NIM with Cash*

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Abstract

$NIM(a_1, \ldots, a_k; n)$ is a 2-player game where initially there are n stones on the board and the players alternate removing either a_1 or $\cdots$ or a_k stones. The first player who cannot move loses. This game has been well studied. For example, it is known that for $NIM(1, 2, 3; n)$ Player II wins if and only if n is divisible by 4. These games are interesting because, despite their simplicity, they lead to interesting win conditions.

We investigate an extension of the game where Player I starts out with d_1 dollars, Player II starts out with d_2 dollars, and a player has to spend a dollars to remove a stones. This game is interesting because a player has to balance out his desire to make a good move with his concern that he may run out of money. This game leads to more complex win conditions then standard NIM. For example, the win condition may depend on *both* what n is congruent to mod some M_1 *and* on what $d_1 - d_2$ is congruent mod some M_2. Some of our results are surprising. For example, there are cases where both players are poor, yet the one with less money wins.

For several choices of $a_1, \ldots, a_k$ we determine for all (n, d_1, d_2) which player wins.

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1 Introduction

NIM is a game involving two players removing stones from a board until there are none left. We define the standard 2-player 1-pile NIM game and then discuss other variants.

Def 1.1 Let $a_1 < \cdots < a_k \in \mathsf{N}$ and $n \in \mathsf{N}$. $NIM(a_1, \ldots, a_k; n)$ is played as follows.

1. Initially there are n stones on the board and their are two players, Player I and Player II.
2. The player's alternate with Player I going first.
3. During a player's turn he removes $a \in \{a_1, a_2, \ldots, a_k\}$ stones from the board.
4. If a player cannot move he loses.

We often say $NIM(a_1, \ldots, a_k)$ when the n will come later in the sentence.

Notation 1.2 The expression *Player I wins* means that Player I has a winning strategy that will work regardless of what Player II does. Similar for *Player II wins*.

NIM is an example of a *combinatorial game*. Such games have a vast literature (see the selected bibliography of Frankel [1]). Variants on the 1-pile version have included letting the number of stones a player can remove depend on how many stones are in the pile [4], having the players have different moves allowed [2], allowing three players [6], viewing the stones as cookies that may spoil [5], and others. Grundy [3] and Sprague [7] showed how to reduce the many-pile NIM games to 1-pile NIM games. These games are appealing because they are easy to explain, yet involve interesting (and sometimes difficult) mathematics to analyze.

We give several examples of known win-loss patterns for NIM-games.

Example 1.3

1. Let $L \geq 1$. Player II wins $NIM(1, \ldots, L)$ iff $n \equiv 0 \pmod{L+1}$.

2. Let $L \geq 2$ and even (the case of L odd is a simple mod-2 pattern so we omit it). Player II wins $NIM(1, L)$ iff $n \equiv 0, 2, 4, \ldots, L-2 \pmod{L+1}$.

3. Let $L \geq 3$ and odd. Player II wins $NIM(1, L-1, L)$ iff $n \equiv 0, 2, 4, \ldots, L-3 \pmod{2L-2}$.

4. Let $L \geq 4$ and even. Player II wins $NIM(1, L-1, L)$ iff $n \equiv 0, 2, 4 \ldots, L-2 \pmod{2L-1}$.

5. Let $L \geq 3$. Player II wins $NIM(2, 3, \ldots, L)$ iff $n \equiv 0, 1 \pmod{L+2}$.

6. Let $L \geq 3$. Player II wins $NIM(L-2, L-1, L)$ iff $n \equiv 0, 1, \ldots, L-3 \pmod{2L-2}$.

We consider a variant of this game where players are also given amounts of money d_1, d_2 and they have to pay i dollars to remove i stones.

Def 1.4 Let $a_1 < \cdots < a_k \in \mathsf{N}$ and $n, d_1, d_2 \in \mathsf{N}$. The $NIM(a_1, \ldots, a_k; n, d_1, d_2)$ is played just like NIM except that

1. If a player removes j stones from the board he must spend j dollars.

2. If a player cannot move he loses; however, this can now happen in one of two ways: either there are $< a_1$ stones on the board or the player does not have enough money to make the move (typically the player is broke).

If we want to specify just $a_1, \ldots, a_k$ but not n, d_1, d_2 then we refer to this game as $NIM(a_1, \ldots, a_k)$ *with cash.*

Our game is interesting because a player has to balance the desire to play an optimal (in the unlimited cash version) with desire to not run out of money. This leads to some surprising results. For example, it is possible that both players are poor yet the one with less money wins. Another surprising scenario are discussed in Section 5.

In Section 2 we define the state of the game which will be an important concept. In Section 3 we prove a general theorem about what happens if both players are *poor*. In Section 4, we give a complete description of who wins $NIM(1, 2, \ldots, L)$ with Cash. This leads to a speculation which we discuss and show is false in Section 5. We then, in Section 6 give a complete description of who wins $NIM(1, 3, 4)$. We then make conjectures and state future research goals in Sections 7 and 8. In the Appendix we discuss the programs used to generate the data that lead to our theorems.

2 State of the Game

Def 2.1 Let $a_1 < \cdots < a_k \in \mathsf{N}$. Assume we are playing $NIM(a_1, \ldots, a_k)$. Let $n, d_1, d_2 \in \mathsf{N}$. We define the state of the game. In all cases it is Player I's turn.

1. The game is in *state* $(n; d_1, d_2)$ if there are n stones on the board, Player I has d_1 dollars, and Player II has d_2 dollars.

2. The game is in *state* $(n, \geq d_1, \leq d_2)$ if there are n stones on the board, Player I has $\geq d_1$ dollars, and Player II has $\leq d_2$ dollars.

3. States $(n, \leq d_1, \geq d_2)$ and other variants are defined similarly.

4. The game is in *state* $(n; d_1, UF)$ if there are n stones on the board, Player I has $\geq d_1$ dollars and Player II has unlimited funds (hence the UF).

5. States (n, UF, d_2), $(n, UF, \leq d_2)$ and other variants are defined similarly.

6. We will sometimes use the notation $(n; d_1, d_2, II)$ to mean that it is Player II's turn.

7. If the game is understood then we use the phrase *Player I wins* (n, d_1, d_2) to mean that Player I has a winning strategy in the understood game starting from that state. Similar for *Player II wins* (n, d_1, d_2).

Def 2.2 If Player I (II) wins from state $(n; d, UF)$ $((n, UF, d))$ then we say that he *wins normally*. Note that he can use the same strategy he uses in Standard NIM. If a Player I (II) wins by always removing a_1 and waiting (with success) for Player II (I) to not be able to move because have $< a_1$ dollars then we say that that he wins *miserly*.

3 Miserly Theorem

Theorem 3.1 *Let* $1 = a_1 < a_2 < \cdots < a_k$. *We assume the game is* $NIM(a_1, \ldots, a_k)$ *with cash.*

1. *For all* $n \geq 1$ *Player I wins* $(n, \geq \lceil \frac{n}{2} \rceil, \leq \lceil \frac{n}{2} - 1 \rceil)$ *by playing a miserly strategy.*

2. *For all* $n \geq 0$ *Player II wins* $(n, \leq \lfloor \frac{n}{2} \rfloor, \geq \lfloor \frac{n}{2} \rfloor)$ *by playing a miserly strategy.*

Proof:

We prove both parts by induction. In both cases the base case is trivial.

1)

Induction Hypothesis: For all $1 \leq n' < n$ the theorem holds. We can assume $n \geq 2$. We can assume the game is in state $(n, ;_1, d_2)$ where $d_1 \geq \lceil \frac{n}{2} \rceil$ and $d_2 \leq \lceil \frac{n}{2} - 1 \rceil$.

Player I's winning strategy is to remove one stone. We can assume Player II removes $a \in \{a_1, \ldots, a_k\}$ stones. The state is now $(n - a - 1, d_1 - 1, d_2 - a)$ which satisfies the induction hypothesis:

$$d_1 - 1 \geq \left\lceil \frac{n}{2} - 1 \right\rceil = \left\lceil \frac{n-2}{2} \right\rceil \geq \left\lceil \frac{n-a-1}{2} \right\rceil.$$

$$d_2 - a \leq \left\lceil \frac{n}{2} - a - 1 \right\rceil = \left\lceil \frac{n-2a}{2} - 1 \right\rceil \leq \left\lceil \frac{n-a-1}{2} - 1 \right\rceil.$$

2)

Induction Hypothesis: For all $1 \leq n' < n$ the theorem holds. We can assume $n \geq 1$. We can assume the game is in state $(n; d_1, d_2)$ where $d_1 \leq \lfloor \frac{n}{2} \rfloor$ and $d_2 \geq \lfloor \frac{n}{2} \rfloor$.

Assume Player I removed $a \in \{a_1, \ldots, a_k\}$. Player II's winning response is to remove one stone. The state is now $(n - a - 1, d_1 - a, d_2 - 1)$ which satisfies the induction hypothesis:

$$d_1 - a \leq \left\lfloor \frac{n}{2} - a \right\rfloor = \left\lfloor \frac{n-2a}{2} \right\rfloor \leq \left\lfloor \frac{n-a-1}{2} \right\rfloor.$$

$$d_2 - 1 \geq \left\lfloor \frac{n}{2} - 1 \right\rfloor = \left\lfloor \frac{n-2}{2} \right\rfloor \geq \left\lfloor \frac{n-a-1}{2} \right\rfloor.$$

∎

4 $NIM(1, 2, \ldots, L)$ with Cash

Throughout this section we assume that $L \geq 2$ and the game is $NIM(1, 2, \ldots, L)$ with Cash.

Def 4.1 Let f and W be the following functions.

1. $f((L+1)k) = (L-1)k$, $W((L+1)k) = II$.
2. For $1 \leq i \leq L$ $f((L+1)k+i) = (L-1)k+i$, $W((L+1)k+i) = I$.

Note that $W(n)$ is who wins (if both sides have unlimited funds). We will see that $f(n)$ is how much $W(n)$ needs to win normally.

In the following subsections we determine who wins (1) when both players have $\geq f(n)$, (2) when one player has $\geq f(n)$ and one has $< f(n)$, (3) when both players have $< f(n)$. We then put these theorems together to obtain a theorem stating exactly who wins when. We only sketch the proofs since they are similar (but easier) then the ones in Section 6.

Theorem 4.2 *Let $n, d \in \mathsf{N}$. Assume $n \leq \left\lfloor \frac{d}{L} \right\rfloor + d$.*

1. *If $n \equiv 0 \pmod{L+1}$ and the game is in state $(n, UF, \geq d)$ then Player II wins normally.*
2. *If $n \equiv 1, 2, 3, \ldots, L \pmod{L+1}$ and the game is in state $(n, \geq d, UF)$ then Player I wins normally.*

Proof sketch:

Whichever player is supposed to win plays the strategy he would play with unlimited funds. An easy induction shows that he has enough money to do this.

∎

Note that if $d_1, d_2 \geq f(n)$ then the player who would win the unlimited game wins normally. Hence we are now concerned with what happens if one or both players has $< f(n)$.

Theorem 4.3 *Let* $n \in \mathsf{N}$. *If* $d < f(n)$ *then Player II wins* $(n, \leq d, \geq d)$.

Proof sketch:

The proof is by induction on n. The base case is easy.

The induction step has two cases.

1. If $d \equiv j \pmod{L}$, $n \equiv j + 1 \pmod{L + 1}$, and Player I removes $i \leq j$ stones then Player II's winning response is to remove $j - i + 1$ stones.

2. In all other cases Player II's response is to remove one stone.

∎

Theorem 4.4 *Let* $n, d \in \mathsf{N}$. *If* $d_2 \leq d_1 - 1, f(n)$ *then Player I wins* (n, d_1, d_2).

Proof sketch:

Player I's strategy is to always remove one stone.

∎

From Theorems 4.2, 4.3, 4.4 we obtain the following complete description of who wins when.

Theorem 4.5 *Let* $n \geq 0$.

1. If $d_1, d_2 \geq f(n)$ *then*

(a) *If* $n \equiv 0 \pmod{L+1}$ *then Player II wins.*

(b) *If* $n \equiv 1, 2, \ldots, L \pmod{L+1}$ *then Player II wins.*

2. *If* $d_1 \geq f(n)$ *but* $d_2 < f(n)$ *then Player I wins.*

3. *If* $d_2 \geq f(n)$ *but* $d_1 < f(n)$ *then Player II wins.*

4. *If* $d_1 d_2 < f(n)$ *then*

   (a) *If* $d_1 \leq d_2$ *then Player II wins.*

   (b) *If* $d_1 > d_2$ *then Player I wins.*

5 An Incorrect Speculation and a Counterexample

The above example, and intuition, would suggest the following speculation:

If both players are rich then the game will go as it usually does, as both can afford their normal strategies. If both players are poor, but equal, then Player II should win since Player I will go first and be at a monetary disadvantage.

This turns out to not be true. We give a simple example:

Consider $NIM(1, 3, 4)$. As noted in Example 1.3 the following is known.

1. If $n \equiv 0, 2 \pmod{7}$ then Player II wins.

2. If $n \equiv 1, 3, 4, 5, 6 \pmod{7}$ then Player I wins.

Consider $NIM(1, 3, 4)$ with cash. We show that in Player I wins $(14; 9, 9)$. Player I's winning first move is to remove 1 stone. Now the game is in state $(13; 8, 9, II)$. If Player II had lots of money then he would remove four stones. But if he did that here he would be to poor to win! If Player II removes three or four stones then Player I can adapt a miserly strategy to win. For the first few pairs of turns Player II has to remove one stone (else he loses) and Player I will respond by removing one stone until the game is in state $(10; 7, 7)$. From here this state it is easy to show that Player I wins.

In the next section we analyze $NIM(1,3,4)$ with cash completely.

6 $NIM(1,3,4)$ with Cash

Throughout this section we assume that the game being played is $NIM(1,3,4)$ with Cash.

Def 6.1 Let f and W be the following functions.

1. $f(7k) = 5k$, $W(7k) = II$.
2. $f(7k+1) = 5k+1$, $W(7k+1) = I$.
3. $f(7k+2) = 5k+1$, $W(7k+2) = II$.
4. $f(7k+3) = 5k+2$, $W(7k+3) = I$.
5. $f(7k+4) = 5k+4$, $W(7k+4) = I$.
6. $f(7k+5) = 5k+4$, $W(7k+5) = I$.
7. $f(7k+6) = 5k+5$, $W(7k+6) = I$.

Note that $W(n)$ is who wins (if both sides have unlimited funds). We will see that $f(n)$ is how much $W(n)$ needs to win normally.

6.1 The Normal Case

The following theorem states when a player can win normally. We omit the easy proof.

Theorem 6.2 *Let* $n \geq 0$.

- *If* $n \equiv 1,3,4,5,6 \pmod 7$ *and* $d_1 \geq f(n)$ *then Player II wins* $(n, UF, \geq f(n))$ *normally.*
- *If* $n \equiv 0,2 \pmod 7$ *and* $d_2 \geq f(n)$ *then Player I wins* $(n, \geq f(n), UF)$ *normally.*

6.2 A Very Useful Table

To completely analyze $NIM(1,3,4)$ with cash we need to know what happens when one or both players have less than $f(n)$. Table 1 below will be useful.

n	a	$n-a-1$	$f(n)$	$f(n-a-1)$	$f(n) \le f(n-a-1)+a(+1)$	$f(n-a-1) \le f(n)-1?$
$7k$	1	$7(k-1)+5$	$5k$	$5k-1$	YES	YES
$7k+1$	1	$7(k-1)+6$	$5k+1$	$5k$	YES	YES
$7k+2$	1	$7k$	$5k+1$	$5k$	YES	YES
$7k+3$	1	$7k+1$	$5k+2$	$5k+1$	YES	YES
$7k+4$	1	$7k+2$	$5k+4$	$5k+1$	NO	NO
$7k+5$	1	$7k+3$	$5k+4$	$5k+2$	$NO(YES)$	NO
$7k+6$	1	$7k+4$	$5k+5$	$5k+4$	YES	YES
$7k$	3	$7(k-1)+3$	$5k$	$5k-3$	YES	YES
$7k+1$	3	$7(k-1)+4$	$5k+1$	$5k-1$	YES	YES
$7k+2$	3	$7(k-1)+5$	$5k+1$	$5k-1$	YES	YES
$7k+3$	3	$7(k-1)+6$	$5k+2$	$5k$	YES	YES
$7k+4$	3	$7k$	$5k+4$	$5k$	$NO(YES)$	YES
$7k+5$	3	$7k+1$	$5k+4$	$5k+1$	YES	YES
$7k+6$	3	$7k+2$	$5k+5$	$5k+1$	$NO(YES)$	YES
$7k$	4	$7(k-1)+2$	$5k$	$5k-4$	YES	YES
$7k+1$	4	$7(k-1)+3$	$5k+1$	$5k-3$	YES	YES
$7k+2$	4	$7(k-1)+4$	$5k+1$	$5k-1$	YES	YES
$7k+3$	4	$7(k-1)+5$	$5k+2$	$5k-1$	YES	YES
$7k+4$	4	$7(k-1)+6$	$5k+4$	$5k$	YES	YES
$7k+5$	4	$7k$	$5k+4$	$5k$	YES	YES
$7k+6$	4	$7k+1$	$5k+5$	$5k+1$	YES	YES

Table 1

6.3 If $d_1 < d_2, f(n)$ then Player II Wins

Theorem 6.3 *Let $n \geq 1$. If $0 \leq d_1 < d_2, f(n)$ then Player II wins $(n; d_1, d_2)$.*

Proof:

We prove this by induction on n.

Base Case: We leave the Base Case of $n = 0, 1, 2, 3, 4, 5$ to the reader.

Induction Hypothesis: For all $1 \leq n' < n$ the theorem holds. We can assume $n \geq 5$. We can assume the game is in state $(n; d_1, d_2)$ where $d_1 < d_2, f(n)$. We can assume that Player I removes $a \in \{1, 3, 4\}$ stones.

Case 1: $f(n) \leq f(n-a-1) + a$. Player II's winning response is to remove one stone resulting in state $(n-a-1, d_1-a, d_2-1)$. This state satisfies the induction hypothesis: (1) Clearly $d_1 - a < d_2 - 1$. (2) Note that

$$d_1 - a < f(n) - a \leq f(n-a-1).$$

Case 2: $f(n) \leq f(n-a-1) + a + 1$.

Case 2.1: $d_1 \leq f(n) - 2$ Player II's winning response is to remove one stone resulting in state $(n-a-1, d_1-a, d_2-1)$. This state satisfies the induction hypothesis: (1) Clearly $d_1 - a < d_2 - 1$. (2)

$$d_1 - a \leq f(n) - a - 2 \leq f(n-a-1) - 1 < f(n-a-1).$$

Case 2.2: $d_1 = f(n) - 1$ and $W(n-a-1) = II$ and $f(n-a-1) \leq f(n) - 1$. Note that since $d_1 < d_2$ we have $d_2 \geq f(n)$. Player II's winning response is to remove one stone resulting in state $(n-a-1, d_1-a, d_2-1)$. This state satisfies the premise of Theorem 6.2 since

$$d_2 - 1 \geq f(n) - 1 \geq f(n-a-1)$$

and $W(n-a-1) = II$.

Case 2.3: $d_1 = f(n) - 1$ and ($W(n-a-1) = I$ or $f(n-a-1) \geq f(n) - 1$). Note that $d_2 \geq f(n)$. By Table 1 there are only two cases that fall into this category.

1. $n = 7k + 4$ and $a = 1$. After Player I removes one stone Player II's winning response is to remove three stones resulting in state $(7k, d_1 - a, d_2 - 3)$. This state satisfies the premise of Theorem 6.2 since

$$d_2 - 3 \geq f(n) - 3 = f(7k + 4) - 3 = 5k + 1 \geq f(7k)$$

   and $W(7k) = II$.

2. $n = 7k + 5$ and $a = 1$. After Player I removes one stone Player II's winning response is to remove four stones resulting in state $(7k, d_1 - a, d_2 - 4)$. This state satisfies the premise of Theorem 6.2 since

$$d_2 - 4 \geq f(n) - 4 = f(7k + 4) - 4 = 5k \geq f(7k)$$

   and $W(7k) = II$.

∎

6.4 If $d_2 < d_1 + 1, f(n)$ then Player I Wins

Table 2 below will be useful.

n	a	$n-a-1$	$f(n)$	$f(n-a-1)$	$f(n) \leq f(n-a-1)+a(+3)$	W
$7k$	1	$7(k-1)+5$	$5k$	$5k-1$	YES	II
$7k+1$	1	$7(k-1)+6$	$5k+1$	$5k$	YES	I
$7k+2$	1	$7k$	$5k+1$	$5k$	YES	II
$7k+3$	1	$7k+1$	$5k+2$	$5k+1$	YES	I
$7k+4$	1	$7k+2$	$5k+4$	$5k+1$	$NO(YES)$	I
$7k+5$	1	$7k+3$	$5k+4$	$5k+2$	$NO(YES)$	I
$7k+6$	1	$7k+4$	$5k+5$	$5k+4$	YES	I
$7k$	3	$7(k-1)+3$	$5k$	$5k-3$	YES	II
$7k+1$	3	$7(k-1)+4$	$5k+1$	$5k-1$	YES	I
$7k+2$	3	$7(k-1)+5$	$5k+1$	$5k-1$	YES	II
$7k+3$	3	$7(k-1)+6$	$5k+2$	$5k$	YES	I
$7k+4$	3	$7k$	$5k+4$	$5k$	$NO(YES)$	I
$7k+5$	3	$7k+1$	$5k+4$	$5k+1$	YES	I
$7k+6$	3	$7k+2$	$5k+5$	$5k+1$	$NO(YES)$	I
$7k$	4	$7(k-1)+2$	$5k$	$5k-4$	YES	II
$7k+1$	4	$7(k-1)+3$	$5k+1$	$5k-3$	YES	I
$7k+2$	4	$7(k-1)+4$	$5k+1$	$5k-1$	YES	II
$7k+3$	4	$7(k-1)+5$	$5k+2$	$5k-1$	YES	I
$7k+4$	4	$7(k-1)+6$	$5k+4$	$5k$	YES	I
$7k+5$	4	$7k$	$5k+4$	$5k$	YES	I
$7k+6$	4	$7k+1$	$5k+5$	$5k+1$	YES	I

Table 2

Theorem 6.4 *Let* $n \geq 1$. *If* $d_2 < d_1 - 1, f(n)$ *then Player I wins* $(n; d_1, d_2)$.

Proof:

We prove this by induction on n.

Base Case: We leave the base case of $n = 1, 2, 3, 4, 5$ to the reader.

Induction Hypothesis: For all $1 \leq n' < n$ the theorem holds. We can assume $n \geq 5$. We can assume the game is in state $(n; d_1, d_2)$ where $d_2 < d_2 - 1, f(n)$.

Case 1: $W(n) = 1$ and $d_1 \geq f(n)$. In this case, by Theorem 6.2, Player I wins.

Case 2: $W(n) \neq I$ or $d_1 \leq f(n) - 1$. Player I's strategy is to remove one stone. We can assume Player II removes $a \in \{1, 3, 4\}$ stones. The game is thus in state $(n - a - 1, d_1 - a, d_2 - a)$. We show that the new state satisfies the induction hypothesis. Clearly $d_1 - a < (d_2 - 1) - 1$. Hence we need only prove that $d_2 - a < f(n - a - 1)$.

Case 2.1: $f(n) \leq f(n - a - 1) + a$. Combine this with $d_2 < f(n)$ to obtain

$$d_2 - a < f(n) - a \leq f(n - a - 1).$$

Case 2.2: $d_1 \leq f(n) - 1$. By Table 2 it is always the case that $f(n) \leq f(n - a - 1) + a + 3$. Since $d_2 < d_1 - 1$ and $d_1 \leq f(n) - 1$ we have $d_2 \leq f(n) - 3$. Hence

$$d_2 - a < f(n) - 3 - a \leq f(n - a - 1) + a + 3 - 3 - a = f(n - a - 1).$$

Case 2.3: $W(n) = II$ and $f(n) \geq f(n - a - 1) + a + 1$. By Table 2 this case never occurs.

∎

6.5 What if $d_1 = d_2 + 1 \leq f(n)$?

Theorem 6.5 *Player I wins* $(n, f(n), f(n) - 1)$.

Proof sketch: Player I's strategy is as follows: If $W(n) = I$ then Player I wins normally. If $W(n) = II$ then Player I will remove one stone. We omit the induction proof that shows that this works. ∎

Lemma 6.6 *Let* $b \geq 1$. *Assume the game starts in state* $(n, f(n) - b, f(n) - b - 1)$.

1. *If on the first move Player I removes three or four stones and then Player II removes one stone, then Player II wins.*

2. *If on the first move Player I removes one stone and then Player II removes three of four stones, then Player I wins.*

Proof:

1) The game is now in state $(n-a-1, f(n)-b-a, f(n)-b-2)$ where $a \in \{3,4\}$. We show that this state satisfies the premise of Theorem 6.3 and hence Player II wins. Clearly we have $f(n)-b-a < f(n)-b-2$. We need $f(n)-b-a < f(n-a-1)$. It suffices to show that $f(n)-a-1 < f(n-a-1)$.

Case 1: $f(n) \leq f(n-a-1)+a$. Hence

$$f(n)-a-1 \leq f(n-a-1)+a-a-1 = f(n-a-1)-1 < f(n-a-1).$$

Case 2: $f(n) = f(n-a-1)+a+1$. By the Table $n \equiv 4, 6 \pmod 7$.

- If $n = 7k+4$ then $f(n) = 5k+4$ hence the current state is $(7k+3-a, 5k+3-a, 5k+2)$. If $a = 3$ then the state is $(7k, 5k, 5k+2)$ and Player II wins by Theorem 6.2. If $a = 4$ then the state is $(7k-1, 5k-1, 5k+2) = (7(k-1)+6, 5(k-1)+4, 5k+2)$. Note that $5(k-1)+4 < f(7(k-1)+6) = 5(k-1)+5$. Hence the state satisfies the premise of Theorem 6.3 so Player II wins.

- If $n = 7k+6$ then $f(n) = 5k+5$ hence the current state is $(7k+5-a, 5k+4-a, 5k+3)$. If $a = 3$ then the state is $(7k+2, 5k+1, 5k+3)$ and Player II wins by Theorem 6.2. If $a = 4$ then the state is $(7k+1, 5k, 5k+3)$. Note that $5k < f(7k+1) = 5k+1$. Hence the state satisfies the premise of Theorem 6.4 so Player II wins.

By the Table there are no more cases.

2) The game is now in state $(n-a-1, f(n)-b-1, f(n)-b-a-1)$ where $a \in \{3,4\}$. We show that this state satisfies the premise of Theorem 6.4 and hence Player I wins. Clearly we have $f(n)-b-a-1 <$

$(f(n)-b-1)-1$. We need $f(n)-a-2 < f(n-a-1)$. We proved $f(n)-a-1 < f(n-a-1)$ in part 1 so we are done. ▮

Theorem 6.7 *Let $b \geq 0$. Let n be such that $f(n)-b-1 \geq 0$. Assume the game starts in state $(n, f(n)-b, f(n)-b-1)$.*

1. *If $b \equiv 0 \pmod 3$ then Player I wins.*
2. *If $b \equiv 1 \pmod 3$ then*
   - *if $n \leq \frac{14(b-1)}{3}+3$ or $n \equiv 0,2,5 \pmod 7$ then Player I wins;*
   - *if $n \geq \frac{14(b-1)}{3}+4$ and $n \equiv 1,3,4,6 \pmod 7$ then Player II wins.*
3. *If $b \equiv 2 \pmod 3$ then*
   - *if $n \leq \frac{14(b-2)}{3}+9$ or $n \equiv 1,3,4,6 \pmod 7$ then Player I wins;*
   - *if $n \geq \frac{14(b-2)}{3}+10$ and $n \equiv 0,2,5 \pmod 7$ then Player II wins.*

Proof: We prove this by induction on $b+n$.

Base Case: If $b+n=1$ then one of the following occurs.

- $b=0$ and $n=1$. The premise holds since $f(n)-b-1 = 1-0-0 \geq 0$. Player I wins by Theorem 6.5.
- $b=1$ and $n=0$. The premise $f(n)-b-1 \geq 0$ does not hold.

Induction Hypothesis: For all (b', n') such that $b'+n' < b+n$ the theorem holds. We prove the theorem for (b,n). We may assume $b \geq 1$ since if $b=0$ then the theorem holds by Theorem 6.5.

Assume the state is $(n, f(n)-b, f(n)-b-1)$. By Lemma 6.6 we can assume that both Players remove one stone (in some cases we will not use this). Hence the new state is $(n-2, f(n)-b-1, f(n)-b-2)$. Note that in the new state the quantity $b+n$ has decreases. Hence, if the new state satisfies the conditions of the theorem, we can use the induction hypothesis. The Table 3 will be helpful.

n	$(n-2, f(n)-b-1, f(n)-b-2) = (n-2, f(n-2)-?, f(n-2)-?-1)$
$7k$	$(7(k-1)+5, 5(k-1)+4-b, 5(k-1)+4-b-1) = (n-2, f(n-2)-b, f(n-2)-b-1)$
$7k+1$	$(7(k-1)+6, 5(k-1)+5-b, 5(k-1)+5-b-1) = (n-2, f(n-2)-b, f(n-2)-b-1)$
$7k+2$	$(7k, 5k-b, 5k-b-1) = (n-2, f(n-2)-b, f(n-2)-b-1)$
$7k+3$	$(7k+1, 5k+1-b, 5k+1-b-1) = (n-2, f(n-2)-b, f(n-2)-b-1$
$7k+4$	$(7k+2, 5k+3-b, 5k+2-b) = (n-2, f(n-2)-(b-2), f(n-2)-(b-2)-1$
$7k+5$	$(7k+3, 5k+2-(b-1), 5k+2-(b-1)-1) = (n-2, f(n-2)-(b-1), f(n-2)-(b-1)-1)$
$7k+6$	$(7k+4, 5k+4-b, 5k+4-b-1) = (n-2, f(n-2)-b, f(n-2)-b-1)$

Table 3

Case 0: $b \equiv 0 \pmod 3$. The state is $(n, f(n)-b, f(n)-b-1)$. By Table 3 note the following:

1. If $n \equiv 0, 1, 2, 3, 6$ then the new state satisfies the condition of the theorem with $b' = b \equiv 0 \pmod 3$ hence Player I wins.

2. If $n \equiv 4 \pmod 3$ then the new state has $b' = b-2 \equiv 1 \pmod 3$ and $n' \equiv 2 \pmod 7$, hence Player I wins. (Since $b \equiv 0 \pmod 3$ and $b \neq 0$ we have $b \geq 3$ so $b-2 \geq 0$.)

3. If $n \equiv 5 \pmod 3$ then the new state has $b' = b-1 \equiv 2 \pmod 3$ and $n' \equiv 3 \pmod 7$, hence Player I wins.

Case 1.1: $b \equiv 1 \pmod 3$ and $n \leq \frac{14(b-1)}{3} + 3$. It is easy to show that in this case $f(n) - b - 1 \geq \lceil n/2 \rceil$ and $f(n) - b - 2 \leq \lceil n/2 \rceil - 1$. Hence, by Theorem 3.1 Player I wins.

Case 1.2: $b \equiv 1 \pmod 3$ and $n \geq \frac{14(b-1)}{3} + 4$.

1. If $n \equiv 0 \pmod 7$ then the new state has $b' = b \equiv 1 \pmod 3$ and $n' \equiv 5 \pmod 7$; hence Player I wins.

2. If $n \equiv 1 \pmod 7$ then the new state has $b' = b \equiv 1 \pmod 3$ and $n' \equiv 6 \pmod 7$. Player II wins

so long as $n' = n - 2 \geq \frac{14(b-1)}{3} + 4$. We have

$$n = 7k + 1 \geq \frac{14(b-1)}{3} + 4.$$

Using mod 7 one can easily show that actually

$$n = 7k + 1 \geq \frac{14(b-1)}{3} + 8.$$

The lower bound on n' follows.

3. If $n \equiv 2 \pmod 7$ then the new state has $b' = b \equiv 1 \pmod 3$ and $n' \equiv 0 \pmod 7$. Hence Player I wins.

4. If $n \equiv 3 \pmod 7$ then the new state has $b' = b \equiv 1 \pmod 3$ and $n' \equiv 1 \pmod 7$. Player II wins so long as $n' = n - 2 \geq \frac{14(b-1)}{3} + 4$. We have

$$n = 7k + 3 \geq \frac{14(b-1)}{3} + 4.$$

Using mod 7 one can easily show that actually

$$n = 7k + 3 \geq \frac{14(b-1)}{3} + 10.$$

The lower bound on n' follows.

5. If $n \equiv 4 \pmod 7$ then the new state has $b' = b - 2 \equiv 2 \pmod 3$ and $n' \equiv 2 \pmod 7$. Player II wins so long as $n' = n - 2 \geq \frac{14(b'-2)}{3} + 10 = \frac{14(b-4)}{3} + 10$. We have

$$n = 7k + 4 \geq \frac{14(b-1)}{3} + 4 = \frac{14(b-4) + 14 \times 3}{3} + 4 = \frac{14(b-4)}{3} + 18$$

Hence clearly

$$n' = n - 2 \geq \frac{14(b-4)}{3} + 16 \geq \frac{14(b-4)}{3} + 10.$$

The lower bound on n' follows.

6. If $n \equiv 5 \pmod 7$ then the new state has $b' = b - 1 \equiv 0 \pmod 3$ and $n' = n - 2 \equiv 3 \pmod 7$. Since $b' \equiv 0 \pmod 3$, Player I wins.

7. If $n \equiv 6 \pmod 7$ then the new state has $b' \equiv b \equiv 1 \pmod 3$ and $n' \equiv 4 \pmod 7$. Player II wins so long as $n' = n - 2 \geq \frac{14(b-1)}{3} + 4$. We have

$$n = 7k + 6 \geq \frac{14(b-1)}{3} + 4.$$

Using mod 7 we one can easily show that actually $n = 7k + 6 \geq \frac{14(b-1)}{3} + 6$.

The lower bounds on n' follows.

Case 2: $b \equiv 2 \pmod 3$. This case is similar to Case 1 (and all its subcases) above.

∎

6.6 What happens when $d_1 = d_2 < f(n)$?

The following theorems have proofs similar in spirit to that of Theorem 6.5 and 6.7 hence we omit the proofs.

Theorem 6.8 *Let $n \geq 1$ and the game is in state $(n, f(n) - 1, f(n) - 1)$. Player I wins iff $n \equiv 0, 2, 5 \pmod 7$.*

Theorem 6.9 *Let $b \geq 0$. Let n be such that $f(n) - b \geq 0$. Assume the game starts in state $(n, f(n) - b, f(n) - b)$.*

1. If $b \equiv 0 \pmod 3$ then

- *if $n \leq \frac{14(b-1)}{3} + 11$ or $n \equiv 1, 3, 4, 6 \pmod 7$ then Player I wins;*
- *if $n \geq \frac{14(b-1)}{3} + 12$ and $n \equiv 0, 2, 5 \pmod 7$ then Player II wins.*

2. *If* $b \equiv 1 \pmod 3$ *then*

   - *if* $n \leq \frac{14(b-1)}{3} + 5$ *or* $n \equiv 0, 2, 5 \pmod 7$ *then Player I wins;*
   - *if* $n \geq \frac{14(b-1)}{3} + 6$ *and* $n \equiv 1, 3, 4, 6 \pmod 7$ *then Player II wins.*

3. *If* $b \equiv 2 \pmod 3$ *then Player II wins.*

6.7 The Complete Theorem

Theorem 6.10 *Let* $n, d_1, d_2 \geq 0$.

1. *If* $d_1, d_2 \geq f(n)$ *then Player I wins iff* $n \equiv 1, 3, 4, 5, 6 \pmod 7$.
2. *If* $d_{3-i} < f(n) \leq d_i$ *then Player* i *wins.*
3. *If* $d_2 < d_1 - 1, f(n)$ *then Player I wins.*
4. *If* $d_1 < d_2$ *then Player II wins.*
5. *Let* $b \geq 0$. *Let* n *be such that* $f(n) - b - 1 \geq 0$. *Assume the game starts in state* $(n, f(n) - b, f(n) - b - 1)$.

   (a) *If* $b \equiv 0 \pmod 3$ *then Player I wins.*

   (b) *If* $b \equiv 1 \pmod 3$ *then*

   - *if* $n \leq \frac{14(b-1)}{3} + 3$ *or* $n \equiv 0, 2, 5 \pmod 7$ *then Player I wins;*
   - *if* $n \geq \frac{14(b-1)}{3} + 4$ *and* $n \equiv 1, 3, 4, 6 \pmod 7$ *then Player II wins.*

   (c) *If* $b \equiv 2 \pmod 3$ *then*

   - *if* $n \leq \frac{14(b-2)}{3} + 9$ *or* $n \equiv 1, 3, 4, 6 \pmod 7$ *then Player I wins;*
   - *if* $n \geq \frac{14(b-2)}{3} + 10$ *and* $n \equiv 0, 2, 5 \pmod 7$ *then Player II wins.*

6. *Let* $b \geq 0$. *Let* n *be such that* $f(n) - b \geq 0$. *Assume the game starts in state* $(n, f(n) - b, f(n) - b)$.

   (a) *If* $b \equiv 0 \pmod 3$ *then*

- *if* $n \le \frac{14(b-1)}{3} + 11$ *or* $n \equiv 1, 3, 4, 6 \pmod 7$ *then Player I wins;*
- *if* $n \ge \frac{14(b-1)}{3} + 12$ *and* $n \equiv 0, 2, 5 \pmod 7$ *then Player II wins.*

(b) If $b \equiv 1 \pmod 3$ *then*

- *if* $n \le \frac{14(b-1)}{3} + 5$ *or* $n \equiv 0, 2, 5 \pmod 7$ *then Player I wins;*
- *if* $n \ge \frac{14(b-1)}{3} + 6$ *and* $n \equiv 1, 3, 4, 6 \pmod 7$ *then Player II wins.*

(c) If $b \equiv 2 \pmod 3$ *then Player II wins.*

7 Our Conjecture

Def 7.1 Let N be a NIM game. $f_N(x)$ is the least d such that

- If Player I wins (n, UF, UF) then he wins $(n; d, UF)$.
- If Player II wins (n, UF, UF) then he wins $(n; UF, d)$.

The following conjecture is true for all of the NIM games discussed in the first section of this paper.

Conjecture: Let N be a NIM game. Let $b_1, \ldots, b_L, M$ such that Player II wins N iff $n \equiv b_i \pmod M$. Then, for all $0 \le i \le M-1$,

- $f(i) \le M - L - 1$.
- For all $k \ge 0$, for all $0 \le i \le M-1$, $f(Mk+i) = (M-L)k + f(i)$

8 Future Directions

We are confident that we can get general theorems for $NIM(1, L)$ and $NIM(1, L-1, L)$. In addition we will prove some general theorems that will make these proofs less detailed then those in this paper. Finally, we are confident that we can get our conjecture proved.

9 Appendix

Standard NIM

```
(x,y) = (input(NAT),input(NAT))
While x>0
        control = input(1,2)
        if control == 1
                (x,y)=(x+10,y-1)
        else
        if control == 2
                (x,y)=(y+17,x-2)
```

9.1 Program for finding $f(n)$

Program Finding $f(n)$ takes as input the takeaway set $\{a_1, \ldots, a_k\}$, the number of stones n, and the win set $\{w_1, \ldots, w_Q\}$. The program produces an array f of size n where each position represents $f(n)$.

Program Finding $f(n)$

```
input(a_1 < ··· < a_k)
input(n)
input(W)
For i = 0 to n
   if (i < a_1)
     then f(i) = 0
   else if (W(i) = II)
      then
            for (j = 0 to a_k)
              if (W(i - a_j) = II)
                 then
                    f(i) = a_j + f(i - a_j)
                    x = a_j + f(i - a_j)
                    break
    else
      f(i) = x
```

10 Acknowledgments

The authors gratefully acknowledge the support of the Maryland Center for Undergraduate Research.

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