |d_3|} \right]. \quad (2.8)$$

Eq. (2.8) and the Stoke's theorem [71] yield  $C = \sum_{\sigma} W_{\sigma}$ .

If  $W_{\sigma} = 0$  for all  $\sigma$ , the complex phase of the pairing order parameter can be gauged away via a non-singular redefinition of the fermion fields and corresponds to a topologically trivial state. This however is impossible if at least one winding number is non-zero. We will call such states time-reversal-symmetry breaking (TRSB) states. The class of TRSB superconductors is larger than and includes that of closely related

topological SCs. If there is just one singly-connected FS, the two types of states are equivalent.

### 2.1.3 General Theorem of the Stability of chiral SC States

Now we examine the stability of TRSB SCs. The order parameter in a certain channel corresponding to a  $d_\Gamma$ -dimensional irreducible representation,  $\Gamma$ , of the group  $\mathbb{T} \otimes \mathbb{G}$  can be written as a linear combination of *real* eigenfunctions of  $\Gamma$ ,  $\phi_a^\Gamma(\mathbf{k})$  (with  $a = 1, \dots, d_\Gamma$ )

$$\Delta_{\mathbf{k}} = \sum_{a=1}^{d_\Gamma} \lambda_a \phi_a^\Gamma(\mathbf{k}). \quad (2.9)$$

In two dimensions, the number of irreducible representations to be considered is highly constrained and includes only 1D and 2D real representations. In particular: (i) For a system with a four-fold rotational symmetry (*e. g.*, arising from a square lattice), the corresponding point group,  $D_4$ , has only one space-inversion-odd irreducible representation,  $E$ , which is two-dimensional; (ii) With a six-fold rotational symmetry (*e. g.*, due to a triangular or hexagonal lattice), there exist three irreducible representations of  $D_6$  odd under space inversion: A 2D representation,  $E_1$  (corresponding to a  $p$ -wave pairing) and two 1D representations,  $B_1$  and  $B_2$  (corresponding to two types of  $f$ -wave pairing). (iii) The continuum group,  $O(2)$ , has an infinite set of 2D real representations, classified by odd orbital momenta,  $l = 1, 3, 5, \dots$

We now consider a pairing channel corresponding to a 2D representation of  $\mathbb{T} \otimes \mathbb{G}$ . There are two real eigenfunctions for this representation:  $\phi_1(\mathbf{k})$  and  $\phi_2(\mathbf{k})$ .

If the order parameter is proportional to either of them, it is real and corresponds to a topologically trivial state with zero winding number. We prove below that such a state is always unstable. The invariance of the Hamiltonian under  $\mathbb{T} \otimes \mathbb{G}$  ensures  $\mathcal{F}_{\text{Non-top}} = \mathcal{F}[\phi_1(\mathbf{k})] = \mathcal{F}[\phi_2(\mathbf{k})]$  (e. g.,  $p_x$ - and  $p_y$ -states have the same energies in continuum).

Let us show that one can always construct a new TRSB state with

$$\phi_{\text{TRSB}}(\mathbf{k}) = \frac{1}{\sqrt{2}}[\phi_1(\mathbf{k}) + i\phi_2(\mathbf{k})]$$

that has a lower free energy than  $\mathcal{F}_{\text{Non-top}}$ . One can see from Eq. (2.5) that  $\mathcal{F}_{II}[\phi_{\text{TRSB}}(\mathbf{k})] = \mathcal{F}_{II}[\phi_1(\mathbf{k})] = \mathcal{F}_{II}[\phi_2(\mathbf{k})]$  because  $\phi_{\text{TRSB}}^2(\mathbf{k}) = \phi_1^2(\mathbf{k})/2 + \phi_2^2(\mathbf{k})/2$ . To handle the less trivial “quasiparticle part” of the free energy (2.4) we take advantage of the Jensen’s inequality which states that for any function with  $f''(x) < 0$ ,  $f(x/2 + y/2) < f(x)/2 + f(y)/2$  for any  $x \neq y$ . The integrand in Eq. (2.4) for  $\mathcal{F}_I$  is a concave function of  $x = |\Delta_{\mathbf{k}}|^2$  and therefore satisfies the Jensen’s inequality (which after integration over momentum becomes a strong inequality for all physically relevant cases). Since  $\phi_{\text{TRSB}}^2(\mathbf{k}) = \phi_1^2(\mathbf{k})/2 + \phi_2^2(\mathbf{k})/2$ , we have proven that

$$\mathcal{F}[\phi_{\text{TRSB}}(\mathbf{k})] < \frac{\mathcal{F}[\phi_1(\mathbf{k})] + \mathcal{F}[\phi_2(\mathbf{k})]}{2} \equiv \mathcal{F}_{\text{Non-top}}. \quad (2.10)$$

This inequality (to which we refer to as “theorem”) represents the main result of our work and proves that a TRSB phase is always energetically favorable within a 2D representation. This is a strong statement that is completely independent of microscopic details, such as hoppings and interactions, and relies only symmetry. It leads, in particular, to the conclusion that any single-band spinless SC (and certain models of spin-polarized SCs) originating from a square lattice with singly-connected

FS must be a  $(p + ip)$ -paired state. Similarly, any SC arising from spinless fermions in continuum must be of a  $(2l + 1) + i(2l + 1)$ -type, which is topologically nontrivial. This includes all continuum models with attractive forces and conceivably some continuum models with weak repulsion that may give rise to pairing via Kohn-Luttinger mechanism [72, 73, 74]. Since a large number of lattice fermion Hamiltonians at low particle densities reduce to an effective single-band continuum model, it means that at least in this low-density regime any paired state is guaranteed to be topological.

## 2.2 Lattice Models

To illustrate how our theorem manifests itself in practice, we examine specific models within a large class of generic tight-binding Hamiltonians on a lattice

$$\hat{\mathcal{H}} = - \sum_{\mathbf{r}, \mathbf{r}'} t_{\mathbf{r}, \mathbf{r}'} \hat{c}_{\mathbf{r}}^\dagger \hat{c}_{\mathbf{r}'} - \mu \sum_{\mathbf{r}} \hat{c}_{\mathbf{r}}^\dagger \hat{c}_{\mathbf{r}} + \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} V_{\mathbf{r}, \mathbf{r}'} \hat{c}_{\mathbf{r}}^\dagger \hat{c}_{\mathbf{r}'}^\dagger \hat{c}_{\mathbf{r}'} \hat{c}_{\mathbf{r}},$$

where  $\hat{c}_{\mathbf{r}}^\dagger/\hat{c}_{\mathbf{r}}$  creates/annihilates a fermion on a lattice site  $\mathbf{r}$ . We note that this real-space Hamiltonian reduces to a more general model (2.1) via a lattice Fourier-transform. For the sake of concreteness, we focus below on the following two models with nearest-neighbor hoppings,  $t_{\mathbf{r}, \mathbf{r}'} = t\delta_{|\mathbf{r}-\mathbf{r}'|, 1}$  and nearest-neighbor attraction,  $V_{\mathbf{r}, \mathbf{r}'} = -g\delta_{|\mathbf{r}-\mathbf{r}'|, 1}$  on (i) a simple square lattice and (ii) a simple triangular lattice.

### 2.2.1 Square Lattice

The square lattice case corresponds to the  $D_4$  symmetry group, which has only a 2D representation. The attractive interaction guarantees that the ground state is a SC [?] and the general theorem (2.10) guarantees that it is topologically non-trivial.



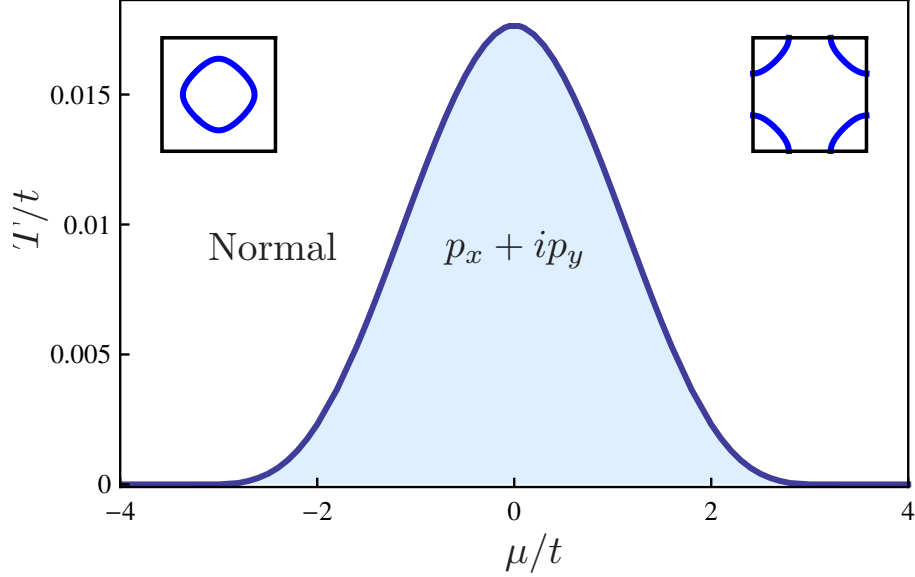


Figure 2.1: The phase diagram for fermions on a square lattice with nearest-neighbor hoppings and attraction ( $g/t=1$ ). The phase boundary separates a normal metal and a topological  $(p_x + ip_y)$ -wave SC. The insets display FSs for  $\mu < 0$  (left) and  $\mu > 0$  (right).

To see how this happens in the specific model, we define two independent order parameters on horizontal and vertical links:  $\Delta_n = g\langle\hat{c}_{\mathbf{r}}\hat{c}_{\mathbf{r}+\mathbf{e}_n}\rangle$ , where  $n = x$  or  $y$  and  $\mathbf{e}_n$  is the corresponding lattice vector (we use units where the lattice constant,  $a = 1$ ). These real-space order parameters are related to the momentum-space definition (2.2) via  $\Delta_{\mathbf{k}} = 2i \sum_{\alpha=x,y} \Delta_{\alpha}\phi_{\alpha}(\mathbf{k})$ , with the BCS interaction being  $\tilde{f}_{\mathbf{k},\mathbf{k}'} = -g \sum_{\alpha=x,y} \phi_{\alpha}(\mathbf{k})\phi_{\alpha}(\mathbf{k}')$ . Here we defined two eigenfunctions of the above-mentioned 2D representation of  $D_4$ :  $\phi_{x,y}(\mathbf{k}) = \sin(\mathbf{k} \cdot \mathbf{e}_{x,y})$ . It is straightforward to calculate the BCS free energy given by Eqs. (2.4) and (2.5) for all possible order parameters encompassed by the linear combinations  $\Delta_{\mathbf{k}} = g[\lambda_x\phi_x(\mathbf{k}) + \lambda_y\phi_y(\mathbf{k})]$ , with arbitrary  $\lambda_{x,y} \in \mathbb{C}$ . We find that a  $p_x + ip_y$ -superconducting state with  $\lambda_x = \pm i\lambda_y$  is selected at

all  $\mu$ . Fig. 2.1 summarizes the phase diagram of the model on the  $\mu - T$  plane. The maximum  $T_c$  within the mean-field treatment occurs at half-filling. The tails of the particle-hole symmetric phase boundary correspond to small “electron” and “hole” densities, and therefore to continuum limit with the isotropic quadratic dispersion,  $\xi_{\mathbf{k}} = (k^2 - k_F^2)/(2m^*)$ , the effective mass,  $m^* = 1/(2ta^2)$ , and the Fermi momentum,  $k_F a = \sqrt{|\mu \pm 4t|/(2t)}$ .

It is useful to consider the continuum limit  $|\mu \pm 4t|/t \rightarrow 0$  in more detail, as it gives a valuable insight into stability of the topological phases. For this purpose, we use standard perturbative expansion [75] in Eqs. (2.4) and (2.5) to derive the Ginzburg-Landau free energy (per unit area):

$$\frac{1}{\mathcal{A}} \mathcal{F}_{\text{GL}}[\Delta_0, \mathcal{S}] = \nu (T/T_c - 1) \Delta_0^2 + \frac{7\zeta(3)\nu}{8\pi^2 T^2} \mathcal{S}^{-1} \Delta_0^4, \quad (2.11)$$

where  $\nu = m^*/(2\pi)$  is the density of states at the FS,  $T_c$  is the BCS transition temperature,  $\zeta$  is the Riemann zeta-function,  $\Delta_0 = g\sqrt{|\lambda_x|^2 + |\lambda_y|^2}$  is the modulus of the order parameter,  $\mathcal{A}$  is the area of the sample, and we introduced a symmetry factor,  $\mathcal{S}$ , as follows [below,  $\theta_{\mathbf{k}} = \tan^{-1}(k_y/k_x)$ ]

$$\mathcal{S}^{-1} = \oint_{\mathbf{k} \in \text{FS}} \frac{d\theta_{\mathbf{k}}}{2\pi} \frac{|\lambda_x \phi_x(\mathbf{k}) + \lambda_y \phi_y(\mathbf{k})|^4}{|\lambda_x|^2 + |\lambda_y|^2}. \quad (2.12)$$

The minimal free energy below  $T_c$  is given by

$$\frac{\mathcal{F}_{\text{GL, min}}}{\mathcal{A}} = -\frac{\mathcal{S}_{\text{max}} [4\pi\nu T_c^2 \ln^2(T/T_c)]}{7\zeta(3)}. \quad (2.13)$$

Therefore, the absolute minimum is achieved by *maximizing the symmetry factor*,  $\mathcal{S}$ .

In the continuum limit  $|\mathbf{k}|a \rightarrow 0$ , we can approximate the normalized eigenfunctions of  $D_4$ , by  $\phi_x(\mathbf{k}) = \sqrt{2} \cos(\theta_{\mathbf{k}})$  and  $\phi_y(\mathbf{k}) = \sqrt{2} \sin(\theta_{\mathbf{k}})$ . Hence, the topologically

trivial  $p_x$ - and  $p_y$ -states lead to  $\mathcal{S}_{p_{x,y}} = \langle 4 \cos^4 \theta_{\mathbf{k}} \rangle_{\text{FS}}^{-1} = 2/3$ , while the topological states  $p_x \pm ip_y$  yield  $\mathcal{S}_{p_x \pm ip_y} = \langle |e^{\pm i\theta_{\mathbf{k}}}|^4 \rangle_{\text{FS}}^{-1} = 1 > 2/3$  and therefore are selected by energetics. This fact is a special case of our general theorem summarized by Eq. (2.10).

We note that the mean-field BCS-type model can formally be considered for the extreme values of the non-interacting chemical potential  $|\mu| > 4t$ , which is not associated with a non-interacting FS. Hence, mean-field paired states in this limit are not topological and correspond to the strong-pairing (Abelian)  $(p + ip)$ -phase considered by Read and Green [19]. While such a mean-field BCS model is sensible in the context of the quantized Hall state, it may be unphysical for fermion lattice models. Indeed, the chemical potential,  $\mu$ , is renormalized by non-BCS interactions or equivalently by superconducting fluctuations originating from the terms with  $\mathbf{q} \neq \mathbf{0}$  in Eq. (2.1). These strong renormalizations are bound to shift  $\mu$  towards the physical values with a reasonable Fermi surface, which in a metal is guaranteed by Luttinger theorem. Hence, it is not clear whether the Abelian  $p_x + ip_y$  superconducting states may survive beyond mean-field. Due to these arguments, we disregard here such case of non-topological  $p_x + ip_y$ -paired states.

We now derive Bogoliubov-de Gennes equations from the lattice model. These equations are often the starting point of discussions on bound states in a vortex core [33, 76] and edge states [77]. To do so, we first present the fermionic mean-field BCS Hamiltonian on a lattice as follows:

$$\hat{\mathcal{H}}^{\text{MF}} = \frac{1}{2} \sum_{\mathbf{r}\mathbf{r}'} (\hat{c}_{\mathbf{r}}^\dagger h_{\mathbf{r}\mathbf{r}'} \hat{c}_{\mathbf{r}'} - \hat{c}_{\mathbf{r}'} h_{\mathbf{r}\mathbf{r}'} \hat{c}_{\mathbf{r}}^\dagger + \Delta_{\mathbf{r}\mathbf{r}'} \hat{c}_{\mathbf{r}} \hat{c}_{\mathbf{r}'} + \text{h.c.})$$

which is a real space version of Eq. (2.3) where  $\Delta_{\mathbf{r}\mathbf{r}'} \equiv g\langle\hat{c}_{\mathbf{r}}\hat{c}_{\mathbf{r}'}\rangle$  is the order parameter on the bond  $(\mathbf{r}\mathbf{r}')$  and  $h_{\mathbf{r}\mathbf{r}'} = -t\delta_{|\mathbf{r}-\mathbf{r}'|,1} - \mu\delta_{\mathbf{r}\mathbf{r}'}$  is the matrix element of the single-particle Hamiltonian. We then follow the standard route and introduce Bogoliubov's transform  $\hat{c}_{\mathbf{r}} = \hat{\gamma}u_{\mathbf{r}} + \hat{\gamma}^\dagger v_{\mathbf{r}}^*$  and the commutation relation  $[\hat{\mathcal{H}}^{\text{MF}}, \hat{\gamma}] = -E\hat{\gamma}$ . This yields gives the desired BdG equations

$$\begin{aligned} Eu_{\mathbf{r}} &= \sum_{\mathbf{r}'} (h_{\mathbf{r}\mathbf{r}'}u_{\mathbf{r}'} + \Delta_{\mathbf{r}\mathbf{r}'}v_{\mathbf{r}'}) \\ Ev_{\mathbf{r}} &= \sum_{\mathbf{r}'} (-\Delta_{\mathbf{r}\mathbf{r}'}^*u_{\mathbf{r}'} - h_{\mathbf{r}\mathbf{r}'}v_{\mathbf{r}'}) \end{aligned} \tag{2.14}$$

In principle, the order parameter  $\Delta_{\mathbf{r}\mathbf{r}'}$  should be determined via solving BdG equation self-consistently. However, we know that in a homogeneous ground state the order parameter has a  $p_x + ip_y$ -wave pairing symmetry, i.e.,  $\Delta_y = \pm i\Delta_x$ . If there are inhomogeneities in the system (e.g., vortices, domain walls) the pairing symmetry (associated with the relative phase between  $\Delta_y$  and  $\Delta_x$  components) is not necessarily  $p_x + ip_y$ . But since this pairing symmetry is selected by energetics, we expect such deviation to be irrelevant for low energy physics. Therefore we can assume that the relation  $\Delta_y = \pm i\Delta_x$  holds for general configurations of order parameter at the mean-field level. This is equivalent to separation of the Cooper pair wave function into parts corresponding to the center-of-mass motion and relative motion.

Now we take the continuum limit of (2.14):  $\sum_{\mathbf{r}'} h_{\mathbf{r}\mathbf{r}'}u_{\mathbf{r}'} \rightarrow \hat{\xi}(-i\nabla)u(\mathbf{r}) = (-\nabla^2/2m^* - \tilde{\mu})u(\mathbf{r})$ , where  $m^*$  is the effective mass and  $\tilde{\mu} = \mu + 4t$  is the chemical potential measured from the bottom of the band. To treat the off-diagonal part, we formally represent the second term in Eq. (2.14.1) as follows  $\sum_{\mathbf{r}'} \Delta_{\mathbf{r}\mathbf{r}'}v_{\mathbf{r}'} = \hat{\Delta}v(\mathbf{r})$ ,

with the gap operator being

$$\hat{\Delta} = \sum_{\mathbf{r}'} \Delta_{\mathbf{r}\mathbf{r}'} e^{(\mathbf{r}' - \mathbf{r}) \cdot \partial_{\mathbf{r}}}. \quad (2.15)$$

The order parameter  $\Delta_{\mathbf{r}\mathbf{r}'}$ , which “lives” on bonds, should be casted into only site-dependent form as follows:

$$\Delta_{\mathbf{r}\mathbf{r}'} = \Delta \left( \frac{\mathbf{r} + \mathbf{r}'}{2} \right) \exp(i\theta_{\mathbf{r}' - \mathbf{r}}) \quad (2.16)$$

where  $\theta_{\mathbf{r}' - \mathbf{r}}$  is the polar angle of  $\mathbf{r}' - \mathbf{r}$ . Then, we expand (2.15) to first order in  $|\mathbf{r}' - \mathbf{r}| = a$  and obtain the familiar BdG equations in continuum:

$$\begin{aligned} Eu(\mathbf{r}) &= \hat{\xi}(-i\nabla)u(\mathbf{r}) + \hat{\Delta}v(\mathbf{r}) \\ Ev(\mathbf{r}) &= \hat{\Delta}^\dagger u(\mathbf{r}) - \hat{\xi}(-i\nabla)v(\mathbf{r}) \end{aligned} \quad (2.17)$$

where the gap operator  $\hat{\Delta} = a\{\Delta(\mathbf{r}), \partial_x + i\partial_y\}$ . An interesting question to be addressed elsewhere is whether fluctuations and in particular deviations of pairing symmetry from  $p_x + ip_y$  play a role in the topological properties.

### 2.2.2 Triangular Lattice

We now address the very interesting case of a simple triangular lattice. Here the  $D_6$  symmetry group has both a 2D representation ( $p$ -wave) and two 1D representations ( $f$ -wave). Therefore, non-topological  $f$ -wave states are allowed. Low “electron” densities correspond to a single circular-shaped Fermi surface and must lead to the  $p + ip$ -wave pairing per the same argument as above. However, the spectrum of the model is not particle-hole symmetric and at large fillings (with  $\mu > \mu^* = 2t$ ), the electron Fermi surface splits into two hole-like Fermi pockets and

maps onto an effective continuum model but with two fermion species:

$$\hat{\mathcal{H}}_{2h,\text{eff}} = \int_{\mathbf{k}} (\xi_{\mathbf{k}} \hat{h}_{+,\mathbf{k}}^\dagger \hat{h}_{+,\mathbf{k}} + \xi_{-\mathbf{k}} \hat{h}_{-,\mathbf{k}}^\dagger \hat{h}_{-,\mathbf{k}}) + \text{interactions}, \quad (2.18)$$

where  $\hat{h}_{\pm,\mathbf{k}}$  are fermion operators near the two pockets labeled by a pseudospin index  $\sigma = \pm$  and the spectrum is asymptotically given by

$$\xi_{\mathbf{k}} = k^2/2m + \alpha(k_x^3 - 3k_x k_y^2) - E_F, \quad (2.19)$$

with  $\mathbf{k}$  measured from the corner points of the hexagonal Brillouin zone. Note that under a  $\pi$  or  $\pm\pi/3$  rotation, the spectrum transforms as  $\xi_{\mathbf{k}} \rightarrow \xi_{-\mathbf{k}}$  and this symmetry is preserved if  $\sigma \rightarrow -\sigma$ . This leads to a pairing analogous to the  $s$ -wave pairing of spin-1/2 fermions, with the order parameter of the inter-pocket pairing defined as  $\Delta_h = g \int_{\mathbf{k}} \langle \hat{h}_{+,\mathbf{k}} \hat{h}_{-,-\mathbf{k}} \rangle$ . However, this is an  $f$ -wave pairing state, because under a  $\pi/3$ -rotation,  $\sigma \rightarrow -\sigma$  and  $\Delta_h(\mathbf{k})$  changes sign. Since the low-density limit leads to a topological phase and the high-density limit leads to an  $f$ -wave topologically trivial state, there must be a quantum phase transition in between. The entire phase diagram can be derived using Eqs. (2.4) and (2.5) and the real-space construction as follows: On a triangular lattice we can define three order parameters on the nearest neighbor bonds corresponding to the three lattice vectors,  $\mathbf{e}_n$  with azimuth angles  $2n\pi/3$  and  $n = 0, 1, 2$ :  $\Delta_n = g \langle \hat{c}_{\mathbf{r}} \hat{c}_{\mathbf{r}+\mathbf{e}_n} \rangle$ . Two different types of pairing channels are formed by these three order parameters: An  $f$ -wave channel with  $\Delta_n = \Delta$  and a  $p$ -wave channel with  $\Delta_n = \Delta e^{\pm 2\pi i n/3}$ .

The resulting phase diagram is shown in Fig. 2.2. As expected, a topological  $p_x + ip_y$ -wave SC state with  $\Delta_n = \Delta e^{\pm 2\pi i n/3}$  is stabilized at low fillings, while an

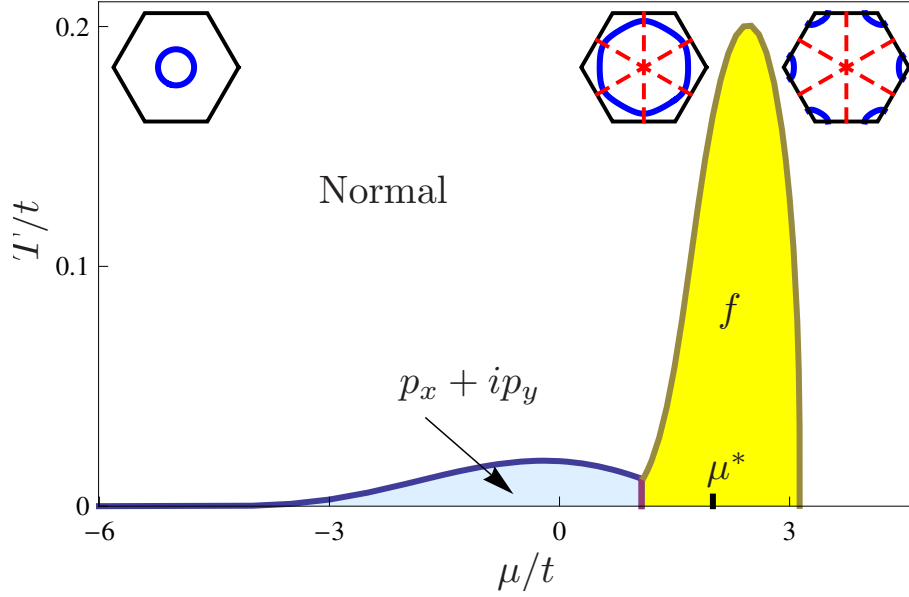


Figure 2.2: The phase diagram for spinless fermions on a triangular lattice with nearest-neighbor hoppings and attraction ( $g/t=1$ ). The bottom of the band is located at  $\mu = -6t$  and the top is at  $\mu = 3t$ ;  $\mu^* = 2t$  corresponds to a van Hove singularity. Two SC phases, with  $(p_x + ip_y)$ - and  $f$ -wave symmetries are present. They are separated by a first-order phase transition at  $\mu_{cr}/t \approx 1.057$ . The insets (left to right) are the FSs for  $\mu < \mu^*$ ,  $\mu \lesssim \mu^*$ , and  $\mu > \mu^*$  and the dashed lines indicate the nodal directions of the  $f$ -wave SC.

$f$ -wave state with  $\Delta_n = \Delta$  is favored at high densities. These phases are separated by a first-order transition. As shown in Fig. 2.2, the van Hove singularity  $\mu = \mu^*$  gives rise to a maximal  $T_c$  and is located inside the  $f$ -wave superconducting dome. This point represents another type of a quantum transition that separates two qualitatively different topologically trivial paired states: (1) For  $\mu < \mu^*$ , there is just one electron-type Fermi pocket that is cut by the nodes of the  $f$ -wave gap in the directions,  $\theta_{\text{node}}^{(m)} = m\pi/3 + \pi/6$ . This gives rise to gapless quasiparticles. (2) For  $\mu > \mu^*$ , no FS can be cut and the nodal quasiparticles disappear. The phase becomes fully gapped and eventually crosses over to the two-specie continuum model (2.18). Experimentally, the two types of  $f$ -wave phases can be distinguished by different  $T$ -dependence of the heat capacity.

We also present the Bogoliubov-de Gennes equations for the  $f$ -wave pairing state in high-density limit  $\mu \rightarrow 3t$ . Their derivation goes along the same lines as that given in Section 2.2.1 for  $p_x + ip_y$  pairing SC. However, in the  $f$ -wave case, the momentum space order parameter is given by  $\Delta_{\mathbf{k}} = \Delta(\sin \mathbf{k} \cdot \mathbf{e}_1 + \sin \mathbf{k} \cdot \mathbf{e}_2 + \sin \mathbf{k} \cdot \mathbf{e}_3)$ . Therefore, the order parameter reads:

$$\Delta_{\mathbf{r}\mathbf{r}'} = \Delta \left( \frac{\mathbf{r} + \mathbf{r}'}{2} \right) \cos(3\theta_{\mathbf{r}'-\mathbf{r}}).$$

Since  $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 = 0$ , the leading term in the expansion is  $\sim a^3$ . With some algebra one can show that the gap operator is

$$\hat{\Delta} = \frac{a^3}{24} \sum_{n=0}^2 \{ \partial_n, \{ \partial_n, \{ \partial_n, \Delta(\mathbf{r}) \} \} \} \quad (2.20)$$

with  $\partial_n \equiv \nabla \cdot \mathbf{e}_n$  and the BdG equation takes the form of Eq. (2.17).



## 2.3 Discussion and Conclusions

In conclusion, we discover that topological superconducting phases breaking time-reversal symmetry emerge naturally within a large class of spinless fermion models. The technique we apply here has a close relation to BCS mean field theory of a spin-triplet superfluid  $^3\text{He}$  [78, 44], which concluded that the B-phase with isotropic gap is stabilized compared to anisotropic A-phase (The A-phase can be stabilized under high pressure. In this case, due to strong spin-fluctuations in liquid  $^3\text{He}$ , the ground state energy departs from the BCS theory, which is not a contradiction to our conclusion). However, we have shown that similar conclusion can be generalized to any band structures, filling factors, and interactions, as long as the system satisfies proper (discrete) rotational group symmetries. More importantly, our proof is insensitive to the existence of the “nodes”. In continuum, it has been argued that a  $p_x$  state is unstable against the  $p_x + ip_y$  pairing state, because the former has nodes thus having smaller condensation energy. However, the stability of a nodeless  $p_x$  state, which could exist in lattice models, was unclear before this our work.

We should also emphasize that although the discussions above focus on spinless fermions, all the conclusions can be generalized to the triplet pairing channels of spin-1/2 fermions, because these pairing channels also correspond to the space-inversion odd representations of the symmetry group. In addition, we note that any pairing state that spontaneously breaks a lattice rotational symmetry must have at least one degenerate state for both spinless and spin-1/2 fermions. Our theorem indicates that

these type of states must have a complex pairing order parameter to be energetically stable.

## Chapter 3

# Majorana Bound States in Topological Defects

In this chapter, we review the analytic solutions of the Bogoliubov-de Gennes equation for Majorana zero modes in a  $p_x + ip_y$  superconductor and at a topological insulator/superconductor interface. From the explicit solutions we deduce the generic  $\mathbb{Z}_2$  classification of Majorana zero-energy modes in superconducting vortices, as well as the  $\mathbb{Z}$  classification for Dirac-type Hamiltonian when an additional chiral symmetry is present. Some of the results are used in the later chapters.

### 3.1 Bound states in $p_x + ip_y$ superconductors

The BdG equation for  $p_x + ip_y$  superconductor has been derived in:

$$\mathcal{H}_{\text{BdG}} \begin{pmatrix} u(\mathbf{r}) \\ v(\mathbf{r}) \end{pmatrix} = E \begin{pmatrix} u(\mathbf{r}) \\ v(\mathbf{r}) \end{pmatrix}, \quad (3.1)$$

where the explicit form of the BdG Hamiltonian in real space is given by

$$\mathcal{H}_{\text{BdG}} = \begin{pmatrix} -\frac{\nabla^2}{2m} - \mu & \frac{1}{k_F} \{\Delta(\mathbf{r}), \partial_x + i\partial_y\} \\ -\frac{1}{k_F} \{\Delta^*(\mathbf{r}), \partial_x - i\partial_y\} & \frac{\nabla^2}{2m} + \mu \end{pmatrix} \quad (3.2)$$

with anti-commutator being defined as  $\{a, b\} = (ab + ba)/2$ .

The particle-hole symmetry of BdG Hamiltonian is represented by  $\Xi = \tau_x K$  with  $K$  being complex conjugation operator and  $\tau_x$  being Pauli matrix in Nambu (particle-

hole) space [76]. That is, if  $\Psi = (u_n, v_n)^T$  is a solution of Eq. (3.1) with eigenvalue  $E_n$ , then  $\Xi\Psi = (v_n^*, u_n^*)^T$  must be a solution with the eigenvalue  $(-E_n)$ . Particularly, a non-degenerate zero energy state must obey the following constraint:  $\Xi\Psi = \lambda\Psi$ . Since  $\Xi^2 = 1$ , it implies that  $\lambda = \pm 1$ . If  $\lambda = -1$ , we can make a global gauge transformation and introduce Nambu spinors as  $\Psi' = i\Psi$  and then  $\Xi\Psi' = \Psi'$ . Thus, a non-degenerate zero energy state could always be made to satisfy  $u^* = v$  in an appropriate gauge.

We will now show that such zero energy states appear in the cores of vortices in chiral p-wave superconductors. The localized states in the vortex cores are known as Caroli-de-Gennes-Matricon states (CdGM) [32]. In conventional s-wave superconductors all CdGM states have non-zero energies [79]. However, due to the chirality of the order parameter  $p_x + ip_y$  superconductors can host zero-energy bound states [33, 79, 80, 76, 81]. Similar to the s-wave superconductors [32, 31], a vortex with vorticity  $l$  (i.e.  $l$  flux quanta  $\frac{hc}{2e}$  is trapped) can be modeled as

$$\Delta(\mathbf{r}) = f(r)e^{il\varphi}, \quad (3.3)$$

where  $\varphi$  is the phase of the order parameter and  $f(r)$  is the vortex profile which can be well approximated by  $f(r) = \Delta_0 \tanh(r/\xi)$  [31]. Here  $\Delta_0$  is the mean-field value of superconducting order parameter and  $\xi = v_F/\Delta_0$  is coherence length. Taking advantage of rotational symmetry, BdG equation can be decoupled into angular momentum channels. The wave function can be written as

$$\Psi_m(\mathbf{r}) = e^{im\varphi}(e^{in\varphi}u_m(r), e^{-in\varphi}v_m(r)), n = \frac{l+1}{2}. \quad (3.4)$$

As argued above, a non-degenerate zero mode requires  $\Xi\Psi = \Psi$  which can only be

satisfied for  $m = 0$ . The singlevalueness of the wavefunction then requires  $l$  to be an odd integer. We thus see that Majorana bound state only exist in vortices with odd vorticity, which justifies the  $\mathbb{Z}_2$  classification of Majorana zero modes in vortices.

The radial part of the BdG equations in  $m = 0$  channel then reads

$$\begin{pmatrix} -\frac{1}{2m}(\partial_r^2 + \frac{1}{r}\partial_r - \frac{n^2}{r^2}) - \mu & \frac{1}{k_F} \left[ f(r)(\partial_r + \frac{1}{2r}) + \frac{f'(r)}{2} \right] \\ -\frac{1}{k_F} f(r) \left[ (\partial_r + \frac{1}{2r}) + \frac{f'(r)}{2} \right] & \frac{1}{2m}(\partial_r^2 + \frac{1}{r}\partial_r - \frac{n^2}{r^2}) + \mu \end{pmatrix} \begin{pmatrix} u_0(r) \\ v_0(r) \end{pmatrix} = 0. \quad (3.5)$$

Given that the radial part of the BdG equation (3.5) is real, one can choose  $u_0(r)$  and  $v_0(r)$  to be real. Then the condition  $\Xi\Psi_0 = \Psi_0$  reduces to  $v_0 = \lambda u_0$  with  $\lambda = \pm 1$ . Using this constraint, the differential equation for  $u_0$  becomes:

$$\left\{ \left( \partial_r^2 + \frac{1}{r}\partial_r - \frac{n^2}{r^2} \right) - 2m\mu - \frac{2\lambda}{v_F} \left[ f \left( \partial_r + \frac{1}{2r} \right) + \frac{f'}{2} \right] \right\} u_0 = 0.$$

One can seek the solution of the above equation in the form

$$u(r) = \chi(r) \exp \left[ \lambda \int_0^r dr' f(r') \right], \quad (3.6)$$

which leads to

$$\chi'' + \frac{\chi'}{r} + \left( 2m\mu - \frac{f^2}{v_F^2} - \frac{n^2}{r^2} \right) \chi = 0. \quad (3.7)$$

Here the profile  $f(r) = \Delta_0 \tanh(r/\xi)$  vanishes at the origin and reaches  $\Delta_0$  away from vortex core region. For our purpose, it's sufficient to consider the behavior of solution outside the core region where  $f(r)$  is equal to its asymptotic bulk value  $\Delta_0$ . It is obvious now that  $\lambda = -1$  yields the only normalizable solution.

When  $\Delta_0^2 < 2m\mu v_F^2$  which is the case for weak-coupling BCS superconductors, Eq.(3.7) becomes first order Bessel equation. Thus, the solution is given by Bessel function of the first kind  $J_n(x)$ :

$$\chi(r) = \mathcal{N}_1 J_1(r\sqrt{2m\mu - \Delta_0^2/v_F^2}), \quad (3.8)$$

where  $\mathcal{N}_1$  is the normalization constant determined by the following equation

$$4\pi \int r dr |u_0(r)|^2 = 1. \quad (3.9)$$

Evaluation of the integral yields:

$$\mathcal{N}_1^2 = \frac{8}{3\pi k_1^2 \xi^4 {}_2F_1\left(\frac{3}{2}, \frac{5}{2}; 3; -k_1^2 \xi^2\right)} \quad (3.10)$$

with its asymptotes given by

$$\mathcal{N}_1^2 \sim \begin{cases} \frac{8}{3\pi k_1^2 \xi^4} & k_1 \xi \ll 1 \\ \frac{k_1}{2\xi} & k_1 \xi \gg 1 \end{cases}. \quad (3.11)$$

In the opposite limit  $\Delta_0^2 > 2m\mu v_F^2$ , the solution of Eq. (3.7) is given by first order imaginary Bessel function:

$$\chi(r) = \mathcal{N}_2 I_1(r \sqrt{\Delta_0^2/v_F^2 - 2m\mu}). \quad (3.12)$$

The function  $I_n(r)$  diverges when  $r \rightarrow \infty$ . But the radial wave function  $u_0(r)$  remains bounded as long as  $\mu > 0$ . This is consistent with the fact that  $\mu = 0$  separates topologically trivial phase ( $\mu < 0$ ) and non-Abelian phase ( $\mu > 0$ ) [19].

We now give expressions for  $\mathcal{N}_2$ . Explicitly, it is given by

$$4\pi \mathcal{N}_2^2 \int_0^\infty r dr I_1^2(k_2 r) e^{-2r/\xi} = 1, \quad (3.13)$$

where  $k_2 = \sqrt{\Delta_0^2/v_F^2 - 2m\mu}$ . Since  $\mu > 0$ ,  $k_2 \xi$  is always smaller than 1. We find  $\mathcal{N}_2$  is given by

$$\mathcal{N}_2^2 = \frac{8}{3\pi k_2^2 \xi^4 {}_2F_1\left(\frac{3}{2}, \frac{5}{2}; 3; k_2^2 \xi^2\right)} \quad (3.14)$$

We summarize this section by providing an explicit expression for zero energy eigenfunction:

$$\Psi_0(\mathbf{r}) = \chi(r) \exp \left[ i \left( \varphi - \frac{\pi}{2} \right) \tau_z - \frac{1}{v_F} \int_0^r dr' f(r') \right], \quad (3.15)$$

where  $\chi(r)$  is given by Eq.(3.8) for  $\Delta_0^2 < 2m\mu v_F^2$  and Eq.(3.12) for  $\Delta_0^2 > 2m\mu v_F^2$ .

Using the zero energy solution obtained for one vortex one can be easily write down wave function for multiple vortices spatially separated so that tunneling effects can be ignored. Assume there are  $2N$  vortices pinned at positions  $\mathbf{R}_i, i = 1, \dots, 2N$ . The superconducting order parameter can be represented as

$$\Delta(\mathbf{r}) = \prod_{i=1}^{2N} f(\mathbf{r} - \mathbf{R}_i) \exp \left[ i \sum_i \varphi_i(\mathbf{r}) \right], \quad (3.16)$$

where  $\varphi_i(\mathbf{r}) = \arg(\mathbf{r} - \mathbf{R}_i)$ . Near the  $k$ -th vortex core, the phase of the order parameter is well approximated by  $\varphi_k(\mathbf{r}) + \Omega_k$  with  $\Omega_k = \sum_{i \neq k} \varphi_i(\mathbf{R}_k)$  which is accurate in the limit of large inter-vortex separation. Then in the vicinity of  $k$ -th vortex core, a zero energy bound state can be found [20]:

$$\Psi_k(\mathbf{r}) = e^{-i\tau_z \frac{\pi}{2}} \chi(r_k) \exp \left[ -\frac{1}{v_F} \int_0^{r_k} dr' f(r') \right] \exp \left[ i \left( \varphi_k + \frac{\Omega_k}{2} \right) \tau_z \right]. \quad (3.17)$$

where  $r_k = |\mathbf{r} - \mathbf{R}_k|$ . Correspondingly, there are  $2N$  Majorana fermion modes localized in the vortex cores.

### 3.2 Bound states in the Dirac fermion model coupled with s-wave superconducting scalar field.

We now discuss the zero energy bound states emerging in the model of Dirac fermions interacting with the superconducting pairing potential. This model is

realized at the interface of a 3D strong topological insulator having an odd number of Dirac cones per surface and an s-wave superconductor [47]. Due to the proximity effect an interesting topological state is formed at the 2D interface between the insulator and superconductor. We will now discuss the emergence of Majorana zero energy states at the TI/SC heterostructure [47]. This model was also considered earlier in the high-energy context by Jackiw and Rossi [82].

Three dimensional time-reversal invariant topological insulators are characterized by an odd number of Dirac cones enclosed by Fermi surface [50, 52, 51]. The metallic surface state is described by the Dirac Hamiltonian. The non-trivial  $\mathbb{Z}_2$  topological invariant ensures the stability of metallic surface states against perturbations which preserve time-reversal symmetry. When chemical potential  $\mu$  is close to the Dirac point the TI/SC heterostructure can be modeled as [47, 83]:

$$\mathcal{H} = \hat{\psi}^\dagger (v\boldsymbol{\sigma} \cdot \mathbf{p} - \mu)\hat{\psi} + \Delta\hat{\psi}_\uparrow^\dagger\hat{\psi}_\downarrow^\dagger + \text{h.c.}, \quad (3.18)$$

where  $\psi = (\psi_\uparrow, \psi_\downarrow)^T$  and  $v$  is the Fermi velocity at Dirac point. The Bogoliubov-de Gennes equations are given by:

$$\mathcal{H}_{\text{BdG}}\Psi(\mathbf{r}) = E\Psi(\mathbf{r}) \quad (3.19)$$

$$\mathcal{H}_{\text{BdG}} = \begin{pmatrix} \sigma \cdot \mathbf{p} - \mu & \Delta \\ \Delta^* & -\sigma \cdot \mathbf{p} + \mu \end{pmatrix}, \quad (3.20)$$

where  $\Psi(\mathbf{r})$  is the Nambu spinor defined as  $\Psi = (u_\uparrow, u_\downarrow, v_\downarrow, -v_\uparrow)^T$ . At  $\mu = 0$  the BdG Hamiltonian above can be conveniently written in terms of the Dirac matrices:

$$H_{\text{BdG}} = \sum_{a=1,2} (\gamma_a p_a + \Gamma_a n_a). \quad (3.21)$$



Here  $\gamma_a$  and  $\Gamma_a$  are  $4 \times 4$  Dirac matrices defined as  $\gamma_1 = \sigma_x \tau_z, \gamma_2 = \sigma_y \tau_z$ , and  $\Gamma_1 = \tau_x, \Gamma_2 = \tau_y$  and  $\mathbf{n} = (\Re\Delta, -\Im\Delta)$ . One can check that these matrices satisfy the following properties:  $\{\gamma_a, \gamma_b\} = \{\Gamma_a, \Gamma_b\} = \delta_{ab}$  and  $\{\Gamma_a, \gamma_b\} = 0$ . The fifth Dirac matrix  $\gamma_5$  is given by  $\gamma_5 = -\gamma_1 \gamma_2 \Gamma_1 \Gamma_2 = \tau_z \sigma_z$ .

As in the case of spinless  $p_x + ip_y$  case, we first discuss the symmetries of Eq.(3.20). The particle-hole symmetry is now  $\Xi = \sigma_y \tau_y K$  where  $\tau$  are Pauli matrices operating in Nambu (particle-hole) space. The difference with the previous case is the presence of time-reversal symmetry:  $\Theta = i\sigma_y K, [\Theta, \mathcal{H}_{\text{BdG}}] = 0$  in this model. Moreover, when  $\mu = 0$  there is additional chiral symmetry in the model which can be expressed as  $\{\gamma^5, \mathcal{H}_{\text{BdG}}\} = 0$ . Bogoliubov quasiparticles are defined from solutions of BdG equations as

$$\hat{\gamma}^\dagger = \sum_{\sigma} \int d^2\mathbf{r} (u_{\sigma}(\mathbf{r}) \hat{\psi}_{\sigma}^{\dagger}(\mathbf{r}) + v_{\sigma}(\mathbf{r}) \hat{\psi}_{\sigma}(\mathbf{r})). \quad (3.22)$$

If we require  $\hat{\gamma}$  to be a Majorana fermion, *i.e.*  $\hat{\gamma} = \hat{\gamma}^\dagger$ , the necessary and sufficient condition is  $v_{\sigma} = u_{\sigma}^*$  up to a global phase.

A vortex with vorticity  $l$  can be introduced in the order parameter as  $\Delta(\mathbf{r}) = f(r)e^{il\varphi}$ . Rotational symmetry allows decomposition of solutions into different angular momentum channels:

$$\Psi_m(\mathbf{r}) = e^{im\varphi} \begin{pmatrix} e^{-i\pi/4} \chi_{\uparrow}(r) \\ e^{i\pi/4} \chi_{\downarrow}(r) e^{i\varphi} \\ e^{-i\pi/4} \eta_{\downarrow}(r) e^{-il\varphi} \\ e^{i\pi/4} \eta_{\uparrow}(r) e^{-i(l-1)\varphi} \end{pmatrix}.$$

We define  $\tilde{\Psi}_0 = (\chi_{\uparrow}, \chi_{\downarrow}, \eta_{\downarrow}, \eta_{\uparrow})^T$  for later convenience.

Similar to the previous analysis, we first look for non-degenerate Majorana zero-energy state. The Majorana condition  $\Xi\Psi \propto \Psi$  fixes the value of  $m$  to be  $\frac{l-1}{2}$  for odd  $l$ . For even  $l$ , there is no integer  $m$  satisfying Majorana condition so no Majorana zero mode exists. The radial part of BdG equation then becomes

$$\begin{pmatrix} \mathcal{H}_r & f(r) \\ f(r) & -\sigma_y \mathcal{H}_r \sigma_y \end{pmatrix} \tilde{\Psi}_0(r) = 0 \quad (3.23)$$

$$\mathcal{H}_r = \begin{pmatrix} -\mu & v \left( \partial_r + \frac{m+1}{r} \right) \\ -v \left( \partial_r - \frac{m}{r} \right) & -\mu \end{pmatrix}. \quad (3.24)$$

Here  $\tilde{\Psi}_0$  is assumed real. Since we are interested in non-degenerate solution,  $\Psi_0$  must be simultaneously an eigenstates of  $\sigma_y \tau_y$  (particle-hole symmetry). This condition implies that  $\eta_\uparrow = -\lambda \chi_\uparrow, \eta_\downarrow = \lambda \chi_\downarrow$  where  $\lambda = \pm 1$ . Taking into account above constraints  $4 \times 4$  BdG equation reduces to

$$\begin{pmatrix} -\mu & v \left( \partial_r + \frac{m+1}{r} \right) + \lambda f \\ -v \left( \partial_r - \frac{m}{r} \right) - \lambda f & -\mu \end{pmatrix} \begin{pmatrix} \chi_\uparrow \\ \chi_\downarrow \end{pmatrix} = 0. \quad (3.25)$$

The solution of the above equation can be easily obtained for  $\mu \neq 0$ :

$$\begin{pmatrix} \chi_\uparrow \\ \chi_\downarrow \end{pmatrix} = \mathcal{N}_3 \begin{pmatrix} J_m\left(\frac{\mu}{v}r\right) \\ J_{m+1}\left(\frac{\mu}{v}r\right) \end{pmatrix} e^{-\lambda \int_0^r dr' f(r')}. \quad (3.26)$$

Obviously, we should take  $\lambda = 1$  to make radial wave functions normalizable. Here  $\mathcal{N}_3$  is the normalization constant, which is determined by

$$4\pi \mathcal{N}_3^2 \int_0^\infty r dr \left[ J_m^2\left(\frac{\mu}{v}r\right) + J_{m+1}^2\left(\frac{\mu}{v}r\right) \right] e^{-2r/\xi} = 1, \quad (3.27)$$

yielding

$$\mathcal{N}_3^2 = \frac{8}{\pi \xi^2 \left[ 8 {}_2F_1\left(\frac{1}{2}, \frac{3}{2}; 1; -\lambda^2\right) + 3\lambda^2 {}_2F_1\left(\frac{3}{2}, \frac{5}{2}; 3; -\lambda^2\right) \right]} \quad (3.28)$$

with  $\lambda = \mu\xi/v$ . It has the following asymptotes:

$$\mathcal{N}_3^2 \sim \begin{cases} \frac{1}{\pi\xi^2} & \lambda \ll 1 \\ \frac{\lambda}{2\xi^2} & \lambda \gg 1. \end{cases} \quad (3.29)$$

The case of  $\mu = 0$  is special due to the presence of an additional symmetry of BdG Hamiltonian - chiral symmetry. For  $l > 0$  the solution of Eq.(3.25) becomes

$$\begin{pmatrix} \chi_\uparrow \\ \chi_\downarrow \end{pmatrix} \propto \begin{pmatrix} r^m \\ 0 \end{pmatrix} e^{-\lambda \int_0^r dr' f(r')}, \quad (3.30)$$

and for  $l < 0$ ,

$$\begin{pmatrix} \chi_\uparrow \\ \chi_\downarrow \end{pmatrix} \propto \begin{pmatrix} 0 \\ r^{-(m+1)} \end{pmatrix} e^{-\lambda \int_0^r dr' f(r')}. \quad (3.31)$$

Again the normalizability requires  $\lambda = 1$ .

Because the chiral symmetry also relates eigenstates with positive energies to those with negative energies which follows from  $\gamma^5 \mathcal{H}_{\text{BdG}} \gamma^5 = -\mathcal{H}_{\text{BdG}}$ , one can always require the zero-energy eigenstates to be eigenstates of  $\gamma^5$ . The wave function in Eq.(3.30) is an eigenstate of  $\gamma^5$  with eigenvalue 1 while wave function (3.31) has eigenvalue  $-1$ . We define eigenstates of  $\gamma^5$  with eigenvalue  $\pm 1$  as  $\pm$  chirality.

To summarize we have obtained the Majorana zero-energy bound state attached to the vortex with odd vorticity:

$$\Psi_0(\mathbf{r}) = e^{i(l-1)\varphi/2} \begin{pmatrix} e^{-i\pi/4} \chi_\uparrow(r) \\ e^{i\pi/4} \chi_\downarrow(r) e^{i\varphi} \\ e^{-i\pi/4} \chi_\downarrow(r) e^{-i\varphi} \\ -e^{i\pi/4} \chi_\uparrow(r) e^{-i(l-1)\varphi} \end{pmatrix} \quad (3.32)$$

Generalization to the case of many vortices is straightforward. Order parameter with  $2N$  vortices pinned at  $\mathbf{R}_i$  is already given in (3.16). Assuming that they are well separated from each other, we can find an approximate zero-energy bound state localized in each vortex core:

$$\Psi_i(\mathbf{r}) = e^{i(l-1)\varphi_i/2} e^{i\Omega_i\tau_z/2} \begin{pmatrix} e^{-i\pi/4}\chi_{\uparrow}(r_i) \\ e^{i\pi/4}\chi_{\downarrow}(r_i)e^{i\varphi_i} \\ e^{-i\pi/4}\chi_{\downarrow}(r_i)e^{-i\varphi_i} \\ -e^{i\pi/4}\chi_{\uparrow}(r_i)e^{-i(l-1)\varphi_i} \end{pmatrix} \quad (3.33)$$

the construction of  $N$  Dirac fermions and  $2^{N-1}$  ground state Hilbert space are the same as the case of spinless  $p_x + ip_y$  superconductors.

### 3.3 Atiyah-Singer-type index theorem

Index theorem provides an intelligent way of understanding the topological stability of zero modes. It is well-known that one can relate the analytical index of an elliptic differential operator (Dirac operator) to the topological index (winding number) of the background scalar field in 2D [84] through the index theorem. Since BdG Hamiltonian for TI/SC system at  $\mu = 0$  can be presented as a Dirac operator (see Eq.(3.21)), we give a brief review of this index theorem, see also recent exposition in Ref. [85]. Specifically, the Hamiltonian for TI/SC heterostructure can be written as

$$\mathcal{H}_D = i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + \boldsymbol{\Gamma} \cdot \mathbf{n}, \quad (3.34)$$

where  $\mathbf{n} = (\Re\Delta, -\Im\Delta)$  field describes the non-trivial configuration of the superconducting order parameter. We assume the following boundary condition for  $\mathbf{n}$  field:

$$|\mathbf{n}(\mathbf{r})| \rightarrow \text{const as } |\mathbf{r}| \rightarrow \infty. \quad (3.35)$$

As mentioned above, this model Hamiltonian has particle-hole symmetry, time-reversal symmetry and chiral symmetry which is given by  $\gamma^5$ . It anticommutes with the Dirac Hamiltonian  $\{\gamma^5, \mathcal{H}_D\} = 0$ . Therefore, all zero modes  $\Psi_0$  of  $\mathcal{H}_D$  are eigenstates of  $\gamma^5$ . Since  $(\gamma^5)^2 = 1$  eigenvalues of  $\gamma^5$  are  $\pm 1$ . We define  $\pm$  chirality of zero modes as  $\gamma^5\Psi_0^\pm = \pm\Psi_0^\pm$ . The analytical index of  $\mathcal{H}_D$  is defined as

$$\text{ind } \mathcal{H}_D = n_+ - n_-, \quad (3.36)$$

where  $n_\pm$  are number of zero modes with  $\pm$  chirality.

The index theorem for the Hamiltonian  $H_D$  states that the analytical index is identical to the winding number of the background scalar field in the two-dimensional space [84]:

$$\text{ind } \mathcal{H}_D = -\frac{1}{2\pi} \int d^i x \epsilon^{ab} \hat{n}_a \partial_i \hat{n}_b, \quad (3.37)$$

where  $\hat{\mathbf{n}} = \mathbf{n}/|\mathbf{n}|$ . According to the index theorem, the number of zero modes is determined by the topology of order parameter at infinity. The right hand side is ensured to be an integer by the fact that the homotopy group  $\pi_1(S^1) = \mathbb{Z}$ . If we have a vortex in the system with vorticity  $l$ , the right hand side of (3.37) evaluates exactly to  $l$ . Thus the index theorem implies that the Dirac Hamiltonian has at least  $l$  zero modes which agrees with explicit solution obtained by Jackiw and Rossi [82]. Our explicit solutions for a single vortex in the previous section also agrees perfectly.

This conclusion can be generalized to the case where multiple vortices are present.

In that case the right hand side is basically the sum of vorticities of all vortices.

The index theorem (3.37) requires chiral symmetry which is broken by presence of a finite chemical potential  $\mu \neq 0$ . Now we argue that when chiral symmetry is broken the Majorana zero modes admit a  $\mathbb{Z}_2$  classification corresponding to even-odd number of zero energy solutions. Generally speaking, a small chiral symmetry breaking term cause coupling between zero modes and split them away from zero energy. However, due to particle-hole symmetry, the number of zero modes that are split by chiral symmetry breaking term must be even. So the parity of the topological index is preserved in the generic case. This is consistent with an explicit solutions of zero mode in TI/SC heterostructure with finite chemical potential. Thus, we conclude that without chiral symmetry the Majorana zero modes bound to vortices are classified by  $\mathbb{Z}_2$  corresponding to even or odd number of zero modes.

## Chapter 4

# Topological Degeneracy Splitting of Majorana

## Zero Modes

In the topological quantum computation scheme based on Majorana non-Abelian vortices in topological superconductors, the quantum information is encoded in the degenerate ground states when there are multiple non-Abelian vortices present. The degeneracy is crucial for the topological protection of the qubits as well as the braiding operations on them. Understanding the fate of ground-state degeneracy of many-anyon system in realistic solid-state structures is a difficult problem of fundamental importance and of relevance to practical realization of topological quantum computing. In this chapter we address one mechanism that may lift the ground state degeneracy associated with the tunneling processes between spatially separated vortices. The presence of the bulk gap protects ground state degeneracy from thermal fluctuations at low temperature leaving out only processes of Majorana fermion quantum tunneling between vortices. Generic features of tunneling of topological charges have been explored recently [86]. The lifting of ground state degeneracy due to intervortex tunneling for a pair of vortices have been studied numerically for  $\nu = 5/2$  quantum Hall state [87, 88],  $p_x + ip_y$  superconductor [89] and Kitaev's honeycomb lattice model [90]. Analytical calculation has been carried

out for the model of spinless  $p_x + ip_y$  superconductors [91].

Generally energy splitting due to intervortex tunneling is determined by the wave function overlap of localized Majorana bound states. In this chapter we calculate the splitting for both spinless  $p_x + ip_y$  superconductor and a model of Dirac fermions interacting with the scalar superconducting pairing potential realized in a TI/SC heterostructure. In both cases, besides the expected exponential decay behavior, it is found that the prefactor exhibits an oscillatory behavior with the intervortex distance which originates from the interference of different bound state wave functions oscillating with the Fermi wave length. This is generic situation for weak coupling superconductors where the Fermi energy  $E_F$  is much larger than the superconducting gap  $\Delta$ . In this chapter, we also consider several cases where the Fermi wavelength is much larger than the coherence length. This scenario is relevant, for example, for TI/SC heterostructure as well as some other systems involving the proximity-induced superconductivity. When chemical potential is tuned to the Dirac point ( $\mu \rightarrow 0$ ), we find indeed that the splitting in TI/SC heterostructure vanishes. This fact can be related to the Atiyah-Singer index theorem an additional symmetry possessed by the system at  $\mu = 0$  - the chiral symmetry.

## 4.1 Degeneracy splitting due to intervortex tunneling

The ground state degeneracy, which is crucial for topological quantum computation with non-Abelian anyons, heavily relies on the assumption that intervortex tunneling is negligible. When tunneling effects are taken into account zero energy



bound states are usually splitted and the ground state degeneracy is lifted. Besides, the sign of energy splitting is important for understanding many-body collective states [92].

We now discuss a general formalism to calculate the energy splitting. We focus on the case of two classical vortices each with vorticity  $l = 1$  located at certain fixed positions  $\mathbf{R}_1$  and  $\mathbf{R}_2$ . To develop a physical intuition, it is useful to view a vortex as a potential well, which may host bound states including zero-energy states, while the regions where superconducting gap is finite play the role of a potential barrier. Therefore, the two-vortex problem resembles the double-well potential problem in single-particle quantum mechanics (sometimes referred to as the Lifshitz problem in the literature [93]). The solution to this simple problem in one-dimensional quantum mechanics is readily obtained [93] by considering symmetric and antisymmetric combinations of single-well wave-functions (which can be taken within the quasiclassical approximation for high barriers) and the overlap of these wave-functions always selects the symmetric state as the ground state in accordance with the elementary oscillation theorem (*i.e.*, the ground state has no nodes). We note that both quasiclassical approximation and the Lifshitz method are not specific to the Schrödinger equation, but actually represent general mathematical methods of solving differential equations of certain types. Moreover, these methods can be applied to rather generic matrix differential operators, and such a generalization has been carried out by one of the authors in a completely different context of magnetohydrodynamics, [94] where interestingly the relevant differential operator appears to be mathematically similar to the BdG Hamiltonian. These considerations suggest

that one can use the generalized Lifshitz method to obtain the splitting of zero modes of the BdG equations, by considering certain linear combinations of the individual Majorana modes in the two vortices and calculating their overlap, which reduces to a boundary integral along a path between the two vortices. Also, if the inter-vortex separation is large, one can use the semiclassical form of the Majorana wave-functions (effectively their large-distance asymptotes) to obtain quantitatively accurate results. Let us note here that apart from a technically more complicated calculation that needs to be carried out for the BdG equation, another important difference between this problem and the simple Lifshitz problem is that we can not rely on any oscillation theorem and there is no way to determine a priori which state has a lower energy. As discussed below, this “uncertainty” is fundamental to this problem and is eventually responsible for a fast-oscillating energy splitting with intervortex separation.

With the two zero-energy eigenstates  $\Psi_1$  and  $\Psi_2$  localized at  $\mathbf{R}_1$  and  $\mathbf{R}_2$  (given by Eq.(3.17) for spinless  $p_x + ip_y$  superconductor and by Eq.(3.32) for TI/SC heterostructure), we can construct approximate eigenstate wave functions in the case of two vortices:  $\Psi_{\pm} = \frac{1}{\sqrt{2}}(\Psi_1 \pm e^{i\alpha}\Psi_2)$  analogous to the symmetric and anti-symmetric wave functions in a double-well problem with energies  $E_{\pm}$ , respectively. The phase factor  $e^{i\alpha}$  can be determined from particle-hole symmetry which requires that new eigenstates  $\Psi_+$  with energy  $E_+ = \delta E$  and  $\Psi_-$  with energy  $E_- = -\delta E$  be related by  $\Xi\Psi_+ = \Psi_-$ . Since  $\Psi_1$  and  $\Psi_2$  are real (Majorana) eigenstates, one finds  $\Xi\Psi_+ = \frac{1}{\sqrt{2}}(\Psi_1 + e^{-i\alpha}\Psi_2) = \Psi_-$ . Thus, one arrives at  $e^{2i\alpha} = -1$  which fixes  $\alpha = \pm\pi/2$ . In the rest of the text we take  $\alpha = \pi/2$  for convenience. The corre-

sponding quasiparticle operator can be identified with the Dirac fermion operator.

We explicitly show this for the case of spinless  $p_x + ip_y$  superconductor:

$$\hat{c} = \frac{1}{\sqrt{2}}(\hat{\gamma}_1 - i\hat{\gamma}_2) = \int d^2\mathbf{r} \left[ \hat{\psi} \frac{u_1^* - iu_2^*}{\sqrt{2}} + \hat{\psi}^\dagger \frac{v_1^* - iv_2^*}{\sqrt{2}} \right]. \quad (4.1)$$

Therefore  $\hat{c}$  ( $\hat{c}^\dagger$ ) annihilates(creates) a quasiparticle on energy level  $E_+$ . The original two fold degeneracy between state with no occupation  $\hat{c}|0\rangle = 0$  and occupied  $|1\rangle = \hat{c}^\dagger|0\rangle$  is lifted by energy splitting  $E_+$ .

To calculate the energy of  $\Psi_+$ , we employ the standard method based on the wave function overlap [93]. Suppose the two vortices are placed symmetrically with respect to  $y$  axis:  $\mathbf{R}_1 = (R/2, 0)$  and  $\mathbf{R}_2 = (-R/2, 0)$ . BdG equations are  $\mathcal{H}_{\text{BdG}}\Psi_+ = E_+\Psi_+$ ,  $\mathcal{H}_{\text{BdG}}\Psi_1 = 0$ . We then multiply the first equation by  $\Psi_1^*$  and second by  $\Psi_+^*$ , subtract corresponding terms, and integrate over region  $\Sigma$  which is the half plane  $x \in (0, \infty), y \in (-\infty, \infty)$  arriving finally at the following expression for  $E_+$ :

$$E_+ = \frac{\int_{\Sigma} d^2\mathbf{r} \Psi_1^\dagger \mathcal{H}_{\text{BdG}} \Psi_+ - \int_{\Sigma} d^2\mathbf{r} \Psi_+^\dagger \mathcal{H}_{\text{BdG}} \Psi_1}{\int_{\Sigma} d^2\mathbf{r} \Psi_1^\dagger \Psi_+}. \quad (4.2)$$

This is the general expression for the energy splitting which is used to evaluate  $E_+$  in  $p_x + ip_y$  SC and TI/SC heterostructure.

#### 4.1.1 Splitting in spinless $p_x + ip_y$ superconductor

We now calculate splitting for two vortices in spinless  $p_x + ip_y$  superconductor. The denominator in Eq.(4.2) can be evaluated quite straightforwardly

$\int_{\Sigma} d^2\mathbf{r} \Psi_1^\dagger \Psi_+ \approx 1/\sqrt{2}$ . With the help of Green's theorem the integral over half plane in the numerator can be transformed into a line integral along the boundary of  $\Sigma$ , namely the  $y$  axis at  $x = 0$  which we denote by  $\partial\Sigma$ :

$$E_+ = \frac{2}{m} \int_{\partial\Sigma} dy \left[ g(s)g'(s) \cos 2\varphi_2 \cos \varphi_2 + \frac{g^2(s)}{s} \sin 2\varphi_2 \sin \varphi_2 - \frac{g^2(s)}{\xi} \right] \quad (4.3)$$

where  $s = \sqrt{(R/2)^2 + y^2}$ ,  $\tan \varphi_2 = 2y/R$ . The function  $g(s)$  is defined as  $g(s) \equiv \chi(s) \exp(-s/\xi)$ .

First we consider the regime where  $\Delta_0^2 < 2m\mu v_F^2$  and radial wave function of Majorana bound state has the form (3.8). We are mainly interested in the behavior of energy splitting at large  $R \gg \xi$  with  $\xi$  being the coherence length, where our tunneling approximation is valid. Another length scales in our problem is the length corresponding to the bound state wave function oscillations  $k = \sqrt{2m\mu - \Delta_0^2}/v_F^2$ .

In the limit  $R \gg \max(k^{-1}, \xi)$  upon evaluating the integral (4.3) we obtain

$$E_+ = \sqrt{\frac{8}{\pi}} \frac{\mathcal{N}_1^2}{m} \left( \frac{\lambda^2}{1 + \lambda^2} \right)^{1/4} \frac{1}{\sqrt{kR}} \exp\left(-\frac{R}{\xi}\right) \left[ \cos(kR + \alpha) - \frac{2}{\lambda} \sin(kR + \alpha) + \frac{2(1 + \lambda^2)^{1/4}}{\lambda} \right], \quad (4.4)$$

where  $\lambda = k\xi$ ,  $2\alpha = \arctan \lambda$  and  $\mathcal{N}_1$  is the normalization constant defined in Eq.(3.8). We find the asymptotes for  $\lambda \gg 1$  and  $\lambda \ll 1$ :

$$\mathcal{N}_1^2 = \begin{cases} \frac{k}{2\xi} & \lambda \gg 1 \\ \frac{8}{3\pi k^2 \xi^4} & \lambda \ll 1 \end{cases}. \quad (4.5)$$

The exponential decay is expected due to the fact that Majorana bound states are localized in vortex core. In addition, the splitting energy  $E_+$  oscillates with intervortex separation  $R$  which can be traced back to interference between the wave functions of the two Majorana bound states since they both oscillates in space.

Of particular importance is the sign of splitting as noted in Ref. [92]. It determines which state is energetically favored when tunneling interaction is present. If  $E_+ > 0$ ,  $|0\rangle$  is favored whereas  $E_+ < 0$  favors  $|1\rangle$ . We note here that the definition of states  $|0\rangle$  and  $|1\rangle$  relies on how we define the Dirac fermion operator  $\hat{c}$  and  $\hat{c}^\dagger$ . Due to the presence of a constant term together with trigonometric function, the sign of splitting can change. To figure out when the sign oscillates, we require the amplitude of the trigonometric part is greater than the constant part which gives

$$\sqrt{1 + \frac{4}{\lambda^2}} > \frac{2(1 + \lambda^2)^{1/4}}{\lambda}.$$

Solving this inequality yields  $\lambda = k\xi > 8$ . Therefore in this parameter regime the sign of splitting changes with distance  $R$ . Otherwise the splitting still shows oscillatory behavior but the sign is fixed to be positive.

In weak-coupling superconductors where  $\Delta_0 \ll \varepsilon_F$  or equivalently  $k_F\xi \gg 1$ , the expression for the energy splitting (4.4) can be considerably simplified. In this case,  $\mu \approx \varepsilon_F$  and  $k \approx k_F$ . Keeping only terms that are leading order in  $(k_F\xi)^{-1}$ , we find

$$E_+ \approx \sqrt{\frac{2}{\pi}} \Delta_0 \frac{\cos(k_F R + \frac{\pi}{4})}{\sqrt{k_F R}} \exp\left(-\frac{R}{\xi}\right), \quad (4.6)$$

which is the expression reported in Ref. [[91]]. A similar expression for splitting of a pair of Majorana bound states on superconductor/2D topological insulator/magnet interface is found in Ref. [95].

Next we consider a different limit  $\Delta_0^2 > 2m\mu v_F^2$  in which the wave function of Majorana bound state for a single vortex doesn't show any spatial oscillations. Thus, we expect that tunneling splitting will show just an exponential decay without

any oscillations. The wave function (3.12) grows exponentially when  $r \rightarrow \infty$ :

$$\chi(r) \sim \frac{1}{\sqrt{r}} e^{k_0 r}$$

with  $k_0 = \sqrt{\Delta_0^2/v_F^2 - 2m\mu}$ . The overall radial wave function decays exponentially  $\sim \exp(-k'r)$  where  $k' = 1/\xi - k_0$ . In this case, the tunneling approximation is only valid for  $k'R \gg 1$  since bound state wave function is localized approximately within distance  $1/k'$  to vortex core. The resulting energy splitting monotonically decays:

$$E_+ \approx \sqrt{\frac{2}{\pi}} \frac{\mathcal{N}_2^2}{m} \left( \frac{3}{k_0 \xi} - 1 \right) \frac{1}{\sqrt{k'R}} \exp(-k'R), \quad (4.7)$$

As  $\mu$  approaches 0 there is a quantum phase transition between the non-Abelian phase and Abelian phase. This transition is accompanied by closing of the gap and the Majorana bound state is no longer localized since  $k' \rightarrow 0$ .

We briefly comment on the degeneracy splitting between vortex zero modes in the ferromagnetic insulator/semiconductor/superconductor hybrid structure proposed by Sau *et. al.* [49] which can be modeled by spin-1/2 fermions with Rashba spin-orbit coupling and s-wave pairing induced by the superconducting proximity effect. Since time-reversal symmetry is broken by the proximity-induced exchange splitting, this system belongs to the same symmetry class as spinless  $p_x + ip_y$  superconductor - class D. The connection between this hybrid structure and spinless  $p_x + ip_y$  can be made more explicit by the following argument: the single particle Hamiltonian after diagonalization yields two bands. Assuming a large band gap (which is actually determined by exchange field), one can project the full Hamiltonian onto the lower band and then the effective Hamiltonian takes exactly the form of spinless  $p_x + ip_y$  superconductor, see, for example, the discussion in Ref.

[61]. Although analytical expression for Majorana bound state in vortex core is not available, the solution behaves qualitatively similar to the one in spinless  $p_x + ip_y$  superconductor. Therefore, we expect that splitting should also resemble that of spinless  $p_x + ip_y$  superconductor.

#### 4.1.2 Splitting in TI/SC heterostructure

In this section we discuss the case of vortex-vortex pair in TI/SC heterostructure. We assume both vortices have vorticity 1. Similar the case of  $p_x + ip_y$  superconductor, one can transform the surface integral over half plane  $\Sigma$  to a line integral along its boundary  $\partial\Sigma$ . Exploiting the explicit expressions for the zero mode solution, we arrive at

$$\begin{aligned} & \int_{\Sigma} d^2\mathbf{r} \Psi_1^\dagger \mathcal{H}_{\text{BdG}} \Psi_+ - \int_{\Sigma} d^2\mathbf{r} \Psi_+^\dagger \mathcal{H}_{\text{BdG}} \Psi_1 \\ & = -2\sqrt{2}v \int_{-\infty}^{\infty} dy \chi_{\uparrow}(s) \chi_{\downarrow}(s) \cos \varphi_2, \end{aligned} \quad (4.8)$$

where  $s = \sqrt{(R/2)^2 + y^2}$ ,  $\cos \varphi_2 = R/2s$ .

First we consider the case with finite  $\mu$ . There are two length scales: Fermi wavelength  $k_F^{-1} = \frac{v}{\mu}$  and coherence length  $\xi = \frac{v}{\Delta_0}$ . We evaluate the integral (4.8) in the limit where  $R$  is large compared to both  $k_F^{-1} = v/\mu$  and  $\xi$ :

$$E_+ \approx \frac{4\mathcal{N}_3^2 v}{\sqrt{\pi} k_F (1 + k_F^2 \xi^2)^{1/4}} \frac{\cos(k_F R + \alpha)}{\sqrt{R/\xi}} \exp\left(-\frac{R}{\xi}\right), \quad (4.9)$$

where  $2\alpha = \arctan(k_F \xi)$ . One can notice that the splitting, including its sign, oscillates with the intervortex separation  $R$  when  $R$  is large. In the limit of large  $\mu$ , say  $k_F \xi \gg 1$ , Eq. (4.9) can be simplified to

$$E_+ \approx \frac{2\Delta_0}{\sqrt{\pi}} \frac{\cos(k_F R + \frac{\pi}{4})}{\sqrt{k_F R}} \exp\left(-\frac{R}{\xi}\right). \quad (4.10)$$

We now turn to the limit where  $\mu$  is very close to Dirac point, *i.e.*  $\mu \rightarrow 0$ ,  $k_F \xi \ll 1$ . We evaluate the integral for  $\xi \ll R \ll k_F^{-1}$ .

$$E_+ \approx -\frac{2\mu}{\sqrt{\pi}} \left(\frac{R}{\xi}\right)^{3/2} \exp\left(-\frac{R}{\xi}\right), \quad (4.11)$$

where we have made use of asymptote of  $\mathcal{N}_3$  in the limit  $k_F \xi \ll 1$ . Eq. (4.11) implies that for fixed  $R$  the splitting vanishes as  $\mu$  approaches Dirac point. Actually this fact can be easily seen from (4.8) without calculating the integral. Because at  $\mu = 0$  either  $\chi_\uparrow$  or  $\chi_\downarrow$  vanishes, the splitting which is proportional to the product of  $\chi_\downarrow$  and  $\chi_\uparrow$  is zero. The same result for splitting at  $\mu = 0$  has also been obtained in Ref. [96].

We now show that vanishing of the splitting at  $\mu = 0$  is a direct consequence of chiral symmetry. At  $\mu = 0$  zero modes carry chirality which labels the eigenvalues of  $\gamma^5$ . More specifically, wave function is an eigenstate of  $\gamma^5$ :  $\gamma^5 \Psi_i = \lambda \Psi_i$ . Consider an arbitrary perturbation represented by  $\mathcal{O}$  to the ground state manifold expanded by these local zero modes. To leading order in perturbation theory its effect is determined by matrix element  $\mathcal{O}_{ij} = \langle \Psi_i | \mathcal{O} | \Psi_j \rangle$ . Now assume that  $\Psi_i$  and  $\Psi_j$  have the same chirality (which means that vortices  $i$  and  $j$  have identical vorticity). If the perturbation  $\mathcal{O}$  preserves chiral symmetry, *i.e.*  $\{\gamma^5, \mathcal{O}\} = 0$ , then

$$\langle \Psi_i | \{\gamma^5, \mathcal{O}\} | \Psi_j \rangle = 2\lambda \langle \Psi_i | \mathcal{O} | \Psi_j \rangle = 0. \quad (4.12)$$

Therefore matrix element  $\langle \Psi_i | \mathcal{O} | \Psi_j \rangle$  vanishes identically. Tunneling obviously preserves chiral symmetry so there is no splitting between two vortices with the same vorticity from this line of reasoning. As discussed below this fact actually holds



beyond perturbation theory and the robustness of zero modes in the presence of chiral symmetry is ensured by an index theorem.

Now we can fit our splitting calculation into the general picture set by index theorem. As being argued above, Majorana zero modes in spinless  $p_x + ip_y$  superconductor is classified by  $\mathbb{Z}_2$ . When there are two vortices in the bulk, the topological index of order parameter is 2 thus there is no zero mode and we find the splitting as expected. The same applies to two vortices in TI/SC heterostructure with  $\mu \neq 0$ . However, as we have seen in the calculation the splitting vanishes for  $\mu = 0$ . This should not be surprising since according to index theorem, there should be at least two zero modes associated with total vorticity 2 which is the case for two vortices.

#### 4.1.3 Comparison with the splitting calculations in other systems.

Recently numerical calculations of the degeneracy splitting have been performed for other systems supporting non-Abelian Ising anyons [87, 88, 90]. In all these calculations it was found that the splitting has qualitatively similar behavior - there is an exponential decay with the oscillating prefactor which stems from the spatial oscillations of Majorana bound states. In the case of Moore-Read quantum Hall state [88], the splitting between two quasiholes exponentially decays and oscillates with the magnetic length  $l_c = \frac{\hbar}{eB}$  since there only one length scale in the problem. The oscillatory behavior is also predicted for pair of vortex excitations in the B-phase of Kitaev's honeycomb lattice model in an external magnetic field [90].

## 4.2 Collective states of many-anyon system

The microscopic calculations of the degeneracy splitting for a pair of vortices are important for understanding the collective states of anyons arising on top of the non-Abelian parent state when many Majorana fermions (Ising anyons) are present [97, 98, 99, 92]. Essentially, the sign of the splitting favors certain fusion channel (  $\mathbf{1}$  or  $\psi$  in the terminology of Ref.[100]) when two vortices carrying Majorana fermions are brought together. These fusion channels correspond to having a fermion ( $\psi$ -channel when  $E_+ < 0$ ) or no fermion ( $\mathbf{1}$ -channel when  $E_+ > 0$ ) left upon fusing of two anyons.

For pedagogical reason we start with the dilute anyon density limit assuming that the average distance between Majorana fermions is large compared with the coherence length  $\xi$ . In this regime, the many anyon state of the system will resemble gas of weakly bound pairs of anyons formed out of two anyons separated by the smallest distance. Because of the exponential dependence of the energy splitting the residual “interactions” with other anyons are exponentially smaller and can be ignored. In this scenario the parent state remains unchanged.

When the density of anyons is increased so that the average distance between them becomes of the order of the Majorana bound state decay length (coherence length  $\xi$  in p-wave superconductors or magnetic length  $l_c$  in Quantum Hall states) the system can form a non-trivial collective liquid (Wigner crystal of anyons or some other incompressible liquid state). This question has been investigated in Refs. [101, 102, 103, 92]. Although our approach used to calculate energy splitting

breaks down in this regime and one should resort to numerical calculations for the magnitude of the splitting, we believe that qualitative form of the splitting will remain the same. It is interesting to discuss the collective state that forms in this regime. Remarkably, it was shown in Ref. [92] that depending on the fusion channel (*i.e.* sign of the splitting) the collective state of anyons may be Abelian or non-Abelian. This result was obtained assuming that the magnitude of the splitting is constant and the sign of the splitting is the same for all anyons (positive or negative). However, because of the prefactor changing rapidly with the Fermi wave length we expect the magnitude of the splitting energy to be random realizing random bond Ising model discussed in Ref. [104, 92].

Finally, we mention that our calculations above and all existing studies of interacting many-anyon systems treat host vortices as classical objects with no internal dynamics. This is a well-defined mathematical framework, which corresponds to the BCS mean-field approximation. In real superconductors, however, there are certainly corrections to it. The order parameter field,  $\Delta(\mathbf{r}, t)$ , which describes a certain vortex configuration has a non-trivial dynamics and fluctuates in both space and time. At low temperatures, when the system is fully gapped, these fluctuation effects are suppressed in the bulk, but they are always significant in the vicinity of the vortex core, where the order parameter vanishes. This dynamics gives rise to an effective motion of a vortex as well as to the dynamics of its shape and the radial profile. The relevant length-scales of these effects certainly exceed the Fermi wave-length, which is the smallest length-scale in the problem in most realistic systems. Even if the vortex is pinned, e.g. by disorder, its motion can be

constrained only up to a mean-free path or another relevant length-scale, which is still much larger than the Fermi wave-length for local superconductivity to exist. These considerations suggest that the intervortex separation between quantum vortices has an intrinsic quantum uncertainty, which is expected to much exceed the inverse Fermi wave-vector. This makes the question of the sign of Majorana mode coupling somewhat ill-defined in the fully quantum problem. Indeed we found the energy splitting to behave as  $\delta E(\mathbf{r}) = |\delta E_0(r)| \cos(k_F r + \alpha)$ , where  $|\delta E_0(r)|$  is an exponentially small magnitude of coupling insensitive to any dynamics of  $\mathbf{r}(t)$ . The cosine-factor, which determines the sign, is however expected to be very much sensitive to quantum dynamics. To derive the actual microscopic model even in the simplest case of two non-Abelian anyons living in the cores of quantum vortices is a tremendously complicated problem, which requires a self-consistent treatment of the vortex order-parameter field and fermionic excitations beyond mean-field. However, one can argue that the outcome of such a treatment would be an effective theory where the  $e^{ik_F r(t)}$  factor that appears in Majorana interactions, should be replaced with a random quantum-fluctuating phase (*c.f.*, Ref. [105]),  $e^{i\theta(t)}$ , whose dynamics is governed by an effective action of type,  $S[\theta] \approx \int d\tau [(\theta - \theta_0)^2 + c(\partial_\tau \theta)^2]$ . This generally resembles a gauge theory, but of an unusual type, and at this stage it is unclear what collective many-anyon state such a theory may give rise to.

### 4.3 Conclusions

In this chapter, we address the problem of topological degeneracy lifting in topological superconductors characterized by the presence of Majorana zero-energy states bound to the vortex cores. We calculate analytically energy splitting of zero-energy modes due to the intervortex tunneling. We consider here canonical model of topological superconductor, spinless  $p_x + ip_y$  superconductor, as well as the model of Dirac fermions coupled to superconducting scalar field. The latter is realized at the topological insulator/s-wave superconductor interface. In the case of spinless  $p_x + ip_y$  superconductor, we find that, in addition to the expected exponential decay, the splitting energy for a pair of vortices oscillates with distance in weak-coupling superconductor and these oscillations become over-damped as the magnitude of the chemical potential is decreased. In the second model, the splitting energy oscillates for finite chemical potential and vanishes at  $\mu = 0$ . The vanishing of splitting energy is a consequence of an additional symmetry, the chiral symmetry, emerging in the model when chemical potential is exactly equal to zero. We show that this fact is not accidental but stems from the index theorem which relates the number of zero modes of the Dirac operator to the topological index of the order parameter. Finally, we discuss the implications of our results for many-anyon systems.

## Chapter 5

# Topological Protection of Majorana-Based Qubits

In this chapter we investigate the effect of finite-temperature thermal fluctuations on three key aspects of topological quantum computation: quantum coherence of the topological qubits, topologically-protected quantum gates and the read-out of qubits. Since the information is encoded in non-local degrees of freedom of the ground state many-body wavefunction, it is important to keep the system close to the ground state. However, any systems realized in the laboratory are operated at a finite temperature  $T > 0$ . To prevent uncontrollable thermal excitations, it is generally accepted that  $T$  has to be way below the bulk excitation gap. However, complications appear when there exist various types of single-particle excitations with different magnitudes of gaps which can change the occupation of the non-local fermionic modes. Note that throughout the chapter we assume that Majorana fermions are sufficiently far away from each other and neglect exponentially small energy splitting due to inter-vortex tunneling. The effect of these processes on topological quantum computing has been discussed elsewhere [91, 106]. Another trivial effect not considered in this work is a situation where the fermion parity conservation is explicitly broken by the Majorana mode being in direct contact with a bath of fermions (electrons and holes) where obviously the Majorana will decay into

the fermion bath, and consequently decohere. Such situations arise, for example, in current topological insulators where the existence of the bulk carriers (invariably present due to the unintentional bulk doping) would make any surface non-Abelian Majorana mode disappear rather rapidly. Another situation that has recently been considered in this context [107] is the end Majorana mode in a one-dimensional nanowire being in contact with the electrons in the non-superconducting part of the semiconductor, leading to a zero-energy Majorana resonance rather than a non-Abelian Majorana bound state at zero energy. The fact that the direct coupling of Majorana modes to an ordinary fermionic bath will lead to its decoherence is rather obvious and well-known, and does not require a general discussion since such situations must be discussed on a case by case basis taking into account the details of the experimental systems. In particular, the reason the quantum braiding operations in Majorana-based systems involves interferometry is to preserve the fermion parity conservation. Our theory in the current work considers the general question of how thermal fluctuations at finite temperatures affect the non-Abelian and the non-local nature of the Majorana mode.

We consider a simple model for two-dimensional chiral  $p_x + ip_y$  superconductor where Majorana zero-energy states are hosted by Abrikosov vortices. The quasiparticle excitations in this system are divided into two categories: a) Caroli-de Gennes-Matricon (CdGM) or so-called midgap states localized in the vortex core with energies below the bulk superconducting gap [32, 33] (the gap that separates the zero-energy state to the lowest CdGM state is called the mini-gap  $\Delta_M$ ); b) extended states with energies above the bulk quasiparticle gap which is denoted

by  $\Delta$ . The natural question arising in this context is how these two types of excitations affect topological quantum computation using the Majorana zero-energy states at finite temperature. This question is very relevant in the context of strontium ruthenate as well as other weak-coupling BCS superconductors where the Fermi energy  $E_F$  is much larger than the superconducting gap  $\Delta$  in which case  $\Delta_M \propto \Delta^2/E_F \ll \Delta$ . We mention in passing here that the semiconductor-based Majorana proposals in nanowires [49, 61, 59, 60] do not have low-lying CdGM states because the one-dimensionality reduces the phase space for the bound states and the minigap  $\Delta_M \sim \Delta$  [108] due to the small Fermi energy in the semiconductor. If the temperature is substantially below the minigap, i.e.  $T \ll \Delta_M$ , obviously all excited states can be safely ignored. However, such low temperatures with  $T \ll \Delta_M$  can be hard to achieve in the laboratory since for typical superconductors  $\Delta/E_F \sim 10^{-3} - 10^{-4}$ . We note that even in the semiconductor two-dimensional sandwich structures the energetics of the subgap states [109] obey the inequality  $\Delta_M < \Delta$  since in general  $E_F > \Delta$  even in the semiconductor-based systems in view of the fact that typically  $\Delta \sim 1$  K. This makes our consideration in this chapter of relevance also to the semiconductor-based topological quantum computing platforms. We investigate the non-trivial intermediate temperature regime  $\Delta_M \ll T < \Delta$ . To make this chapter more pedagogical, we will use a simple physical model that captures the relevant physics. We find that the presence of the excited midgap states localized in the vortex core does not effect braiding operations. However, the midgap states do affect the outcome of the interferometry experiments.

We also study the quantum dynamical evolution and obtain equations of mo-



tion for the reduced density matrix assuming that the finite temperature is set by a bosonic bath (e.g. phonons). We find that the qubit decay rate  $\lambda$  is given by the rate of changing fermion parity in the system and is exponentially suppressed (i.e.  $\lambda \propto \exp(-\Delta/T)$ ) at low temperatures in a fully-gapped  $p_x + ip_y$  superconductor. In this context, we make some comments about Refs.[110] claiming to obtain different results regarding the effect of thermal fluctuations.

## 5.1 Non-Abelian Braiding in the Presence of Midgap States

In this section we address the question of how the midgap states affect the non-Abelian statistics at finite temperature. The usual formulation of the non-Abelian statistics as unitary transformation of the ground states does not apply, since at finite temperature the system has to be described as a mixed state. We need to generalize the notion of the non-Abelian braiding in terms of physical observables [111]. This can be done as the following: consider a topological qubit made up by four vortices labeled by  $a = 1, 2, 3, 4$ . Each of them carries a Majorana zero-energy state, whose corresponding quasiparticle is denoted by  $\hat{\gamma}_{a0}$  which satisfies  $\hat{\gamma}_{a0}^2 = 1, \hat{\gamma}_{a0} = \hat{\gamma}_{a0}^\dagger$ . There are other midgap states in the vortex core which are denoted by  $\hat{d}_{ai}, i = 1, 2, \dots, m$ . (Actually the number of midgap states is huge and the midgap spectrum eventually merges with the bulk excitation spectrum. However, since we are interested in  $T \ll \Delta$ , we can choose an energy cutoff  $\Lambda$  such that  $T \ll \Lambda \ll \Delta$  and only include those midgap states that are below  $\Lambda$ .) It is convenient to write  $\hat{d}_{ai} = \hat{\gamma}_{a,2i-1} + i\hat{\gamma}_{a,2i}$ , so each vortex core carries odd number of

Majorana fermions  $\hat{\gamma}_{ai}, i = 0, 1, \dots, 2m$ .

Having the notations set up, we now define a generalized Majorana operator  $\hat{\Gamma}_a = i^m \prod_{i=0}^{2m} \hat{\gamma}_{ai}$ . It is straightforward to check that  $\{\hat{\Gamma}_a, \hat{\Gamma}_b\} = 2\delta_{ab}$ . We then define the fermion parity shared by a pair of vortices  $\hat{\Sigma}_{ab} = i\hat{\Gamma}_a\hat{\Gamma}_b$ . The topological qubit can be uniquely specified by a set of measurements of the expectation value of the following Pauli matrices  $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ :

$$\hat{\sigma}_x = \hat{\Sigma}_{32}, \hat{\sigma}_y = \hat{\Sigma}_{13}, \hat{\sigma}_z = \hat{\Sigma}_{21}. \quad (5.1)$$

The non-Abelian braiding can be represented as the transformation of  $\langle \hat{\boldsymbol{\sigma}} \rangle$ .

Now we list the key assumptions to establish the non-Abelian properties of the vortices:

1. The fermion parity  $\hat{\Sigma}_{ab}$  is a physical observable that can be measured by suitable interferometry experiments, even at finite temperature.
2. All the bound states remain localized together with the zero-energy state when the vortices are transported. Therefore they can be considered as one composite system.
3. The tunneling processes of fermions between different vortices and transitions to the gapped continuum are exponentially suppressed due to the presence of the bulk superconducting gap. This condition needs to be satisfied in the first place to ensure the existence of (nearly) zero modes in the topological phase.

Under these conditions, the only local dynamical processes are the transitions of fermions between the localized bound states, e.g. scattering by collective excitations

like phonons. However, such processes necessarily conserve  $\hat{\Gamma}_a$ , therefore also the parities  $\hat{\Sigma}_{ab}$ .

To see this explicitly, the state of the qubit is described by the density matrix  $\hat{\rho}(t)$ . Because we are truncating the whole Hilbert space to include only those below our cutoff  $\Lambda$ , it is necessary to use the time-dependent instantaneous basis [112]. **At low-energies, The occupations of the various subgap states can be changed by four-fermion scattering processes or coupling to bosonic bath.** To be specific, we write down the Hamiltonian of the system:

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}. \quad (5.2)$$

Here  $\hat{H}_0$  is the Hamiltonian of the BCS superconductor with vortices, whose positions  $\mathbf{R}_i$  are time-dependent. At each moment of time  $\hat{H}_0$  can be diagonalized, yielding a set of complete eigenbasis which are represented by the time-dependent generalization of the aforementioned Bogoliubov quasiparticles  $\hat{\gamma}_{a0}(t), \hat{d}_{ai}(t)$ .  $\hat{H}_{\text{int}}$  describes all kinds of perturbations that are allowed under the assumptions.

Without going into the details of microscopic calculations, we write down the general Lindblad form of the master equation [113] governing the time-evolution of the density matrix:

$$\frac{d\hat{\rho}}{dt} = \frac{\partial\hat{\rho}}{\partial t} - i[\hat{H}_t(t), \hat{\rho}] + \hat{S}\hat{\rho}\hat{S}^\dagger - \frac{1}{2}\{\hat{S}^\dagger\hat{S}, \hat{\rho}\}. \quad (5.3)$$

The  $\frac{\partial\hat{\rho}}{\partial t}$  denotes the change of  $\hat{\rho}$  solely due to the change of basis states. Here  $\hat{H}_t$  describes the (effective) unitary evolution of the density matrix due to transitions between different fermionic states and the Lindblad superoperators  $\hat{S}$  corresponds

to non-unitary evolution induced by system-environment coupling. Our assumption on the locality of the interactions in the system implies that

$$[\hat{H}_t, \hat{\Sigma}_{ab}] = 0, [\hat{S}, \hat{\Sigma}_{ab}] = 0. \quad (5.4)$$

The time evolution of the expectation values of  $\boldsymbol{\sigma}(t)$  is given by

$$\frac{d\langle\boldsymbol{\sigma}\rangle}{dt} = \frac{d}{dt}\text{Tr}\boldsymbol{\sigma}\hat{\rho} = \text{Tr}\frac{\partial\boldsymbol{\sigma}}{\partial t}\hat{\rho}(t) + \text{Tr}\boldsymbol{\sigma}\frac{d\hat{\rho}}{dt}. \quad (5.5)$$

With (5.4), it is straightforward to check that

$$\text{Tr}\boldsymbol{\sigma}[\hat{H}, \hat{\rho}] = 0, \text{Tr}\boldsymbol{\sigma}(\hat{S}\hat{\rho}\hat{S}^\dagger - \frac{1}{2}\{\hat{S}^\dagger\hat{S}, \hat{\rho}\}) = 0. \quad (5.6)$$

Therefore we have

$$\frac{d\langle\boldsymbol{\sigma}(t)\rangle}{dt} = \text{Tr}\frac{\partial\boldsymbol{\sigma}(t)}{\partial t}\hat{\rho}(t) + \text{Tr}\boldsymbol{\sigma}(t)\frac{\partial\hat{\rho}(t)}{\partial t} = \partial_t\text{Tr}[\boldsymbol{\sigma}(t)\hat{\rho}(t)]. \quad (5.7)$$

As we have defined,  $\partial_t$  means that all changes come from the change in the basis  $\{\hat{\gamma}_{ai}(t)\}$ . Since after the braiding the system returns to its initial configuration, the operators  $\hat{\gamma}_{ia}$  undergo unitary transformations. So if the braiding starts at  $t = t_i$  and ends at  $t = t_f$ , we have the simple result  $\langle\boldsymbol{\sigma}(t_i)\rangle = \langle\boldsymbol{\sigma}(t_f)\rangle$ . However, the operators  $\hat{\Gamma}(t_f)$  are different from  $\hat{\Gamma}(t_i)$ . One can easily verify that the operators  $\hat{\Gamma}_a$  satisfy Ivanov's rule [35, 111] under braiding of vortices  $a$  and  $b$ :

$$\hat{\Gamma}_a \rightarrow \hat{\Gamma}_b, \hat{\Gamma}_b \rightarrow -\hat{\Gamma}_a. \quad (5.8)$$

And the transformation of  $\langle\hat{\boldsymbol{\sigma}}\rangle$  is identical to the case without any midgap states. In conclusion, in terms of physically measurable quantities, the non-Abelian statistics is well-defined in the presence of excited midgap states localized in the vortex core.

We also notice that in this formulation, the Abelian phase in the braiding is totally out of reach. It is very likely that occupations of the midgap states cause dephasing of this Abelian phase.

This result is thoroughly non-obvious because it may appear on first sight that arbitrary thermal occupancies of the mid-gap excited states would completely suppress the non-Abelian nature of the system since the Majorana mode resides entirely at zero energy and not in the excited mid-gap states.

We now briefly discuss how the condition of fermion parity conservation are satisfied in realistic systems. It requires that no local physical processes that can change fermion parity are present in the system. This is indeed the case in a superconductor since the presence of bulk superconducting gap suppresses single-particle excitations at low energies and results in even-odd effect in fermion number. Therefore in a fully gapped superconductor the fermion parity is indeed well-defined at equilibrium and there are no parity-violation processes intrinsic to the superconductor. On the other hand, if the topological qubits are in contact with gapless fermions, the fermion parity is apparently not a good quantum number (see [107] for a detailed discussion of related issues and possible resolutions). Therefore the topological qubits have to be separated galvanically from external sources of unpaired electrons, which can also be achieved experimentally (e.g. in superconducting charge qubits [114]).

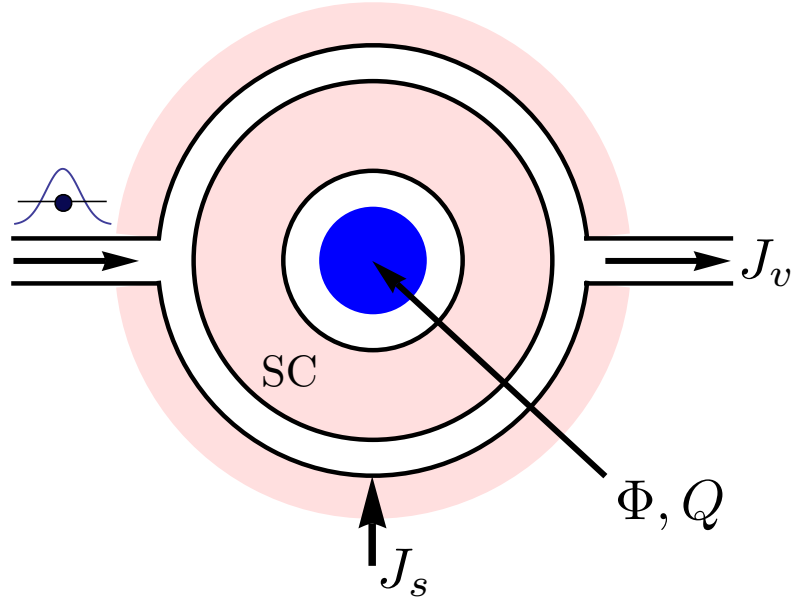


Figure 5.1: Mach-Zehnder interferometer proposed in Ref. [122] for topological qubit detection. Due to the Aharonov-Casher effect, the vortex current is sensitive to the charge enclosed. Long Josephson junction between two topological superconductors carries allows for Josephson vortices (fluxons) that carry Majorana zero-energy modes.

## 5.2 Interferometry in the Presence of Midgap States

We now discuss the effect of the midgap states in interferometry experiments designed for the qubit read-out [115, 116, 117, 118]. There is a number of recent proposals for interferometry experiments in topological superconductors [119, 120, 121]. Here we use an example of the Mach-Zehnder interferometer proposed by Grosfeld and Stern [122] based on Aharonov-Casher (AC) effect. In this proposal, a Josephson vortex (fluxon) is driven by supercurrent  $J_s$  to circumvent a superconducting island with charge  $Q$  and flux  $\Phi$ , see Fig. 5.1. The fluxon appearing at the interface

between two topological  $p_x + ip_y$  superconductors (represented by the shaded region in Fig. 5.1) carries a zero-energy Majorana modes, and behaves as a non-Abelian anyon. Therefore, the vortex current around the central superconductor is sensitive to the topological content of the enclosed superfluid. (We refer the reader to Ref. [122] for more details.) Indeed, the vortex current is proportional to the total tunneling amplitude:

$$\begin{aligned}
J_v &\propto |(t_L \hat{U}_L + t_R \hat{U}_R)|\Psi_0\rangle|^2 \\
&= |t_L|^2 + |t_R|^2 + 2\text{Re}\{t_L^* t_R \langle \Psi_0 | \hat{U}_L^{-1} \hat{U}_R | \Psi_0 \rangle\} \\
&= |t_L|^2 + |t_R|^2 + 2\text{Re}\{t_L^* t_R e^{i\varphi_{AC}} \langle \Psi_0 | \hat{M} | \Psi_0 \rangle\}.
\end{aligned} \tag{5.9}$$

Here  $|\Psi_0\rangle$  is the initial state of the system and  $\hat{U}_L$  and  $\hat{U}_R$  are the unitary evolution operators for the fluxon taking the two respective paths.  $\varphi_{AC}$  is the Aharonov-Casher phase accumulated by the fluxon:  $\varphi_{AC} = \pi Q/e$ . Here  $Q$  is the total charge enclosed by the trajectory of the fluxon, including the offset charge  $Q_{\text{ext}}$  set by external gate and the fermion parity  $n_p$  of the low-energy fermionic states:

$$Q = Q_{\text{ext}} + en_p. \tag{5.10}$$

$\hat{M}$  encodes the transformation solely due to the braiding statistics of the non-Abelian fluxon around  $n$  non-Abelian vortices. If the superconducting island contains no vortices, then  $\hat{M} = 1$  and the interference term is solely determined by the AC phase. The magnitude of the vortex current shows an oscillation:

$$J_v = J_{v0} \left[ 1 + \zeta \cos \left( \frac{\pi Q}{e} \right) \right]. \tag{5.11}$$

Here  $\zeta$  is the visibility of the interference.

When  $n$  is odd, there is no interference because  $\hat{M}|\Psi_0\rangle$  and  $|\Psi_0\rangle$  have different fermion parity, implying  $\langle\Psi_0|\hat{M}|\Psi_0\rangle = 0$ . To see this explicitly, let us consider  $n = 1$  and denote the Majorana zero mode in the vortex by  $\hat{\gamma}_1$ . When the Majorana fermion  $\hat{\gamma}_0$  in the fluxon is taken around  $\hat{\gamma}_1$ , a unitary transformation  $\hat{M} = \exp(\pm i\frac{\pi}{2}\hat{\gamma}_1\hat{\gamma}_0) = \pm i\hat{\gamma}_1\hat{\gamma}_0$  is acted upon the ground state of the system. Thus the matrix element  $\langle\Psi_0|\hat{M}|\Psi_0\rangle = 0$ . The vortex current becomes independent of the charge encircled. Therefore, the disappearance of the interference can be used as a signature of the non-Abelian statistics of the vortices.

We now consider a situation where the non-Abelian fluxon has midgap states other than the Majorana bound state. The internal state of the fluxon then also depends on the occupation of these midgap states. As we have argued in the previous section, as far as braiding is concerned the non-Abelian character is not affected at all by the presence of midgap states. So the interference still vanishes when there are odd numbers of non-Abelian vortices in the island. On the other hand, when there are no vortices in the island, transitions to the midgap states can significantly reduce the visibility of the interference term  $\zeta$ .

To understand quantitatively how the visibility of the interference pattern is affected by the midgap state, let us consider the following model of the fluxon. Since we are interested in the effect of midgap states, we assume there is only one midgap state and model the probe vortex by a two-level system, or spin 1/2, with the Hilbert space  $\{|0\rangle, |1\rangle\}$ . Here  $|1\rangle$  denotes the state with the midgap state occupied. We also assume that the charge enclosed by the interference trajectory  $Q = 0$  so we can



neglect the AC phase. The Hamiltonian is then given by

$$\hat{H} = |L\rangle\langle L| \otimes \hat{H}_L + |R\rangle\langle R| \otimes \hat{H}_R. \quad (5.12)$$

where  $\hat{H}_{L,R}$  is given by

$$\hat{H}_\eta = \frac{\Delta}{2}\sigma_z + \sigma_x \sum_k g_k (\hat{a}_{\eta,k}^\dagger + \hat{a}_{\eta,k}) + \sum_k \omega_{\eta,k} \hat{a}_{\eta,k}^\dagger \hat{a}_{\eta,k}. \quad (5.13)$$

Here  $\eta = L, R$ . Here  $\hat{a}_k$  are annihilation operators for a bosonic bath labeled by  $k$ .

The form of the coupling between the internal degree of freedom and the bosonic bath are motivated on very general grounds. In fact, Hermiticity requires that coupling between Majorana mode and any other fermionic modes have to take the following form:

$$H_{\text{coupling}} = i\hat{\gamma}_0(z\hat{d} + z^*\hat{d}^\dagger). \quad (5.14)$$

Here in this context  $\hat{\gamma}_0$  is the zero-energy Majorana operator in the fluxon and  $\hat{d}$  is the annihilation operator for the midgap fermion,  $z$  is a bosonic degree of freedom. We then use the mapping between Majorana operators and spin operators:  $\sigma_z = 2\hat{d}^\dagger\hat{d} - 1$ ,  $\sigma_x = i\hat{\gamma}_0(\hat{d} + \hat{d}^\dagger)$ ,  $\sigma_y = \hat{\gamma}_0(\hat{d}^\dagger - \hat{d})$  to rewrite the above coupling term as:

$$H_{\text{coupling}} = \text{Re}(z)\sigma_x + \text{Im}(z)\sigma_y. \quad (5.15)$$

If we take  $z \sim \hat{a} + \hat{a}^\dagger$ , we recover the coupling term in (5.13). Eq. (5.14) can arise from, e.g. electron-phonon interaction.

We also assume the bath couples to the fluxon locally so we introduce two independent baths for  $L$  and  $R$  paths. The unitary evolution at time  $t$  is then factorizable:

$$\hat{U}(t) = |L\rangle\langle L| \otimes \hat{U}_L(t) + |R\rangle\langle R| \otimes \hat{U}_R(t). \quad (5.16)$$

Given initial state  $\hat{\rho}(0) = \hat{\rho}_{\text{path}} \otimes \hat{\rho}_s \otimes \hat{\rho}_{\text{bath}}$ , we can find the off-diagonal component of the final state  $\hat{\rho}(t) = \hat{U}(t)\hat{\rho}(0)\hat{U}^\dagger(t)$ , corresponding to the interference, as

$$\begin{aligned}\lambda_{LR} &= \text{Tr} \left[ \hat{U}_L(t)\hat{U}_R^\dagger(t)\hat{\rho}_s \otimes \hat{\rho}_{\text{bath}} \right] \\ &= \text{Tr} \left[ \rho_s \text{Tr}_L[\hat{\rho}_{\text{bath},L}\hat{U}_L(t)] \text{Tr}_R[\hat{\rho}_{\text{bath},R}\hat{U}_R(t)] \right].\end{aligned}\tag{5.17}$$

Now we evaluate  $\hat{W}_\eta(t) = \text{Tr}_\eta[\hat{\rho}_{\text{bath},\eta}\hat{U}_\eta(t)]$  (notice  $\hat{W}_\eta$  is still an operator in the spin Hilbert space). We drop the  $\eta$  index in this calculation. First we switch to interaction picture and the evolution operator  $\hat{U}(t)$  can be represented formally as  $\hat{U}(t) = \mathcal{T} \exp\{-i \int_0^t dt' \hat{H}_1(t')\}$  where

$$\hat{H}_1(t) = \sum_k g_k (\sigma^+ e^{i\Delta t/2} + \sigma^- e^{-i\Delta t/2}) (\hat{a}_k^\dagger e^{i\omega_k t} + \hat{a}_k e^{-i\omega_k t}).\tag{5.18}$$

Following the derivation of the master equation for the density matrix, we can derive a ‘‘master equation’’ for  $\hat{W}(t)$  under the Born-Markovian approximation:

$$\frac{d\hat{W}}{dt} = -\gamma(\bar{n} + 1/2 + \sigma_z/2)\hat{W},\tag{5.19}$$

where  $\gamma = \pi \sum_k g_k^2 \delta(\omega_k - \Delta)$ ,  $\bar{n} = \gamma^{-1} \sum_k g_k^2 \bar{n}_k \delta(\omega_k - \Delta)$ .

Therefore, the visibility of the interference, proportional to the trace of  $\hat{W}$ , is given by

$$\zeta \propto \text{Tr}[\hat{W}(t)\rho_s] \propto e^{-\gamma\bar{n}t} = e^{-\gamma\bar{n}L/v}.\tag{5.20}$$

Here  $L$  is the length of the interferometer and  $v$  is the average velocity of the fluxon.

We notice that the model we have used is of course a simplification of the real fluxon. We only focus on the decoherence due to the midgap states and assume that only one such state is present. In reality, there could be many midgap states

which in principle lead to a stronger suppression of visibility. The approach taken here can be easily generalized to the case where more than one midgap states.

The above interferometer is able to detect the existence of non-Abelian vortices which requires that the Josephson vortex (i.e. fluxon) also has Majorana midgap states. To fully read out a topological qubit, one needs to measure the fermion parity of the qubit. This can also be done using interferometry experiments with flux qubits, essentially making use of the AC effect of Josephson vortices [39, 38].

Another relevant question is whether the thermal excitations of the (non-Majorana) midgap states localized in the vortex core have any effects on the interferometry. Since the interferometry is based on AC effect where vortex acquires a geometric phase after circling around some charges, one might naively expect that the interferometric current might depend on the occupation of the midgap states due to the charge associated with the midgap states (i.e. for a midgap state whose Bogoliubov wavefunctions are  $(u, v)$ , its charge is given by  $Q = e \int d\mathbf{r} (|u|^2 - |v|^2)$ ). The situation is more subtle, however, once one takes into account the screening effect due to the superfluid condensate. The kinetics of the screening process is beyond the scope of this chapter. However, assuming equilibrium situation, we now show that the geometric phases acquired by the Josephson vortices only depends on the total fermion parity in the low-energy midgap states (even if they are not Majorana zero-energy modes) and the offset charge set by the external gate voltage.

We follow here the formalism developed in the context of AC effect for flux qubits [39]. We assume that a superconducting island with several midgap fermionic states, labeled by  $\hat{d}_m^\dagger$ , is coupled to a flux qubit. In the low-energy regime well below

the bulk superconducting gap and the plasma frequency, the only degrees of freedom of this system are the superconducting phase  $\phi$  and the midgap fermions. We also assume that the phase varies slowly so the fermionic part of the system follows the BCS mean-field Hamiltonian with superconducting phase  $\phi$ .

We want to know the geometric phase associated with vortex tunneling in the presence of midgap fermions. It can be derived by calculating the transition amplitude  $\mathcal{A}_{fi}$  associated with a time-dependent phase  $\phi = \phi(t)$ :

$$\mathcal{A}_{fi} = \langle \phi_f | \hat{Q}_f \hat{U}(t_f, t_i) \hat{Q}_i^\dagger | \phi_i \rangle, \quad (5.21)$$

where  $|\phi\rangle$  denotes the BCS ground state with superconducting phase  $\phi$  and  $\phi_f - \phi_i = 2w\pi$ .  $\hat{Q}^\dagger = \prod_m (\hat{d}_m^\dagger)^{n_m}$  denote the occupation of the midgap fermionic states with  $n_m = 0, 1$ .

The midgap fermionic operators  $\hat{d}_m^\dagger$  are explicitly expressed in terms of Bogoliubov wavefunctions  $u_m$  and  $v_m$ :

$$\hat{d}_m^\dagger(t) = e^{-i\varepsilon_m t} \int d\mathbf{r} [u_m(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}) e^{i\phi/2} + v_m(\mathbf{r}) \hat{\psi}(\mathbf{r}) e^{-i\phi/2}]. \quad (5.22)$$

Therefore,

$$\hat{U}(t_f, t_i) \hat{d}_m^\dagger(t_i) \hat{U}^\dagger(t_f, t_i) = \hat{d}_m^\dagger(t_f) e^{i\pi w n_m}. \quad (5.23)$$

So the transition amplitude is evaluated as

$$\begin{aligned} \mathcal{A}_{fi} &= \langle \phi_f | \hat{Q}_f \hat{U}(t_f, t_i) \hat{Q}_i^\dagger \hat{U}^\dagger(t_f, t_i) \hat{U}(t_f, t_i) | \phi_i \rangle \\ &= e^{i\pi w n} e^{-i \sum_m n_m \varepsilon_m (t_f - t_i)} \langle \phi_f | \hat{Q}_f \hat{Q}_f^\dagger \hat{U}(t_f, t_i) | \phi_i \rangle \\ &= e^{i\pi w n} e^{-i \sum_m n_m \varepsilon_m (t_f - t_i)} \langle \phi_f | \hat{U}(t_f, t_i) | \phi_i \rangle \end{aligned} \quad (5.24)$$

We conclude that the geometric phase is precisely  $\pi wn = \frac{n}{2}(\phi_f - \phi_i)$ . Physically this reflects the fact that one fermion is “half” of a Cooper pair. The vortex tunneling causes the phase of the Cooper pair condensate changes by  $2\pi$  and correspondingly the fermionic states obtain  $\pi$  phases. Notice that the phase  $\sum_m \varepsilon_m(t_f - t_i)$  is simply the overall dynamical phase of the whole system due to its finite energy and does not contribute to the interference at all.

### 5.3 Depolarization of Qubits at Finite Temperature

We now study the coherence of the topological qubit itself. From our discussion on the effect of bound states in the vortex core, it is clear that decoherence only occurs when the qubit is interacting with a macroscopically large number of fermionic degrees of freedom, a fermionic bath. An example of such a bath is provided by the continuum of the gapped quasiparticles, which are unavoidably present in any realsuperconductors. Once the Majorana fermion is coupled to the bath via a tunneling Hamiltonian, the fermion occupation in the qubit can leak into the environment, resulting in the depolarization of the qubit. It is then crucial to have a fully gapped quasiparticle spectrum to ensure that such decoherence is exponentially small, as will be shown below.

To study the decay of a Majorana zero mode, we consider two such modes,  $\hat{\gamma}_1$  and  $\hat{\gamma}_2$ , forming an ordinary fermion  $\hat{c} = \hat{\gamma}_1 + i\hat{\gamma}_2$ . The gapped fermions are coupled locally to  $\hat{\gamma}_1$ , without any loss of generality. The coupling is mediated by a bosonic

bath. The Hamiltonian then reads

$$\hat{H} = i\varepsilon\hat{\gamma}_1\hat{\gamma}_2 + \sum_k \epsilon_k \hat{d}_k^\dagger \hat{d}_k + \sum_l \omega_l \hat{a}_l^\dagger \hat{a}_l + i \sum_{kl} g_{kl} \hat{\gamma}_1 (\hat{d}_k^\dagger + \hat{d}_k) (\hat{a}_l^\dagger + \hat{a}_l). \quad (5.25)$$

Here  $\hat{d}_k$  is the annihilation operator of the gapped fermions with quantum number  $k$  and energy  $\varepsilon_k$ .  $\hat{a}_l$  is the annihilation operators of the bosonic bath. The last term in the model Hamiltonian, representing the coupling between the Majorana zero mode and the gapped fermions mediated by the bosonic bath, has been justified in the previous section.

Since we are interested in the qubit only, we will derive the master equation for the reduced density matrix  $\hat{\rho}_r$ , tracing out the bosonic bath and the gapped fermions. The density matrix of the whole system evolves according to the equation of motion:

$$\frac{d\hat{\rho}}{dt} = i[\hat{H}_I, \hat{\rho}]. \quad (5.26)$$

Notice that we will be working in interaction picture in the following. Here the coupling Hamiltonian

$$\hat{H}_I(t) = i \sum_{kl} g_{kl} \hat{\gamma} \hat{\eta}_k(t) \hat{\phi}_l(t). \quad (5.27)$$

where

$$\begin{aligned} \hat{\eta}_k(t) &= \hat{d}_k e^{i\varepsilon_k t} + \hat{d}_k^\dagger e^{-i\varepsilon_k t}, \\ \hat{\phi}_l(t) &= \hat{a}_l e^{i\omega_l t} + \hat{a}_l^\dagger e^{-i\omega_l t}. \end{aligned} \quad (5.28)$$

Assume the coupling between the qubit and the bath is weak, we integrate the equation of motion for a time interval  $\Delta t$ :

$$\frac{\Delta \hat{\rho}_r}{\Delta t} = -\frac{1}{\Delta t} \int_t^{t+\Delta t} dt_1 \int_t^{t_1} dt_2 \text{Tr}_B[\hat{H}_I(t_1), [\hat{H}_I(t_2), \hat{\rho}(t_2)]]. \quad (5.29)$$

The first-order term vanishes due to the fact that  $\langle \hat{\phi}(t) \rangle = \langle \hat{\eta}(t) \rangle = 0$ . Now we make the Born approximation for the bath: assume that the bath is so large that it relaxes very quickly to thermal equilibrium. The density matrix of the whole system can be factorized as  $\hat{\rho}(t) = \hat{\rho}_r(t) \otimes \hat{\rho}_B$ . Here the bath includes with the gapped fermionic bath and the bosonic bath.

The commutator on the right-hand side of (5.29) can be evaluated:

$$\begin{aligned} & \text{Tr}_B[\hat{H}_I(t_1), [\hat{H}_I(t_2), \hat{\rho}(t_2)]] \approx \\ & [\hat{\rho}_r(t) - \hat{\gamma}(-1)^{\hat{n}} \hat{\rho}_r(t) (-1)^{\hat{n}} \hat{\gamma}] \left\{ \langle \hat{\eta}_k(t_1) \hat{\eta}_k(t_2) \rangle \langle \hat{\phi}_l(t_1) \hat{\phi}_l(t_2) \rangle + \langle \hat{\eta}_k(t_2) \hat{\eta}_k(t_1) \rangle \langle \hat{\phi}_l(t_2) \hat{\phi}_l(t_1) \rangle \right\} \end{aligned} \quad (5.30)$$

The factor  $(-1)^{\hat{n}}$  appears because of the anti-commutation relation between fermionic operators. The correlators of the bath are easily calculated:

$$\begin{aligned} \langle \hat{\eta}_k(t_1) \hat{\eta}_k(t_2) \rangle &= n_k^f e^{i\varepsilon_k(t_1-t_2)} + (1 - n_k^f) e^{-i\varepsilon_k(t_1-t_2)} \\ \langle \hat{\phi}_l(t_1) \hat{\phi}_l(t_2) \rangle &= n_l^b e^{i\omega_l(t_1-t_2)} + (n_l^b + 1) e^{-i\omega_l(t_1-t_2)} \end{aligned} \quad (5.31)$$

Here  $n_k^f = 1/(e^{\varepsilon_k/T} + 1)$ ,  $n_l^b = 1/(e^{\omega_l/T} - 1)$  are the Fermi and Bose distribution functions.

Performing the integral over  $t_1$  and  $t_2$ , we finally arrive at

$$\frac{d\hat{\rho}_r}{dt} = -\lambda [\hat{\rho}_r - \hat{\gamma}_1(-1)^{\hat{n}} \hat{\rho}_r(t) (-1)^{\hat{n}} \hat{\gamma}_1]. \quad (5.32)$$

Here  $\lambda$  is given by

$$\lambda = 2 \sum_{kl} g_{kl}^2 [(1 - n_k^f) n_l^b + n_k^f (n_l^b + 1)] \delta(\varepsilon_k - \omega_l). \quad (5.33)$$

Notice that at low temperatures  $T \ll \Delta$ , due to energy conservation, both  $n_l^b$  and  $n_k^f$  are suppressed by the Gibbs factor  $e^{-\Delta/T}$ . Therefore, the rate  $\lambda \sim e^{-\Delta/T}$ .

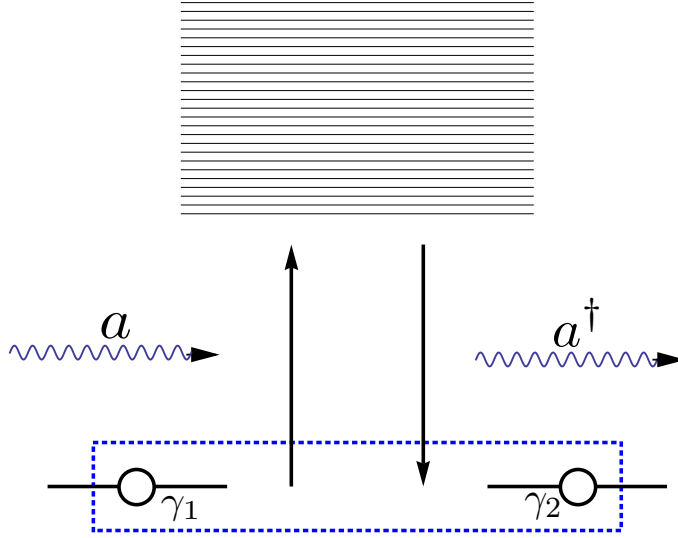


Figure 5.2: Schematic illustration of the topological qubit coupled to thermal bath, modeled by a collection of harmonic oscillators.

Then the polarization of the qubit  $\langle \sigma_z \rangle = \text{Tr}[\sigma_z \hat{\rho}_r]$  satisfies  $d_t \langle \sigma_z \rangle = -2\lambda \langle \sigma_z \rangle$ . Therefore the lifetime of the topological qubit is given by  $T_1 \sim \lambda^{-1}$ . Physically, this is reasonable since we introduce tunneling term between the Majorana fermion and the gapped fermionic environment so the fermion parity of the qubit is no longer conserved. It is expected that  $\lambda$  is determined by the exponential factor  $e^{-\Delta/T}$  when  $T \ll \Delta$ . Therefore, this provides a quantitative calibration of the protection of the topological qubit at finite temperature. In the high-temperature limit  $T \gg \Delta$ , the distribution function scales linearly with  $T$  so the decay rate is proportional to  $T$ . This is quite expected since  $T \gg \Delta$ , the gap does not play a role. We note that a recent work by Goldstein and Chamon [110] studying the decay rate of Majorana zero modes coupled to classical noise essentially corresponds to the high-temperature limit of our calculation  $T \gg \Delta$  and, as such, does not



apply to any realistic system where the temperature is assumed to be low, i.e.  $T \ll \Delta$ . In fact, in the trivial limit of  $\Delta \ll T$ , the Majorana decoherence is large and weakly temperature dependent because the fermion parity is no longer preserved and the fermions can simply leak into the fermionic bath [123]. By definition, this classical limit of  $T \gg \Delta$  is of no interest for the topological quantum computation schemes since the topological superconductivity itself (or for that matter, any kind of superconductivity) will be completely absent in this regime. Our result makes sense from the qualitative considerations: quantum information is encoded in non-local fermionic modes and changing fermion parity requires having large thermal fluctuations or external noise sources with finite spectral weight at frequencies  $\omega \sim \Delta$ . Furthermore, it is important to notice that such relaxation can only occur when the qubit is coupled to a continuum of fermionic states which renders the fermion parity of the qubit undefined. Intuitively, the fermion staying in the qubit can tunnel to the continuum irreversibly, which is accounted for by the procedure of “tracing out the bath” in our derivation of the master equation. It is instructive to compare this result to a different scenario, where the zero-energy fermionic state is coupled to a fermionic state (or a finite number of them) instead of a continuum. In that case, due to hybridization between the states the fermion number oscillates between the two levels with a period (recurrence time) determined by the energy difference  $\Delta E$  between them. The expectation value of the fermion number (or spectral weight) in the zero-energy state is depleted and oscillatory in time, but will not decay to zero.

The above derivation can be straightforwardly generalized to  $N > 2$  Majorana

fermions, each coupled locally to gapped fermions and bosonic bath.

$$\frac{d\hat{\rho}_r}{dt} = - \sum_{i=1}^N \lambda_i [\hat{\rho}_r - \hat{\gamma}_i (-1)^{\hat{n}_i} \hat{\rho}_r (-1)^{\hat{n}_i} \hat{\gamma}_i], \quad (5.34)$$

The depolarization of the qubit can be calculated in the same fashion.

## 5.4 Conclusion and Discussion

We study quantum coherence of the Majorana-based topological qubits. We analyze the non-Abelian braiding in the presence of midgap states, and demonstrate that when formulating in terms of the physical observable (fermion parity of the qubit), the braiding statistics is insensitive to the thermal occupation of the midgap states. We also clarify here the conditions for such topological protection to hold. Our conclusion applies to the case of localized midgap states in the vortex core which are transported along with the Majorana zero states during the braiding operations. If there are spurious (e.g. impurity-induced [124, 125, 126]) midgap bound states spatially located near the Majorana zero-energy states but are not transported together with them, they could strongly affect braiding operations. For example, during braiding the fermion in the qubit has some probability (roughly determined by the non-adiabaticity of the braiding operation) to hybridize with the other bound states near its path leading to an error. If the disorder is weak and short-ranged, such low-energy states are unlikely to occur unless the bulk superconducting gap is significantly suppressed at some spatial points (e.g. vortices) as it is well-known that for a single short-range impurity the energy of such a bound state is close to the bulk excitation gap [127, 128]. Thus, well-separated impurity-induced

bound states are typically close to the gap edge and would not affect braiding operations. If the concentration of impurities is increased, then it is meaningful to discuss the probability distribution of the lowest excited bound state in the system [129]. The distribution of the first excited states determining the minigaps depends on many microscopic details (e.g. system size, concentration of the disorder). Since the magnitude of the minigaps is system-specific, one should evaluate the minigap for a given sample. As a general guiding principle, it is important to reduce the effect of the disorder which limits the speed of braiding operations. However, we note here that physically moving anyons for braiding operations might not be necessary and there are alternative measurement-only approaches to topological quantum computation [130] where the issue of the low-lying localized bound states is not relevant.

We also consider the read-out of topological qubits via interferometry experiments. We study the Mach-Zehnder interferometer based on Aharonov-Casher effect and show that the main effect of midgap states in the Josephson vortices is the reduction of the visibility of the read-out signal. We also consider the effect of thermal excitations involving midgap states of Abrikosov vortices localized in the bulk on the interferometry and find that such processes do not effect the signal provided the system reaches equilibrium fast enough compared to the tunneling time of the Josephson vortices.

Finally, we address the issue of the quantum coherence of the topological qubit itself coupled to a gapped fermionic bath via quantum fluctuations. We derive the master equation governing the time evolution of the reduced density matrix of the topological qubit using a simple physical model Hamiltonian. The decoherence rate

of the qubit is exponentially suppressed at low temperatures  $T \ll \Delta$ . Since topological protection assumes that fermion parity in the superconductor is preserved, our result is very intuitive.

We conclude that the Majorana-based qubits are indeed topologically well-protected at low temperatures as long as the experimental temperature regime is well below the superconducting gap energy.

## Chapter 6

# Non-adiabaticity in the Braiding of Majorana

## Fermions

Mathematical definition of quantum statistics necessarily builds upon the concept of Berry phase of many-body wavefunctions. This implies that the adiabaticity of braiding is an essential ingredient for non-Abelian statistics, since the quantum state has to stay in the ground state manifold during the entire process of the braiding [131, 132]. In the real world, however, braidings are necessarily performed within a finite time interval, *i.e.*, they are always non-Adiabatic. As known from the adiabatic perturbation theory, Berry phase is the leading-order term in the adiabatic perturbative expansion. [133, 134, 135] Given the fundamental role played by adiabatic braiding in TQC, it is therefore important to understand quantitatively the higher order corrections arising from non-adiabatic evolution,

In this chapter, we present a systematic study of the non-adiabatic corrections to the braiding of non-Abelian anyons and develop formalism to describe their dynamical aspects. In our treatment, braidings are considered as dynamical evolutions of the many-body system, essentially using the time-dependent Schrödinger equation of the BCS condensate whose solutions are derived from time-dependent Bogoliubov-de Gennes (BdG) equation. Generally, adiabaticity may break down in

three different ways: (a) tunneling of non-Abelian anyons when there are multiples of them, which splits the degenerate ground state manifold and therefore introduces additional dynamical phases in the evolution; (b) transitions to excited bound states outside the Hilbert space of zero-energy states, in this case topologically protected braidings have to be defined within an enlarged Hilbert space; (c) transitions to the continuum of extended states which render the fermion parity in the low-energy Hilbert space ill-defined. These non-adiabatic effects are possible sources of errors for quantum gates in TQC. The main goal of this chapter is to quantitatively address these effects and their implications on quantum computation.

Our work is the first systematic attempt to study non-adiabaticity in the anyonic braiding of non-Abelian quantum systems. Given that the braiding of non-Abelian anyons is the unitary gate operation [136] in topological quantum computation [100], understanding the dynamics of braiding as developed in this work is one of the keys to understanding possible errors in topological quantum computation. The other possible source of error in topological quantum computation is the lifting of the ground state anyonic degeneracy due to inter-anyon tunneling, which we have studied elsewhere [91, 106]. Although we study the braiding non-adiabaticity in the specific context of the topological chiral p-wave superconductors using the dynamical BdG equations within the BCS theory, our work should be of general validity to all known topological quantum computation platforms, since all currently known non-Abelian anyonic platforms in nature are based on the  $SU(2)_2$  conformal field theory of Ising anyons, which are all isomorphic to the chiral p-wave topological superconductors [100]. As such, our work, with perhaps some

minor modifications in the details, should apply to the fractional quantum Hall non-Abelian qubits [136], real p-wave superconducting systems based on solids [46] and quantum gases [137], topological insulator-superconductor heterostructures [47], and semiconductor-superconductor sandwich structures [49] and nanowires [59, 125]. Our results are quite general and are independent, in principle, of the detailed methods for the anyonic braiding which could vary from system to system in details. However, it is worthy to point out that the susceptibility of the systems to the non-adiabatic effects is sensitive to the microscopic details, such as the size of the bulk gap, the overlap between the various eigenstates which will become further clarified later.

## 6.1 Quantum Statistics of Majorana Fermions

### 6.1.1 Quantum Statistics and Adiabatic Evolution

We first briefly review how quantum statistics is formulated mathematically in terms of the adiabatic evolution of many-body wavefunctions, following a recent exposition in Ref. [138]. Consider the general many-body Hamiltonian  $\hat{\mathcal{H}}[\mathbf{R}_1(t), \dots, \mathbf{R}_n(t)]$  where parameters  $\{\mathbf{R}_i\}$  represent positions of quasiparticles. We assume the existence of well-defined, localized excitations which we call quasiparticles. At each moment  $t$ , there exists a subspace of instantaneous eigenstates of  $\hat{\mathcal{H}}[\mathbf{R}_1(t), \dots, \mathbf{R}_n(t)]$  with degenerate energy eigenvalues. Instantaneous eigenstates in the subspace are labeled as  $|\alpha(t)\rangle \equiv |\alpha(\{\mathbf{R}_i(t)\})\rangle$ . We constraint our discussion in the ground state subspace with zero energy eigenvalue.

The adiabatic exchange of any two particles can be mathematically implemented by adiabatically changing the positions of two particles, say  $\mathbf{R}_i$  and  $\mathbf{R}_j$ , in such a way that in the end they are interchanged. This means that

$$\mathbf{R}_i(T) = \mathbf{R}_j(0), \mathbf{R}_j(T) = \mathbf{R}_i(0). \quad (6.1)$$

According to the adiabatic theorem [139], it results in a unitary transformation within the subspace: if the system is initially in state  $|\psi(0)\rangle$ , then  $|\psi(T)\rangle = \hat{U}|\psi(0)\rangle$ . To determine  $\hat{U}$ , we first consider initial states  $|\psi(0)\rangle = |\alpha(\{\mathbf{R}_i(0)\})\rangle$ . Under this evolution the final state can be written as

$$|\psi_\alpha(T)\rangle = \hat{U}_0|\alpha(T)\rangle. \quad (6.2)$$

Here the matrix  $\hat{U}_0$  is the non-Abelian Berry phase:[133, 140, 141]

$$\hat{U}_0 = \mathcal{P} \exp \left( i \int_0^T dt \hat{\mathcal{M}}(t) \right), \quad (6.3)$$

where  $\mathcal{P}$  denotes path ordering and matrix element of the Berry's connection  $\hat{\mathcal{M}}$  is given by

$$\hat{\mathcal{M}}_{\alpha\beta}(t) = i\langle\alpha(t)|\dot{\beta}(t)\rangle. \quad (6.4)$$

Although the exchange defines a cyclic trajectory in the parameter space of Hamiltonian, the final basis states can be different from the initial ones (e.g,  $|\alpha(\{\mathbf{R}_i\})\rangle$  can be multivalued functions of  $\mathbf{R}_i$ , which is allowed if we are considering quasiparticles being collective excitations of many-body systems). The only requirement we impose is that the instantaneous eigenstates  $\{\alpha(t)\}$  are continuous in  $t$ . Therefore, we have another matrix  $\hat{B}$  defined as  $\hat{B}_{\alpha\beta} \equiv \langle\alpha(0)|\beta(T)\rangle$ , relating



$\{\alpha(T)\}$  to  $\{\alpha(0)\}$ :  $|\alpha(T)\rangle = \hat{B}_{\alpha\beta}|\beta(0)\rangle$ . Combining with (6.2), we now have the expression for  $\hat{U}$ :

$$|\psi(T)\rangle = \hat{U}_0 \hat{B} |\psi(0)\rangle. \quad (6.5)$$

Therefore

$$\hat{U} = \hat{U}_0 \hat{B} = \mathcal{P} \exp \left( i \int_0^T dt \hat{\mathcal{M}}(t) \right) \hat{B}. \quad (6.6)$$

In fact, the factorization of  $\hat{U}$  into  $\hat{U}_0$  and  $\hat{B}$  is somewhat arbitrary and gauge-dependent. However, their combination  $\hat{U}$  is gauge-independent provided that the time-evolution is cyclic in parameter space (positions of particles). The unitary transformation  $\hat{U}$  defines the statistics of quasiparticles.

### 6.1.2 Non-Abelian Majorana Fermions

We now specialize to the non-Abelian statistics of Majorana fermions in topological superconductors, carefully treating the effect of Berry phases. We mainly use spinless superconducting fermions as examples of topological superconductors in both 1D and 2D since all known topological superconducting systems supporting non-Abelian excitations essentially stem from spinless chiral p-wave superconductors. [19, 61, 47]

In the BCS mean-field description of superconductors, the Hamiltonian is particle-hole symmetric due to  $\mathbb{U}(1)$  symmetry breaking. In terms of Nambu spinor  $\hat{\Psi}(\mathbf{r}) = (\hat{\psi}(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}))^T$ , the BCS Hamiltonian is expressed as  $\hat{\mathcal{H}}_{\text{BCS}} = \frac{1}{2} \int d^2\mathbf{r} \hat{\Psi}^\dagger(\mathbf{r}) H_{\text{BdG}} \hat{\Psi}(\mathbf{r})$ .

The Bogoliubov-de Gennes Hamiltonian  $\mathbf{H}_{\text{BdG}}$  takes the following form [20, 19]

$$\mathbf{H}_{\text{BdG}} = \begin{pmatrix} \mathbf{h} & \Delta \\ \Delta^\dagger & -\mathbf{h}^T \end{pmatrix}, \quad (6.7)$$

where  $\mathbf{h}$  is the single-particle Hamiltonian [for spinless fermions it is simply  $\mathbf{h} = (-\frac{1}{2m}\partial_{\mathbf{r}}^2 - \mu)\delta(\mathbf{r} - \mathbf{r}')$ ] and  $\Delta$  is the gap operator. The BCS Hamiltonian can be diagonalized by Bogoliubov transformation

$$\hat{\gamma}^\dagger = \int d^2\mathbf{r} [u(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}) + v(\mathbf{r})\hat{\psi}(\mathbf{r})]. \quad (6.8)$$

Here the wavefunctions  $u(\mathbf{r})$  and  $v(\mathbf{r})$  satisfy BdG equations:

$$\mathbf{H}_{\text{BdG}} \begin{pmatrix} u(\mathbf{r}) \\ v(\mathbf{r}) \end{pmatrix} = E \begin{pmatrix} u(\mathbf{r}) \\ v(\mathbf{r}) \end{pmatrix}. \quad (6.9)$$

Throughout this work, we adopt the convention that operators which are hatted are those acting on many-body Fock states while bold ones denote matrices in “lattice” space.

The single-particle excitations  $\hat{\gamma}$ , known as Bogoliubov quasiparticles, are coherent superpositions of particles and holes. The particle-hole symmetry implies that the quasiparticle with eigenenergy  $E$  and that with eigenenergy  $-E$  are related by  $\hat{\gamma}_{-E} = \hat{\gamma}_E^\dagger$ . Therefore,  $E = 0$  state corresponds to a Majorana fermion  $\hat{\gamma}_0 = \hat{\gamma}_0^\dagger$  [76]. The existence of such zero-energy excitations also implies a non-trivial degeneracy of ground states: when there are  $2N$  such Majorana fermions, they combine pair-wisely into  $N$  Dirac fermionic modes which can either be occupied or unoccupied, leading to  $2^N$ -fold degenerate ground states. The degeneracy is further

reduced to  $2^{N-1}$  by fermion parity [100]. Since these fermionic modes are intrinsically non-local, any local perturbation can not affect the non-local occupancy and thus the ground state degeneracy is topologically protected. This non-locality lies at the heart of the idea of topological qubits.

We are mostly interested in Majorana zero-energy states that are bound states at certain point defects (*e.g.* vortices in 2D, domain walls in 1D). In fact, Majorana bound states are naturally hosted by defects because that zero-energy states only appear when gap vanishes. Defects can be moved along with the Majorana fermions bound to them. Braidings of such Majorana fermions realizes very non-trivial non-Abelian statistics.

We now apply the general theory of quantum statistics as previously discussed in Sec. 6.1.1 to the case of Majorana fermions in topological superconductors. The simplest setting where non-trivial statistics can be seen is the adiabatic braiding of two spatially separated Majorana fermions  $\hat{\gamma}_1$  and  $\hat{\gamma}_2$ . We denote the two bound state solutions of the BdG equation by  $\Psi_{01}$  and  $\Psi_{02}$ . When  $\mathbf{R}_1$  and  $\mathbf{R}_2$  vary with time they become instantaneous zero-energy eigenstates of BdG Hamiltonian. We choose their phases in such a way that the explicit analytical continuation of BdG wavefunction leads to the following basis transformation under exchange [35, 142]

$$\begin{aligned}\Psi_{01}(T) &= s\Psi_{02}(0) \\ \Psi_{02}(T) &= -s\Psi_{01}(0),\end{aligned}\tag{6.10}$$

where  $s = \pm 1$ . The value of  $s$  depends on the choice of wavefunctions and we choose the convention that  $s = 1$  throughout this work. In the case of Majorana fermions in vortices, the additional minus sign originates from branch cuts introduced to define

the phase of wavefunctions. This transformation actually gives the  $\hat{B}$  matrix in the general theory. Equivalently in terms of quasiparticle operator, we have

$$\begin{aligned}\hat{\gamma}_1 &\rightarrow \hat{\gamma}_2 \\ \hat{\gamma}_2 &\rightarrow -\hat{\gamma}_1.\end{aligned}\tag{6.11}$$

If we define the non-local fermionic mode  $\hat{d}^\dagger = \frac{1}{\sqrt{2}}(\hat{\gamma}_1 + i\hat{\gamma}_2)$ , the states with even and odd fermion parity are given by  $|g\rangle$  and  $\hat{d}^\dagger|g\rangle$ . Due to the conservation of fermion parity, the two states are never coupled. However, the non-Abelian statistics still manifest itself in the phase factor acquired by the two states after an adiabatic exchange. To see this, first we notice that under exchange, the analytical continuation(or basis transformation) gives the following transformation of the two states:

$$\begin{aligned}|g\rangle &\rightarrow e^{i\varphi}|g\rangle \\ \hat{d}^\dagger|g\rangle &\rightarrow e^{i\frac{\pi}{2}}e^{i\varphi}\hat{d}^\dagger|g\rangle.\end{aligned}\tag{6.12}$$

Here the  $\frac{\pi}{2}$  phase difference is reminiscence of non-Abelian statistics.

So far we have obtained the basis transformation matrix  $\hat{B}$ . To know the full quantum statistics we also need to calculate the adiabatic evolution  $\hat{U}_0$ . We now show by explicit calculation that  $\hat{U}_0 \propto \hat{1}$  up to exponentially small corrections. This requires knowledge of Berry connection accompanying adiabatic evolutions of BCS states. Fortunately, for BCS superconductor the calculation of many-body Berry phase can be done analytically [34]. The ground state  $|g\rangle$  has the defining property that it is annihilated by all quasiparticle operators  $\gamma_n$ . All other states can be obtained by populating Bogoliubov quasiparticles on the ground state  $|g\rangle$ . Let us consider a state with  $M$  quasiparticles  $|n_1, n_2, \dots, n_M\rangle = \hat{\gamma}_{n_1}^\dagger \hat{\gamma}_{n_2}^\dagger \cdots \hat{\gamma}_{n_M}^\dagger |g\rangle$ . The

Berry connection of this state then reads [34]

$$\langle n_1, \dots, n_M | \partial | n_1, \dots, n_M \rangle = \langle g | \partial | g \rangle + \sum_{i=1}^M (u_{n_i}^*, v_{n_i}^*) \partial \begin{pmatrix} u_{n_i} \\ v_{n_i} \end{pmatrix}. \quad (6.13)$$

So the difference between the Berry phase of a state with quasiparticles and the ground state is simply the sum of ‘‘Berry phase’’ of the corresponding BdG wavefunctions. Since the Berry phase of ground state  $|g\rangle$  can be eliminated by a global  $U(1)$  transformation, only the difference has physical meaning.

According to (6.13), the relevant term to be evaluated is

$$(u_{01}^* - iu_{02}^*, v_{01}^* - iv_{02}^*) \partial \begin{pmatrix} u_{01} + iu_{02} \\ v_{01} + iv_{02} \end{pmatrix} = 2\text{Re}(u_1^* \partial u_1 + u_2^* \partial u_2) + 2i\text{Re}(u_1^* \partial u_2 - u_2^* \partial u_1), \quad (6.14)$$

where we have made use of the Majorana condition  $v = u^*$ . The first term in (6.14) vanishes because  $\int u^* \partial u$  must be purely imaginary. The second term has a non-vanishing contribution to the total Berry phase. However, due to the localized nature of zero-energy state, the overlap between  $u_1$  and  $u_2$  is exponentially small:

$$\int_0^T dt \text{Re}(u_1^* \partial_t u_2 - u_2^* \partial_t u_1) \sim e^{-|\mathbf{R}_1 - \mathbf{R}_2|/\xi}. \quad (6.15)$$

Therefore the Berry phase can be neglected in the limit of large separation  $R$ . This completes our discussion of non-Abelian statistics. The above calculation can be easily generalized to the case of many anyons.

## 6.2 Time-dependent Bogoliubov-de Gennes Equation

In this section we derive the formalism to track down time-evolution of BCS condensate within mean-field theory. For BCS superconductivity arising from interactions, the pairing order parameter has to be determined self-consistently, which makes the mathematical problem highly nonlinear. In the situations that we are interested in, it is not critical where the pairing comes from. In some systems that are believed to be experimentally accessible, e.g. semiconductor/superconductor heterostructure, superconductivity is induced by proximity effect [47, 49] and there is no need to keep track of the self-consistency. We will take the perspective that order parameter is simply a external field in the Hamiltonian.

The time-dependent BdG equation [143, 144] has been widely used to describe dynamical phenomena in BCS superconductors. To be self-contained here we present a derivation of the time-dependent BdG equation highlighting its connection to quasiparticle operators. It can also be derived by methods of Heisenberg equation of motion or Green's function. Suppose we have a time-dependent BdG Hamiltonian  $H_{\text{BdG}}(t)$ . The unitary time-evolution of the many-body system is formally given by

$$\hat{U}(t) = \mathcal{T} \exp \left[ -i \int_0^t dt' \hat{\mathcal{H}}_{\text{BCS}}(t') \right]. \quad (6.16)$$

To obtain an explicit form of  $\hat{U}(t)$ , we define the time-dependent Bogoliubov quasiparticle operator as

$$\hat{\gamma}_n(t) = \hat{U}(t) \hat{\gamma}_n \hat{U}^\dagger(t), \quad (6.17)$$

where  $\hat{\gamma}_n$  is the quasiparticle operator for  $\hat{\mathcal{H}}_{\text{BCS}}(0)$  and the corresponding BdG

wavefunction is  $u_n(\mathbf{r}), v_n(\mathbf{r})$ . We adopt the normalization condition

$$\int d^2\mathbf{r} |u_n(\mathbf{r})|^2 + |v_n(\mathbf{r})|^2 = 1, \quad (6.18)$$

which means  $\{\hat{\gamma}_n, \hat{\gamma}_n^\dagger\} = 1$ .

$\hat{\gamma}_n(t)$  by definition satisfies the following equation of motion

$$i \frac{d\hat{\gamma}_n(t)}{dt} = [\hat{\mathcal{H}}_{\text{BCS}}(t), \hat{\gamma}_n(t)]. \quad (6.19)$$

In fact, by direct calculation one can show that the equation of motion is solved by

$$\hat{\gamma}_n^\dagger(t) = \int d\mathbf{r} \left[ u_n(\mathbf{r}, t) \hat{\psi}(\mathbf{r}) + v_n(\mathbf{r}, t) \hat{\psi}^\dagger(\mathbf{r}) \right], \quad (6.20)$$

where the wavefunction  $u_n(\mathbf{r}, t)$  and  $v_n(\mathbf{r}, t)$  are solutions of time-dependent BdG equation:

$$i \frac{d}{dt} \begin{pmatrix} u_n(\mathbf{r}, t) \\ v_n(\mathbf{r}, t) \end{pmatrix} = \mathbf{H}_{\text{BdG}}(t) \begin{pmatrix} u_n(\mathbf{r}, t) \\ v_n(\mathbf{r}, t) \end{pmatrix} \quad (6.21)$$

together with initial condition (another way of saying  $\hat{\gamma}_n(0) = \hat{\gamma}_n$ )

$$u_n(\mathbf{r}, 0) = u_n(\mathbf{r}), v_n(\mathbf{r}, 0) = v_n(\mathbf{r}). \quad (6.22)$$

As long as the solutions of the time-dependent BdG equation (6.21) are obtained, we can construct the operators  $\{\hat{\gamma}_n(t)\}$ .

We now derive an explicit formula of  $\hat{U}$  when the time-evolution is cyclic (*i.e.*,  $\hat{\mathcal{H}}_{\text{BCS}}(T) = \hat{\mathcal{H}}_{\text{BCS}}(0)$ ). In that case, it is always possible to express  $\gamma_n(T)$  as a linear combination of  $\hat{\gamma}_n \equiv \hat{\gamma}_n(0)$  and  $\hat{\gamma}_n^\dagger$ . Since particle number is not conserved, it is more convenient to work with Majorana operators. We thus write  $\hat{\gamma}_n = \hat{c}_{2n-1} + i\hat{c}_{2n}$  where  $c_m$  are Majorana operators. Suppose the BdG matrix has totally  $2N$  eigenvectors

so  $n = 1, 2, \dots, N$ . Assume that by solving time-dependent BdG equation we obtain the transformation of  $c_m$  as follows:

$$\hat{c}_k(T) = \sum_l \mathbf{V}_{kl} \hat{c}_l, \quad (6.23)$$

where  $\mathbf{V} \in \mathbb{S}\mathbb{O}(2N)$  as required by unitarity and the conservation of fermion parity. The matrix  $\mathbf{V}$  can be calculated once we know the BdG wavefunctions. Then we can write down an explicit expression of  $\hat{U}(T)$  in terms of  $\hat{c}_m$  [28]:

$$\hat{U}(T) = \exp \left( \frac{1}{4} \sum_{mn} \mathbf{D}_{mn} \hat{c}_m \hat{c}_n \right), \quad (6.24)$$

where the matrix  $\mathbf{D}$  is defined by the relation  $e^{-\mathbf{D}} = \mathbf{V}$ . Here  $\mathbf{D}$  is necessarily a real, skew-symmetric matrix. We notice that the usefulness of (6.24) is actually not limited to cyclic evolution. In fact, (6.24) is purely an algebraic identity that shows any  $\mathbb{S}\mathbb{O}(2N)$  rotation of  $2N$  Majorana operators can be implemented by a unitary transformation.

In the following we outline the method to solve the time-dependent BdG equation. To make connection with the previous discussion of Berry phase, we work in the “instantaneous” eigenbasis of time-dependent Hamiltonian. At each moment  $t$ , the BdG Hamiltonian  $\mathbf{H}_{\text{BdG}}(t)$  can be diagonalized yielding a set of orthonormal eigenfunctions  $\{\Psi_n(\mathbf{r}, t)\}$ . A remark is right in order: because of particle-hole symmetry, the spectrum of BdG Hamiltonian is symmetric with respect to zero energy and the quasiparticle corresponding to negative energy are really “holes” of positive energy states. However, at the level of solving BdG equation mathematically, both positive and negative energy eigenstates have to be retained to form a complete



basis. The most general form of BdG wavefunction can be expanded as

$$\Psi(\mathbf{r}, t) = \sum_n c_n(t) \Psi_n(\mathbf{r}, t). \quad (6.25)$$

Plugging into time-dependent BdG equation, we obtain

$$i\dot{c}_n + \sum_m \mathbf{M}_{nm}(t) c_m = E_n(t) c_n, \quad (6.26)$$

where  $\mathbf{M}_{nm}(t) = i\langle \Psi_n(t) | \partial_t | \Psi_m(t) \rangle$ .

Assume that starting from initial condition  $c_n(0) = \delta_{mn}$  (roughly the quasiparticle is in the  $\Psi_m$  state at  $t = 0$ ), we obtain the solutions of (6.26) at  $t = T$  denoted by  $c_n^m(T)$ . The transformation of basis states themselves is given by the matrix  $\hat{B}$ .

Combining these two transformations we find

$$\hat{\gamma}_n \rightarrow \sum_{kl} c_k^n \hat{B}_{kl} \hat{\gamma}_l, \quad (6.27)$$

from which the linear transformation  $\mathbf{V}$  can be directly read off. Then by taking the matrix log of  $\hat{V}$  we can obtain the evolution operator. This is the procedure that we will use to solve the (cyclic) dynamics of BCS superconductors.

We will not be attempting to obtain the most general solution, since it depends heavily on the microscopic details. Instead, we focus on two major aspects of non-adiabaticity: (a) finite splitting of ground state degeneracy which only becomes appreciable when the braiding time is comparable to the “tunneling” time of Majorana fermions. (b) excited states outside the ground state subspace. In both cases, the non-adiabaticity caused by the finite speed of transporting the anyons enters through the Berry matrix  $\mathbf{M}$ . The explicit forms of the matrix  $\mathbf{M}$  will be presented in the following analysis, but several general remarks are in order. Since the matrix

element of  $\mathbf{M}$  is given by  $\mathbf{M}_{nm} = i\langle\Psi_n|\partial_t|\Psi_m\rangle$ , and the time dependence only enters in the parameters  $\{\mathbf{R}_i(t)\}$  in the basis eigenstates, we can rewrite it as

$$\mathbf{M}_{nm} = \sum_i \dot{\mathbf{R}}_i \cdot i\langle\Psi_n|\nabla_{\mathbf{R}_i}|\Psi_m\rangle, \quad (6.28)$$

where  $i\langle\Psi_n|\nabla_{\mathbf{R}_i}|\Psi_m\rangle$  is time-independent. Therefore, the degree of the non-adiabaticity is characterized by  $|\dot{\mathbf{R}}| \sim \bar{R}\omega$  where  $\bar{R}$  measures the average distance between the two anyons that are braided and  $\omega$  measures the instantaneous angular velocity. In general, the speed of the anyons can vary with time. But if we assume that the variation of the speed is not significant, then it is reasonable to characterize the non-adiabaticity by the average value of  $\omega$  and neglect its variation. We will make this approximation throughout our work. In this sense, we can relate  $\omega$  to the total time  $T$  of the braiding operation by  $\bar{\omega} = \frac{2\pi}{T}$ .

The path  $\{\mathbf{R}_i(t)\}$  can be arbitrary as long as they form a braid. To illustrate the physics in the simplest setting, we assume that the two vortices travel on a circle whenever we have to specify the trajectory. Mathematically, the positions of the two anyons are

$$\mathbf{R}_1(t) = -\mathbf{R}_2(t) = R(\cos(\omega t + \theta_0), \sin(\omega t + \theta_0)). \quad (6.29)$$

Here  $\omega = \frac{2\pi}{T}$ . The choice of the path makes the Berry matrix  $\mathbf{M}$  independent of time which simplifies our calculation. In realistic situations, the Berry matrix may acquire time-dependence from the variation of the speed of the anyons varies with time, but we expect this level of complication has only minor quantitative changes to our results.

### 6.2.1 Effect of Tunneling Splitting

The derivation of transformation rule (6.11) assumes that the two Majorana bound states have vanishing energies so there is no dynamical phase accumulated. The assumption is only true when tunneling splitting of zero-energy states is neglected. It has been established that the finite separation between anyons always leads to a non-zero splitting of zero-energy states [19, 91, 145] although the splitting is exponentially suppressed due to the existence of bulk gap. As a result, the two ground states acquire different dynamical phases during the time-evolution. Here we take into account all the non-universal microscopic physics including dynamical phase induced by tunneling splitting and non-Abelian Berry phase.

In the framework of time-dependent BdG equation, the two basis states are  $\Psi_{\pm} = \frac{1}{\sqrt{2}}(\Psi_{01} \pm i\Psi_{02})$ . The energy splitting between two zero-energy states in vortices in a spinless  $p_x + ip_y$  superconductor has been calculated in the limit of large separation [91, 106]:

$$E_+ = -E_- \approx \sqrt{\frac{2}{\pi}} \Delta_0 \frac{\cos(k_F R + \pi/4)}{\sqrt{k_F R}} e^{-R/\xi}, \quad (6.30)$$

with  $\Delta_0$  being the amplitude of the bulk superconducting gap,  $k_F$  the Fermi momentum and  $\xi$  the coherence length. The exponential decay of splitting is universal for all non-Abelian topological phase and is in fact the manifestation of the topological protection.

The Berry matrix  $\mathbf{M}$  can be evaluated:

$$\begin{aligned} M_{++} = M_{--} &= \frac{i}{2} \left( \langle \Psi_1 | \dot{\Psi}_1 \rangle + \langle \Psi_2 | \dot{\Psi}_2 \rangle \right) \\ M_{+-} &= \frac{1}{2} \left( i \langle \Psi_1 | \dot{\Psi}_1 \rangle - i \langle \Psi_2 | \dot{\Psi}_2 \rangle + \langle \Psi_1 | \dot{\Psi}_2 \rangle - \langle \Psi_2 | \dot{\Psi}_1 \rangle \right). \end{aligned} \quad (6.31)$$

From (??) we have  $\langle \Psi_{01} | \dot{\Psi}_{01} \rangle = \langle \Psi_{02} | \dot{\Psi}_{02} \rangle = 0$ . So  $\mathbf{M}$  only has off-diagonal elements. Furthermore, we can show that  $M_{+-}$  must be a real number following the Majorana condition  $\Psi^* = \Psi$ . Write  $\Psi = (u, u^*)^T$ , we have

$$\langle \Psi_{01} | \dot{\Psi}_{02} \rangle = \int d^2\mathbf{r} (u_1^* \dot{u}_2 + u_1 \dot{u}_2^*), \quad (6.32)$$

from which we can easily see  $\langle \Psi_{01} | \dot{\Psi}_{02} \rangle \in \mathbb{R}$ . The same for  $\langle \Psi_{02} | \dot{\Psi}_{01} \rangle$ . The integral in  $M_{+-}$  can be further simplified:

$$M_{+-} = \omega \alpha, \alpha = \frac{1}{2} \int d^2\mathbf{r} \Psi^\dagger(\mathbf{r} + \mathbf{R}) \Psi(\mathbf{r}), \quad (6.33)$$

where  $\mathbf{R} = \mathbf{R}_1 - \mathbf{R}_2$ . The form of  $\alpha$  is not important apart from the fact that  $|\alpha| \sim e^{-R/\xi}$ .

Therefore the matrix  $\mathbf{M}$  takes the following form:

$$\mathbf{M} = \omega \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}. \quad (6.34)$$

Since both  $\alpha$  and  $E_\pm$  are functions of  $R$ , they are time-independent. We now have to solve essentially the textbook problem of the Schrödinger equation of a spin 1/2 in a magnetic field, the solution of which is well-known:

$$\begin{pmatrix} c_+(T) \\ c_-(T) \end{pmatrix} = \begin{pmatrix} \cos \mathcal{E}T - \frac{iE_+}{\mathcal{E}} \sin \mathcal{E}T & \frac{i\omega\alpha}{\mathcal{E}} \sin \mathcal{E}T \\ \frac{i\omega\alpha}{\mathcal{E}} \sin \mathcal{E}T & \cos \mathcal{E}T + \frac{iE_+}{\mathcal{E}} \sin \mathcal{E}T \end{pmatrix} \begin{pmatrix} c_+(0) \\ c_-(0) \end{pmatrix}, \mathcal{E} = \sqrt{E_+^2 + \omega^2\alpha^2}. \quad (6.35)$$

We can translate the results into transformation of Majorana operators:

$$\begin{aligned} \hat{\gamma}_1 &\rightarrow \left( \cos \mathcal{E}T + \frac{i\omega\alpha}{\mathcal{E}} \sin \mathcal{E}T \right) \hat{\gamma}_2 - \frac{E_+}{\mathcal{E}} \sin \mathcal{E}T \hat{\gamma}_1 \\ \hat{\gamma}_2 &\rightarrow -\frac{E_+}{\mathcal{E}} \sin \mathcal{E}T \hat{\gamma}_2 - \left( \cos \mathcal{E}T + \frac{i\omega\alpha}{\mathcal{E}} \sin \mathcal{E}T \right) \hat{\gamma}_1 \end{aligned} \quad (6.36)$$

Because  $E_+ \sim \Delta_0 e^{-R/\xi}$ ,  $|\alpha| \sim e^{-R/\xi}$  and  $\omega \ll \Delta_0$ , by order of magnitude we can safely assume  $|E_+| \gg \omega|\alpha|$ . In the limiting case  $\omega \rightarrow 0$  we find

$$\begin{aligned}\hat{\gamma}_1 &\rightarrow \cos \mathcal{E}T \hat{\gamma}_2 - \sin \mathcal{E}T \hat{\gamma}_1 \\ \hat{\gamma}_2 &\rightarrow -\sin \mathcal{E}T \hat{\gamma}_2 - \cos \mathcal{E}T \hat{\gamma}_1\end{aligned}\tag{6.37}$$

which can be compactly written as  $\hat{\gamma}_i \rightarrow \hat{U} \hat{\gamma}_i \hat{U}^\dagger$  where

$$\hat{U} = \exp \left[ \left( \frac{\pi}{4} - \frac{\mathcal{E}T}{2} \right) \hat{\gamma}_2 \hat{\gamma}_1 \right].\tag{6.38}$$

Physically it simply means that the two ground states with different fermion parity pick up different dynamical phases due to the energy splitting. When  $\mathcal{E}T$  becomes  $O(1)$ , the dynamical phase becomes appreciable so a unnegligible error has been introduced. Physically, this means that the braiding is carried out so slowly that the two ground states can not be considered as being degenerate.

When we also take into account the terms containing  $\omega$ , the transformation matrix is no longer in  $\mathbb{S}\mathbb{O}(2)$ . This implies that the two-dimensional Hilbert space spanned by the two Majorana zero-energy bound states is not sufficient to describe the full time evolution. However, this contribution is small in both  $\omega$  and  $\alpha$  compared to the dynamical phase correction and can be safely neglected.

The above calculation is carried out for the case of two vortices, where the two degenerate ground states belong to different fermion parity sectors and can never mix. Four vortices are needed to have two degenerate ground states in the same fermion parity sector. But the applicability of the result (6.36) and (6.38) is not limited to only two vortices. We expect that due to the tunneling splitting, the ground states with different fermion parities in each pair of vortices acquires

different dynamical phases, which interferes with the non-Abelian transformation.

## 6.2.2 Effects of Excited Bound States

The concept of quantum statistics is built upon the adiabatic theorem claiming that in adiabatic limit, quantum states evolve within the degenerate energy subspace. Going beyond adiabatic approximation, we need to consider processes that can cause transitions to states outside the subspace which violates the very fundamental assumption of adiabatic theorem. In the case of the braiding of Majorana fermions in superconductors, there are always extended excited states in the spectrum which are separated from ground states by roughly the superconducting gap. In addition, there may be low-lying bound states within the bulk gap, such as the CdGM states in vortices. We call them subgap states. Extended states and subgap bound states apparently play different roles in the braiding of Majorana fermions. To single out their effects on the braiding we consider them separately and in this subsection we consider excited bound states first. Since the energy scale involved here is the superconducting gap, we neglect the exponentially small energy splitting whose effect has been considered in the previous subsection.

The BdG wavefunctions of excited bound states in each defect are denoted by  $\Psi_{\lambda i}$ ,  $i = 1, 2$  where  $i$  labels the defects, with energy eigenvalues  $\varepsilon_{\lambda}$ . If no other inhomogeneities are present, the wavefunctions are all functions of  $\mathbf{r} - \mathbf{R}_i$ . Assuming  $|\mathbf{R}_i - \mathbf{R}_j| \gg \xi$ , we have approximately  $\langle \Psi_{\lambda i} | \Psi_{\lambda' j} \rangle = \delta_{ij}$  up to exponentially small corrections. Therefore, the BdG equation decouples for the two defects since the

tunneling amplitudes between them are all negligible. So it is sufficient to consider one defect and we will omit the defect label  $i$  in the following. We write the solution to time-dependent BdG equation as

$$\Psi(t) = c_0(t)\Psi_0(t) + \sum_{\lambda} [c_{\lambda}(t)\Psi_{\lambda}(t) + \bar{c}_{\lambda}(t)\bar{\Psi}_{\lambda}(t)], \quad (6.39)$$

where we have defined  $\bar{\Psi}_{\lambda}$  as the particle-hole conjugate state of  $\Psi_{\lambda}$ , with energy eigenvalue  $-\varepsilon_{\lambda}$ . We will focus on the minimal case where only one extra excited state is taken into account. Actually, in the case of bound states in vortices, due to the conservation of angular momentum, zero-energy state is only coupled to one excited bound state and to the leading order we can neglect the couplings of the zero-energy states to other excited states as well as those between excited states.

The time-dependent BdG equation reduces to

$$\begin{aligned} i\dot{c}_0 &= -\beta c_{\lambda} + \beta^* \bar{c}_{\lambda} \\ i\dot{c}_{\lambda} &= (\varepsilon_{\lambda} - \alpha)c_{\lambda} - \beta^* c_0, \\ i\dot{\bar{c}}_{\lambda} &= -(\varepsilon_{\lambda} + \alpha)\bar{c}_{\lambda} - \beta c_0 \end{aligned} \quad (6.40)$$

where we have defined the components of the Berry matrix as

$$\beta_{\lambda} = i\langle \Psi_0 | \dot{\Psi}_{\lambda} \rangle, \alpha_{\lambda} = i\langle \Psi_{\lambda} | \dot{\Psi}_{\lambda} \rangle. \quad (6.41)$$

In the following we suppress the subscript  $\lambda$  and shift the energy of excited level to eliminate  $\alpha_{\lambda}$ :  $\varepsilon_{\lambda} \rightarrow \varepsilon_{\lambda} + \alpha_{\lambda}$ . The quasiparticle operator corresponding to the excited level is denoted by  $\hat{d}$ , as defined in. For technical convenience, we write it as  $\hat{d} = \frac{1}{\sqrt{2}}(\hat{\xi} + i\hat{\eta})$  where  $\hat{\xi}$  and  $\hat{\eta}$  are both Majorana operators. Solving the BdG

equation (6.40), we find

$$\begin{aligned}
\hat{\gamma} &\rightarrow \left(1 - \frac{4\beta^2}{E^2} \sin^2 \frac{Et}{2}\right) \hat{\gamma} + \frac{2\sqrt{2}\beta\varepsilon \sin^2 \frac{Et}{2}}{E^2} \hat{\xi} - \frac{\sqrt{2}\beta \sin Et}{E} \hat{\eta} \\
\hat{\xi} &\rightarrow \frac{2\sqrt{2}\varepsilon\beta \sin^2 \frac{Et}{2}}{E^2} \hat{\gamma} + \left(\cos Et + \frac{4\beta^2 \sin^2 \frac{Et}{2}}{E^2}\right) \hat{\xi} + \frac{\varepsilon \sin Et}{E} \hat{\eta} \\
\hat{\eta} &\rightarrow \frac{\sqrt{2}\beta \sin Et}{E} \hat{\gamma} - \frac{\varepsilon \sin Et}{E} \hat{\xi} + \cos Et \hat{\eta}
\end{aligned} \tag{6.42}$$

which should be followed up by the basis transformation  $\hat{B}$ . Here we have defined  $E = \sqrt{\varepsilon^2 + 4\beta^2}$ .

By using (6.24) we can work out explicitly how the ground state wavefunctions transform. Physically, the non-adiabatic process causes transitions of quasiparticles residing on the zero-energy level to the excited levels. Superficially these transitions to excited states significantly affect the non-Abelian statistics, since the parity of fermion occupation in the ground state subspaces is changed as well as the quantum entanglement between various ground states [20, 146]. This can also be directly seen from (6.42) since starting from  $|g\rangle$  the final state is a superposition of  $|g\rangle$  and  $\hat{d}_0^\dagger \hat{d}_\lambda^\dagger |g\rangle$ . So we might suspect that errors are introduced to the gate operations.

However, noticing that the excited states are still localized, they are always transported together with the zero-energy Majorana states. Therefore the parity of the total fermion occupation in the ground state subspace and local excited states are well conserved. This observation allows for a redefinition of the Majorana operators to properly account for the fermion occupation in local excited states, as being done in [111]. We therefore have to represent the fermion parity in the following way:

$$\hat{\mathcal{P}}_{12} = -i\hat{\gamma}_1 \hat{\xi}_1 \hat{\eta}_1 \hat{\gamma}_2 \hat{\xi}_2 \hat{\eta}_2 = i\hat{\gamma}_1 \hat{\gamma}_2 \prod_{i=1,2} (1 - 2\hat{d}_i^\dagger \hat{d}_i) \tag{6.43}$$

shared by defects 1 and 2. Accordingly, we define the generalized Majorana operators



$\hat{\Gamma}_i = i\hat{\gamma}_i\hat{\xi}_i\hat{\eta}_i, i = 1, 2$ . Since the couplings between the two vortices are exponentially small, we treat their dynamics independently.  $\hat{\Gamma}_i$  is invariant under local unitary evolution, which can be checked explicitly using (6.42). An even general proof proceeds as following: according to (6.23), the three Majorana operators  $\hat{\gamma}_i, \hat{\xi}_i, \hat{\eta}_i$  must transform by a  $\mathbb{SO}(3)$  matrix  $\mathbf{V}$ . Then it is straightforward algebra to show that  $\hat{\Gamma}_i \rightarrow \det \mathbf{V} \cdot \hat{\Gamma}_i$ . Since  $\mathbf{V} \in \mathbb{SO}(3)$ ,  $\det \mathbf{V} = 1$  which means that  $\hat{\Gamma}_i$  is unchanged. Thus the effect of the braiding comes only through the basis transformation matrix  $\hat{B}$ . As a result, the fermion parities, being the expectation values of (6.43), transform exactly according to the Ivanov's rule under the braiding. This result can be easily generalized to the case where many bound states exist in the vortex core.

From the perspective of measurement, to probe the status of a topological qubit it is necessary and sufficient to measure the fermion parity as defined in (6.43). It is practically impossible (and unnecessary) to distinguish between the fermion occupations in ground state subspace and excited states as long as they are both localized and can be considered as a composite qubit.

To conclude, non-Adiabatic population of fermions onto the low-lying excited bound states has no effect on the non-Abelian statistics due to the fact that the fermion parity shared by a pair of vortices is not affected by such population.

### 6.2.3 Effects of Excited Extended States

We next consider the effect of excited states that are extended in space. Usually such states form a continuum. The scenario considered here may not be very

relevant to Majorana fermions in 2D  $p_x + ip_y$  superconductor since bound states in vortices often dominate at low energy. However, for Majorana fermions in one-dimensional systems, zero-energy state is the only subgap state and coupling between zero-energy state and continuum of excited states may become important.

Again we write time-dependent BdG equation in the instantaneous eigenbasis

$$\begin{aligned}
i\dot{c}_{0,1} &= -\sum_{\lambda} (\beta_{1\lambda} c_{\lambda} - \beta_{1\lambda}^* \bar{c}_{\lambda}) \\
i\dot{c}_{0,2} &= -\sum_{\lambda} (\beta_{2\lambda} c_{\lambda} - \beta_{2\lambda}^* \bar{c}_{\lambda}) \\
i\dot{c}_{\lambda} &= (\varepsilon_{\lambda} - \alpha_{\lambda}) c_{\lambda} - \beta_{1\lambda}^* c_{0,1} - \beta_{2\lambda}^* c_{0,2} \\
i\dot{\bar{c}}_{\lambda} &= -(\varepsilon_{\lambda} - \alpha_{\lambda}) \bar{c}_{\lambda} + \beta_{1\lambda} c_{0,1} + \beta_{2\lambda} c_{0,2}
\end{aligned} \tag{6.44}$$

Here we still use  $\lambda$  to label the excited states. As in the case of bound states, we ignore the coupling between excited states since these only contribute higher order terms to the dynamics of zero-energy states. In another word, we treat each excited state individually and in the end their contributions are summed up.

To proceed we need to determine matrix elements  $\beta_{1\lambda}$  and  $\beta_{2\lambda}$ . We will consider the spinless one-dimensional p-wave superconductor as an example. The zero-energy states localize at two ends of the 1D system which lie in the interval  $[0, L]$ . We also assume reflection symmetry with respect to  $x = L/2$ . Without worrying about the tunneling splitting, we can consider the two ends near  $x = 0$  and  $x = L$  independently. Then we make use of the fact that BdG equations near  $x = 0$  and  $x = L$  are related by a combined coordinate and gauge transformation  $x \rightarrow L - x$ ,  $\Delta(x) \rightarrow -\Delta(L - x)$ . Therefore, the bound state and the local part of the extended states near  $x = 0$  and  $x = L$  are related by gauge transformations. Based on these

considerations, we should have  $\beta_{1\lambda} = \beta_{2\lambda} \equiv \beta_\lambda$ , up to exponentially small corrections. Similar argument also applies to vortices in 2D  $p_x + ip_y$  superconductors.

As mentioned above, we will make the approximation that each excited state can be treated independently. So we consider the effect of one of the excited states first and omit the label  $\lambda$  temporarily. Again we write  $\hat{d} = \frac{1}{\sqrt{2}}(\hat{\xi} + i\hat{\eta})$ . Without loss of generality we also assume that  $\beta$  is real. We find from the solution of time-dependent BdG equation that

$$\begin{aligned}
\hat{\gamma}_1 &\rightarrow \frac{4\beta^2}{E^2} \sin^2 \frac{Et}{2} \hat{\gamma}_1 + \left(1 - \frac{4\beta^2}{E^2} \sin^2 \frac{Et}{2}\right) \hat{\gamma}_2 + \frac{2\sqrt{2}\beta\varepsilon \sin^2 \frac{Et}{2}}{E^2} \hat{\xi} + \frac{\sqrt{2}\beta \sin Et}{E} \hat{\eta} \\
\hat{\gamma}_2 &\rightarrow -\left(1 - \frac{4\beta^2}{E^2} \sin^2 \frac{Et}{2}\right) \hat{\gamma}_1 - \frac{4\beta^2}{E^2} \sin^2 \frac{Et}{2} \hat{\gamma}_2 + \frac{2\sqrt{2}\beta\varepsilon \sin^2 \frac{Et}{2}}{E^2} \hat{\xi} + \frac{\sqrt{2}\beta \sin Et}{E} \hat{\eta} \\
\hat{\xi} &\rightarrow \frac{2\sqrt{2}\beta\varepsilon \sin^2 \frac{Et}{2}}{E^2} (-\hat{\gamma}_1 + \hat{\gamma}_2) + \left(\cos Et + \frac{8\beta^2 \sin^2 \frac{Et}{2}}{E^2}\right) \hat{\xi} - \frac{\sqrt{2}\varepsilon \sin Et}{E} \hat{\eta} \\
\hat{\eta} &\rightarrow \frac{2\sqrt{2}\beta \sin^2 \frac{Et}{2}}{E} (-\hat{\gamma}_1 + \hat{\gamma}_2) + \frac{\varepsilon \sin Et}{E} \hat{\xi} + \cos Et \hat{\eta}
\end{aligned} \tag{6.45}$$

Here again  $E = \sqrt{\varepsilon^2 + 4\beta^2}$ .

At first glance the physics here is very similar to what has been discussed for local bound states: non-adiabatic transitions cause changes of fermion parity in the ground state subspace. The crucial difference between local bound states and a continuum of extended states is that, in the former case, local fermion parity is still conserved as long as we count fermion occupation in the excited states while in the latter, it is impossible to keep track of the number of fermions leaking into the continuum so the notion of local fermion parity breaks down. These non-adiabatic effects may pose additional constraints on manufacturing of topological qubits. Let's consider to what extent the braiding statistics is affected. A useful quantity to look

at here is the expectation value of the fermion parity operator in the ground state subspace, namely  $\langle \hat{\mathcal{P}}_0 \rangle = \langle i\hat{\gamma}_1\hat{\gamma}_2 \rangle$ .

Suppose at  $t = 0$  we start from the ground state  $|g\rangle$  with even fermion parity  $\langle g|\hat{\mathcal{P}}_0|g\rangle = 1$  and the excited level is unoccupied, too. After the braiding at time  $T$  the expectation value of  $\hat{\mathcal{P}}_0$  becomes

$$\langle \hat{\mathcal{P}}_0(T) \rangle = 1 - \frac{8\beta^2}{E^2} \sin^2 \frac{ET}{2}. \quad (6.46)$$

where  $\hat{\mathcal{P}}_0(T) = \hat{U}^\dagger(T)\hat{\mathcal{P}}_0\hat{U}(T)$ . This confirms that fermion parity is not conserved anymore. For  $|\beta| \ll \varepsilon$ , the coupling to excited state can be understood as a small perturbation.  $\langle \hat{\mathcal{P}}_0 \rangle$  only slightly deviates from the non-perturbed value. In the opposite limit  $|\beta| \gg \varepsilon$ ,  $\langle \hat{\mathcal{P}}_0 \rangle$  can oscillate between 1 and  $-1$  so basically fermion parity is no longer well-defined.

Now we can sum up the contributions from each excited state and (6.46) is replaced by:

$$\langle \hat{\mathcal{P}}_0(T) \rangle = 1 - \sum_{\lambda} \frac{8|\beta_{\lambda}|^2}{E_{\lambda}^2} \sin^2 \frac{E_{\lambda}T}{2}. \quad (6.47)$$

The sum over the continuum states can be replaced by an integral over energy. We assume that the couplings  $\beta_{\lambda}$  depends only weakly on the energy  $\varepsilon_{\lambda}$  so it can be factored out as  $\beta_{\lambda} \approx \beta$ . Then we obtain

$$\langle \hat{\mathcal{P}}_0(T) \rangle = 1 - 8|\beta|^2 \int_{\Delta_0}^{\infty} d\varepsilon \frac{\nu(\varepsilon)}{\varepsilon^2 + 4|\beta|^2} \sin^2 \frac{\varepsilon T}{2}. \quad (6.48)$$

The density of states  $\nu(\varepsilon)$  depends on the microscopic details of the underlying superconducting phase. For simplicity we take the typical BCS-type density of

states:

$$\nu(\varepsilon) = \frac{2\nu_0\varepsilon}{\sqrt{\varepsilon^2 - \Delta_0^2}} \Theta(\varepsilon - \Delta_0). \quad (6.49)$$

Here  $\nu_0$  is the normal-state density of states and  $\Delta_0$  is the bulk superconducting gap. We consider the limit  $\Delta_0 T \gg 1$ . The long-time asymptotic behavior of the integral is given by

$$\langle \hat{\mathcal{P}}_0(T) \rangle \approx 1 - \frac{8\nu_0|\beta| \sinh^{-1} \frac{2|\beta|}{\Delta_0}}{\sqrt{4|\beta|^2 + \Delta_0^2}} + O\left(\frac{1}{\sqrt{\Delta_0 T}}\right). \quad (6.50)$$

Therefore, the non-adiabatic coupling to the excited continuum causes finite depletion of the fermion parity in the zero-energy ground state subspace, which can be regarded as the dissipation of the topological qubit. The depletion becomes comparable to 1 if  $\nu_0 \left(\frac{|\beta|}{\Delta_0}\right)^2 \sim 1$ , rendering the qubit undefined. We notice that our calculation breaks down for large  $|\beta|$  since then the excited states can not be treated as being independent. They are coupled through second-order virtual processes via the zero-energy state, which is weighted by  $\left(\frac{|\beta|}{\Delta_0}\right)^2$  perturbatively. Thus our results should be regarded as the leading-order correction in the non-adiabatic perturbation theory.

### 6.3 Discussion and Conclusion

In conclusion, we have considered the braiding of non-Abelian anyons as a dynamical process and calculated the corrections to non-Abelian evolutions due to non-adiabatic effects. We discuss several sources of non-adiabaticity: first of all, tunneling between non-Abelian anyons results in splitting of the degenerate ground states. The Abelian dynamical phase accumulated in the process of braiding

modifies Ivanov’s rule of non-Abelian statistics. Since the bulk of the superconductor is fully gapped such corrections are exponentially small. In the context of TQC such deviations from Ivanov’s rule are sources of errors in single-qubit quantum gates.

Second, we consider dynamical transitions of Majorana fermions in the zero-energy ground states to excited states. The effects of such non-adiabatic transitions strongly rely on whether these excited states are bound states with discrete spectrum and localized at the same positions with the Majorana bound states, or they extend through the whole bulk and form a continuum. Generally speaking, non-adiabatic transitions mix the zero-energy ground states with other excited states and it is questionable whether the quantum entanglement crucial to non-Abelian statistics is still preserved. In the former case where excited states are localized, we are still able to define conserved fermion parity stored in these low-energy bound states. Non-Abelian statistics can be generalized once we enlarge the Hilbert space to include all local bound states. In the latter case, the situation is dramatically different because the notion of fermion parity in the low-energy Hilbert space no longer makes sense once extended states above the bulk gap are involved. We characterize the loss of fermion parity in such non-adiabatic transitions by the expectation value of the “local fermion parity” operator. This can be viewed as the dissipation of topological qubit resulting from couplings to a continuum of fermionic states. We have thus quantified the expectation that a zero-energy Majorana mode will decay if it is put in contact with a continuum of fermionic states(*e.g.*, electrons).

Although the underlying technological motivation for topological quantum computation is that quantum error correction against continuous decoherence is un-

necessary as a matter of principle in topological systems since decoherence due to local coupling to the environment is eliminated, other errors, such as non-adiabaticity considered in this work, would invariably occur in all quantum systems in the presence of time-dependent quantum gate operations. In addition, braiding is the cornerstone of the strange quantum statistical properties which distinguish non-Abelian anyons from ordinary fermions and bosons. Our work, involving the non-adiabatic corrections to anyonic braiding, is therefore relevant to all current considerations in the subject of Ising anyons whether it is in the context of the observation of the non-Abelian statistics or the implementation of topological quantum computation. In particular, non-adiabaticity in the Majorana braiding in the specific context of non-Abelian topological superconductors as discussed in this chapter, may be relevant to various recently proposed Majorana interferometry experiments involving vortices in 2D [147, 148, 120, 121].

We now speculate about the physical sources of non-adiabaticity of braidings. Throughout our work we have focused on the intrinsic non-adiabaticity originating from the fact that braidings are done during a finite interval of time. For such effects to be appreciable, the time-scale of braidings has to be comparable to  $\Delta t \sim \frac{\hbar}{\Delta E}$  where  $\Delta E$  is the energy gap protecting the anyonic Majorana modes. For the corrections from the tunneling of Majorana fermions,  $\Delta t$  is exponentially large so it is usually legitimate to neglect the tunneling effect. On the other hand, if we are interested in corrections from states above the gap, then the relevant time scale is  $\hbar/\Delta$ . Using (6.47) one can estimate the non-adiabatic error rate in performing gate operations.

These considerations also apply to other types of non-adiabatic perturbations as long as they have non-zero matrix elements between zero-energy states and excited states. In particular disorder scattering, which is unavoidable in solid state systems, can be a source of non-adiabaticity [146]. As the non-Abelian anyons are moving, the disorder potential seen by the anyons changes randomly with time so it can be modeled as a time-dependent noise term in the Hamiltonian which may cause dephasing. Other possible perturbations in solid state systems include collective excitations, such as phonons and plasmons(or phase fluctuations). We leave the investigation of these effect for future work. The formalism developed in the current work can, in principle, be used to study these dephasing errors.



## Chapter 7

# Majorana Zero Modes Beyond BCS Mean-Field

## Theory

The BCS theory of superconductivity [149], which all our theoretical study of topological superconductors is based on, is a mean-field theory of the many-body effect originating from four-fermion interaction. Although it has been proved to be enormously successful in describing superconductivity, fluctuation effects beyond the mean-field theory do arise in certain circumstances. For example, in the neighborhood of the superconducting phase transition where the mean-field order parameter is very small, the fluctuation effect can be dominant in various thermodynamical quantities. Another scenario where fluctuations can not be neglected is low-dimensional systems, where fluctuation effects are actually most prominent. A celebrated theorem proved by Mermin and Wagner [150], states that under very generic conditions (e.g. short-range interactions) no spontaneous continuous symmetry breaking can occur in one dimension ( $1 + 1$  space-time dimension) even at zero temperature. The same is true in two dimensions at any finite temperature. In both cases, the off-diagonal long-range order [151], which defines the spontaneous symmetry breaking, is smeared out by strong quantum or thermal fluctuations and becomes quasi-long-range order characterized by the algebraic decay of order pa-

parameter correlations.

It is therefore an important question to understand the fluctuation effects on the topological aspects, particularly the Majorana zero modes in TSC, since they do live in one or two dimensions. From a more general perspective, the interplay between interaction (since fluctuations are essentially caused by interactions) and topological classification of non-interacting systems is a fundamental problem which we only began to understand quite recently. A remarkable progress is that the topological classification of one-dimensional non-interacting fermionic systems with time-reversal symmetry is dramatically changed by interactions [152, 153, 154]. Several theoretical studies on the effects of interactions on Majorana fermions in proximity-induced TSC have been performed recently [155, 156, 157, 158], confirming the stability of Majorana fermions against weak and moderate interactions.

In this chapter we present an attempt to understand the fate of Majorana zero modes when quantum fluctuations are strong enough that only quasi-long-range superconducting order can exist. We consider a generic theoretical model of spinless fermions on two-chain ladders. The model generalizes the simplest one-dimensional TSC, namely spinless fermions with  $p$ -wave pairing (also known as Majorana chain) [28], to interacting two-chain systems. Instead of introducing pairing by proximity effect, the effective field theory includes inter-chain pair tunneling with inter-chain single-particle tunneling being suppressed. Therefore the fermion parity on each chain is conserved. When the pair-tunneling interaction drives the system to strong coupling, localized Majorana zero-energy states are found on the boundaries, which represents a nontrivial many-body collective state of the underlying fermions.

We then demonstrate that in a finite-size system the Majorana edge states lead to (nearly) degenerate ground states with different fermion parity on each chain, thus revealing its analogy with the Majorana edge states in non-interacting TSC. The degeneracy is shown to be robust to any weak intra-chain perturbations, but inter-chain single-particle tunneling and backscattering can possibly lift the degeneracy. We also discuss a lattice model where such field theory is realized at low energy.

## 7.1 Field-Theoretical Model

We start from an effective field-theoretical description of the model for the purpose of elucidating the nature of the Majorana edge states. We label the two chains by  $a = 1, 2$ . The low-energy sector of spinless fermions on each chain is well captured by two chiral Dirac fermions  $\hat{\psi}_{L/R,a}(x)$ . The non-interacting part of the Hamiltonian is simply given by  $\hat{H}_0 = \int dx \hat{\mathcal{H}}_0(x)$  where

$$\hat{\mathcal{H}}_0 = -iv_F \sum_a \left( \hat{\psi}_{Ra}^\dagger \partial_x \hat{\psi}_{Ra} - \hat{\psi}_{La}^\dagger \partial_x \hat{\psi}_{La} \right). \quad (7.1)$$

Four-fermion interactions can be categorized as intra-chain and inter-chain interactions. Intra-chain scattering processes (e.g., forward and backward scattering) are incorporated into the Luttinger liquid description of spinless fermions and their effects on the low-energy physics are completely parameterized by the renormalized velocities  $v_a$  and the Luttinger parameters  $K_a$ . We assume that the filling of the system is incommensurate so Umklapp scattering can be neglected. For simplicity we assume the two chains are identical so  $v_1 = v_2 = v$ ,  $K_1 = K_2 = K$ .

We now turn to inter-chain interactions. Those that can be expressed in terms

of the densities of the chiral fermions can be absorbed into the Gaussian part of the bosonic theory after a proper change of variables(see below) and we do not get into the details here. We have to consider the pair tunneling and the inter-chain backscattering:

$$\begin{aligned}\hat{\mathcal{H}}_{\text{pair}} &= -g_{\text{p}}(\hat{\psi}_{R2}^\dagger \hat{\psi}_{L2}^\dagger \hat{\psi}_{L1} \hat{\psi}_{R1} + \text{h.c.}) \\ \hat{\mathcal{H}}_{\text{bs}} &= g_{\text{bs}}(\hat{\psi}_{L1}^\dagger \hat{\psi}_{R1} \hat{\psi}_{R2}^\dagger \hat{\psi}_{L2} + 1 \leftrightarrow 2).\end{aligned}\tag{7.2}$$

The microscopic origin of such terms is highly model-dependent which will be discussed later. The motivation of studying pair tunneling is to “mimic” the BCS pairing of spinless fermions without explicitly introducing superconducting pairing order parameter.

The Hamiltonian of the effective theory is then expressed as  $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_{\text{bs}} + \hat{\mathcal{H}}_{\text{pair}}$ . Notice that total fermion number  $\hat{N} = \hat{N}_1 + \hat{N}_2$  is conserved by the Hamiltonian, but  $\hat{N}_1$  and  $\hat{N}_2$  themselves fluctuate due to the tunneling of pairs. However, their parities  $(-1)^{\hat{N}_a}$  are still separately conserved. Due to the constraint that  $(-1)^{\hat{N}_1} \cdot (-1)^{\hat{N}_2} = (-1)^{\hat{N}}$ , we are left with an overall  $\mathbb{Z}_2$  symmetry. Therefore we define the fermion parities  $\hat{P}_a = (-1)^{\hat{N}_a}$ , the conservation of which is crucial for establishing the existence and stability of the Majorana edge states and ground state degeneracy. In the following we refer to this overall  $\mathbb{Z}_2$  fermion parity as single-chain fermion parity. It is important to notice that the conservation of the single-chain fermion parity relies on the fact that there is no inter-chain single-particle tunneling in our Hamiltonian. We will address how this is possible when turning to the discussion of lattice models.

We use bosonization [159, 160] to study the low-energy physics of the model.

The standard Abelian bosonization reads

$$\hat{\psi}_{r,a} = \frac{\hat{\eta}_{r,a}}{\sqrt{2\pi a_0}} e^{i\sqrt{\pi}(\theta_a + r\varphi_a)} \quad (7.3)$$

where  $a_0$  is the short-distance cutoff,  $r = +/-$  for  $R/L$  movers and  $\hat{\eta}_{r,a}$  are Majorana operators which keep track of the anti-commuting character of the fermionic operators. We follow the constructive bosonization as being thoroughly reviewed in [160]. The two bosonic fields  $\varphi_a$  and  $\theta_a$  satisfy the canonical commutation relation:

$$[\partial_x \varphi_a(x), \theta_a(x')] = i\delta(x - x'). \quad (7.4)$$

The  $\varphi_a$  field is related to the charge density on chain  $a$  by  $\rho_a = \frac{1}{\sqrt{\pi}} \partial_x \varphi_a$ , and  $\theta_a$  is its conjugate field, which can be interpreted as the phase of the pair field.

It is convenient to work in the bonding and anti-bonding basis:

$$\varphi_{\pm} = \frac{1}{\sqrt{2}}(\varphi_1 \pm \varphi_2), \quad \theta_{\pm} = \frac{1}{\sqrt{2}}(\theta_1 \pm \theta_2). \quad (7.5)$$

The resulting bosonized Hamiltonian decouples as  $\hat{\mathcal{H}} = \hat{\mathcal{H}}_+ + \hat{\mathcal{H}}_-$ :

$$\begin{aligned} \hat{\mathcal{H}}_+ &= \frac{v_+}{2} [K_+ (\partial_x \theta_+)^2 + K_+^{-1} (\partial_x \varphi_+)^2], \\ \hat{\mathcal{H}}_- &= \frac{v_-}{2} [K_- (\partial_x \theta_-)^2 + K_-^{-1} (\partial_x \varphi_-)^2] \\ &+ \frac{g_p}{2(\pi a_0)^2} \cos \sqrt{8\pi} \theta_- + \frac{g_{bs}}{2(\pi a_0)^2} \cos \sqrt{8\pi} \varphi_-. \end{aligned} \quad (7.6)$$

Here  $a_0$  is the short-distance cutoff. This decoupling of the bonding and the anti-bonding degrees of freedom is analogous to the spin-charge separation of electrons in one dimension. Without any inter-chain forward scattering, we have  $K_{\pm} = K$ ,  $v_{\pm} = v$ .

The bonding sector is simply a theory of free bosons. The Hamiltonian in the anti-bonding sector can be analyzed by the perturbative Renormalization

Group(RG) method, assuming the bare couplings  $g_p$  and  $g_{bs}$  are weak. RG flow of the coupling constants are governed by the standard Kosterlitz-Thouless equations [161]:

$$\begin{aligned}\frac{dy_p}{dl} &= (2 - 2K_-^{-1})y_p \\ \frac{dy_{bs}}{dl} &= (2 - 2K_-)y_{bs} \\ \frac{d \ln K_-}{dl} &= 2K_-^{-1}y_-^2,\end{aligned}\tag{7.7}$$

where  $y_- = \frac{g_p}{\pi v_-}$ ,  $y_{bs} = \frac{g_{bs}}{\pi v_-}$  are the dimensionless coupling constants and  $l = \ln \frac{a}{a_0}$  is the flow parameter. When  $K_- > 1$  (corresponding to attractive intra-chain interaction),  $y_p$  is relevant and flows to strong-coupling under RG flow, indicating gap formation in the anti-bonding sector, while  $y_{bs}$  is irrelevant so can be neglected when considering long-wavelength, low-energy physics. Semiclassically, the  $\theta_-$  is pinned in the ground state. From now on, we will assume  $K_- > 1$  and neglect the irrelevant coupling  $y_{bs}$ .

## 7.2 Majorana zero-energy edge states

To clarify the nature of the gapped phase in the anti-bonding sector, we study the model at a special point  $K_- = 2$ , known as the Luther-Emery point [162], where the sine-Gordon model is equivalent to free massive Dirac fermions. First we rescale the bosonic fields:

$$\tilde{\varphi}_- = \frac{\varphi_-}{\sqrt{K_-}}, \quad \tilde{\theta}_- = \sqrt{K_-}\theta_-, \tag{7.8}$$

and define the chiral fields by  $\tilde{\varphi}_{r-} = \frac{1}{2}(\tilde{\varphi}_- + r\tilde{\theta}_-)$ . Neglecting the irrelevant backscattering term,  $\hat{\mathcal{H}}_-$  is reformionized to

$$\hat{\mathcal{H}}_- = -iv_-(\hat{\chi}_R^\dagger \partial_x \hat{\chi}_R - \hat{\chi}_L^\dagger \partial_x \hat{\chi}_L) + im(\hat{\chi}_R^\dagger \hat{\chi}_L^\dagger - \hat{\chi}_L \hat{\chi}_R), \quad (7.9)$$

where the Dirac fermionic fields  $\hat{\chi}_r$  are given by

$$\hat{\chi}_r = \frac{1}{\sqrt{2\pi a_0}} \hat{\xi}_r e^{ir\sqrt{4\pi}\tilde{\varphi}_{r-}}, \quad (7.10)$$

with the fermion mass  $m = \frac{g_p}{\pi a_0}$ .  $\hat{\xi}_r$  are again Majorana operators. It is quite clear that effective theory (7.9) also describes the continuum limit of a Majorana chain, which is known to support Majorana edge states [28].

However, caution has to be taken here when dealing with open boundary condition(OBC). We impose open boundary condition at the level of underlying lattice fermionic operators [163]:

$$\hat{c}_{ia} \approx \sqrt{a_0} [\hat{\psi}_{Ra}(x) e^{ik_F x} + \hat{\psi}_{La}(x) e^{-ik_F x}], \quad (7.11)$$

where  $\hat{c}_{ia}$  are annihilation operators of fermions and  $x = ia_0$ . Since the chain terminates at  $x = 0$  and  $x = L$ , we demand  $\hat{c}_0 = \hat{c}_{N+1} = 0$  where  $N = L/a_0$  is the number of sites on each chain. Let us focus on the boundary  $x = 0$ . Thus the chiral fermionic fields have to satisfy  $\hat{\psi}_{Ra}(0) = -\hat{\psi}_{La}(0)$ . Using the bosonization identity, we find  $\varphi_a(0) = \frac{\sqrt{\pi}}{2}$ , from which we can deduce the boundary condition of the anti-bonding field:

$$\varphi_-(0) = 0. \quad (7.12)$$

Therefore, we obtain the boundary condition of the Luther-Emery fermionic fields as  $\hat{\chi}_R(0) = \hat{\chi}_L(0)$ . The Hamiltonian is quadratic in  $\hat{\chi}$  and can be exactly

diagonalized by Bogoliubov transformation. We find that the Luther-Emery fields have the following representation:

$$\begin{pmatrix} \hat{\chi}_R(x) \\ \hat{\chi}_L(x) \end{pmatrix} = \sqrt{\frac{m}{v_-}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-mx/v_-} \hat{\gamma} + \dots \quad (7.13)$$

Here ... denotes the gapped quasiparticles whose forms are not of any interest to us. The  $\hat{\gamma}$  is a Majorana field (i.e.,  $\hat{\gamma} = \hat{\gamma}^\dagger$ ) and because  $[\hat{\mathcal{H}}_-, \hat{\gamma}] = 0$ , it represents a zero-energy excitation on the boundary.

Now suppose the system has finite size  $L \gg \xi = v_-/m$ . The same analysis implies that we would find two Majorana fermions localized at  $x = 0$  and  $x = L$  respectively, denoted by  $\hat{\gamma}_1$  and  $\hat{\gamma}_2$ . As in the case of TSC, the two Majorana modes have to be combined into a (nearly) zero-energy Dirac fermionic mode:  $\hat{c} = \frac{1}{\sqrt{2}}(\hat{\gamma}_1 + i\hat{\gamma}_2)$ . Occupation of this mode gives rise to two degenerate ground states. Tunneling of quasiparticles causes a non-zero splitting of the ground state degeneracy:  $\Delta E \approx me^{-L/\xi}$  [91, 106].

We notice that very similar technique was previously applied to the spin-1/2 edge excitations [164, 163, 165] in the Haldane phase of spin-1 Heisenberg chain, the  $\mathbb{SO}(n)$  spinor edge states in the  $\mathbb{SO}(n)$  spin chain [166] and also the edge state in an attractive one-dimensional electron gas [167, 168].

To understand the nature of the Majorana edge state, we have to explicitly keep track of the Klein factors which connect states with different fermion numbers. Therefore we separate out the so-called zero mode in the bosonic field  $\phi_{r,a}$  and write  $\hat{\psi}_{ra} = \frac{1}{\sqrt{2\pi a_0}} \hat{\eta}_{ra} \hat{F}_{ra} e^{ir\sqrt{4\pi}\phi_{ra}}$  where the Klein factors  $\hat{F}_{ra}$  are bosonic operators that decrease the numbers of  $r$ -moving fermions on chain  $a$  by one [160]. Bosonized form



of (7.2) has a product of the Klein factors  $\hat{F}_{R2}^\dagger \hat{F}_{L2}^\dagger \hat{F}_{L1} \hat{F}_{R1}$  in it. Since this term is to be refermionized as  $\sim \hat{\chi}_L \hat{\chi}_R$ , we are naturally led to define new Klein factors  $\hat{F}_r = \hat{F}_{r2}^\dagger \hat{F}_{r1}$  for  $\hat{\chi}_r$ . Notice that so-defined Klein factors satisfy  $\{\hat{P}_a, \hat{F}_r\} = 0$ , *i.e.*  $\hat{F}_r$  change single-chain fermion parity. Then the fermionic fields that refermionize the sine-Gordon theory at the Luther-Emery point should take the form

$$\hat{\chi}_r = \frac{1}{\sqrt{2\pi a_0}} \hat{\xi}_r \hat{F}_r e^{ir\sqrt{4\pi}\tilde{\varphi}_{r-}}. \quad (7.14)$$

Thus one can identify that  $\hat{\chi}_r$  corresponds to inter-chain single-particle tunneling. The ground state  $|G\rangle$  of the Hamiltonian (7.9) can be schematically expressed as

$$|G\rangle = \exp \left[ \int dx_1 dx_2 \hat{\chi}^\dagger(x_1) g(x_1, x_2) \hat{\chi}^\dagger(x_2) \right] |\text{vac}\rangle, \quad (7.15)$$

where  $g(x_1, x_2)$  is the Cooper-pair wave function of the spinless  $p$ -wave superconductor and  $|\text{vac}\rangle$  is the vacuum state of  $\hat{\chi}$  fermion. With the definition (7.14), it is easy to check that  $|G\rangle$  is a coherent superposition of Fock states having the same single-chain fermion parity, thus an eigenstate of  $\hat{P}_a$ . On the other hand, the Majorana fermion  $\hat{\gamma}$ , being a superposition of  $\hat{\chi}$  and  $\hat{\chi}^\dagger$ , changes the single-chain fermion parity:  $\{\hat{\gamma}, \hat{P}_a\} = 0$ . As a result, the two degenerate ground states  $|G\rangle$  and  $\hat{c}^\dagger|G\rangle$  have different single-chain fermion parity which is the essence of the Majorana edge states. If the total number of fermions  $N$  is even, then the two (nearly) degenerate ground states correspond to even and odd number of fermions on each chain, respectively.

So far all the conclusions are drawn at the Luther-Emery point  $K_- = 2$ . Once we move away from the Luther-Emery point, the theory is no longer equivalent to free massive fermions. An intuitive way to think about the situation is that if we

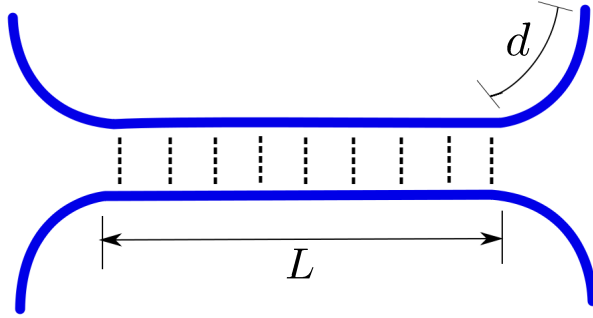


Figure 7.1: Schematic view of the two chains coupled by pair tunneling(denoted by dashed lines). The chains are bended near the two ends to avoid the single-particle tunneling.

move away from the Luther-Emery point, the  $\hat{\chi}$  fermions start to interact with each other. Since the Majorana edge states are protected by the bulk gap as well as the single-chain fermion parity [155, 156, 157, 158], we expect the qualitative features hold for the whole regime  $K_- > 1$  based on adiabatic continuity.

Notice that the bonding sector remains gapless. In our field-theoretical model, the bonding and anti-bonding degrees of freedom are completely decoupled so the gaplessness of the bonding boson does not affect the degeneracy in the anti-bonding sector.

### 7.3 Stability of the Degeneracy

We now examine whether the ground state degeneracy we have found has a topological nature. Here we define a topological degeneracy of the ground states by the following criteria: the two degenerate ground states are not distinguishable by any local order parameters (i.e. the difference of the expectation values of any local order parameters in the two ground state must be exponentially small in system size). By local, we mean local operators in the *original* fermionic operators, otherwise we can easily find such an operator in the bosonic representation. For example, in the model (7.6) the operator  $\mathcal{O}(x) = \cos \sqrt{2\pi}\theta_-(x)$  can distinguish the two degenerate ground states. But the operator itself is highly non-local in terms of the original fermionic operators.

First of all, by analogy with Majorana chain it is quite obvious that any local operators that involve even numbers of fermion operators on each chain are not able to distinguish the two ground states because such operators always commute with single-chain fermion parity operator. Therefore we only have to consider operators that consist of odd number of single-chain fermion operators. They change the single-chain fermion parity and thus presumably connect the two degenerate ground states. Since all such operators can be decomposed into products of single-particle inter-chain tunneling and backscattering operators, it is sufficient to consider these single-particle operators.

Let us start with single-particle inter-chain tunneling

$$\mathcal{O}_T = \sum_{r=R,L} (\hat{\psi}_{2r}^\dagger \hat{\psi}_{1r} + \text{h.c.}). \quad (7.16)$$

Its bosonic representation is

$$\mathcal{O}_T = \frac{2}{\pi a_0} \cos \sqrt{2\pi} \varphi_- \cos \sqrt{2\pi} \theta_-. \quad (7.17)$$

First let us consider the case when the operator is taken in the bulk of the chain away from any of the boundaries. Because  $\theta_-$  is pinned in the ground states,  $\varphi_-$  gets totally disordered and therefore  $\langle \mathcal{O}_T \rangle \propto \langle \cos \sqrt{2\pi} \varphi_- \rangle = 0$ , which is just equivalent to the fact that the Luther-Emery fermions are gapped. However, this is no longer true as one approaches the ends of the chains, since there exists zero-energy edge states. Let us focus on the left boundary  $x = 0$ . The boundary condition of the anti-bonding boson field  $\varphi_-$  has been derived:  $\varphi_-(0) = 0$ . With the boundary condition, we proceed with Luther-Emery solution at  $K_- = 2$  and find  $\mathcal{O}_T(0) \sim \hat{\chi}(0) + \hat{\chi}^\dagger(0)$ . Thus  $\mathcal{O}_T(0)$  has non-vanishing matrix element between the two ground states, independent of the system size. As a result, the two-fold degeneracy is splitted.

We now turn to the inter-chain backscattering

$$\begin{aligned} \mathcal{O}_B &= \hat{\psi}_{2R}^\dagger \hat{\psi}_{1L} + \hat{\psi}_{2L}^\dagger \hat{\psi}_{1R} + \text{h.c.} \\ &= \frac{2}{\pi a_0} \cos \sqrt{2\pi} \varphi_+ \cos \sqrt{2\pi} \theta_-. \end{aligned} \quad (7.18)$$

An analysis similar to the single-particle tunneling leads to the conclusion that backscattering at the ends also splits the degeneracy. However, even if the backscattering occurs in the middle of the chain, it still causes a splitting of the ground states decaying as a power law in system size  $L$ . To see this, let us consider a single impurity near the middle of the chain, modeled by  $\mathcal{O}_B(x)$  where  $x \approx L/2$ . We assume that the backscattering potential is irrelevant under RG flow and study its conse-

quence. The splitting is then proportional to  $\langle \cos \sqrt{2\pi} \varphi_+(x) \rangle$  since  $\cos \sqrt{2\pi} \theta_-$  has different expectation values on the two ground states. Because  $\varphi_+$  is pinned at  $x = 0$ ,  $\langle \cos \sqrt{2\pi} \varphi_+(x) \rangle \sim 1/x^{K_+}$ . Therefore the splitting of the ground states due to a single impurity in the middle of the system scales as  $1/L^{K_+}$ .

We thereby conclude that the ground state degeneracy is spoiled by the single-particle inter-chain tunneling near the boundaries and the backscattering processes in the bulk. To avoid the unwanted tunneling processes near the ends, one can put strong tunneling barriers between the two chains near the ends, or the chains can be bended outwards so that the two ends are kept far apart, as depicted in Fig. 7.1.

## 7.4 Lattice Model

We now show that the field theory (7.6) can be realized in lattice models of fermions. We consider the model of two weakly coupled chains of spinless fermions [169, 170, 159, 171]. The Hamiltonian reads

$$\hat{H} = -t \sum_{i,a} (\hat{c}_{i+1,a}^\dagger \hat{c}_{ia} + \text{h.c.}) + \sum_{i,a,r} V(r) \hat{n}_{ia} \hat{n}_{i+r,a} - t_\perp \sum_i (\hat{c}_{i2}^\dagger \hat{c}_{i1} + \text{h.c.}). \quad (7.19)$$

Here  $a = 1, 2$  labels the two chains. We assume the filling is incommensurate to avoid complications from Umklapp scatterings.  $V(r)$  is an intra-chain short-range attractive interaction between two fermions at a distance  $r$  (in units of lattice spacing). Thus without inter-chain coupling, each chain admits a Luttinger liquid description with two control parameters: charge velocity  $v$  and Luttinger parameter  $K$  (we assume  $V$  is not strong enough to drive the chain to phase separation).

We bosonize the full Hamiltonian and write the theory in the bonding and anti-

bonding basis. Hamiltonian in the bonding sector is just a theory of free bosons. In the anti-bonding sector, it reads

$$\hat{\mathcal{H}} = \frac{v}{2} \left[ K(\partial_x \theta)^2 + \frac{1}{K}(\partial_x \varphi)^2 \right] + \frac{2t_{\perp}}{\pi a_0} \cos \sqrt{2\pi} \varphi \cos \sqrt{2\pi} \theta. \quad (7.20)$$

The bosonic fields  $\varphi$  and  $\theta$  are in the anti-bonding basis. The perturbation ( $t_{\perp}$ ) term has *nonzero* conformal spin which implies that two-particle processes are automatically generated by RG flow even when they are absent in the bare Hamiltonian. Therefore, one has to include two-particle perturbations in the RG flow

$$\hat{\mathcal{H}}_2 = \frac{g_1}{(\pi a_0)^2} \cos \sqrt{8\pi} \varphi + \frac{g_2}{(\pi a_0)^2} \cos \sqrt{8\pi} \theta. \quad (7.21)$$

The RG flow equations for weak couplings have been derived by Yakovenko [169] and Nersesyan *et al.* [170]. Here we cite their results [159]:

$$\begin{aligned} \frac{dz}{dl} &= \left( 2 - \frac{K + K^{-1}}{2} \right) z \\ \frac{dy_1}{dl} &= (2 - 2K)y_1 + (K - K^{-1})z^2 \\ \frac{dy_2}{dl} &= (2 - 2K^{-1})y_2 + (K^{-1} - K)z^2 \\ \frac{dK}{dl} &= \frac{1}{2}(y_2^2 - y_1^2 K^2) \end{aligned}, \quad (7.22)$$

where the dimensionless couplings are defined as  $z = \frac{t_{\perp} a}{2\pi v}$  and  $y_{1,2} = \frac{g_{1,2}}{\pi v}$ .

Since we are interested in the phase where the pair tunneling dominates at low energy, we assume  $K > 1$  so  $y_1$  is irrelevant and can be put to 0. Also we neglect renormalization of  $K$ . Integrating the RG flow equations with initial conditions  $z(0) = z_0 \ll 1, y_2(0) = 0$  we obtain

$$y_2(l) = z_0^2 \frac{K^{-1} - K}{2\alpha} \left[ e^{2(1-\alpha)l} - e^{2(1-K^{-1})l} \right], \quad (7.23)$$

where  $\alpha = \frac{1}{2}(K + K^{-1} - 2)$ . Assume  $K^{-1} < \alpha$ , then the large- $l$  behavior of  $y_2$  is dominated by  $e^{2(1-K^{-1})l}$ .  $y_2$  becomes of order of 1 at  $l^* \approx -\ln z_0/(1 - K^{-1})$ , where the flow of  $z$  yields  $z(l^*) \approx z_0^{(\alpha-K^{-1})/(1-K^{-1})} \ll 1$  given  $z_0 \ll 1$ . This means that if  $K > \sqrt{2} + 1$  (so  $K^{-1} < \alpha$ ), then  $y_2$  reaches strong-coupling first. Thus the strong-coupling field theory is given by (7.6).

## Chapter 8

# Conclusion and Outlook

We first present a summary of the findings in this dissertation and then discuss open problems that can be pursued in the future. In the first Chapter, we reviewed the theory of non-Abelian topological superconductors, focusing on the topological classification and the non-Abelian statistics of quasiparticle excitations and briefly introduced the topological quantum computation scheme based on non-Abelian superconductors. In Chapter 2 we established a general condition under which topological superconductivity can arise in lattice models of interacting spinless fermions, within the framework of BCS mean-field theory. We showed that due to the particular convexity property of the BCS free energy, superconducting order parameters with a chiral pairing symmetry are naturally selected by energetics when the point symmetry group has only multidimensional irreducible representations. We also studied the phase diagram and topological phase transitions in other lattice models which do not satisfy the condition. In Chapter 3 we reviewed the solutions of Bogoliubov-de Gennes equation and derived the analytical expressions of the zero-energy Majorana bound states in vortices. We also made the connection to the general index theorem and provided a physical argument that the Majorana zero modes in vortices admit a  $\mathbb{Z}_2$  classification. In Chapter 4 we considered the



effect of quasiparticle tunneling on the topological degeneracy that is fundamental to the realization of topological qubits, and calculated the energy splitting of the degenerate states using a generalized WKB method. We found the energy splitting exhibits an oscillatory behavior with the inter-vortex distance, apart from the well-known exponential suppression. The presence of these oscillations has important implications for topological quantum computation, since the energy splitting determines the fusion channel of two non-Abelian vortices. In Chapter 5 we turn to the question of thermal effects on the topological quantum computation scheme based on Majorana quasiparticles. We distinguished two types of fermionic excitations that can possibly spoil the topological protection of qubits, the localized midgap states and extended states above the gap, and considered their effects on the braiding, read out and the lifetime of the qubits. We exploited a density matrix formulation based on physical observables and found the topological braiding remains intact in the presence of thermal excitations. However, thermally excited midgap states do result in decoherence in the read out of topological qubits based on vortex interferometry and we derived an analytical expression for the deduction in the interference visibility using a simplified but still physical model. In Chapter 6 we consider the effect of non-adiabaticity on vortex braiding in a microscopic model of a spinless  $p_x + ip_y$  superconductor. We developed a time-dependent Bogoliubov-de Gennes equation approach to describe time evolution of BCS superconductors. With the help of this formalism, we studied the robustness of the braiding operations when non-adiabaticity is taken into account and calculated the corrections to the Ivanov's rule perturbatively. In Chapter 7, we addressed the question of whether

Majorana zero modes can survive under strong quantum fluctuations, especially in one dimension. We first considered a continuum field theory of spinless fermions on a two-leg ladder with pair tunneling, in the presence of quasi-long-range superconducting order. Using bosonization technique we analyze non-perturbatively the strong-coupling phase and found interesting degeneracies of low-energy states that can be interpreted as Majorana zero-energy edge states. We discussed the stability of these degeneracies under various perturbations. Then we proposed a possible lattice realization of this field theory.

We now discuss possible future research directions. It is interesting to explore the possible vortex lattice phase in a topological superconductor where low-energy physics can be described by Majorana fermions hopping on the lattice with hopping amplitudes determined by the energy splitting calculated in Chapter 3. More work needs to be done to fully understand the robustness of topological qubits, including the effect of disorder and possible low-energy impurity bound states, and how they affect the braiding and the read out schemes. The effect of quantum fluctuations on Majorana zero modes in higher dimensions remains an open problem although the one-dimensional case has been rather well understood. It would be very interesting to generalize the bosonization approach to higher dimensions.

## Appendix A

# Derivation of the Pfaffian Formula for the Chern Parity

In this appendix we derive the Pfaffian formula for the parity of the Chern number in a class D topological superconductor, defined by BdG Hamiltonian  $\mathcal{H}(\mathbf{k})$  satisfying

$$\Xi^{-1}\mathcal{H}(\mathbf{k})\Xi = -\mathcal{H}^*(-\mathbf{k}). \quad (\text{A.1})$$

Assume that  $\mathcal{H}(\mathbf{k})$  is a  $2N \times 2N$  matrix. After diagonalizing we get  $2N$  bands  $\varepsilon_m(\mathbf{k})$ . Due to particle-hole symmetry energy eigenvalues come in pairs so we label the bands as  $\varepsilon_{-m}(-\mathbf{k}) = \varepsilon_m(\mathbf{k})$ . We choose a gauge such that the eigenvectors  $u_m(\mathbf{k})$  satisfy

$$u_{-m}(-\mathbf{k}) = \Xi u_m^*(\mathbf{k}).$$

The topological invariant for class D superconductors is the Chern number:

$$\mathcal{C} = \sum_{m < 0} \frac{1}{2\pi} \int_{1BZ} d^2\mathbf{k} \mathcal{F}_m(\mathbf{k}) = \sum_{m < 0} \frac{1}{2\pi} \oint_{1BZ} d\mathbf{k} \cdot \mathcal{A}_m(\mathbf{k}) \in \mathbb{Z}. \quad (\text{A.2})$$

Here the Berry connection is defined as

$$\mathcal{A}_{mn}(\mathbf{k}) = -i \langle u_m(\mathbf{k}) | \nabla_{\mathbf{k}} | u_n(\mathbf{k}) \rangle,$$

and we denote  $\mathcal{A}_m \equiv \mathcal{A}_{mm}$ . And the Berry curvature follows:

$$\mathcal{F}_m(\mathbf{k}) = [\nabla_{\mathbf{k}} \times \mathcal{A}_m(\mathbf{k})]_z.$$

Notice that we choose the first Brillouin zone  $k_x \in [-\pi, \pi], k_y \in [-\pi, \pi]$ .

We now start simplifying the expression for  $\mathcal{C}$ . The particle-hole symmetry implies

$$\mathcal{A}_m(\mathbf{k}) = \mathcal{A}_{-m}(-\mathbf{k}), \mathcal{F}_m(\mathbf{k}) = \mathcal{F}_{-m}(-\mathbf{k}).$$

Therefore

$$\begin{aligned} \sum_{m<0} \int_{-\pi}^0 dk_x \int_{-\pi}^0 dk_y \mathcal{F}_m(\mathbf{k}) &= \sum_{m<0} \int_{-\pi}^0 dk_x \int_{-\pi}^0 dk_y \mathcal{F}_{-m}(-\mathbf{k}) = \sum_{m>0} \int_0^\pi dk_x \int_0^\pi dk_y \mathcal{F}_m(\mathbf{k}), \\ \sum_{m<0} \int_0^\pi dk_x \int_{-\pi}^0 dk_y \mathcal{F}_m(\mathbf{k}) &= \sum_{m>0} \int_{-\pi}^0 dk_x \int_0^\pi dk_y \mathcal{F}_m(\mathbf{k}). \end{aligned}$$

So altogether we obtain

$$\sum_{m<0} \int_{-\pi}^\pi dk_x \int_{-\pi}^0 dk_y \mathcal{F}_m(\mathbf{k}) = \sum_{m>0} \int_{-\pi}^\pi dk_x \int_0^\pi dk_y \mathcal{F}_m(\mathbf{k}),$$

and the Chern number is expressed as

$$\begin{aligned} \mathcal{C} &= \frac{1}{2\pi} \sum_{m<0} \left( \int_{-\pi}^\pi dk_x \int_{-\pi}^0 dk_y + \int_{-\pi}^\pi dk_x \int_0^\pi dk_y \right) \mathcal{F}_m(\mathbf{k}) \\ &= \frac{1}{2\pi} \sum_m \int_{-\pi}^\pi dk_x \int_0^\pi dk_y \mathcal{F}_m(\mathbf{k}) \end{aligned}$$

We now choose a gauge in which  $\mathcal{A}_y$  is single-valued, and let  $a_m(\mathbf{k}) = -i \langle u_m(\mathbf{k}) | \partial_{k_x} | u_m(\mathbf{k}) \rangle$ .

$$\begin{aligned} \mathcal{C} &= \frac{1}{2\pi} \sum_m \int_{-\pi}^\pi dk_x \int_0^\pi dk_y \frac{\partial a_m(\mathbf{k})}{\partial k_y} \\ &= \frac{1}{2\pi} \sum_m \left( \int_{-\pi}^\pi dk_x a_m(k_x, \pi) - \int_{-\pi}^\pi dk_x a_m(k_x, 0) \right). \end{aligned}$$

Using the same trick, we can further deduce

$$\begin{aligned} \sum_m \int_{-\pi}^\pi dk_x a_m(k_x, \pi) &= \sum_m \int_{-\pi}^0 dk_x a_m(k_x, \pi) + \sum_m \int_0^\pi dk_x a_m(k_x, \pi) \\ &= \sum_m \int_{-\pi}^0 dk_x a_{-m}(-k_x, -\pi) + \sum_m \int_0^\pi dk_x a_m(k_x, \pi) \\ &= 2 \sum_m \int_0^\pi dk_x a_m(k_x, \pi), \end{aligned}$$

which allows further simplification of  $\mathcal{C}$ :

$$\mathcal{C} = \frac{1}{\pi} \sum_m \left( \int_0^\pi dk_x a_m(k_x, \pi) - \int_0^\pi dk_x a_m(k_x, 0) \right).$$

We can relate the gauge fields  $a_m$  to the eigenstates of the Hamiltonian. Define the unitary matrix  $U(\mathbf{k})$  as

$$U^\dagger(\mathbf{k})\mathcal{H}(\mathbf{k})U(\mathbf{k}) = D(\mathbf{k}),$$

where  $D(\mathbf{k})$  is the diagonal matrix of eigenvalues ordered in descending order of value. It is easy to show that

$$\sum_m \mathcal{A}_m(\mathbf{k}) = -i\nabla_{\mathbf{k}} \ln \det U(\mathbf{k}).$$

Therefore

$$\sum_m a_m(k, \theta) = -i\partial_k \ln \det U(k, \theta).$$

As a result,

$$\mathcal{C} = \frac{1}{\pi i} \left[ \int_0^\pi dk \partial_k \ln \det U(k, \pi) - \int_0^\pi dk \partial_k \ln \det U(k, 0) \right] = \frac{1}{\pi i} \ln \frac{\det U(\pi, \pi) \det U(0, 0)}{\det U(0, \pi) \det U(\pi, 0)}.$$

It can be written in a more natural form:

$$e^{i\pi\mathcal{C}} = \frac{\det U(\pi, \pi) \det U(0, 0)}{\det U(0, \pi) \det U(\pi, 0)}. \quad (\text{A.3})$$

which is the desired result.

Although in the derivation we choose a certain gauge, the result is certainly gauge-invariant.

We now further simplify this results. (A.3) basically means that the parity of Chern number of a 2D particle-hole symmetric insulator is determined by the

product of  $\det U(\mathbf{\Gamma})$  where  $\mathbf{\Gamma}$  is a particle-hole symmetric momentum ( $\mathbf{\Gamma} = -\mathbf{\Gamma} + \mathbf{G}$  where  $\mathbf{G}$  is reciprocal lattice vector). We can relate the  $\det U$  to the Hamiltonian. Define a matrix  $W(\mathbf{\Gamma}) = \mathcal{H}(\mathbf{\Gamma})\Xi$ . First we can show that it is anti-symmetric:

$$W^T(\mathbf{\Gamma}) = \Xi^T \mathcal{H}^T(\mathbf{\Gamma}) = -\Xi^T \Xi^{-1} \mathcal{H}(-\mathbf{\Gamma}) \Xi = -\mathcal{H}(\mathbf{\Gamma}) \Xi = W(\mathbf{\Gamma}).$$

On the other hand, we have

$$W(\mathbf{\Gamma}) = \mathcal{H}(\mathbf{\Gamma})\Xi = U(\mathbf{\Gamma})D(\mathbf{\Gamma})U^\dagger(\mathbf{\Gamma})\Xi.$$

Notice that  $\Xi U(\mathbf{k}) = U^*(\mathbf{k})\Lambda$ , so

$$U(\mathbf{\Gamma})D(\mathbf{\Gamma})U^\dagger(\mathbf{\Gamma})\Xi = U(\mathbf{\Gamma})D(\mathbf{\Gamma})\Lambda U^T(\mathbf{\Gamma}).$$

One can show that  $D(\mathbf{\Gamma})\Lambda$  is anti-symmetric. Therefore

$$\text{Pf } W(\mathbf{\Gamma}) = \text{Pf } [U(\mathbf{\Gamma})D(\mathbf{\Gamma})\Lambda U^T(\mathbf{\Gamma})] = \det U(\mathbf{\Gamma}) \text{Pf } [D(\mathbf{\Gamma})\Lambda].$$

Combining, we finally obtain

$$e^{i\pi C} = \prod_{\mathbf{\Gamma}} \text{Pf } W(\mathbf{\Gamma}).$$

# List of Publications

Publications this thesis is based on:

1. Meng Cheng, Roman Lutchyn and S. Das Sarma, *Topological Protection of Majorana Qubits*, Phys. Rev. B. **85**, 165124 (2012)
2. Meng Cheng and Hong-Hao Tu, *Majorana Edge States in Interacting Two-chain Ladder of Fermions*, Phys. Rev. B. **84**, 094503 (2011)
3. Meng Cheng, Victor Galitski and S. Das Sarma, *Non-Adiabatic Effects in the Braiding of Non-Abelian Anyons in Topological Superconductors*, Phys. Rev. B **84**, 104529(2011)
4. Meng Cheng, Roman Lutchyn, Victor Galitski, and S. Das Sarma, *Tunneling of anyonic Majorana excitations in topological superconductors*, Phys. Rev. B. **82**, 094504 (2010)
5. Meng Cheng, Kai Sun, Victor Galitski and S. Das Sarma, *Stable topological superconductivity in a family of two-dimensional fermion models*, Phys. Rev. B. **81**, 024504 (2009)
6. Meng Cheng, Roman Lutchyn, Victor Galitski, and S. Das Sarma, *Splitting of Majorana-fermion modes due to intervortex tunneling in a  $p_x + ip_y$  superconductor*, Phys. Rev. Lett. **103**, 107001 (2009)

Other publications and preprints that I contributed to:

1. Xiong-Jun Liu, Zheng-Xin Liu and Meng Cheng, *Manipulating Topological Edge Spins in One-Dimensional Optical Lattice*, arXiv:1209.2990
2. Meng Cheng, *Superconducting Proximity Effect on the Edge of Fractional Topological Insulators*, arXiv:1204.6085
3. Meng Cheng and Roman Lutchyn, *Josephson Current through a Semiconductor Nanowire: effect of strong spin-orbit coupling and Zeeman splitting*, Phys. Rev. B **86**, 134522 (2012)



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