ABSTRACT

Title of dissertation: BEYOND ORIENTABILITY AND COMPACTNESS: NEW RESULTS ON THE DYNAMICS OF FLAT SURFACES

Rodrigo Treviño, Doctor of Philosophy, 2012

Dissertation directed by: Professor Giovanni Forni
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In the first part, we prove the non-uniform hyperbolicity of the Kontsevich-Zorich cocycle for a measure supported on abelian differentials which come from non-orientable quadratic differentials. The proof uses Forni's criterion for non-uniform hyperbolicity of the cocycle for $SL(2, \mathbb{R})$-invariant measures. We apply these results to the study of deviations in homology of typical leaves of the vertical and horizontal (non-orientable) foliations and deviations of ergodic averages.

In the second part, we prove an ergodic theorem for flat surfaces of finite area whose Teichmuller orbits are recurrent to a compact set of $SL(2, \mathbb{R})/SL(S, \alpha)$, where $SL(S, \alpha)$ is the Veech group of the surface. In this setting, this means that the translation flow on a flat surface can be renormalized through its Veech group. This result applies in particular to flat surfaces of infinite genus and finite area, and we apply our result to existing surfaces in the literature to prove that the corresponding foliations of the surface corresponding to a periodic or recurrent Teichmuller orbit are ergodic.

Beyond orientability and compactness:
new results on the dynamics of flat surfaces

by

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To my dad, my first science hero.

Acknowledgments

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Chapter 1

Introduction

Flat surfaces are weird objects. At first glance, they seem rigid and boring. Luckily, the apparent rigidity is a manifestation of very deep and beautiful properties they possess. This thesis is about flat surfaces and some of their beautiful and deep properties.

It has two parts and they are completely independent of each other and self-contained. The first part, Chapter 1, was a wrinkle in the now-fully-developed and well-understood theory of dynamics of compact flat surfaces. It seemed like nobody wanted to bother to straighten out this small wrinkle enough to do it, so they waited for a graduate student to come around looking for a problem to straighten it out. Still, the results are cute and a bit surprising, so it is not an entirely boring wrinkle to iron out and it seems to be one of the last missing pieces of a puzzle to be put in place.

A harder puzzle to figure out is a theory about dynamics of non-compact flat surfaces, which really means flat surfaces of infinite genus. This has become quite a popular research trend in the last few years, and yet everyone remains clueless as to what such a theory should look like. This being so, the second part of the thesis, Chapter 2, seems more exciting as the results are at the forefront of the state of the art and they expand the extent of what is known about the dynamics of flat surfaces of infinite genus. In fact, the point of view and results of the second part have yielded many further questions and research directions, making it harder for this thesis to die in oblivion any time in the near future.

Teichmüller theory and *Teichmüller dynamics* are now synonymous with the study of flat surfaces and the different dynamical systems that can be associated to them. They are named after Oswald Teichmüller, a German mathematician from the first half of the twentieth century. He proved a beautiful theorem concerning maps on surfaces while taking the first steps in uncovering the rich and deep structure which is carried by flat surfaces and for this the theory carries his name. Although he was a brilliant mathematician, sadly, he was a terrible human being (see [41, page 442]). I wish there was a different name for the field – or that somehow his very impressive mathematical legacy could carry the stains of his personal one. Alas, I could not figure out a clever way to do so, but I hope someone else does.

Chapter 2

The non-uniform hyperbolicity of the Kontsevich-Zorich cocycle

It is well known that the properties of a geodesic foliation (or flow) on a flat 2-torus are completely characterized by its slope, whereas for a flat surface of higher genus the situation is far from similar. Such Riemann surface M of genus greater than one with a flat metric outside finitely many singularities can be given a pair of transverse, measured foliations (in the sense of Thurston). If such foliations are orientable, Zorich [44] detected numerically that homology classes of segments of typical leaves of the foliation deviate from the asymptotic cycle (which is defined as the limit of normalized segments of leaves) in an unprecedented way, and that the rate of deviations are given by the positive Lyapunov exponents of the Kontsevich-Zorich cocycle. Based on numerical experiments, the *Kontsevich-Zorich conjecture* was formulated, which claimed that for Lebesgue-almost all classes of conformally equivalent flat metrics with orientable foliations, the exponents are all distinct and non-zero. In other words, the cocycle is non-uniformly hyperbolic and has a simple spectrum. It was also conjectured that there should be similar deviation phenomena for ergodic averages of functions in some space of functions.

The first proof of the non-uniform hyperbolicity of the Kontsevich-Zorich cocycle came from Forni [17], but the simplicity question remained open for surfaces of genus greater than 2. The full conjecture was finally proved through methods completely different from those of Forni by Avila and Viana [3]. In [17], a complete picture is painted on the deviations of ergodic averages along the straight line flows given by vector fields tangent to the foliations on the flat surface. The rate of divergence of such deviations are also described by all of the Lyapunov exponents of the Kontsevich-Zorich cocycle.

In this chapter we study the same phenomena for the case of non-orientable foliations on flat surfaces. Although there is no vector field to speak of, we can still describe deviations of integrals of functions along leaves of the foliation. Our work has been made substantially easier by the recent criterion of Forni [18], where the proof of non-uniform hyperbolicity in [17] has been condensed and generalized to apply to special $SL(2,\mathbb{R})$-invariant measures in the moduli space of abelian differentials. Note that if one has a flat surface with a non-orientable foliation, one can always pass to a double cover whereon the lift of the foliation becomes orientable. The measure on the moduli space of abelian differentials which is supported on differentials which are the pullback of non-orientable differentials is shown here to satisfy Forni's criterion. Thus most of the work is done in studying how information of the original surface is related to the information on covering surface, which is a solved problem by the works of Zorich and Forni.

The crucial ingredient in Forni's criterion is to show there that exists a point in the support of an $SL(2,\mathbb{R})$-invariant probability measure with a completely periodic foliation whose homology classes of closed leaves span a Lagrangian subspace of the first homology space. We overcome this by a much stronger statement, showing that these special points are in fact *dense* in the moduli space. We are very interested to see what the tools from generalized permutations can say to this end.

There is a canonically defined involution on the orienting double cover corresponding to the choice of orientation of the covering foliations. The involution splits the bundle on which the Kontsevich-Zorich cocycle acts into invariant and anti-invariant sub-bundles, corresponding to eigenvalues ± 1 of map induced by the involution. The Kontsevich-Zorich cocycle respects such splitting, defines two cocycles by its restriction to the invariant and anti-invariant sub-bundles, and thus the spectrum of the cocycle can be written as the spectrum of those two cocycles. Unlike the case for abelian differentials, *the exponents which describe the deviations in homology are not the same exponents which describe the deviations of ergodic averages*, and vice-versa. Specifically, the Lyapunov exponents of the cocycle restricted to the invariant sub-bundle describe the deviations in homology of typical leaves of non-orientable foliations while the exponents of the cocycle restricted to the anti-invariant sub-bundle describe the deviations of averages of functions along leaves of non-orientable foliations. Since for any genus g surface the anti-invariant sub-bundle can have arbitrarily large dimension (due to the presence of simple poles), there are non-orientable foliations on a genus g surface on which the deviation of the ergodic averages along its leaves are described by arbitrarily many parameters.

Like in the original proof for abelian differentials, the proof here cannot address the question of simplicity of the Lyapunov spectrum of the cocycle. Since the restriction of the cocycle to the invariant part is equivalent to the cocycle over the moduli space of non-orientable quadratic differentials and since the anti-invariant sub-bundle describes the deviations of ergodic averages, there is no reason a-priori of why the spectrum of the cocycle over the moduli space of non-orientable quadratic differentials describes the deviations of averages of functions along leaves of non-orientable foliations defined by such quadratic differentials. Thus, unless there is some repetition of exponents across the invariant/anti-invariant division, the cocycle over the space of non-orientable quadratic differentials does not say anything about such averages. In our own numerical experiments we have found strong evidence that the spectrum of the cocycle is in fact simple.

The chapter is organized as follows. In Section 2.1 we review the necessary material for quadratic differentials, the double cover construction and the absolutely continuous $SL(2,\mathbb{R})$-invariant ergodic probability measure defined on each stratum of the moduli space of quadratic differentials. In Section 2.2 we define the Kontsevich-Zorich cocycle and state Forni's criterion for the non-uniform hyperbolicity of the cocycle. In Section 2.3 we show that the measure supported on abelian differentials which come from non-orientable differentials through the double cover construction satisfy Forni's criterion and thus that the Kontsevich-Zorich cocycle is non-uniformly hyperbolic with respect to that measure. In Section 2.4 with study the applications to deviation phenomena of homology classes and ergodic averages. Finally, in the appendix, we summarize our experimental findings of approximating numerically the Lyapunov exponents for different strata, which strongly suggest the simplicity of the cocycle.

2.1 Quadratic Differentials and Flat Surfaces

Let M be a closed, orientable surface of genus g and let $\Sigma_\kappa = \{p_1, \ldots, p_\tau\}$ be a set of points on M with $\kappa = \{n_1, \ldots, n_\tau\}$, $\sum_i n_i = 4g - 4$, and $n_i \in \{-1\} \cup \mathbb{N}$. M is a *half-translation surface* if transitions between charts on $M \backslash \Sigma_\kappa$ are given by functions of the form $\varphi(z) = \pm z + c$ for some constant c. On $M \backslash \Sigma_\kappa$ there is a flat metric for which the points Σ_κ are singularities of order n_i at p_i. On any such surface, we can place a pair of orthogonal foliations $\mathcal{F}^v$ and $\mathcal{F}^h$ which are defined everywhere on $M \backslash \Sigma_\kappa$ and have singularities at Σ_κ.

The same information is carried by a *quadratic differential* on M. A holomorphic quadratic differential assigns to any local coordinate z a quadratic form $q = \phi(z) dz^2$ where $\phi(z)$ has poles of order n_i at p_i. If we represent it as $\phi'(w)$ with respect to another coordinate chart w, then it satisfies $\phi'(w) = \phi(z)(dz/dw)^2$. The foliations are then defined by integrating the distributions $\phi(z)dz^2 > 0$ and $\phi(z)dz^2 < 0$, respectively. In other words,

$$\mathcal{F}_q^v = \ker \operatorname{Re} q^{1/2} \qquad \text{and} \qquad \mathcal{F}_q^h = \ker \operatorname{Im} q^{1/2}$$

are, respectively, the *vertical* and *horizontal* foliations defined by a quadratic differential q. They are measured foliations in the sense of Thurston with respective transverse measures $|\operatorname{Re} q^{1/2}|$ and $|\operatorname{Im} q^{1/2}|$. The flat metric comes from the adapted local coordinates

$$\zeta = \int_p^z \sqrt{\phi(w)}\, dw$$

around any point $p \in M \backslash \Sigma_\kappa$.

A measured foliation $\mathcal{F}$ on a compact surface is called *periodic* if the set of non-closed leaves has measure zero. A quadratic differential whose horizontal foliation is periodic is called a *periodic quadratic differential*. A *saddle connection* is a leaf of the foliation joining two singularities. In the literature, periodic quadratic differentials also go by the name of *Strebel* quadratic differentials.

If a quadratic differential is globally the square of an abelian differential, i.e., a holomorphic 1-form, then the foliations $\mathcal{F}_q^h$ and $\mathcal{F}_q^v$ are orientable and change of coordinates are given by maps of the form $\varphi(z) = z + c$. In this case we speak of a *translation surface*.

Let $\mathcal{H}_g$ be the *moduli space* of abelian differentials on a genus g surface, which is the set of conformally equivalent classes of abelian differentials for a surface M of genus g. The singularities in this case satisfy $\sum_i n_i = 2g - 2$ and the complex dimension of this space is $2g + \tau - 1$. The space $\mathcal{H}_g$ is stratified by the singularity pattern $\kappa = \{n_1, \ldots, n_\tau\}$. As such, the set

$$\mathcal{H}_\kappa = \mathcal{H}_g \cap \{\text{abelian differentials with singularity pattern } \kappa\}$$

is the stratum of all abelian differentials on a genus g surface with singularity pattern $\kappa = \{n_1, \ldots, n_\tau\}$ and $\sum_i n_i = 2g - 2$. We will interchangeably use the terms *abelian differential*, *quadratic differential which is a square of an abelian*, and *orientable quadratic differential* since a quadratic differential q with $\mathcal{F}_q^{v,h}$ orientable is necessarily the square of an abelian differential α and thus we can identify q with α. Note that an orientable quadratic differential has two square roots. Since they are part of the same $SL(2, \mathbb{R})$ orbit, it does not matter which square root, $+$ or $-$, we consider and thus we will by convention always pick $+$. Thus the space of quadratic differentials which are squares of abelian is equally stratified.

The *moduli space of quadratic differentials* $\mathcal{H}_g \coprod \mathcal{Q}_g$ on a Riemann surface M of genus $g \geq 1$ is the quotient of the *Teichmüller space of meromorphic quadratic differentials* with at most simple poles

$$\mathcal{M}_g \equiv \{\text{meromorphic quadratic differentials}\}/\mathrm{Diff}_0^+(M)$$

with respect to the action of the mapping class group Γ_g, where Diff_0^+ denotes the set of orientation preserving diffeomorphisms isotopic to the identity. The subset $\mathcal{Q}_g$ denotes the set of meromorphic quadratic differentials which are *not* the square of abelian differentials. These sets are equally stratified: for some singularity pattern $\kappa = \{n_1, \ldots, n_\tau\}$ with $\sum_i n_i = 4g-4$, $\mathcal{Q}_\kappa$ denotes the set of quadratic differentials on a surface of genus g with singularity pattern κ. Elements of $\mathcal{Q}_\kappa$ will be sometimes called *non-orientable* quadratic differentials since they induce a half-translation structure on M, i.e., non-orientable foliations $\mathcal{F}_q^{v,h}$. Clearly it is necessary for all quadratic differentials in $\mathcal{H}_\kappa$ to have each singularity be of even order, but it is not sufficient. In fact, a result of Masur and Smillie [37] states that for any $\kappa = \{n_1, \ldots, n_\tau\}$ with $\sum_i n_i = 4g-4$ there is a non-orientable quadratic differential $q \in \mathcal{Q}_\kappa$ with such singularity pattern with two exceptions ($\kappa = \{-1, 1\}$ or $\varnothing$) in genus 1 and two exceptions ($\kappa = \{4\}$ or $\{1, 3\}$) in genus two. Additionally, each stratum of $\mathcal{H}_g$ or $\mathcal{Q}_g$ is not necessarily connected. Kontsevich and Zorich [31] have achieved a complete classification of the connected components of each stratum of abelian differentials while Lanneau [33] has classified the connected components of the strata of non-orientable differentials. The space $\mathcal{Q}_\kappa$ has complex dimension $2g + \tau - 2$.

Given any quadratic differential $q \in \mathcal{Q}_\kappa$ on a genus g surface M one can construct a canonical double cover $\pi_\kappa : \hat{M} \to M$ with $\hat{M}$ connected if and only if q is *not* the square of an abelian differential. Moreover, $\pi_\kappa^* q = \hat{\alpha}^2$, where $\hat{\alpha}$ is an abelian differential on $\hat{M}$. The construction can be summarized as follows for a non-orientable differential q. Let (U_i, ϕ_i) be an atlas for $M \backslash \Sigma_\kappa$. For any U_i define $g_i^\pm(z) = \pm\sqrt{\phi_i(z)}$ on the open sets $V_i^\pm$ which are each a copy of U_i. The charts $\{V_i^\pm\}$ can then be glued together in a compatible way and after filling in the holes given by Σ_κ we get the surface $\hat{M}$ with a quadratic differential $\hat{\alpha}^2 = \pi_\kappa^* q$. The surface $\hat{M}$ is an orienting double cover since $\mathcal{F}_q^{v,h}$ for $q \in \mathcal{Q}_g$ lifts to an orientable foliation on $\hat{M}$.

Let κ be written as $\kappa = \{n_1, \ldots, n_\nu, n_{\nu+1}, \ldots, n_\tau\}$ where n_i is odd for $1 \leq i \leq \nu$ and even for $\nu < i \leq \tau$ with $n_1 \leq \cdots \leq n_\nu$. Then the double cover construction gives a local embedding of $\mathcal{Q}_\kappa$ for $\kappa = \{n_1, \ldots, n_\nu, n_{\nu+1}, \ldots, n_\tau\}$ into $\mathcal{H}_{\hat{\kappa}}$, where

$$\hat{\kappa} = \left\{ n_1 + 1, \ldots, n_\nu + 1, \frac{1}{2}n_{\nu+1}, \frac{1}{2}n_{\nu+1}, \ldots, \frac{1}{2}n_\tau, \frac{1}{2}n_\tau \right\}.$$

In the double cover construction, the preimages of the poles become marked points, the odd zeros of q are critical points of π_κ (ramification points) and each even singularity of q has two preimages. The genus $\hat{g}$ of $\hat{M}$ can be computed by the Riemann-Hurwitz formula and satisfies $2\hat{g} = \nu + 4g - 2$.

There is an involution $\sigma : \hat{M} \to \hat{M}$ mapping $\sigma : V_i^\pm \to V_i^\mp$ (that is, interchanging the points on each fiber) and clearly fixing $\pi_\kappa^{-1}\Sigma_\kappa$ as a set. Let

$\hat{\Sigma}_\kappa \equiv \pi_\kappa^{-1}\Sigma_\kappa \backslash \pi_\kappa^{-1}(\{p_1, \ldots, p_{\tau_{-1}}\})$, where $p_1, \ldots, p_{\tau_{-1}}$ are simple poles of the quadratic differential q. The involution induces a splitting on the relative homology and cohomology of $\hat{M}$ into invariant and anti-invariant subspaces. Specifically, there is the following symplectic decomposition

$$H_1(\hat{M}, \hat{\Sigma}_\kappa; \mathbb{R}) = H_1^+(\hat{M}, \hat{\Sigma}_\kappa; \mathbb{R}) \oplus H_1^-(\hat{M}, \hat{\Sigma}_\kappa; \mathbb{R}) \tag{2.1}$$

where the splitting corresponds to the eigenvalues ± 1 of σ_*. There is also a similar symplectic splitting in $H^1(\hat{M}, \hat{\Sigma}_\kappa; \mathbb{R})$:

$$H^1(\hat{M}, \hat{\Sigma}_\kappa; \mathbb{R}) = H^1_+(\hat{M}, \hat{\Sigma}_\kappa; \mathbb{R}) \oplus H^1_-(\hat{M}, \hat{\Sigma}_\kappa; \mathbb{R}).$$

We will denote by $P^\pm = \frac{1}{2}(\mathrm{Id} \pm \sigma_*) : H_1(\hat{M}, \hat{\Sigma}_\kappa; \mathbb{Q}) \to H_1^\pm(\hat{M}, \hat{\Sigma}_\kappa; \mathbb{Q})$ and $P^\pm = \frac{1}{2}(\mathrm{Id} \pm \sigma^*) : H^1(\hat{M}, \hat{\Sigma}_\kappa; \mathbb{Q}) \to H^1_\pm(\hat{M}, \hat{\Sigma}_\kappa; \mathbb{Q})$ the projection to the corresponding eigenspaces in both cases.

A small neighborhood of $[\hat{\alpha}]$ in $H^1_-(\hat{M}, \hat{\Sigma}_\kappa; \mathbb{C})$ gives a local coordinate chart of a regular point q in $\mathcal{Q}_\kappa$. In other words, elements of $H^1_-(\hat{M}, \hat{\Sigma}_\kappa; \mathbb{C})$ are abelian differentials which come from the pull-back of non-orientable quadratic differentials, $[\hat{\alpha}] \in H^1_-(\hat{M}, \hat{\Sigma}; \mathbb{C})$, where $\hat{\alpha} = \sqrt{\pi_\kappa^* q}$. The local charts are given by the period map $q \mapsto [\sqrt{\pi_\kappa^* q}] \in H^1_-(\hat{M}, \hat{\Sigma}_\kappa; \mathbb{C})$.

There is a canonical absolutely continuous invariant measure μ_κ on any stratum $\mathcal{Q}_\kappa$ of the moduli space $\mathcal{Q}_g$ defined as the Lebesgue measure on $H^1_-(\hat{M}, \hat{\Sigma}_\kappa; \mathbb{C})$ normalized so that the quotient torus $H^1_-(\hat{M}, \hat{\Sigma}_\kappa; \mathbb{C})/H^1_-(\hat{M}, \hat{\Sigma}_\kappa; \mathbb{Z} \oplus i\mathbb{Z})$ has volume one. We remark that an analogous canonical absolutely continuous invariant measure ν_κ can be defined for the moduli space $\mathcal{H}_\kappa$ of squares of abelian differentials. Since the period map $q \mapsto [q^{1/2}] \in H^1(M, \Sigma_\kappa; \mathbb{C})$ gives local coordinates to $\mathcal{H}_\kappa$, it is defined in the same way and has the same properties as the measure μ_κ defined on strata of the moduli space of non-orientable quadratic differentials.

The group $SL(2,\mathbb{R})$ acts on quadratic differentials $q \in (\mathcal{H}_g \coprod \mathcal{Q}_g)$ by left multiplication on the (locally defined) vector $(\operatorname{Re} q^{1/2}, \operatorname{Im} q^{1/2})$. More precisely, since local coordinates are given by

$$H^1_-(\hat{M}, \hat{\Sigma}_\kappa; \mathbb{C}) \cong \mathbb{R}^2 \otimes H^1_-(\hat{M}, \hat{\Sigma}_\kappa; \mathbb{R})$$

($H^1(M, \Sigma_\kappa; \mathbb{C})$ in the case of an orientable differential), $SL(2,\mathbb{R})$ acts on $\mathcal{Q}_\kappa$ by multiplication on the first factor. Thus, the measures μ_κ and ν_κ respectively defined on $\mathcal{Q}_\kappa$ and $\mathcal{H}_\kappa$ are $SL(2,\mathbb{R})$-invariant.

The local embedding $i_\kappa : \mathcal{Q}_\kappa \hookrightarrow \mathcal{H}_{\hat{\kappa}}$ defined by the double cover construction induces a map which maps the measure μ_κ to the measure

$$\hat{\mu}_\kappa \equiv (i_\kappa)_* \mu_\kappa \tag{2.2}$$

on $\mathcal{H}_{\hat{\kappa}}$. In general, we expect the measure $\hat{\mu}_\kappa$ is singular with respect to $\nu_{\hat{\kappa}}$ since the support of $\hat{\mu}_\kappa$ is the sub-variety of $\mathcal{Q}_{\hat{\kappa}}$ which is the preimage of the subspace $H^1_-(\hat{M}, \hat{\Sigma}_\kappa; \mathbb{C})$ under the period map. Only in the case of hyperelliptic surfaces we have $\hat{\mu}_\kappa = \nu_{\hat{\kappa}}$. The measure (2.2) is clearly $SL(2,\mathbb{R})$-invariant.

2.2 The Kontsevich-Zorich Cocycle

The action of diagonal subgroup

$$g_t \equiv \left\langle \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\rangle \leq SL(2,\mathbb{R})$$

on $\mathcal{H}_\kappa$ or $\mathcal{Q}_\kappa$ is the *Teichmüller flow* and plays a central role in the study of quadratic differentials. It is was proved by Masur [34] for the principal stratum $\kappa = \{1, \ldots, 1\}$ and then for any stratum by Veech [43] that the Teichmüller flow acts ergodically on each connected component of a stratum with respect to the measure μ_κ (respectively, ν_κ) when restricted to a hypersurface $\mathcal{Q}_\kappa^{(A)} \subset \mathcal{Q}_\kappa$ of quadratic differentials on a surface of area A (respectively, the hypersurface $\mathcal{H}_\kappa^{(A)} \subset \mathcal{H}_\kappa$ of abelian differentials of norm A) and that the measure $\mu_\kappa^{(A)} \equiv \mu_\kappa|_{\mathcal{Q}_\kappa^{(A)}}$ (respectively, $\nu_\kappa^{(A)} \equiv \nu_\kappa|_{\mathcal{H}_\kappa^{(A)}}$) is finite.

The Teichmüller flow g_t admits two invariant foliations $\mathcal{W}^\pm$ on $\mathcal{H}_g$. For an abelian differential $\alpha \in \mathcal{H}_g$, the foliations are locally defined by

$$\begin{aligned}
\mathcal{W}^+(\alpha) &= \{\alpha' \in \mathcal{H}_g : \operatorname{Im}\alpha' \in \mathbb{R}^+ \cdot \operatorname{Im}\alpha\} = \{\alpha' \in \mathcal{H}_g : \mathcal{F}^h_{\alpha'} = [\mathcal{F}^h_\alpha]\} \\
\mathcal{W}^-(\alpha) &= \{\alpha' \in \mathcal{H}_g : \operatorname{Re}\alpha' \in \mathbb{R}^+ \cdot \operatorname{Re}\alpha\} = \{\alpha' \in \mathcal{H}_g : \mathcal{F}^v_{\alpha'} = [\mathcal{F}^v_\alpha]\}.
\end{aligned}$$

Let $\mathcal{W}^\pm_\kappa(\alpha)$ be the intersection of $\mathcal{W}^\pm(\alpha)$ with the stratum $\mathcal{H}_\kappa$. For any open set $\mathcal{U} \subset \mathcal{H}_\kappa$, define the local, invariant foliations $\mathcal{W}^\pm_\mathcal{U}$ as the unique, connected component of the intersection $\mathcal{W}^\pm_\kappa(\alpha) \cap \mathcal{U}$ which contains the abelian differential $\alpha \in \mathcal{U}$.

2.2.1 Definition of the Cocycle

We briefly recall the definition of a cocycle as used in this chapter.

Let X be a metric space and $\phi : X \times \mathbb{G} \to X$ a group action on X by a group $\mathbb{G}$. For $p : V \to X$ a real vector bundle over X of dimension D, a *linear cocycle over* ϕ is a map $\varphi : V \to V$ defined on the base space by ϕ and on fibers by $\varphi : v \mapsto A \cdot v$

where $A : X \times \mathbb{G} \to GL(D, \mathbb{R})$ satisfies the *cocycle condition*

$$A(x, g_1 + g_2) = A(\phi(x, g_1), g_2)A(x, g_1)$$

for any $g_1, g_2 \in \mathbb{G}$.

Let $\mathcal{M}_g$ be the Teichmüller space of meromorphic quadratic differentials on a Riemann surface M of genus $g > 1$. The Kontsevich-Zorich cocycle G_t, introduced in [30], is the quotient cocycle, with respect to the mapping class group Γ_g, of the trivial cocycle

$$g_t \times \mathrm{id} : \mathcal{M}_g \times H^1(M; \mathbb{R}) \longrightarrow \mathcal{M}_g \times H^1(M; \mathbb{R})$$

acting on the orbifold vector bundle

$$\mathcal{H}^1_g(M; \mathbb{R}) \equiv (\mathcal{M}_g \times H^1(M; \mathbb{R}))/\Gamma_g$$

over the moduli space $Q_g \equiv (\mathcal{H}_g \coprod \mathcal{Q}_g) = \mathcal{M}_g/\Gamma_g$ of meromorphic quadratic differentials. Note that we can identify fibers of close points using the Gauss-Manin connection. The projection of the cocycle G_t coincides with the Teichmüller flow g_t on the moduli space Q_g.

By the Oseledets Multiplicative Ergodic Theorem for linear cocycles [28], for a g_t-invariant probability measure μ supported on some stratum of Q_g there is a decomposition μ-almost everywhere of the cohomology bundle $H^1_q(M; \mathbb{R}) = E^+(q) \oplus E^-(q) \oplus E_0(q)$ where

$$E^\pm(q) = E^\pm_1(q) \oplus \cdots \oplus E^\pm_{s^\pm}(q) \tag{2.3}$$

and Lyapunov exponents $\lambda^+_1 > \cdots > \lambda^+_{s^+} > 0 > \lambda^-_1 > \cdots > \lambda^-_{s^-}$ which describe the exponential rate of expansion and contraction of elements in such sub-bundles under

G_t. Elements of E_0 have zero exponential expansion or contraction. The dimension of each sub-bundle $E_i^{\pm}$ in (2.3) is exactly the multiplicity of $\lambda_i^{\pm}$.

It follows from the fact that G_t is a symplectic cocycle that the Lyapunov spectrum of the cocycle G_t, with respect to any g_t-invariant ergodic probability measure, is symmetric. In other words, if λ is a Lyapunov exponent of G_t, so is $-\lambda$ and $\dim E^+ = \dim E^-$. Thus, the Lyapunov exponents for the Kontsevich-Zorich cocycle satisfy

$$1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_g \geq 0 \geq -\lambda_g = \lambda_{g+1} \geq \cdots \geq \lambda_{2g-1} \geq \lambda_{2g} = -1. \qquad (2.4)$$

Since the period map identifies the tangent space of Q_g to the cohomology space, there is a relationship between the Lyapunov exponents of the Kontsevich-Zorich cocycle and those of the tangent cocycle of the Teichmüller flow. Since we can express the local trivialization of the tangent bundle as $TQ_\kappa = Q_\kappa \times H^1(M, \hat{\Sigma}_\kappa; \mathbb{C})$ ($Q_\kappa \times H^1_-(M, \Sigma_\kappa; \mathbb{C})$ when G_t acts on strata of non-orientable differentials), then by the isomorphism of the vector bundles

$$\mathcal{H}^1_\kappa(M, \mathbb{C}) \cong \mathbb{C} \otimes H^1(M; \mathbb{R}) \cong \mathbb{R}^2 \otimes H^1(M; \mathbb{R})$$

induced by the isomorphism on each fiber, the projection of Tg_t to the absolute cohomology can be expressed in terms of the Kontsevich-Zorich cocycle as

$$Tg_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \otimes G_t \qquad \text{acting on} \qquad \mathbb{R}^2 \otimes H^1(M; \mathbb{R}).$$

Thus, the Lyapunov exponents of the Teichmüller flow with respect to the canonical, absolutely continuous measures μ_κ or ν_κ can be written as

$$\begin{aligned} 2 &\geq (1+\lambda_2) \geq \cdots \geq (1+\lambda_g) \geq \overbrace{1 = \cdots = 1}^{\tau-1} \geq (1-\lambda_g) \\ &\geq \cdots \geq (1-\lambda_2) \geq 0 \geq -(1-\lambda_2) \geq \cdots \geq -(1-\lambda_g) \\ &\geq \underbrace{-1 = \cdots = -1}_{\tau-1} \geq -(1+\lambda_g) \geq \cdots \geq -(1+\lambda_2) \geq -2. \end{aligned}$$

where the $\tau - 1$ trivial exponents come from cycles relative to Σ_κ.

The trivial exponents of the tangent cocycle Tg_t are neglected by G_t since the bundle $\mathcal{H}^1_g$ neglects cocycles in $H^1(M, \Sigma_\kappa; \mathbb{C})$ which are dual to cycles relative to Σ_κ, from which we get such trivial exponents. The non-uniform hyperbolicity of the tangent cocycle for the Teichmüller flow is equivalent to the spectral gap of the Kontsevich-Zorich cocycle, i.e., that $\lambda_1 > \lambda_2$. This was proved by Veech [43] for the canonical measure and then by Forni in [17] for any Teichmüller invariant ergodic probability measure μ in $\mathcal{H}_g$.

Let $\hat{q} = i_\kappa(q) \in \mathcal{H}_{\hat{\kappa}}$ be an orientable quadratic differential which is obtained by the double cover construction. The splitting $H^1(\hat{M}; \mathbb{R}) = H^1_+ \oplus H^1_-$ is equivariant with respect to the Gauss-Manin connection. Since both H^1_+ and H^1_- are symplectic subspaces, the restriction of the Kontsevich-Zorich cocycle to either the invariant or anti-invariant sub-bundles defines another symplectic cocycle. Thus we get symmetric Lyapunov spectra

$$\lambda_1^+ \geq \lambda_2^+ \geq \cdots \geq \lambda_g^+ \geq 0 \geq -\lambda_g^+ = \lambda_{g+1}^+ \geq \cdots \geq \lambda_{2g}^+$$

and

$$\lambda_1^- \geq \lambda_2^- \geq \cdots \geq \lambda_{g+n-1}^- \geq 0 \geq -\lambda_{g+n-1}^- = \lambda_{g+n}^- \geq \cdots \geq \lambda_{2g+2n-2}^-$$

which are, respectively, the Lyapunov exponents of the symplectic cocycles of the invariant and anti-invariant sub-bundles.

It follows from the double cover construction that the action of g_t commutes with i_κ. Moreover, since π_κ^* is an isomorphism between $H^1(M;\mathbb{R})$ and $H^1_+(\hat{M};\mathbb{R})$,

$$(i_\kappa \times \pi_\kappa^*) \circ (g_t|_{\mathcal{Q}_\kappa} \times \mathrm{id}) = (g_t|_{\mathcal{H}_{\hat{\kappa}}} \times \mathrm{id}) \circ (i_\kappa \times \pi_\kappa^*), \tag{2.5}$$

and thus the Lyapunov spectrum of the Kontsevich-Zorich cocycle on the bundle over $i_\kappa(\mathcal{Q}_\kappa)$ restricted to the invariant sub-bundle is the same as the Lyapunov spectrum of the Kontsevich-Zorich cocycle on the bundle over $\mathcal{Q}_\kappa$.

2.2.2 A Criterion for Non-Uniform Hyperbolicity

The non-uniform hyperbolicity of the Kontsevich-Zorich cocycle for the canonical, absolutely continuous measure on $\mathcal{H}_\kappa$ was first proved by Forni in [17]. Recently, the proof of such result has been generalized in [18] to apply to *any* $SL(2,\mathbb{R})$-invariant ergodic probability measure on $\mathcal{H}_\kappa$ which have special points in their support. In this section we review the necessary material to state Forni's criterion.

Definition 1. An open set $\mathcal{U} \subset \mathcal{H}_\kappa$ is of *product type* if for any $(\omega^+, \omega^-) \in \mathcal{U} \times \mathcal{U}$ there is an abelian differential $\omega \in \mathcal{U}$ and an open interval $(a,b) \subset \mathbb{R}$ such that

$$\mathcal{W}^+_{\mathcal{U}}(\omega^+) \cap \mathcal{W}^-_{\mathcal{U}}(\omega^-) = \bigcup_{t=a}^{b} \{g_t(\omega)\}.$$

Define for an open subset $\mathcal{U} \subset \mathcal{H}_\kappa$ of product type and any subset $\Omega \subset \mathcal{U}$,

$$\mathcal{W}^\pm_{\mathcal{U}}(\Omega) \equiv \bigcup_{\omega \in \Omega} \mathcal{W}^\pm_{\mathcal{U}}(\omega).$$

Definition 2. A Teichmüller-invariant measure μ supported on $\mathcal{H}_\kappa$ has *product structure* on an open subset $\mathcal{U} \subset \mathcal{H}_\kappa$ of product type if for any two Borel subsets $\Omega^\pm \subset \mathcal{U}$,

$$\mu(\Omega^-) \neq 0 \text{ and } \mu(\Omega^+) \neq 0 \qquad \text{implies} \qquad \mu\left(\mathcal{W}^+_{\mathcal{U}}(\Omega^+) \cap \mathcal{W}^-_{\mathcal{U}}(\Omega^-)\right) \neq 0.$$

A Teichmüller-invariant measure μ on $\mathcal{H}_\kappa$ has *local product structure* if *every* abelian differential $\omega \in \mathcal{H}_\kappa$ has an open neighborhood $\mathcal{U}_\omega \subset \mathcal{H}_\kappa$ of product type on which μ has a product structure.

Definition 3. The *homological dimension* of a completely periodic measured foliation $\mathcal{F}$ on an orientable surface M of genus $g > 1$ is the dimension of the isotropic subspace $\mathcal{L}(\mathcal{F}) \subset H_1(M;\mathbb{R})$ generated by the homology classes of closed leaves of the foliation $\mathcal{F}$. A completely periodic measured foliation $\mathcal{F}$ is *Lagrangian* if $\dim \mathcal{L}(\mathcal{F}) = g$, that is, if the subspace in $H_1(M;\mathbb{R})$ generated by classes of closed leaves of the foliation is a Lagrangian subspace with respect to the intersection form.

A periodic measured foliation is Lagrangian if and only if it has g distinct leaves $\gamma_1, \ldots, \gamma_g$ such that $\tilde{M} = M \backslash (\gamma_1 \cup \cdots \cup \gamma_g)$ is homeomorphic to a sphere minus $2g$ paired, disjoint disks.

Definition 4. A Teichmüller-invariant probability measure on a stratum $\mathcal{H}_\kappa$ is *cuspidal* if it has local product structure and its support contains a holomorphic differential with a completely periodic horizontal or vertical foliation. The *homological dimension* of a Teichmüller-invariant measure is the maximal homological dimension of a completely periodic vertical or horizontal foliation of a holomorphic differential in its support. A Teichmüller-invariant probability measure is *Lagrangian* if it has maximal homological dimension, i.e., its support contains a holomorphic differential whose vertical or horizontal foliation is Lagrangian.

As far as the author is aware, all known $SL(2,\mathbb{R})$-invariant measures on $\mathcal{H}_g$ (and in particular the measure (2.2)) are cuspidal. We can now state Forni's criterion for the non-uniform hyperbolicity of the Kontsevich-Zorich cocycle with respect to some $SL(2,\mathbb{R})$-invariant measure.

Theorem (Forni's Criterion [18])**.** *Let μ be an $SL(2,\mathbb{R})$-invariant ergodic probability measure on a stratum $\mathcal{H}_\kappa \subset \mathcal{H}_g$ of the moduli space of abelian differentials. If μ is cuspidal Lagrangian, the Kontsevich-Zorich cocycle is non-uniformly hyperbolic μ-almost everywhere. The Lyapunov exponents $\lambda_1^\mu \geq \cdots \geq \lambda_{2g}^\mu$ form a symmetric subset of the real line in the following way:*

$$1 = \lambda_1^\mu > \lambda_2^\mu \geq \cdots \geq \lambda_g^\mu > 0 > \lambda_{g+1}^\mu = -\lambda_g^\mu \geq \cdots \geq \lambda_{2g-1}^\mu = -\lambda_2^\mu > \lambda_{2g}^\mu = -1.$$

The spectral gap $\lambda_1^\mu > \lambda_2^\mu$ is an easier result than the entire proof of non-uniform hyperbolicity. In fact, in [17] the spectral gap was proved for *any* g_t-invariant probability measure. It follows from this result that both $E_1^+(q)$ and $E_{2g}^-(q)$ in the decomposition (2.3) are one-dimensional. In fact, for an Oseledets-regular point $q \in \mathcal{H}_\kappa$, $E_1^+(q) = [\mathrm{Re}\, q^{1/2}] \cdot \mathbb{R}$ and $E_{2g}^-(q) = [\mathrm{Im}\, q^{1/2}] \cdot \mathbb{R}$, and their dual bundles (in the sense of Poincaré duality) in $H_1(M;\mathbb{R})$ are generated, respectively, by the Schwartzman asymptotic cycles (which will be defined in section 2.4) for the horizontal and vertical foliations, $\mathcal{F}_q^{v,h}$.

2.3 Non-Uniform Hyperbolicity for Quadratic Differentials

In this section we apply Forni's criterion to the $SL(2,\mathbb{R})$-invariant measure (2.2) on $\mathcal{H}_{\hat{\kappa}}$ coming from non-orientable quadratic differentials by the double cover construction detailed in section 2.1. The non-trivial property to show is that the support of such measure in every stratum contains a completely periodic quadratic differential q on M whose vertical or horizontal foliation lifts to a Lagrangian foliation on $\hat{M}$, since for any surface M of genus g, the anti-invariant space $H_1^-(\hat{M};\mathbb{R})$ can have arbitrarily large dimension. In this section we will prove a much stronger statement, Proposition 1, which states that such quadratic differentials are dense in every stratum $\mathcal{Q}_\kappa$, which will suffice in order to apply the criterion.

2.3.1 Construction of convenient basis of homology

Following [32, §4.1], we make some remarks about the structure of $\pi_\kappa : \hat{M} \to M$ and the canonical basis on homology one can construct from it. Note that

$$\pi_\kappa : \hat{M}\backslash\{\text{ramification points}\} \to M\backslash\{\text{odd singularities}\}$$

is a regular covering space with group of deck transformations $\mathbb{Z}_2$. As such, and denoting $\dot{M} = M\backslash\{\text{odd singularities}\}$, the monodromy representation $\pi_1(\dot{M}) \to \mathbb{Z}_2$ factors through $H_1(\dot{M};\mathbb{Z})$ (and even through $H_1(\dot{M};\mathbb{Z}_2)$) since $\mathbb{Z}_2$ is Abelian. Let $m : H_1(\dot{M};\mathbb{Z}_2) \to \mathbb{Z}_2$ denote such map. Starting with a standard symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for $H_1(M;\mathbb{Z}_2)$ with $a_i \cap b_i = 1$ and all other intersections zero, it is possible to construct the following (symplectic) basis on $H_1(\hat{M};\mathbb{Q})$, using that $[\gamma] \in \ker(m)$ if and only if the loop γ lifts to two loops on $\hat{M}$.

Suppose that M has no singularities of odd degree. In this case $\pi_\kappa : \hat{M} \to M$ is a regular covering space and as such σ has no fixed points and the holonomy of a curve depends only on its homology class. Starting with a standard symplectic basis $\{\bar{a}_1, \bar{b}_1, \ldots, \bar{a}_g, \bar{b}_g\}$ of $H_1(M; \mathbb{Z})$ we can make a change of basis to obtain a "nice" basis of $H_1(\hat{M}; \mathbb{Z})$. By assumption, q is not the square of an Abelian differential, so there is at least one cycle of our symplectic basis with non-trivial monodromy, which we can assume is $\bar{b}_g$. For $1 \leq i < g$, let $a_i = \bar{a}_i + \bar{b}_g$ if $m(\bar{a}_i) = 1$ and otherwise $a_i = \bar{a}_i$, and construct b_i in a similar way. Then any loop γ_{a_i} or γ_{b_i} representing the new basis $\{a_i,\, b_i\}$ lifts to two disjoint loops $\gamma_{a_i}^{\pm}$ and $\gamma_{b_i}^{\pm}$ for $1 \leq i < g$ with $[\gamma_{a_i}^{\pm}] = a_i^{\pm}$ and $[\gamma_{b_i}^{\pm}] = b_i^{\pm}$. We can assign the labels $\pm$ such that $a_i^+ \cap b_i^+ = a_i^- \cap b_i^- = 1$ and all other intersections are zero for $1 \leq i < g$. Because of the prescribed symplectic structure, $P^{\pm} a_i^+ \neq 0 \neq P^{\pm} b_i^+$ for $1 \leq i < g$ and moreover they span a symplectic subspace of $H_1(\hat{M}; \mathbb{Q})$ of dimension $4g - 4$ (codimension 2).

Let b_g^+ be half the homology class of a lift of a curve representing $2b_g$ on M and similarly for a lift a_g^+ of a_g, independent of the value of $m(a_g)$. Then

$$\{a_1^+, b_1^+, a_1^-, b_1^-, \ldots, a_{g-1}^+, b_{g-1}^+, a_{g-1}^-, b_{g-1}^-, a_g^+, b_g^+\} \tag{2.6}$$

is a basis of $H_1(\hat{M}; \mathbb{Q})$. Moreover we have $a_i^- = \sigma_* a_i^+$ and $b_i^- = \sigma_* b_i^+$ for $1 \leq i < g$, and $\sigma_* b_g^+ = b_g^+$. Furthermore we have $\pi_{\kappa *} a_i^{\pm} = a_i$ and $\pi_{\kappa *} b_i^{\pm} = b_i$.

The cycles on $H_1(\hat{M}; \mathbb{Z})$ which come through modified cycles on $H_1(M; \mathbb{Z})$ can be modified by subtracting b_g^+ to give a symplectic basis for $H_1(\hat{M}; \mathbb{Z})$, which we can explicitly write in terms of the invariant and anti-invariant subspaces in homology:

$$\begin{aligned} H_1^+(\hat{M};\mathbb{Q}) &= \langle P^+a_1^+, P^+b_1^+, \ldots, P^+a_g^+, P^+b_g^+\rangle \\ H_1^-(\hat{M};\mathbb{Q}) &= \langle P^-a_1^-, P^-b_1^-, \ldots, P^-a_{g-1}^-, P^-b_{g-1}^-\rangle \end{aligned}$$

In these coordinates, $P^+a_i^+ \cap P^+b_i^+ = P^-a_i^- \cap P^-b_i^- \neq 0$ and all other intersections are zero. Thus H_1^+ and H_1^- are symplectically orthogonal.

Suppose that M has some singularities of odd order, which by necessity has to be an even number of them, $2n$, and label the odd singularities $p_1, \ldots, p_{2n}$. Consider a standard symplectic basis $a_1, b_1, \ldots, a_g, b_g$ of $H_1(M;\mathbb{R})$. Note that two loops representing homology classes can be different in $H_1(\dot{M};\mathbb{Z})$ while being homologous in $H_1(M;\mathbb{Z})$. This happens, for example, when the loops have different monodromy. Thus any loop representing a basis element of $H_1(M;\mathbb{Z})$ with non-trivial monodromy can be modified slightly to change its monodromy while staying in the same homology class. This is done by "taking a detour" to go around an odd singularity, say p_{2n}. By making such modifications to representatives of a_i and b_i we can suppose that every loop representing a basis element of $H_1(M;\mathbb{Z})$ lifts to two loops on $\hat{M}$, $\gamma_{a^\pm}$ and $\gamma_{b^\pm}$ with $[\gamma_{a_i^\pm}] = a_i^\pm$ and $[\gamma_{b_i^\pm}] = b_i^\pm$. By considering the intersections of curves representing the basis of $H_1(M)$ and their lifts, we can assign the $\pm$ labels to the lifts so that we get a collection of cycles in $H_1(\hat{M};\mathbb{Q})$

$$\{a_1^+, b_1^+, a_1^-, b_1^-, \ldots, a_g^+, b_g^+, a_g^-, b_g^-\} \tag{2.7}$$

such that $a_i^+ \cap b_i^+ = a_i^- \cap b_i^- = 1$ for $i \leq i \leq g$ and all other intersections are zero. Moreover we have $a_i^- = \sigma_* a_i^+$ and $b_i^- = \sigma_* b_i^+$ for $1 \leq i \leq g$. Furthermore we have $\pi_{\kappa *}a_i^{\pm} = a_i$ and $\pi_{*\kappa}b_i^{\pm} = b_i$. Because of the prescribed symplectic structure, $P^{\pm}a_i^{+} \neq 0 \neq P^{\pm}b_i^{+}$ for $1 \leq i \leq g$ and these cycles span a $4g$-dimensional symplectic subspace of $H_1(\hat{M};\mathbb{Q})$. Thus we can explicitly write the basis for the invariant and anti-invariant subspaces in homology:

$$\begin{aligned} H_1^{+}(\hat{M};\mathbb{Q}) &= \langle P^{+}a_1^{+}, P^{+}b_1^{+}, \ldots, P^{+}a_g^{+}, P^{+}b_g^{+}\rangle \\ H_1^{-}(\hat{M};\mathbb{Q}) &= \langle P^{-}a_1^{-}, P^{-}b_1^{-}, \ldots, P^{-}a_g^{-}, P^{-}b_g^{-}\rangle \end{aligned} \tag{2.8}$$

with the corresponding intersections, making them symplectically orthogonal. By the Riemann-Hurwitz formula, $\dim H_1^{-}(\hat{M},\mathbb{R}) = 2g+2n-2$, so in the case of $n=1$ we have constructed a basis for the homology of the covering surface. For $n>1$, the other $2n-2$ cycles on $\hat{M}$ which are basis elements of $H_1(\hat{M};\mathbb{R})$ are constructed in a way reminiscent of the way one constructs basis elements on a hyperelliptic surface.

Consider a series of paths l_i joining p_i to p_{i+1} for $1 \leq i \leq 2n-2$. We can chose these paths so that they have no intersection with the cycles a_i or b_i and that the line $\bigcup_{i=1}^{2n-2} l_i$ does not have self intersections. For ε sufficiently small, take an ε-tubular neighborhood E_i of l_i and consider the oriented boundary ∂E_i which we can identify with a cycle $\bar{c}_i$. This cycle clearly has trivial monodromy and, as such, lifts to two different paths on $\hat{M}$. Pick one of these and label it c_i. Thus we get the cycles $c_1,\ldots,c_{2n-2}$ on $\hat{M}$ with $c_j \cap c_{j+1} = 1$ for $1 \leq j \leq 2n-3$ and $\sigma_* c_j = -c_j$. Let $\mathcal{C} \subset H_1(\hat{M};\mathbb{Z})$ be the subspace spanned by the cycles c_i. This space is symplectically orthogonal to the subspaces spanned by $P^{\pm}a_i^{\pm}$ and $P^{\pm}b_i^{\pm}$. The subspace $\mathcal{C}$ can be thought of absolute homology classes of the covering surface which are represented by lifts of curves which are homologous to zero. We will denote by $P^{\mathcal{C}} : H_1(\hat{M};\mathbb{R}) \to \mathcal{C}$ the projection of a cycle to $\mathcal{C}$.

For the case when q has at least two odd singularities, we adopt from now on the following notation. Let $H_1^-(\hat{M};\mathbb{Z}) = \hat{H}_1^-(\hat{M};\mathbb{Z}) \oplus \mathcal{C}$ be the anti-invariant eigenspace, i.e., the projection $P^- H_1(\hat{M};\mathbb{Z})$. Then we can write the homology of the covering surface, which represents the (symplectic) orthogonal splitting, as:

$$H_1(\hat{M};\mathbb{R}) = H_1^+(\hat{M};\mathbb{R}) \oplus \hat{H}_1^-(\hat{M};\mathbb{R}) \oplus \mathcal{C}. \tag{2.9}$$

Similarly, there is a splitting in cohomology:

$$H^1(\hat{M};\mathbb{R}) = H^1_+(\hat{M};\mathbb{R}) \oplus \hat{H}^1_-(\hat{M};\mathbb{R}) \oplus \mathcal{C}^*.$$

Note that when $n > 1$, $\hat{H}_1^-(\hat{M};\mathbb{R})$ is *not* the entire anti-invariant eigenspace, but the projection to the negative eigenspace of the cycles on $\hat{M}$ which come from basis elements of $H_1(M;\mathbb{Z})$.

2.3.2 Verification of Forni's criterion

We now relate structure of periodic foliations induced by quadratic differentials to the above discussion of the relationship between the homology of the half-translation surface M carrying a quadratic differential and its orienting double cover $\hat{M}$. By removing saddle connections and singularities, a half-translation surface carrying a periodic quadratic differential q decomposes M into the disjoint union of cylinders $\{c_q^1, \ldots, c_q^s\}$ composed of closed leaves of the foliation. Each cylinder c_q^i has a waistcurve $|a_q^i|$ whose homology class $a_q^i = [|a_q^i|]$ represents the homology class of all other closed leaves in c_q^i.

Since it does not make a difference whether we speak of the vertical of horizontal foliation, **whenever it is not specified whether we consider the horizontal or vertical foliation defined by a quadratic differential, we shall assume it is the horizontal foliation**.

For any measured foliation $\mathcal{F}_q$ on M, denote by $\hat{\mathcal{F}}_q$ the measured foliation given by $\mathcal{F}_{\pi_\kappa^* q}$ on $\hat{M}$, i.e., the lift of $\mathcal{F}_q$ to $\hat{M}$. As such, we have that $\mathcal{F}_q$ is periodic if and only if $\hat{\mathcal{F}}_q$ is periodic. Let α be an Abelian differential on a translation surface M which, for the next lemma, we do not assume is the pullback of a quadratic differential. Let S_α be the union of all saddle connections in the periodic foliation given by a holomorphic 1-form α. By convention, we also assume the singularities of α are contained in S_α. Then $M\backslash S_\alpha$ is a disjoint union of cylinders $c^1_\alpha, \ldots, c^s_\alpha$.

Lemma 1. *Let α be an Abelian differential on a translation surface M whose horizontal foliation is periodic with cylinders $\{c^1_\alpha, \ldots, c^s_\alpha\}$ with respective waistcurves $\{|a^1_\alpha|, \ldots, |a^s_\alpha|\}$ and heights $\{h^i_\alpha\}$. Let $\gamma : [0,1] \longrightarrow M$ be a simple curve with $\gamma(0), \gamma(1) \in S_\alpha$. Then*

$$\Im\left(\int_\gamma \alpha\right) = \sum_{i=1}^s h^i_\alpha \left([\gamma_i] \cap a^i_\alpha\right), \tag{2.10}$$

where $[\gamma_i] \equiv [\gamma \cap c^i_\alpha] \in H_1(c^i_\alpha, \partial c^i_\alpha; \mathbb{Z})$.

Proof. Since M decomposes into cylinders,

$$\int_\gamma \Im(\alpha) = \sum_{i=1}^s \int_{\gamma\cap c^i_\alpha} \Im(\alpha).$$

Moreover, in each cylinder $\Im(\alpha)$ can be written in local coordinates as dy_i. Thus

$$\int_\gamma \Im(\alpha) = \sum_{i=1}^s \int_{\gamma\cap c^i_\alpha} dy_i,$$

from which (2.10) follows. $\square$

Note that in Lemma 1 we did not require γ to be closed. The lemma thus yields information of the intersection properties of curves γ with waistcurves of cylinders of M defined by a periodic Abelian differential. It follows that any periodic $\hat{\mathcal{F}}_q$ is given by a holomorphic 1-form α with the property that $P^{-1}\alpha = \sum_{i=1}^{s} h_\alpha^i a_\alpha^i$, where $h_\alpha^i > 0$ is the height of the cylinder c_α^i, a_α^i is the homology class represented by its oriented waistcurve $|a_\alpha^i|$ (with respect to the orientation of the foliation), and P is the (symplectic) isomorphism given by Poincaré duality.

Let q be a completely periodic quadratic differential. Let $I(q)$ and $I(\alpha)$ denote the maximal isotropic subspaces of $H_1(M;\mathbb{Q})$ and $H_1(\hat{M};\mathbb{Q})$, respectively, spanned by closed leaves of the foliation $\mathcal{F}_q$ and of $\hat{\mathcal{F}}_q$. Consider a completion of $I(\alpha)$ to a symplectic basis of $H_1(\hat{M};\mathbb{Q})$ as in (2.9) in Section 2.3.1. Let $\mathcal{C}^\perp = H_1^+(\hat{M};\mathbb{Q}) \oplus \hat{H}_1(\hat{M};\mathbb{Q})$ and denote by $P^\perp : H_1(\hat{M};\mathbb{Q}) \to \mathcal{C}^\perp$ the projection. Denote by $I^+(\alpha) \equiv P^+ I(\alpha)$, $I^-(\alpha) \equiv P^\perp P^- I(\alpha)$, $I^{\mathcal{C}}(\alpha) \equiv P^{\mathcal{C}} I(\alpha)$.

Lemma 2. *Let q be a periodic quadratic differential. Then*

$$\dim I^+(\alpha) \geq \dim I^-(\alpha), \tag{2.11}$$

equality holds if q has at least two odd singularities.

Proof. Let $\{a_1, \ldots, a_k\}$ be a basis of $I(q)$, where the a_i are homology classes of waistcurves of cylinders. Every waistcurve $|a_i|$ from this set is an isometrically embedded flat cylinder, hence has trivial monodromy and therefore lifts to two different waistcurves $|a_i^\pm|$ and hence $\dim\langle a_1^+, \ldots, a_k^+, a_1^-, \ldots, a_k^- \rangle \leq 2k$. By changing basis through $P^{\pm}$,

$$\dim \operatorname{span}\{P^+a_1^+, \dots, P^+a_k^+\} + \dim \operatorname{span}\{P^-a_1^+, \dots, P^-a_k^+\} \leq 2k. \tag{2.12}$$

If a_j is any homology class of a waistcurve of a cylinder given by q, then $a_j = \sum_{i=1}^k t_i^j a_i$ for some $t^j \in \mathbb{Z}^k$ since $\{a_1, \dots, a_k\}$ is a basis of $I(q)$. Suppose then that for some j we have $a_j^+ = \sum_{i=1}^k t_i^j a_i^+ + e$, where, for $1 \leq i \leq k$, a_i^+ is the homology class of a lift of a representative of a_i and let $e^+ \equiv P^+e$.

We claim $e^+ = 0$. Otherwise $\sum_{i=1}^k t_i a_i + \pi_{\kappa *}(e^+) = a_j = \sum_{i=1}^k t_i a_i$, a contradiction since $\pi_{\kappa *}$ restricted to $H_1^+(\hat{M}; \mathbb{Q})$ is an isomorphism onto $H_1(M; \mathbb{Q})$. Thus,

$$\operatorname{span}\{P^+a_1^+, \dots, P^+a_k^+\} = I^+(\alpha).$$

Combinining this with (2.12) we obtain that

$$\dim I^+(\alpha) \geq \dim \operatorname{span}\{P^-a_1^+, \dots, P^-a_k^+\}. \tag{2.13}$$

We address the case of odd singularities to show that $\dim\langle P^-a_1^+, \dots, P^-a_k^+\rangle = k = \dim I^+(\alpha)$. We first show that $a_i^+ \neq \pm a_i^-$ for all $i \in \{1, \dots, k\}$. Suppose $a_i^+ = \pm a_i^-$ holds for some i. Then $\hat{M} \backslash (|a_i^+| \cup |a_i^-|)$ is a disjoint union of punctured Riemann surfaces $S_1 \coprod S_2$, each of which maps to itself under σ since q has odd singularities and thus σ has fixed points. This implies that $M \backslash |a_i|$ is disconnected, or $a_i = 0$, a contradiction. Thus the lift of each waistcurve satisfies $P^{\pm}a_i^+ \neq 0$ for all i.

Now we follow the basis construction from Section 2.3.1. Let $\{\bar{a}_1, b_1, \dots, \bar{a}_g, b_g\}$ be a completion of a basis of $I(q)$ to a symplectic basis of $H_1(M; \mathbb{Z})$ with $\bar{a}_i = a_q^i$ for $1 \leq i \leq k$, $\bar{a}_i \cap b_j = \delta_i^j$, and $\bar{a}_i \cap \bar{a}_j = b_i \cap b_j = 0$ for all i, j. Suppose such basis is represented by simple closed curves $\gamma_{\bar{a}_i}$ and γ_{b_i} with trivial holonomy, which can be assumed since there are at least two odd singularities. Then each such curve has two lifts $\gamma^{\pm}_{\bar{a}_i}$ and $\gamma^{\pm}_{b_i}$ with $[\gamma^{\pm}_{\bar{a}_i}] = a^{\pm}_i$ and $[\gamma^{\pm}_{b_i}] = b^{\pm}_i$. We can assign the $\pm$ labels such that $a^+_i \cap b^+_i = a^-_i \cap b^-_i \neq 0$ and all other intersections are zero. Indeed, starting with a symplectic basis such that $a_i \cap b_i = 1$ then there exist two closed curves γ_{a_i} and γ_{b_i} with trivial monodromy representing, respectively, a_i and b_i, and intersecting only once on $M\backslash\Sigma_\mu$. The point of intersection has two lifts, which means there are two intersections on $\hat{M}\backslash\hat{\Sigma}$, from which the $\pm$ labels are assigned so that $a^+_i \cap b^+_i = a^-_i \cap b^-_i \neq 0$ and all other intersections are zero. By repeating this procedure one obtains a basis for $H_1(\hat{M};\mathbb{Q})$ with the desired intersection properties. Since $\mathcal{C}$ is symplectically orthogonal to $I^+(\alpha)$ and $I^-(\alpha)$ we do not worry about the intersection with cycles in $\mathcal{C}$.

Suppose $\dim\langle P^-a^+_1, \ldots, P^-a^+_k\rangle < k$. Without loss of generality we can assume there exists a $c \in \mathbb{Q}^{k-1}$ such that

$$P^-a^+_1 = \frac{1}{2}(a^+_1 - \sigma_* a^+_1) = \sum_{i=2}^{k} c_{i-1}(a^+_i - \sigma_* a^+_i).$$

Since $P^-b^+_1$ is the symplectic dual of $P^-a^+_1$,

$$0 \neq P^-b^+_1 \cap P^-a^+_1 = \sum_{i=2}^{k} c_{i-1} P^-b^+_1 \cap (a^+_i - \sigma_* a^+_i) = 0,$$

a contradiction since the right hand side involves a sum of intersections which are all zero. Therefore, when q has at least two odd singularities,

$$\dim\langle P^-a^+_1, \ldots, P^-a^+_k\rangle = k. \tag{2.14}$$

It remains to show that

$$\langle P^- a_1^+, \dots, P^- a_k^+ \rangle = I^-(\alpha).$$

Let q have at least two odd singularities. We consider a completion of $I(\alpha)$ to a symplectic basis as in Section 2.3.1: let $\{\bar{a}_1, \dots, \bar{a}_k\}$ be a basis of $I(q)$ consisting of homology classes of waistcurves of cylinders. Consider its completion to a symplectic basis $\{a_1, b_1, \dots, a_g, b_g\}$ of $H_1(M;\mathbb{Z})$ with $a_i = \bar{a}_i$ for $1 \leq i \leq k$, $a_i \cap b_j = \delta_i^j$. We can lift this basis to a symplectic basis $\{a_1^+, b_1^+, \dots, a_g^+, b_g^+, a_1^-, b_1^-, \dots, a_g^-, b_g^-\}$ of a symplectic subspace of $H_1(\hat{M};\mathbb{Q})$ with $a_i^\pm \cap b_j^\pm = \delta_i^j$ and all other intersections zero.

Suppose $\langle P^- a_1^+, \dots, P^- a_k^+ \rangle \neq I^-(\alpha)$. Let $\hat{c}^*$ be a cylinder of $\hat{\mathcal{F}}_q$ such that $P^- \hat{a}^* \notin \operatorname{span}\{P^- a_1^+, \dots, P^- a_k^+\}$ and $0 \neq a^* \equiv \pi_* \hat{a}^* \in H_1(M;\mathbb{Q})$. Then $\hat{a}^* = \sum_{i=i}^k t_i^* a_i^+ + e^-$, where $\sigma_* e^- = -e^-$ and $t^* \in \mathbb{Z}^k$ is defined by $a^* = \sum_{i=1}^k t_i^* a_i$.

We claim that there is some element $E \in \{a_{k+1}^+, b_{k+1}^+, \dots, a_g^+, b_g^+\}$ such that $E \cap \hat{a}^* = E \cap e^- \neq 0$. Indeed, if there was no such element, then $E \cap \hat{a}^* = 0$ for all $E \in \{a_{k+1}^+, b_{k+1}^+, \dots, a_g^+, b_g^+\}$. Since $\langle a_1^\pm, b_1^\pm, \dots, a_k^\pm, b_k^\pm \rangle$ and $\langle a_{k+1}^\pm, b_{k+1}^\pm, \dots, a_g^\pm, b_g^\pm \rangle$ are symplectically orthogonal and $P^- a_i^+ = -P^- a_i^-$, $P^- b_i^+ = -P^- b_i^-$ for all i, and since $P^- \hat{a}^* \notin \langle P^- a_1^+, \dots, P^- a_k^+ \rangle$, $P^- \hat{a}^*$ would have a non-zero component $P^- b_i^+$ for some $i \leq k$ implying that $\hat{a}^*$ would have non-zero intersection with a_i^+, which cannot happen since the $a_i^\pm$ are represented by waistcurves of cylinders of the foliation. Therefore, the claim holds. Projecting onto M,

$$0 \neq \pi_* E \cap a^* = \sum_{i=1}^k t_i^* (\pi_* E \cap a_i) = 0,$$

a contradiction. Therefore, if q has at least two off singularities,

$$\operatorname{span}\{P^- a_1^+, \dots, P^- a_k^+\} = I^-(\alpha),$$

and, combining this with (2.14), the proof follows for the case of q having at least two odd singularities.

The case of q having no odd singularities is practically the same. Let q have no odd singularities and consider a completion of $I(\alpha)$ as in Section 2.3.1: let $\{\bar{a}_1, \ldots, \bar{a}_k\}$ be a basis of $I(q)$ consisting of homology classes of waistcurves of cylinders. Consider its completion to a symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ of $H_1(M;\mathbb{Z})$ with $a_i = \bar{a}_i$ for $1 \leq i \leq k$ and $a_i \cap b_j = \delta_i^j$. As in (2.6), we can lift this basis to a symplectic basis

$$\{a_1^+, b_1^+, a_1^-, b_1^-, \ldots, a_{g-1}^+, b_{g-1}^+, a_{g-1}^-, b_{g-1}^-, a_g^+, b_g^+\}$$

of $H_1(\hat{M};\mathbb{Q})$.

Let $\hat{c}^*$ be a cylinder of $\hat{\mathcal{F}}_q$ such that $P^-\hat{a}^* \notin \operatorname{span}\{P^-a_1^+, \ldots, P^-a_k^+\}$ and $a^* = \pi_*\hat{a}^* \neq 0$. Then $\hat{a}^* = \sum_{i=i}^k t_i^* a_i^+ + e^-$, where $\sigma_* e^- = -e^-$ and $t^* \in \mathbb{Z}^k$ is defined by $a^* = \sum_{i=1}^k t_i^* a_i$.

We claim that there is some element $E \in \{a_{k+1}^+, b_{k+1}^+, \ldots, a_{g-1}^+, b_{g-1}^+\}$ such that $E \cap \hat{a}^* = E \cap e^- \neq 0$. Indeed, if there was no such element, then $E \cap \hat{a}^* = 0$ for all $E \in \{a_{k+1}^+, b_{k+1}^+, \ldots, a_{g-1}^+, b_{g-1}^+\}$. Since $\langle a_1^\pm, b_1^\pm, \ldots, a_k^\pm, b_k^\pm\rangle$ and $\langle a_{k+1}^\pm, b_{k+1}^\pm, \ldots, a_{g-1}^\pm, b_{g-1}^\pm\rangle$ are symplectically orthogonal and $P^-a_i^+ = -P^-a_i^-$, $P^-b_i^+ = -P^-b_i^-$ for all i, and since $P^-\hat{a}^* \notin \langle P^-a_1^+, \ldots, P^-a_k^+\rangle$, $P^-\hat{a}^*$ would have a non-zero component $P^-b_i^+$ for some $i \leq k$ implying that $\hat{a}^*$ would have non-zero intersection with a_i^+, which cannot happen since the $a_i^\pm$ are represented by waistcurves of cylinders of the foliation. Therefore, the claim holds. Projecting onto M,

$$0 \neq \pi_* E \cap a^* = \sum_{i=1}^k t_i^* (\pi_* E \cap a_i) = 0,$$

a contradiction. Therefore, span $\{P^- a_1^+, \ldots, P^- a_k^+\} = I^-(\alpha)$.

Combining this with (2.13), the proof follows. $\square$

Let $\mathcal{L}_\kappa^{h,v}$ be the set of quadratic differentials $q \in \mathcal{Q}_\kappa$ for which the foliation $\hat{\mathcal{F}}_q^{h,v}$ is Lagrangian.

Proposition 1. *The set $\mathcal{L}_\kappa^{h,v}$ is dense in $\mathcal{Q}_\kappa$.*

We remark that [17, Lemma 4.4] proves this statement in the case of q being the square of an abelian differential. Thus this proof follows closely the ideas of that proof, making slight modifications. We briefly review the idea for abelian differentials. One begins with a periodic horizontal foliation given by a holomorphic 1-form. Since these foliations are dense in the moduli space, the proof is completed by showing that given any such periodic foliation, one can make an arbitrary small perturbation to this form (keeping the vertical foliation fixed) to obtain a 1-form whose horizontal foliation is periodic and whose isotropic span has larger dimension than that of the unperturbed foliation. By making finitely many perturbations (no more than the genus of the surface) one obtains a Lagrangian foliation.

For a quadratic differential $q \in \mathcal{Q}_\kappa$ the idea is similar but one has to proceed carefully. Since local coordinates of $\mathcal{Q}_\kappa$ are given by periods in $H^1_-(\hat{S}, \hat{\Sigma}_\kappa; \mathbb{C})$, we can only make perturbations of $\alpha = \sqrt{\pi_\kappa^* q}$ in the anti-invariant subspace of $H^1(\hat{S}, \hat{\Sigma}_\kappa; \mathbb{C})$ by an anti-invariant holomorphic 1-form. From here, by virtue of Lemma 2, we can proceed as in [17] when there are at least two odd singularities. When there are no odd singularities, the space $H^1_-(\hat{M}; \mathbb{C})$ is too small to give enough perturbations to grow isotropically to a Lagrangian foliation, so we perturb our holomorphic 1-form with anti-invariant *relative* cocycles, i.e., exact forms of the form df which are non-zero elements of $H^1(\hat{S}, \hat{\Sigma}; \mathbb{C})$ and satisfy $\sigma^* df = -df$. We will show that perturbing with these exact forms we may continue growing-out until we get a Lagrangian foliation.

Proof. We will consider two different cases: quadratic differentials with and without odd singularities.

Case 1 (Quadratic differentials with at least two odd singularities)**.** Since periodic quadratic differentials form a dense subset of $\mathcal{Q}_\kappa$, when $\mathcal{Q}_\kappa$ is a stratum of quadratic differentials with at least two odd singularities, we will show that there is a Lagrangian foliation arbitrarily close to a periodic one which is not Lagrangian.

Suppose q is a quadratic differential with at least two odd singularities such that its horizontal foliation is periodic and that for $\alpha = \sqrt{\pi_\kappa^* q}$ we have $g > \dim I^+(\alpha) = k$. Let $\{|a_q^1|, \ldots, |a_q^k|\}$ be the waistcurves of the cylinders of the periodic foliation on M whose homology classes span $I(q)$. Then

$$\dot{\mathcal{M}} \equiv M \backslash (|a_q^1| \cup \cdots \cup |a_q^k|)$$

is a genus $(g-k)$ connected surface with $2k$ paired punctures. Let $\gamma_c : [0,1] \to \dot{\mathcal{M}}$ be a smooth simple closed curve which represents a cycle which is not homologous to a linear combination of boundary cycles and has empty intersection with the singularity set Σ_κ.

Denoting by $i : \dot{\mathcal{M}} \hookrightarrow M$ the inclusion map, then $\gamma \equiv i \circ \gamma_c : [0,1] \to M$, by construction, satisfies the following properties. If we define the non-zero homology class $h \equiv [\gamma] \in H_1(M;\mathbb{Z})$, then $h \notin I(q)$, $h \cap b = 0$ for any $b \in I(q)$, $\gamma \cap |a_q^1| = \cdots = \gamma \cap |a_q^t| = \varnothing$ and $\gamma \cap \Sigma_\kappa = \varnothing$. Furthermore, we can assume $m(h) = 0$, since we can always modify γ_c slightly to go around an odd singularity of q in order to force $m(h) = 0$. Since each γ has trivial monodromy, it has two lifts $\gamma^\pm$ to $\hat{M}$ with $[\gamma^-] = \sigma_*[\gamma^+]$. Let $h^\pm = [\gamma^\pm] \in H_1(\hat{M};\mathbb{Z})$, which by construction satisfies $h^\pm \cap b = 0$ for any $b \in I^\pm(\alpha)$ and $h^\pm \notin I^\pm(\alpha)$.

We claim $h^+ \neq \pm h^-$. Suppose $h^+ = \pm h^-$. Then $\hat{M} \backslash (\gamma^+ \cup \gamma^-)$ is a disjoint union of punctured Riemann surfaces $S_1 \coprod S_2$, each of which maps to itself under σ since q has odd singularities and thus σ has fixed points. This implies that $M \backslash \gamma$ is disconnected, or $h = 0$, a contradiction. For the two lifts $\gamma^\pm$ on $\hat{M}$ of the cycle γ, we have $\gamma^\pm \cap |\hat{a}_\alpha^1| = \cdots = \gamma^\pm \cap |\hat{a}_\alpha^{\hat{t}}| = \varnothing$ and $\gamma^\pm \cap \hat{\Sigma}_\kappa = \varnothing$.

Let $\mathcal{V}^\pm(\gamma^\pm) \subset\subset \mathcal{U}^\pm(\gamma^\pm)$ be sufficiently small open tubular neighborhoods of $\gamma^\pm$ in $\hat{M}$ such that

$$\overline{\mathcal{U}^+(\gamma^+)} \cap \overline{\mathcal{U}^-(\gamma^-)} = \varnothing, \qquad \overline{\mathcal{U}^\pm(\gamma^\pm)} \cap (|\hat{a}_\alpha^1| \cup \cdots \cup |\hat{a}_\alpha^{\hat{t}}|) = \varnothing, \qquad \overline{\mathcal{U}^\pm(\gamma^\pm)} \subset \hat{M} \backslash \hat{\Sigma}_\kappa \tag{2.15}$$

and $\mathcal{U}^-(\gamma^-) = \sigma(\mathcal{U}^+(\gamma^+))$, $\mathcal{V}^-(\gamma^-) = \sigma(\mathcal{V}^+(\gamma^+))$. Let $\mathcal{U}_\epsilon^\pm(\gamma^\pm)$, $\epsilon \in \{0,1\}$, be the two connected components of $\mathcal{U}^\pm(\gamma^\pm) \backslash \gamma^\pm$ and $\mathcal{V}_\epsilon^\pm(\gamma^\pm) = \mathcal{V}^\pm(\gamma^\pm) \cap \mathcal{U}_\epsilon^\pm(\gamma^\pm)$. Let $\phi^\pm : \mathcal{U}^\pm \to \mathbb{R}$ be a smooth function such that

$$\phi^\pm(x) = \begin{cases} 0 & \text{for } x \in \mathcal{U}_0^\pm(\gamma^\pm) \backslash \mathcal{V}_0^+(\gamma^+), \\ 1 & \text{for } x \in \mathcal{U}_1^\pm(\gamma^\pm) \end{cases}$$

and define the closed 1-forms

$$\lambda^\pm = \begin{cases} 0 & \text{on } \hat{M} \backslash \mathcal{U}^\pm(\gamma^\pm) \\ d\phi^\pm & \text{on } \mathcal{U}^\pm(\gamma^\pm) \end{cases}, \qquad \eta^- = P^-\lambda^+. \tag{2.16}$$

We claim that $0 \neq [\eta^-] \in H^1(\hat{M};\mathbb{Q})$. Indeed, since λ^+ is dual to h^+ and $\sigma^*\lambda^+$ is dual to h^-, it follows from the fact that $h^+ \neq h^-$.

The horizontal foliation given by $\alpha'_r = \alpha + ir\eta^-$ for $r \in \mathbb{Q}$ sufficiently small is periodic and satisfies, by construction, the property that every waistcurve of $\mathcal{F}_\alpha$ is homologous to a waistcurve of $\mathcal{F}_{\alpha'_r}$ and therefore $I(\alpha) \subset I(\alpha'_r)$. This is a strict inclusion, since $P^{-1}\alpha'_r = P^{-1}\alpha + P^-h^+$ and by construction $h^\pm \notin I(\alpha)$. Therefore, $\dim I(\alpha) < \dim I(\alpha'_r)$.

After finitely many iterations of this perturbation procedure we obtain a form α^- with $I^-(\alpha^-)$ a Lagrangian subspace of $H_1^-(\hat{M};\mathbb{Q})$. As in [17], one may continue with the perturbation procedure to obtain a form α^* with a Lagrangian subspace $I^{\mathcal{C}}(\alpha^*)$ of the symplectic subspace $\mathcal{C}$ by making similar perturbations in $\mathcal{C}^*$. Then, since $I^-(\alpha^*)$ and $I^{\mathcal{C}}(\alpha^*)$ are Lagrangian subspaces, by Lemma 2, $I^+(\alpha^*)$ is also a Lagrangian subspace of $\hat{H}_1^+(\hat{M};\mathbb{Q})$. Thus the case of a quadratic differential with at least two odd singularities is proved.

Case 2 (Quadratic differentials with no odd singularities)**.** Suppose q is a periodic quadratic differential with no odd singularities. In this case the only shortcoming is that the space $H^1_-(\hat{M};\mathbb{Q})$ is not big enough to provide enough perturbations to create a Lagrangian subspace in $H_1^+(\hat{M};\mathbb{Q})$. Specifically, since in this case $\dim H_1^-(\hat{M};\mathbb{Q}) = 2g-2$, if we begin with a periodic quadratic differential with $\dim I(q) = k < g$ after $g-k-1$ iterations of the perturbative procedure described in the previous case we may get an isotropic subspace in $H_1^+(\hat{M};\mathbb{Q})$ of dimension $g-1$. At this point we are unable to perturb in $H^1_-(\hat{M};\mathbb{Q})$, so we perturb with elements of $H^1_-(\hat{M},\hat{\Sigma}_\kappa;\mathbb{Q})$ since it is this space which gives local coordinates to $\mathcal{Q}_\kappa$. As in the case of periodic quadratic differentials with odd singularities, it will be sufficient to show there is one with a Lagrangian foliation which is arbitrarily close.

Suppose $q \in \mathcal{Q}_\kappa$ is a periodic quadratic differential on the genus g surface M in a stratum with no odd singularities and $k^- = \dim I^-(\alpha) < g-1$. Let $h \in H_1^-(\hat{M};\mathbb{Q})$ be a cycle such that $h \notin I^-(\alpha)$ and $h \cap b = 0$ for all $b \in I^-(\alpha)$. Let $\bar{h} \in H_1^-(\hat{M};\mathbb{Z})$ be the unique (up to a sign) primitive integer multiple of h.

We can proceed to perturb α by the Poincaré dual to $\bar{h}$ (which by construction is an element of $H^1_-(\hat{M};\mathbb{Q})$) as in (2.15) and (2.16). In this case, we do not have to worry about making sure the perturbation is done by the dual of an element in $H_1^-(\hat{M};\mathbb{Q})$ since we have guaranteed this by construction in the preceding paragraph. Thus we obtain a new form α' with $\dim I^-(\alpha') > \dim I^-(\alpha)$. Indeed, the argument which showed isotropic growth in Case 1 relied on the type of the perturbation (namely, being a perturbation in $H^1_-(\hat{M};\mathbb{Q})$), which is independent of the type of stratum to which α belongs. By construction in the preceding paragraph, we have chosen the right kind of perturbation and the same arguments for isotropic growth from Case 1 apply here. After finitely many iterations of the previous perturbative procedure, each time with the Poincaré dual of an $\bar{h}$ as in the preceding paragraph, one can end up with a periodic foliation on $\hat{M}$ given by the Abelian differential α with $\dim I^+(\alpha) = \dim I^-(\alpha) = g-1$. It could also happen that we obtain an Abelian differential with $g = \dim I^+(\alpha) > \dim I^-(\alpha) = g-1$ at which point the proposition would be proved for quadratic differentials with no odd singularities. In what follows, we treat the case $\dim I^{\pm}(\alpha) = g-1$.

Let $\{|a_1^+|, \ldots, |a_{g-1}^+|, |a_1^-|, \ldots, |a_{g-1}^-|\}$ be waistcurves of cylinders of the foliation given by α which represent a basis in homology for $I^+(\alpha) \oplus I^-(\alpha)$. Then

$$\dot{\mathcal{M}} \equiv \hat{M} \backslash (|a_1^+| \cup \cdots \cup |a_{g-1}^+| \cup |a_1^-| \cup \cdots \cup |a_{g-1}^-|)$$

is topologically a torus with $2g-2$ paired punctures coming from removing the waistcurves of cylinders. Let p_1^+ be a zero of α and $p_1^- = \sigma(p_1^+)$.

Let $\gamma_1^+ : [0,1] \to \hat{M} \backslash \mathcal{N}_\delta^1$, where $\mathcal{N}_\delta^1$ is a δ neighborhood of the punctures for some $\delta > 0$, be a path on $\hat{M}$ such that

$$\gamma_1^+(0) = p_1^-, \;\; \gamma_1^+(1) = p_1^+, \;\; \hat{\Sigma}_\kappa \cap \{\gamma_1^+(t)\}_{t\in(0,1)} = \varnothing, \;\text{ and }\; 0 \neq [\pi_\kappa \gamma_1^+] \in H_1(M;\mathbb{Z}). \tag{2.17}$$

Denote by $\gamma_1^- = \sigma(\gamma_1^+)$ its image path satisfying $\gamma_1^{\pm}(\epsilon) = \gamma_1^{\mp}(1-\epsilon)$, $\epsilon \in \{0,1\}$. Note that $0 \neq [\gamma_1^+ \cup \gamma_1^-] \in H_1(\hat{M};\mathbb{Q})$ and $0 \neq P^-[\gamma_1^+] \in H_1(\hat{M}, \hat{\Sigma};\mathbb{Q})$.

Let $\mathcal{U}_\varepsilon^{\pm} = B(p_1^{\pm}, \varepsilon/2)$ be two open $\varepsilon/2$-balls around p_1^+ and p_1^- and $\mathcal{V}_1^\varepsilon$ a ε-tubular neighborhood around $\gamma_1^+ \cup \gamma_1^-$. Let f_1 be a smooth function compactly supported in $\mathcal{V}_1^\varepsilon$ such that

$$f_1(x) = \begin{cases} 0 & \text{on } \mathcal{U}_\varepsilon^- \\ 1 & \text{on } \mathcal{U}_\varepsilon^+ \end{cases} \tag{2.18}$$

and $f_1^{\pm} \equiv P^{\pm} f_1$.

Let $\alpha'_{r_1} = \alpha + i r_1 \cdot d f_1^-$ for $r_1 \in \mathbb{Q}$ sufficiently small. Since f_1 is constant inside $\mathcal{U}_\varepsilon^{\pm}$, $df_1^- = 0$ in a neighborhood of $p_1^{\pm}$, α'_{r_1} is still an Abelian differential with a periodic foliation. Moreover, since $\gamma_1^{\pm}$ is disjoint from the waistcurves $|a_\alpha^i|$ for ε sufficiently small, the waistcurves $|a_\alpha^i|$ persist under the perturbation and are close and homologous to the waistcurves $|a^j_{\alpha'_{r_1}}|$ of the Abelian differential α'_{r_1}.

We claim not only that the horizontal foliation given by α'_{r_1} has more cylinders than the one given by α, but that the waistcurve of at least one of these cylinders has non-zero intersection with γ_1^+. Since

$$\Im\left(\int_{\gamma_1^+} \alpha'_{r_1}\right) \neq 0, \tag{2.19}$$

the claim follows from Lemma 1. At this point either $\dim I(\alpha'_{r_1}) > \dim I(\alpha)$ or $\dim I(\alpha'_{r_1}) = \dim I(\alpha)$. If the former occurs, since $H_1^+(\hat{M};\mathbb{Q})$ and $H_1^-(\hat{M};\mathbb{Q})$ are symplectically orthogonal, $\dim H_1^-(\hat{M};\mathbb{Q}) = 2g-2$, and $\dim I^-(\alpha) = g-1$, this is equivalent to $\dim I^+(\alpha'_{r_1}) > \dim I^+(\alpha)$, and this completes the proof for quadratic differentials with no odd singularities.

Suppose $\dim I(\alpha'_{r_1}) = \dim I(\alpha)$. Let $c^*_{\alpha'_{r_1}}$ be a cylinder of the foliation given by α'_{r_1} such that $a^*_{\alpha'_{r_1}} \cap [\gamma_1^+] \neq 0$ in the sense of Lemma 1. Clearly we have $a^*_{\alpha'_{r_1}} \cap a^i_\alpha = 0$ for any other waistcurve a^i_α of the foliation given by α.

Let $\mathcal{M}$ be a torus obtained by inserting $2g-2$ copies $\{D_i\}_{i=1}^{2g-2}$ of the two-disk to the punctures of $\dot{\mathcal{M}}$. Let θ_1 be the *closed* 1-form on $\mathcal{M}$ defined as

$$\theta_1 = \begin{cases} \alpha'_{r_1} & \text{on } \dot{\mathcal{M}} \\ \omega_i & \text{on } D_i \end{cases}, \tag{2.20}$$

where the ω_i are smooth forms outside finitely many singularities in the interior of each D_i and are defined such that (2.20) defines a smooth, closed form outside finitely many points. Then θ_1 defines an orientable foliation on $\mathcal{M}$ which coincides with α'_{r_1} outside the inserted disks D_i. It follows from the Poincaré-Hopf index formula that if a simply connected, planar domain bounded by a periodic orbit of a vector field contains finitely many fixed points, the sum of the indices at every fixed point in the interior is equal to 1. In other words, denoting by $\iota_p(\theta)$ the index of the vector field (foliation) given by θ at the singularity p, we have

$$\sum_{p\in \mathrm{int}(D_i)} \iota_p(\theta) = 1$$

for any i since D_i is a simply connected, bounded planar domain. If $\dim I(\alpha'_{r_1}) = \dim I(\alpha)$ both the waistcurve $|a^*_{\alpha'_{r_1}}|$ and its image $\sigma|a^*_{\alpha'_{r_1}}|$ each bound a simply connected domain on $\mathcal{M}$. By (2.19), p_1^+ is contained in the interior of one of the two domains B_1^+ and p_1^- in the other B_1^-. We claim that this finishes the proof for all differentials $q \in \mathcal{Q}_\kappa$ for $\kappa = \{4g-4\}$ for any $g > 1$. Indeed, since $p_1^\pm$ were the only singularities of α and each was of negative index, by the Poincaré-Hopf index theorem,

$$0 = \chi(\mathcal{M}) = \sum_{p\in B_1^\pm} \iota_p(\theta_1) + \sum_{p\in(\mathcal{M}\setminus B_1^\pm)} \iota_p(\theta_1) = 2 + \sum_{p\in(\mathcal{M}\setminus B_1^\pm)} \iota_p(\theta_1) \geq 2, \tag{2.21}$$

a contradiction. Thus neither $|a^*_{\alpha'_{r_1}}|$ or its image $\sigma|a^*_{\alpha'_{r_1}}|$ bound a simply connected domain, i.e., $\dim I(\alpha'_{r_1}) > \dim I(\alpha)$ and the proof is concluded in this case.

After finitely many iterations of the above argument we can reach the same contradiction for any quadratic differential with no odd singularities. In fact, if $q \in Q_\kappa$ with $\kappa = \{n_1, \dots, n_\tau\}$ has no odd singularities, after no more than τ iterations, we reach the same contradiction. We show the argument for $\kappa = \{n_1, n_2\}$ with n_1, n_2 even and $n_1 + n_2 = 4g - 4$ for some $g > 1$. For $\tau > 2$, the argument is the same.

If after one iteration we do not reach a contradiction, we pick two other singularities p_2^+ and $p_2^- = \sigma(p_2^+)$ of α'_{r_1} which are not in the interior of $B_1^\pm$ (if there are no such singularities, we reach the same contradiction through (2.21)). Define a path $\gamma_2^+ : [0,1] \to \mathcal{M}\backslash\mathcal{N}_\delta^2$ as in (2.17) where $\mathcal{N}_\delta^2$ is a δ neighborhood of the set $\{D_i\}_{i=1}^{2g-2} \cup B_1^\pm \cup \mathcal{V}_1^\varepsilon$ for δ small enough. Let f_2 and $f_2^\pm$ be defined as in (2.18) for $p_2^\pm$ and let $\theta_2 = \theta_1 + ir_2 \cdot df_2^-$ for a small enough $r_2 \in \mathbb{Q}$. Then

$$\Im\left(\int_{\gamma_2^+} \theta_2\right) \neq 0$$

which, by Lemma 1, implies there is a new cylinder given by the horizontal foliation which intersects γ_2^+. Note that we obtain the same form θ_2 if we substitute the form $\alpha'_{r_2} = \alpha + i(r_1 \cdot df_1^- + r_2 \cdot df_2^-)$ for α'_{r_1} in (2.20), thus the new waistcurve given by θ_2 also corresponds to a new waistcurve on $\hat{M}$ given by α'_{r_2}.

If the waistcurve $|a_{\theta_2}|$ of this new cylinder represents a cycle which is homologous to zero, that is, if $\dim I(\alpha'_{r_2}) = \dim I(\alpha)$, then $|a_{\theta_2}|$ and its image $\sigma|a_{\theta_2}|$ bound simply connected domains B_2^+ and B_2^- containing p_2^+ and p_2^-, respectively, on $\mathcal{M}$. As in (2.21),

$$\begin{aligned} 0 = \chi(\mathcal{M}) = \sum_{i\in\{1,2\}} \sum_{p\in B_i^\pm} \iota_p(\theta_2) \quad &+ \sum_{p\in(\mathcal{M}\backslash(B_1^\pm \cup B_2^\pm))} \iota_p(\theta_2) \\ = 2 \quad &+ \sum_{p\in(\mathcal{M}\backslash(B_1^\pm \cup B_2^\pm))} \iota_p(\theta_2) \geq 2, \end{aligned}$$

since the only singularities of θ_2 of negative index were in $B_1^\pm$ and $B_2^\pm$. Thus we get the same contradiction as in (2.21). For an arbitrary stratum with no odd singularities, we can continue the same perturbation procedure with different anti-invariant relative cocycles which are dual to relative cycles connecting paired zeros at every step. After finitely many perturbations (no more than τ) each zero of α (singularity of negative index) is contained in a simply connected domain of the foliation, which leads to a contradiction through the Poincaré-Hopf index formula. Thus, at some point of the perturbative procedure with relative, anti-invariant cycles, we obtain $\dim I^+(\alpha'_{r_i}) > \dim I^+(\alpha)$ and thus a Lagrangian foliation on $\hat{M}$. $\square$

Finally we can prove the main theorem of this chapter.

Theorem 1. *The Kontsevich-Zorich cocycle is non-uniformly hyperbolic $\hat{\mu}_\kappa$-almost everywhere on $\mathcal{H}_{\hat{\kappa}}$, where $\hat{\mu}_\kappa$ is the measure (2.2) supported on abelian differentials which come from non-orientable quadratic differentials through the double cover construction. The Lyapunov exponents satisfy*

$$1 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_g > 0 > \lambda_{g+1} = -\lambda_g \geq \cdots \geq \lambda_{2g-1} = -\lambda_2 > \lambda_{2g} = -1. \tag{2.22}$$

Since the Kontsevich-Zorich cocycle defines two cocycles on the bundle over $i_\kappa(\mathcal{Q}_\kappa) \subset \mathcal{H}_{\hat{\kappa}}$, namely, the restriction of the cocycle to the invariant and anti-invariant sub-bundles (which are each invariant under the action of the cocycle), Theorem 1 implies we can express the Lyapunov exponents of the Kontsevich-Zorich cocycle of the invariant and anti-invariant sub-bundles as

$$1 > \lambda_1^+ \geq \lambda_2^+ \geq \cdots \geq \lambda_g^+ > 0 > -\lambda_g^+ = \lambda_{g+1}^+ \geq \cdots \geq \lambda_{2g}^+ > -1$$

and

$$1 = \lambda_1^- > \lambda_2^- \geq \cdots \geq \lambda_{g+n-1}^- > 0 > -\lambda_{g+n-1}^- = \lambda_{g+n}^- \geq \cdots > \lambda_{2g+2n-2}^- = -1$$

since, by the remark following the statement of Forni's criterion, the sub-bundles corresponding to the simple, extreme exponents are respectively generated by $[\mathrm{Re}\sqrt{\pi_\kappa^* q}] \cdot \mathbb{R}$ and $[\mathrm{Im}\sqrt{\pi_\kappa^* q}] \cdot \mathbb{R}$.

The question about the simplicity of the Lyapunov spectrum remains open. Examples of non-simple spectrum (in fact, degenerate spectrum, i.e., $\lambda_i = 0$ for all $i \neq 1$) for other measures usually involve a certain set of symmetries (see [19] for a thorough discussion and examples) which are not present for Lebesgue almost all non-orientable quadratic differentials. The involution σ splits the cocycle into two symplectic cocycles and it would be very surprising to find strong enough symmetries from the involution which would imply non-simplicity of the spectrum (2.22). Numerical experiments indeed show strong evidence for a simple spectrum. Thus we conjecture that for $\hat{\mu}_\kappa$-almost all quadratic differentials, the Kontsevich-Zorich cocycle has simple spectrum. We have approximated numerically the values of the exponents for several strata, which we summarize in the appendix.

We remark that Proposition 1 is stronger than needed to prove the result, as Forni's criterion needs *one* Lagrangian differential in the support of the measure. It is thus possible to prove Theorem 1 through other methods by showing there is at least one Lagrangian differential in the support of the canonical measure such that not only $\mathcal{F}_q$ is Lagrangian on M, but also that $\hat{\mathcal{F}}_q$ is Lagrangian on $\hat{M}$. It seems that the tools from generalized permutations (see for example [5]) could be used to obtain such results, although we believe in such case it the hardest task would be to obtain a Lagrangian subspace $I^{\mathcal{C}}$ in the symplectic subspace $\mathcal{C}$ in the case of many odd singularities. In the same case, showing that $I^{\pm}$ are Lagrangian would not be a difficult task since, by Lemma 2, it suffices to obtain a Lagrangian foliation on M. The case of quadratic differentials with no odd singularities would most likely also have to be treated as a special case as well. We would be very interested to see whether Theorem 1 can be proved in such way (the tools and results of [14] look particularly promising for this task).

Proof of Theorem 1. Since the measure (2.2) is the push-forward of a canonical measure which is locally equivalent to Lebesgue by the period map, it is easy to see that it has local product structure. By Proposition 1, quadratic differentials q such that $\hat{\mathcal{F}}_q$ is Lagrangian are dense in every stratum of $\mathcal{Q}_\kappa$ and thus the measure $\hat{\mu}_\kappa$ on $\mathcal{H}_{\hat{\kappa}}$ is cuspidal Lagrangian. The theorem then follows from Forni's criterion. $\square$

2.4 Deviation Phenomena

Let M be a smooth, closed manifold and X a smooth vector field on M which generates a flow φ_t. For a point $p \in M$, let $c_T(p) \in H_1(M;\mathbb{R})$ be the cycle represented by closing the segment $\varphi_T(p)$ by a shortest path joining $\varphi_T(p)$ to p. For an ergodic measure μ, invariant under X, and a point p the support of μ, the *Schwartzman asymptotic cycle* [40] is defined as

$$c^*_\mu \equiv \lim_{T\to\infty} \frac{c_T(p)}{T} \in H_1(M;\mathbb{R}).$$

The cycle c^*_μ is a sort of topological invariant of the flow X with respect to the measure μ which can be regarded as a generalization of a rotation number since it coincides with the usual notion of rotation number for a minimal flow on a torus.

In the case when M is an closed, orientable surface of genus $g > 1$ endowed with a flat metric outside finitely many singular points and X generates a (uniquely ergodic) translation flow (in other words, straight-line flow on a translation surface) on M, Zorich [44] observed the following unexpected deviation phenomena through computational experiments. There are g numbers $1 = \lambda_1 > \cdots > \lambda_g > 0$ and a filtration of subspaces

$$\langle c^* \rangle = F_1 \subset \cdots \subset F_g \subset H_1(\hat{M}; \mathbb{R})$$

with $\dim F_i / F_{i-1} = 1$ such that, for $\phi \in \mathrm{Ann}(F_i)$ but $\phi \notin \mathrm{Ann}(F_{i+1})$,

$$\limsup_{T \to \infty} \frac{\log \| \langle \phi, c_T \rangle \|}{\log T} = \lambda_{i+1} \tag{2.23}$$

Cycles which generate the subspaces F_i are called *Zorich cycles*. It was also proved that the numbers λ_i actually coincide with the Lyapunov exponents (2.4) of the Kontsevich-Zorich cocycle. In fact, he proved the following conditional statement.

Theorem ([44]). *Suppose the Kontsevich-Zorich cocycle is non-uniformly hyperbolic ν-almost everywhere, where ν is the canonical, absolutely continuous g_t-invariant probability measure on some stratum of Abelian differentials, i.e., $\lambda_1 \geq \cdots \geq \lambda_g > 0$ for ν-almost every Abelian differential α. Let $\lambda_1' > \cdots > \lambda_s' > 0$ be the different Lyapunov exponents. Then for ν-almost every Abelian differential there exists a filtration of subspaces in $H_1(M; \mathbb{R})$*

$$\langle c^* \rangle = F_1 \subseteq \cdots \subseteq F_s \subset H_1(\hat{M}; \mathbb{R})$$

with $\dim F_i / F_{i-1} =$ multiplicity of λ_i', $\dim F_s = g$ such that (2.23) holds. Moreover $[c_T(p)]$ remains within bounded distance of F_s for almost every point p.

We remark that this result of Zorich has been recently generalized to *any* g_t-invariant probability measure on a stratum of Abelian differentials by Delecroix, Hubert and Lelièvre [12].

Based on the computer experiments, it was conjectured by Kontsevich and Zorich that for the canonical measure on the moduli space of orientable quadratic differentials, the Kontsevich-Zorich cocycle is non-uniformly hyperbolic and has a simple spectrum. This became known as the *Kontsevich-Zorich conjecture* [30]. It was also conjectured that similar deviations should hold for ergodic averages of smooth functions. Specifically, it was conjectured that for a smooth function f and large T,

$$\left| \int_0^T f \circ \varphi_t(p)\, dt \right| \approx T^{\lambda_{i+1}} \tag{2.24}$$

for almost every p on a codimension i subspace in some space of functions.

The non-uniform hyperbolicity of the Kontsevich-Zorich cocycle was first proved in [17]. There it was proved that the deviation of ergodic averages is in fact described by the exponents of the Kontsevich-Zorich cocycle and that $\lambda_g > 0$, but the simplicity of the spectrum was not proved for surfaces of genus greater than two. The full conjecture, that is, that the spectrum of the cocycle is simple and that $\lambda_g > 0$, was proved by Avila and Viana [3]. We now recall the precise results on deviations of ergodic averages from [17, §6-§9].

Let X_α be a vector field on a surface M of genus g which is tangent to the horizontal foliation of an abelian differential α. Let $\mathcal{I}^1_{X_\alpha}(M)$ denote the vector space of X_α-invariant distributions (in the sense of Schwartz), i.e., distributional solutions $\mathcal{D} \in H^{-1}(M)$ of the equation $X_\alpha \mathcal{D} = 0$, where $H^{-1}(M)$ is the dual space of the Sobolev space $H^1(M)$.

Theorem ([17])**.** *For Lebesgue-almost all abelian differentials α the space $\mathcal{I}^1_{X_\alpha}(M)$*

has dimension g and there exists a splitting

$$\mathcal{I}^1_{X_\alpha}(M) = \mathcal{I}^1_{X_\alpha}(\lambda'_1) \oplus \cdots \oplus \mathcal{I}^1_{X_\alpha}(\lambda'_s)$$

where $\dim \mathcal{I}^1_{X_\alpha}(\lambda'_i)$ = *multiplicity of* λ'_i *for the* i^{th} *distinct Lyapunov exponent of Kontsevich-Zorich cocycle. Denoting by* φ_t *the flow of* X_α*, for any function* $f \in H^1(M)$ *such that*

$$\mathcal{D}f = 0 \qquad \textit{for all} \qquad \mathcal{D} \in \mathcal{I}^1_{X_\alpha}(\lambda'_1) \oplus \cdots \oplus \mathcal{I}^1_{X_\alpha}(\lambda'_i),$$

and if there exists a $\mathcal{D}_{i+1} \in \mathcal{I}^1_{X_\alpha}(\lambda'_{i+i}) \backslash \{0\}$ *such that* $\mathcal{D}_{i+1} f \neq 0$*, then, if* $0 < i < s$*, for almost every* $p \in M$*,*

$$\limsup_{T\to\infty} \frac{\log |\int_0^T f \circ \varphi_t(p)\, dt|}{\log T} = \lambda'_{i+1}.$$

If $\mathcal{D}f = 0$ *for all* $\mathcal{D} \in \mathcal{I}^1_{X_\alpha}$*, then for any* p *not contained in a singular leaf,*

$$\limsup_{T\to\infty} \frac{\log |\int_0^T f \circ \varphi_t(p)\, dt|}{\log T} = 0.$$

A *basic current* C for $\mathcal{F}$ is a current (in the sense of de Rham) of dimension and degree equal to one such that for all vector fields X tangent to $\mathcal{F}$ we have

$$i_X C = \mathcal{L}_X C = 0.$$

Let $\mathcal{B}^s_q$ be the space of currents for $\mathcal{F}^h_q$ of order s. It was proved in [17] that the space $\mathcal{I}^1_{X_\alpha}$ is in bijection with the subspace $\mathcal{B}^1_{q,+} \subset \mathcal{B}^1_q$ of *closed* currents which are not exact. In fact, $C \in \mathcal{B}^1_{q,+}$ if and only if $C \wedge [\operatorname{Im} \alpha] \in \mathcal{I}^1_{X_\alpha}$. There is an analogous splitting of the space $\mathcal{B}^1_{q,+}$:

$$\mathcal{B}^1_{q,+} = \mathcal{B}^1_{q,+}(\lambda'_1) \oplus \cdots \oplus \mathcal{B}^1_{q,+}(\lambda'_s)$$

with respect to the Lyapunov spectrum of the Kontsevich-Zorich cocycle. Let

$$\Pi_q^i : \mathcal{B}_q^1 \longrightarrow \mathcal{B}_{q,+}^1(\lambda_i')$$

be the projection to the i^{th} summand of the splitting. The invariant distributions which generate each $\mathcal{I}_{X_\alpha}^1(\lambda_i')$ are constructed from the asymptotic currents as follows. There is a sequence of times $T_k \to \infty$ such that

$$\mathcal{D}_i \equiv \lim_{k\to\infty} \frac{\Pi_q^i \, \ell_{T_k} \wedge [\operatorname{Im}\alpha]}{|\Pi_q^i \, \ell_{T_k}|_{-1}} = \lim_{k\to\infty} \frac{\Pi_q^i \, \ell_{T_k}}{|\Pi_q^i \, \ell_{T_k}|_{-1}} \wedge [\operatorname{Im}\alpha] = C_i \wedge [\operatorname{Im}\alpha] \in \mathcal{I}_{X_\alpha}^1(\lambda_i'), \quad (2.25)$$

are the invariant distributions, where ℓ_T is the current defined by a segment of a leaf of $\mathcal{F}_q^h$ (a chain) of length T. Furthermore,

$$\limsup_{T\to\infty} \frac{\log |\Pi_q^i \, \ell_T|_{-1}}{\log T} = \lambda_i'. \quad (2.26)$$

Thus, the basic currents C_i in (2.25) are the Zorich cycles which generate the subspaces F_i in Zorich's theorem. In fact, there is a representation theorem of Zorich cycles which states that all Zorich cycles are represented by basic currents of order 1 [17, Theorem 8.3].

We remark that although Forni's theorem on deviation of ergodic averages is stated for almost every abelian differential with respect to the canonical, abosolutely continuous invariant probability measure on a stratum of moduli space of abelian differentials, the statement holds for any $SL(2,\mathbb{R})$-invariant measure. Indeed, this requirement is a consequence of the fact that the non-uniform hyperbolicity of the Kontsevich-Zorich cocycle in [17] was only proved for the canonical measure. The proof of deviation of ergodic averages generalizes without modifications to any $SL(2,\mathbb{R})$-invariant measure for which the Kontsevich-Zorich cocycle is non-uniformly hyperbolic. See [17, Chapters 8 and 9], [15], and [8].

Any element of the spectrum of the Kontsevich-Zorich cocycle for the canonical measure in the moduli space of abelian differentials describes deviations of both homology cycles as well as that of ergodic averages. For the case of non-orientable quadratic differentials, it is surprisingly not the same.

2.4.1 Deviations in homology for quadratic differentials

Let $q \in \mathcal{Q}_\kappa$ be a quadratic differential on M which is an Oseledets-regular point with respect to the measure (2.2) for the Kontsevich-Zorich cocycle. Let $\hat{M}$ the orienting double cover and $\alpha = \sqrt{\pi_\kappa^* q}$. For a point $p \in M$ on a minimal leaf ℓ of $\mathcal{F}_q$ and picking a local direction, we can follow a segment of length T, ℓ_T, of the leaf ℓ in such direction. Let $c_T \in H_1(M;\mathbb{R})$ be the cycle obtained by closing the chain ℓ_T by a short path.

For a point $\hat{p} \in \pi_\kappa^{-1}(p)$, following a leaf $\hat{\ell}_T$ of length T of the foliation $\hat{\mathcal{F}}_q$ such that $\pi_\kappa \hat{\ell}_T = \ell_T$, let $\hat{c}_T \in H_1(\hat{M};\mathbb{R})$ be the cycle obtained by closing the chain $\hat{\ell}_T$ by a short path. Then

$$\hat{c}_q^* \equiv \lim_{T\to\infty} \frac{\hat{c}_T}{T} \in H_1^-(\hat{M};\mathbb{R})$$

is the Schwartzman asymptotic cycle. It is anti-invariant with respect to σ_* since it can be shown to be the Poincaré dual of the cohomology class defining the foliation, in this case either $[\mathrm{Re}(\alpha)]$ or $[\mathrm{Im}(\alpha)]$. By construction, $\pi_{\kappa *}\hat{c}_q^* = c_q^*$. By Theorem 1, the Kontsevich-Zorich cocycle is non-uniformly hyperbolic with respect to the measure $\hat{\mu}_\kappa$ supported on $i_\kappa(\mathcal{Q}_\kappa) \subset \mathcal{H}_{\hat{\kappa}}$. Let $1 = \lambda_1^- > \lambda_2^- \geq \cdots \geq \lambda_{2g+n-1}^- > 0$ and

$\lambda_1^+ \geq \cdots \geq \lambda_g^+ > 0$ be the positive Lyapunov exponents of the restriction of the cocycle to the anti-invariant and invariant sub-bundles, respectively. By Zorich's theorem, for large T we (intuitively) have,

$$\hat{c}_T \approx \underbrace{\hat{c}_q^* T + c_2^- T^{\lambda_2^-} + \cdots}_{\text{coming from } H_1^-(\hat{M};\mathbb{R})} + \underbrace{c_1^+ T^{\lambda_1^+} + c_2^+ T^{\lambda_2^+} + \cdots}_{\text{coming from } H_1^+(\hat{M};\mathbb{R})}.$$

Since $\pi_{\kappa *}\hat{c}_T = c_T$ and $\ker \pi_{\kappa *} = H_1^-(\hat{M};\mathbb{R})$,

$$c_T \approx \pi_{\kappa *}(c_1^+)T^{\lambda_1^+} + \pi_{\kappa *}(c_2^+)T^{\lambda_2^+} + \cdots . \tag{2.27}$$

If we define the Schwartzman asymptotic cycle for the non-orientable foliations on M as

$$c_q^* \equiv \lim_{T\to\infty} \frac{c_T}{T} \in H_1(M;\mathbb{R}),$$

then, by (2.27), it is well-defined and equal to zero. Thus the deviation of homology classes is sublinear and described completely by invariant behavior. The result is summarized in the following theorem, which follows from Theorem 1 and the generalization of Zorich's theorem in [12].

Theorem 2 (Deviations in homology for a typical leaf of a quadratic differential)**.** *For Lebesgue-almost all quadratic differentials $q \in \mathcal{Q}_g$ on M, there exists a filtration of subspaces*

$$F_1 \subset \cdots \subset F_s \subset H_1(M;\mathbb{R})$$

with $\dim F_i/F_{i-1} =$ *multiplicity of* λ_i^+ *and* F_s *a Lagrangian subspace, such that, for* $\phi \in \mathrm{Ann}(F_i)$ *but* $\phi \notin \mathrm{Ann}(F_{i+1})$,

$$\limsup_{T\to\infty} \frac{\log \|\langle \phi, c_T\rangle\|}{\log T} = \lambda_{i+1}^+$$

where c_T is obtained by closing a non-singular leaf ℓ_T of length T by a short segment and $\lambda_1^+ > \cdots > \lambda_s^+ > 0$ are the distinct Lyapunov exponents of the Kontsevich-Zorich cocycle with respect to the measure coming from quadratic differentials, restricted to the invariant sub-bundle $H^1_+(\hat{M};\mathbb{R})$.

2.4.2 Deviation of ergodic averages for quadratic differentials

Let $q \in \mathcal{Q}_\kappa$ be a quadratic differential on M which is an Oseledets-regular point with respect to the measure (2.2) for the Kontsevich-Zorich cocycle. Let $\hat{M}$ be the orienting double cover and $\alpha = \sqrt{\pi_\kappa^* q}$. For a point $p \in M$ on a minimal leaf of $\mathcal{F}_q$, let $\varphi_t(p)$ be the "flow" obtained by integrating the distribution defining the horizontal foliation in a chosen direction and starting at p. As such, $\bigcup_{t=0}^T \varphi_t(p)$ is a segment ℓ_T of a leaf of the horizontal foliation $\mathcal{F}_q$ of length T with an endpoint p. Then for a smooth function f,

$$\int_0^T f \circ \varphi_s(p)\, ds \tag{2.28}$$

is well defined. Let $\hat{f} = \pi_\kappa^* f$ be a smooth function on $\hat{M}$. Then

$$\int_0^T f \circ \varphi_s(p)\, ds = \int_0^T \hat{f} \circ \hat{\varphi}_s(\hat{p})\, ds,$$

for the flow $\hat{\varphi}_t(p)$ defined by the orientable horizontal foliation $\hat{\mathcal{F}}_q$ for a point $\hat{p} \in \pi_\kappa^{-1}(p)$. Moreover,

$$\int_0^T f \circ \varphi_s(p)\, ds = \int_0^T \hat{f} \circ \hat{\varphi}_s(\hat{p})\, ds = \langle \ell_T, \hat{f} \cdot \operatorname{Im}\alpha \rangle = \int_{\ell_T} \hat{f} \cdot \operatorname{Im}\alpha. \tag{2.29}$$

For the space of invariant distributions $\mathcal{I}_q^1(\hat{M})$, let $\mathcal{I}_q^\pm \equiv P^\pm \mathcal{I}_q^1(\hat{M})$. By [17], there is a splitting of the closed, non-exact basic currents of order one

$$\mathcal{B}_q^1 = \mathcal{B}_q^+ \oplus \mathcal{B}_q^- = \mathcal{B}_q^+(\lambda_1^+) \oplus \cdots \oplus \mathcal{B}_q^+(\lambda_{s^+}^+) \oplus \mathcal{B}_q^-(\lambda_1^-) \oplus \cdots \oplus \mathcal{B}_q^-(\lambda_{s^-}^-)$$

into the components corresponding to the Lyapunov exponents coming from the restriction of the cocycle to the invariant and anti-invariant sub-bundles, respectively. Let $\Pi_\pm^i : \mathcal{B}_q^1 \longrightarrow \mathcal{B}_q^\pm(\lambda_i^\pm)$. For an invariant distribution $\mathcal{D} = C \wedge \mathrm{Im}\,[\alpha]$, since $[\alpha] \in H_-^1(\hat{M}, \hat{\Sigma}_\kappa; \mathbb{R})$, $\mathcal{D} \in \mathcal{I}_q^\pm$ if and only if $C \in \mathcal{B}_q^\mp$.

If $\mathcal{D} \in \mathcal{I}_q^-$, $\mathcal{D}(\hat{f}) = 0$ for $\hat{f} = \pi_\kappa^* f$. Then, by (2.25), (2.26), and (2.29), for large T,

$$\begin{aligned}\int_0^T f \circ \varphi_s(p)\, ds &\approx \sum_{i=1}^{s^-} \langle \Pi_-^i\, \ell_T, \hat{f} \cdot \mathrm{Im}\,\alpha \rangle \cdot T^{\lambda_i^-} + \sum_{i=1}^{s^+} \langle \Pi_+^i\, \ell_T, \hat{f} \cdot \mathrm{Im}\,\alpha \rangle \cdot T^{\lambda_i^+} \\ &= \sum_{i=1}^{s^-} \langle \Pi_-^i\, \ell_T, \hat{f} \cdot \mathrm{Im}\,\alpha \rangle \cdot T^{\lambda_i^-},\end{aligned}$$

and thus the deviation of ergodic averages are described by anti-invariant behavior. If $H^1(M)$ denotes the standard Sobolev space of functions on M, then it is clear to see that $\pi_\kappa^* H^1(M) \subset H^1(\hat{M})$. The results of [17], [15], [8], and Theorem 1 imply the following.

Theorem 3 (Deviations of ergodic averages for quadratic differentials)**.** *For Lebesgue-almost all non-orientable differentials q on a genus g surface M there is a space $\mathcal{I}_q^1(M)$ of dimension $2g + 2n - 2$ of distributions defined as the push-forward of the space of invariant distributions $\mathcal{I}_q^+$ on $\hat{M}$ which splits as*

$$\mathcal{I}_q^1(M) = \pi_{\kappa *}\mathcal{I}_q^+(\lambda_1') \oplus \cdots \oplus \pi_{\kappa *}\mathcal{I}_q^+(\lambda_{s^-}')$$

where $\dim \mathcal{I}_q^+(\lambda_i')$ = *multiplicity of* λ_i' *for the* i^{th} *distinct Lyapunov exponent of Kontsevich-Zorich cocycle restricted to the anti-invariant sub-bundle. Denoting by* φ_t *the local flow of* $\mathcal{F}_q^h$ *as in (2.28), for any function* $f \in H^1(M)$ *such that*

$$\mathcal{D}f = 0 \qquad \text{for all} \qquad \mathcal{D} \in \pi_{\kappa *}\mathcal{I}_q^+(\lambda_1') \oplus \cdots \oplus \pi_{\kappa *}\mathcal{I}_q^+(\lambda_i'),$$

and if there exists a $\mathcal{D}_{i+1} \in \pi_{\kappa *}\mathcal{I}_q^+(\lambda_{i+i}')\backslash\{0\}$ *such that* $\mathcal{D}_{i+1}f \neq 0$*, then, if* $0 < i < s^-$*, for almost every* $p \in M$*,*

$$\limsup_{T\to\infty} \frac{\log |\int_0^T f \circ \varphi_t(p)\, dt|}{\log T} = \lambda_{i+1}'.$$

If $\mathcal{D}f = 0$ *for all* $\mathcal{D} \in \pi_{\kappa *}\mathcal{I}_q^+$*, then for any* p *not contained in a singular leaf,*

$$\limsup_{T\to\infty} \frac{\log |\int_0^T f \circ \varphi_t(p)\, dt|}{\log T} = 0.$$

It is a consequence of a result of Masur and Smillie [37] that the anti-invariant sub-bundle can be arbitrarily large for a fixed genus g surface. Consequently, by the above theorem, there are non-orientable foliations for which the space of invariant distributions $\mathcal{I}_q^1(M)$ can have arbitrarily large dimension and the deviation of ergodic averages are described by arbitrarily many parameters.

By (2.5), the Kontsevich-Zorich cocycle over $\mathcal{Q}_\kappa$ describes only the Lyapunov exponents of the invariant sub-bundle over $i_\kappa(\mathcal{Q}_\kappa) \subset \mathcal{H}_{\hat{\kappa}}$. Thus, by the above theorem, there seems to be no *a-priori* reason for the Lyapunov exponents of the cocycle over $\mathcal{Q}_\kappa$ to describe the deviation of averages of functions along leaves of the foliation: only if there is repetition of exponents across the invariant and anti-invariant sub-bundles does the cocycle over $\mathcal{Q}_\kappa$ describe the deviation behavior of ergodic integrals.

2.5 Approximating the Lyapunov exponents numerically

The Kontsevich-Zorich cocycle is a continuous-time version of a discrete, matrix-valued cocycle, the Rauzy-Veech-Zorich cocycle. Thus one can try to numerically compute the Lyapunov exponents for this discrete cocycle. In fact, this was how Zorich originally conjectured a simple spectrum for the case of Abelian differentials. We will not go into details behind the discrete theory of (half-)translation surfaces, that of interval exchange transformations, zippered rectangles, Rauzy-Veech induction, Zorich acceleration, generalized permutations, et cetera. We have written this section assuming the reader is acquainted with these concepts. We will give references for the unfamiliar but interested reader.

The language of generalized permutations [5] is the right discrete language in which to study the dynamics of the discrete cocycle on a surface carrying a non-orientable quadratic differential. Not surprisingly, one can pass to the orienting double cover and study the dynamics of the Rauzy-Veech-Zorich cocycle for an interval exchange transformation through analogues of the already-developed tools for interval exchange transformations. The concept of *interval exchange transformation with involution*, first introduced in [2], is the right analogue of interval exchange transformations for Abelian differentials which are the pull-back of non-orientable ones. Although the explicit connection between generalized permutations and interval exchange transformations with involution, as well as explicit expressions for all the cocycles involved on the orienting cover, are not found in the literature, it is not hard to work them out from [5] and [2]. Having computed the matrix-valued cocycle expressions for the interval exchange transformations with involution, we have approximated the Lyapunov exponents for such cocycles numerically, following [13, §V.C]. The code and instructions on how to use it can be found in the appendix.

Below is a table of all the strata of quadratic differentials for which the Lyapunov exponents were approximated numerically. Recall that we always have $\lambda_1^- = 1$. According to [33], some strata are not connected and in some cases we have computed the exponents for different components of such strata. Note that the result for $\mathcal{Q}(2,-1,-1)$ has actually been proved in [4, Theorem 1.7]. The results for all strata examined suggest a simple spectrum, so we conjecture that this is true for $\hat{\mu}_\kappa$-almost all quadratic differentials for any singularity pattern κ.

Stratum	Geni	Invariant Exponents	Anti-Invariant Exponents
$\mathcal{Q}(2,-1,-1)$	$g=1,\ \hat{g}=2$	$\lambda_1^+ = \frac{1}{2}$	$\lambda_1^- = 1$
$\mathcal{Q}(2,1,-1^3)$	$g=1,\ \hat{g}=3$	$\lambda_1^+ = \frac{1}{2}$	$\lambda_2^- = \frac{1}{3}$
$\mathcal{Q}(8)$	$g=3,\ \hat{g}=5$	$\lambda_1^+ = 0.660189$ $\lambda_2^+ = 0.3973745$ $\lambda_3^+ = 0.142043$	$\lambda_2^- = 0.2000206$
$\mathcal{Q}(-1,3,3,3)^{adj}$	$g=3,\ \hat{g}=4$	$\lambda_1^+ = 0.778654$ $\lambda_2^+ = 0.47222$ $\lambda_3^+ = 0.229875$	$\lambda_2^- = 0.551526333$ $\lambda_3^- = 0.233913333$ $\lambda_4^- = 0.097543$
$\mathcal{Q}(-1,3,3,3)^{irr}$	$g=3,\ \hat{g}=4$	$\lambda_1^+ = 0.597168$ $\lambda_2^+ = 0.402619$ $\lambda_3^+ = 0.200314$	$\lambda_2^- = 0.327950333$ $\lambda_3^- = 0.190083$ $\lambda_4^- = 0.083007333$
$\mathcal{Q}(-1,3,6)^{adj}$	$g=3,\ \hat{g}=3$	$\lambda_1^+ = 0.601297$ $\lambda_2^+ = 0.3795885$ $\lambda_3^+ = 0.1677125$	$\lambda_2^- = 0.30827666$ $\lambda_3^- = 0.1406165$
$\mathcal{Q}(-1,3,6)^{irr}$	$g=3,\ \hat{g}=3$	$\lambda_1^+ = 0.767285$ $\lambda_2^+ = 0.445894$ $\lambda_3^+ = 0.190788$	$\lambda_2^- = 0.524996$ $\lambda_3^- = 0.17866075$

Stratum	Geni	Invariant Exponents	Anti-Invariant Exponents
$\mathcal{Q}(-1,9)^{adj}$	$g=3,\ \hat{g}=3$	$\lambda_1^+ = 0.607201$ $\lambda_2^+ = 0.346005$ $\lambda_3^+ = 0.135734$	$\lambda_2^- = 0.281791$ $\lambda_3^- = 0.080341$
$\mathcal{Q}(-1,9)^{irr}$	$g=3,\ \hat{g}=3$	$\lambda_1^+ = 0.742725$ $\lambda_2^+ = 0.3902795$ $\lambda_3^+ = 0.139563$	$\lambda_2^- = 0.4617$ $\lambda_3^- = 0.082813$
$\mathcal{Q}(12)^I$	$g=4,\ \hat{g}=7$	$\lambda_1^+ = 0.6639145$ $\lambda_2^+ = 0.45256$ $\lambda_3^+ = 0.2278785$ $\lambda_4^+ = 0.089465$	$\lambda_2^- = 0.303482$ $\lambda_3^- = 0.119673$
$\mathcal{Q}(12)^{II}$	$g=4,\ \hat{g}=7$	$\lambda_1^+ = 0.7476805$ $\lambda_2^+ = 0.49137$ $\lambda_3^+ = 0.2437355$ $\lambda_4^+ = 0.0893735$	$\lambda_2^- = 0.443258$ $\lambda_3^- = 0.12827975$
$\mathcal{Q}(4,4)$	$g=3,\ \hat{g}=2$	$\lambda_1^+ = 0.704425$ $\lambda_2^+ = 0.4367675$ $\lambda_3^+ = 0.1917245$	$\lambda_2^- = 0.33313725$

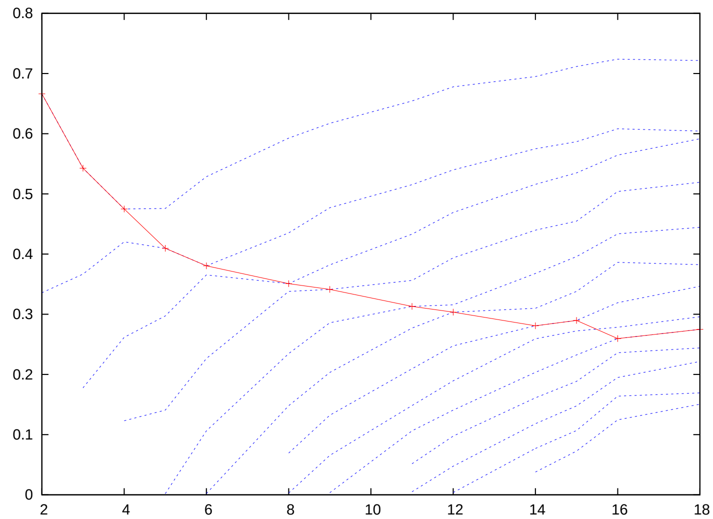

Figure 2.1: Positive spectra of the Kontsevich-Zorich cocycle for the strata $\mathcal{Q}(-1^n, 1^n)$ for most values of $n \in \{2, \ldots, 18\}$. The red lines indicate the Lyapunov exponent corresponding to the invariant sub-bundle of each stratum.

Consider the stratum $\mathcal{Q}((-1)^n, 1^n)$ for some $n \geq 2$ which consists of surfaces of genus n which cover the torus. The Kontsevich-Zorich cocycle on any such stratum has only one invariant Lyapunov exponent, independent of n, which we denote by λ_n^+. Vincent Delecroix asked me the following question: what is the behavior of λ_n^+ as $n \to \infty$?

My intuitive guess was that, as $n \to \infty$, $\lambda_n^+ \to 0$. The reasoning is that since λ_n^+ describes the (sublinear) speed at which non-orientable foliations commit to an asymptotic direction, as you put more and more obstacles which turn you around on the torus (i.e., as you put more poles) then it takes you a longer time to commit to any asymptotic direction. Numerical experiments agree with this point of view (see Figure 2.1), although a proof of this fact is not readily available.

Chapter 3

The ergodicity of flat surfaces of finite area

A flat surface is a two-dimensional oriented manifold M endowed with a flat metric everywhere except on a set of "bad points" Σ or singularities, which is forced to exist by the topology of the surface if the surface is of genus greater than one. Flat surfaces are inextricably connected to quadratic differentials since the latter give a Riemann surface a flat metric and a pair of transverse, measured foliations, called the vertical and horizontal foliations. If the foliations are orientable, which is not always the case, by considering them as flows we suddenly have a dynamical system, called the translation flow, given by an analytic object, i.e., by a given quadratic differential or, since the foliations are orientable, by a holomorphic 1-form or an Abelian differential α. Thus one can try to derive dynamical and ergodic properties of the flow by studying properties of the Abelian differential. Although we can get two different flows by considering the horizontal or vertical foliations, from now on we shall assume the flow corresponds to the horizontal foliation. The translation flow is defined along a global direction $\theta_\alpha \in S^1$, locally defined as the argument of the holomorphic coefficient of the Abelian differential α. This flow preserves an absolutely continuous measure μ_α, singular at Σ, which is also defined by the Abelian differential. For a very thorough background on flat surfaces, see [38, 45].

In the case when the surface is compact, this point of view is rather favorable, as the "right" space of *all* quadratic differentials on a fixed Riemann surface of genus g is a finite dimensional space. This "right" space is the moduli space of quadratic differentials, or moduli space for short. It is the "right" space because it is the space of classes of conformally-equivalent flat metrics on a Riemann surface and is a topological space homeomorphic to an open ball of dimension $6g-6$, where g is the genus of the surface, equipped with an absolutely continuous $SL(2,\mathbb{R})$-invariant probability measure, a fact proved independently by Masur and Veech [34, 43]. Properties of the translation flow on a compact flat surface can be derived from an associated dynamical system on moduli space, namely, the action of the diagonal subgroup of $SL(2,\mathbb{R})$ on the moduli space, also known as the Teichmüller flow.

The question of ergodicity of the translation flow on a flat surface is addressed by studying the beautiful interplay between the dynamics on the flat surface and that of the Teichmüller flow on the moduli space of quadratic differentials. The relationship between the dynamics of the translation flow on a flat Riemann surface and that of the dynamics of the Teichmüller flow on the moduli space of quadratic differentials is given by a famous result of Masur, known as **Masur's criterion**: if the translation flow on a flat Riemann surface is minimal and not uniquely ergodic, then the Teichmüller orbit (of the class of that flat metric on which our translation flow is defined) leaves every compact set of the moduli space [34, 35]. In fact, for almost every $\theta \in S^1$, the translation flow generated by $\alpha_\theta = e^{i\theta}\alpha$ is uniquely ergodic [29] and the set of non-ergodic directions has Hausdorff dimension at most $\frac{1}{2}$ [36, 35].

There are very special flat surfaces whose $SL(2,\mathbb{R})$ orbit is three-dimensional.

These flat surfaces are called Veech surfaces and what makes them special is a large collection of "symmetries" which preserve the flat structure. These symmetries renormalize the translation flow via the action of the diagonal subgroup of $SL(2,\mathbb{R})$, i.e., the action of the Teichmüller flow. For these special surfaces it suffices to study its $SL(2,\mathbb{R})$ orbit to derive dynamical properties of the translation flow on it. In particular, the phase space of the orbit is a three-dimensional manifold, regardless of the genus of the surface, which is in high contrast with the dimension of the phase space in the typical case, which grows linearly with the genus of the surface. In these special cases, Masur's criterion can be expressed as follows: if the translation flow on a Veech surface is minimal but not uniquely ergodic, then its Teichmüller orbit leaves every compact subset of $SL(2,\mathbb{R})/SL(S)$, where $SL(S)$ is the large collection of symmetries already mentioned (and defined in §3.1.2), called the *Veech group* of the surface S. Veech surfaces have the additional property of satisfying the *Veech dichotomy*: the translation flow in any direction θ on a Veech surface is either completely periodic or uniquely ergodic. By completely periodic we mean that all orbits which do not emanate from singularities are closed.

Since the dynamics of finite-genus flat surfaces are by now very well understood, there has been a recent surge in the study of the dynamics of the translation flow on flat surfaces of infinite genus [9, 27, 21, 7, 22, 23, 20, 39]. In this case, all nice structure from the finite-genus theory is lost. In particular, there is no well-defined notion of moduli spaces which allow us to carry out an analogous study and thus most results so far about the ergodicity of the translation flow on a flat surface of infinite genus are done in a case-by-case scenario. A common approach for all of the examples known and studied is the genus-independent approach already used in the finite genus case, that is, by exploiting the properties given by the Veech group of the surface.

There are two types of infinite-genus flat surfaces that can be considered: flat surfaces with finite area and those with infinite area. At the moment, it seems there are more results for the ergodicity of the translation flow in the case of infinite area flat surfaces of infinite genus. Most of these surfaces are $\mathbb{Z}^d$ branched coverings of surfaces of finite area and one can recover some information about the dynamics on the cover from the dynamics on the finite-genus surface being covered. In particular we should mention the results of [20], where a criterion for the non-ergodicity of the translation flow for a full measure set of directions is established for a large class of flat surfaces of infinite genus, which seems to be the most general result concerned with flat surfaces of infinite genus and infinite area. There are some infinite genus flat surfaces of finite area in the literature with non-trivial Veech groups, but there has been no unifying approach in these cases to prove ergodicity of the translation flow, although the results of [23] are a step in this direction.

In this chapter we give a general proof of the ergodicity of the translation flow for infinite genus flat surfaces of finite area with sufficiently large Veech group. In spirit, our theorem is very much like Masur's criterion. The main result is the following.

Theorem 4. *Let S be a flat surface of finite area whose Teichmüller orbit does not leave every compact set of $SL(2,\mathbb{R})/SL(S)$. Then the translation flow is ergodic* *with respect to Lebesgue measure.*

In style, however, our theorem is different from Masur's criterion since the methods are quite different. In particular, it is not clear from this approach that unique ergodicity can be proved. What we gain is that we obtain an ergodicity theorem by weakening the hypotheses by removing the minimality requirement in Masur's criterion.

Theorem 4 applies to all of the known flat surfaces of infinite genus and finite area with non-trivial Veech groups [9, 7, 23]. For some of these examples our result gives ergodicity for the translation flow on surfaces where other methods could not and thus proves to be useful as a general tool, readily applicable to any new examples of flat surfaces of finite area and infinite genus with a non-trivial Veech group. We are particularly interested to see whether it can be applied to the family of surfaces in [11]. Comparing the results in this chapter with others in the literature, it is reasonable to conjecture that there are flat surfaces of infinite genus and finite area whose translation flows are ergodic but not uniquely ergodic.

Theorem 4 applies to any flat surface of finite area with nontrivial Veech group. For a Veech surface (which is defined in Section 3.1.2), the proof of the Veech dichotomy relies on Masur's criterion to make the conclusion about unique ergodicity. Therefore we can replace Masur's criterion with Theorem 4 in the proof of the Veech dichotomy to obtain a weaker version, but one which holds for non-compact Veech surfaces.

Theorem 5 (Weak Veech dichotomy)**.** *Let S be a Veech surface of finite area. Then* *the translation flow in any direction θ is either ergodic or completely periodic.*

It is unknown whether there exist flat surfaces of finite area and infinite genus for which this dichotomy holds.

Section 1 gives background on flat surfaces from a geometric and analytic point of view, as well as background on Veech groups. Section 2 deals with proving the main result, Theorem 4, followed by a discussion of the weak Veech dichotomy. In Section 3 we apply the main result to surfaces of infinite genus and finite area.

3.1 Flat Surfaces and Veech Groups

3.1.1 Flat structures

Let S be a Riemann surface with no boundary and $\Sigma \subset \bar{S}$ a discrete set of points. S is a flat surface if it carries an atlas $\{(\mathcal{U}_i, \varphi_i)\}_i$ with $\mathcal{U}_\alpha, \mathcal{U}_\beta \subset S \backslash \Sigma$ such that for any two charts $(\mathcal{U}_\alpha, \varphi_\alpha)$ and $(\mathcal{U}_\beta, \varphi_\beta)$, $\varphi_\alpha \circ \varphi_\beta^{-1}(z) = \pm z + c_{\alpha\beta}$ for $z \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$. If $\varphi_\alpha \circ \varphi_\beta^{-1}(z) = z + c_{\alpha\beta}$ for all z and α, β, then S is a *translation surface*. Otherwise it is a *half-translation surface*. Here we will only be interested in translation surfaces since half-translation surfaces can be studied by passing to an appropriate double cover where they become translation surfaces.

The points which make up Σ are the singularities of S. Any compact translation surface S of genus greater that one must, by the Gauss-Bonnet theorem, have overall negative curvature. Since a translation surface has a flat metric everywhere on $S \backslash \Sigma$, any surface of genus greater than one must have its negative curvature concentrated on Σ. Thus at a point $p \in \Sigma$ the metric can be written in polar coordinates (r, θ) as $\sqrt{dr^2 + (ar\, d\theta)^2)}$, where $2\pi a$ is the cone angle at p.

The complex structure of a translation surface can also be completely obtained by an *Abelian differential*, i.e., a holomorphic 1-form. In local coordinates away from Σ any Abelian differential can be written as $\alpha = \phi(z)\, dz$, with ϕ a holomorphic function, and the metric can be written locally as $R_\alpha = |\alpha||dz|$ while the area form is given on $S \backslash \Sigma$ by $\omega_\alpha = \Re(\alpha) \wedge \Im(\alpha)$. Any Abelian differential α comes with a pair of transverse measured foliations, the horizontal and vertical foliations, $\mathcal{F}^h$ and $\mathcal{F}^v$. They are the foliations generated by the distributions $\operatorname{Ker} \Im(\alpha)$ and $\operatorname{Ker} \Re(\alpha)$, respectively.

A flat surface will be denoted as (S, α) which emphasizes the metric and foliated structure imposed on the topological surface S by the Abelian differential α. The flat surface (S, α_θ), where $\alpha_\theta = e^{i\theta}\alpha$, carries the same metric as the flat surface (S, α), but their foliations differ. The foliations on (S, α_θ) are simply obtained by "rotating" the foliations on (S, α) by the angle θ. Sometimes we may refer to S as a flat surface without specification of any Abelian differential. In such case, we mean that we are considering (S, α_θ) for some α and all $\theta \in S^1$.

A *regular* leaf for the vertical or horizontal foliation is a leaf which does not limit to a point in Σ, i.e., a singularity of α. Otherwise it is called *singular*. A *saddle connection* of α is a singular leaf of the vertical or horizontal foliation which connects two singularities. We remark that in the case of non-compact surfaces, the set of singularities also includes the ideal boundary of S, a feature that is not present for compact surfaces. As such, in some cases, saddle connections may be arbitrarily long, even of infinite length. By the Poincaré recurrence theorem, the set of saddle connections on a flat surface of finite area has zero measure.

We denote the set of saddle connections of an Abelian differential on a surface S by $\mathrm{SC}(S, \alpha)$. The horizontal or vertical foliations of an abelian differential are *periodic* if all but perhaps the singular leaves are closed. In such case, by considering $S \backslash \mathrm{SC}(S, \alpha)$, the surface decomposes into a union of cylinders bounded by saddle connections and each cylinder is foliated by homotopically-equivalent closed leaves of the foliation. It may be possible for a surface $S \backslash \mathrm{SC}(S, \alpha)$ to decompose as the disjoint union of periodic components (cylinders), and minimal components.

In this chapter we deal with flat surfaces of infinite genus and finite area. For such surfaces the set of singularities Σ not only consists of finite-angle singularities as in the compact case, but in addition of singularities of infinite angle. This will be of no consequence in the present analysis. Examples of such surfaces are found in section 3.3. These surfaces also carry a translation structure just as in the finite genus case and therefore a (singular) flat metric given by an Abelian differential α. The requirement that the surface have finite area is equivalent to the requirement that the norm of the Abelian differential, $\|\alpha\| = \int_S |\alpha|^2 \, \omega_\alpha$, is finite.

The vector fields X and Y of norm 1 which, respectively, are tangent to the foliations $\mathcal{F}^{h,v}$, commute and in addition have the following properties [16]:

1. $\{X, Y\}$ is an orthonormal frame for the tangent bundle TS on $S \backslash \Sigma$ with respect to the metric R_α.

2. X and Y preserve the smooth area form ω_α, thus $\eta_X \equiv \imath_X \omega_\alpha$ and $\eta_Y \equiv -\imath_Y \omega_\alpha$ are closed, smooth 1-forms on $S \backslash \Sigma$.

3. η_X and η_Y generate the measured foliations $\mathcal{F}^{h,v}$ on $S\backslash\Sigma$.

The complex structure provided by the Abelian differential α also defines spaces of functions compatible with the induced foliations and the vector fields X and Y. We define

$$L^2_\alpha(S) = \left\{ u : \int_S |u|^2 \, \omega_\alpha \equiv \|u\|^2 < \infty \right\} \tag{3.1}$$

to be the weighted L^2 spaces of S. These spaces have a natural structure of Hilbert spaces with inner product $(\cdot,\cdot)_\alpha$ defined as

$$(u,v)_\alpha \equiv \int_S u\bar{v}\, \omega_\alpha$$

which satisfies, by the invariance of the of ω_α under X and Y,

$$(Xu,v)_\alpha = -(u,Xv)_\alpha \text{ and } (Yu,v)_\alpha = -(u,Yv)_\alpha. \tag{3.2}$$

Define the $s-$norm to be

$$\|u\|_s^2 \equiv \sum_{i+j\le s} \|X^iY^ju\|^2. \tag{3.3}$$

Let $H^s_\alpha(S)$ be the completion of the set of smooth functions with finite $\|\cdot\|_s$ norm. We denote by $H^{-s}_\alpha(S)$ the dual space of $H^s_\alpha(S)$. From the vector fields X and Y, we construct the *Cauchy-Riemann operators*

$$\partial^\pm_\alpha \equiv X \pm iY, \tag{3.4}$$

the kernels of which contain the meromorphic, respectively anti-meromorphic, functions which are elements of $L^2_\alpha(S)$. As shown in [16, Proposition 3.2], it follows from (3.2) that $(\partial^\pm_\alpha)^*$ are extensions of $-\partial^\mp_\alpha$. It follows by Hilbert space theory that we have the orthogonal splitting

$$L^2_\alpha(S) = \text{Range}(\partial^\pm_\alpha) \oplus^\perp \text{ Ker}\,(\partial^\mp_\alpha). \tag{3.5}$$

Finally, the Dirichlet form $Q_\alpha : H^1_\alpha(S) \times H^1_\alpha(S) \to \mathbb{C}$ is defined as

$$Q_\alpha(u, v) = (Xu, Xv)_\alpha + (Yu, Yv)_\alpha = (\partial^\pm_\alpha u, \partial^\pm_\alpha v)_\alpha.$$

The *Dirichlet norm* of a function u is defined to be $Q_\alpha(u) \equiv Q_\alpha(u, u)$.

3.1.2 $SL(2, \mathbb{R})$ action

Let (S, α) denote a surface S with a complex structure given by an Abelian differential α. There is a well-defined action of the group $SL(2, \mathbb{R})$ on (S, α). For $A \in SL(2, \mathbb{R})$, we define $A \cdot (S, \alpha)$ to be the surface (S, α) with charts post-composed with the action of A on $\mathbb{R}^2$.

The stabilizer of this action is denoted by $Stab(S, \alpha)$ and its image in $PSL(2, \mathbb{R})$ is called the *Veech group* of (S, α). It is usually denoted by $SL(S, \alpha)$ or $Aff(S, \alpha)$ since it coincides with the group of derivatives of affine diffeomorphisms (with respect to α) of S. In other words, if $r \in SL(S, \alpha)$, then there exists a unique affine diffeomorphism f_r with constant derivative Df_r such that the action of Df_r on the complex structure of (S, α) coincides with that of r. Such diffeomorphisms will be called *Teichmüller maps*.

When S is compact, the Veech group $SL(S, \alpha)$ is always a discrete subgroup and, when $SL(S, \alpha)$ is a lattice, (S, α) is called a *Veech surface*. Usually one expects the Veech group of a surface to be trivial. Thus, surfaces with non-trivial Veech groups turn out to be quite interesting (and are hard to find). The $SL(2, \mathbb{R})$-orbit of (S, α), denoted by $D_{(S,\alpha)}$, is isometric to the unit tangent bundle of the Poincaré disk $\mathbb{H}$, and is called the *Teichmüller disk* of (S, α). The Veech group $SL(S, \alpha)$ acts on $D_{(S,\alpha)}$ by isometries of the hyperbolic metric. The quotient of the Teichmüller disk of (S,α) by its Veech group is denoted by

$$H_{(S,\alpha)} \equiv \mathbb{H}/SL(S,\alpha),$$

where $\mathbb{H} = SL(2,\mathbb{R})/SO(2,\mathbb{R})$. The projection map will be denoted by

$$\Pi_{(S,\alpha)} : D_{(S,\alpha)} \to H_{(S,\alpha)}.$$

The disk $H_{(S,\alpha)}$ has finite area if, and only if, (S,α) is a Veech surface. However, if (S,α) is compact, $H_{(S,\alpha)}$ is never compact. It is not known whether there exists a flat surface of finite area and infinite genus whose Veech group is not discrete.

It is natural to talk about *elliptic, parabolic,* and *hyperbolic* elements of $SL(2,\mathbb{R})$ corresponding, respectively, to elements with zero, one, and two distinct real eigenvalues. Elliptic elements are conjugate to the elements of the subgroup $SO(2,\mathbb{R})$ while parabolic elements are, in a conveniently-rotated coordinate system, of the form

$$h^t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad h_s = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$$

for $s,t \in \mathbb{R}$. Parabolic elements generate both parabolic elements and hyperbolic elements. Associated to every parabolic element there corresponds a unique invariant direction corresponding to its eigenvector and we say that the parabolic element *fixes* this direction. Any direction invariant by a hyperbolic element is also said to be fixed by it.

The diagonal subgroup

$$g_t \equiv \left\langle \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix} : t \in \mathbb{R} \right\rangle$$

is an important subgroup of $SL(2,\mathbb{R})$. Its action on the Teichmüller disk of a flat surface is called the *Teichmüller geodesic flow* since it minimizes distances between two points in the Teichmüller disk of a flat surface. Its action on the complex structure of (S,α) is also referred to as Teichmüller deformation.

3.2 Ergodicity

Recall that the complex structure of any translation surface is given by an Abelian differential α which defines a commuting pair of vector fields X and Y of norm 1. Let

$$\partial_t^{\pm} \equiv e^t X \pm i e^{-t} Y = X_t \pm i Y_t \tag{3.6}$$

be the one-parameter family of Cauchy-Riemann operators defined for the complex structure given by the Abelian differential

$$\alpha_t = e^{-t}\Re(\alpha) + i e^{t}\Im(\alpha).$$

In other words, one can think of the operators $\partial_t^{\pm}$ as the Cauchy-Riemann operators of the surface $g_t \cdot (S,\alpha) = (S,\alpha_t)$. To be consistent with the notation of (3.4), we make the identification $\partial_t^{\pm} \equiv \partial_{\alpha_t}^{\pm}$.

Note that the volume form ω_{α_t} given by α_t is invariant, i.e., $\omega_{\alpha_t} = \omega_\alpha$ for all t, and thus the Hilbert space of square integrable functions with respect to ω_{α_t} is the same for all t (and so are all derived Sobolev spaces $H^s_{\alpha_t}(S)$). There is, however, a one-parameter splitting of L^2_α as in (3.5):

$$L^2_\alpha(S) = \mathrm{Range}(\partial_t^\pm) \oplus_t^\perp \ \mathrm{Ker}\,(\partial_t^\mp). \tag{3.7}$$

Thus, for any function $u \in L^2(S)$ and for any $t \in \mathbb{R}$ there exist functions $v_t^\pm \in H^1_\alpha(S)$, meromorphic functions m_t^- and anti-meromorphic functions m_t^+ such that

$$u = \partial_t^+ v_t^+ + m_t^- = \partial_t^- v_t^- + m_t^+. \tag{3.8}$$

If the surface is compact, the spaces $\mathrm{Ker}\,(\partial_t^\pm)$ are finite dimensional by the Riemann-Roch theorem. For surfaces of infinite genus this is not the case necessarily, but this fact is irrelevant in the discussion.

If we chose each $v_t^\pm$ to have zero average, then the one parameter families $v_t^\pm$ are smooth. So we will assume this without loss of generality. Finally, it is easy to verify that

$$\overline{\partial_t^\pm f} = \partial_t^\mp \bar{f}. \tag{3.9}$$

Remark 1. As remarked in [10, Proposition 2.5], since our surfaces have finite area, by [38, Lemma 1.8], the horizontal and vertical foliations are minimal if there are no saddle connections, since that result only depends on the finiteness of area. It also follows from the proof of [38, Lemma 1.8] that the trajectory of a point which is not contained in a saddle connection is dense in an open subset, i.e., its minimal component has non-empty interior and therefore has positive measure.

To address the issue of ergodicity of the flows (or foliations) generated by X and Y, we are interested in studying functions which are X-invariant (or Y-invariant). Note that if $u \in L^2(S)$ is an X-invariant function, i.e., $Xu = 0$, by considering its real part we can study X-invariant functions while assuming they are real valued. We also assume that if $u \in L^2$ is a real-valued, invariant function, then $u \in L^\infty$. Indeed, for any invariant u, the set $A_r \equiv \{x \in S : |u(x)| > r\}$ is invariant for any r, so for our purposes we can work with the function $u' = \chi_{S\setminus A_r} u + \chi_{A_r}$ for some r, which implies $\|u'\|_\infty < \infty$.

Lemma 3. *Let u be a zero-average, real-valued, X-invariant function on a flat surface of finite area. Then, writing u as in (3.8), we have that v_t^+ (and thus v_t^-) is purely imaginary and that*

$$v_t^+ = \overline{v_t^-} \tag{3.10}$$

$$m_t^+ = \overline{m_t^-}. \tag{3.11}$$

Proof. By applying $\partial_t^\pm$ to the decomposition (3.8) we obtain that

$$\begin{aligned} \triangle_t v_t^+ &= (\partial_t^-)^2 v_t^- + \partial_t^- m_t^+ \\ \triangle_t v_t^- &= (\partial_t^+)^2 v_t^+ + \partial_t^+ m_t^- \end{aligned} \tag{3.12}$$

in $H_\alpha^{-1}(S)$, where $\triangle_t = \partial_t^\pm \partial_t^\mp$ is the Laplacian with respect to the complex structure given by α_t. Since $X_t = \frac{1}{2}(\partial_t^+ + \partial_t^-)$, then $Xu = 0$ implies, by (3.8),

$$(\partial_t^+)^2 v_t^+ + \partial_t^+ m_t^- + \triangle_t v_t^+ = 0 \qquad \text{and} \qquad (\partial_t^-)^2 v_t^- + \partial_t^- m_t^+ + \triangle_t v_t^- = 0. \tag{3.13}$$

Putting (3.12) and (3.13) together,

$$\triangle_t v_t^+ - \partial_t^- m_t^+ = -\triangle_t v_t^- - \partial_t^- m_t^+,$$

which implies $\triangle_t(v_t^+ + v_t^-) = 0$. In other words, $\delta v_t \equiv v_t^+ + v_t^- \in H_\alpha^1(S)$ is a harmonic function. Moreover, since $v_t^\pm \in H_\alpha^1(S)$, $Q_\alpha(\delta v_t) = -(\partial_t^\pm \delta v_t, \partial_t^\pm \delta v_t) = 0$, i.e., the Dirichlet norm of δv_t is zero. Since the kernel of Q_α consists of constant functions and $v_t^\pm$ can be chosen to be of zero average (without loss of generality),

$$v_t^+ = -v_t^-. \tag{3.14}$$

Since u is real-valued, using (3.8) and (3.9),

$$u = \partial_t^+ v_t^+ + m_t^- = \overline{\partial_t^- v_t^-} + \overline{m_t^+} = \partial_t^+ \bar{v}_t^- + \overline{m_t^+} = \bar{u},$$

which, by (3.7), $m_t^+ = \overline{m_t^-}$. From the equation above and the fact that $v_t^+ = -v_t^-$ it also follows that $\Re(v_t^+) \in \mathrm{Ker}(\partial_t^+)$. Using the same equation above and (3.9) again, it follows that $\Re(v_t^-) \in \mathrm{Ker}(\partial_t^-)$. Since $\Re(v_t^+) = -\Re(v_t^-) \in \mathrm{Ker}(\partial_t^+) \cap \mathrm{Ker}(\partial_t^-)$, $\Re(v_t^\pm)$ is constant and, since $v_t^\pm$ has zero average, it vanishes. □

Given these relations, we can compute the evolution of the norm of $m_t^\pm$.

Lemma 4. *Under the splitting (3.8) for a real-valued, X-invariant function u on a flat surface of finite area, the evolution of the norm of $m_t^\pm$ is described by*

$$\frac{d}{dt}\|m_t^\pm\|^2 = 4\|\Im(m_t^+)\|^2.$$

Proof. We first note that $\frac{d}{dt}\partial_t^\pm = \partial_t^\mp$. We perform the calculation m_t^+; the case for m_t^- is essentially the same.

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|m_t^+\|^2 &= \mathrm{Re}\left(\frac{d}{dt}m_t^+, m_t^+\right) = \mathrm{Re}\left(\frac{d}{dt}(u - \partial_t^- v_t^-), m_t^+\right) \\
&= -\mathrm{Re}\left(\frac{d}{dt}\partial_t^- v_t^-, m_t^+\right) = -\mathrm{Re}\left(\partial_t^+ v_t^- - \partial_t^- \dot{v}_t^-, m_t^+\right) \\
&= \mathrm{Re}\,(\partial_t^+ v_t^+, m_t^+) = \mathrm{Re}\,(u - m_t^-, m_t^+) \\
&= \|m_t^+\|^2 - \mathrm{Re}\int_S (m_t^+)^2\,\omega_\alpha = 2\|\Im(m_t^+)\|^2.
\end{aligned}$$

□

Definition 5. The g_t orbit of (S, α) is *recurrent* if for any $\varepsilon > 0$ there is an $s_\varepsilon \in \mathbb{R}^+$ and an element $r \in SL(S, \alpha)$ such that the distance between $g_{s_\varepsilon} \cdot (S, \alpha)$ and $r \cdot (S, \alpha)$ is less than ε.

This definition gives the usual definition, from the point of view of topological dynamics, of a recurrent orbit on $H_{(S,\alpha)}$. We use this definition since it will be more useful in the proof of ergodicity.

Remark 2. If (S, α) is g_t-recurrent, then for any sequence of $\varepsilon_i \to 0$ there is a sequence of angles θ_i and times $t_i \to \infty$ such that the distance between $g_{t_i} \cdot (S, \alpha)$ and $g_{t_i} r_{\theta_i}(S, \alpha)$ is less than ε_i and $g_{t_i} r_{\theta_i} \in SL(S, \alpha)$. As such, it follows that $r_{\theta_i} \to \mathrm{Id}$, i.e., $\theta_i \to 0$. Indeed, since the $SL(2, \mathbb{R})$ orbit of (S, α) isometric to the unit tangent bundle of the Poincaré disk, i.e., a simply connected surface with constant sectional curvature $\kappa = -4$, it follows from the hyperbolic law of sines that

$$|\sin(\theta_i)| \leq \frac{\sinh(\varepsilon_i)}{\sinh(2t_i)}.$$

Moreover, since $g_{t_i} r_{\theta_i} \in SL(S, \alpha)$, there exists a sequence of affine diffeomorphisms f_i such that $g_{t_i} r_{\theta_i} = Df_i \in SL(S, \alpha)$.

For a flat surface (S, α) with a recurrent g_t orbit, we will call a sequence of quadruples

$$\{(t_i, \theta_i, \varepsilon_i, f_i)\}_{i=1}^{\infty} \in (\mathbb{R}^+ \times S^1 \times \mathbb{R}^+ \times SL(S, \alpha))^{\mathbb{N}}$$

as in the above remark the *recurrent data* of (S, α). We can assume without loss of generality that $\varepsilon_{i+1} < \varepsilon_i$ for all i.

Lemma 5. *Let (S, α) be a flat surface of finite area whose g_t-orbit is recurrent. Then no component of $S \backslash SC(S, \alpha)$ is a cylinder.*

Proof. Suppose there is a component $C \subset S \backslash \mathrm{SC}(S, \alpha)$ which is a cylinder. Let w_C and A_C be the waisturve and area of C, respectively. The Teichmüller maps $f_i \in SL(S, \alpha)$ in the recurrent data are affine and therefore take cylinders to cylinders. Define $C_0 = C$ and $C_i = f_i^{-1}(C)$ for $i > 0$. By applying the Teichmüller deformation g_{t_i} and the Teichmüller map f_i^{-1} we see that the length w_{C_i} of the waistcurve of cylinder C_i is $e^{-t_i} w_C$. Note that the angle θ_i between the waistcurves of C and C_i satisfies $\sin(\theta_i) \leq \sinh(\varepsilon_i)/\sinh(2t_i)$. By passing to the appropriate subsequences, we can control how fast the length of the waistcurves of the C_i diminish as well as how small the angle is between waistcurves.

We claim that $\omega_\alpha(C_i \cap C_j) = 0$ for all $i \neq j$. Indeed, let us consider C_1. Since the waistcurve of C_1 is exponentially smaller than that of C and the angle between the two foliations exponentially small (as remarked above, this can be done by passing to a subsequence if necessary), it follows that the trajectories foliating C_1 cannot close up if $\omega_\alpha(C \cap C_1) \neq 0$. By the same token, $\omega_\alpha(C \cap C_2) = 0$ and for the same reasons in fact $\omega_\alpha(C_1 \cap C_2) = 0$. Considering this for any i, we have that $\omega_\alpha(C_i \cap C_j) = 0$ for all $j < i$. But if these cylinders do not overlap and their area is the same since the Teichmüller maps preserve area, it is impossible to fit them all in S since the total area is finite. It therefore follows that there is no component which is a cylinder. $\square$

Lemma 6. *Let (S, α) be a flat surface of finite area whose g_t orbit is recurrent and*

$u \in L^2_\alpha(S)$ be a real-valued, X-invariant function. Then there is a sequence of times $\{t_i\}$ and a sequence of affine diffeomorphisms $\{f_i\} \subset SL(S,\alpha)$ such that the family of functions $\mathbb{F} = \{(f_i^{-1})^* m^+_{t_i}\}_{i=0}^\infty$, where m_t^+ is the α_t-meromorphic part of u as in (3.8), is a normal family on $S \backslash \Sigma$.

Proof. Since $g_t(S,\alpha)$ is recurrent, it follows that for the compact set

$$\mathcal{K}_{(S,\alpha)} = \bigcup_{\substack{\theta \in S^1 \\ t \in [0,\varepsilon_1]}} g_t r_\theta(S,\alpha)$$

in the Teichmüller disk of (S,α), we have $\Pi_{(S,\alpha)}(g_{t_i}(S,\alpha)) \in \Pi_{(S,\alpha)}(\mathcal{K}_{(S,\alpha)})$. As such, for each i there exists a $\varphi_i \in S^1$ such that the function $F_i \equiv (f_i^{-1})^* m^+_{t_i}$ is meromorphic on $g_{\varepsilon_i} r_{\varphi_i}(S,\alpha)$.

Let $K \subset S \backslash \Sigma$ be a compact set. Since every point in the compact set $\mathcal{K}_{(S,\alpha)}$ represents a deformation of the conformal structure of S, the quantity

$$\delta_K \equiv \min d(K, \Sigma),$$

where the minimum is taken over conformal deformations of S corresponding to points in $\mathcal{K}_{(S,\alpha)}$ and the distance d is taken with respect to α, is well defined. Since $\mathcal{K}_{(S,\alpha)}$ is a compact family of deformations, it follows from the Cauchy integral formula (see for example [24, Theorem 1.2.4]) that for any neighborhood K' of K in $S \backslash \Sigma$ there exists a constant $M_{K'}$ such that for all F_i we have that

$$|F_i(z)| \leq M_{K'} \|F_i\|_{L^1_\alpha(K')} \leq M_{K'} \|F_i\| \leq M_{K'} \|u\|$$

for any $z \in K$ and therefore the functions in $\mathbb{F}$ are uniformly bounded on K.

Let $p \in K$ and consider a disk D_i of radius $\delta_K/2$ in the conformal structure given by $g_{\varepsilon_i} r_{\varphi_i}(S, \alpha)$ centered at p. It follows by the Cauchy integral formula that

$$|F_i(z_1) - F_i(z_2)| \leq \frac{16 M_K \|u\|}{\delta_K} |z_1 - z_2| \tag{3.15}$$

for any two points z_1, z_2 in a disk D_i^* of radius $\delta_K/4$ in the conformal structure given by $g_{\varepsilon_i} r_{\varphi_i}(S, \alpha)$ centered at p. Let $D_K(p)$ be a disk of radius $e^{-\varepsilon_1}\delta_K/4$ in the conformal structure of (S, α). Then $D_K(p) \subset D_i^*$ for all i. Therefore, by (3.15), $\mathbb{F}$ is equicontinuous in $D_K(p)$ and thus on K since K can be covered by finitely many disks of radius $e^{-\varepsilon_1}\delta_K/4$. The statement then follows from the Arzela-Ascoli theorem. □

Since the norm of $m_t^{\pm}$ is always bounded, i.e., $0 \leq \|m_t^{\pm}\| \leq \|u\|$, then certainly

$$\liminf \frac{d}{dt} \|m_t^{\pm}\|^2 = \liminf \|\Im(m_t^{\pm})\|^2 = 0$$

by Lemma 4. It will be crucial that along our the sequence of recurrent times $\frac{d}{dt}\|m_{t_i}^{\pm}\|^2 = 4\|\Im(m_{t_i}^{\pm})\|^2 \to 0$. The following lemma shows that we can always find a sequence of recurrent times for which this is possible.

Lemma 7. *Let $u = \partial_t^{\pm} v_t^{\pm} + m_t^{\mp}$ be a real-valued, X-invariant function on a flat surface of finite area (S, α) whose g_t orbit is recurrent. Then there is a sequence $\{t_i\}$ of recurrent times as in Remark 2 such that $\|\Im(m_{t_i}^{\pm})\| \longrightarrow 0$ as $t_i \longrightarrow \infty$.*

Proof. By Remark 2 we have recurrent data $\{\varepsilon_i, t_i, \theta_i\}$ such that $\mathrm{dist}(g_{t_i}, g_{t_i} r_{\theta_i}) \leq \varepsilon_i \to 0$. If our sequence has the desired property, we are done. Otherwise suppose there is a subsequence t_{i_j}, $j \in \mathbb{N}$, and a number $\delta > 0$ such that $\|\Im(m_{t_{i_j}}^{\pm})\| \geq \delta$ for all j.

Since $4\|\Im(m_t^\pm)\|^2 = \frac{d}{dt}\|m_t^\pm\|^2$ is continuous and $\|m_t^\pm\|^2$ bounded, there exists a sequence $\tau_n \to \infty$ and a further subsequence $t_{i_{j_n}}$, such that

$$|\tau_n - t_{i_{j_n}}| \leq \frac{1}{\sqrt{n}} \qquad \text{and} \qquad \|\Im(m_{\tau_n}^\pm)\|^2 < \frac{1}{\sqrt{n}}.$$

Then

$$\begin{aligned} \operatorname{dist}(g_{\tau_n}, g_{t_{i_{j_n}}} r_{\theta_{i_{j_n}}}) &\leq \operatorname{dist}(g_{\tau_n}, g_{t_{i_{j_n}}}) + \operatorname{dist}(g_{t_{i_{j_n}}}, g_{t_{i_{j_n}}} r_{\theta_{i_{j_n}}}) \\ &\leq \frac{1}{\sqrt{n}} + \varepsilon_{i_{j_n}} \equiv \hat{\varepsilon}_n \longrightarrow 0. \end{aligned}$$

Since $g_{t_i} r_{\theta_i} \in SL(S,\alpha)$ for all i, g_{τ_i} is another sequence of recurrent times with the desired property. $\square$

Lemma 8. *Let (S,α) be flat surface of finite area with a recurrent g_t orbit and recurrent data $\{(t_i,\theta_i,\varepsilon_i,f_i)\}_{i=1}^\infty$. Then*

$$f_i^*\partial_0^\pm = \frac{1}{\cos\theta_i}\left[\partial_{t_i}^\pm f_i^* - f_i^*(e^{2t_i}Y \mp ie^{-2t_i}X)\sin\theta_i\right]. \tag{3.16}$$

Moreover it follows that if $u = \partial_t^- v_t^- + m_t^+$ is a real, X-invariant function, then

$$\partial_0^+\left((f_i^{-1})^* m_{t_i}^+\right) \longrightarrow 0$$

weakly.

Proof. Let $\zeta \in H^1(S)$. We will now drop the indices for a while to avoid tedious notation and work under the assumption that t is large and ε, θ are small. Since we know exactly how the derivative of f acts, we have

$$\partial_t^\pm f^*\zeta = f^*\left[\cos\theta\partial_0^\pm + (e^{2t}Y \pm ie^{-2t}X)\sin\theta\right]\zeta,$$

from which (3.16) follows. Using this:

$$\begin{aligned}((f^{-1})^* m_t^+, \partial_0^- \zeta) &= (m_t^+, f^* \partial_0^- \zeta) \\ &= \sec\theta (m_t^+, \partial_t^- f^* \zeta - \sin\theta f^*(e^{2t} Y + ie^{-2t} X)\zeta) \\ &= -\sec\theta (m_t^+, \sin\theta f^*(e^{2t} Y + ie^{-2t} X)\zeta).\end{aligned}$$

Using the estimate from Remark 2:

$$\begin{aligned}|((f^{-1})^* m_t^+, \partial_0^- \zeta)| &= e^{2t} |\sin\theta| |((f^{-1})^* m_t^+, (Y + ie^{-4t} X)\zeta)| \sec\theta \\ &\leq \frac{\sinh(\varepsilon)}{\sinh(2t)} e^{2t} |((f^{-1})^* m_t^+, (Y + ie^{-4t} X)\zeta)| \sec\theta \\ &\leq \sinh(\varepsilon) \|m_t^+\| \|\zeta\|_1 \\ &\leq \sinh(\varepsilon) \|u\| \|\zeta\|_1.\end{aligned}$$

Since $\varepsilon_i \to 0$, the claim follows.

□

From Lemmas 4, 7, and 8, we can get the following crucial result.

Proposition 2. *Let (S, α) be a flat surface of finite area which is g_t-recurrent and $u \in L^2_\alpha(S)$ a real valued, X-invariant function of zero average. Then there exists a sequence $t_i \to \infty$ such that $f_i^* m_{t_i}^+ \rightharpoonup 0$ weakly, where m_t^+ is the meromorphic part of u as in (3.8), and $f_i \in SL(S, \alpha)$ are Teichmüller maps associated to the recurrent data of (S, α).*

Proof. Let u be a real-valued, X-invariant function of zero average. Writing it as in (3.8),

$$u = \partial_t^+ v_t^+ + m_t^- = \partial_t^- v_t^- + m_t^+.$$

Note that the norm $\|m_t^\pm\|$ is always bounded by the norm of u and by Lemma 4 is a non-decreasing function of t. Let $\{c_i\} \equiv \left\{(f_i^{-1})^* m_{t_i}^+\right\}$ for $i \in \mathbb{N}$ be a sequence of functions on (S, α_0), where the f_i are as in Remark 2. Since $\left\|(f_i^{-1})^* m_{t_i}^+\right\| = \|m_{t_i}^+\| \leq \|u\|$ for all i, the sequence $\{c_i\}$ is bounded. Therefore, there exists a function m_*^+ and weakly convergent subsequence $\{c_{i_j}\}$ such that $(f_{i_j}^{-1})^* m_{t_{i_j}}^+ \rightharpoonup m_*^+$.

We can assume, by Lemma 7, that (since f_i^* is unitary)

$$\|\Im\left((f_i^{-1})^* m_{t_i}^+\right)\| \longrightarrow 0 \tag{3.17}$$

as $i \to \infty$. By Lemma 8, m_*^+ is meromorphic and by (3.17) it has zero imaginary part and thus it is a constant. Since u has zero average, $\int_S m_t\, \omega_\alpha = 0$ for all t. It follows from this that

$$\int_S (f_i^{-1})^* m_{t_i}^+\, \omega_\alpha = 0$$

for all i since the Jacobian of f_i is identically 1 for all i. Thus, since m_*^+ is a constant of zero average, it is identically zero. $\square$

Proposition 3. *Let (S, α) be a flat surface of finite area whose g_t orbit is recurrent. Then the translation flow is ergodic.*

Proof. Let u be a real-valued, X-invariant function of zero average. Writing it as in (3.8),

$$u = \partial_t^+ v_t^+ + m_t^- = \partial_t^- v_t^- + m_t^+.$$

Consider an exhaustion $K_0 \subset K_1 \subset K_2 \subset \cdots$ of $S \backslash \Sigma = \bigcup K_n$ by compact sets K_n such that $\omega_\alpha(K_n) \geq 1 - \frac{1}{n}$ and consider the sequences of functions $F_k^n = (f_{n_k}^{-1})^* m_{t_{n_k}}$ which, by Lemma 6, converge uniformly on K_n. By Proposition 2, for each n, $F_k^n \to 0$ uniformly as $k \to \infty$ on K_n. Therefore, for every n and $\delta > 0$, there exists an N_δ such that

$$\|(f_{n_k}^{-1})^* m_{t_{n_k}}^+\|_{L_\alpha^\infty(K_n)} < \delta \tag{3.18}$$

for all $k > N_\delta$. Equivalently,

$$B_{n,k} \equiv \|m_{t_{n_k}}^+\|_{L_\alpha^\infty(f_{n_k}(K_n))} < \delta.$$

Now let $\varepsilon > 0$ and choose n big enough so that $\omega_\alpha(S \backslash K_n) < \varepsilon$. Then

$$\begin{aligned}
\|m_{t_{n_k}}^+\|^2 &= \int_{K_n} (f_{n_k}^{-1})^*(m_{t_{n_k}}^+ u)\,\omega_\alpha + \int_{S\backslash K_n} (f_{n_k}^{-1})^*(m_{t_{n_k}}^+ u)\,\omega_\alpha \\
&= \int_{f_{n_k}(K_n)} m_{t_{n_k}}^+ u\,\omega_\alpha + \int_{f_{n_k}(S\backslash K_n)} m_{t_{n_k}}^+ u\,\omega_\alpha \\
&\leq B_{n,k}\|u\|_{L_\alpha^1(K_n)} + \|u\|_\infty \int_{f_{n_k}(S\backslash K_n)} |m_{t_{n_k}}^+|\,\omega_\alpha \\
&\leq \varepsilon\|u\|_{L_\alpha^1(K_n)} + \omega_\alpha(S\backslash K_n)^{\frac{1}{2}}\|m_{t_{n_k}}^+\|\|u\|_\infty \\
&\leq \sqrt{\varepsilon}(\sqrt{\varepsilon} + \|u\|_\infty)\|u\|
\end{aligned}$$

for $k > N_\varepsilon$ as in (3.18). This implies that $\|m_t^+\|$ can be arbitrarily small for arbitrarily large values of t. It follows by Lemma 4 that $m_t^+ \equiv 0$ for all t. Moreover we have $u = \partial_t^\pm v_t^\pm$ for some $v_t^\pm \in H_\alpha^1(S)$. Since u is real and, by Lemma 3, $v_t^\pm$ imaginary, $u = \partial_t^\pm v_t^\pm = X_t v_t^\pm \pm i Y_t v_t^\pm$ implies that $v_t^\pm$ is X-invariant.

For a point $p \in S\backslash\Sigma$ and $w, h > 0$, a (w,h)-*rectangle* for p is defined as

$$K_p(w,h) = \bigcup_{\substack{s\in(-w,w)\\ t\in(-h,h)}} \varphi_s^X \circ \varphi_t^Y(p),$$

where $\varphi^{X,Y}$ are the respective flows generated by X and Y. It is well defined for any $p \in S\backslash\Sigma$ if w and h are chosen small enough. Since $v_t^\pm$ is X-invariant, its restriction to any (w,h)-rectangle $K_p(w,h)$ for some point p is a function of one variable, namely, the Y-coordinate. Thus by the Sobolev embedding theorem we have that since $v_t^\pm \in H^1_\alpha(K_p(w,h))$, $v_t^\pm$ is a continuous function on $\overline{K_p(w,h)}$.

Let $\{K_{p_i}(w_i,h_i)\}_{i\in\mathbb{N}}$ be an open cover of $S\backslash\Sigma$. By the Sobolev embedding theorem $v_t^\pm$ is continuous on each $\overline{K_{p_i}(w_i,h_i)}$, and so it is continuous on S. If there are no saddle connections, by Remark 1, the flow φ_t^X is minimal. Since $v_t^\pm$ is X-invariant and continuous, and φ_t^X minimal, $v_t^\pm$ is constant and thus $u=0$ and we conclude that the flow is ergodic.

Otherwise suppose that there is at least one saddle connection and thus we cannot guarantee minimality. By Lemma 5, there is no periodic component and $S\backslash\mathrm{SC}(S,\alpha)$ decomposes into countably many minimal components of positive area (see Remark 1). Since $v_t^\pm$ is a continuous function on S which is constant on countably many components, it assumes countably many values. So $v_t^\pm$ is constant and $u=0$. $\square$

We can now prove the main result.

Proof of Theorem 4. If the g_t orbit of the flat surface (S,α) is recurrent, then the theorem is proved by Proposition 3. Therefore it remains to prove the theorem for flat surfaces who are not recurrent but nonetheless have a limit point ℓ in a compact set $\Lambda \subset H_{(S,\alpha)}$.

Consider a fundamental domain $\hat{\Lambda}$ of the action of $SL(S,\alpha)$ on $SL(2,\mathbb{R})/SO(2,\mathbb{R})$, let $\hat{\ell}$ be the point on this domain which projects to ℓ: $\ell = \Pi_{(S,\alpha)}(\hat{\ell})$ and consider $S \in \hat{\Lambda}$ representing (S,α). There exist numbers $s,t\in\mathbb{R}$ such that $h^t g_s S = \hat{\ell}$. Consider the flat surface $h^t g_s(S,\alpha)$. It has the following properties:

1. It is in the stable horocycle of the g_T orbit of (S, α). Therefore, the distance on $SL(2, \mathbb{R})/SL(S, \alpha)$ between $g_T(S, \alpha)$ and $g_T h^t g_s(S, \alpha)$ goes to zero as $T \to \infty$ since $SL(S, \alpha)$ acts by isometries.

2. The horizontal foliation of (S, α) and that of $h^t g_s(S, \alpha)$ are the same. This follows from the fact that g_s and h^t parametrize the stable horocycle of any point in $SL(2, \mathbb{R})$, meaning that the horizontal foliation of any point in the stable horocycle limits to the same projective horizontal foliation under the geodesic (Teichmüller) flow.

It follows from property (i) above that the g_T orbit of $h^t g_s(S, \alpha)$ is recurrent. Indeed, since there is a sequence of times $\{T_i\}_0^\infty$ such that $g_{T_i} \to \ell$ and $g_T h^t g_s$ is asymptotic to g_T as $T \to \infty$, it follows that $h^t g_s(S, \alpha)$ has a recurrent g_T orbit. Therefore, by Proposition 3, the horizontal foliation of $h^t g_s(S, \alpha)$ is ergodic. Moreover, by property (ii) above, the horizontal foliation of $h^t g_s(S, \alpha)$ is the same as that of (S, α). Therefore, since it is ergodic for $h^t g_s(S, \alpha)$, it is ergodic for (S, α). $\square$

Definition 6. The g_t orbit of (S, α) is *periodic* if there exists an s such that $g_s \in SL(S, \alpha)$. The number s is the *period* of (S, α).

Suppose (S, α) is g_t periodic with period T. Then there exists a unique affine diffeomorphism $f : S \to S$ such that Df can be identified with $r \equiv g_T \in SL(S, \alpha)$. Any periodic orbit $g_t(S, \alpha)$ is obviously recurrent and thus by Theorem 4 has an ergodic horizontal foliation. By considering the orbit $g_{-t} r_{\pi/2}(S, \alpha)$ for $t \geq 0$ and (3.16), then the same theorem gives us the ergodicity of the vertical foliation.

Corollary 1. *Let (S, α) be a flat surface of infinite genus and finite area which is g_t-periodic. Then the flows generated by X and Y on S are ergodic.*

3.2.1 The Veech dichotomy

Veech [42] was the first to notice that if the group of affine automorphisms of a surface (now known as the Veech group) is big enough, then the translation flow on it is reminiscent to the case on the flat torus: it is either completely periodic or uniquely ergodic. This dichotomy is referred to as *the Veech dichotomy.* More specifically, for a Veech surface, that is, for a closed flat surface of finite genus for which $SL(S, \alpha)$ is a lattice in $SL(2, \mathbb{R})$, this dichotomy holds.

A modern proof of the Veech dichotomy hardly relies on the fact that it is coming from a surface of finite genus. It does, however, depend on the size of the singular set $\Sigma \subset S$. Suppose that the Veech group of a flat surface of infinite genus and finite area (S, α) is a lattice and that $|\Sigma| < \infty$. If the g_t orbit does not leave every compact set of $SL(2, \mathbb{R})/SL(S, \alpha)$, the horizontal foliation of (S, α) is ergodic by Theorem 4.

If the g_t orbit leaves every compact subset of $H_{(S,\alpha)}$, then $g_t(S, \alpha)$ limits to a cusp of $H_{(S,\alpha)}$ and, therefore (see [26, §1.3-1.4]), the horizontal foliation of (S, α) is preserved by a parabolic element of $h \in SL(S, \alpha)$. We claim that, in this case, all singular leaves are saddle connections. Indeed, suppose there is a singular leaf l which is not a saddle connection. Then it is dense in a minimal component A of positive measure (see Remark 1). No power H^k of the parabolic automorphism H

(with $DH = h$) which preserves the horizontal direction sends l to itself as it would otherwise restrict to the identity on the minimal component of positive area on which this leaf is dense, contradicting the fact that the automorphism H is parabolic. By the same token, no power of H sends A to itself. But if $H^i(A) \cap H^j(A) = \varnothing$ for all $i \neq j$ (since A is a minimal component), $\omega_\alpha(A) > 0$ and H preserves ω_α, l cannot be dense in an open set since the area of (S, α) is finite. Therefore l is a saddle connection. The fact that regular leaves are closed follows from [26, Lemma 4]. Therefore we have a weak Veech dichotomy.

Theorem (Weak Veech dichotomy)**.** *Let (S, α) be a flat surface of finite area whose Veech group is a lattice. Then the horizontal foliation is either ergodic or completely periodic.*

The requirement that the singular set be finite is not unusual for flat surfaces with non-trivial Veech groups. Indeed, as far as we know, all known infinite genus flat surfaces of finite area whose Veech group is non-trivial have finitely many singularities. These examples will be discussed in the next section. It is not clear whether the assumption on $|\Sigma|$ can be dropped while retaining the conclusion of the theorem. More importantly, it is unknown whether there exist infinite genus flat surfaces of finite area whose Veech group is a lattice.

3.3 Applications

In this section we go over examples of flat surfaces of infinite genus and finite area to which Theorem 4 applies. The first is a family of surfaces constructed by

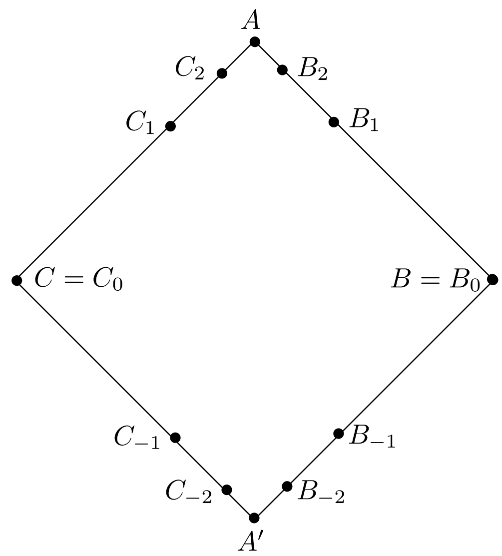

Figure 3.1: Construction of the surface S_p.

Chamanara. The second example is the Arnoux-Bowman-Yoccoz surface, whose Veech group contains no parabolic elements and whose Teichmüller orbit has only one periodic orbit. The third example is an application to a construction of Hooper which produces flat surfaces of infinite genus out of infinite graphs. We conclude the chapter with a discussion of "staircase" surfaces of finite area.

3.3.1 Chamanara's surface

The infinite genus surface introduced in [9] is constructed as follows (see Figure 3.1). Let $\mathcal{S} = ABA'C$ be a square centered at the origin in $\mathbb{C}$ such that its sides have length one and the diagonal BC is on the real line. Set $B_0 = B$ and $C_0 = C$. For $i \geq 1$ define B_i (respectively B_{-i}, C_i and C_{-i}) to be the point on the interval BA (respectively BA', CA and CA') such that the length of AB_i (respectively $A'B_{-i}$, AC_i, and $A'C_{-i}$) is p^i for some $0 < p < 1$. The sides B_iB_{i+1} and $C_{-(i+1)}C_{-i}$ are identified by a translation. This identifies all the points of the form B_{2k+1} and C_{2k} and the points of the form B_{2k} and C_{2k+1}. We denote the identification map by Q_p. The resulting surface obtained from the above is denoted by $S_p = Q_p(\mathcal{S})$ and it is clear that it is a flat surface of finite area. It is shown in [9, Proposition 9] that it is an infinite genus surface with one end. It is also easy to see that it is the geometric limit of finite genus surfaces: let $\mathcal{S}^n$ be the subset of $\mathcal{S}$ bounded from above by C_nB_n and below by $C_{-n}B_{-n}$. Then for each n, $S_p^n = Q_p(\mathcal{S})$ is a translation surface of genus n with two singularities of order $n-1$. Then limiting surface $S_p^n \longrightarrow S_p$ is our infinite genus surface with singularities of infinite order.

For any n, let λ_p^n be the direction of the line joining C to B_n on S_p. We denote by (S_p, α_p^n) the flat surface S_p with an Abelian differential with horizontal foliation in the direction of λ_p^n. We present now the main properties of the surface S_p and its Veech group $SL(S_p, \alpha_p)$ from [9].

Theorem. *Let S_p be the flat surface constructed as above. Then*

- *S_p is a Riemann surface of the first kind.*
- *For any rational number $p \in (0,1)$ and $n \in \mathbb{Z}$ there is a cyclic subgroup of $SL(S_p, \alpha_p^n)$ consisting of parabolic automorphisms in $SL(S_p, \alpha_p^n)$ with invariant direction λ_p^n.*
- *For any direction λ that makes an angle of more than $\frac{\pi}{4}$ with the horizontal direction there is no parabolic automorphism of S_p with invariant direction λ.*
- *$SL(S_p, \alpha_p)$ is a Fuchsian group of the second kind and thus $H_{(S_p,\alpha_p)}$ has infinite*

hyperbolic area.

- *When $p = 1/n$, $H_{(S_{1/n},\alpha_{1/n})}$ is a surface of genus zero with two cusps and one hole such that the length of the closed geodesic representing the homotopy class of the hole is* $\ln n$.

- *If p is rational and not of the form $1/n$ then $H_{(S_p,\alpha_p)}$ has infinitely many cusps.*

By the theorem above, since $H_{(S_p,\alpha_p)}$ has infinite volume, it is hard to find recurrent directions. It seems that there is not much known about directions with slope greater than $\frac{\pi}{4}$. However, directions which are fixed by hyperbolic elements which are products of parabolic elements of $SL(S_p, \alpha_p)$ give periodic g_t orbits and thus to these directions we can apply Proposition 1. Thus we have the following new result.

Theorem 6. *For rational $p \in (0, 1)$ there is a countably infinite set D_p of directions for which the translation flow on S_p is ergodic, given by the periodic and recurrent directions of hyperbolic elements of $SL(S_p, \alpha_p)$.*

3.3.2 The Arnoux-Bowman-Yoccoz surface

In the early 80's, Arnoux and Yoccoz [1] constructed a family of flat surfaces, one of every genus $g \geq 3$. These served as examples of surfaces carrying pseudo Anosov maps, which where not well understood as the theory was still in its infancy. It was eventually shown that the Veech groups of these surfaces are quite peculiar: they do not contain parabolic elements [25]. One usually expects that if the Veech group of a flat surface has an infinite subgroup of hyperbolic automorphisms, then it is generated by parabolic elements. For the Arnoux-Yoccoz family of surfaces, this was shown not to be the case.

The recent work of J. Bowman [7] has taken the geometric limit of this family of surfaces as the genus goes to infinity. The limiting surface will be referred to as the *Arnoux-Bowman-Yoccoz surface*, and it is depicted in Figure 3.2. This surface has finite area and, much like its finite-genus "subsurfaces", the Veech group of this surface contains no parabolic elements. In fact Bowman showed that the Veech group of this surface is isomorphic to $\mathbb{Z}\times\mathbb{Z}_2$, where the infinite subgroup is generated by the map which expands the horizontal direction by a factor of 2 while contracting the vertical by a factor of $\frac{1}{2}$ (as shown in figure 3.2). Note that this direction gives a g_t periodic orbit of period $\log 2$ and that the vertical foliation contains saddle connections.

In [7], the ergodicity of the vertical and horizontal foliations is claimed without proof. Here it follows from Theorem 4 and the description of the Veech group given in [7].

Corollary 2. *The vertical and horizontal foliations on the Arnoux-Bowman-Yoccoz surface, as depicted in Figure 3.2, are ergodic.*

It is not known whether these foliations are uniquely ergodic or not.

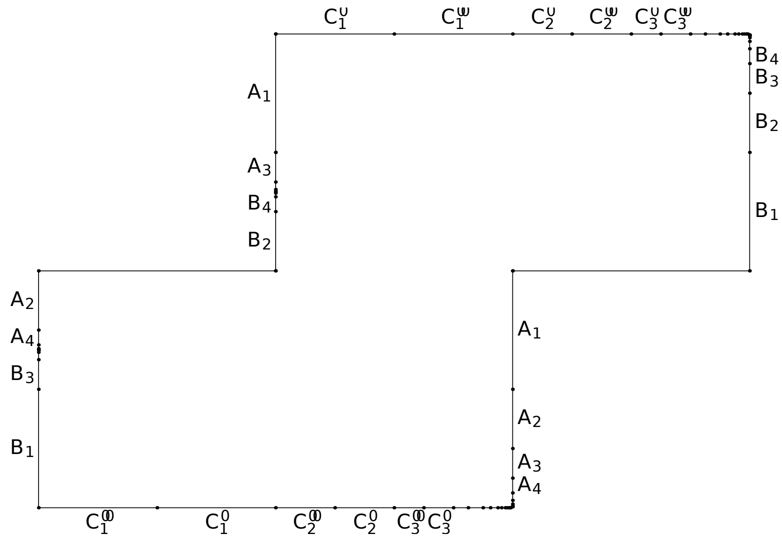

Figure 3.2: The Arnoux-Bowman-Yoccoz surface. The lengths of the identified sides are $\frac{1}{2^n}$, where n is the index of the side.

3.3.3 Hooper's Surfaces

Let S be a flat surface and $C \subset S$ a cylinder. The *modulus* of C is the ratio $\frac{width}{circumference}$. The following terminology is from [23].

A *cylinder decomposition* $\mathcal{C}(\theta)$ *of* S in a direction $\theta \in S^1$ is a description of S as a union of cylinders with boundaries parallel to θ and disjoint interiors: $S = \bigcup_{i \in \mathcal{I}} C_i$. A cylinder decomposition is *twistable* if there exists a positive constant $\kappa_{\mathcal{C}(\theta)}$ such that $\kappa_{\mathcal{C}(\theta)} m_i \in \mathbb{Z}$, where m_i is the modulus of C_i. The existence of a twistable cylinder decomposition implies the existence of a non-trivial parabolic element of the Veech group of S, namely

$$M_{\mathcal{C}(\theta)} = r_\theta \circ h^{\kappa_{\mathcal{C}(\theta)}} \circ r_{-\theta}$$

or, if θ is parallel to the horizontal foliation of (S, α), then $M_{\mathcal{C}} = h^{\kappa_{\mathcal{C}(\theta)}}$.

Suppose $\mathcal{C}(\theta_1)$ and $\mathcal{D}(\theta_2)$ are two twistable cylinder decompositions of S and

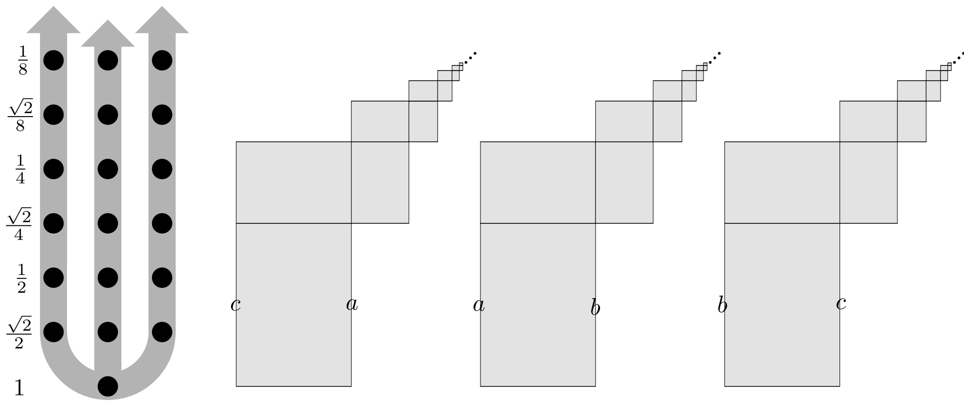

Figure 3.3: An infinite genus flat surface of finite area from Hooper's construction. The twistable cylinder decompositions are evident (from [23]).

let G be the group generated by $M_{\mathcal{C}(\theta_1)}$ and $M_{\mathcal{D}(\theta_2)}$. Define the collection of $(\mathcal{C},\mathcal{D})$-*renormalizable directions* to be those elements of the set

$$\Lambda = \{\theta \in S^1 : \text{ there exist } c > 0 \text{ such that } |M\theta| < c \text{ for infinitely many } M \in G\}$$

which

1. θ is not fixed by a parabolic element in the group G
2. θ is not the endpoint of an open interval in $S^1 \backslash \Lambda$.

The main theorem in [23] concerning this chapter is the following.

Theorem. *Let S be an infinite genus flat surface of finite area and θ_1, $\theta_2 \in S^1$ directions for which S admits twistable cylinder decompositions $\mathcal{C}(\theta_1)$ and $\mathcal{D}(\theta_2)$ with $\theta_1 \neq \pm\theta_2$. Assume moreover that every C_i crosses at least two cylinders in $\mathcal{D}$, and vice-versa. Then the translation flow in every $(\mathcal{C},\mathcal{D})$-renormalizable direction is uniquely ergodic.*

Suppose that for cylinder decompositions $(\mathcal{C}(\theta_1), \mathcal{D}(\theta_2))$, there is a cylinder C_i such that it crosses only one cylinder of $\mathcal{D}$. It is possible to overcome this obstruction to apply the above theorem by subdividing C_i in two cylinders of equal size and modulus to obtain a new cylinder decomposition $\hat{\mathcal{C}}(\theta_1)$ with the desired crossing properties. The price paid for this is that the the constant of the parabolic automorphism which generates the set of renormalizable directions satisfies $\kappa_{\hat{\mathcal{C}}(\theta_1)} = 2\kappa_{\mathcal{C}(\theta_1)}$.

It is not hard to see that Hooper's renormalizable directions correspond to directions of foliations of g_t-recurrent surfaces as in Definition 5. In many cases, the set of directions of the translation flow which are uniquely ergodic as a consequence of the above theorem is much smaller than the directions which are ergodic under Theorem 4 since the renormalizable directions of Hooper's theorem are generated by smaller subgroups of the Veech group.

In [23], a generalization was given of a construction of Thurston in which one starts from a connected, bipartite, ribbon graph and gets a translation surface with a non-trivial Veech group containing hyperbolic elements. Hooper gave a general way of constructing flat surfaces of infinite genus with large Veech groups containing hyperbolic elements from infinite graphs (see Figure 3.3 for an example). Surfaces in his constructions come automatically with twistable cylinder decompositions which give renormalizable directions and infinitely many uniquely ergodic directions for the translation flow when such a surface has finite area. Theorem 4 applies to surfaces in Hooper's construction and the set of directions for which the translation flow is ergodic is larger than the set or directions for which it is uniquely ergodic as a

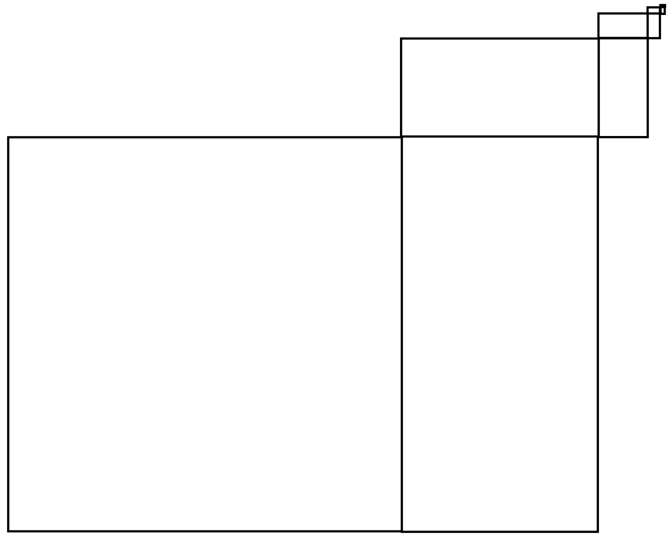

Figure 3.4: The crunched staircase $S^+_{\frac{1}{2}}$ constructed from rectangles of decreasing sides. Parallel sides are identified in the obvious way.

consequence of the above theorem.

3.3.4 Staircases

Recently there has been a considerable amount of attention given to a type of flat surface of infinite genus called *infinite staircases*. This flat surface of infinite genus has infinite area. In this section, we consider finite-area versions of it. The construction is as follows.

Let $p = a/b \in (0,1) \cap \mathbb{Q}$, where $gcd(a,b) = 1$. Starting with the unit square, we glue a rectangle of width p and height 1 to the right side of the square and glue parallel sides. We now glue a rectangle R_2 of width p and height p^2 to our starting rectangle by identifying the bottom edge of R_2 to $\{(x,1) : x \in [1, 1+p]\}$, the top edge of R_2 with $\{(0,x) : x \in (1, 1+p)\}$, and the left and right edges of R_2.

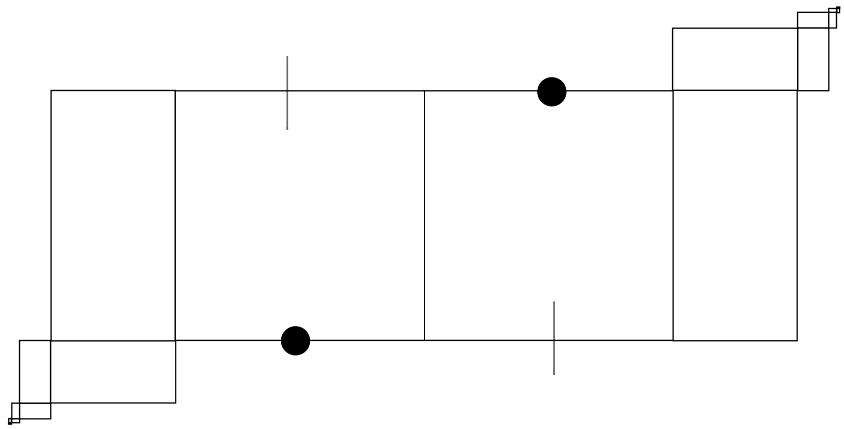

Figure 3.5: The double cover $\hat{S}^+_{\frac{1}{2}}$ of the surface $S^+_{\frac{1}{2}}$ in Figure 3.4.

The next step is to glue the rectangle R_3 of width p^3 and height p^2 by identifying the left edge of R_3 with $\{(1+p, y) : y \in [1, 1+p^2])\}$, the right edge of R_3 with $\{(1, y) : y \in [1, 1+p^2]\}$, and the top and bottom edges of R_3. Carrying out this construction by attaching infinitely many rectangles we obtain the surface S^+_p of infinite genus and finite area (see Figure 3.4).

The surface S^+_p naturally decomposes into vertical and horizontal cylinders. All but one of the vertical cylinders have modulus $\frac{p}{1+p^2}$ while the one coming from the unit square has modulus 1. All but one of the horizontal cylinders have modulus $\frac{p}{1+p^2}$ while one has modulus $\frac{1}{1+p^2}$. From this it is not difficult to work out a non-trivial subgroup of the Veech group of S^+_p generated by twists in the vertical and horizontal directions.

We can consider the surface $\hat{S}^+_p$ which is a double cover of the surface S^+_p. Figure 3.5 illustrates the construction. In this case, it is easy to see that the Veech group contains a subgroup of order two. This automorphism switches two cylinders in $\hat{S}^+_p$ covering the same cylinder in S^+_p. The same is true for the surface S_p, given by a similar construction, and illustrated in Figure 3.6.

Consider the parabolic matrices

$$\Phi^h(a,b) = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \Phi^v(a,b) = \begin{bmatrix} 1 & 0 \\ a+b & 1 \end{bmatrix},$$

and consider the discrete subgroups of $SL(2,\mathbb{R})$ generated by these matrices: $G(a,b) = \langle \Phi^h(a,b), \Phi^v(a,b) \rangle \subset SL(2,\mathbb{R})$. We summarize the results for the staircase surfaces in the next proposition.

Proposition 4. *For $p = a/b \in (0,1) \cap \mathbb{Q}$, we have that $G(a,b) \subset SL(S_p^+)$, $G(2a,2b) \subset SL(\hat{S}_p^+)$, and $G(a,b) \subset SL(S_p)$. Therefore, by Theorem 4, there is a countably infinite set of ergodic directions of the translation flow on S_p^+.*

Although $G(a,b)$ is a subgroup of the Veech groups of both S_p^+ and S_p, $G(a,b) \subset SL(S_p)$ can also be generated by $\Phi^h(a,b)$ and an element of order 2. Therefore the set of ergodic directions may differ for these two surfaces, although it is not clear by how much. The proof of the Proposition above follows from the fact that, by construction, these surfaces decompose into twistable cylinders. See [26] for details.

Surfaces with "obvious" orthogonal cylinder decompositions, such as the staircase surfaces above and surfaces coming from Hooper's construction, exhibit a high degree of self similarity. They can be obtained as geometric limits: by means of connected sums, one such surface is obtained by gluing onto a finite genus flat surface a smaller version of itself. This construction preserves the twistable directions and moduli of the cylinders. Therefore, each finite genus flat surface which limits

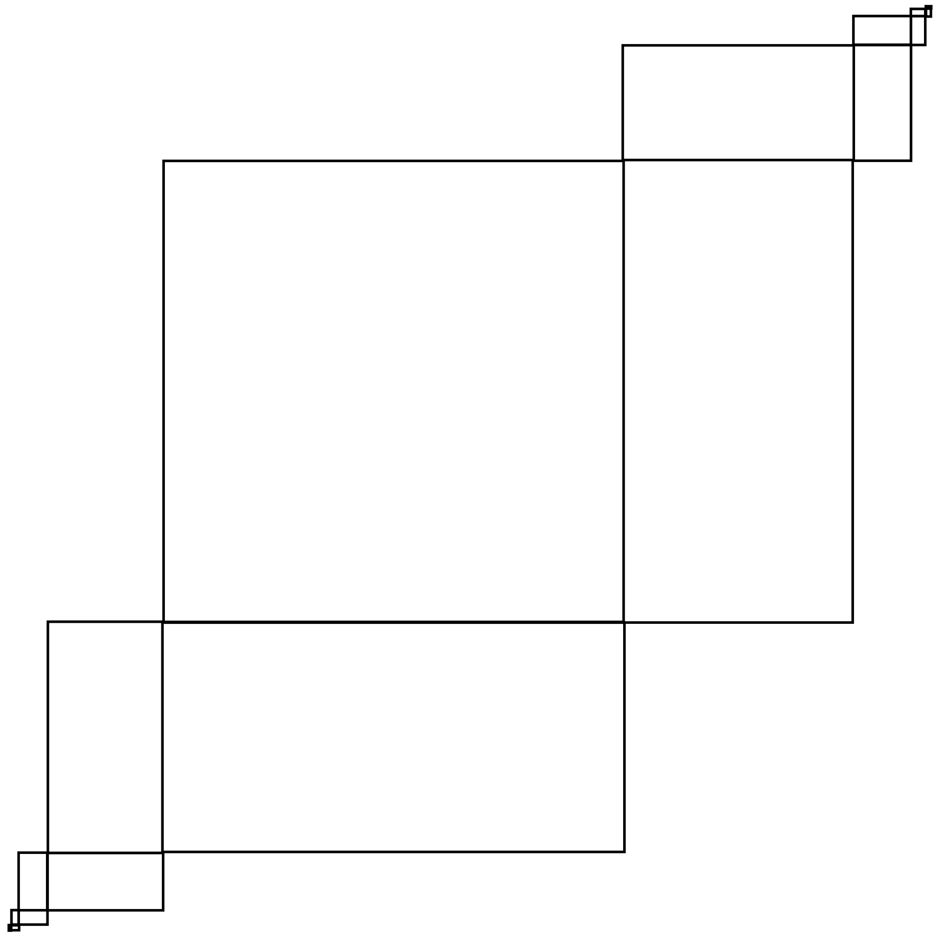

Figure 3.6: The infinite staircase $S_{\frac{1}{2}}$.

to an infinite genus flat surface of this kind has the same set of ergodic directions for the translation flow which are generated by the same twists, as a consequence of Theorem 4. This set of ergodic directions prevails in the infinite genus limit.

Appendix A

Code for computing Lyapunov exponents for the Kontsevich-Zorich cocycle for quadratic differentials

A.1 Description

The purpose of this piece of code is to compute the Lyapunov exponents for the Kontsevich-Zorich cocycle. It is done through the Rauzy-Veech-Zorich cocycle and writen for the case of quadratic differentials, but it should work for the case of abelian differentials.

The code was writen on a linux machine and needs the g++ compiler and The Numerical Recipes Library, which can be found at: http://www.nr.com/codefile.php?nr3

A.2 How the code works

First of all, I will assume the reader is familiar with the language of interval exchange transformations, generalized permutations, flat surfaces, et cetera.

The code computes the Lyapunov exponents for the Kontsevich-Zorich cocycle via its discrete version, the Rauzy-Veech-Zorich cocycle. In particular, we use the Rauzy-Veech induction step for intereval exchange transformations with involution, as introduced by Avila and Resende [2].

One should think of this as starting with a quadratic differential giving a non-orientable foliation on a flat surface. The dynamics of such surfaces are studied in the "non-orientable world" by generalized permutations (see Boissy and Lanneau's work [6]). Alternatively, one can also pass to an orienting double cover (and to the "orientable world") whereon the foliation becomes orientable and study the cocycle on this cover, which is what we do here, by putting together the languages of generalized permutations and interval exchange transformation with involution. Since the cocycle is non-uniformly hyperbolic (see the first part of this thesis), all exponents are non-zero.

Since the cocycle is defined on the homology (or cohomology) bundle and there is an involution on the surface, the bundles split into invariant and anti-invariant sub-bundles corresponding to 1 or -1 eigenvalues of the induced action by the involution. In the code we compute the entire cocycle and the cocycle restricted to the anti-invariant sub-bundle, the difference of which gives the cocycle restricted to the invariant sub-bundle. So we know which exponents come from the invariant part and which from the anti-invariant part.

Instead of writing a formal and technical description of how to use the code, we will illustrate with two examples, which is probably the best way to get you started using the code. If you want to compute the exponents for some stratum, you need to have an explicit generalized permutation which represents a surface in that stratum. This you can do with Anton Zorich's software (http://perso.univ-rennes1.fr/anton.zorich/Software/software_en.html).

Let's take, for example, a surface in $\mathcal{Q}(2,-1,-1)$. A generalized permutation for this is

```
(1 2 2 3)
(3 4 4 1)
```

by going to the double cover, we "lift" this generalized permutation to get to subdivided intervals. For this generalized permutation, the lifted interval looks like

```
1 4 4 3 . 1 2 2 3
```

where the dot (.) is called the "marker" and it marks where the intervals on the double cover split, i.e., it tells you how the information on the covering surface corresponds to the information og the generalized permutation. After you do this, rename the alphabet from 0 to D-1 (we start at zero to get used to the indexing conventions used the code, forced by the indexing conventions of the syntax of C++). So we get:

```
(*)     1 4 4 3 . 1 2 2 3
        0 1 2 3 . 4 5 6 7
```

Now we need to get the involution vector, which really defines our interval exchange transformation with involution. The involution vector for this example is

```
involution[0] = 4;
involution[1] = 2;
involution[2] = 1;
involution[3] = 7;
involution[4] = 0;
involution[5] = 6;
involution[6] = 5;
involution[7] = 3;
```

since we are pairing the indices of the alphabet of the top row of (*). This vector can also be defined as

```
involution = {4,2,1,7,0,6,5,3};
```

Finally we need to define the initial position of the marker, which in this case is 4.

So, the information you need to compute the exponents for a stratum are:

- The involution vector defined on the double cover (which you can construct as in the above examples).
- The initial position of the marker (which may change as the cocycle evolves)
- The size of the alphabet defining your generalized permutation (or interval exchange transformation with involution). In the code, we call it D and is defined globally at the begninning of the code (for the example above D = 8).
- The number of experiments you want to do per run as well as the number of iterations of the cocycle you want to do per experiment.

The involution vector and initial marker position are defined in the constructor for the IET class.

We close with another example to make sure we are clear. A representative generalized permutation for a surface in $\mathcal{Q}(-1, 2, 3)$ is

```
1 1 2 3 2 4 5
3 4 6 5 6
```

so we get

```
6 5 6 4 3 . 1 1 2 3 2 4  5
0 1 2 3 4 . 5 6 7 8 9 10 11
```

which gives the involution vector

```
invol = {2, 11, 0, 10, 8, 6, 5, 9, 4, 7, 3, 1};
```

so the starting data is

```
noRuns = 20;

D = 12;

marker = 5;

invol = {2, 11, 0, 10, 8, 6, 5, 9, 4, 7, 3, 1};
```

and this should compute the Lyapunov exponents for such stratum.

Be sure to add at the end of the code a routine which performs the QR-factorization the way it is done in Numerical Recipes. In the code, the class

```
QRdcmp
```

is very much the same class as in Numerical Recipes and defined the same way. We do will not write include the Numerical Recipes' QR-decomposition routine here since I am sure there is some sort of copyright issue.

Once you have changed the starting involution, D, marker, and number of iterations of the Zorich cocycle, the number of experiments per run, and included an QR-factorization function that works, save the file and compile making sure to link to the appropriate libraries, such as the Numerical Recipes library if you included their QR-decomposition routine.

The program starts with a random point and computes the Lyapunov exponents of the cocycle starting at this random point. The program generates two files: 'dataP.dat' and 'dataM.dat' which is the data of the evolution of the Lyapunov exponents of the full cocycle and its restriction to the anti-invariant part, respectively.

These you can plot using a program such as gnuplot. After the program runs, the last chunk of the output should look something like

```
The average Lyapunov exponents, after 20 runs, are:


1 1
0.5836283714563749 0.2015752359502962
0.317831871467477 -inf
0.201687189307008 -0.001237484552094726
0.0002042856236410213 -0.2015938286117917
4.953772744522572e-05 -inf
-6.434889398229047e-05 -inf
-8.404635272592056e-05 -inf
-0.2015420824123736 -inf
-0.317868437665994 -inf
-0.5836091719808862 -inf
-1.000233168276001 -inf
```

the first column corresponds to the Lyapunov exponents, in decreasing order, of the general cocycle while the second one to the restriction to the anti-invariant sub-bundle (NOTE: I know there are some issues sometimes with the anti-invariant cocycle, but for the most part you can still figure out which eigenvalues correspond to which eigenspaces).

The WARNING messages come up when one is on the edge of moduli space: this happens when the size of one of the intervals (or two since the involution pairs up intervals) is much larger than the rest. When this happens one needs to perform a large number of Rauzy-Veech induction steps before performing one step of Zorich acceleration (in other words, the warning messages state that one has had the same "type" for 50,000 or more iterations of Rauzy-Veech induction).

NOTE: This should work for for the classical Rauzy-Veech induction for IETs corresponding to abelian differentials. I have not checked this, but I see no reason why it should not work.

A.3 The code

```
#include <nr3.h>

#include <iostream>

#include <math.h>

#include <time.h>

#include <stdlib.h>

#include <stdio.h>

#include <iomanip>

#include <fstream>


#define noRuns 20         // # of experiments

#define iterations 50000  // Iterations of the cocycle per experiment

#define D 12              // Number of intervals on the top

using namespace std;


class IET {               // This is our main object, our IET with involution

public:


  IET();                  // This is the constructor

  double lambda[D];       // The lambda, i.e., vector defining the intervals

  int invol[D];           // involution

  int marker;             // The marker is where the intervals split by lifting a generalized

                          //                   permutation to an IET with involution

  int pi[D];              // The map from an alphabet on D = 2*d letters to the integers {0,...,D-1}

  int piInv[D];           // The inverse of the map

  double cocyclePlus[D][D];  // The Rauzy Veech cocycle matrix

  double cocycleMinus[D][D]; // The Rauzy Veech cocycle matrix, restricted to the anti-invariant subspace

  int type;               // The type, either zero or one, depending on who's the winner and who's loser

  int zorichStep;         // This is one if the type has changed from the previous step of induction,

                          //                                                             zero if not

  void induction();       // This updated all the values of this class which represents one

                          // iteration of Rauzy-Veech induction
```

```
};

struct QRdcmp {           // This is what does the QR factorization

  Int n;

  MatDoub qt, r;

  Bool sing;

  QRdcmp(MatDoub_IO &a);

};


int main(){

  int d=D/2;

  int count, zorichTime = 0;

  ofstream dataPlus, dataMinus;

  dataMinus.open("dataM.dat");  // names of files where the data is written to

  dataPlus.open("dataP.dat");

  dataPlus.precision(16);

  dataMinus.precision(16);

  cout.precision(16);

  IET iet;                      // This initializes the object IET

  double avgPlus[D], avgMinus[D], zeros = 0.0, etaPlus[D], etaMinus[D], TotalPlus[D], TotalMinus[D];

  MatDoub oldQPlus(D,D,zeros), oldQMinus(D,D,zeros);

  for(int i = 0; i < D; i ++){                    // Starting up the cocycles

    avgPlus[i] = avgMinus[i] = etaPlus[i] = etaMinus[i] = 0.0;

    oldQPlus[i][i] = oldQMinus[i][i] = 1.0;

    TotalPlus[i] = 0;

    TotalMinus[i] = 0;

  }

  for(int run = 0; run < noRuns; run++){

    for(int i = 1; zorichTime < iterations; i++){  // Loop for steps of Rauzy Veech induction,

                                                   // but stops after some definite Zorich

                                                   // steps have been taken

      MatDoub TPlus(D,D,zeros), TMinus(D,D,zeros);

      double TbarP[D][D], TbarM[D][D], rCheck[D][D];

      iet.induction();                         // COMPUTES THE NEW PERMUTATION AND NEW LAMBDA.

                                               // i.e., A STEP OF RAUZY-VEECH INDUCTION
```

```
zorichTime += iet.zorichStep;            // Records a step of Zorich acceleration if
                                         // the type has changed
for(int j = 0; j < D; j++){
  for(int k = 0; k < D; k++){
    for(int l = 0; l < D; l++){         // Define the matrices which will be decomposed
      TPlus[j][k] += iet.cocyclePlus[j][l]*oldQPlus[l][k];
      TMinus[j][k] += iet.cocycleMinus[j][l]*oldQMinus[l][k];
    }
  }
}
QRdcmp plusPart(TPlus), minusPart(TMinus);  // Computes the QR decomposition of
                                            // the matrix T defined as T = cocycle*oldQ
for(int j = 0; j < D; j++){              // Makes the new Q matrix the old one
  for(int k = 0; k < D; k++){
    oldQPlus[j][k] = plusPart.qt[k][j];
    oldQMinus[j][k] = minusPart.qt[k][j];
  }
}
for(int j = 0; j < D; j++){   // zorichTime^{th} step approximation of the jth Lyap. exponent
  etaPlus[j] += log(fabs(plusPart.r[j][j])); // (you need to divide by time, i.e., zorichTime)
  etaMinus[j] += log(fabs(minusPart.r[j][j]));
}
if(iet.zorichStep){                          // We only record steps of Zorich acceleration,
                                             // not steps of Rauzy-Veech induction
  count = 0;
  dataPlus << zorichTime << " ";
  dataMinus << zorichTime << " ";
  if((zorichTime%(iterations/2) == 0)){
    cout<<(zorichTime*100/iterations)<<" percent done of run number "<<run+1<<" out of " << noRuns << endl;
  }
  for(int j = 0; j < D; j++){   // Writes out the normalized Lyapunov exponents to file
    dataPlus << etaPlus[j]/etaPlus[0] << " ";
    dataMinus << etaMinus[j]/etaPlus[0] << " ";
    if(zorichTime > iterations/4){              // Keeps track of the exponents to calculate the
```

```
          avgPlus[j] += etaPlus[j]/etaPlus[0];       // average, but only does in the second half of
          avgMinus[j] += etaMinus[j]/etaPlus[0];     // the trajectory
        }
      }
      dataPlus << endl;
      dataMinus << endl;
    }
    count++;
    if(count == 50000){ cout << "Warning: we have been stuck on the edge for 50,000 iterations" << endl; }
    if(count == 100000){ cout << "Warning: we have been stuck on the edge for 100,000 iterations" << endl; }
    if(count == 500000){ cout << "Warning: we have been stuck on the edge for 500,000 iterations" << endl; }
    if(count == 1000000){ cout << "Warning: we have been stuck on the edge for 1,000,000 iterations" << endl; }
  }
  cout << endl << endl << "The average Lyapunov exponents for this run are:" <<  endl << endl;
  for(int i = 0; i < D; i++){
    cout << 4*avgPlus[i]/(3*iterations) << " " << 4*avgMinus[i]/(3*iterations) << endl;
  }
  for(int j = 0; j < D; j++){
    TotalPlus[j] += 4*avgPlus[j]/(3*iterations);
    TotalMinus[j] += 4*avgMinus[j]/(3*iterations);
    for(int k = 0; k < D; k++){
      oldQPlus[j][k] = 0.0;
      oldQMinus[j][k] = 0.0;
    }
  }
  for(int i = 0; i < D; i ++){                    // Starting up the cocycles
    avgPlus[i] = avgMinus[i] = etaPlus[i] = etaMinus[i] = 0.0;
    oldQPlus[i][i] = oldQMinus[i][i] = 1.0;
  }
  zorichTime = 0;
}
cout << endl << endl << "The average Lyapunov exponents, after " << noRuns <<" runs, are:" <<  endl << endl;
for(int i = 0; i < D; i++){
  cout << TotalPlus[i]/noRuns << " " << TotalMinus[i]/noRuns << endl;
```

```
  }
  return 0;
}


IET::IET() {
  for (int i = 0; i < D; i++){pi[i] = piInv[i] = i;}
  // SETTING UP THE FIRST RANDOM INTERVAL
  srand ( time(NULL) );
  double intLength= 0.0;
  type = 0;
  double leftLength = 0.0, rightLength = 0.0;
  int doubleLeft, doubleRight;
  marker = 5;
  invol = {2, 11, 0, 10, 8, 6, 5, 9, 4, 7, 3, 1};
  // Get a double letter on each side
  for(int i = 0; i < marker; i++){
    if(pi[invol[piInv[i]]] < marker){doubleLeft = piInv[i]; break;}
  }
  for(int i = marker; i < D; i++){
    if(pi[invol[piInv[i]]] >= marker){doubleRight = piInv[i]; break;}
  }
  // Assign lenghts to all letters except the double letters on each side
  for(int i = 0; i < marker; i++){
    if(piInv[i] != doubleLeft && invol[piInv[i]] != doubleLeft){
      lambda[i] = lambda[invol[i]] = rand();
      while(lambda[i] <= 0.0){ lambda[i] = lambda[invol[i]]= rand(); }
         // (In case we get the length of an interval to be zero)
    }
  }
  for(int i = marker; i < D; i++){
    if(piInv[i] != doubleRight && invol[piInv[i]] != doubleRight){
      lambda[i] = lambda[invol[i]] = rand();
      while(lambda[i] <= 0.0){ lambda[i] = lambda[invol[i]]= rand(); }
    }
```

```
}
for(int i = 0; i < marker; i++){
  leftLength += lambda[piInv[i]];
}
for(int i = marker; i < D; i++){
  rightLength += lambda[piInv[i]];
}
if(leftLength > rightLength){
    lambda[doubleLeft] = lambda[invol[doubleLeft]] = rand();
    lambda[doubleRight] = lambda[invol[doubleRight]] = (leftLength + 2*lambda[doubleLeft] - rightLength)/2.0;
}
else{
    lambda[doubleRight] = lambda[invol[doubleRight]] = rand();
    lambda[doubleLeft] = lambda[invol[doubleLeft]] = (rightLength + 2*lambda[doubleRight] - leftLength)/2.0;
}
double leftsum = 0.0;
double rightsum = 0.0;
for(int i = 0; i < marker; i++){
  leftsum += lambda[piInv[i]];
}
for(int i = marker; i < D; i++){
  rightsum += lambda[piInv[i]];
}
for(int i = 0; i<D; i++){intLength += lambda[i];}
for(int i = 0; i<D; i++){lambda[i] /= intLength/2.0;}
leftsum = 0.0;
rightsum = 0.0;
for(int i = 0; i < marker; i++){
  leftsum += lambda[piInv[i]];
}
for(int i = marker; i < D; i++){
  rightsum += lambda[piInv[i]];
}
// DONE SETTING UP
```

```
}
void IET::induction(){
  int winner, loser, newPiInv[D], oldType;
  double newLambda[D];
  oldType = type;
  // COMPUTE THE TYPE
  if(lambda[piInv[0]] > lambda[piInv[D-1]]){type = 0; winner = piInv[0]; loser = piInv[D-1];}
  else{type = 1; winner = piInv[D-1]; loser = piInv[0];}

  if(type == 0){
    for(int i = 0; i < D; i++){     // Getting the new permutation
      if(i <= pi[invol[winner]]){ newPiInv[i] = piInv[i]; }
      else if(i == (pi[invol[winner]] + 1) ){ newPiInv[i] = piInv[D-1]; }
      else{ newPiInv[i] = piInv[i-1]; }
    }
  }
  else{
    for(int i = 0; i < D; i++){     // Getting the new permutation
      if(i >= pi[invol[winner]]){ newPiInv[i] = piInv[i]; }
      else if(i == (pi[invol[winner]] - 1) ){ newPiInv[i] = piInv[0]; }
      else{ newPiInv[i] = piInv[i+1]; }
    }
  }

  if(type == oldType){   // Figure out whether we've done a step of Zorich acceleration or not
    zorichStep = 0;
  }
  else{ zorichStep = 1; }

  if(winner == piInv[0]){
    if(pi[invol[winner]] < marker){marker += 1;} // This means if the winner and the involution
  }                                              // of the winner are on the same side. In such
  else{                                          // cases you have to move the mid-point marker.
    if(pi[invol[winner]] >= marker){marker -= 1;}
```

```
}

for(int i=0; i < D; i++){     // Get pi
  piInv[i] = newPiInv[i];
  pi[piInv[i]] = i;
}
// Update lambda
double intLength = 0.0;
newLambda[winner] = newLambda[invol[winner]] = lambda[winner] - lambda[loser];
for(int i = 0; i < D; i++){
  if(i != winner && i != invol[winner]){
    newLambda[i] = newLambda[invol[i]] = lambda[i];
  }
}
for(int i = 0; i < D; i++){
  intLength += newLambda[i];
}
            // THIS GIVES YOU THE (NORMALIZED) NEW LAMBDA
for(int i = 0; i < D; i++){ lambda[i] = lambda[invol[i]] = 2.0*newLambda[i] /intLength; }
   // Ok so here we compute the matrix at every step of induction from which
   // the cocycle is built.
for(int i = 0; i < D; i++){
  for (int j = 0; j < D; j++){
    cocyclePlus[i][j] = 0.0;
    cocycleMinus[i][j] = 0.0;
  }
}
for(int i = 0; i < D; i ++){
  cocyclePlus[i][i] = 1.0;
  cocycleMinus[i][i] = cocycleMinus[i][invol[i]] = 0.5;
}
cocyclePlus[invol[loser]][winner] = cocyclePlus[loser][invol[winner]] = 1.0;
cocycleMinus[invol[loser]][winner] = cocycleMinus[loser][invol[winner]] = 0.5;
cocycleMinus[loser][winner] = cocycleMinus[invol[loser]][invol[winner]] = 0.5;
```

```
  // Finally, we adjust the lambda so that the trajectory remains in the
  // hypersurface containing IET's which come from generalized permutations.
  int doubleLeft, doubleRight;
  double leftLength = 0.0;
  double rightLength = 0.0;
  for(int i = 0; i < marker; i++){
    if(pi[invol[piInv[i]]] < marker){doubleLeft = piInv[i]; break;}
  }
  for(int i = marker; i < D; i++){
    if(pi[invol[piInv[i]]] >= marker){doubleRight = piInv[i]; break;}
  }
  for(int i = 0; i < marker; i++){
    leftLength += lambda[piInv[i]];
  }
  for(int i = marker; i < D; i++){
    rightLength += lambda[piInv[i]];
  }
  if(leftLength > rightLength){
    lambda[doubleRight] += (leftLength-rightLength)/2.0;
    lambda[invol[doubleRight]] += (leftLength-rightLength)/2.0;
  }
  else{
    lambda[doubleLeft] += (rightLength-leftLength)/2.0;
    lambda[invol[doubleLeft]] += (rightLength-leftLength)/2.0;
  }
  for(int i = 0; i<D; i++){lambda[i] /= leftLength;}
}
```

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