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Appendix A: Proofs of Theorems 1 through 6

<u>Proof of Theorem 1</u>. The proof for Part (a) is quite straightforward, so we only give the proof for Part (b) here. Suppose N_1^* and N_2^* are optimal solution to (2) such that $N_1^* < N_2^*$. Let $N_1 = N_1^* + x$ and $N_2 = N_2^* - r'x$. We want to show that there exists some x such that (i) $N_1 \geq N_2 \geq 0$, and (ii)

$$\frac{r}{N_1} + \frac{1/r}{N_2} - \frac{2\rho}{N_1} \leq \frac{r}{N_1^*} + \frac{1/r}{N_2^*} - \frac{2\rho}{N_2^*},$$

which implies that N_1 and N_2 are also optimal solution to (2).

To satisfy (i), we only need

$$\frac{N_2^* - N_1^*}{1 + r'} \leq x \leq \frac{N_2^*}{r'}.$$

For (ii), we have

$$-\frac{2\rho}{N_1} \leq -\frac{2\rho}{N_2^*}$$

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if $x \leq N_2^* - N_1^*$ (note that $\rho > 0$), and

$$\frac{r}{N_1} + \frac{1/r}{N_2} \leq \frac{r}{N_1^*} + \frac{1/r}{N_2^*},$$

if

$$x \leq \frac{rN_2^* - \sqrt{r'}N_1^*}{\sqrt{r'} + r'r}.$$

It is also easy to verify that if $\sqrt{r'} \leq r$, then

$$\frac{N_2^* - N_1^*}{1 + r'} \leq \frac{rN_2^* - \sqrt{r'}N_1^*}{\sqrt{r'} + r'r}.$$

Putting the above together, we conclude that (i) and (ii) hold when

$$\frac{N_2^* - N_1^*}{1 + r'} \leq x \leq \min(N_2^* - N_1^*, \frac{rN_2^* - \sqrt{r'}N_1^*}{\sqrt{r'} + r'r}).$$

This concludes the proof. □

Proof of Theorem 2. Denote

$$\begin{aligned} \Upsilon^* &= \max \lambda_i/\alpha_i, \\ \Omega_1 &= \{i \in \Omega | \lambda_i/\alpha_i = \Upsilon^*\}, \\ \Omega_2 &= \{i \in \Omega_1 | N_1^* < N_i^*\}, \\ \Omega_3 &= \{i \in \Omega_1 | N_1^* \geq N_i^*\}. \end{aligned}$$

Suppose the result does not hold, then $\Omega \setminus \Omega_1$ is not empty. Let

$$\begin{aligned} N'(\varepsilon) &= N^* + \varepsilon, \\ N_i'(\varepsilon) &= \begin{cases} N_1^* + \varepsilon & i = 1, \\ N_i^* & i \in \Omega_2, \\ N_i^* + \varepsilon & i \in \Omega_3, \\ N_i^* - C_0\varepsilon & i \in \Omega \setminus \Omega_1, \end{cases} \end{aligned}$$

where $C_0 = (\sum_{i \in \Omega_3} b_i + b_1 + b_0)/(\sum_{i \in \Omega \setminus \Omega_1} b_i)$, $0 < \varepsilon < \delta = \max\{\min_{i \in \Omega_2}(N_i - N_1), \min_{i \in \Omega_3}(N_i/C_0)\}$. Note that we still have $\sum_i b_i N_i'(\varepsilon) = T$.

For $i \in \Omega_2$, we have

$$\left(\frac{\sigma_i^2 - 2C_{1i}}{N_i} + \frac{\sigma_1^2}{N_1 + \varepsilon}\right) - \left(\frac{\sigma_i^2 - 2C_{1i}}{N_i} + \frac{\sigma_1^2}{N_1}\right) = -\frac{\sigma_1^2}{N_1(N_1 + \varepsilon)}\varepsilon < 0,$$

and for $i \in \Omega_3$, we have

$$\begin{aligned} &\left(\frac{\sigma_1^2 - 2C_{1i}}{N_1 + \varepsilon} + \frac{\sigma_i^2}{N_i + \varepsilon}\right) - \left(\frac{\sigma_1^2 - 2C_{1i}}{N_1} + \frac{\sigma_i^2}{N_i}\right) \\ = &-\left[\frac{\sigma_1^2 - 2C_{1i}}{N_1(N_1 + \varepsilon)} + \frac{\sigma_i^2}{N_i(N_i + \varepsilon)}\right]\varepsilon \leq -\frac{\sigma_1^2 + \sigma_i^2 - 2C_{1i}}{N_1(N_1 + \varepsilon)} < 0. \end{aligned}$$

Denote

$$\begin{aligned}
\Upsilon_1'(\varepsilon) &= \max_{i\in\Omega_2\bigcup\Omega_3}\{\lambda_i/\alpha_i|\sum_i b_i N_i'(\varepsilon)=T\} \\
\Upsilon_2'(\varepsilon) &= \max_{i\in\Omega\backslash\Omega_1}\{\lambda_i/\alpha_i|\sum_i b_i N_i'(\varepsilon)=T\},
\end{aligned}$$

then $\Upsilon_1'(\varepsilon) < \Upsilon^*$, for $\varepsilon < \delta$. In addition, because $\Upsilon_2'(0) < \Upsilon^*$ and it is an continuous function of ε, we can take ε small enough such that $\Upsilon_2'(\varepsilon) < \Upsilon^*$. Therefore, $\{N_i'(\varepsilon), i = 1,\dots,k\}$ is a better solution than $\{N_i^*, i = 1,\dots,k\}$, which is contradictory to the fact that $\{N_i^*, i = 1,\dots,k\}$ is the optimal solution. This completes the proof. □

<u>Proof of Theorem 3</u>. For $N_1^* \geq N_i^*$, we have

$$\frac{\sigma_1^2 - 2C_{1i}}{N_1^*} + \frac{\sigma_i^2}{N_i^*} = \Upsilon^*\alpha_i,$$

which leads to

$$N_i^* = \frac{\sigma_i^2}{M\alpha_i - \sigma_1^2 + 2C_{1i}} N_1^* \quad\Rightarrow\quad \frac{\sigma_i^2}{M\alpha_i - \sigma_1^2 + 2C_{1i}} \leq 1 \quad\Rightarrow\quad A_i \leq M.$$

For $N_1^* \leq N_i^*$, we have

$$\frac{\sigma_1^2}{N_1^*} + \frac{\sigma_i^2 - 2C_{1i}}{N_i^*} = \Upsilon^*\alpha_i,$$

which leads to

$$N_i^* = \frac{\sigma_i^2 - 2C_{1i}}{M\alpha_i - \sigma_1^2} N_1^* \quad\Rightarrow\quad \frac{\sigma_i^2 - 2C_{1i}}{M\alpha_i - \sigma_1^2} \geq 1 \quad\Rightarrow\quad A_i \geq M.$$

The reverse of the above also holds. In other words,

$$\begin{aligned}
A_i \geq M &\quad\Leftrightarrow\quad N_1^* \leq N_i^* \\
A_i \leq M &\quad\Leftrightarrow\quad N_1^* \geq N_i^*.
\end{aligned}$$

In the following, we prove $2\rho_{1i} \geq \sigma_i/\sigma_1 \Rightarrow N_i^* \leq N_1^*$. We denote $I' = \{i \in I|N_i^* > N_1^*\}, J' = \{i \in J|N_i^* > N_1^*\}, \Omega' = \{i \in \Omega|N_i^* \leq N_1^*\}$. If the result does not hold, then J' is not empty, and let

$$\begin{aligned}
N'(\varepsilon) &= N^* + \varepsilon, \\
N_i'(\varepsilon) &= \begin{cases} N_1^* + \varepsilon & i = 1, \\ N_i^* & i \in I', \\ N_i^* + \varepsilon & i \in \Omega', \\ N_i^* - C_3\varepsilon & i \in J', \end{cases}
\end{aligned}$$

where $C_3 = (\sum_{i\in\Omega'} b_i + b_1 + b_0)/(\sum_{i\in J'} b_i)$, $0 < \varepsilon < \delta = \max\left\{\min_{i\in I'}(N_i - N_1), \min_{i\in J'} \frac{N_i - N_1}{1+C_3}\right\}$. We still have $\sum_i b_i N_i'(\varepsilon) = T$.
For $i \in I'$,

$$\left(\frac{\sigma_i^2 - 2C_{1i}}{N_i^*} + \frac{\sigma_1^2}{N_1^* + \varepsilon}\right) - \left(\frac{\sigma_i^2 - 2C_{1i}}{N_i^*} + \frac{\sigma_1^2}{N_1^*}\right) = -\frac{\sigma_1^2}{N_1^*(N_1^* + \varepsilon)}\varepsilon < 0.$$

For $i \in \Omega'$,

$$\begin{aligned} & \left(\frac{\sigma_i^2 - 2C_{1i}}{N_i^*} + \frac{\sigma_1^2}{N_1^* + \varepsilon}\right) - \left(\frac{\sigma_i^2 - 2C_{1i}}{N_i^*} + \frac{\sigma_1^2}{N_1^*}\right) \\ = & -\left[\frac{\sigma_1^2 - 2C_{1i}}{N_1^*(N_1^* + \varepsilon)} + \frac{\sigma_i^2}{N_i^*(N_i^* + \varepsilon)}\right]\varepsilon \leq -\frac{\sigma_1^2 + \sigma_i^2 - 2C_{1i}}{N_1^*(N_1^* + \varepsilon)} < 0. \end{aligned}$$

For $i \in J'$,

$$\begin{aligned} & \left(\frac{\sigma_i^2 - 2C_{1i}}{N_i^*} + \frac{\sigma_1^2}{N_1^* + \varepsilon}\right) - \left(\frac{\sigma_i^2 - 2C_{1i}}{N_i^*} + \frac{\sigma_1^2}{N_1^*}\right) \\ = & -\left[\frac{(2C_{1i} - \sigma_i^2)C_3}{N_i^*(N_i^* - C_3\varepsilon)} + \frac{\sigma_1^2}{N_1^*(N_1^* + \varepsilon)}\right]\varepsilon < 0. \end{aligned}$$

Therefore, $\{N_i'(\varepsilon), i = 1, \ldots, k\}$ is a better solution to (OP_3) than $\{N_i^*, i = 1, \ldots, k\}$, which contradicts to the fact that $\{N_i^*, i = 1, \ldots, k\}$ is the optimal solution. This completes the proof. □

Remark. Note that Theorem 3 implies that $2\rho_{1i} \geq \sigma_i/\sigma_1 \Rightarrow A_i \leq M$, which also follows by applying $N_i \leq N_1$ to (OP_3).

Proof of Theorem 4. Let Υ^*, N_i^* denote the optimal solution. From Theorem 3, for $i \in I$, we have

$$\Upsilon^*\alpha_i = \frac{\sigma_1^2}{N_1^*} + \frac{\sigma_i^2}{N_i^*} - \frac{2C_{1i}}{\max(N_1^*, N_i^*)} \geq \frac{\sigma_1^2}{N_1^*} + \frac{\sigma_i^2 - 2C_{1i}}{\max(N_1^*, N_i^*)} > \frac{\sigma_1^2}{N_1^*},$$

hence, we have $M > \sigma_1^2/\alpha_i$, which further leads to $M \geq A_0$. Denote $x = \Upsilon N_1$. Under (4), we only need to consider $x \geq A_0$ and

$$N_i = \begin{cases} \dfrac{\sigma_i^2}{x\alpha_i - \sigma_1^2 + 2C_{1i}} N_1 \leq N_1 & \text{if } A_i \leq x, \\ \dfrac{\sigma_i^2 - 2C_{1i}}{x\alpha_i - \sigma_1^2} N_1 \geq N_1 & \text{otherwise.} \end{cases}$$

Substituting (7) and (15) into (5), we have

$$\left[(b_1 + b_0\mathbf{1}_{K(x)}(1)) + \sum_{i\in J\cup I\setminus I(M)} \frac{b_i\sigma_i^2}{\alpha_i M - \sigma_1^2 + 2C_{1i}} + \sum_{i\in I(M)} \frac{(b_i + b_0\mathbf{1}_{K(x)}(i))(\sigma_i^2 - 2C_{1i})}{\alpha_i M - \sigma_1^2}\right] N_1^* = T, \tag{15}$$

where

$$j \in K(M) = \{i | N_i(M) = \max\{N_m(M), m = 1, \ldots, k\}\}$$

and

$$\left[(b_1 + b_0 \mathbf{1}_{K(x)}(1)) + \sum_{i \in J \cup I \setminus I(x)} \frac{b_i \sigma_i^2}{\alpha_i x - \sigma_1^2 + 2C_{1i}} + \sum_{i \in I(x)} \frac{(b_i + b_0 \mathbf{1}_{K(x)}(i))(\sigma_i^2 - 2C_{1i})}{\alpha_i x - \sigma_1^2} \right] N_1 = T, \tag{16}$$

where

$$j \in K(x) = \{i | N_i(x) = \max\{N_m(x), m = 1, \ldots, k\}\}.$$

In addition, $\forall j \in K(x)$, $h(x)$ is constant (so it is well defined). (15) and (16) are equivalent to

$$\frac{h(M)}{T} = \frac{M}{N_1^*} = \Upsilon^* \quad \text{and} \quad \frac{h(x)}{T} = \frac{x}{N_1} = \Upsilon,$$

respectively. Therefore,

$$\frac{h(M)}{T} = \Upsilon^* \le \Upsilon = \frac{h(x)}{T}.$$

This completes the proof. □

Proof of Theorem 5. Let

$$\begin{aligned}
d_i^{'} &= \begin{cases} b_i(j_n^m)\sigma_i^2(\sigma_1^2 - 2C_{1i})/\alpha_i^2, & i \in \Omega \setminus I_n, \\ b_i(j_n^m)\sigma_1^2(\sigma_i^2 - 2C_{1i})/\alpha_i^2, & i \in I_n; \end{cases} \\
d_i &= \begin{cases} (\sigma_1^2 - 2C_{1i})/\alpha_i, & i \in \Omega \setminus I_n, \\ \sigma_1^2/\alpha_i, & i \in I_n; \end{cases} \\
[h_n^{j_n^m}(x)]^{'} x^2 &= b_1(j_n^m) x^2 - \sum_{i=2}^{k} d_i^{'} - \frac{2 d_i^{'} d_i (x - d_i/2)}{(x - d_i)^2}.
\end{aligned}$$

Differentiating $[h_n^{j_n^m}(x)]^{'} x^2$, we obtain

$$\{[h_n^{j_n^m}(x)]^{'} x^2\}^{'} = 2 b_1(j_n^m) x + \sum_{i=2}^{k} 2 d_i^{'} d_i \frac{x}{(x - d_i)^3}.$$

In addition, for $i \in I_n$, by the definition of A_0, we have $A_{(n)}^{(m)} > A_0 \ge d_i$, and for $i \in \Omega \setminus I_n$, by the definition of A_i. we have $A_{(n)}^{(m)} \ge A_i > d_i$. For all $i \in \Omega$, we have $x > d_i$, $x \in [A_{(n)}^{(m)}, A_{(n)}^{(m+1)}]$, and $\{[h_n^{j_n^m}(x)]^{'} x^2\}^{'} > 0$. Therefore, $[h_n^{j_n^m}(x)]^{'} x^2 = 0$ has at most one solution in $x \in [A_{(n)}^{(m)}, A_{(n)}^{(m+1)}]$. This completes the proof. □

Proof of Theorem 6. If $x > \max_{i\in\Omega}\{A_i + C(\sigma_1^2 \vee |\sigma_1^2 - 2C_{1i}|)/4\alpha_i\}$, for any $i \in \Omega$, we have both

$$\alpha_i x - \sigma_1^2 + 2C_{1i} > \sigma_i^2 + C|\sigma_1^2 - 2C_{1i}|/4 \geq \sqrt{C\sigma_i^2|\sigma_1^2 - 2C_{1i}|},$$

and

$$\alpha_i x - \sigma_1^2 > \sigma_i^2 - 2C_{1i} + C\sigma_1^2/4 \geq \sqrt{C\sigma_1^2(\sigma_i^2 - 2C_{1i})}.$$

Then, for any $i \in \Omega \setminus K(x)$, we have

$$\left|\frac{b_i\sigma_i^2(\sigma_1^2 - 2C_{1i})}{(\alpha_i x - \sigma_1^2 + 2C_{1i})^2}\right| < \frac{b_1 b_i}{\sum_{i\in\Omega} b_i + b_0}$$

and

$$\left|\frac{b_i\sigma_1^2(\sigma_i^2 - 2C_{1i})}{(\alpha_i x - \sigma_1^2)^2}\right| < \frac{b_1 b_i}{\sum_{i\in\Omega} b_i + b_0}$$

For $i_0 \in K(x)$, we have

$$\left|\frac{\sigma_1^2(\sigma_{i_0}^2 - 2C_{1i_0})(b_{i_0} + b_0\mathbf{1}_{K(x)}(i_0))}{(\alpha_{i_0} x - \sigma_1^2)^2}\right| < \frac{b_1(b_0 + b_{i_0})}{\sum_{i\in\Omega} b_{i_0} + b_0}.$$

Therefore

$$\begin{aligned} h'(x) > b_1 &- \left|\frac{\sigma_1^2(\sigma_{i_0}^2 - 2C_{1i_0})(b_0 + b_{i_0})}{(\alpha_i x - \sigma_1^2)^2}\right| - \left|\sum_{i\in\Omega\setminus I(x)} \frac{b_i\sigma_i^2(\sigma_1^2 - 2C_{1i})}{(\alpha_i x - \sigma_1^2 + 2C_{1i})^2}\right| \\ &- \left|\sum_{i\in I(x)\setminus\{i_0\}} \frac{b_i\sigma_1^2(\sigma_i^2 - 2C_{1i})}{(\alpha_i x - \sigma_1^2)^2}\right| \\ &> b_1 - \frac{b_1(b_0 + b_{i_0})}{\sum_{i\in\Omega} b_i + b_0} - \sum_{i\in\Omega\setminus I(x)} \frac{b_1 b_i}{\sum_{i\in\Omega} b_i + b_0} - \sum_{i\in I(x)\setminus\{i_0\}} \frac{b_1 b_i}{\sum_{i\in\Omega} b_i + b_0} = 0. \qquad \square \end{aligned}$$

Appendix B: More Numerical Results

Additional tests with Example 1 ($\tilde{J}_{im} \sim N(10-i, 6^2), i = 1, 2, \cdots, 10$) for $n_0 = 5, 10, 15, 20, 25$. In all scenarios, GBA outperforms the other methods for the case of unequal and sharing simulation budget, in particular, when correlations are not too low, and the PCS for GBA is relatively insensitive to changes in parameters $\{b_i\}$ compared with the other methods. As can be seen in Tables 1 through 10 provided here, the OCBA algorithms work well as long as n_0 is not too small.

We also tested four cases with negative correlation in Example 1, which include Case 1: $\rho = -0.1$, Case 2: $\rho_{1,i} = -0.2,\ \rho_{i,j} = 0.2,\ i, j = 2, .., k,\ i \neq j$, Case 3: $\rho_{1,i} = -0.5,\ \rho_{i,j} = 0.5,\ i, j = 2, .., k,\ i \neq j$, and Case 4: $\rho_{1,i} = -0.9,\ \rho_{i,j} = 0.9,\ i, j = 2, .., k,\ i \neq j$. The results shown in Tables 6 through 10 are very similar to the cases with positive correlation (see Tables 1 through 5). Although the performance of all OCBA algorithms deteriorates for negative correlation, GBA still remains the best among the four methods.

Table 1: Estimated PCS with $n_0 = 5$ for Example 1

	$b_0 = 0$				$b_0 = 1$				$b_0 = 2$			
$\rho =$	*0*	*0.2*	*0.5*	*0.9*	*0*	*0.2*	*0.5*	*0.9*	*0*	*0.2*	*0.5*	*0.9*
	$b_i = 1.5,\ i = 1,..,10$											
GBA	.711	.733	.801	.937	.688	.714	.771	.927	.660	.696	.760	.926
CBA	.716	.733	.793	.938	.680	.720	.772	.928	.658	.694	.760	.922
IBA	.702	.726	.772	.854	.677	.705	.745	.833	.662	.685	.732	.825
EBA	.703	.745	.816	.985	.702	.737	.809	.983	.699	.728	.796	.976
	$b_{2i-1} = 2,\ b_{2i} = 1,\ i = 1,..,5$											
GBA	.721	.741	.795	.940	.690	.715	.785	.925	.666	.702	.757	.921
CBA	.702	.726	.781	.919	.679	.697	.760	.882	.648	.686	.737	.864
IBA	.696	.723	.753	.848	.673	.693	.730	.810	.646	.676	.719	.804
EBA	.707	.735	.779	.862	.685	.709	.761	.841	.678	.693	.743	.826
	$b_{2i-1} = 1,\ b_{2i} = 2,\ i = 1,..,5$											
GBA	.707	.737	.800	.941	.688	.714	.777	.933	.654	.698	.765	.924
CBA	.725	.756	.796	.899	.696	.716	.766	.878	.662	.696	.749	.855
IBA	.728	.742	.772	.840	.701	.713	.751	.818	.665	.691	.724	.785
EBA	.724	.753	.789	.857	.710	.735	.766	.842	.693	.706	.752	.829
	$b_i \sim U(1,2),\ i = 1,..,10$											
GBA	.673	.703	.756	.923	.690	.722	.780	.936	.670	.695	.759	.927
CBA	.658	.689	.747	.896	.675	.710	.769	.923	.659	.686	.749	.905
IBA	.661	.679	.727	.813	.677	.700	.743	.839	.660	.679	.723	.820
EBA	.687	.717	.770	.892	.703	.729	.791	.900	.684	.724	.768	.892

Table 2: Estimated PCS with $n_0 = 10$ for Example 1

	$b_0 = 0$				$b_0 = 1$				$b_0 = 2$			
$\rho =$	*0*	*0.2*	*0.5*	*0.9*	*0*	*0.2*	*0.5*	*0.9*	*0*	*0.2*	*0.5*	*0.9*
	$b_i = 1.5,\ i = 1, .., 10$											
GBA	.778	.811	.868	.974	.756	.777	.848	.969	.730	.766	.828	.962
CBA	.773	.812	.866	.973	.759	.782	.844	.965	.723	.761	.828	.965
IBA	.775	.799	.853	.914	.755	.770	.828	.909	.728	.757	.809	.896
EBA	.704	.748	.817	.984	.701	.733	.811	.983	.697	.727	.799	.976
	$b_{2i-1} = 2,\ b_{2i} = 1,\ i = 1, .., 5$											
GBA	.783	.818	.864	.974	.748	.781	.837	.964	.731	.767	.826	.963
CBA	.760	.806	.846	.940	.735	.764	.822	.910	.711	.753	.799	.895
IBA	.769	.802	.839	.901	.741	.767	.807	.874	.712	.744	.787	.860
EBA	.705	.729	.781	.854	.691	.716	.763	.842	.671	.702	.743	.838
	$b_{2i-1} = 1,\ b_{2i} = 2,\ i = 1, .., 5$											
GBA	.784	.810	.875	.970	.746	.782	.847	.970	.733	.760	.827	.963
CBA	.797	.807	.857	.932	.751	.776	.823	.908	.727	.754	.800	.882
IBA	.798	.810	.850	.897	.752	.778	.816	.873	.723	.743	.788	.846
EBA	.720	.749	.784	.848	.696	.731	.777	.847	.683	.718	.760	.829
	$b_i \sim U(1, 2),\ i = 1, .., 10$											
GBA	.787	.816	.870	.971	.752	.782	.848	.968	.738	.773	.839	.965
CBA	.784	.807	.865	.961	.749	.774	.841	.949	.729	.757	.822	.934
IBA	.780	.808	.851	.912	.746	.777	.818	.894	.736	.754	.799	.880
EBA	.709	.740	.801	.908	.702	.732	.781	.899	.692	.706	.777	.890

Table 3: Estimated PCS with $n_0 = 15$ for Example 1

	$b_0 = 0$				$b_0 = 1$				$b_0 = 2$			
$\rho =$	*0*	*0.2*	*0.5*	*0.9*	*0*	*0.2*	*0.5*	*0.9*	*0*	*0.2*	*0.5*	*0.9*
	$b_i = 1.5,\ i = 1,..,10$											
GBA	.801	.831	.896	.982	.776	.799	.870	.982	.753	.791	.852	.980
CBA	.798	.834	.894	.982	.773	.803	.866	.982	.752	.786	.854	.979
IBA	.792	.830	.879	.939	.777	.795	.853	.939	.743	.781	.834	.934
EBA	.709	.748	.822	.982	.693	.738	.811	.982	.685	.727	.797	.975
	$b_{2i-1} = 2,\ b_{2i} = 1,\ i = 1,..,5$											
GBA	.797	.835	.890	.982	.772	.812	.868	.981	.748	.786	.848	.978
CBA	.796	.827	.879	.944	.767	.800	.839	.917	.743	.765	.819	.907
IBA	.792	.827	.869	.923	.760	.793	.841	.899	.742	.766	.818	.892
EBA	.704	.732	.778	.857	.684	.713	.766	.841	.667	.695	.746	.831
	$b_{2i-1} = 1,\ b_{2i} = 2,\ i = 1,..,5$											
GBA	.806	.834	.895	.985	.764	.801	.868	.979	.750	.802	.852	.980
CBA	.809	.837	.877	.939	.764	.796	.846	.917	.743	.774	.815	.906
IBA	.804	.839	.872	.923	.765	.794	.839	.893	.743	.771	.808	.878
EBA	.727	.744	.795	.856	.703	.731	.777	.846	.684	.712	.758	.831
	$b_i \sim U(1,2),\ i = 1,..,10$											
GBA	.813	.835	.895	.984	.778	.813	.875	.981	.757	.785	.857	.983
CBA	.808	.832	.886	.969	.775	.805	.860	.958	.752	.777	.838	.946
IBA	.813	.834	.881	.940	.780	.803	.851	.925	.749	.771	.833	.917
EBA	.709	.739	.794	.912	.694	.727	.790	.898	.679	.712	.774	.891

Table 4: Estimated PCS with $n_0 = 20$ for Example 1

	$b_0 = 0$				$b_0 = 1$				$b_0 = 2$			
$\rho =$	*0*	*0.2*	*0.5*	*0.9*	*0*	*0.2*	*0.5*	*0.9*	*0*	*0.2*	*0.5*	*0.9*
	$b_i = 1.5,\ i = 1,..,10$											
GBA	.797	.837	.898	.989	.778	.808	.870	.988	.753	.785	.856	.985
CBA	.798	.840	.894	.991	.774	.809	.876	.989	.747	.784	.853	.986
IBA	.802	.839	.885	.959	.783	.804	.863	.960	.750	.788	.850	.962
EBA	.702	.746	.819	.984	.700	.741	.810	.979	.690	.726	.799	.977
	$b_{2i-1} = 2,\ b_{2i} = 1,\ i = 1,..,5$											
GBA	.802	.836	.893	.989	.767	.805	.878	.989	.757	.780	.851	.984
CBA	.798	.828	.881	.944	.765	.798	.845	.925	.744	.771	.824	.909
IBA	.799	.826	.873	.938	.768	.801	.842	.916	.746	.768	.821	.903
EBA	.701	.728	.777	.859	.684	.716	.768	.843	.669	.700	.738	.830
	$b_{2i-1} = 1,\ b_{2i} = 2,\ i = 1,..,5$											
GBA	.806	.829	.897	.991	.775	.806	.871	.990	.746	.787	.855	.986
CBA	.805	.826	.877	.941	.769	.799	.840	.920	.743	.770	.821	.904
IBA	.804	.831	.877	.931	.768	.796	.843	.910	.735	.770	.813	.897
EBA	.712	.745	.784	.858	.705	.729	.772	.841	.690	.716	.765	.833
	$b_i \sim U(1,2),\ i = 1,..,10$											
GBA	.816	.846	.902	.990	.776	.816	.881	.988	.751	.788	.850	.988
CBA	.813	.843	.896	.976	.769	.802	.869	.961	.750	.788	.839	.948
IBA	.811	.843	.894	.951	.778	.806	.865	.943	.753	.784	.839	.933
EBA	.706	.736	.806	.913	.706	.727	.787	.898	.684	.713	.783	.894

Table 5: Estimated PCS with $n_0 = 25$ for Example 1

	$b_0 = 0$				$b_0 = 1$				$b_0 = 2$			
$\rho =$	*0*	*0.2*	*0.5*	*0.9*	*0*	*0.2*	*0.5*	*0.9*	*0*	*0.2*	*0.5*	*0.9*
	$b_i = 1.5,\ i = 1, .., 10$											
GBA	.792	.823	.889	.994	.754	.790	.862	.992	.735	.766	.835	.987
CBA	.784	.826	.887	.993	.757	.789	.859	.991	.725	.762	.831	.986
IBA	.787	.822	.884	.978	.756	.792	.851	.976	.727	.761	.830	.980
EBA	.703	.745	.819	.984	.702	.735	.806	.983	.691	.727	.803	.978
	$b_{2i-1} = 2,\ b_{2i} = 1,\ i = 1, .., 5$											
GBA	.798	.821	.890	.993	.759	.792	.858	.993	.721	.764	.837	.987
CBA	.790	.823	.873	.942	.760	.784	.842	.937	.727	.761	.816	.925
IBA	.795	.819	.872	.943	.756	.785	.836	.934	.729	.767	.815	.932
EBA	.702	.725	.782	.867	.681	.710	.747	.839	.662	.695	.731	.835
	$b_{2i-1} = 1,\ b_{2i} = 2,\ i = 1, .., 5$											
GBA	.792	.823	.889	.992	.760	.792	.863	.990	.732	.770	.833	.988
CBA	.794	.821	.869	.938	.755	.786	.837	.922	.719	.761	.806	.918
IBA	.797	.827	.873	.932	.757	.776	.838	.923	.719	.761	.808	.919
EBA	.733	.751	.797	.854	.704	.723	.777	.842	.692	.714	.760	.826
	$b_i \sim U(1, 2),\ i = 1, .., 10$											
GBA	.802	.834	.894	.994	.768	.799	.869	.992	.733	.773	.848	.988
CBA	.805	.836	.882	.972	.760	.795	.856	.965	.733	.767	.827	.954
IBA	.802	.835	.884	.961	.762	.798	.860	.955	.734	.760	.829	.952
EBA	.714	.747	.801	.911	.695	.734	.778	.894	.688	.723	.776	.887

We define four cases for negative correlation.

Case 1: $\rho = -0.1$;

Case 2: $\rho_{1,i} = -0.2,\ \rho_{i,j} = 0.2,\ i, j = 2, .., k,\ i \neq j$;

Case 3: $\rho_{1,i} = -0.5,\ \rho_{i,j} = 0.5,\ i, j = 2, .., k,\ i \neq j$;

Case 4: $\rho_{1,i} = -0.9,\ \rho_{i,j} = 0.9,\ i, j = 2, .., k,\ i \neq j$.

Table 6: Estimated PCS with $n_0 = 5$ for Example 1

	$b_0 = 0$				$b_0 = 1$				$b_0 = 2$			
Case	*1*	*2*	*3*	*4*	*1*	*2*	*3*	*4*	*1*	*2*	*3*	*4*
	$b_i = 1.5,\ i = 1, .., 10$											
GBA	.703	.717	.729	.773	.673	.694	.710	.742	.655	.677	.684	.729
CBA	.703	.712	.728	.766	.670	.685	.709	.749	.652	.671	.683	.723
IBA	.707	.714	.710	.697	.679	.677	.681	.676	.661	.665	.658	.657
EBA	.695	.726	.731	.757	.694	.711	.725	.739	.683	.703	.717	.730
	$b_{2i-1} = 2,\ b_{2i} = 1,\ i = 1, .., 5$											
GBA	.704	.718	.728	.764	.675	.691	.705	.742	.661	.678	.683	.720
CBA	.690	.712	.721	.763	.657	.673	.699	.733	.638	.657	.674	.704
IBA	.691	.694	.701	.695	.661	.669	.671	.660	.646	.658	.655	.647
EBA	.692	.703	.716	.727	.674	.695	.711	.726	.659	.673	.695	.709
	$b_{2i-1} = 1,\ b_{2i} = 2,\ i = 1, .., 5$											
GBA	.705	.715	.733	.766	.675	.695	.709	.744	.655	.670	.693	.726
CBA	.719	.734	.738	.774	.687	.704	.715	.742	.656	.677	.691	.721
IBA	.723	.713	.718	.708	.693	.696	.686	.680	.658	.678	.668	.658
EBA	.709	.735	.743	.750	.695	.711	.727	.730	.673	.701	.713	.719
	$b_i \sim U(1, 2),\ i = 1, .., 10$											
GBA	.709	.718	.740	.776	.670	.699	.719	.745	.659	.678	.689	.731
CBA	.703	.720	.735	.781	.665	.684	.708	.747	.644	.669	.682	.719
IBA	.705	.702	.708	.708	.667	.683	.676	.668	.651	.664	.650	.657
EBA	.702	.722	.733	.754	.692	.707	.713	.733	.675	.693	.707	.728

Table 7: Estimated PCS with $n_0 = 10$ for Example 1

	$b_0 = 0$				$b_0 = 1$				$b_0 = 2$			
Case	*1*	*2*	*3*	*4*	*1*	*2*	*3*	*4*	*1*	*2*	*3*	*4*
	$b_i = 1.5,\ i = 1, .., 10$											
GBA	.759	.784	.792	.811	.739	.760	.759	.787	.724	.741	.737	.770
CBA	.760	.785	.792	.810	.738	.754	.762	.788	.714	.732	.739	.760
IBA	.757	.772	.773	.759	.736	.746	.742	.730	.720	.731	.719	.714
EBA	.689	.720	.736	.743	.693	.715	.722	.744	.671	.706	.718	.734
	$b_{2i-1} = 2,\ b_{2i} = 1,\ i = 1, .., 5$											
GBA	.771	.783	.786	.808	.745	.748	.764	.786	.724	.747	.736	.758
CBA	.763	.778	.780	.807	.732	.738	.752	.775	.706	.733	.730	.746
IBA	.760	.767	.759	.748	.738	.730	.730	.717	.709	.725	.715	.699
EBA	.688	.712	.720	.742	.672	.693	.698	.709	.662	.677	.691	.714
	$b_{2i-1} = 1,\ b_{2i} = 2,\ i = 1, .., 5$											
GBA	.757	.778	.791	.819	.740	.755	.757	.788	.719	.735	.740	.774
CBA	.773	.791	.795	.826	.744	.760	.755	.786	.721	.736	.746	.767
IBA	.775	.783	.775	.767	.743	.749	.740	.735	.725	.725	.725	.730
EBA	.703	.729	.734	.745	.688	.712	.713	.731	.673	.689	.707	.713
	$b_i \sim U(1, 2),\ i = 1, .., 10$											
GBA	.772	.788	.790	.823	.751	.761	.760	.792	.712	.743	.747	.779
CBA	.772	.783	.793	.820	.745	.748	.761	.786	.705	.734	.744	.764
IBA	.765	.782	.773	.765	.743	.741	.730	.733	.702	.726	.728	.714
EBA	.707	.723	.735	.750	.690	.701	.719	.736	.677	.687	.701	.728

Table 8: Estimated PCS with $n_0 = 15$ for Example 1

	$b_0 = 0$				$b_0 = 1$				$b_0 = 2$			
Case	*1*	*2*	*3*	*4*	*1*	*2*	*3*	*4*	*1*	*2*	*3*	*4*
	$b_i = 1.5,\ i = 1,..,10$											
GBA	.789	.803	.808	.827	.753	.775	.782	.799	.739	.754	.761	.779
CBA	.791	.801	.811	.823	.757	.776	.781	.797	.728	.753	.763	.772
IBA	.791	.796	.783	.783	.766	.768	.766	.753	.731	.746	.746	.741
EBA	.703	.718	.734	.748	.690	.717	.717	.743	.673	.693	.713	.735
	$b_{2i-1} = 2,\ b_{2i} = 1,\ i = 1,..,5$											
GBA	.793	.796	.802	.822	.755	.776	.784	.788	.735	.748	.768	.769
CBA	.778	.796	.810	.817	.746	.768	.763	.783	.726	.748	.755	.767
IBA	.782	.788	.785	.773	.741	.764	.755	.739	.725	.730	.739	.733
EBA	.702	.707	.720	.730	.674	.705	.706	.714	.656	.687	.689	.712
	$b_{2i-1} = 1,\ b_{2i} = 2,\ i = 1,..,5$											
GBA	.790	.807	.810	.831	.763	.782	.790	.803	.735	.760	.758	.779
CBA	.797	.807	.812	.827	.759	.781	.779	.798	.731	.748	.752	.766
IBA	.797	.806	.793	.789	.762	.777	.763	.753	.733	.743	.734	.736
EBA	.712	.730	.737	.738	.683	.711	.724	.739	.672	.701	.706	.727
	$b_i \sim U(1,2),\ i = 1,..,10$											
GBA	.799	.808	.809	.832	.756	.779	.783	.803	.739	.756	.764	.779
CBA	.791	.804	.816	.835	.759	.779	.775	.792	.735	.747	.756	.773
IBA	.794	.799	.792	.789	.761	.771	.763	.754	.735	.748	.745	.730
EBA	.702	.718	.730	.753	.686	.709	.715	.737	.674	.699	.716	.717

Table 9: Estimated PCS with $n_0 = 20$ for Example 1

	$b_0 = 0$				$b_0 = 1$				$b_0 = 2$			
Case	*1*	*2*	*3*	*4*	*1*	*2*	*3*	*4*	*1*	*2*	*3*	*4*
	$b_i = 1.5,\ i = 1, .., 10$											
GBA	.786	.807	.810	.830	.762	.780	.776	.793	.730	.755	.758	.772
CBA	.793	.812	.813	.823	.762	.772	.776	.791	.729	.753	.753	.769
IBA	.793	.802	.800	.790	.751	.774	.772	.770	.730	.752	.746	.748
EBA	.690	.731	.739	.757	.693	.715	.727	.742	.674	.702	.709	.730
	$b_{2i-1} = 2,\ b_{2i} = 1,\ i = 1, .., 5$											
GBA	.789	.807	.799	.824	.762	.777	.786	.789	.735	.756	.753	.770
CBA	.793	.805	.800	.816	.754	.774	.781	.785	.733	.746	.748	.762
IBA	.789	.802	.789	.784	.749	.768	.771	.753	.731	.744	.737	.736
EBA	.693	.703	.726	.737	.679	.694	.706	.711	.662	.684	.691	.705
	$b_{2i-1} = 1,\ b_{2i} = 2,\ i = 1, .., 5$											
GBA	.795	.814	.817	.828	.750	.771	.784	.797	.736	.750	.760	.771
CBA	.793	.814	.807	.820	.761	.770	.782	.784	.732	.751	.753	.767
IBA	.792	.803	.808	.797	.752	.767	.763	.759	.727	.743	.743	.746
EBA	.715	.728	.737	.749	.692	.718	.718	.735	.679	.697	.713	.722
	$b_i \sim U(1, 2),\ i = 1, .., 10$											
GBA	.795	.823	.811	.834	.760	.785	.785	.796	.730	.761	.766	.774
CBA	.800	.815	.815	.836	.766	.779	.785	.796	.730	.760	.762	.779
IBA	.801	.811	.801	.803	.761	.783	.772	.774	.742	.756	.749	.745
EBA	.699	.728	.725	.743	.692	.713	.709	.733	.673	.692	.703	.731

Table 10: Estimated PCS with $n_0 = 25$ for Example 1

	$b_0 = 0$				$b_0 = 1$				$b_0 = 2$			
Case	*1*	*2*	*3*	*4*	*1*	*2*	*3*	*4*	*1*	*2*	*3*	*4*
	$b_i = 1.5,\ i = 1,..,10$											
GBA	.782	.796	.801	.820	.750	.760	.770	.779	.719	.740	.750	.758
CBA	.774	.786	.804	.813	.740	.759	.770	.781	.720	.741	.744	.760
IBA	.781	.786	.789	.786	.746	.763	.763	.762	.719	.739	.743	.749
EBA	.690	.726	.732	.745	.684	.704	.726	.739	.682	.698	.715	.727
	$b_{2i-1} = 2,\ b_{2i} = 1,\ i = 1,..,5$											
GBA	.780	.800	.806	.807	.737	.759	.764	.774	.715	.739	.741	.753
CBA	.781	.802	.795	.801	.741	.756	.766	.777	.721	.737	.739	.748
IBA	.775	.796	.790	.791	.734	.754	.750	.761	.715	.740	.731	.739
EBA	.688	.718	.721	.734	.675	.693	.702	.716	.661	.681	.691	.703
	$b_{2i-1} = 1,\ b_{2i} = 2,\ i = 1,..,5$											
GBA	.768	.793	.802	.813	.739	.754	.771	.786	.718	.728	.746	.763
CBA	.779	.798	.798	.807	.734	.758	.761	.785	.713	.725	.734	.757
IBA	.788	.795	.793	.784	.733	.762	.759	.760	.711	.721	.741	.742
EBA	.715	.731	.736	.744	.693	.708	.722	.727	.682	.697	.707	.727
	$b_i \sim U(1, 2),\ i = 1,..,10$											
GBA	.789	.801	.803	.817	.752	.756	.773	.783	.711	.744	.755	.763
CBA	.787	.805	.807	.816	.750	.763	.772	.784	.718	.742	.749	.761
IBA	.788	.800	.798	.800	.750	.763	.762	.770	.715	.737	.746	.748
EBA	.704	.728	.735	.754	.686	.711	.714	.732	.680	.692	.712	.731