ABSTRACT

Title of dissertation: TORELLI ACTIONS AND SMOOTH STRUCTURES ON 4-MANIFOLDS

Jack Samuel Calcut III, Doctor of Philosophy, 2004

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In the theory of Artin presentations, a smooth four manifold is already determined by an Artin presentation of the fundamental group of its boundary. Thus, one of the central problems in four dimensional smooth topology, namely the study of smooth structures on these manifolds and their Donaldson and Seiberg-Witten invariants, can be approached in an entirely new, *exterior*, purely group theoretic manner.

The main purpose of this thesis is to explicitly demonstrate how to change the smooth structure in this manner, while preserving the underlying continuous topological structure. These examples also have physical relevance.

We also solve some related problems. Namely, we study knot and link theory in Artin presentation theory, give a group theoretic formula for the Casson invariant, study the combinatorial group theory of Artin presentations, and state some important open problems.

TORELLI ACTIONS AND SMOOTH STRUCTURES ON 4-MANIFOLDS

by

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DEDICATION

To my wife, Margot.

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LIST OF ABBREVIATIONS

r	Artin presentation
$\mathcal{R}_n$	group of Artin presentations on n generators
$A(r)$	exponent sum matrix of r
t	Torelli
Ω_n	compact 2-disk with n holes
$h(r)$	self homeomorphism of Ω_n fixed on $\partial\Omega_n$
$M^3(r)$	closed, connected, orientable 3-manifold
$W^4(r)$	smooth, compact, connected, simply connected 4-manifold
k_i	canonical knots determined by open book construction
G_i	knot group of k_i
m_i, l_i	peripheral structure of k_i

CHAPTER 1

Introduction

Artin Presentation theory (AP theory) is a discrete, purely group theoretic theory of smooth, compact, simply connected 4-manifolds, their boundaries, and knots and links therein [**W**],[**CW**].

By definition, an Artin presentation r is a finite presentation:

$$\langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$$

satisfying the following equation in F_n (the free group on $x_1, \ldots, x_n$):

$$x_1 x_2 \cdots x_n = \left(r_1^{-1} x_1 r_1\right) \left(r_2^{-1} x_2 r_2\right) \cdots \left(r_n^{-1} x_n r_n\right).$$

The name, given by González-Acuña in 1975, was well chosen as Emil Artin first considered such presentations in 1925 [**A**], p.416-441, regarding his theory of braids.

Details and proofs of the following statements are in [**W**],[**CW**], and they appear in Chapter 2 for the sake of completeness.

For $n > 0$, $\mathcal{R}_n$ will denote the set of Artin presentations on n generators. $\mathcal{R}_n$ forms a group canonically isomorphic to $P_n \times \mathbb{Z}^n$, where P_n is the classical pure braid group on n strands. Let Ω_n denote the compact 2-disk with n holes. An Artin presentation $r \in \mathcal{R}_n$ determines:

$$\begin{aligned}
\pi(r) &= \text{the group presented by } r, \\
A(r) &= \text{an } n \times n \text{ symmetric integer matrix}, \\
h(r) &= \text{a self homeomorphism of } \Omega_n \text{ that is the identity} \\
&\quad\ \text{on } \partial\Omega_n \text{ and is unique up to isotopy rel } \partial\Omega_n, \\
M^3(r) &= \text{a closed, connected, orientable 3-manifold}, \\
W^4(r) &= \text{a smooth, compact, connected,} \\
&\quad\ \text{simply connected 4-manifold.}
\end{aligned}$$

The manifold $M^3(r)$ is the open book with planar page Ω_n defined by $h(r)$, has fundamental group isomorphic to $\pi(r)$, and bounds $W^4(r)$. The matrix $A(r)$ is the exponent sum matrix of the presentation r, is a presentation matrix of $H_1(M^3(r);\mathbb{Z})$, and represents the quadratic form of $W^4(r)$. In particular, $M^3(r)$ is a rational homology 3-sphere if and only if $\det A(r) \neq 0$ and is an integral homology 3-sphere if and only if $\det A(r) = \pm 1$. We have:

Theorem (González-Acuña [**GA**]). *Every closed, connected, orientable 3-manifold is homeomorphic to some $M^3(r)$.*

Thus, Artin Presentations characterize the fundamental groups of closed, orientable 3-manifolds.

Artin presentations whose exponent sum matrices are identically zero are called Torelli and are usually denoted by t. The set of all Torelli in $\mathcal{R}_n$ forms a subgroup canonically isomorphic to $[P_n, P_n]$, the commutator subgroup of P_n. The Torelli in AP theory are a subgroup of the classical Torelli group consisting of elements of the mapping class group of the closed, orientable genus n surface that act trivially on the first homology group of the surface.

A is a group homomorphism. If $r, r' \in \mathcal{R}_n$ and $\cdot$ denotes the group operation in $\mathcal{R}_n$ then:

$$A(r' \cdot r) = A(r') + A(r).$$

It follows that $A(t \cdot r) = A(r)$ for all Torelli t.

The Torelli subgroup acts on $\mathcal{R}_n$ by left translation. This action preserves the integer homology of both $M^3(r)$ and $W^4(r)$, although it can change the topology and the knot and linking theory of the manifolds concerned.

Observe that Theorem I of [**W**] reveals a somewhat surprising tie in of Donaldson's theorem with *discrete* pure group theory.

Theorem (Winkelnkemper [**W**], Th. I, p.240)**.** *Let r be an Artin Presentation whose exponent sum matrix $A(r)$ is definite, but not congruent to $\pm I$ over $\mathbb{Z}$.*

Then, the group $\pi(r)$ *is nontrivial. In fact, there is a nontrivial representation of* $\pi(r)$ *into* $SU(2)$.

This theorem is a consequence of 4D AP theory and Taubes' augmentation of Donaldson's theorem [**W**], p.240, and is *purely group theoretic*, despite the fact that Donaldson's theorem is arrived at via smooth, differential geometric methods.

In fact, there also exists a nontrivial, purely group theoretic theory of Donaldson *invariants* [**CW**].

One of the central problems of modern differential topology and physics is the study of the smooth structures of simply connected, compact 4-manifolds. The theory is of utmost importance and considerable subtlety, not only mathematically but also physically. See for example, Fintushel and Stern [**FS**] and Witten [**Wi1**],[**Wi2**].

The main purpose of this thesis is to show that one can change the smooth structure of a compact, smooth, simply connected 4-manifold while preserving the underlying continuous topology in an *entirely different manner* than that of [**FS**] and others, namely with the Torelli action as follows.

Let r be an Artin presentation such that $\det A(r) = \pm 1$, then $W^4(r)$ is a compact, smooth, simply connected 4-manifold whose boundary is an oriented, integral homology 3-sphere. Suppose t is a Torelli such that $M^3(r)$ and $M^3(t \cdot r)$ are orientation preserving homeomorphic. Then, since $A(r) = A(t \cdot r)$, the

4-manifolds $W^4(r)$ and $W^4(t \cdot r)$ will be homeomorphic by Freedman's theorem [**FrQ**], see also [**GS**], p.448. However, the important question arises whether they are diffeomorphic also.

Using work of Akbulut [**Ak1**],[**Ak2**] we show that:

Theorem 5 (Chapter 3 below)**.** *There exists Artin Presentations* $r \in \mathcal{R}_n$ *and Torelli* $t \in \mathcal{R}_n$ *for all* $n \geq 10$ *such that* $W^4(r)$ *and* $W^4(t \cdot r)$ *are homeomorphic but not diffeomorphic.*

The common boundaries of these 4-manifolds is the simplest hyperbolic integral homology 3-sphere, namely the 1/2 Dehn sphere of the figure eight knot of S^3 (see Section 3.2).

Thus, smooth structures on an underlying topological 4-manifold can be changed in a general, 'exterior', purely group theoretic manner, as opposed to the more 'internal', classical surgery methods of [**FS**] and others. It is pure group theory that generates new 4D smooth structures, i.e. structures that are ultimately at the foundation of General Relativity.

An intriguing question arises, further discussion of which we defer to other papers. The *global* consequences of solving the 4D quantum Yang-Mills mass gap ('Millennium') problem [**JWi**], p.6, are closely related to the behavior of Donaldson invariants on algebraic surfaces [**Wi2**], p.25. Since generalizing Witten's work [**Wi1**] on this subject from the Kähler case to the general case involves serious *analytical* obstructions, it is natural to hope that developing our purely group theoretic Donaldson invariants, where analytic problems of moduli are absent, could be a promising attack.

Another problem of a difficult nature is, since AP theory has an analogue of Donaldson theory, to find an analogue of Floer theory in AP theory. A hint that this is possible is that the Casson invariant of an integral homology 3-sphere is the Euler characteristic of the Floer homology. In Chapter 5 we show how to compute the Casson invariant of $M^3(r)$ with $\det A(r) \neq 0$ in function of r purely group theoretically. This already shows that at least all 3D Seiberg-Witten invariants can be computed *group theoretically* in AP theory by Lim's result [**Lim**]. The analogous 4D problem is open.

An important strength of AP theory is its canonical group theoretic knot/link theory [**W**], p.226,227, which plays a key role in computing the Casson invariant. Fix $r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle \in \mathcal{R}_n$. Then, there are $n+1$ distinguished knots, $k_0, k_1, \ldots, k_n$, in $M^3(r)$. These knots are the boundary of Ω_n in the open book construction. The knot groups G_i of the knots k_i are presented by ([**W**], p.226,227 and [**CW**], Section 2.1):

$$\begin{aligned} G_0 &= \langle x_1, \ldots, x_n \mid r_1 = r_2 = \cdots = r_n \rangle, \\ G_i &= \langle x_1, \ldots, x_n \mid r_1, r_2, \ldots r_{i-1}, r_{i+1}, \ldots, r_n \rangle, \ i \neq 0. \end{aligned}$$

In Chapter 4 we prove that these knots are sufficiently general. In particular, every link in every closed, connected, orientable 3-manifold M^3 appears as a sublink of the union of the $k_i\left(r\right)$s for some Artin presentation r such that $M^3\left(r\right)$ is homeomorphic to M^3. We are indebted to González-Acuña for this fundamental, unpublished result.

Furthermore [**W**], p.226,227, if $A\left(r\right)$ is unimodular (i.e. $M^3\left(r\right)$ is an integral homology 3-sphere), then the peripheral structures m_i, l_i of the knots k_i are given by: m_0 =any r_i, $l_0 = x_1 x_2 \cdots x_n m_0^{-s}$ where s is the sum of all elements in $A\left(r\right)^{-1}$, and for $i \neq 0$, $m_i = r_i$, $l_i = x_i m_i^{-b_i}$ where $b_i = \left[A\left(r\right)^{-1}\right]_{ii}$. Thus, using a computer algebra system, such as MAGMA, one can systematically explore link theory in closed, orientable 3-manifolds.

Artin presentation knot theory does not use skein methods, is functorial with respect to the Torelli action, and framings are not put in 'by hand.'

Recall that relationships between Alexander polynomials of knots and smooth invariants have already surfaced [**MeTa**],[**FS**]. Thus, it is natural to expect that smooth invariants of the 4-manifolds $W^4\left(r\right)$ are related to the Alexander polynomials of the canonical knots $k_i\left(r\right)$.

In Chapter 6 we discuss combinatorial group theoretic aspects of Artin presentations. Using only combinatorial group theory, we characterize Artin presentations in $\mathcal{R}_2$, we show that the j-reduction of an Artin presentation is an

Artin presentation, and as a corollary we obtain a new, purely group theoretic proof of the symmetry of $A(r)$ for any Artin presentation.

We close this introduction by stating some fundamental open problems which can be attacked with AP theory.

First, there is the 11/8 conjecture (see [**GS**], p.16). If X^4 is a smooth, closed, simply connected 4-manifold with indefinite, even intersection form Q, then Q is isomorphic to $2kE_8 \oplus lH$ for some integers k and l (see [**GS**], p.14-17). It is known that $l \geq 2|k| + 1$; the 11/8 conjecture states that $l \geq 3|k|$. This problem is computer approachable due to AP theory as follows. Choose a unimodular, symmetric integer matrix A that satisfies the above conditions but contradicts the 11/8 conjecture. Construct an Artin presentation r such that $A(r) = A$ (this is always possible [**W**], p.248). Now, using a computer algebra system such as MAGMA, compose r with many Torelli hoping to find a Torelli t such that $\pi(t \cdot r) = 1$.

Second, is every closed, smooth, connected, simply connected 4-manifold obtained as a $W^4(r)$ union D^4 where r is an Artin presentation of S^3? No counterexamples are known, see [**GS**], p.344. An affirmative answer, plus the truth of the Poincaré conjecture, would imply that the study of such 4-manifolds embeds into the theory of Artin presentations of the trivial group.

Finally, is every integral homology 3-sphere Σ^3 homeomorphic to $M^3(r)$ where r is an Artin presentation such that its exponent sum matrix $A(r)$ equals I, the identity matrix? For these spheres, González-Acuña discovered a beautiful, purely group theoretic, formula for the Rohlin invariant (see Chapter 5 below). Our formula for the Casson invariant in Chapter 5 generalizes González-Acuña's formula.

CHAPTER 2

Artin Presentations

Recall that by definition an Artin presentation r is a finite presentation:

$$\langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$$

satisfying the following equation in F_n (the free group on $x_1, \ldots, x_n$):

$$x_1 x_2 \cdots x_n = \left(r_1^{-1} x_1 r_1\right)\left(r_2^{-1} x_2 r_2\right) \cdots \left(r_n^{-1} x_n r_n\right).$$

This equation means that the right hand side freely reduces to the word $x_1 x_2 \cdots x_n$ in F_n. We will always assume, and it is natural to do so, that the words r_i are freely reduced in an Artin presentation.

The purpose of this chapter is to provide the details and proofs of statements made in the previous chapter. We will discuss homeomorphisms of the punctured 2-disk, pure braids, 3-manifolds, and 4-manifolds.

2.1. Homeomorphisms of the punctured 2-disk

Artin presentations arise naturally as follows. Let Ω_n denote the compact 2-disk with $n > 0$ holes. Let $\partial_0, \partial_1, \ldots, \partial_n$ denote the boundary components of Ω_n

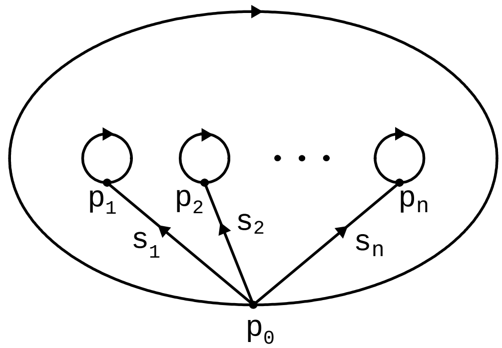


Figure 2.1. Ω_n the compact 2-disk with n holes. Also depicted are oriented boundary components $\partial_0, \ldots, \partial_n$ with basepoints $p_0, \ldots, p_n$ and oriented segments s_i from p_0 to p_i, $i = 1, \ldots, n$.

oriented clockwise with a basepoint p_i on each ∂_i as in Figure 2.1. Also depicted in the figure are oriented segments s_i from p_0 to p_i, $i = 1, \ldots, n$. For $i = 1, \ldots, n$ the loops $s_i \partial_i s_i^{-1}$ define generators x_i of the fundamental group $\pi_1(\Omega_n, p_0)$ which is isomorphic to F_n the free group on $x_1, \ldots, x_n$. Let $x_0 = x_1 x_2 \cdots x_n$ denote the element of $\pi_1(\Omega_n, p_0)$ determined by ∂_0.

Let h be any self homeomorphism of Ω_n that restricts to the identity on the boundary $\partial\Omega_n$. This induces the isomorphism $h_\# : \pi_1(\Omega_n, p_0) \to \pi_1(\Omega_n, p_0)$. Let $r_i = r_i(h) \in F_n$ be defined by the loop $s_i h\left(s_i^{-1}\right)$. Notice that $h_\#(x_i) = r_i^{-1} x_i r_i$

since:

$$\begin{aligned}
h_{\#}(x_i) &= \left[h\left(s_i \partial_i s_i^{-1}\right)\right] \\
&= \left[h(s_i)\,\partial_i h\left(s_i^{-1}\right)\right] \\
&= \left[h(s_i)\,s_i^{-1} s_i \partial_i s_i^{-1} s_i h\left(s_i^{-1}\right)\right] \\
&= \left[h(s_i)\,s_i^{-1}\right]\left[s_i \partial_i s_i^{-1}\right]\left[s_i h\left(s_i^{-1}\right)\right] \\
&= r_i^{-1} x_i r_i.
\end{aligned}$$

Hence:

$$\begin{aligned}
x_1 x_2 \cdots x_n &= x_0 \\
&= h_{\#}(x_0) \;\; (\text{since } h \text{ is identity on } \partial\Omega_n) \\
&= \left(r_1^{-1} x_1 r_1\right)\left(r_2^{-1} x_2 r_2\right) \cdots \left(r_n^{-1} x_n r_n\right),
\end{aligned}$$

and so such an h determines $r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ an Artin presentation.

The converse is also true and was implicitly known to Artin in 1925 [**A**], p.416-441. Namely, an Artin presentation $r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ determines a self homeomorphism $h = h(r)$ of Ω_n that is the identity on $\partial\Omega_n$ and is unique up to isotopy rel $\partial\Omega_n$. We will prove this result in the following section.

The remainder of this section discusses representatives of isotopy classes of homeomorphisms of Ω_n. To be succinct, we tacitly assume all self homeomorphisms of Ω_n are the identity on $\partial\Omega_n$ and all isotopies of such homeomorphisms are rel $\partial\Omega_n$.

Theorem 1. *Every self homeomorphism h of Ω_n has a PL representative h' in its isotopy class. Moreover, if h_1, h_2 are two self homeomorphisms of Ω_n that agree on any of the segments s_i, then PL representatives h_1', h_2' for the isotopy classes of h_1, h_2 respectively can be chosen that also agree on those segments.*

Proof. This is classical and follows from results in Moise [**Mo**]. We sketch the proof. Triangulate Ω_n as a PL subset of $\mathbb{R}^2$. By assumption, h is the identity on $\partial\Omega_n$ and so is PL there. Now, one uses the techniques in the proof of Theorem 7, p.73-76 of [**Mo**] to isotop h rel $\partial\Omega_n$ so that it is PL on the 1-skeleton of Ω_n. This is not difficult noting that the homeomorphisms h_ν and h_a described in Moise's proof each have support in the interior of a nice PL 2-cell N_ν,N_a respectively. Then, each individual homeomorphism h_ν or h_a is isotopic to the identity rel $\mathbb{R}^2 - intN_\nu$ or rel $\mathbb{R}^2 - intN_a$ respectively.

Now, we have h PL on the 1-skeleton of Ω_n. Let σ be a 2-simplex in Ω_n. Then, $h\left(\partial\sigma\right)$ is PL and so by Theorem 2, p.18 of [**Mo**], $\left|h\left(\sigma\right)\right|$ is a combinatorial 2-cell. Thus, Theorem 4, p.43 of [**Mo**] implies that h restricted to $\partial\sigma$ extends PL homeomorphically to σ. As this PL extension to the interior of σ equals h on the boundary, the extension is isotopic (rel boundary) to h using Alexander's trick (see

[**Mo**], Theorem 1, p.81, and use uniqueness of disk embedding as in Chapter 5 of [**Mo**]). This completes the proof of the first statement in the theorem. The second part follows similarly with a little care. □

Corollary 2. *A self homeomorphism h of Ω_n is completely determined up to isotopy by the images $h(s_i)$, $i = 1, 2, \ldots, n$.*

Proof. Let h_1 and h_2 be two self homeomorphisms of Ω_n that agree on all s_i. By the previous theorem, we may assume these homeomorphisms are PL and still agree on the s_i. By standard PL techniques, we may further assume h_1 and h_2 agree in some closed regular neighborhood N of $\partial\Omega_n \cup \{s_i \mid i = 1, 2, \ldots, n\}$. Then, restricting to $D = \Omega_n - intN$, h_1 and h_2 are two PL homeomorphic embeddings of PL 2-cells D into $\mathbb{R}^2$ that agree on ∂D. Hence, they are isotopic to one another rel ∂D using Alexander's trick. Extending by the identity on N gives the desired isotopy of h_1 to h_2. □

The above PL results imply the corresponding DIFF (smooth) formulations (e.g. see [**T**], Section 3.10).

Therefore, while smoothness of a homeomorphism h of Ω_n need not be be postulated, it is uniquely inherited for free.

2.2. Pure braids

In this section we will show that the set of all Artin presentations $\mathcal{R}_n$ on n generators forms a group canonically isomorphic to $P_n \times \mathbb{Z}^n$ where P_n is the classical n strand pure braid group.

Recall that the classical braid group B_n has a faithful representation as a group of automorphisms of F_n as shown in Birman [**B**], p.25, and so we regard $B_n \subset AutF_n$. There is also a canonical representation of B_n to the symmetric group on n letters $Sym\,(n)$; the kernel of this homomorphism is the pure braid group which we regard as $P_n \subset B_n \subset AutF_n$.

Notice that the group of isotopy classes (rel $\partial\Omega_n$) of homeomorphisms of Ω_n that are the identity on the boundary is canonically isomorphic to $P_n \times \mathbb{Z}^n$. To see this, extend such a homeomorphism to all of D^2 by the identity inside the inner boundary components $\partial_1, \ldots, \partial_n$. This extension is isotopic to the identity (rel ∂_0) say by $F : D^2 \times I \to D^2$. Extending this map to $f : D^2 \times I \to D^2 \times I$ by $f\,(x, t) = (F\,(x, t)\,, t)$ one immediately sees the boundary components $\partial_1, \ldots, \partial_n$ trace out an n strand pure braid. The $\mathbb{Z}^n$ factor comes from how many times each individual ∂_i twists completely around. For a concrete identification, let $(\beta, v) \in P_n \times \mathbb{Z}^n$. Let h be the homeomorphism of Ω_n obtained by taking Ω_n and sliding it up β; during this slide the boundary components ∂_i intertwine with each other, but should not twist themselves. Next compose this homeomorphism with one that simply twists each ∂_i the number of complete twists given by v_i, where a positive number means to twist clockwise and a negative number means to twist counterclockwise.

An Artin presentation $r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ determines a homomorphism $\beta : F_n \to F_n$ defined by $x_i \mapsto r_i^{-1} x_i r_i$. That r is Artin is exactly what is needed to show:

Lemma 3. *β is a pure braid automorphism of F_n.*

Proof. A detailed proof is in Birman [**B**], p.30-32. The main idea is that since r is Artin the word:

$$\left(r_1^{-1} x_1 r_1\right) \left(r_2^{-1} x_2 r_2\right) \cdots \left(r_n^{-1} x_n r_n\right)$$

must freely reduce to $x_1 x_2 \cdots x_n$ in F_n. Analysis of this reduction shows that precomposing β with some generator of the braid automorphisms $B_n \subset AutF_n$ gives a similar presentation that is shorter. Repeating this process, one obtains a braid automorphism $\alpha \in B_n$ that precomposed with β is the identity. This implies that β itself is a braid automorphism that must be pure since β is defined by $x_i \mapsto r_i^{-1} x_i r_i$. □

From this it follows that every Artin presentation determines a unique pure braid, compare Birman [**B**], p.32-34. Thus, we identify $\mathcal{R}_n$ with $P_n \times \mathbb{Z}^n$ by associating to r the pure braid as just shown and the ith component of the vector in $\mathbb{Z}^n$ given by the exponent of x_i in the abelianized r_i. Thus we have shown:

Theorem 4. *An Artin presentation* $r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ *determines a self homeomorphism* $h = h(r)$ *of* Ω_n *that is the identity on* $\partial\Omega_n$ *and is unique up to isotopy rel* $\partial\Omega_n$.

In this way, we see that $\mathcal{R}_n$ is in fact a group. Let $r, r' \in \mathcal{R}_n$ and let R_i be obtained by substituting $r_j^{-1} x_j r_j$ for each x_j in r_i. Then, $r'' = r' \cdot r$ is given by $\langle x_1, \ldots, x_n \mid r_1' R_1, \ldots, r_n' R_n \rangle$. This composition law is consistent with those on $P_n \times \mathbb{Z}^n$ and the group of homeomorphisms of Ω_n described above.

The above is summarized by the short exact sequence of groups, which splits:

$$0 \to \mathbb{Z}^n \to \mathcal{R}_n \to P_n \to 0.$$

2.3. The 3-manifolds $M^3(r)$

Let r be an Artin presentation and $h = h(r)$ a self homeomorphism of Ω_n determined by r (h restricts to the identity on $\partial\Omega_n$). One obtains a closed, connected, orientable 3-manifold $M^3(r)$ by the open book construction [**GA**],[**W**]. Namely, let $\Omega(h)$ denote the mapping torus of h; the boundary of $\Omega(h)$ is $(\partial\Omega_n) \times S^1$. $M^3(r)$ is obtained by glueing on $(\partial\Omega_n) \times D^2$ by the identity on

$(\partial\Omega_n) \times S^1$. Here Ω_n is called the page and $\partial\Omega_n$ is called the binding in the open book construction.

The fundamental group of $M^3(r)$ is isomorphic to the group $\pi(r)$ presented by r (see [**W**], p.247).

Every closed, connected, orientable 3-manifold is homeomorphic to some $M^3(r)$ [**GA**], see also Chapter 4 below.

The $n \times n$ integer matrix $A(r)$ is defined to be the exponent sum matrix of r. This matrix $A(r)$ is a presentation matrix of $H_1(M^3(r);\mathbb{Z})$. Namely, let $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$ be the homomorphism induced by $A(r)$, then:

$$H_1\left(M^3(r);\mathbb{Z}\right) \cong \mathbb{Z}^n \diagup \operatorname{Im}\varphi.$$

To see this, note that the first integral homology group is isomorphic to $\pi(r) \diagup [\pi(r), \pi(r)]$. This latter group is abelian and by [**MKS**], Sec.2.1, admits the presentation:

$$Ab(r) = \left\langle x_1, \ldots, x_n \mid r_1, \ldots, r_n, x_i x_j x_i^{-1} x_j^{-1}, 1 \le i < j \le n \right\rangle.$$

On the other hand, one can start with the presentation:

$$FA_n = \left\langle x_1, \ldots, x_n \mid x_i x_j x_i^{-1} x_j^{-1}, 1 \le i < j \le n \right\rangle,$$

which presents the free abelian group $\mathbb{Z}^n$ of rank n. In $\mathbb{Z}^n$, presented by FA_n, the words r_i generate a normal subgroup which we denote by N'. Again by [**MKS**], Sec. 2.1, the factor group $\mathbb{Z}^n / N'$ admits the presentation $Ab(r)$. Hence, by [**MKS**], Sec. 1.2, $\pi(r) / [\pi(r), \pi(r)]$ is isomorphic to $\mathbb{Z}^n / N'$. It is easy to see that $\mathbb{Z}^n / N'$ is isomorphic to $\mathbb{Z}^n / \operatorname{Im}\varphi$.

$A(r)$ is always symmetric for Artin presentations [**W**], p.248-250. There are multiple geometric proofs of this interesting fact; a new proof using only combinatorial group theory is in Chapter 6.

Thus, $A(r)$ contains all of the homological information about $M^3(r)$. It follows that $H_1(M^3(r); \mathbb{Z})$ is finite if and only if $\det A(r) \neq 0$, and in this case has order equal to $|\det A(r)|$. $M^3(r)$ is an integral homology 3-sphere if and only if $\det A(r) = \pm 1$ and is a rational homology 3-sphere if and only if $\det A(r) \neq 0$.

Thus, as Winkelnkemper points out in [**W**], $\det A(r) \neq \pm 1$ is an abelian condition preventing $\pi(r)$ from being trivial, and his Theorem I [**W**], p.240, is another such abelian condition (see Chapter 1 above). It is interesting to ask what other such abelian conditions exist.

We note that $M^3(r)$ can also be obtained by performing integral surgery on the closure of the pure braid determined by r with surgery coefficients given by the diagonal of $A(r)$ (see [**CW**]).

We close this section with some simple examples. When $n = 1$, $A(r)$ clearly determines r and the 3-manifold corresponding to $A(r) = k$ is the Lens space $L(k, 1)$. When $n = 2$, $A(r)$ also determines r; this follows geometrically from braid considerations, but it also can be proved purely group theoretically (see Chapter 6 below). Let $r \in \mathcal{R}_2$ be given by:

$$\begin{aligned} r_1 &= x_1^3 (x_1 x_2)^{-2}, \\ r_2 &= x_2^5 (x_1 x_2)^{-2}, \end{aligned}$$

then $M^3(r)$ is the Poincaré homology 3-sphere, $\pi(r) = \pi_1(M^3(r)) = I(120)$ and:

$$A(r) = \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix}.$$

2.4. The 4-manifolds $W^4(r)$

Let r be an Artin presentation and $h = h(r)$ an associated self homeomorphism of Ω_n. As shown in Section 2.1, we can, and do, choose h to be smooth. One obtains a smooth, compact, connected, simply connected 4-manifold $W^4(r)$ with connected boundary $M^3(r)$ as follows [**W**], p.250. Embed Ω_n in S^2 and let $c\Omega$ denote the closure of $S^2 - \Omega_n$. Extend h to all of S^2, then extend to all of D^3 by a diffeomorphism $H : D^3 \to D^3$ (which is unique up to isotopy rel

$\partial D^3 = S^2$). Let $W(H)$ denote the mapping torus of H. Then, $c\Omega \times S^1 \subset \partial W(H)$ and one obtains $W^4(r)$ by glueing on $c\Omega \times D^2$ in the canonical way.

As shown in [**W**], p.250, $W^4(r)$ is simply connected and its intersection form is represented by $A(r)$.

In [**CW**], we showed that $W^4(r)$ can also be obtained by attaching n 2-handles to D^4 along the framed link given by the closure of the pure braid determined by r with framings given by the diagonal of $A(r)$. As pointed out in [**GS**], "…the complexity of a 4-dimensional handlebody is mainly due to the 2-handles."

If the boundary $M^3(r)$ of $W^4(r)$ is homeomorphic to S^3 then it is natural to view $W^4(r)$ as a *closed*, smooth, simply connected 4-manifold (close up with a 4-handle). An immediate question is: which *closed*, smooth, simply connected 4-manifolds are $W^4(r)$s? This is an open problem (see [**GS**], p.344). It is possible that all such manifolds are obtained as $W^4(r) \cup D^4$.

It is easy to see that the following appear in AP theory: $\mathbb{C}P^2$, $\overline{\mathbb{C}P}^2$, $S^2 \times S^2$ (see [**GS**], p.127), and S^4 (considered as the empty Artin presentation $\langle | \rangle$). Clearly the connected sum of manifolds appearing also appears.

More interesting 4-manifolds are known to appear. In particular, in [**CW**] we showed that all elliptic surfaces $E(n)$ admit Artin presentations, in particular the Kummer surface $K3 = E(2)$. These were the first known examples of 4-manifolds in AP theory with nontrivial Donaldson and Seiberg-Witten invariants. The following chapter provides further examples.

CHAPTER 3

Torelli Actions

In this chapter, the main chapter of this thesis, we will show how the purely group theoretic action of the Torelli can change smooth structures on 4-manifolds.

Let $r \in \mathcal{R}_n$ be an Artin presentation with $A(r)$ unimodular, so $M^3(r)$ is an integral homology 3-sphere. Note that $W^4(r)$ and $M^3(r)$ are both oriented (view them in terms of 2-handlebodies). Suppose $t \in \mathcal{R}_n$ is a Torelli such that $M^3(t \cdot r)$ is orientation preserving homeomorphic to $M^3(r)$. Then, Freedman's theorem (extended form with oriented boundary a fixed homology 3-sphere) implies that $W^4(t \cdot r)$ and $W^4(r)$ are homeomorphic 4-manifolds [**FrQ**] (see also [**GS**], p.448). In case, $W^4(t \cdot r)$ and $W^4(r)$ are not diffeomorphic we say the Torelli t 'juggles' the smooth structure of the 4-manifold $W^4(r)$.

Figure 3.1 contains two framed, pure braids $s1$ and $s2$ on ten strands. In [**CW**], we gave an explicit way to construct Artin presentations from framed, pure braids. Thus, we also let $s1$ and $s2$ denote the Artin presentations in $\mathcal{R}_{10}$ corresponding to these framed, pure braids where no confusion should arise. We have:

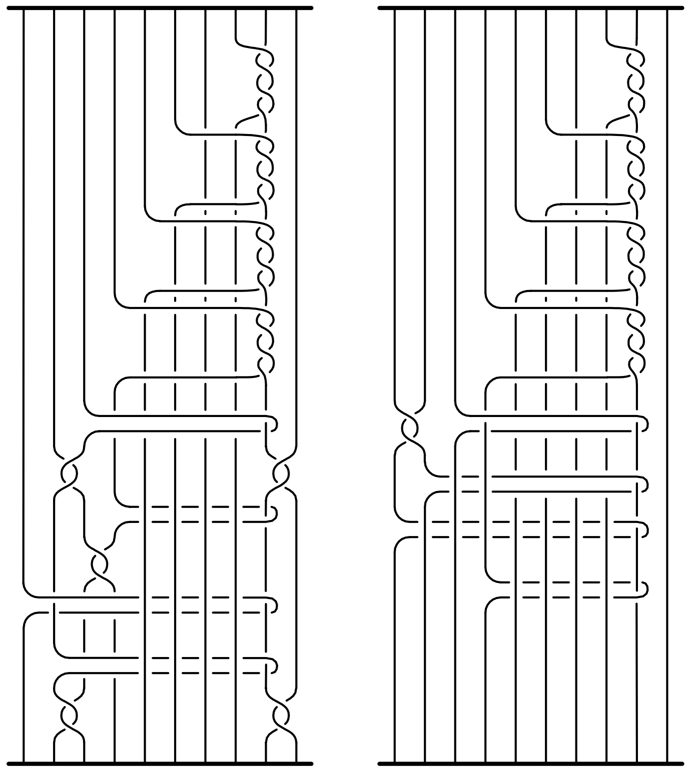


Figure 3.1. Pure braids $s1$ (left) and $s2$ (right) each with framings $-1, -2, -1, -2, -1, -1, -1, -1, -23, -1$ from left to right.

Theorem 5. *The Artin presentations $s1$ and $s2$ differ by multiplication by a Torelli t. Furthermore, $W^4(s1)$ and $W^4(s2)$ are homeomorphic but not diffeomorphic.*

Before proceeding to the proof of this theorem (Section 3.1 below), we will discuss some properties of these Artin presentations and some broader aspects of Torelli juggling.

The matrices $A(s1)$ and $A(s2)$ are equal and of determinant one (see Figure 3.2). Hence, $M^3(s1)$ and $M^3(s2)$ are integral homology 3-spheres and $s1$ and $s2$ differ by multiplication by the Torelli $t = s2^{-1} \cdot s1$. The inverse matrix $A(s1)^{-1}$ in

$$\begin{array}{cccccccccc}
-1 & & & & & & & & -1 & \\
 & -2 & & & & & & & -1 & \\
 & & -1 & & & & & & -1 & \\
 & & & -2 & & & & & +1 & \\
 & & & & -1 & & & & -2 & \\
 & & & & & -1 & & & -2 & \\
 & & & & & & -1 & & -2 & \\
 & & & & & & & -1 & -2 & \\
-1 & -1 & -1 & +1 & -2 & -2 & -2 & -2 & -23 & \\
 & & & & & & & & & -1
\end{array}$$

Figure 3.2. Matrix $A(s1) = A(s2)$ of determinant one.

Figure 3.3 gives the peripheral structures of the knots k_i. That $M^3(s1)$ and $M^3(s2)$ are orientation preserving homeomorphic follows from Akbulut [**Ak1**],[**Ak2**] (see also [**GS**], p.449) along with our construction of these pure braids in following section.

In Section 3.2 below, we identify the 3-manifolds $M^3(s1)$ and $M^3(s2)$ as the simplest hyperbolic, integral homology 3-sphere, namely the 1/2 Dehn sphere of the figure eight knot of S^3.

The above discussion of Freedman's theorem implies $W^4(s1)$ and $W^4(s2)$ are homeomorphic. In Section 3.1 below we show that they are not diffeomorphic. In fact, this remains true stably (after blowing up with finitely many $\overline{\mathbb{C}P}^2 s$ [**Ak1**],[**Ak2**], see also Section 3.1) and hence one easily obtains similar examples of Torelli juggling in $\mathcal{R}_n$ for all $n \geq 10$.

−3	1	−3	2	−2	−2	−2	−2	1	0
1	−1	0	0	0	0	0	0	0	0
−3	0	−11	7	−6	−6	−6	−6	3	0
2	0	7	−5	4	4	4	4	−2	0
−2	0	−6	4	−5	−4	−4	−4	2	0
−2	0	−6	4	−4	−5	−4	−4	2	0
−2	0	−6	4	−4	−4	−5	−4	2	0
−2	0	−6	4	−4	−4	−4	−5	2	0
1	0	3	−2	2	2	2	2	−1	0
0	0	0	0	0	0	0	0	0	−1

Figure 3.3. Matrix $A\,(s1)^{-1} = A\,(s2)^{-1}$.

These examples are entirely new. They show how multiplying an Artin presentation by a Torelli in the discrete group $\mathcal{R}_n$ can represent changing the smooth structure of a compact, simply connected 4-manifold while preserving its continuous topology. In fact, this action changes the Donaldson and Seiberg-Witten invariants of the manifolds.

The only Torelli in $\mathcal{R}_1$ and $\mathcal{R}_2$ are the trivial ones, namely the identity in each of these groups. However, in $\mathcal{R}_3$ the Torelli subgroup is already infinitely generated. Thus, it is natural to think that the Torelli action is in fact very effective in changing smooth structures. One expects to find more examples of Torelli juggling where $3 \leq n < 10$ and/or the boundary 3-manifolds are simply connected.

Our examples are meant to show that the Torelli can and do juggle the smooth structures on 4-manifolds purely group theoretically. In the future, one would hope to compute smooth invariants (Donaldson and Seiberg-Witten) group theoretically in function of r so as to obtain much broader juggling.

The lengths of the individual relations in $s1$ and $s2$ are:

Relation	s1	s2
1	3187	1723
2	13506	734
3	8103	243
4	7132	5624
5	323	1787
6	269	1733
7	251	1715
8	245	1709
9	7475	8215
10	4891	1

Thus, $s1$ has total relator length 45382 and $s2$ has total relator length 23484. Notice that even though $s2$ splits off a $\overline{\mathbb{C}P}^2$ summand, it is the manifold giving nontrivial Donaldson invariants [**Ak1**],[**Ak2**]; it seems curious that $s2$ is the tighter presentation. The Kummer surface $K3$ also has nontrivial smooth invariants and one of our presentations for $K3$ in $\mathcal{R}_{22}$ is of total relator length 4562 [**CW**].

We remark that of the knots $k_i\,(s1)$ and $k_i\,(s2)$ in the 3-manifolds $M^3\,(s1)$ and $M^3\,(s2)$, the only ones whose knot groups we recognize are $k_9\,(s1)$ and $k_{10}\,(s2)$. The group of $k_9\,(s1)$ is isomorphic to the group of the 5_2 knot in S^3. It is easy to see that $k_{10}\,(s2)$ is the trivial knot in $M^3\,(s2)$ from the braid $s2$. The Torelli t takes the knot $k_9\,(s1)$ to the knot $k_9\,(s2)$, the latter of which has a huge presentation and Alexander polynomial $(t^2 - t + 1)^2$.

The knots $k_0\,(s1)$ and $k_0\,(s2)$ both have Alexander polynomials of degree 108 (when normalized so the lowest degree term is t^0). However, the Alexander polynomial of $k_0\,(s1)$ is irreducible while that of $k_0\,(s2)$ factors into the product of:

$$t^2 - t + 1,$$

$$t^6 - t^3 + 1,$$

$$t^8 - t^7 + t^5 - t^4 + t^3 - t + 1,$$

$$t^8 + t^7 - t^5 - t^4 - t^3 + t + 1,$$

$$t^{18} - t^9 + 1,$$

$$t^{24} - t^{21} + t^{15} - t^{12} + t^9 - t^3 + 1, \text{ and}$$

$$t^{24} + t^{21} - t^{15} - t^{12} - t^9 + t^3 + 1.$$

The reader is reminded of our comment in the introduction on the relationship between smooth invariants and Alexander polynomials.

Finally, the lengths of the individual relations in the Torelli t are:

Relation	t
1	4764
2	21430
3	12778
4	11724
5	196
6	196
7	196
8	196
9	10400
10	6850

Thus, the Torelli t has total relator length 68730.

3.1. Proof of Theorem 5

Our starting point is the two manifolds $\overline{Q}_1$ and $\overline{Q}_2$ shown in Figure 3.4. These interesting manifolds were originally discovered by Akbulut [**Ak1**],[**Ak2**]. In particular, by reversing each crossing and changing the sign of the framing from -1

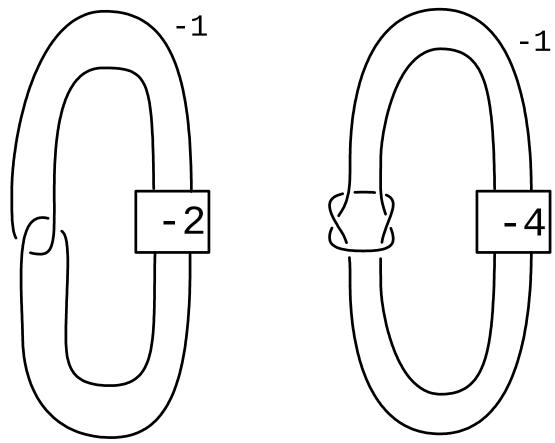


Figure 3.4. Two 4-manifolds $\overline{Q}_1$ (left) and $\overline{Q}_2$(right).

to $+1$, one obtains Akbulut's manifolds Q_1, Q_2 respectively (see [**Ak2**], p.357). We are following standard convention and letting $\overline{M}$ denote the oriented manifold obtained from the oriented manifold M by changing the orientation on every component (all of our manifolds will have just one component).

Akbulut showed in [**Ak2**] that Q_1 and Q_2 are homeomorphic but not diffeomorphic. Hence, $\overline{Q}_1$ and $\overline{Q}_2$ are homeomorphic but not diffeomorphic. His proof relies on the computation of a Donaldson invariant in [**Ak1**]. The same result follows from different considerations in [**GS**], p.448,449.

For our purposes, we need the above result to be true stably. This is already implicit in Akbulut's work:

Claim 6. *$\overline{Q}_1 \#_k \overline{\mathbb{C}P}^2$ and $\overline{Q}_2 \#_k \overline{\mathbb{C}P}^2$ are homeomorphic but not diffeomorphic for all $k \geq 0$.*

Proof of claim. We will follow the notation of [**Ak2**]. The manifolds in question are homeomorphic since $\overline{Q}_1$ and $\overline{Q}_2$ are homeomorphic. In [**Ak1**], Akbulut constructed a smooth, compact, connected, simply connected 4-manifold M_1 with $\partial M_1 = \partial Q_1 = \partial Q_2$. In [**Ak2**], p.359, Akbulut showed that Q_2 splits off a $\mathbb{C}P^2$ summand, that is $Q_2 = W_1 \# \mathbb{C}P^2$ where W_1 is a smooth, compact, contractible 4-manifold with $\partial W_1 = \partial Q_2$. Thus, $\overline{Q}_2 = \overline{W}_1 \# \overline{\mathbb{C}P}^2$ splits off a $\overline{\mathbb{C}P}^2$.

Assume $\overline{Q}_1 \#_k \overline{\mathbb{C}P}^2$ and $\overline{Q}_2 \#_k \overline{\mathbb{C}P}^2$ are diffeomorphic. Since $\overline{Q}_2 = \overline{W}_1 \# \overline{\mathbb{C}P}^2$, it follows that there are $k+1$ disjoint smoothly embedded 2-spheres in $\overline{Q}_1 \#_k \overline{\mathbb{C}P}^2$ each of self intersection number -1. Thus, $\overline{Q}_1 \#_k \overline{\mathbb{C}P}^2 = \overline{V} \#_{k+1} \overline{\mathbb{C}P}^2$ for some smooth, contractible 4-manifold V with $\partial V = \partial Q_1$ (see [**GS**], p.46). Let $\widetilde{M} = M_1 \cup_\partial \overline{Q}_1$ and $M' = M_1 \cup_\partial \overline{V}$. Then, we have:

$$\begin{aligned}
\widetilde{M} \#_k \overline{\mathbb{C}P}^2 &= \left(M_1 \cup_\partial \overline{Q}_1\right) \#_k \overline{\mathbb{C}P}^2 \\
&= M_1 \cup_\partial \left(\overline{Q}_1 \#_k \overline{\mathbb{C}P}^2\right) \\
&= M_1 \cup_\partial \left(\overline{V} \#_{k+1} \overline{\mathbb{C}P}^2\right) \\
&= M' \#_{k+1} \overline{\mathbb{C}P}^2.
\end{aligned}$$

This contradicts [**Ak2**], p.358 Property (2), and the claim follows. □

We remark that the previous claim can also be deduced using uniqueness of minimal models of surfaces of general type. We thank Professor Bob Gompf for pointing this out.

The remainder of the proof of Theorem 5 will consist of blowing up $\overline{Q}_1$ and $\overline{Q}_2$ with finitely many $\overline{\mathbb{C}P}^2 s$ and using isotopy to obtain the closure of two pure braids with equal linking matrices. It is not difficult to blow up a knot or link and apply isotopy and handle slides to obtain a pure link; the difficulty lies in doing this to two different links with the ultimate goal of obtaining equal linking matrices.

Below we show how to blow up $\overline{Q}_1$ and $\overline{Q}_2$ each with 9 $\overline{\mathbb{C}P}^2 s$ and apply isotopy to obtain the closure of the pure braids in Figure 3.1 with equal linking matrices given by Figure 3.2.

For an excellent reference on the Kirby calculus see Gompf and Stipsicz [**GS**], particularly Chapters 4 and 5.

We begin by modifying $\overline{Q}_1$. Blowing up $\overline{Q}_1$ (from Figure 3.4) we obtain the first diagram in Figure 3.5. The rest of the figure modifies $\overline{Q}_1\#\overline{\mathbb{C}P}^2$ by isotopy. Blowing up again produces the first diagram in Figure 3.6. The rest of Figure 3.6 modifies $\overline{Q}_1\#2\overline{\mathbb{C}P}^2$ by isotopy.

One blows up once, then once more, to obtain the first and second diagrams in Figure 3.7 respectively.

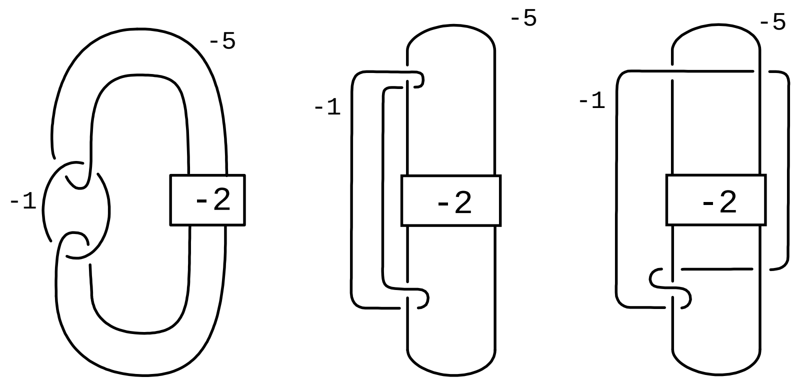


Figure 3.5. Two isotopies of $\overline{Q}_1\#\overline{\mathbb{C}P}^2$.

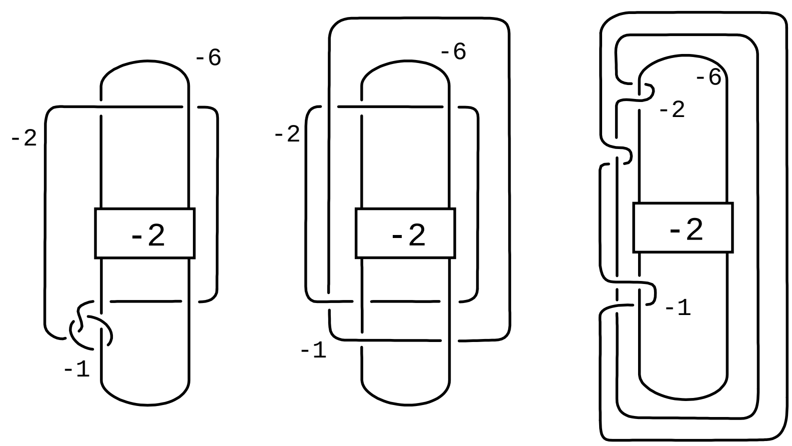


Figure 3.6. Blow up of $\overline{Q}_1\#\overline{\mathbb{C}P}^2$ and two isotopies.

By an isotopy of the second diagram in Figure 3.7, one obtains the first diagram in Figure 3.8; another isotopy yields the second diagram in Figure 3.8. Figure 3.9 is then obtained by blowing up.

Now, we describe a useful operation to blow up and eliminate twists. The first diagram in Figure 3.10 represents a local picture of a single knot, where the top two free strands connect elsewhere, and similarly the bottom two strands connect

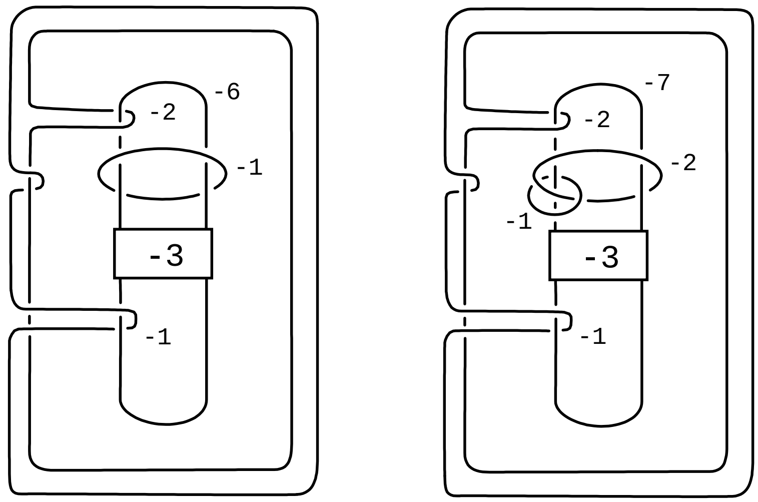


Figure 3.7. Two blow ups of $\overline{Q}_1\#2\overline{\mathbb{C}P}^2$.

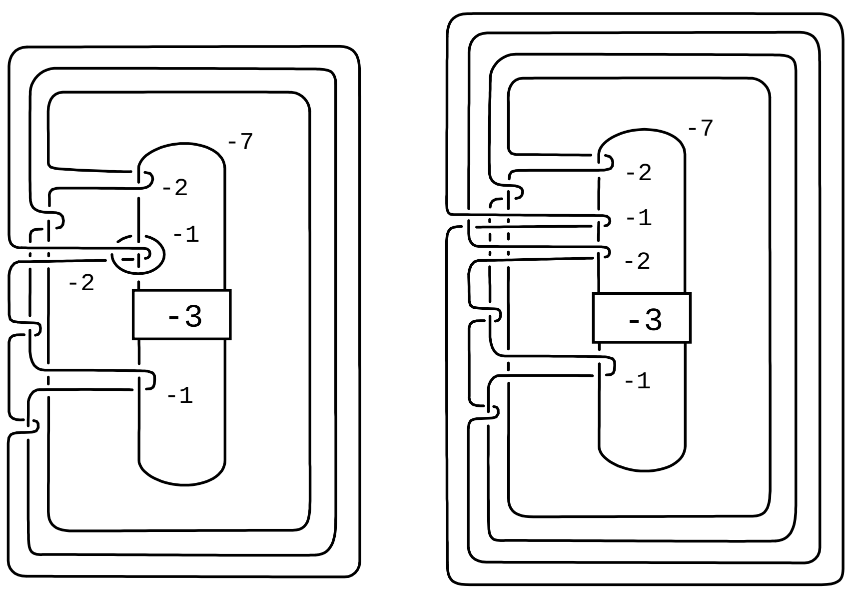


Figure 3.8. Two isotopies of $\overline{Q}_1\#4\overline{\mathbb{C}P}^2$.

elsewhere. This knot's framing coefficient equals d. We only change this diagram locally. The box with -1 in it represents a single twist, as shown by the second diagram in Figure 3.10. We blow that up as shown with $\overline{\mathbb{C}P}^2$ obtaining the third

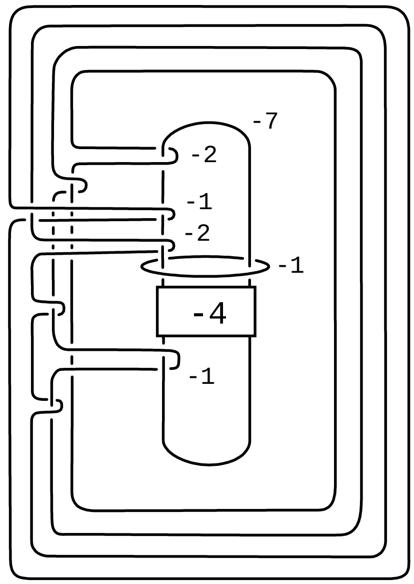


Figure 3.9. Blow up of $\overline{Q}_1\#4\overline{\mathbb{C}P}^2$.

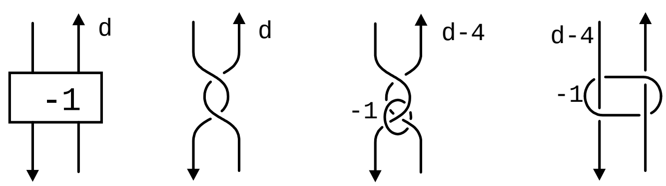


Figure 3.10. Blowing up to remove a twist in a single component.

diagram, and finally a simple local isotopy produces the final diagram. The framing d changes to $d-4$ as shown in [**GS**], p.152.

It is useful to see diagrammatically how to apply this operation and then perform isotopy to obtain pure links. In Figure 3.11 we show how to remove multiple twists by blowing up. Note that the thickened lines represent parts of the link diagram that do not change at all.

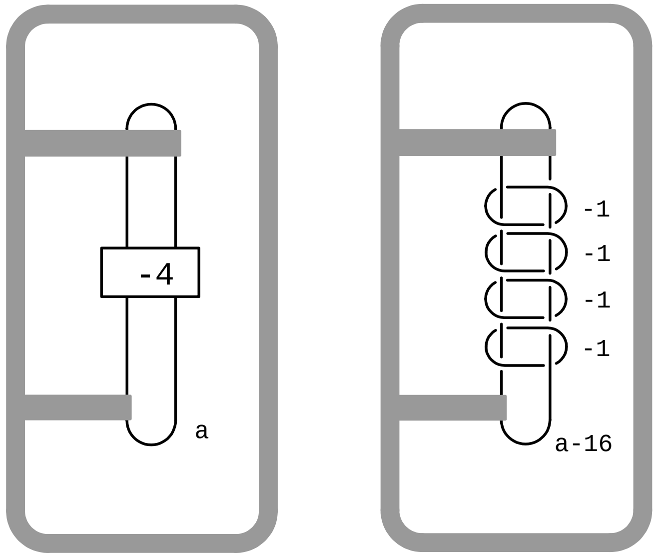


Figure 3.11. Four blow ups to remove four twists.

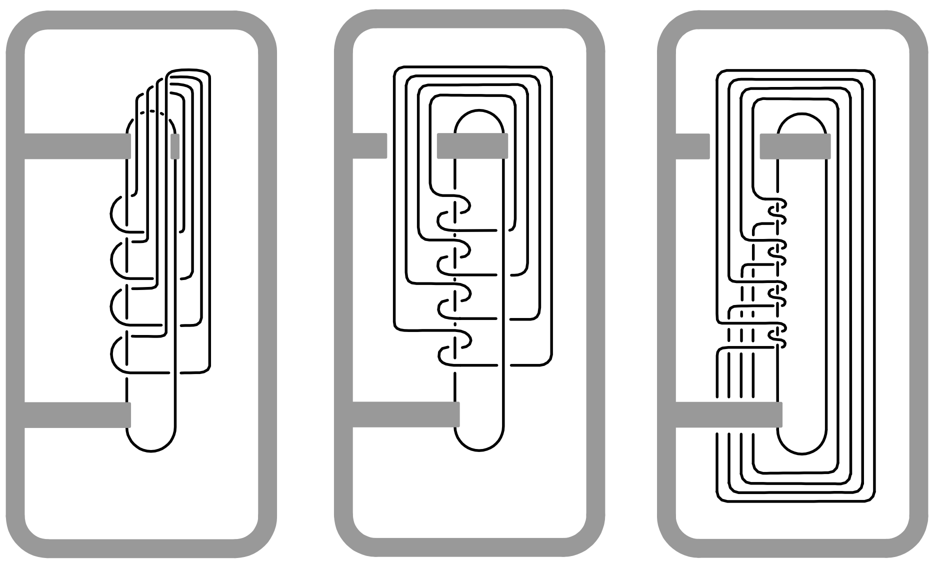


Figure 3.12. Isotopy of untwist operation to braid.

Figure 3.12 shows how to perform isotopy to the second diagram in Figure 3.11 in order to put the new components in pure link form; no framings change here.

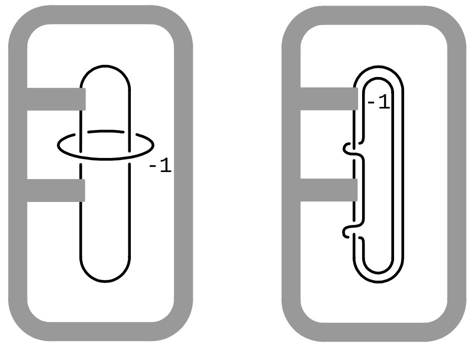


Figure 3.13. Isotopy of loop to braid.

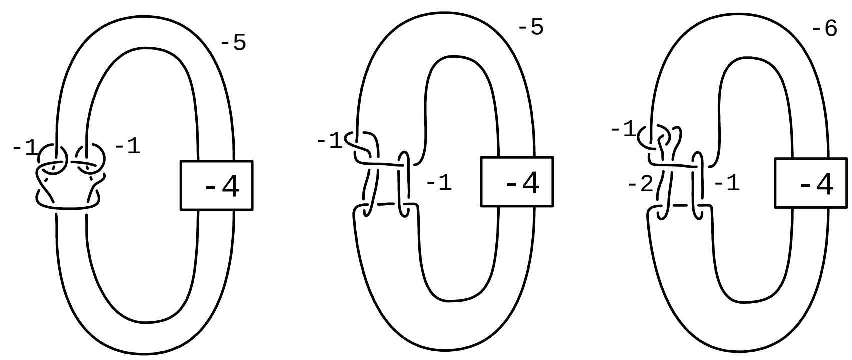


Figure 3.14. Isotopy of $\overline{Q}_2 \# 2\overline{\mathbb{C}P}^2$ followed by blow up.

One obtains the pure braid $s1$ in Figure 3.1 as follows. The -1 framed circle C (just above the '-4' box) in Figure 3.9 should be thought of as laying in the upper thickened line in Figure 3.11 that cuts across the shown knot twice. Perform the operations shown in Figures 3.11 and 3.12, and then perform the operation in Figure 3.13 on the -1 framed circle C just described. Now, take the portion of the link that was in the upper thickened line and slide it up and all the way around to the bottom of the diagram. This produces the pure braid $s1$.

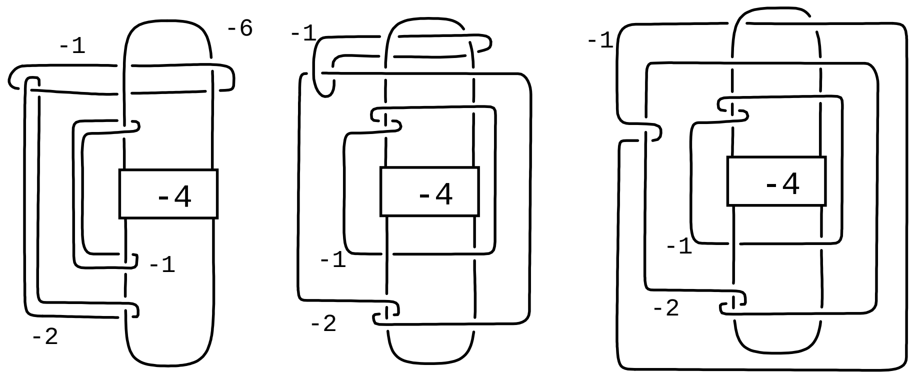


Figure 3.15. Isotopies of $\overline{Q}_2\#3\overline{\mathbb{C}P}^2$.

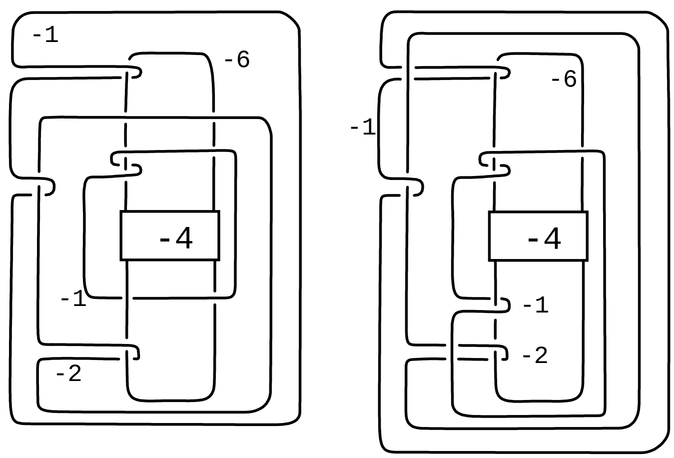
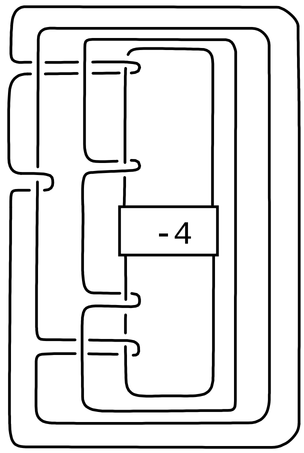


Figure 3.16. Isotopies of $\overline{Q}_2\#3\overline{\mathbb{C}P}^2$.

Now, we proceed to $\overline{Q}_2$. Blow up $\overline{Q}_2$ (from Figure 3.4) twice to obtain the first diagram in Figure 3.14. Perform a simple isotopy and then blow up again to obtain the rest of Figure 3.14. Figures 3.15 and 3.16 contain straightforward isotopies.

The first diagram in Figure 3.17 is obtained by isotopy, and the second by blowing up. Another isotopy yields Figure 3.18. Perform the operations in Figures 3.11 and 3.12 to Figure 3.18. Take the portion of the link that was in the upper



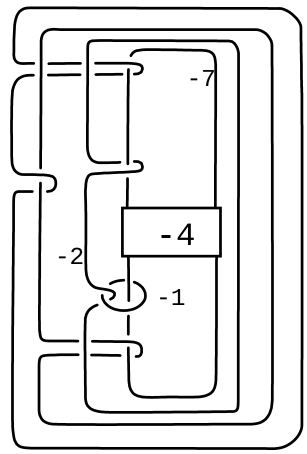


Figure 3.17. Isotopy of $\overline{Q}_2\#3\overline{\mathbb{C}P}^2$ followed by blow up.

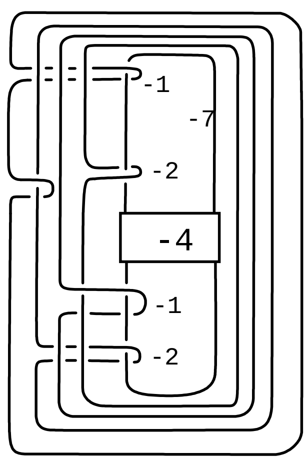


Figure 3.18. Isotopy of $\overline{Q}_2\#4\overline{\mathbb{C}P}^2$.

thickened line in the operation in Figures 3.11 and 3.12 and slide it up and all the way around to the bottom of the diagram. Blow up once more and leave this as the trivial tenth strand (not linking anything). This produces the second pure braid $s2$ in Figure 3.1.

This completes the proof of Theorem 5.

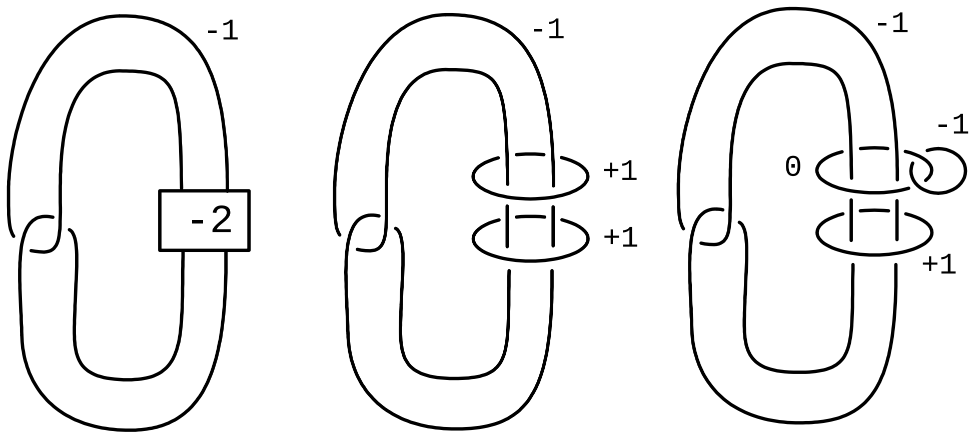


Figure 3.19. Blow ups of $\overline{Q}_1$, first with two $\mathbb{C}P^2s$, then with one $\overline{\mathbb{C}P}^2$.

3.2. Identifying Boundaries

In this section we show that the 3-manifolds $M^3(s1)$ and $M^3(s2)$ are both homeomorphic to the 1/2 Dehn sphere of the figure eight knot of S^3, the simplest hyperbolic integral homology 3-sphere. It suffices to identify $\partial\overline{Q}_1$ from Figure 3.4 as this Dehn sphere, since the 3-manifolds $M^3(s1)$, $M^3(s2)$, $\partial\overline{Q}_1$, and $\partial\overline{Q}_2$ are all homeomorphic by above discussions. Here, one can blow up with $\overline{\mathbb{C}P}^2$ and $\mathbb{C}P^2$ since we are only concerned with the 3-manifolds.

As in Figure 3.19, blow up $\overline{Q}_1$ with two $\mathbb{C}P^2s$ to remove the two twists, then blow up with $\overline{\mathbb{C}P}^2$ as shown. Next, slide the 0 framed component over the +1 framed component in a very simple way (use a trivial band) to obtain the first diagram in Figure 3.20. The second diagram in Figure 3.20 is then obtained by blowing down the right +1 framed component, and the third diagram is obtained

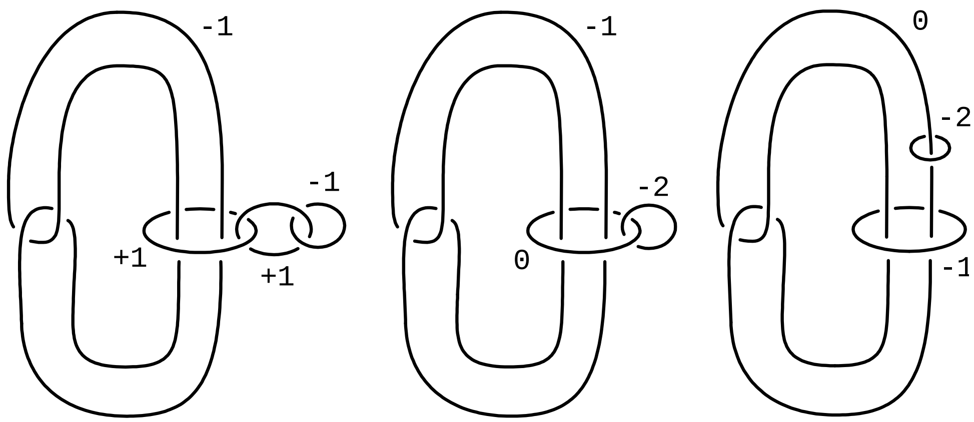


Figure 3.20. A 2-handle slide, a blow down (of right +1), and an isotopy.

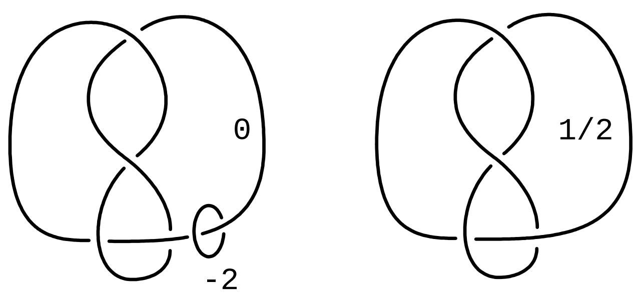


Figure 3.21. Blow down and slam dunk.

by interchanging the 0 and -1 framed components by isotopy (exactly as one does with a Whitehead link). Now, take the third diagram in Figure 3.20, rotate it 90 degrees clockwise, then blow down the -1 framed component to introduce a twist; the result is the first diagram in Figure 3.21. Finally, perform a slam dunk (see [**GS**], p.163-164) to obtain 1/2 surgery on the figure eight knot as in Figure 3.21, as desired.

CHAPTER 4

Knots and Links

In the knot theory of AP theory, one need not use skein methods, projections into the plane, categorification, etc., and knot/link groups and peripheral structures are obtained without putting in framings 'by hand.'

Fix $r \in \mathcal{R}_n$. Then, there are $n+1$ distinguished knots, $k_0, k_1, \ldots, k_n$, in $M^3(r)$ given by the boundary components of the planar page Ω_n in the open book construction. The knot groups G_i of the knots k_i are presented by ([**W**], p.226,227 and [**CW**], Section 2.1):

$$\begin{aligned} G_0 &= \langle x_1, \ldots, x_n \mid r_1 = r_2 = \cdots = r_n \rangle, \\ G_i &= \langle x_1, \ldots, x_n \mid r_1, r_2, \ldots r_{i-1}, r_{i+1}, \ldots, r_n \rangle, \ i \neq 0. \end{aligned}$$

Moreover, if $A(r)$ is unimodular (i.e. $M^3(r)$ is an integral homology 3-sphere) then the peripheral structures m_i, l_i of the knots k_i are given by: $m_0 =$any r_i, $l_0 = x_1 x_2 \cdots x_n m_0^{-s}$ where s is the sum of all elements in $A(r)^{-1}$, and for $i \neq 0$, $m_i = r_i$, $l_i = x_i m_i^{-b_i}$ where $b_i = \left[A(r)^{-1}\right]_{ii}$.

The proof of the following fundamental theorem is due to González-Acuña (unpublished). In particular, by taking L to be the empty link we obtain

González-Acuña's result [**GA**] that every closed, connected, orientable 3-manifold is homeomorphic to $M^3(r)$ for some Artin presentation r.

Theorem 7. *Let L be a link in a closed, connected, orientable 3-manifold M^3. Then, (M^3, L) is homeomorphic to $(M^3(r), K)$ for some Artin presentation r, where K is the sublink $k_1, \ldots, k_m$ of the boundary of Ω_n.*

Proof. Let $l_1, \ldots, l_m$ be the components of L. Let Y be the subset of M^3 obtained from a tubular neighborhood $T(L)$ of L by connecting each component of $T(L)$ to a disjoint 3-disk $D^3 \subset M^3 - T(L)$ with an embedded 1-handle. Y is homeomorphic to the standard, orientable handlebody H_m of genus m. By attaching finitely more 1-handles to D^3 in M^3 (disjoint from $Y - D^3$) one obtains $Z \subset M^3$ such that Z is homeomorphic to H_g, $g \geq m$, and also $W = M^3 - intZ$ is homeomorphic to another copy H_g' of the standard handlebody; this follows from Morse/handle theory [**GS**], Chapter 4.

Following Lickorish [**L1**],[**L2**], the homeotopy group of ∂H_g is generated by Dehn twists about the simple curves $a_1, \ldots, a_g$, $b_1, \ldots, b_g$, and $c_1, \ldots, c_{g-1}$ in ∂H_g where the $a_i s$ are not contractible in H_g. Then, Z is homeomorphic to the standardly embedded H_g in $\mathbb{R}^3$ such that l_i is parallel to a_i by our construction above. Moreover, M^3 is homeomorphic $H_g \cup_f H_g'$ for some homeomorphism f that is isotopic to a product of finitely many Dehn twists $De_1, \ldots, De_k$, where each e_i is one of the curves $a_1, \ldots, a_g$,$b_1, \ldots, b_g$,$c_1, \ldots, c_{g-1}$ (see [**L2**]). We may assume that

$e_i = a_i$ for $i = 1, 2, \ldots, m$, since if we perform the Dehn twists $Da_1, \ldots, Da_m$, $Da_1^{-1}, \ldots, Da_m^{-1}$, and then $De_1, \ldots, De_k$, the resulting homeomorphism of ∂H_g is isotopic to f.

As Lickorish showed [**L1**] each Dehn twist Dx can be accomplished by performing ± 1 surgery on a knot in the interior of H_g that is parallel x. Let s_i be a knot in the interior of H_g that is parallel to a_i for $i = 1, \ldots, m$ such that l_i is a longitude of s_i that does not link s_i. Since s_i has framing ± 1, one can slide l_i over s_i by isotopy so that l_i becomes a meridian of s_i. Each of the remaining Dehn twists contributes a knot to be surgered; these are all disjoint and each is disjoint from a neighborhood of each s_i that contains l_i as a meridian. Let β be the union of all the knots to be surgered (including the s_i). It follows from [**L3**], p.418-419, or [**R**], p.279,340,341, that β is isotopic to the closure of a pure braid. The result follows since each component l_i of our link L is a meridian of a component of β. $\square$

There are other canonical knot groups resulting from an Artin presentation. Fix $r \in \mathcal{R}_n$. Let $\beta = \beta(r)$ denote the framed pure braid associated to r with components β_i framed with $a_i = [A(r)]_{ii}$ for $i = 1, \ldots, n$. Let $M^3(\beta_1, \ldots, \beta_k)$ denote the closed, orientable 3-manifold obtained by just performing surgery on the closure of the first k components of β. Notice that by performing j-reduction (see discussion before Lemma 14 in Chapter 6 below) on r for $j = k+1, k+2, \ldots, n$ one obtains an Artin presentation s such that $M^3(s)$ is homeomorphic to

$M^3(\beta_1, \ldots, \beta_k)$. Notice further that the closure of β_k is a knot in $M^3(\beta_1, \ldots, \beta_{k-1})$ whose knot group is presented by:

$$H_i = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_{i-1}, x_{i+1}, \ldots, x_n \rangle .$$

This follows from the HNN construction (see [**W**], p.247).

The knot groups G_i and H_i will both be used in the following chapter pertaining to the computation of the Casson invariant.

CHAPTER 5

The Casson Invariant in AP Theory

An important theme of AP theory is that invariants of the 3- and 4-manifolds $M^3(r)$ and $W^4(r)$ should be computed *group theoretically* in function of r. The purpose of this chapter is to show how to compute the Casson invariant of any rational homology 3-sphere $M^3(r)$ (i.e. $\det A(r) \neq 0$) in such a way.

First, let us recall the beautiful formula of González-Acuña for the Rohlin invariant of an integral homology 3-sphere $M^3(r)$, where for simplicity we assume $A(r) = I$ (see [**GA**]). Let Δ be the Alexander polynomial of the associated presentation (which clearly abelianizes to $\mathbb{Z}$):

$$\langle x_1, \ldots, x_n \mid x_1 r_1 = r_1 x_2, x_2 r_2 = r_2 x_3, \ldots, x_{n-1} r_{n-1} = r_{n-1} x_n \rangle .$$

Let $d = \Delta(-1)$. Then:

$$\mu\left(M^3(r)\right) = \frac{d^2 - 1}{8} \bmod 2.$$

Our formula for the Casson invariant of $M^3(r)$ with $A(r) = I$ is as follows. For $i = 1, \ldots, n$, let H_i be the presentation:

$$H_i = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_{i-1}, x_{i+1}, \ldots, x_n \rangle ,$$

described in the end of the previous chapter and let Δ_i denote the Conway normalized Alexander polynomial of the group presented by H_i. Recall that Conway normalized means $\Delta_i(1) = 1$ and $\Delta_i(t) = \Delta_i(t^{-1})$. Notice that Δ_i can be computed group theoretically in function of H_i (which is in function of r) using the Fox free calculus and MAGMA. We let $\Delta_i''(1)$ denote the second derivative of Δ_i evaluated at 1. Then, we have:

$$\lambda\left(M^3(r)\right) = \frac{1}{2}\sum_{i=1}^{n} \Delta_i''(1).$$

This formula follows from the discussion at the end of the previous chapter and [**AkMc**].

The above two formulas for μ and λ have particularly nice forms, which is largely due to the fact that $A(r) = I$. In case $A(r)$ is a diagonal matrix with $[A(r)]_{ii} = \epsilon_i = \pm 1$, our formula for λ is exactly as above except:

$$\lambda\left(M^3(r)\right) = \frac{1}{2}\sum_{i=1}^{n} \epsilon_i \Delta_i''(1).$$

The general case where $\det A(r) \neq 0$ is more involved. First, we need a definition. Let A be an $n \times n$ integer matrix. Let $A_{1\cdots k}$ denote the upper left $k \times k$ minor of A. We say A is permissible provided $\det A_{1\cdots k} \neq 0$ for $k = 1, \ldots, n$.

Notice that if $r \in \mathcal{R}_n$ is an Artin presentation and $A(r)$ is permissible, then $M^3(\beta_1, \ldots, \beta_k)$ is a rational homology 3-sphere for $i = 1, \ldots, n$, in particular $M^3(r)$ is a rational homology 3-sphere. So, we can obtain $M^3(r)$ by a sequence of surgeries on $\beta_1, \ldots, \beta_n$ and at each stage we will have a rational homology 3-sphere. This agrees with Walker's notion of permissible in [**Wa**], p.96. In this case, one can compute $\lambda(M^3(r))$ using the Alexander polynomials Δ_i described above and Walker's formula [**Wa**], p.95,96. Notice that the homological data in Walker's formula is strictly in function of $A(r)$ and a computer can be programmed to compute these numbers.

Finally, suppose $r \in \mathcal{R}_n$ is an Artin presentation with $\det A(r) \neq 0$, but with $A(r)$ not permissible. Walker [**Wa**], p.105,106, describes a method of circumventing difficulties in this case, however, his method is unduly complicated. Our goal is to modify r in a simple way so we can use our formula for the permissible case. We thank Henry King for discussions pertaining to the following technical lemma. Let ϵ denote a diagonal $n \times n$ matrix where each diagonal entry $\epsilon_i = \pm 1$ or 0. Let $A + \epsilon_{1 \cdots k}$ denote the result of adding A to the diagonal matrix with entries $\epsilon_1, \ldots, \epsilon_k, 0, \ldots, 0$ where we have $n - k$ zeroes.

Claim 8. *Suppose A is an $n \times n$ integer matrix with* $\det A \neq 0$ *that is not permissible. Then, for some choice of ϵ the matrix $A + \epsilon$ is permissible and* $\det(A + \epsilon_{1 \cdots k}) \neq 0$ *for $k = 1, \ldots, n$.*

Proof. Here is a constructive way to choose ϵ. Choose ϵ_1 such that $\det(A_{1\ldots1}+\epsilon_{1\ldots1})\neq 0$ and $\det(A+\epsilon_{1\ldots1})\neq 0$. Having chosen $\epsilon_1,\ldots,\epsilon_{k-1}$, choose ϵ_k so that $\det(A_{1\ldots k}+\epsilon_{1\ldots k})\neq 0$ and $\det(A+\epsilon_{1\ldots k})\neq 0$.

One can inductively choose the ϵ_i to satisfy these requirements since at each stage one will encounter two linear equations to satisfy and one has three choices for ϵ_k (0 and ±1). Inspection of the linear equations shows an appropriate choice can be made.

Having chosen ϵ in this way, the result follows immediately. □

Returning to where we have $A(r)$ of nonzero determinant and not permissible, let ϵ be given by the claim. We know $M^3(r)$ has a surgery diagram given by closure of the pure braid β. For each $i=1,\ldots,n$, if $\epsilon_i\neq 0$ then introduce a meridian to β_i with framing ∞ in the surgery diagram of $M^3(r)$. This does not change the 3-manifold; notice that the meridian to β_i is in fact $k_i=k_i(r)$ (a main point). Now, perform a Rolfsen twist ([**R**], p.264-267 or see [**GS**], p.162,163) in the correct direction (+ or - depending on ϵ_i) that simply changes framings in our diagram: the framing $a_i=[A(r)]_{ii}$ of β_i becomes $a_i+\epsilon_i$ and the framing ∞ of k_i becomes ϵ_i. One obtains $M^3(r)$ by surgering (in this order) $\beta_1,\ldots,\beta_n$ with new framings and then $k_n,\ldots,k_1$ with framings ϵ_i; here one skips any k_i where $\epsilon_i=0$. Then, by our choice of ϵ from the claim, we have a rational homology 3-sphere at each stage. Moreover, all of the knot groups in this series of successive surgeries is either a G_i or an H_i and so is determined by r.

Details of this process, along with a computer program to carry it out in practice, will appear elsewhere.

CHAPTER 6

Combinatorial Group Theory

Relationships between Artin presentations and topology provide topological proofs of many interesting properties of Artin presentations. The purpose of this chapter is to give purely combinatorial group theoretic proofs of some of these properties. The methods of proof are elementary and it is hoped that the techniques will lead to deeper studies of Artin presentations using combinatorial group theory and possibly computer aided proofs.

Recall that an Artin presentation r is a finite presentation:

$$\langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$$

satisfying the following equation in F_n (the free group on $x_1, \ldots, x_n$):

$$x_1 x_2 \cdots x_n = \left(r_1^{-1} x_1 r_1\right)\left(r_2^{-1} x_2 r_2\right) \cdots \left(r_n^{-1} x_n r_n\right).$$

Also, $\mathcal{R}_n$ denotes the set of Artin presentations on n generators $x_1, x_2, \ldots, x_n$. By convention, one may assume the empty presentation $\langle | \rangle$ is the unique Artin presentation in $\mathcal{R}_0$. For the remainder of this chapter we assume $n > 0$. We always assume that the words r_i are freely reduced in an Artin presentation (free reduction in F_n is discussed below in Section 6.1). Thus, it is clear that:

$$\mathcal{R}_1 = \left\{ \langle x_1 \mid r_1 \rangle \mid r_1 = x_1^k \text{ for some } k \in \mathbb{Z} \right\}.$$

Associated to an Artin presentation $r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ is the exponent sum matrix $A(r)$, namely the $n \times n$ integer matrix given by:

$$[A(r)]_{ij} = \text{exponent of } x_j \text{ in abelianized } r_i.$$

Of course, one can define such a matrix for non-Artin presentations.

A main result is:

Theorem 9. *If r is any Artin presentation then $A(r)$ is symmetric.*

This will follow from a technical result (j-reduction) and a characterization of Artin presentations on two generators.

Theorem 10. *Artin presentations $r = \langle x_1, x_2 \mid r_1, r_2 \rangle$ in $\mathcal{R}_2$ are characterized by the following:*

$$\begin{aligned} r_1 &= x_1^a (x_1 x_2)^c, \ \textit{and} \\ r_2 &= x_2^b (x_1 x_2)^c, \ \textit{for some } a, b, c \in \mathbb{Z}. \end{aligned}$$

Notice that if $r \in \mathcal{R}_2$ is as above with $a, b, c \in \mathbb{Z}$ then:

$$A(r) = \begin{bmatrix} a+c & c \\ c & b+c \end{bmatrix},$$

is symmetric.

Characterizing Artin presentations for larger n is an open problem and is discussed below in Section 6.4.

6.1. Basic Properties of Free Groups

In this section we recall basic notions about free groups and fix some notation. Two excellent references are Magnus, Karrass and Solitar [**MKS**] and Stillwell [**St**].

The free group $F_n = \langle x_1, \ldots, x_n \rangle$ is defined combinatorially in [**St**], Sec. 0.5.2-0.5.6, and [**MKS**], Sec. 1.2 and 1.4. We will abuse notation and write w to mean both a word in the generators $x_1, \ldots, x_n$ and the equivalence class it represents in $F_n = \langle x_1, \ldots, x_n \rangle$, which is common practice [**MKS**], p.19, and [**St**], p.42; the context should make clear which is actually meant. A simple free reduction on a word w in F_n is the removal of a term $x_i x_i^{-1}$ or $x_i^{-1} x_i$ for some $1 \leq i \leq n$. A word is freely reduced if no such cancellation is possible. As shown in [**St**], p.94, performing simple free reductions on w as far as possible and in any order always produces the same freely reduced word denoted $\rho(w)$. This process solves the word problem in F_n. As Stillwell states, "This confirms the commonsense impression that one decides whether a given element equals 1 in F_n simply by cancelling as much as possible." Stillwell proves this result using the Cayley diagram of a free group. A purely group theoretic proof of this result is in [**MKS**], pp.34-35. In fact, Magnus, Karrass and Solitar give a concrete process, also denoted ρ, for producing the unique free reduction $\rho(w)$ of a word w in F_n. From here on ρ will denote this concrete process.

Given any two words u, v in F_n we write $u = v$ in case they are identically equal when written out as products of $x_i^{\pm 1}$, $1 \leq i \leq n$, without performing any free reductions. Thus, we regard $u = x_1^2$ and $v = x_1 x_1$ as being equal, and $u = x_1^{-1} x_1$ and $v = 1$ as not being equal. Above we defined a simple free reduction; a free insertion on a word w in F_n is the inverse process, namely the insertion of a term $x_i x_i^{-1}$ or $x_i^{-1} x_i$ for some $1 \leq i \leq n$. Two words u, v in F_n are freely equal, written $u \approx v$, provided one can be obtained from the other by free reductions and insertions. Thus, the following are equivalent: u and v determine the same element in F_n, $u \approx v$, and $\rho(u) = \rho(v)$.

We note that in previous chapters we used $=$ instead of $\approx$. Only in this chapter are we so pedantic.

The definition of an Artin presentation can be rephrased using the notation above. Let r_i, $1 \leq i \leq n$, be freely reduced words in F_n. Then the presentation:

$$r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$$

is an Artin presentation if and only if:

$$x_1 x_2 \cdots x_n \approx \left(r_1^{-1} x_1 r_1\right)\left(r_2^{-1} x_2 r_2\right) \cdots \left(r_n^{-1} x_n r_n\right),$$

which is equivalent to:

$$x_1 x_2 \cdots x_n = \rho\left(\left(r_1^{-1} x_1 r_1\right)\left(r_2^{-1} x_2 r_2\right) \cdots \left(r_n^{-1} x_n r_n\right)\right).$$

We refer to either of these equivalent conditions as AC, the ‘Artin condition’.

It is important to note that given n words r_i, $1 \leq i \leq n$, one can easily check if the Artin condition AC is satisfied using the solution of the word problem in F_n stated above. For large words r_i one can use a computer algebra system such as MAGMA to quickly check whether AC is satisfied.

It will be useful to perform substitutions on words in F_n. Let w be a word in F_n. We write $w = w\left(x_\mu\right)$ to emphasize that w is a word in the letters x_μ, $1 \leq \mu \leq n$. Let y_μ, $1 \leq \mu \leq n$, be any expressions. Then, we let $w\left(y_\mu\right)$ denote the result of substituting y_μ for x_μ in $w\left(x_\mu\right)$. It is implicit that y_μ^{-1} is substituted for x_μ^{-1}. Notice that no free reduction takes place in this definition. For example, let $w\left(x_\mu\right)$ be the word $x_1 x_2 x_1^{-1}$ in F_2. Let $y_1 = x_1$ and $y_2 = 1$. Then, $w\left(y_\mu\right) = x_1 1 x_1^{-1} = x_1 x_1^{-1}$. Of course, removing appearances of 1 in a nontrivial expression (except if the expression is identically equal to 1) is allowed.

We present some lemmas that will be needed later.

Lemma 11. *Let u_i, $1 \leq i \leq k$, be any words in F_n. Then:*

$$\rho\left(\prod_{i=1}^{k} u_i\right) = \rho\left(\prod_{i=1}^{k} \rho\left(u_i\right)\right).$$

Proof. For $1 \leq i \leq k$ we have $u_i \approx \rho\left(u_i\right)$ and so $\prod_{i=1}^{k} u_i \approx \prod_{i=1}^{k} \rho\left(u_i\right)$. The result follows by applying ρ to this last equation. $\square$

Lemma 12. *Let u be a freely reduced word in F_n and suppose $u^{-1}x_iu \approx x_i$ for some $1 \leq i \leq n$, then $u = x_i^k$ for some integer k.*

Proof. Commuting elements in F_n are powers of a common word [**MKS**], p.42. So, $x_i \approx w^m$ and $u \approx w^k$ for some freely reduced word w and integers m and k. We claim that $x_i \approx w^m$ implies $w = x_i^{\pm 1}$ and $m = \pm 1$. Notice that the lemma follows immediately from this claim. Without loss, we may assume $m > 1$ and $w \neq 1$ in proving the claim. As w is freely reduced and $x_i \approx w^m$, we must have $w = x_j^{\pm 1} w' x_j^{\mp 1}$. Let a denote the longest initial segement of w that equals the initial segment of w^{-1}. It is easy to see that we must have $w = aca^{-1}$ for freely reduced $a, c \neq 1$. Furthermore, our choice of a implies that the initial letter of c does not equal the initial letter of c^{-1}; it follows that c^m is freely reduced. Thus, $w^m \approx ac^m a^{-1}$ and the latter is freely reduced, and hence equal to x_i. This is a contradiction, proving the claim. $\square$

Along the same lines, we point out that:

Corollary 13. *If $r \in \mathcal{R}_n$ is an Artin presentation such that $r_i = w$ for all $i = 1, \ldots, n$, then $w = (x_1 x_2 \cdots x_n)^k$ for some integer k.*

Proof. By the Artin condition:

$$\begin{aligned} x_1 x_2 \cdots x_n &\approx \left(r_1^{-1} x_1 r_1\right)\left(r_2^{-1} x_2 r_2\right) \cdots \left(r_n^{-1} x_n r_n\right) \\ &\approx w^{-1} x_1 x_2 \cdots x_n w. \end{aligned}$$

By [**MKS**], p.42, we have $x_1 x_2 \cdots x_n \approx u^k$ and $w \approx u^j$ for some freely reduced word u and integers k and j. Clearly $k \neq 0$, and if $k = \pm 1$ the result follows. Without loss, assume $k > 1$. Again, write $u = aca^{-1}$ for the longest possible initial segment a of u. Since $k > 1$, we must have $a, c \neq 1$. As in the previous lemma, c^k and $u = aca^{-1}$ are freely reduced. It follows that $x_1 x_2 \cdots x_n = ac^k a^{-1}$, a contradiction proving the corollary. □

Another class of basic Artin presentations is: let $a = (a_1, a_2, \ldots, a_n)$ be an element of $\mathbb{Z}^n$ and define $r_i = x_i^{a_i}$. Then $r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ is an Artin presentation with $A(r)$ the diagonal matrix with diagonal equal to a.

We close this section with a technical result. Recall the notion of the j-reduction of an Artin presentation [**W**], p.227,251. Let $r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ be an Artin presentation and let $1 \leq j \leq n$. The idea is that by deleting r_j, setting $x_j = 1$ in the other r_i, freely reducing these resulting words and renumbering, one obtains an Artin presentation in $\mathcal{R}_{n-1}$. It was noted in [**W**], p.251, that the result is in fact an Artin presentation by topological considerations. We present a purely group theoretic proof of this fact. We choose not to renumber simply for notational reasons; our result immediately implies that the j-reduction given in [**W**] is an Artin presentation. With $r \in \mathcal{R}_n$ and $1 \leq j \leq n$ fixed, we define:

$$\begin{aligned} y_\mu &= x_\mu, \text{ for } 1 \leq i \leq n \text{ and } \mu \neq j, \\ y_j &= 1, \\ u_i &= r_i\left(y_\mu\right), \text{ for } 1 \leq i \leq n \text{ and } i \neq j, \\ u_j &= 1, \\ s_i &= \rho\left(u_i\right), \text{ for } 1 \leq i \leq n. \end{aligned}$$

Lemma 14. *With $r \in \mathcal{R}_n$ and y_μ, u_i, and s_i as directly above, we have that:*

$$x_1 \cdots x_{j-1} x_{j+1} \cdots x_n \approx \left(s_1^{-1} x_1 s_1\right) \cdots \left(s_{j-1}^{-1} x_{j-1} s_{j-1}\right)\left(s_{j+1}^{-1} x_{j+1} s_{j+1}\right) \cdots \left(s_n^{-1} x_n s_n\right).$$

Notice that the free reductions required in the above equation occur in $F_{n-1} = \langle x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n \rangle$ since no x_j appear anywhere.

Proof. First, we claim that if $w = w(x_\mu)$ is any word in F_n then:

$$\rho(w(y_\mu)) = \rho([\rho(w(x_\mu))](y_\mu)).$$

Intuitively, this means that setting $x_j = 1$ in w and then freely reducing produces exactly the same freely reduced word as freely reducing w, setting all $x_j = 1$ and then freely reducing again. To see this, let $F_{n-1} = \langle x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n \rangle$ and define the homomorphism $\psi : F_n \to F_{n-1}$ by $x_i \mapsto x_i$, $i \neq j$, and $x_j \mapsto 1$. Since $w(x_\mu) \approx \rho(w(x_\mu))$, the well definition of ψ implies that $\psi(w(x_\mu)) \approx \psi(\rho(w(x_\mu)))$. So, we have that:

$$\begin{aligned} w(y_\mu) &= \psi(w(x_\mu)) \\ &\approx \psi(\rho(w(x_\mu))) \\ &= [\rho(w(x_\mu))](y_\mu). \end{aligned}$$

Hence, $\rho(w(y_\mu)) = \rho([\rho(w(x_\mu))](y_\mu))$, as desired.

Now, let $w(x_\mu) = \left(r_1^{-1}x_1r_1\right)\left(r_2^{-1}x_2r_2\right)\cdots\left(r_n^{-1}x_nr_n\right)$. Since r is an Artin presentation, we have $\rho(w(x_\mu)) = x_1x_2\cdots x_n$. Thus, we compute:

$$\begin{aligned} x_1\cdots x_{j-1}x_{j+1}\cdots x_n &= \rho\left(\left[x_1x_2\cdots x_n\right](y_\mu)\right) \\ &= \rho\left(\left[\rho\left(w\left(x_\mu\right)\right)\right](y_\mu)\right) \\ &= \rho\left(w\left(y_\mu\right)\right), \end{aligned}$$

where the last equality follows from the claim. We also have:

$$\begin{aligned} w(y_\mu) &= \prod_{i=1}^{n}\left(\left[r_i^{-1}x_ir_i\right](y_\mu)\right) \\ &= \left[\prod_{i=1}^{j-1}u_i^{-1}x_iu_i\right]\left[u_j^{-1}1u_j\right]\left[\prod_{i=j}^{n}u_i^{-1}x_iu_i\right] \\ &\approx \prod_{i=1,\ i\neq j}^{n}u_i^{-1}x_iu_i. \end{aligned}$$

Applying ρ and using Lemma 11 gives:

$$\begin{aligned} \rho\left(w\left(y_\mu\right)\right) &= \rho\left(\prod_{i=1,\ i\neq j}^{n}\rho\left(u_i^{-1}\right)x_i\rho\left(u_i\right)\right) \\ &= \rho\left(\prod_{i=1,\ i\neq j}^{n}s_i^{-1}x_is_i\right), \end{aligned}$$

and the result follows. □

6.2. Proof of Theorem 10

First, suppose that $r = \langle x_1, x_2 \mid r_1, r_2 \rangle$ and:

$$\begin{aligned} r_1 &= x_1^a (x_1 x_2)^c, \text{ and} \\ r_2 &= x_2^b (x_1 x_2)^c, \text{ for some } a, b, c \in \mathbb{Z}. \end{aligned}$$

Then, an easy computation shows that r is in fact Artin.

The following two lemmas will be useful in proving the converse in our characterization of $\mathcal{R}_2$. If w is a freely reduced word in F_n then we let $\#w$ denote the length of w. That is, write $w = x_{i1}^{\epsilon_{i1}} \cdots x_{ik}^{\epsilon_{ik}}$ where $\epsilon_{ij} = \pm 1$ for each $j = 1, \ldots, k$, then $\#w = k \geq 0$. If r is an Artin presentation then we define $\#r$ to be $\#r_1 + \cdots + \#r_n$.

Lemma 15. *If u, v are freely reduced words in $F_2 = \langle x_1, x_2 \rangle$ and $uv \approx x_1 x_2$ then $\#u$ is equal to either $\#v - 2$, $\#v$, or $\#v + 2$.*

Proof. We proceed by induction on $k = \#u + \#v$. Since $\#(uv) \leq k$, $uv \approx x_1 x_2$ clearly implies $k \geq 2$. If $k = 2$ then either $u = 1$ and $v = x_1 x_2$, $u = x_1 x_2$ and $v = 1$, or $u = x_1$ and $v = x_2$, which satisfy the lemma.

So, we may assume $k \geq 3$ and $\#u, \#v \geq 1$. Since $uv \approx x_1x_2$ and u, v are freely reduced, we must have:

$$\begin{aligned} u &= u'x_i^{\epsilon}, \text{ and} \\ v &= x_i^{-\epsilon}v', \end{aligned}$$

where $i = 1$ or 2, $\epsilon = \pm 1$, and u', v' are freely reduced (either possibly equal to 1). Thus:

$$\begin{aligned} x_1x_2 &\approx uv \\ &= u'x_i^{\epsilon}x_i^{-\epsilon}v' \\ &\approx u'v', \end{aligned}$$

where $k' = \#u' + \#v' = \#u - 1 + \#v - 1 = k - 2$. Furthermore, $x_1x_2 \approx u'v'$ implies $k - 2 \geq 2$, and so the result follows easily by induction. $\square$

Lemma 16. *If u, v are freely reduced words in $F_2 = \langle x_1, x_2 \rangle$ and $uv \approx x_1x_2$, then either:*

i) u begins with x_1x_2,

ii) v ends with x_1x_2, or

iii) u begins with x_1 and v ends with x_2.

Proof. The previous lemma implies that $\#u$ equals either $\#v - 2$, $\#v$, or $\#v + 2$. The same method of proof implies that $\#u = \#v - 2$ if and only if *ii* holds, $\#u = \#v$ if and only if *iii* holds, and $\#u = \#v + 2$ if and only if *i* holds. $\square$

We now return to the proof proper of Theorem 10. Fix $r = \langle x_1, x_2 \mid r_1, r_2 \rangle$ in $\mathcal{R}_2$. In particular, r_1, r_2 are freely reduced. The proof is by induction on $\#r = \#r_1 + \#r_2$.

If $\#r = 0$, then we are done. If $\#r = 1$, we have four cases:

$$
\begin{aligned}
i)\ r_1 &= x_1^{\pm 1} \text{ and } r_2 = 1, \\
ii)\ r_1 &= 1 \text{ and } r_2 = x_2^{\pm 1}, \\
iii)\ r_1 &= x_2^{\pm 1} \text{ and } r_2 = 1, \\
iv)\ r_1 &= 1 \text{ and } r_2 = x_1^{\pm 1}.
\end{aligned}
$$

Cases i and ii yield Artin presentations and are of the desired form, while cases iii and iv do not give Artin presentations. Hence, we may assume $\#r \geq 2$.

Suppose $r_1 = 1$. Then $x_1 x_2 \approx x_1 r_2^{-1} x_2 r_2$, which implies that $r_2^{-1} x_2 r_2 \approx x_2$, and so $r_2 = x_2^m$ for some $m \in \mathbb{Z}$ by Lemma 12. These presentations are Artin and of the desired form; a similar result follows if $r_2 = 1$. Hence, we may assume $r_i \neq 1$, $i = 1, 2$.

Suppose r_1 begins with $x_1^{\pm 1}$. Let $s_1 = \rho\left(x_1^{\mp 1} r_1\right)$ and $s_2 = r_2$. Then $s = \langle x_1, x_2 \mid s_1, s_2 \rangle$ is an Artin presentation and $\#s = \#r - 1$. By induction, s has the desired form, and so r has the desired form as well. The same result holds if r_2 begins with $x_2^{\pm 1}$. Hence, we may assume that r_i does not begin with a nonzero power of x_i.

Thus, we have $r_i \neq 1$, $i = 1, 2$, and:

$$\begin{aligned} r_1 &= x_2^{\alpha} w_1, \\ r_2 &= x_1^{\beta} w_2, \end{aligned}$$

where α, β are nonzero integers, w_1, w_2 are freely reduced words (either may equal 1), w_1 does not begin with a nonzero power of x_2, and w_2 does not begin with a nonzero power of x_1. Let $A = r_1^{-1} x_1 r_1$ and $B = r_2^{-1} x_2 r_2$, which are both freely reduced. Thus, $AB \approx x_1 x_2$ and so $\#A$ equals either $\#B$ or $\#B \pm 2$ by Lemma 15. This implies $\#r_1$ equals either $\#r_2$ or $\#r_2 \pm 1$. Suppose $\#r_1 = \#r_2$, then since

A, B are freely reduced and $AB \approx x_1x_2$, we must have that $r_1r_2^{-1} \approx 1$. But this implies that $r_1 \approx r_2$ which is a contradiction since r_1 and r_2 are freely reduced and begin with different letters. So, $\#r_1 = \#r_2 \pm 1$.

We claim that $|\alpha| = |\beta| = 1$. To see this, suppose $|\alpha| \geq 2$. We have:

$$\begin{aligned} \#r_1 &= |\alpha| + \#w_1, \text{ and} \\ \#r_2 &= |\beta| + \#w_2. \end{aligned}$$

There are two cases:

$$\begin{aligned} i)\ \#r_1 &= \#r_2 + 1, \text{ and} \\ ii)\ \#r_1 &= \#r_2 - 1. \end{aligned}$$

In case ii we have $|\alpha| + \#w_1 = |\beta| + \#w_2 - 1$. But, again inspection of $AB \approx x_1x_2$ shows that cancelling the last $|\alpha| + \#w_1$ letters in A with the first $|\alpha| + \#w_1$ letters in B (which must cancel for $AB \approx x_1x_2$ to hold) gives:

$$x_1x_2 \approx \left(r_1^{-1}x_1\right)\left(x_1^{\epsilon}x_2r_2\right),$$

and so we must have $\epsilon = -1$ and:

$$x_1x_2 \approx \left(w_1^{-1}x_2^{-\alpha}\right)\left(x_2x_1^{\beta}w_2\right).$$

This implies $\alpha > 0$ and:

$$x_1x_2 \approx \left(w_1^{-1}x_2^{-\alpha+1}\right)\left(x_1^{\beta}w_2\right),$$

where $-\alpha+1, \beta$ are nonzero and both words in parentheses are freely reduced. This is a contradiction. Similar contradictions arise in case i and when $|\beta| \neq 1$. The claim follows.

Taking stock, we have reduced to the situation where:

$$\begin{aligned} r_1 &= x_2^{\alpha}w_1, \\ r_2 &= x_1^{\beta}w_2, \end{aligned}$$

and $|\alpha| = |\beta| = 1$, w_1, w_2 are freely reduced words (either may equal 1), w_1 does not begin with a nonzero power of x_2, w_2 does not begin with a nonzero power of x_1, and $\#r_1 = \#r_2 \pm 1$.

Case 1. $\#r_1 = \#r_2 - 1$. This implies $\#A = \#B - 2$ and so B ends with x_1x_2 by Lemma 16. Since $B = r_2^{-1}x_2r_2 = w_2^{-1}x_1^{-\beta}x_2x_1^{\beta}w_2$, there are two subcases: either $w_2 = x_2$ or $w_2 \neq x_2$.

Case 1.1. $w_2 = x_2$. This implies $\beta = 1$, $r_2 = x_1x_2$, $\#r_1 = 1$, and $r_1 = x_2^{\alpha}$. The Artin condition implies $\alpha = 1$. This presentation is Artin and corresponds to the desired form with $a = -1$, $b = 0$ and $c = 1$.

Case 1.2. $w_2 \neq x_2$. This implies $w_2 = wx_1x_2$ where w is freely reduced and does not end in x_1^{-1}. The word w contains some nonzero power of x_2 since we assumed w_2 did not begin with a nonzero power of x_1. Recalling that $\#r_1 = \#r_2 - 1$, which see that $\#w_1 = \#w_2 - 1 = \#w + 2 - 1 \geq 2$. Therefore, $AB \approx x_1x_2$ implies w_1 ends in x_1x_2. Define $s = \langle x_1, x_2 \mid s_1, s_2 \rangle$ by:

$$\begin{aligned} s_1 &= \rho\left(r_1x_2^{-1}x_1^{-1}\right), \\ s_2 &= \rho\left(r_2x_2^{-1}x_1^{-1}\right). \end{aligned}$$

Then, we compute:

$$\begin{aligned} \left(s_1^{-1}x_1s_1\right)\left(s_2^{-1}x_2s_2\right) &\approx x_1x_2x_2^{-1}x_1^{-1}\left(s_1^{-1}x_1s_1\right)x_1x_2x_2^{-1}x_1^{-1}\left(s_2^{-1}x_2s_2\right)x_1x_2x_2^{-1}x_1^{-1} \\ &= x_1x_2\left(r_1^{-1}x_1r_1\right)\left(r_2^{-1}x_2r_2\right)x_2^{-1}x_1^{-1} \\ &\approx x_1x_2\left(x_1x_2\right)x_2^{-1}x_1^{-1} \\ &\approx x_1x_2, \end{aligned}$$

and so s is an Artin presentation. Also, $\#s \leq \#r - 4$ and so s is of the desired form by induction. This implies r is of the desired form.

Case 2. $\#r_1 = \#r_2 + 1$. This implies $\#A = \#B + 2$ and so A starts with x_1x_2 by Lemma 16. The proof follows in the same way as case 1.

This completes the proof of Theorem 10.

6.3. Proof of Theorem 9

Let $r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ be an Artin presentation in $\mathcal{R}_n$. We will show $A(r)$ is symmetric by induction on n. If $n = 1$ there is nothing to show. If $n = 2$ the result follows by the characterization of $\mathcal{R}_2$ in Theorem 10. So, assume $n \geq 3$ and the result holds for all Artin presentations on $n - 1$ generators.

Fix $j = n$ and define y_μ, u_i, and s_i exactly as they were defined preceding Lemma 14 on j-reduction. Lemma 14 implies that $s = \langle x_1, \ldots, x_{n-1} \mid s_1, \ldots, s_{n-1} \rangle$ is an Artin presentation in $\mathcal{R}_{n-1}$. Hence, $A(s)$ is symmetric by induction. We claim that for all $1 \leq \alpha, \beta \leq n - 1$:

$$[A(r)]_{\alpha,\beta} = [A(s)]_{\alpha,\beta}\,.$$

To see this note that:

$$\begin{aligned} [A(r)]_{\alpha,\beta} &= \text{exponent of } x_\beta \text{ in abelianized } r_\alpha, \text{ and} \\ [A(s)]_{\alpha,\beta} &= \text{exponent of } x_\beta \text{ in abelianized } s_\alpha. \end{aligned}$$

Now, $[A(r)]_{\alpha,\beta}$ also equals the exponent of x_β in abelianized u_α since u_α is obtained from r_α by setting $x_n = 1$. Moreover, $s_\alpha = \rho(u_\alpha)$ and each simple free reduction in passing from u_α to s_α preserves the exponent sum of every x_μ. Hence, the exponent of x_β in abelianized u_α equals the exponent of x_β in abelianized s_α, and the claim follows.

The claim implies that the upper left $(n-1) \times (n-1)$ block of $A(r)$ is symmetric. Repeating this process for $j = n-1$ and $j = n-2$ (this is where we need $n = 2$ as a basecase) shows that $A(r)$ is symmetric, as desired.

6.4. Characterizing the r_i

One hopes to characterize the words r_i in Artin presentations in a useful, combinatorial group theoretic manner. Two questions arise. Which words can be an r_i in some Artin presentation? Given an Artin presentation $r \in \mathcal{R}_n$, how can one extend r to an Artin presentation $r' \in \mathcal{R}_{n+1}$ such that the j-reduction of r' for $j = n+1$ is exactly r? These are open problems. We close with some observations related to these problems.

The Jordan curve theorem restricts the words r_i in an Artin presentation r in the following way. Every $r_i = x_i^k r_i'$ for some integer k and freely reduced word r_i' such that r_i' does not begin with $x_i^{\pm 1}$, adjacent letters in r_i' are distinct and they appear to the power of ± 1. This restriction follows geometrically by considering Ω_n (see Corollary 2). Can one prove this result algebraically?

There are further restrictions, though. The word $r_1 = x_2 x_1^{-1}$ satisfies the above restriction, but r_1 cannot be the first defining word in an Artin presentation in $\mathcal{R}_2$ by our characterization of $\mathcal{R}_2$ Theorem 10.

Characterizing the words in Artin presentations should be useful for ordering Artin presentations and attacking the problem of whether the Gassner representation of P_n is faithful using AP theory, see [**B**], p.133 and [**W**], p.266.

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