ABSTRACT

Title of dissertation: PROGRESS TOWARD CLASSIFYING TEICHMÜLLER DISKS WITH COMPLETELY DEGENERATE KONTSEVICH-ZORICH SPECTRUM

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We present results toward resolving a question posed by Eskin-Kontsevich-Zorich and Forni-Matheus-Zorich. They asked for a classification of all $\mathrm{SL}_2(\mathbb{R})$-invariant ergodic probability measures with completely degenerate Kontsevich - Zorich spectrum. Let $\mathcal{D}_g(1)$ be the subset of the moduli space of Abelian differentials $\mathcal{M}_g$ whose elements have period matrix derivative of rank one. There is an $\mathrm{SL}_2(\mathbb{R})$-invariant ergodic probability measure ν with completely degenerate Kontsevich-Zorich spectrum, i.e. $\lambda_1 = 1 > \lambda_2 = \cdots = \lambda_g = 0$, if and only if ν has support contained in $\mathcal{D}_g(1)$. We approach this problem by studying Teichmüller disks contained in $\mathcal{D}_g(1)$. We show that if (X, ω) generates a Teichmüller disk in $\mathcal{D}_g(1)$, then (X, ω) is completely periodic. Furthermore, we show that there are no Teichmüller disks in $\mathcal{D}_g(1)$, for $g = 2$, and the known example of a Teichmüller disk in $\mathcal{D}_3(1)$ is the only one. Finally, we present an idea that might be able to fully resolve the problem.

PROGRESS TOWARD CLASSIFYING TEICHMÜLLER DISKS WITH COMPLETELY DEGENERATE KONTSEVICH-ZORICH SPECTRUM

by

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Dedication

To my loving wife, Claudine.

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Chapter 1

Introduction and Foundations

1.1 Introduction

In [20], Kontsevich and Zorich introduced the Kontsevich-Zorich cocycle as a cocycle on the Hodge bundle over the moduli space of Riemann surfaces, denoted G_t^{KZ}, which is a continuous time version of the Rauzy-Veech-Zorich cocycle. They showed that this cocycle has a spectrum of $2g$ Lyapunov exponents with the property

$$1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_g \geq -\lambda_g \geq \cdots \geq -\lambda_2 \geq -\lambda_1 = -1.$$

These exponents have strong implications about the dynamics of flows on Riemann surfaces, interval exchange transformations, rational billiards, and related systems. They also describe how generic trajectories of an Abelian differential distribute over a surface [40]. Furthermore, Zorich [40] proved that they fully describe the non-trivial exponents of the Teichmüller geodesic flow, denoted G_t. Veech [34] proved $\lambda_2 < 1$, which implies that G_t is non-uniformly hyperbolic. Since then, the study of the Lyapunov spectrum of the Kontsevich-Zorich cocycle has become of widespread interest. Forni [10] proved the first part of the Kontsevich-Zorich conjecture [20]: $\lambda_g > 0$ for the canonical $\mathrm{SL}_2(\mathbb{R})$-invariant ergodic measure in the moduli space of holomorphic quadratic differentials. His result implies G_t^{KZ} is also non-uniformly hyperbolic. Avila and Viana [1] then used independent techniques to show that the spectrum is simple for the canonical measures on the strata of Abelian differentials, i.e. $\lambda_k > \lambda_{k+1}$, for all k.

Throughout this thesis, the spectrum of Lyapunov exponents of the Kontsevich-Zorich cocycle will be referred to as the Kontsevich-Zorich spectrum (KZ-spectrum). Veech asked to what extent the KZ-spectrum could be degenerate. Forni [11] found an example of an $\mathrm{SL}_2(\mathbb{R})$-invariant measure supported on the Teichmüller disk of a genus three surface with completely degenerate KZ-spectrum, i.e. $\lambda_1 = 1 > \lambda_2 = \lambda_3 = 0$. In the literature, the genus three surface generating Forni's example, denoted here by (M_3, ω_{M_3}), is known as the Eierlegende Wollmilchsau for its numerous remarkable properties [17]. Forni and Matheus [12] then found an example generated by a genus four surface, denoted here by (M_4, ω_{M_4}), with $\lambda_1 = 1 > \lambda_2 = \lambda_3 = \lambda_4 = 0$. Both surfaces are Veech surfaces and in particular, square tiled cyclic covers. They will be defined and depicted in Section 3.2. By relating Teichmüller and Shimura curves, Möller [30] proved that these two examples are the only examples of Veech surfaces generating Teichmüller disks supporting a measure with completely degenerate KZ-spectrum except for possible examples in certain strata of Abelian differentials in genus five. In a paper of Forni, Matheus and Zorich [14], they proved that the two examples are the only square-tiled cyclic cover surfaces generating Teichmüller disks supporting a measure with completely degenerate KZ-spectrum. In the recent work of [7], it was shown that there are no regular $\mathrm{SL}_2(\mathbb{R})$-invariant suborbifolds with completely degenerate Kontsevich-Zorich spectrum for $g \geq 7$. It was recently announced by Eskin and Mirzakhani [8], that the closure of every Teichmüller disk is an $\mathrm{SL}_2(\mathbb{R})$-invariant suborbifold. However, it is an open conjecture that every $\mathrm{SL}_2(\mathbb{R})$-invariant suborbifold is regular [7][Section 1.5]. Hence, the result of [7][Corollary 5] does not yet imply some of the results of this thesis.

Both [7] and [14] asked if the two known examples generate the only Teichmüller disks whose closures support an $\mathrm{SL}_2(\mathbb{R})$-invariant ergodic probability measure with completely degenerate Kontsevich-Zorich spectrum. In this thesis we present

progress toward answering this question. Let $\mathcal{D}_g(1)$ denote the subset of the moduli space of Abelian differentials, where the derivative of the period matrix has rank one. We address a potentially stronger problem and ask for a classification of all Teichmüller disks in $\mathcal{D}_g(1)$. Furthermore, we present an idea in the final section that has the potential to reduce the problem to Möller's conjecture that there are no genus five square-tiled surfaces generating a Teichmüller disk in the $\mathcal{D}_5(1)$.

Theorem 1.1.1. *If the Teichmüller disk D generated by (X, ω) is contained in $\mathcal{D}_g(1)$, then (X, ω) is completely periodic. Furthermore, there are no Teichmüller disks in $\mathcal{D}_2(1)$, and the surface (M_3, ω_{M_3}) generates the only Teichmüller disk in $\mathcal{D}_3(1)$.*

The main techniques used in this thesis include degenerating surfaces under the Deligne-Mumford compactification of the moduli space of Riemann surfaces, and an analysis of the derivative of the period matrix under such deformations. This concept has already been used successfully in [10]. Several other authors have also used this concept in other guises such as the second fundamental form of the Hodge bundle [13] and the Kodaira-Spencer map in the work of Möller and his coauthors [30, 3, 4].

To prove this theorem we show first that any surface generating a Teichmüller disk in $\mathcal{D}_g(1)$ is completely periodic, cf. Theorem 2.1.5. Then we show that degenerating surfaces in the closure of a Teichmüller disk in $\mathcal{D}_g(1)$ must have a very specific configuration, cf. Lemma 2.1.10. Proving the results requires some technical lemmas demonstrating convergence of the derivative of the period matrix, cf. Section 1.3.2, and a variation of a theorem of Masur [27][Theorem 2] to a more general setting, cf. Lemma 1.4.3. These results quickly yield some applications, cf. Proposition 2.2.4.

Next we show that the closure of every Teichmüller disk in $\mathcal{D}_g(1)$ must contain a (possibly degenerate) surface that is a Veech surface, cf. Theorem 3.1.4. This leads to an analysis of punctures on a Veech surface with the goal of excluding more and more configurations of the punctures until the remainder of the results follow. Theorem 1.1.1 summarizes Theorem 2.1.5, Proposition 2.2.4, and Theorem 3.2.9.

1.2 Preliminaries

1.2.1 The Moduli Space of Riemann Surfaces

Let X be a Riemann surface of genus g with n punctures (i.e. marked points). Let $R(X)$ denote the *Teichmüller space of* X or simply $R_{g,n}$ when X is understood. The surface X admits a *pants decomposition*, $X = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_{3g-3+n}$, into $3g-3+n$ pairs of pants, where each pair of pants is homeomorphic to the sphere with a total of three punctures and disjoint boundary curves. The Fenchel-Nielsen coordinates for Teichmüller space describe surfaces in terms of the lengths and twists of curves in a pants decomposition of X. A point in Teichmüller space is given by $(\ell_1, \ldots, \ell_{3g-3+n}, \theta_1, \ldots, \theta_{3g-3+n}) \in \mathbb{R}_+^{3g-3+n} \times \mathbb{R}^{3g-3+n}$.

Let $\mathrm{Diff}^+(X)$ be the group of orientation preserving diffeomorphisms on X. Let $\mathrm{Diff}_0^+(X)$ denote the normal subgroup of $\mathrm{Diff}^+(X)$ whose elements are isotopic to the identity. Then the *mapping class group* is the quotient

$$\Gamma(X) = \mathrm{Diff}^+(X)/\mathrm{Diff}_0^+(X).$$

The *moduli space of genus g surfaces with n punctures* is defined to be

$$\mathcal{R}_{g,n} = R(X)/\Gamma(X).$$

Deligne and Mumford [6] introduced a compactification of the moduli space denoted $\overline{\mathcal{R}_{g,n}}$ of Riemann surfaces within the more general setting of compactifying the space of stable curves. Every neighborhood of a point on a *Riemann surface with nodes* is either conformally equivalent to the unit complex disc, or to the set $\{(x, y) \in \mathbb{C}^2 | xy = 0\}$. The point mapped to $(0, 0)$ with the latter property is called a *node*. We regard this as the contraction or pinching of a simple closed curve on a surface to a point. Removing a node results in two punctures on either side of the node. This may or may not disconnect the surface. After removing all nodes, each of the connected components of the punctured degenerate surface is called a *part*. A *pair of punctures*, denoted (p, p'), will specifically refer to the punctures created by removing a node. We will assume this deconstruction throughout and say that pinching a curve results in a pair of punctures unless we say otherwise. Theorem B.1 in Appendix B of [19] describes the compactification of the moduli space in terms of the Fenchel-Nielson coordinates (or equivalently, a choice of pants decomposition) for Teichmüller space. By [19][Theorem B.1], the boundary of the moduli space $\overline{\mathcal{R}_{g,n}}$ under the Deligne-Mumford compactification is given by letting one or more of the lengths ℓ_i in the Fenchel-Nielson coordinates be zero.

1.2.2 Abelian and Quadratic Differentials

1.2.2.1 Abelian Differentials

Let K be the cotangent bundle over X. A section ω of K is a complex 1-form called an *Abelian differential*. An Abelian differential ω on X is given in local coordinates by $\omega = \phi(z)\,dz$, where $\phi(z)$ is a holomorphic function on the punctured surface possibly having poles of finite order at the punctures. Furthermore, ω obeys the change of coordinates formula

$$\phi(\sigma(z))\,d\sigma(z) = \phi(\sigma(z))\sigma'(z)\,dz.$$

The zeros and poles of ω are called *singularities* and all other points are called *regular*. The Chern formula relates the total number of zeros and poles counting multiplicity, by

$$\sharp(\text{zeros}) - \sharp(\text{poles}) = 2g - 2.$$

An Abelian differential ω determines an orientable horizontal and vertical foliation of a surface given by $\{\Im(\omega) = 0\}$ and $\{\Re(\omega) = 0\}$, respectively. Equivalently, the foliations can be defined by a pullback of the horizontal and vertical lines in the complex plane under the local coordinate chart on the surface. The Abelian differential ω determines a flat structure on the surface away from the singularities. A maximal connected subset of a foliation is called a *leaf*. If a leaf is compact and it does not pass through a singularity of ω, then it is called a *closed regular trajectory*. A closed connected subset σ of a leaf with endpoints at zeros of ω whose interior consists entirely of regular points of ω is called a *saddle connection*. Given a closed regular trajectory γ, the closure of the maximal set of parallel closed regular trajectories homotopic to γ form a *cylinder*. By definition, the boundaries of a cylinder consist of a union of saddle connections. We say that two cylinders are *homologous* (resp. *parallel*) if their core curves are homologous (resp. parallel). If every leaf of a foliation is compact, the foliation is *periodic*.

Lemma 1.2.1. *If C_1 and C_2 are homologous cylinders on a surface (X, ω), then C_1 and C_2 are parallel.*

Proof. Let γ_1 and γ_2 be the core curves of C_1 and C_2, respectively. Without loss of generality, assume that γ_1 is a closed curve of the vertical foliation on X by ω. Then by the definition of homologous

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega.$$

Thus

$$\int_{\gamma_1} \omega = \int_{\gamma_1} \Re(\omega) + i\Im(\omega) = i \int_{\gamma_1} \Im(\omega).$$

The last equality follows because γ_1 lies exactly in the vertical foliation so it has no horizontal holonomy. However, this implies

$$\int_{\gamma_2} \omega = i \int_{\gamma_1} \Im(\omega),$$

which implies

$$\int_{\gamma_2} \Re(\omega) = 0.$$

Therefore, γ_2 has no horizontal holonomy either, so it must be parallel to γ_1. $\square$

Call $\phi(z)$ or ω *holomorphic* if it can be continued holomorphically across all punctures of X. When $\phi(z)$ is holomorphic it naturally determines a flat metric on the surface. The length of a curve γ in this metric is given by

$$\int_{\gamma} |\phi(z)dz|.$$

Furthermore, there is an area form given by

$$A(\omega) = \frac{i}{2}\int_X \omega \wedge \bar{\omega}.$$

In the case of meromorphic differentials, the metric is still defined on compact subsets away from the punctures at which the differential has a pole though the area form is infinite.

Let $T_{g,n}$ be the Teichmüller space of Riemann surfaces carrying Abelian differentials. Define *the moduli space of Abelian differentials on Riemann surfaces of genus g with n punctures* by $\mathcal{M}_{g,n} = T_{g,n}/\Gamma(X)$. Define $\mathcal{M}_g := \mathcal{M}_{g,0}$ and $\mathcal{M}^{(1)}_{g,n} := \{(X,\omega) \in \mathcal{M}_g | A(\omega) = 1\}$.

Given a holomorphic differential ω on X, the sum of the orders of the zeros of ω is $2g-2$. This determines a stratification of the moduli space of holomorphic differentials by the multiplicities of the zeros of the Abelian differential. Denote the strata by $\mathcal{H}(\kappa)$, where κ is a vector corresponding to a partition of $2g-2$. In the case of meromorphic differentials, we list the orders of the poles in the vector κ so that the sum of the components of the vector remains $2g-2$.

The moduli space of Abelian differentials can be expanded so that limits of convergent sequences of Abelian differentials lying on degenerating surfaces exist on nodal surfaces [16]. An Abelian differential ω on a nodal Riemann surface is holomorphic everywhere except possibly at the punctures arising from removing the nodes, where ω is meromorphic with at most simple poles. At each pair of punctures (p, p'), ω satisfies

$$\mathrm{Res}_p(\omega) = -\mathrm{Res}_{p'}(\omega).$$

Let $\overline{\mathcal{M}_g}$ denote the moduli space of meromorphic Abelian differentials over the compactified base space $\overline{\mathcal{R}_g}$.

There is a natural action by $\mathbb{R}^*$ on the bundle of Abelian differentials. Let $r \in \mathbb{R}^*$ and $(X, \omega) \in \overline{\mathcal{M}_g}$, then

$$r \cdot (X, \omega) := (X, r\omega).$$

For the remainder of the thesis, we abuse notation and assume that the moduli space $\mathcal{M}_g$ is always quotiented by $\mathbb{R}^*$ unless we say otherwise. Furthermore, it will often be useful to choose a representative differential of the coset $(X, \omega)[\mathbb{R}^*]$. For instance, if ω is holomorphic and nonzero, we may choose the representative so that its area form is one and if ω is not holomorphic, we may choose a representative such that the modulus of the largest residue is one. This will be called *area normalization* or *residue normalization*, respectively.

The advantage of this projectivized moduli space of Abelian differentials is that it guarantees that for every sequence of Abelian differentials converging to an Abelian differential on a degenerate surface that there is at least one part of the degenerate surface on which the limiting Abelian differential is not identically zero. Without the projectivization, no such guarantee can be made. Let $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ be a sequence of surfaces carrying holomorphic Abelian differentials converging to a degenerate surface (X', ω') in $\overline{\mathcal{M}_g}$. Since X_n has finite genus, there are finitely many pinching curves. We can assume that ω_n is band bounded [38][Definition 1] on the annulus around each pinching curve. If we multiply ω_n by r_n so that the constant M in the definition of band bounded is uniformly bounded away from zero and infinity for all n, then Lemma 1.2.2 follows from [38][Lemma 2].

Lemma 1.2.2. *Given a sequence $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ such that the sequence $\{X_n\}_{n=0}^{\infty}$ converges to a degenerate surface X', there exists an Abelian differential ω' on X' such that ω' is the limit of the sequence $\{\omega_n\}_{n=0}^{\infty}$ in $\overline{\mathcal{M}_g}/\mathbb{R}^*$ and ω' is not identically zero on every part of X'.*

1.2.2.2 Quadratic Differentials

Let K be the cotangent bundle over X. The sections of the bundle $K \otimes_{\mathbb{C}} K$ are complex 2-forms called *quadratic differentials*. A quadratic differential is given in local coordinates by $q = \phi(z)\, dz^2$ and obeys the change of coordinates formula

$$\phi(\sigma(z))\, d\sigma(z)^2 = \phi(\sigma(z))(\sigma'(z))^2\, dz^2.$$

Singularities and regular points are defined as before and in this case the Chern formula reads

$$\sharp(\text{zeros}) - \sharp(\text{poles}) = 4g - 4.$$

A quadratic differential determines a horizontal and vertical foliation of a surface given by $\{\Im(\sqrt{\phi(z)}) = 0\}$ and $\{\Re(\sqrt{\phi(z)}) = 0\}$, respectively. These foliations are not necessarily orientable. If they are, q is called an *orientable quadratic differential.* If a quadratic differential is holomorphic everywhere except for at most a finite set of simple poles, then it is called an *integrable quadratic differential.* Denote the *Teichmüller space of integrable quadratic differentials* by $Q_{g,n}$ and the corresponding *moduli space of integrable quadratic differentials* by $\mathcal{Q}_{g,n} := Q_{g,n}/\Gamma_{g,n}$.

There is a natural way of associating all quadratic differentials to Abelian differentials. If q is non-orientable, then there is a connected double covering $\pi : \hat{X} \to X$ defined as follows. For each chart U of X, let $q = \phi_U(z)\, dz^2$ and define two charts $V^{\pm}$ of $\hat{X}$ each of which maps homeomorphically to U under π and $V^{\pm}$ carry the local differentials $\pm\sqrt{\phi_U(z)}\, dz$. This lift is compatible across charts and defines a quadratic differential ω^* with the property $\hat{q} = h^2$, where h is an Abelian differential. This lifting procedure is called the *orientating double cover construction*, and it can be used to translate the terms defined for Abelian differentials above (metrics, etc.) to non-orientable quadratic differentials.

As above, the bundle of quadratic differentials can be extended to the boundary of the moduli space of Riemann surfaces as defined by the Deligne-Mumford compactification. By admitting quadratic differentials with at most double poles, limits of sequences of integrable quadratic differentials on non-degenerate surfaces exist on degenerate surfaces. Define the residue of a quadratic differential q to be the coefficient of the term $1/z^2$ in its Taylor expansion. Given a quadratic differential q on a degenerate surface X with a pair of punctures (p, p'), the residues of q obey the relation

$$\mathrm{Res}_p(q) = \mathrm{Res}_{p'}(q).$$

Let $\overline{\mathcal{Q}_{g,n}}$ denote the moduli space of regular quadratic differentials on the compactified base space of Riemann surfaces $\overline{\mathcal{R}_{g,n}}$.

1.2.3 The $\mathrm{SL}_2(\mathbb{R})$ Action

We define the $\mathrm{SL}_2(\mathbb{R})$ action on quadratic differentials. It is clear that this definition applies to Abelian differentials as well. Let q be an integrable quadratic differential. Let h (resp. v) denote the horizontal (resp. vertical) foliation of q. The action by $A \in \mathrm{SL}_2(\mathbb{R})$ on an integrable quadratic differential q is defined by

$$\begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} h \\ v \end{bmatrix}$$

and denoted by $A \cdot (X, q)$. The action is well-defined on and between charts of X. Thus it defines an action by A globally on (X, q). It was stated in [3][Section 11] that the action is also well-defined on meromorphic Abelian differentials with at most simple poles. Furthermore, [3][Proposition 11.1] says that the action of $\mathrm{GL}_2^+(\mathbb{R})$ extends continuously to the boundary of $\overline{\mathcal{M}_g}$. We point out to the reader that the action by $\mathrm{GL}_2^+(\mathbb{R})$ on $\overline{\mathcal{M}_g}$ without the action by $\mathbb{R}^*$ is the same as considering the action of $\mathrm{SL}_2(\mathbb{R})$ on $\overline{\mathcal{M}_g}/\mathbb{R}^*$ because the action by $\mathbb{R}$ commutes with everything.

Definition. *Given a surface* $(X, q) \in \mathcal{Q}_{g,n}$*, the* Teichmüller disk *of* (X, q) *is the orbit of* (X, q) *in* $\mathcal{Q}_{g,n}$ *under the action by* $SL_2(\mathbb{R})$.

The *Teichmüller geodesic flow*, denoted G_t, on the bundle of quadratic differentials is the action by diagonal matrices:

$$G_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

We note for the convenience of the reader that the residue of the simple pole of an Abelian differential differs from the holonomy vector by a factor of $2\pi i$.

Lemma 1.2.3. *Let* ω *be an Abelian differential on a surface* X *with residue* $c = a+ib$ *at* $p \in X$*. Let* c_{G_t} *denote the residue at* p *after acting by* G_t *on* (X, ω)*. Then*

$$c_{G_t} = ae^{-t} + ibe^t.$$

Proof. Without loss of generality, let $p = 0$ in local coordinates about p. By [32][Theorem 6.3], it suffices to look at how the differential $c\,dz/z$ changes under the action by G_t. To do this, convert to polar coordinates and integrate the differential around the curve γ defined by $r = 1$. Let $c = a + ib$. Then

$$\frac{c\,dz}{z} = (a + ib)\left(\frac{dr}{r} + i\,d\theta\right) = \frac{a\,dr}{r} - b\,d\theta + i\left(b\frac{dr}{r} + a\,d\theta\right).$$

Furthermore, $dr = 0$ because $r = 1$. So this simplifies to $(-b + ia)\,d\theta$ and acting by G_t we get $(-be^t + iae^{-t})\,d\theta$. Therefore,

$$c_{G_t} = \frac{1}{2\pi i}\int_0^{2\pi} (-be^t + iae^{-t})\,d\theta = ae^{-t} + ibe^t.$$

$\square$

Definition. *A number $c \in \mathbb{C}$ is* ε-nearly imaginary *if* $|\arg(c) \pm \pi/2| < \varepsilon$.

Lemma 1.2.4. *Let (X', ω') be a degenerate surface carrying an Abelian differential with simple poles and residues $\{c_1, \ldots, c_m\}$. Given $\varepsilon > 0$, there exists $A \in SL_2(\mathbb{R})$ such that if c_j' is a residue of $A \cdot (X', \omega')$, for $1 \leq j \leq m$, then c_j' is ε-nearly imaginary.*

Proof. It is possible that ω' has some real residues. If so, multiply ω' by a complex unit ζ so that $\zeta\omega'$ has no real residues. Given a residue ζc_j of $\zeta\omega'$, after acting on ζc_j by G_t, the real part of the resulting residue is $e^{-t}\Re(\zeta c_j)$ by Lemma 1.2.3. Hence, there exists T such that $|e^{-T}\Re(\zeta c_j)| < \varepsilon$. $\square$

Lemma 1.2.5. *Let $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ be a sequence of surfaces containing cylinders $C_n \subset X_n$ with core curves γ_n. Let w_n and h_n denote the flat length with respect to ω_n of the circumference and height of C_n, respectively. If the ratio h_n/w_n tends to infinity with n, then the hyperbolic length of γ_n converges to zero.*

Proof. The modulus of the cylinder C_n is exactly the quotient h_n/w_n. By [23][Lemma 3],

$$\mathrm{Ext}_x(\gamma_n) \leq \frac{1}{\mathrm{Mod}_x(\gamma_n)}.$$

By [24][Corollary 2], $\mathrm{Ext}_x(\gamma_n)$ goes to zero with the hyperbolic length of γ_n. $\square$

Corollary 1.2.6. *Let (X, ω) admit a cylinder with core curve γ such that γ lies in the vertical foliation of X by ω. Then for all divergent sequences of times $\{t_n\}_{n=1}^{\infty}$ for which the limit*

$$\lim_{n \to \infty} G_{t_n} \cdot (X, \omega) = (X', \omega'),$$

exists, γ degenerates to a node of X'.

Proof. Let $C \subset X$ denote the cylinder with core curve γ and let w and h denote the circumference and height of C, respectively. After time t_n, the circumference and height are given by $e^{-t_n} w$ and $e^{t_n} h$. Since

$$\lim_{n\to\infty} \frac{e^{t_n} h}{e^{-t_n} w} = \infty,$$

γ pinches as n tends to infinity, by Lemma 1.2.5. $\square$

Lemma 1.2.7. *Let D be a Teichmüller disk in $\overline{\mathcal{M}_g}/\mathbb{R}^*$. Given a sequence $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ in D converging to a degenerate surface (X', ω'), there exists a degenerate surface (X'', ω'') in the closure of D such that ω'' is not holomorphic on every part of X''. Furthermore, X'' is reached from X' by pinching additional curves of X'.*

Proof. By Lemma 1.2.2, we assume that there is a part $S \subset X'$ such that ω' is not identically zero on S. If ω' has simple poles on X', then we are done, so assume otherwise. By [27][Theorem 2], there is a cylinder C_1 on S. Degenerate S under the Teichmüller geodesic flow by pinching the core curve of C_1. All punctures of X' are obviously preserved under the $\mathrm{SL}_2(\mathbb{R})$ action. The new limit ω_1' carries an Abelian differential which is not identically zero everywhere by Lemma 1.2.2. If ω_1' is holomorphic on every part we can repeat the argument. Since the genus is finite, the repetition of this argument will terminate when we reach a differential that is not holomorphic or when the surface degenerates to a sphere, which does not carry holomorphic differentials. Since the punctures of X' are preserved under the

$\mathrm{SL}_2(\mathbb{R})$ action, X'' is reached from X' by pinching additional curves. Furthermore, it follows from the continuity of the $\mathrm{SL}_2(\mathbb{R})$ action [3][Proposition 11.1] that X'' is in the closure of D. $\square$

1.3 Lyapunov Exponents and the Rank One Locus

In the first subsection, we give the precise formulation of the problem answered in this thesis. In the second subsection we present all of the technical lemmas related to the derivative of the period matrix that will be used throughout the remainder of this thesis.

1.3.1 Lyapunov Exponents of the KZ-Cocycle

Let X be a Riemann surface of genus g. Consider the cocycle defined by the Teichmüller geodesic flow as follows

$$G_t \times \mathrm{Id} : T_g \times H^1(X, \mathbb{C}) \to T_g \times H^1(X, \mathbb{C}).$$

The mapping class group preserves the real and imaginary parts of
$T_g \times H^1(X, \mathbb{C})$. The *Kontsevich-Zorich cocycle* is the quotient cocycle

$$G_t^{KZ} : \Re\left((T_g \times H^1(X, \mathbb{C}))/\Gamma_g\right) \to \Re\left((T_g \times H^1(X, \mathbb{C}))/\Gamma_g\right)$$

restricted to the real part.

Let ν denote a finite $\mathrm{SL}_2(\mathbb{R})$-invariant ergodic measure on $\mathcal{M}_g$. The cocycle G_t^{KZ} admits a spectrum of $2g$ Lyapunov exponents with respect to ν. The natural symplectic structure on $H^1(X, \mathbb{C})$ induces a symplectic structure on the entire bundle $\Re\left((T_g \times H^1(X,\mathbb{C}))/\Gamma_g\right)$, which forces a symmetry of the $2g$ Lyapunov exponents.

$$1 = \lambda_1^\nu \geq \lambda_2^\nu \geq \cdots \geq \lambda_g^\nu \geq -\lambda_g^\nu \geq \cdots \geq -\lambda_2^\nu \geq -\lambda_1^\nu = -1.$$

We refer to these $2g$ numbers as the *spectrum of Lyapunov exponents of the Kontsevich-Zorich cocycle* or the *KZ-spectrum* for short. If $\lambda_k^\nu = 0$, for some k, then the spectrum is called *degenerate*. If $\lambda_k^\nu = 0$ for all $k > 1$, then the KZ-spectrum is *completely degenerate*.

Kontsevich and Zorich [20] as well as Forni [10] gave a formula for the sum of these exponents in terms of the eigenvalues of a Hermitian form. These eigenvalues were reinterpreted through the second fundamental form of the Hodge bundle [13]. Let $(X, \omega) \in \mathcal{M}_g$. Let $L^2_\omega(X)$ be the Hilbert space of complex-valued functions on X that are L^2 with respect to ω. Let $\langle \cdot, \cdot \rangle_\omega$ be the inner product on $L^2_\omega(X)$. Let $M^\pm_\omega \subset L^2_\omega(X)$ be the subspaces of meromorphic and anti-meromorphic functions, respectively. Define the orthogonal projections

$$\pi^\pm_\omega : L^2_\omega(X) \to M^\pm_\omega.$$

For two meromorphic functions $m_1^+, m_2^+ \in M^+_\omega$,

$$H_\omega(m_1^+, m_2^+) = \langle \pi^-_\omega(m_1^+), \pi^-_\omega(m_2^+) \rangle_\omega.$$

The eigenvalues of $H_\omega(\cdot,\cdot)$ are given by the functionals $\Lambda_k(\omega) : \mathcal{M}_g^{(1)} \to \mathbb{R}$, which are continuous for all k and ω, and obey the inequalities

$$1 \equiv \Lambda_1(\omega) \geq \Lambda_2(\omega) \geq \cdots \geq \Lambda_g(\omega) \geq 0.$$

In [11], Forni introduced a filtration of sets

$$\mathcal{D}_g(1) \subset \mathcal{D}_g(2) \subset \cdots \subset \mathcal{D}_g(g-1),$$

where

$$\mathcal{D}_g(k) = \{(X,\omega) \in \mathcal{M}_g | \Lambda_{k+1}(\omega) = \cdots = \Lambda_g(\omega) = 0\},$$

and $\mathcal{D}_g(k)$ is called the *rank k locus.* The set $\mathcal{D}_g(g-1) = \mathcal{D}_g$ is the *determinant locus* introduced in [10].

Let ν be an $\mathrm{SL}_2(\mathbb{R})$-invariant measure on a connected component $\mathcal{C}_\kappa$ of the stratum $\mathcal{H}(\kappa) \subset \mathcal{M}_g$ of Abelian differentials. Corollary 5.3 of [10] gives the following identity:

$$\lambda_2^\nu + \cdots + \lambda_g^\nu = \frac{1}{\nu(\mathcal{C}_\kappa)} \int_{\mathcal{C}_\kappa} \Lambda_2(\omega) + \cdots + \Lambda_g(\omega)\, d\nu.$$

In [11], Forni notes that this formula can be extended to any $\mathrm{SL}_2(\mathbb{R})$-invariant ergodic probability measure, from which the lemma follows.

Lemma 1.3.1 (Forni [11], Cor. 7.1)**.** *Let ν be a finite $SL_2(\mathbb{R})$-invariant ergodic measure on the moduli space $\mathcal{M}_g$. The KZ-spectrum with respect to ν is completely degenerate if and only if for almost every $(X,\omega) \in supp(\nu)$, H_ω has rank one, i.e. $supp(\nu) \subset \mathcal{D}_g(1)$.*

We introduce the derivative of the period matrix, which will be the focus of this thesis. Let $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ be a basis for the first homology group $H_1(X,\mathbb{C})$. Let $\{\theta_j\}_{j=1}^g$ be a basis of the complex vector space of holomorphic Abelian differentials on X normalized so that

$$\int_{a_i} \theta_j = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. Under this choice of basis of Abelian differentials, the *period matrix* $\Pi(X)$ is the symmetric matrix with positive definite imaginary part whose components are given by

$$b_{ij} = \int_{b_i} \theta_j.$$

The space of Beltrami differentials, $B(X)$ is dual to the cotangent space of quadratic differentials. Every Abelian differential ω uniquely determines a Beltrami differential

$$\mu_\omega = \frac{\bar{\omega}}{\omega},$$

which is defined everywhere except at the zeros and poles of ω of which there are only finitely many. In the Teichmüller space $R(X)$ the space $B(X)$ represents the tangent space and $\mu \in B(X)$ a tangent vector at X. In $R(X)$, μ determines a direction in which we can take a derivative of $\Pi(X)$. The derivative of the period matrix at X in direction μ is denoted by $d\Pi(X)/d\mu$. Let $\omega = h(z)\,dz$ and $\theta_k = f_k(z)\,dz$, for all k. Rauch's formula, [19][Proposition A.3], gives a concise formula for the components of the derivative of the period matrix.

$$\frac{d\Pi_{ij}(X)}{d\mu_\omega} = \int_X \theta_i \theta_j \, d\mu_\omega = \int_X f_i f_j \frac{\bar{h}}{h} dz \wedge d\bar{z}$$

In the proof of Lemma 4.1 of [10], Forni defines a complex bilinear form on holomorphic Abelian differentials ω_1, ω_2 by

$$B_\omega(\omega_1, \omega_2) = \left\langle \frac{\omega_1}{\omega}, \frac{\bar{\omega}_2}{\omega} \right\rangle_\omega.$$

It was proven in [10] that $H_\omega = B_\omega B_\omega^*$ (and a typo in the equation in [10] was corrected in [13]). It is possible to choose a basis of Abelian differentials $\{\phi_1, \ldots, \phi_g\}$

on X such that

$$\frac{d\Pi_{ij}(X)}{d\mu_\omega} = B_\omega(\phi_i, \phi_j).$$

Hence, H_ω has rank one if and only if $d\Pi(X)/d\mu_\omega$ has rank one. For this reason it suffices to regard $\mathcal{D}_g(1)$ as the set where $d\Pi(X)/d\mu_\omega$ has rank one for the remainder of this thesis.

Since ν is an $\mathrm{SL}_2(\mathbb{R})$-invariant measure, $\mathrm{supp}(\nu)$ must be an $\mathrm{SL}_2(\mathbb{R})$-invariant set. Consider $(X, \omega) \in \mathrm{supp}(\nu)$. Let D be the Teichmüller disk generated by (X, ω). Then $D \subset \mathrm{supp}(\nu)$, and if the KZ-spectrum with respect to ν is completely degenerate, then $D \subset \mathcal{D}_g(1)$. This is precisely the problem that we address in this thesis.

Problem. *Classify all Teichmüller disks D such that $D \subset \mathcal{D}_g(1)$.*

1.3.2 The Derivative of the Period Matrix

One of the most important techniques in this thesis is the use of estimates for the derivative of the period matrix near the boundary of the moduli space $\mathcal{M}_g$. In this section we introduce plumbing coordinates for a Riemann surface and express Abelian differentials in terms of those plumbing coordinates using the exposition of [37]. Unfortunately, it will not be possible to guarantee convergence of the derivative of the period matrix in every possible scenario, but it will be possible for all cases relevant to this thesis. Lemma 1.3.2 below is a stronger statement than that of [10][Lemma 4.2] because it applies to any sequence satisfying a relatively lax set of assumptions. These convergence lemmas motivate and justify defining the rank of the derivative of the period matrix for surfaces in the boundary of $\overline{\mathcal{M}_g}$.

Plumbing coordinates have been used extensively from [26] to [10], among others. They have been used to write explicit formulas for differentials near the boundary of the moduli space. Wolpert [38] reworked the foundations of differentials on families of degenerating surfaces using the language of sheaves, and expressed the differentials on degenerating surfaces in terms of plumbing coordinates. We copy the language and notation of [10][Section 4] and [38, 37], as appropriate. Let X' be a degenerate Riemann surface in the boundary of $\overline{\mathcal{R}_g}$. Let X' have $1 \leq m \leq 3g-3$ pairs of punctures $\{(p_i, p_i')\}$, for $1 \leq i \leq m$. Let $\tau \in \mathbb{C}^{3g-3-m}$ denote the local coordinates for a neighborhood of X' in the Teichmüller space of X'. We denote surfaces in a neighborhood of $X' \in \mathcal{R}_g$ by $X(0,\tau)$. We refer the reader to [37][Section 3], where the coordinates are specifically chosen to correspond to small deformations of the complex structure on X'. For our purposes, it suffices to know that such a coordinate τ exists. Let $(U_i(0,\tau), z_i)$ and $(V_i(0,\tau), w_i)$ be coordinate charts around p_i and p_i', respectively, such that $z_i(p_i) = w_i(p_i') = 0$. Following [38, 37], let c', c'' be positive constants, $V = \{|z| < c', |w| < c''\}$, $D = \{|t| < c'c''\}$, and $\pi : V \to D$ be the singular fibration with projection $\pi(z,w) = zw = t$, where $t \in \mathbb{C}$. Let $t = (t^{(1)}, \ldots, t^{(m)}) \in D^m$. Let $c < 1$ be a small positive constant. For $|t^{(i)}| < c^4$ and $1 \leq i \leq m$, remove the discs $\{|z_i| \leq c^2\}$ and $\{|w_i| \leq c^2\}$ from $X'(0,\tau)$ to get an open surface X_τ^*. For each i, identify a point $u_0 \in \{u | c^2 < |z_i(u)| < c\} \subset X_\tau^*$ to the point $(z_i(u_0), t^{(i)}/z_i(u_0))$ in the fiber of a k^{th} factor of $\pi : V \to D$, and identify a point $v_0 \in \{v | c^2 < |w_i(v)| < c\} \subset X_\tau^*$ to the point $(t^{(i)}/w_i(v_0), w_i(v_0))$ in the fiber of a k^{th} factor of $\pi : V \to D$. This implies that we can write $X(t,\tau)$ to fully coordinatize a neighborhood of the degenerate surface $X' := X(0, \tau_\infty) \in \overline{\mathcal{R}_g}$.

In [26] and [10], the identification of the annuli is made directly so that if we translate their language to Wolpert's, we get

$$(z_i(u_0), t^{(i)}/z_i(u_0)) = (t^{(i)}/w_i(v_0), w_i(v_0))$$

and identify along the curve $|w_i(v_0)| = |z_i(u_0)| = \sqrt{|t^{(i)}|}$. It suffices to follow this convention throughout this thesis. However, it may be useful to know that such an identification is possible for the sake of future work, so we present the identification in its full generality here. Let $0 < q < 1$. The usual identification occurs when $q = 1/2$. We choose an identification whereby, $|z_i(u_0)| = |t^{(i)}|^{1-q}$ and $|w_i(v_0)| = |t^{(i)}|^q$. Following the notation of [38], we define annuli with respect to this identification for fixed q. Let

$$R_z(t^{(i)}) := \{|t^{(i)}|^{1-q}/c'' < |\zeta_i| < c'\} \subset \{|t^{(i)}|/c'' < |\zeta_i| < c'\}$$

and

$$R_w(t^{(i)}) := \{|t^{(i)}|^{q}/c' < |\zeta_i| < c''\} \subset \{|t^{(i)}|/c' < |\zeta_i| < c''\}.$$

Let $c = c' = c''$ and define

$$X^*(t, \tau) =: X^*_\tau \cup \bigcup_{i=1}^{m} R_z(t^{(i)}) \cup R_w(t^{(i)}).$$

Next we consider Abelian differentials on Riemann surfaces. Let $D_1 \times \cdots \times D_m = D^m$ denote the m copies of D above. Following [38], every Abelian differential can be expressed in terms of local coordinates on D_j. This is done by considering the coordinate ζ_j on an annulus and the map $\zeta_j \mapsto (\zeta_j, t^{(j)}/\zeta_j)$ (resp. $\zeta_j \mapsto (t^{(j)}/\zeta_j, \zeta_j)$).

As $t^{(j)}$ tends to zero this yields the convergence of the differential in local coordinates about the degenerating annuli resulting in the map $\zeta_j \mapsto (\zeta_j, 0)$ (resp. $\zeta_j \mapsto (0, \zeta_j)$).

It follows from a form the Cartan-Serre theorem with parameters or [26][Proposition 4.1], that there is a basis of Abelian differentials

$$\{\theta_1(t,\tau), \ldots, \theta_g(t,\tau)\}$$

on $X(t,\tau)$, for all small t, such that $\{\theta_1(0,\tau_\infty), \ldots, \theta_g(0,\tau_\infty)\}$ spans the space of Abelian differentials on X'. We assume such a fixed basis in a neighborhood of a degenerate surface throughout this thesis. Let $t' = (t^{(1)}, \ldots, t^{(j-1)}, t^{(j+1)}, \ldots, t^{(m)})$. In local coordinates on D_j, let $\theta_i(t', \tau, \zeta_j, t^{(j)}/\zeta_j) = 2 f_i(t', \tau, \zeta_j, t^{(j)}/\zeta_j)\, d\zeta_j/\zeta_j$, where

$$f_i(t', \tau, \zeta_j, t^{(j)}/\zeta_j) = \sum_{k,\ell \geq 0} a_{k\ell}(t', \tau) \zeta_j^k (t^{(j)}/\zeta_j)^\ell,$$

by [38].

Let $\{(X_n, \omega_n)\}_{n=0}^\infty$ be a sequence of surfaces carrying Abelian differentials converging to a degenerate surface (X', ω'). Without loss of generality, we can ignore the beginning of the sequence so that every element of the sequence can be expressed in terms of the local coordinates established above. Thus, let $X_n = X(t_n, \tau_n)$ and $X' = X(0, \tau_\infty)$. Let

$$\omega_n = 2A_n(t', \tau, \zeta_j, t^{(j)}/\zeta_j)\, \frac{d\zeta_j}{\zeta_j},$$

be local coordinates on D_j. Contrary to the coefficients f_i in the basis of Abelian differentials, note the dependence of the function A_n on n.

Lemma 1.3.2. *We follow the notation established above. Let $\{(X(t_n, \tau_n), \omega_n)\}_{n=0}^\infty$ be a sequence of surfaces converging to a degenerate surface $(X(0, \tau_\infty), \omega')$. For each*

n, let $\{\theta_1(t_n,\tau_n),\ldots,\theta_g(t_n,\tau_n)\}$ be a basis for the space of Abelian differentials on $X(t_n,\tau_n)$. Given i,j, for all k, if one of the following is true:

(1) Either $f_i(0,\tau_\infty,0,0)=0$ on D_k or $f_j(0,\tau_\infty,0,0)=0$ on D_k, or

(2) $A_\infty(0,\tau_\infty,0,0)\neq 0$ on D_k, then

$$\lim_{n\to\infty}\left(\frac{d\Pi_{ij}(X(t_n,\tau_n))}{d\mu_{\omega_n}}-\int_{X^*(t_n,\tau_\infty)}\theta_i(0,\tau_\infty)\theta_j(0,\tau_\infty)\,d\mu_{\omega'}\right)=0.$$

Proof. It follows from [39] that the limit converges on compact subsets of the complement of the punctures. Hence, it suffices to prove convergence on each annulus $R_z(t_n^{(k)})$ and $R_w(t_n^{(k)})$. To get convergence on $R_w(t_n^{(k)})$, it suffices to show convergence on $R_z(t_n^{(k)})$ because the only property of $1-q$ relevant to the argument below is $0<1-q<1$, and it is certainly also true that $0<q<1$. Using Rauch's formula, we explicitly write the expression to be estimated as t_n tends to zero in $\mathbb{C}^n$. That the following integral makes sense and proves the desired convergence follows from [38][Lemma 2].

$$4\int_{R_z(t_n^{(k)})}\left(\frac{f_i(t_n',\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)}{\zeta_k}\,\frac{f_j(t_n',\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)}{\zeta_k}\,\frac{\overline{A_n(t_n',\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)/\zeta_k}}{A_n(t_n',\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)/\zeta_k}\right.$$
$$\left.-\frac{f_i(0,\tau_\infty,\zeta_k,0)}{\zeta_k}\,\frac{f_j(0,\tau_\infty,\zeta_k,0)}{\zeta_k}\,\frac{\overline{A_\infty(0,\tau_\infty,\zeta_k,0)/\zeta_k}}{A_\infty(0,\tau_\infty,\zeta_k,0)/\zeta_k}\right)d\zeta_k\wedge d\overline{\zeta_k}.$$

Following the proof of [10][Lemma 4.2], we split the difference in the integrand into the following three terms:

$$\begin{aligned}
&\text{(I)} \quad 4\int_{R_z(t_n^{(k)})} \left(\tfrac{f_i(t'_n,\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)}{\zeta_k} - \tfrac{f_i(0,\tau_\infty,\zeta_k,0)}{\zeta_k}\right) \\
&\qquad \tfrac{f_j(t'_n,\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)}{\zeta_k} \tfrac{\overline{A_n(t'_n,\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)/\zeta_k}}{A_n(t'_n,\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)/\zeta_k} d\zeta_k \wedge d\overline{\zeta_k} \\
&\text{(II)} \quad 4\int_{R_z(t_n^{(k)})} \left(\tfrac{f_j(t'_n,\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)}{\zeta_k} - \tfrac{f_j(0,\tau_\infty,\zeta_k,0)}{\zeta_k}\right) \\
&\qquad \tfrac{f_i(0,\tau_\infty,\zeta_k,0)}{\zeta_k} \tfrac{\overline{A_n(t'_n,\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)/\zeta_k}}{A_n(t'_n,\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)/\zeta_k} d\zeta_k \wedge d\overline{\zeta_k} \\
&\text{(III)} \quad 4\int_{R_z(t_n^{(k)})} \left(\tfrac{f_i(0,\tau_\infty,\zeta_k,0)}{\zeta_k} \tfrac{f_j(0,\tau_\infty,\zeta_k,0)}{\zeta_k}\right) \\
&\qquad \left(\tfrac{\overline{A_n(t'_n,\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)/\zeta_k}}{A_n(t'_n,\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)/\zeta_k} - \tfrac{\overline{A_\infty(0,\tau_\infty,\zeta_k,0)/\zeta_k}}{A_\infty(0,\tau_\infty,\zeta_k,0)/\zeta_k}\right) d\zeta_k \wedge d\overline{\zeta_k}.
\end{aligned}$$

Regardless of whether Case 1) or 2) holds, convergence of the expressions (I) and (II) is guaranteed. Consider the difference

$$f_*(t'_n, \tau_n, \zeta_k, t_n^{(k)}/\zeta_k) - f_*(0, \tau_\infty, \zeta_k, 0),$$

where $*$ indicates that the choice of subscript i or j does not matter here as long as the subscript is the same on both functions. By [38], f_* is holomorphic in all variables, hence, there is a constant $C_0 > 0$ such that

$$\frac{2}{|\zeta_k|}\left|f_*(t'_n, \tau_n, \zeta_k, t_n^{(k)}/\zeta_k) - f_*(0, \tau_\infty, \zeta_k, 0)\right| \le C_0 \frac{|t_n^{(k)}|}{|\zeta_k|^2}$$

and

$$2\left|\frac{f_*(t'_n, \tau_n, \zeta_k, t_n^{(k)}/\zeta_k)}{\zeta_k}\right| \le C_0 \frac{1}{|\zeta_k|}.$$

Using Hölder's inequality, there is a constant $C_1 > 0$ such that the following inequalities hold

$$\begin{aligned}
|\text{(I)}| \quad &\le 4\|(f_i(t'_n, \tau_n, \zeta_k, t_n^{(k)}/\zeta_k) - f_i(0, \tau_\infty, \zeta_k, 0))/\zeta_k\|_{L^2(R_z(t_n^{(k)}))} \\
&\qquad \|f_j(t'_n, \tau_n, \zeta_k, t_n^{(k)}/\zeta_k)/\zeta_k\|_{L^2(R_z(t_n^{(k)}))} \\
&\le \|C_0 \tfrac{|t_n^{(k)}|}{|\zeta_k|^2}\|_{L^2(R_z(t_n^{(k)}))} \|C_0 \tfrac{1}{|\zeta_k|}\|_{L^2(R_z(t_n^{(k)}))} \\
&\le C_1 \tfrac{|t_n^{(k)}|}{|t_n^{(k)}|^{1-q}} (\log|t_n^{(k)}|)^{1/2} = C_1 |t_n^{(k)}|^q (\log|t_n^{(k)}|)^{1/2}
\end{aligned}$$

and

$$\begin{aligned}|(\mathrm{II})| &\leq 4\|(f_j(t'_n,\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)-f_j(0,\tau_\infty,\zeta_k,0))/\zeta_k\|_{L^2(R_z(t_n^{(k)}))}\\ &\qquad \|f_i(0,\tau_\infty,\zeta_k,0)/\zeta_k\|_{L^2(R_z(t_n^{(k)}))}\\ &\leq \|C_0\tfrac{|t_n^{(k)}|}{|\zeta_k|^2}\|_{L^2(R_z(t_n^{(k)}))}\|C_0\tfrac{1}{|\zeta_k|}\|_{L^2(R_z(t_n^{(k)}))}\leq C_1|t_n^{(k)}|^q(\log|t_n^{(k)}|)^{1/2}.\end{aligned}$$

The convergence for (III) remains to be shown. We split this into two cases that are resolved by Lemmas 1.3.3 and 1.3.4. Note that in Case 2), it suffices to assume that $f_i(0,\tau_\infty,0,0)\neq 0$ and $f_j(0,\tau_\infty,0,0)\neq 0$. Otherwise, Case 2) is subsumed by Case 1). □

Lemma 1.3.3. *Given k, if $f_i(0,\tau_\infty,0,0)=0$ on D_k or $f_j(0,\tau_\infty,0,0)=0$ on D_k, then (III) converges to zero as n tends to infinity.*

Proof. By the assumption that at most one of f_i and f_j has a simple pole, we have

$$|(\mathrm{III})|\leq 4\left|\int_{R_z(t_n^{(k)})}(f_i(0,\tau_\infty,\zeta_k,0)f_j(0,\tau_\infty,\zeta_k,0))\right.$$
$$\left.\left(\frac{\overline{A_n(t'_n,\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)/\zeta_k}}{A_n(t'_n,\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)/\zeta_k}-\frac{\overline{A_\infty(0,\tau_\infty,\zeta_k,0)/\zeta_k}}{A_\infty(0,\tau_\infty,\zeta_k,0)/\zeta_k}\right)\frac{d\zeta_k\wedge d\overline{\zeta_k}}{\zeta_k}\right|.$$

Since f_i and f_j are holomorphic, they are bounded on $R_z(t_n^{(k)})$. This implies that there is a constant $C>0$ such that

$$|(\mathrm{III})|\leq C\int_{R_z(t_n^{(k)})}\left|\frac{\overline{A_n(t'_n,\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)}}{A_n(t'_n,\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)}-\frac{\overline{A_\infty(0,\tau_\infty,\zeta_k,0)}}{A_\infty(0,\tau_\infty,\zeta_k,0)}\right|\frac{d\zeta_k\wedge d\overline{\zeta_k}}{|\zeta_k|}.$$

This converges by the dominated convergence theorem because the integrand is bounded by the integrable function $2/|\zeta_k|$ for all n. □

Lemma 1.3.4. *Given k, if $f_i(0,\tau_\infty,0,0)\neq 0$, $f_j(0,\tau_\infty,0,0)\neq 0$, and $A_\infty(0,\tau_\infty,0,0)\neq 0$ on D_k, then (III) converges to zero as n tends to infinity.*

Proof. By assumption, there exists N such that $A_n(0,\tau_\infty,0,0)\neq 0$ for all $n\geq N$. Since $A_\infty(0,\tau_\infty,0,0)\neq 0$, there exists $r>0$ such that $A_n(t'_n,\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)\neq 0$ and $\overline{A_n(t'_n,\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)}/A_n(t'_n,\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)$ is a real analytic function in the polydisk $\{|t_n^{(k)}|<r,|\zeta_k|<r\}\subset\mathbb{C}^2$. Therefore, there exists a constant $C_2>0$ such that in the annulus $\{|t_n^{(k)}|^{1-q}<|\zeta_k|<r/2\}$, we have

$$\left|\frac{\overline{A_n(t'_n,\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)/\zeta_k}}{A_n(t'_n,\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)/\zeta_k}-\frac{\overline{A_\infty(0,\tau_\infty,\zeta_k,0)/\zeta_k}}{A_\infty(0,\tau_\infty,\zeta_k,0)/\zeta_k}\right|$$

$$=\left|\frac{\overline{A_n(t'_n,\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)}}{A_n(t'_n,\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)}-\frac{\overline{A_\infty(0,\tau_\infty,\zeta_k,0)}}{A_\infty(0,\tau_\infty,\zeta_k,0)}\right|\leq C_2\frac{|t_n^{(k)}|}{|\zeta_k|}\leq C_2|t_n^{(k)}|^q.$$

Exactly as in the proof of [10][Lemma 4.2], there exists a constant $C_3>0$ such that

$$|(\mathrm{III})|\leq -C_3|t_n^{(k)}|^q\log|t_n^{(k)}|+4\left|\int_{|\zeta_k|\geq r/2}\left(\frac{f_i(0,\tau_\infty,\zeta_k,0)}{\zeta_k}\frac{f_j(0,\tau_\infty,\zeta_k,0)}{\zeta_k}\right)\right.$$
$$\left.\left(\frac{\overline{A_n(t'_n,\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)/\zeta_k}}{A_n(t'_n,\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)/\zeta_k}-\frac{\overline{A_\infty(0,\tau_\infty,\zeta_k,0)/\zeta_k}}{A_\infty(0,\tau_\infty,\zeta_k,0)/\zeta_k}\right)d\zeta_k\wedge d\overline{\zeta_k}\right|.$$

Since the domain of integration in the right-hand integral does not depend on t, the domain of integration is compact and the integrand is bounded by an integrable function for all n. This proof is completed by applying the dominated convergence theorem to the sequence as n tends to infinity. □

Definition. *Define the extension of the rank k locus to the boundary of $\mathcal{M}_g$ to be the closure of $\mathcal{D}_g(k)$ in $\overline{\mathcal{M}_g}$ and denote it by $\overline{\mathcal{D}_g(k)}$.*

Remark. *Since $\mathcal{D}_g(k)$ is already a closed set in $\mathcal{M}_g$, we would never need to write $\overline{\mathcal{D}_g(k)}$ to mean the closure of $\mathcal{D}_g(k)$ in $\mathcal{M}_g$.*

Lemma 1.3.5. *If $(X', \omega') \in \overline{\mathcal{D}_g(k)}$, ω' is holomorphic on X', and $\omega' \not\equiv 0$ on any part of X', then*

$$\mathit{Rank}\left(\frac{d\Pi(X')}{d\mu_{\omega'}}\right) \leq k.$$

Proof. This is clear for $(X', \omega') \subset \mathcal{D}_g(k)$, so we assume $(X', \omega') \in \overline{\mathcal{D}_g(k)} \cap \partial\overline{\mathcal{M}_g}$. By definition, $\overline{\mathcal{D}_g(k)}$ is the closure of $\mathcal{D}_g(k)$ in $\overline{\mathcal{M}_g}$, so there exists a sequence $\{(X_n, \omega_n)\}_{n=1}^\infty$ in $\mathcal{D}_g(1)$ converging to (X', ω'). Let X' be a surface of genus $g' < g$. Let $\{\theta_1^{(n)}, \ldots, \theta_{g'}^{(n)}, \ldots \theta_g^{(n)}\}$ be a basis of Abelian differentials on X_n ordered so that

$$\lim_{n\to\infty} \theta_m^{(n)} = \theta_m,$$

for $1 \leq m \leq g'$, and the set $\{\theta_1, \ldots, \theta_{g'}\}$ is a basis for the space of holomorphic Abelian differentials on X'. Note that for each m, $1 \leq m \leq g'$, $\{\theta_m^{(n)}\}_{n=1}^\infty$ is a sequence of holomorphic differentials converging to a holomorphic differential. Let $A_n = (A_n)_{ij}$ denote the minor of $d\Pi(X_n)/d\mu_{\omega_n}$ defined by

$$A_{ij}^{(n)} = \int_{X_n} \theta_i^{(n)} \theta_j^{(n)} d\mu_{\omega_n},$$

for $1 \leq i, j \leq g'$, and let A denote the derivative of the period matrix of (X', ω'). Since we restricted our attention to the basis of differentials that are holomorphic on X' and ω' is holomorphic, A_n converges to A component-wise by Lemma 1.3.2. For any sequence of matrices $\{A_n\}_{n=1}^\infty$ converging to a matrix A component-wise, there exists an $\varepsilon > 0$ such that if $\|A_n - A\| < \varepsilon$, where $\|A\|$ denotes the sum of the absolute values of the components of A, then $\mathrm{Rank}(A_n) \geq \mathrm{Rank}(A)$. Also, given a matrix M with minor B, $\mathrm{Rank}(M) \geq \mathrm{Rank}(B)$. The lemma follows by letting

$M = d\Pi(X_n)/d\mu_{\omega_n}$ and $B = A_n$, so that

$$k \geq \operatorname{Rank}\left(\frac{d\Pi(X_n)}{d\mu_{\omega_n}}\right) \geq \operatorname{Rank}(A_n) \geq \operatorname{Rank}(A) = \operatorname{Rank}\left(\frac{d\Pi(X')}{d\mu_{\omega'}}\right).$$

□

Lemma 1.3.6. *Let $\{(X(t_n,\tau_n),\omega_n)\}_{n=0}^{\infty}$ be a sequence of surfaces converging to a surface $(X',\omega') \in \overline{\mathcal{M}_g}$. For all i,j and $n \geq 0$, there exists a constant $C > 0$, such that*

$$\left|\int_{X^*_{\tau_\infty}} \theta_i(0,\tau_\infty)\theta_j(0,\tau_\infty)\frac{\bar{\omega}'}{\omega'}\right| < C.$$

Proof. The differentials $\theta_i(0,\tau_\infty)$ are holomorphic on the compact set $\overline{X^*_{\tau_\infty}}$, for all i, by the definition of $X^*_{\tau_\infty}$. Hence, $|\theta_i(0,\tau_\infty)| < C'$ for some constant C' and all i. This implies

$$\left|\int_{X^*_{\tau_\infty}} \theta_i(0,\tau_\infty)\theta_j(0,\tau_\infty)\frac{\bar{\omega}'}{\omega'}\right| \leq \int_{X^*_{\tau_\infty}} |\theta_i(0,\tau_\infty)\theta_j(0,\tau_\infty)| \leq C'^2 = C.$$

□

Lemma 1.3.7. *Let $D_\varepsilon = \{z \big| |\varepsilon| \leq |z| \leq 1\} \subset \mathbb{C}$. For all $N \geq 0$ and $\varepsilon > 0$,*

$$\int_{D_\varepsilon} \frac{z^N}{\bar{z}} dz \wedge d\bar{z} = 0.$$

Proof. Convert to polar coordinates by letting $z = re^{i\theta}$. For all $\varepsilon > 0$

$$\int_{D_\varepsilon} z^N/\bar{z}\, dz \wedge d\bar{z} = -2i \int_0^{2\pi}\int_\varepsilon^1 \frac{r^N e^{iN\theta}}{re^{-i\theta}} r\, dr d\theta = -2i \int_0^{2\pi}\int_\varepsilon^1 r^N e^{i(N+1)\theta}\, dr d\theta.$$

This expression integrates to zero, for all $N \geq 0$. □

Lemma 1.3.8. *Let $D_\varepsilon = \{z \big| |\varepsilon| \leq |z| \leq 1\} \subset \mathbb{C}$. For all $N \in \mathbb{Z}$, $K \geq 0$ and $\varepsilon > 0$, there exists $C > 0$ such that*

$$\left|\int_{D_\varepsilon} z^N \bar{z}^K\, dz \wedge d\bar{z}\right| < C.$$

Proof. Convert to polar coordinates by letting $z = re^{i\theta}$. Then for all $\varepsilon > 0$

$$\begin{aligned}\int_{D_\varepsilon} z^N \bar{z}^K \, dz \wedge d\bar{z} &= -2i \int_0^{2\pi} \int_\varepsilon^1 r^N e^{iN\theta} r^K e^{-iK\theta} r \, dr d\theta \\ &= -2i \int_0^{2\pi} \int_\varepsilon^1 r^{N+1+K} e^{i(N-K)\theta} \, dr d\theta\end{aligned}$$

If $N - K \neq 0$, this expression integrates to zero. Otherwise, this equals

$$2 \left| \int_0^{2\pi} \int_\varepsilon^1 r^{2K+1} \, dr d\theta \right| < \frac{2\pi}{K+1} + O(\varepsilon) < C,$$

for some $C > 0$. $\square$

We state the following two results for the annulus $R_z(t_n^{(k)})$ and remark that the same results hold for $R_w(t_n^{(k)})$.

Lemma 1.3.9. *We follow the notation established above. Let $\{(X(t_n, \tau_n), \omega_n)\}_{n=0}^\infty$ be a sequence of surfaces converging to a degenerate surface $(X(0, \tau_\infty), \omega')$. For each n, let $\{\theta_1(t_n, \tau_n), \ldots, \theta_g(t_n, \tau_n)\}$ be a basis for the space of Abelian differentials on $X(t_n, \tau_n)$. Given i, j, k, if either $f_i(0, \tau_\infty, 0, 0) = 0$ on D_k or $f_j(0, \tau_\infty, 0, 0) = 0$ on D_k, then there exists $C > 0$ such that for all $n \geq 0$*

$$\left| \int_{R_z(t_n^{(k)})} \theta_i(0, \tau_\infty) \theta_j(0, \tau_\infty) d\mu_{\omega'} \right| < C.$$

In particular, if $f_i(0, \tau_\infty, 0, 0) = 0$, $f_j(0, \tau_\infty, 0, 0) = 0$ on D_k, and $A_\infty(0, \tau_\infty, 0, 0) \neq 0$ on D_k, then

$$\lim_{n \to \infty} \int_{R_z(t_n^{(k)})} \theta_i(0, \tau_\infty) \theta_j(0, \tau_\infty) \, d\mu_{\omega'} = 0.$$

Proof. There are three cases to consider in the first claim of the lemma. It suffices to consider the case where exactly one of the differentials $\theta_i(0, \tau_\infty)$ or $\theta_j(0, \tau_\infty)$ has a simple pole. Without loss of generality, assume that $\theta_j(0,\tau_\infty)$ is holomorphic. Fix a choice of coordinates ζ_k in $R_z(t_n^{(k)})$ so that by [32][Theorem 6.3], there exists $K \geq -1$ and $c \in \mathbb{C}$ such that $\omega' = c\zeta_k^K \, d\zeta_k$. Let $\theta_i(0,\tau_\infty) = (c_i/\zeta_k + h_i(\zeta_k))\, d\zeta_k$ and $\theta_j(0,\tau_\infty) = h_j(\zeta_k)\, d\zeta_k$, where h_i and h_j are holomorphic in ζ_k. This yields

$$\left|\int_{R_z(t_n^{(k)})} \theta_i(0,\tau_\infty)\theta_j(0,\tau_\infty)\, d\mu_{\omega'}\right| = \left|\int_{R_z(t_n^{(k)})} (c_i/\zeta_k + h_i(\zeta_k))\, h_j(\zeta_k)\frac{\overline{c\zeta_k^K}}{c\zeta_k^K}\, d\zeta_k \wedge d\overline{\zeta_k}\right|$$

$$\leq \left|\int_{R_z(t_n^{(k)})} h_j(\zeta_k)\frac{c_i\overline{c\zeta_k^K}}{c\zeta_k^{K+1}}\, d\zeta_k \wedge d\overline{\zeta_k}\right| + \left|\int_{R_z(t_n^{(k)})} h_i(\zeta_k)h_j(\zeta_k)\frac{\overline{c\zeta_k^K}}{c\zeta_k^K}\, d\zeta_k \wedge d\overline{\zeta_k}\right|$$

$$\leq \left|\int_{R_z(t_n^{(k)})} h_j(\zeta_k)\frac{c_i\overline{\zeta_k^K}}{\zeta_k^{K+1}}\, d\zeta_k \wedge d\overline{\zeta_k}\right| + \left|\int_{R_z(t_n^{(k)})} |h_i(\zeta_k)h_j(\zeta_k)|\, d\zeta_k \wedge d\overline{\zeta_k}\right|.$$

By Lemma 1.3.7 or 1.3.8, depending on the value of K, the right-hand side of the inequality is bounded.

In the particular case when $K = -1$, we have

$$\left|\int_{R_z(t_n^{(k)})} \theta_i(0,\tau_\infty)\theta_j(0,\tau_\infty)\, d\mu_{\omega'}\right| = \left|\int_{R_z(t_n^{(k)})} (c_i/\zeta_k + h_i(\zeta_k))\, h_j(\zeta_k)\frac{\overline{c}\zeta_k}{c\overline{\zeta_k}}\, d\zeta_k \wedge d\overline{\zeta_k}\right|$$

$$\leq \left|\int_{R_z(t_n^{(k)})} h_j(\zeta_k)\frac{c_i}{\overline{\zeta_k}}\, d\zeta_k \wedge d\overline{\zeta_k}\right| + \left|\int_{R_z(t_n^{(k)})} h_i(\zeta_k)h_j(\zeta_k)\frac{\zeta_k}{\overline{\zeta_k}}\, d\zeta_k \wedge d\overline{\zeta_k}\right|.$$

By Lemma 1.3.7, both terms on the right-hand side of the inequality are zero. □

Lemma 1.3.10. *We follow the notation established above. Let $\{(X(t_n,\tau_n),\omega_n)\}_{n=0}^\infty$ be a sequence of surfaces converging to a degenerate surface $(X(0,\tau_\infty),\omega')$. For each n, let $\{\theta_1(t_n,\tau_n),\ldots,\theta_g(t_n,\tau_n)\}$ be a basis for the space of Abelian differentials on $X(t_n,\tau_n)$. Given i,j,k, if $f_i(0,\tau_\infty,0,0) = c_i \neq 0$, $f_j(0,\tau_\infty,0,0) = c_j \neq 0$, and $A_\infty(0,\tau_\infty,0,0) = c \neq 0$ on D_k, then for sufficiently large n,*

$$\int_{R_z(t_n^{(k)})} \theta_i(0,\tau_\infty)\theta_j(0,\tau_\infty)\, d\mu_{\omega'} = c_ic_j\frac{\overline{c}}{c}(1-q)4\pi\sqrt{-1}\log|t_n^{(k)}| + O(1).$$

Proof. We have

$$\int_{R_z(t_n^{(k)})} \theta_i(0,\tau_\infty)\theta_j(0,\tau_\infty)\, d\mu_{\omega'}$$

$$= \int_{R_z(t_n^{(k)})} \left(c_i/\zeta_k + h_i(\zeta_k)\right)\left(c_j/\zeta_k + h_j(\zeta_k)\right) \frac{\zeta_k}{\overline{\zeta_k}} \left(\overline{c}/c + H(\zeta_k, \overline{\zeta_k})\right)\, d\zeta_k \wedge d\overline{\zeta_k},$$

where h_i and h_j are holomorphic, H is analytic in both variables, and $H(0,0) = 0$. It follows from Lemmas 1.3.7 and 1.3.8 that every term is bounded uniformly for all n with the exception of

$$c_i c_j \frac{\overline{c}}{c} \int_{R_z(t_n^{(k)})} \frac{1}{|\zeta_k|^2}\, d\zeta_k \wedge d\overline{\zeta_k} = -2c_i c_j \frac{\overline{c}}{c}\sqrt{-1} \int_0^{2\pi} \int_{|t_n^{(k)}|^{1-q}/c''}^{c'} \frac{1}{r^2} r\, dr\, d\theta$$

$$= -4\pi c_i c_j \frac{\overline{c}}{c}\sqrt{-1}\left(\log(c') - \log(|t_n^{(k)}|^{1-q}/c'')\right)$$

$$= c_i c_j \frac{\overline{c}}{c}(1-q)4\pi\sqrt{-1}\log|t_n^{(k)}| + O(1).$$

□

1.4 Surgery on Abelian Differentials

We introduce a surgery on holomorphic Abelian differentials that associates them to integrable quadratic differentials. Define $\mathcal{Q}_{g,n}^{(s)}$ to be the moduli space of integrable quadratic differentials with marked line segments of finite length, called *slits*, on the surfaces in $\mathcal{Q}_{g,n}$. The surgery associates elements of $\mathcal{M}_g$ to $\cup_i \mathcal{Q}_{g_i',n_i'}^{(s)}$, where $g_i' \leq g$ and $n_i' \geq 0$, for all i.

Definition. *Let $(X,\omega) \in \mathcal{M}_g$. Let $S = \{\gamma_1, \ldots, \gamma_n\}$ be a set of pairwise non-homotopic closed regular trajectories of the vertical foliation of X by ω, with $n \geq 0$. Let C_i be the cylinder defined by the closure of the maximal set of closed regular leaves homotopic to γ_i. Let $X^* = X \setminus \cup_i C_i$. Choose antipodes with respect to the flat metric* *induced by ω in each of the $2n$ holes of $\overline{X^*}$ so that the antipodes lie at regular points of (X, ω) and for each hole identify the two semicircles. This identification results in marked line segments called* slits*. We call this procedure the* cylinder surgery *and let $\tilde{X}$ denote the (possibly disconnected) surface with slits resulting from performing the cylinder surgery.*

The surgery is well-defined up to a choice of antipodes. In this thesis we will only be concerned with the relationship between the $\mathrm{SL}_2(\mathbb{R})$ action and the cylinder surgery. Since the foliation in which each slit is a subset of a leaf is invariant under the choice of antipodes, the choice of antipodes will not matter to us. It is possible that the vertical foliation of ω is periodic in which case the cylinder surgery results in the empty set. We exclude this case, whenever we perform the cylinder surgery because the surgery does not provide us with any useful information in this case. Now we show that ω naturally induces a quadratic differential $\tilde{q}$ on $\tilde{X}$. If $S = \emptyset$, then $\tilde{X} = X$ and let $\tilde{q} = \omega$.

Lemma 1.4.1. *If $\tilde{X} \neq \emptyset$, then ω induces a non-zero integrable quadratic differential on $\tilde{X}$ denoted by $\tilde{q}$.*

Proof. Assume that $S \neq \emptyset$. There is a natural inclusion $i : X^* \to X$ by the identity. Let $\mathcal{F}$ denote the vertical foliation of ω on X. This naturally pulls back to a foliation on X^* under i. Since the boundary of every hole of X^* is a union of saddle connections by definition of the cylinder surgery, gluing opposite sides of the slits of X^* results in a foliation of $\tilde{X}$, denoted $\tilde{\mathcal{F}}$, that is identical to $i^*(\mathcal{F})$ away from the slits. In a sufficiently small neighborhood of any point p in a slit which is not an endpoint or a zero of ω, the foliation looks like the vertical (or horizontal) foliation of dz. By definition of the cylinder surgery, the ends of the slits locally look like the vertical foliation of dz^2/z.

We define a double cover $\pi : \hat{X} \to \tilde{X}$, which the reader will recognize as the classical orientating double cover construction for a quadratic differential. Let Σ denote the union of the set of zeros of ω and the set of antipodes chosen in the cylinder surgery. Let (U_i, ϕ_i) be an atlas for $\tilde{X} \setminus \Sigma$. For each U_i define $g_i^{\pm}(z) = \pm\sqrt{\phi_i(z)}$ on the open sets $V_i^{\pm}$ which are each a copy of U_i. The charts $\{V_i^{\pm}\}$ can be glued together in a compatible way after filling in the holes of Σ. This defines a surface $\hat{X}$ with a foliation $\hat{\mathcal{F}}$. The reader will easily see that the foliation about the endpoints of the slits of $\tilde{X}$, at which the foliation induced on $\tilde{X}$ locally has the foliation of the simple pole of a quadratic differential, lifts to the foliation about a regular point on $\hat{X}$.

By [18][Main Theorem], the foliation $\hat{\mathcal{F}}$ induces a quadratic differential $\hat{q}$ on $\hat{X}$. (Hubbard and Masur [18] state their Main Theorem in terms of a horizontal foliation, but it can be stated for a vertical foliation as well simply by considering $\sqrt{-1}\,\hat{q}$.) This allows us to view the construction above as the orientating double cover construction, which implies that $\hat{q}$ defines a quadratic differential $\pi_*(\hat{q}) = \tilde{q}$ on $\tilde{X}$ by pushforward. It is obvious that $\tilde{q}$ is not the zero differential. $\square$

Assume that $\tilde{X} \neq \emptyset$. The cylinder surgery defines two maps: an injection $i : \overline{X^*} \hookrightarrow X$, which extends to a map on the Abelian differentials, and a "gluing map" $g : (\overline{X^*}, \omega^*) \to (\tilde{X}, \tilde{q})$, where $g\mid_{X^*} = \mathrm{id}$ and g maps ∂C_j to the slits of

$\tilde{X}$ as prescribed by the cylinder surgery for all j. We abuse notation and write $i : (\overline{X^*}, \omega^*) \hookrightarrow (X, \omega)$, where ω^* is the restriction of ω to $\overline{X^*}$. This allows us to define a "cylinder surgery map" P such that the diagram commutes. Furthermore, given i and g, P can be inverted, so the cylinder surgery can be regarded as a set of maps $\{i, g\}$ associated to (X, ω).

$$\begin{tikzcd}
(\overline{X^*}, \omega^*) \arrow[r, "i"] \arrow[dr, "g"'] & (X, \omega) \arrow[d, dashed, "P"] \\
& (\tilde{X}, \tilde{q})
\end{tikzcd}$$

Lemma 1.4.2. *If $\tilde{X} \neq \emptyset$, then*

$$G_t \cdot P(X, \omega) = P \circ G_t \cdot (X, \omega).$$

Proof. To prove this, we construct the following diagram and prove it commutes.

$$\begin{tikzcd}
(\overline{X^*}, \omega^*) \arrow[rr, "i"] \arrow[dr, "g"] \arrow[dd, "\hat{f}_t"] & & (X, \omega) \arrow[dl, "P"'] \arrow[dd, "f_t"] \\
& (\tilde{X}, \tilde{q}) \arrow[dd, "\tilde{f}_t"] & \\
(\overline{X^*}_t, \omega^*_t) \arrow[rr, "i_t"] \arrow[dr, "g_t"] & & (X_t, \omega_t) \arrow[dl, "P_t"'] \\
& (\tilde{X}_t, \tilde{q}_t) &
\end{tikzcd}$$

The cylinder surgery denoted by P is completely determined without any ambiguity by the maps i and g defined above. The action of G_t is well-defined on all three surfaces in the upper commutative triangle and induces quasiconformal maps on the surfaces which we denote by $\hat{f}_t$, $\tilde{f}_t$, and f_t as indicated in the large diagram.

The map g_t is well-defined because it is induced by the map g, which dictates a choice of antipodes. In charts away from the holes of $\overline{X^*}$, or slits of $\tilde{X}$, $g = g_t = \mathrm{Id}$ and the square trivially commutes. Since there exists a quasiconformal map which is well-defined across the slit, namely $\tilde{f}_t$, the map $\hat{f}_t$ commutes with g and g_t by construction.

The map i_t is well-defined because compact leaves of the horizontal and vertical foliations of (X, ω) are preserved under the action by G_t. By quasiconformal continuation [22], the map $i_t \circ \hat{f}_t$ can be continued to a quasiconformal map $f'_t : (X, \omega) \to (X_t, \omega_t)$, such that $f'_t \mid_{i(\overline{X^*})} = f_t$. Hence $i_t \circ \hat{f}_t = f_t \circ i$. Since the cylinder surgery is completely determined by i and g, the map P_t is induced by i_t and g_t, and the diagram commutes.

$\square$

Proposition 1.4.3. *If $\tilde{X} \neq \emptyset$, then there exists $\theta \in \mathbb{R}$ such that $(\tilde{X}, e^{i\theta}\tilde{q})$ has a closed regular trajectory that does not pass through the slits of $\tilde{X}$.*

Proof. Since the slits lie in the vertical foliation of $\tilde{q}$ by definition, the lengths of the slits tend to zero in this direction as t tends to infinity. Let $\{t_n\}_{n=1}^{\infty}$ be a divergent sequence of times such that

$$\lim_{n \to \infty} G_{t_n} \cdot (\tilde{X}, e^{i\theta}\tilde{q}) = (\tilde{X}', \tilde{q}')$$

has a limit in $\overline{\mathcal{Q}_{g',n'}}$. There are two cases to consider. Either $\tilde{q}'$ has a double pole or it does not. If $\tilde{q}'$ does have a double pole, then there is a cylinder with height tending to infinity with t_n. Hence, there is a closed regular trajectory on this cylinder and it cannot cross a slit since the length of the slits are going to zero while the height of the cylinder is tending to infinity.

On the other hand, if $\tilde{q}'$ is integrable, then we claim it is holomorphic. Since the cylinders are maximal sets, every slit will contain at least one zero of $\tilde{q}$. Then $\tilde{q}'$ must be holomorphic because all of the simple poles of $\tilde{q}$ lie at the ends of slits by definition and every simple pole of $\tilde{q}$ has been contracted to a zero of $\tilde{q}$. Since $\tilde{q}'$ is holomorphic, we can apply [27][Theorem 2] to find a dense set of closed trajectories on $(\tilde{X}', \tilde{q}')$. Each of these trajectories correspond to a cylinder. Let C denote one such cylinder. Then C also represents a cylinder on $G_{t_n} \cdot (\tilde{X}, e^{i\theta}\tilde{q})$ for large n. Choose $N > 0$ so that C has height h and the total length of the slits is $\varepsilon > 0$ and $h >> \varepsilon$. There is a regular trajectory corresponding to a waist curve γ of C at time t_N such that γ does not intersect any of the slits. $\square$

Chapter 2

The Main Theorems and Direct Applications

2.1 Complete Periodicity and the Connectivity Graph in $\mathcal{D}_g(1)$

The key results of this section are Theorem 2.1.5 and Lemma 2.1.10. They form the foundation on which the remainder of this thesis rests. The former result proves that every surface generating a Teichmüller disk in the rank one locus must be completely periodic, while the latter result describes the configuration of the parts of a degenerate surface in the closure of a Teichmüller disk contained in the rank one locus. We begin by recalling some basic definitions from graph theory.

Let G be a graph consisting of a vertex set $V(G)$ and an edge set $E(G)$. A *path* is a graph with vertex set $\{v_1, \ldots, v_n\}$ such that there is an edge from v_i to v_{i+1}, for all $1 \leq i \leq n-1$. A *cycle* is a path with an additional edge connecting v_1 to v_n. Consider the set of all cycles contained in G. This set forms a finite dimensional vector space over the field $\mathbb{F}_2$ called the *cycle space of* G. Denote the dimension of the cycle space by $\dim^C(G)$. All the graphs in the discussion below may be multigraphs, i.e. we permit multiple edges between the same pair of vertices and there may be edges from a vertex to itself.

Definition. *Let $G(X')$ be a multigraph associated to X' or simply G when the surface is understood. There is a bijection sending $V(G)$ to the parts of X' by $v_i \mapsto S_i$. For all i, j and all pairs of punctures (p, p') from parts S_i to S_j of X',* *with i not necessarily distinct from j, there is a unique edge of G from v_i to v_j representing (p, p'). The graph G is called the* connectivity graph. *Let $G^P(X', \omega')$ be the subgraph of $G(X')$ such that $V(G^P) = V(G)$ and the edges of G^P correspond to the pairs of punctures at which ω' has simple poles.*

Remark. *We will be using Lemma 1.3.2 implicitly throughout this section. It is extremely important to note that nowhere in these results do we require that every component of the derivative of the period matrix has a limit as we take sequences in $\mathcal{M}_g$ converging to a degenerate surface. We are very careful to choose minors of the derivative of the period matrix such that the limit exists. This will suffice to provide the requisite lower bounds on the rank of the derivative of the period matrix near the boundary of the moduli space.*

Throughout this section, it will be advantageous to choose a basis of Abelian differentials with very specific properties depending on the surface to which a sequence of Abelian differentials is converging. Most importantly, the choice of basis we make in the following lemma will facilitate the application of the convergence lemmas from Section 1.3.2.

Lemma 2.1.1. *Given a degenerate surface $X(0, \tau_\infty)$ in the boundary of $\overline{\mathcal{R}_g}$, there exists a set of Abelian differentials $\{\theta_1(0, \tau_\infty), \ldots, \theta_g(0, \tau_\infty)\}$ on $X(0, \tau_\infty)$ such that for all $t = (t_1, \ldots, t_m)$, with $t_j \neq 0$ for all j, $\{\theta_1(t, \tau), \ldots, \theta_g(t, \tau)\}$ is a basis for the space of holomorphic Abelian differentials on $X(t, \tau)$. Moreover, this set can be constructed so that $\{\theta_1(0, \tau_\infty), \ldots, \theta_g(0, \tau_\infty)\}$ has the following properties:*

(1) For some $1 \leq g_1 \leq g$, $\theta_i(0, \tau_\infty)$ is holomorphic if and only if $1 \leq i \leq g_1$.

(2) For all (p_i, p_i') such that $(p_i, p_i') \in S$ for some part $S \subset X'$, $\theta_i(0, \tau_\infty)$ has simple poles at (p_i, p_i'), $\theta_i(0, \tau_\infty)$ is holomorphic across all other punctures of S, and $\theta_i(0, \tau_\infty) \equiv 0$ on $X' \setminus S$.

(3) For each cycle $C_i \in G(X')$ consisting of more than one edge, $\theta_i(0, \tau_\infty)$ has poles at the pairs of punctures corresponding to the edges of C_i and $\theta_i \equiv 0$ for all $S \subset X'$ such that S does not correspond to a vertex of C_i.

(4) For any puncture $p \in X'$ and for all i, j, if $Res_p(\theta_i) \neq 0$ and $Res_p(\theta_j) \neq 0$, then $Res_p(\theta_i) = Res_p(\theta_j) = \pm 1$.

Proof. The first claim follows from the Cartan-Serre theorem or [26][Proposition 4.1]. We proceed by explicitly constructing a basis of Abelian differentials on X' with the desired properties. The first g_1 differentials can be taken as a union of the bases of holomorphic differentials on each part such that if θ_i is an element of the basis of Abelian differentials on a part $S \subset X'$, then define $\theta_i \equiv 0$ on $X' \setminus S$.

Let the parts of X' be given by $S_1 \sqcup \cdots \sqcup S_n$. By [9][Theorem II.5.1 b.], given two punctures (p, p') on a connected Riemann surface S, there exists a meromorphic Abelian differential on S which is holomorphic everywhere on S and across all punctures of S except p and p', where it can be expressed as dz/z and $-dw/w$, in terms of local coordinates z and w, respectively. Hence, for each part S_j carrying a pair of punctures (p, p') we can take a basis element to be a differential which has simple poles only at those two punctures and is zero on every other part. Let the basis of Abelian differentials on X' consist of g_2 such differentials with exactly two simple poles, where $0 \leq g_2 \leq g$.

Finally, let G_1 be the subgraph of $G(X', \omega')$ such that G_1 has no edges from a vertex to itself. We claim $\dim^C(G_1) = g - g_1 - g_2$. This follows because each basis differential on X' corresponds to a closed horizontal homology curve on a surface near X' in the interior of the moduli space $\mathcal{R}_g$. The only horizontal homology curves that have not been accounted for in the description above are those that split over several parts. Define the remaining basis differentials as follows. For each j, with $0 \leq j \leq g - g_1 - g_2$, let C_j be an element of the cycle basis of G. Define θ_j to be zero on every part which does not correspond to a vertex of C_j. Each vertex v of C_j corresponds to a part S of X' such that S has two punctures p_1 and p_2 corresponding to edges of C_j incident to v. The punctures p_1 and p_2 are not paired. By [9][Theorem II.5.1 b.], there is a meromorphic differential holomorphic everywhere on S and across all punctures of S except for p_1 and p_2 at which it has simple poles with residues 1 and -1, respectively. Define the differential θ_j to have two poles on each part corresponding to a vertex in the cycle C_j. The only restriction is given by the rule that if the residue of the simple pole at p_1 is ± 1, then the residue of the simple pole at p_1' is ∓ 1. This construction completes the proof that such a basis exists.

By construction, the residues of each differential at every pole are ± 1. In order to satisfy the final property, it may be necessary to multiply some of the differentials by -1 so that the residues at each puncture are equal. □

Lemma 2.1.2. *Let $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ be a sequence of surfaces in a Teichmüller disk D converging to a degenerate surface (X', ω'). Let $S \subset X'$ be a part of X'. If ω' has*

k_1 pairs of poles on S, then

$$\sup_n Rank\left(\frac{d\Pi(X_n)}{d\mu_{\omega_n}}\right) \geq k_1.$$

Proof. We show that a single pair of poles on X' corresponds to a divergent diagonal term of $d\Pi(X_n)/d\mu_{\omega_n}$ as n tends to infinity, while the off-diagonal terms in the row and column of that unbounded diagonal term are bounded for all n. Let $b_{ij}^{(n)}$ be the ij component of $d\Pi(X_n)/d\mu_{\omega_n}$. Let (p_i, p_i') be a pair of punctures on S such that ω' has a pair of poles at (p_i, p_i'), for $1 \leq i \leq k_1$. As in Lemma 2.1.1, let θ_i have a pair of poles with residue ± 1 at (p_i, p_i') and let θ_i be holomorphic everywhere else on X', for $1 \leq i \leq k_1$. We consider the $k_1 \times k_1$ minor of $d\Pi(X_n)/d\mu_{\omega_n}$ given by $(b_{ij}^{(n)})$, for $1 \leq i, j \leq k_1$, and show that it has full rank for sufficiently large n. By Lemmas 1.3.6 and 1.3.9, all of the off-diagonal terms $b_{ji}^{(n)} = b_{ij}^{(n)}$ are bounded, for all n, because θ_i and θ_j do not have any poles at the same pair of punctures for $i \neq j$. Furthermore, for each i, the contribution of the integral in Rauch's formula to the diagonal term $b_{ii}^{(n)}$ is bounded everywhere outside of the discs around p_i and p_i' by Lemmas 1.3.6 and 1.3.9. By Lemma 1.3.10, the contribution to the integral in Rauch's formula on $R_z(t_n^{(k)})$ diverges with n. Recall that if ω' has residue c at p_i, then it has residue $-c$ at p_i'. Since the quotient $\bar{c}/c = -\bar{c}/-c$, the sum of the two divergent terms coming from Lemma 1.3.10 do not cancel and $b_{ii}^{(n)}$ diverges to infinity with n. □

Lemma 2.1.3. *Let $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ be a sequence of surfaces in a Teichmüller disk D converging to a degenerate surface (X', ω'). Let G'^P be the subgraph of G^P formed* *by removing all edges from each vertex to itself. Let* $k_2 = \min(\dim^C(G'^P), 2)$*. Then*

$$\sup_n \mathit{Rank}\left(\frac{d\Pi(X_n)}{d\mu_{\omega_n}}\right) \geq k_2.$$

Proof. If $\dim^C(G'^P) = 0$, we are done. If $\dim^C(G'^P) = 1$, then we claim that $d\Pi(X_n)/d\mu_{\omega_n}$ is not the zero matrix, for some choice of n. Let θ_1 be the differential with poles along the cycle of G'^P. Let (p_1, p_1') be a pair of poles of ω' in the cycle. The claim follows from Lemma 1.3.10 by letting $c_1 = \pm 1$, $\lim_{n\to\infty} c^{(n)} = c_1 = \pm 1$, where $c^{(n)}$ is the residue of ω_n in local coordinates about p_1, and considering the $1,1$ component of $d\Pi(X_n)/d\mu_{\omega_n}$.

Assume $\dim^C(G'^P) \geq 2$. Let $C \subset G'^P$ be a cycle. Using Lemma 1.2.4 assume that the residues of ω' are δ-nearly imaginary. It can be shown that given $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $c \in \mathbb{C}$ that are δ-nearly imaginary

$$\left|\frac{\bar{c}}{c} + 1\right| < \varepsilon.$$

Hence, the coefficients of the unbounded $\log|t_n^{(k)}|$ terms in Lemma 1.3.10, for all k, differ from each other by at most 2ε.

By Lemma 2.1.1, there is a basis $\{\theta_1, \ldots, \theta_g\}$ such that for all $1 \leq i \leq g$, θ_i has residue ± 1 at all of its simple poles. Without loss of generality, let θ_1 be an element of the basis of Abelian differentials that has pairs of simple poles corresponding to all of the edges of C. Again, let $b_{ij}^{(n)}$ denote the ij component of the derivative of the period matrix on X_n with respect to ω_n. By Lemma 1.3.6, the integral in Rauch's formula for the derivative of the period matrix is bounded outside of all discs around the punctures of X'. However, it is possible that two different elements in the basis of differentials have simple poles at the same pairs of punctures at which ω' has a simple pole.

Let $C' \subset G'^P$ be a cycle distinct from C (though it may have non-trivial intersection with C). Let θ_2 be the differential with poles at the pairs of punctures corresponding to edges of C'. Every edge of both C and C' corresponds to a pair of poles of ω'. (Note that Lemma 1.3.2 guarantees that we can apply all of the lemmas of Section 1.3.2 to the 2×2 minor $(b_{ij}^{(n)})$, for $1 \leq i, j \leq 2$, because ω' has poles at every puncture where θ_1 or θ_2 have poles.) We claim that for all n sufficiently large, $|b_{11}^{(n)}| > |b_{12}^{(n)}| = |b_{21}^{(n)}|$. Lemma 1.3.10 implies that each of these three terms is a sum of divergent terms. However, $\sharp(E(C \cap C')) < \sharp(E(C))$ implies that $b_{12}^{(n)}$ is a sum of fewer divergent terms than $b_{11}^{(n)}$, and there is no cancellation between the divergent terms by the δ-nearly imaginary assumption. For the exact same reason, $|b_{22}^{(n)}| > |b_{12}^{(n)}| = |b_{21}^{(n)}|$. Thus the diagonal term of each row and column is strictly larger than the off-diagonal terms in its row and column, for n sufficiently large. This implies that the derivative of the period matrix has a 2×2 minor of full rank. $\square$

Lemma 2.1.4. *Let D be a Teichmüller disk contained in $\mathcal{D}_g(1)$. If (X', ω') is a degenerate surface in the closure of D and ω' is not holomorphic, then $G^P(X', \omega')$ is the union of a cycle (possibly on one or two vertices) and a finite (possibly empty) set of isolated vertices.*

Proof. Since every Abelian differential with a simple pole on a Riemann surface S has at least two simple poles on S, no vertex in G^P has degree one. Using the notation of Lemmas 2.1.2 and 2.1.3, we must have $k_1 + k_2 \leq 1$. The case where $k_1 + k_2 = 0$ is excluded by the assumption that ω' in not holomorphic, so we assume $k_1 + k_2 = 1$. If $k_1 = 1$, then G^P has a vertex with an edge forming a loop and Lemmas 2.1.2 and 2.1.3 imply that there are no other edges. If $k_2 = 1$, then G^P contains a cycle C. However, we claim G^P cannot contain any other edges. There are no additional paths in G^P between any two vertices in C because $k_2 = 1$. Since $k_1 = 0$ implies there are no vertices from an edge to itself, there are no additional paths emanating from a vertex in C because any such path would have to end in a vertex of degree one. Hence, $k_2 = 1$ implies $E(G^P) = E(C)$. □

Definition. *Given (X, ω), let $\mathcal{F}_\theta$ denote the vertical foliation of $(X, e^{i\theta}\omega)$. If, for all $\theta \in \mathbb{R}$, $\mathcal{F}_\theta$ has a closed regular trajectory implies that $\mathcal{F}_\theta$ is periodic, then (X, ω) is* completely periodic.

Theorem 2.1.5. *If the Teichmüller disk D generated by (X, ω) is contained in $\mathcal{D}_g(1)$, then (X, ω) is completely periodic.*

Proof. By [27][Theorem 2], there exists a real number θ such that $(X, e^{i\theta}\omega)$ admits a cylinder in the vertical foliation. Without loss of generality, let (X, ω) admit a cylinder C_1 in its vertical foliation. Performing the cylinder surgery on (X, ω) results in the set $\tilde{X}$. By contradiction, assume that $\tilde{X} \neq \emptyset$, i.e. the vertical foliation of X by ω is not periodic. Then the cylinder surgery results in a surface $(\tilde{X}, \tilde{q})$ carrying an integrable quadratic differential with slits. Recall that $\tilde{q}$ has simple poles at the ends of the slits. Since the slits correspond to cylinders in X with parallel core curves, the slits themselves are parallel, in other words, they are leaves of the same foliation. By acting on $(\tilde{X}, \tilde{q})$ by the Teichmüller geodesic flow, the slits, which lie in the vertical foliation, contract at the maximum rate e^{-t} under the area normalization. Consider the one parameter family of surfaces $G_t \cdot (\tilde{X}, \tilde{q})$ for all $t \geq 0$. From this family, choose a sequence of times $\{t_n\}_{n=0}^{\infty}$, with $t_0 = 0$, such that

$$\lim_{n\to\infty} G_{t_n} \cdot (\tilde{X}, \tilde{q}) = (\tilde{X}', \tilde{q}'),$$

where $\tilde{X}'$ is either a degenerate Riemann surface or $\tilde{q}'$ is holomorphic. If $\tilde{q}'$ is holomorphic, this implies that the slits contracted to points. By the definition of the cylinder surgery, $\tilde{q}$ will always have a zero on every slit. In other words, every simple pole of $\tilde{q}$ converged to a zero, resulting in a holomorphic quadratic differential. Let

$$(\tilde{X}_n, \tilde{q}_n) = G_{t_n} \cdot (\tilde{X}, \tilde{q}).$$

Equivalently, consider the sequence $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ defined by

$$(X_n, \omega_n) = G_{t_n} \cdot (X, \omega).$$

By Lemma 1.4.2, we can freely pass between (X_n, ω_n) and $(\tilde{X}_n, \tilde{q}_n)$ for all n. Let (X', ω') denote the degenerate surface to which $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ converges as n tends to infinity. Let $\{C_1^{(n)}\}_{n=0}^{\infty}$ denote the sequence of cylinders such that $C_1^{(0)} = C_1$ and $C_1^{(n)}$ is the corresponding cylinder in X_n after action by G_{t_n}. By Corollary 1.2.6, the core curve of $C_1^{(n)}$ pinches as n tends to infinity. Let C_1' denote the cylinder to which $C_1^{(n)}$ converges as n tends to infinity, if such a cylinder exists. It is possible that ω' is holomorphic at the pair of punctures resulting from pinching the core curves of the cylinder $C_1^{(n)}$, in which case the surface (X', ω') does not have an infinite cylinder at that pair of punctures, i.e. C'_1 does not exist. On the other hand, if C'_1 exists, then it is an infinite cylinder.

We claim that there is a sequence of cylinders $\{C_2^{(n)}\}_{n=0}^{\infty}$ such that $C_2^{(n)} \subset \tilde{X}_n$, for all $n \geq 0$, and the sequence converges to a cylinder $C'_2 \subset \tilde{X}'$ such that C'_2 does not intersect the slits of $\tilde{X}'$ and C'_2 has finite circumference. It suffices to find a cylinder $C'_2 \subset \tilde{X}'$ with the desired properties. We consider four cases. In the first case, the slits have contracted to points and $\tilde{q}'$ is holomorphic. In this case we can find a cylinder C'_2 on $(\tilde{X}', \tilde{q}')$ by [27][Theorem 2]. If the slits have contracted to points and $\tilde{q}'$ has double poles, then we let C'_2 be the infinite cylinder corresponding to a pair of double poles. If the slits have positive length and $\tilde{q}'$ is integrable, then Proposition 1.4.3, guarantees that we can find a cylinder C'_2 not intersecting the slits. Finally, if the slits have positive length and $\tilde{q}'$ has double poles, then, as before, we let C'_2 be the infinite cylinder corresponding to a pair of double poles.

We claim that $\tilde{q}'$, as defined above, can never have simple poles, though it may have double poles. In other words, the length of every slit on $\tilde{X}$ will always converge to zero as n tends to infinity regardless of our choice of normalization. By contradiction, assume that the lengths of the slits on $\tilde{q}'$ have nonzero length. Let $w_j^{(n)}$ denote the circumference of $C_j^{(n)}$, and w'_j denote the circumference of C'_j, for $j = 1, 2$. In this case $w'_j > 0$ for $j = 1, 2$. Consider the ratio $w_1^{(n)}/w_2^{(n)}$. By assumption,

$$\lim_{n\to\infty} w_1^{(n)}/w_2^{(n)} > C > 0.$$

Pass to a subsequence of times $\{t_n\}_{n=0}^{\infty}$ such that there is a constant C^L satisfying

$0 < C^L \leq w_1^{(n)}/w_2^{(n)}$, for all n. Recall that under the area normalization, the length of the slit contracts by e^{-t_n} for each n. Therefore,

$$\lim_{n\to\infty} e^{t_n} w_1^{(n)} = w_1' < \infty,$$

and $w_1' > 0$ by assumption. This implies $w_2' < \infty$ because e^{t_n} is the maximal rate of expansion and

$$w_2' \leq \lim_{n\to\infty} e^{t_n} w_2^{(n)} < \infty.$$

Hence, the core curves of the cylinders $C_1^{(n)}$ and $C_2^{(n)}$ contract for all n at the maximal rate under the area normalization. This is only possible if the core curves of $C_1^{(n)}$ and $C_2^{(n)}$ are parallel for all n. Otherwise, there would be an $N > 0$ sufficiently large, such that $w_2^{(n)}$ increases exponentially for all $n \geq N$. However, it was assumed above that $C_1^{(0)}$ and $C_2^{(0)}$ are not parallel because all cylinders parallel to $C_1^{(0)}$ were removed from the surface. This contradiction implies that the length of every slit must indeed converge to zero.

Since each slit converges to a point, the cylinder C_1' does not exist. If C_2' has finite height, pinch the core curve of the cylinder C_2' under the Teichmüller geodesic flow while normalizing the largest residue. The new degenerate surface, denoted (X', ω') by abuse of notation, either has (Case A:) poles resulting from an infinite cylinder C_2', or (Case B:) neither C_1' nor C_2' exist. By the continuity of the $\mathrm{SL}_2(\mathbb{R})$ action to the boundary of the moduli space [3][Proposition 11.1], there is a sequence $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ in D converging to (X', ω'). We address Cases A and B in the course of the remainder of the proof.

By Lemma 1.2.1, $C_1^{(0)}$ is not homologous to $C_2^{(0)}$ because $C_1^{(0)}$ is not parallel to $C_2^{(0)}$. Since the $\mathrm{SL}_2(\mathbb{R})$ action preserves homology, $C_1^{(n)}$ is not homologous to $C_2^{(n)}$ for all $n \geq 0$. The remainder of this proof is dedicated to finding a degenerate surface (X', ω') in the closure of D such that $G^P(X', \omega')$ contradicts the conclusion of Lemma 2.1.4.

Consider the case when ω' has one or more pairs of simple poles arising from pinching a set of cylinders that are pairwise homologous. In this case, let C_2' be an infinite cylinder, while C_1' does not exist because the circumferences of the cylinders in the sequence $\{C_1^{(n)}\}_{n=0}^{\infty}$ converge to zero. Given $\varepsilon' > 0$, we can find a surface $(X^{(n)}, \omega^{(n)}) \in D$, where n depends on ε', such that $(X^{(n)}, \omega^{(n)})$ has two non-homologous cylinders of equal circumference at most $\sqrt{\varepsilon'}$ and the moduli of the cylinders tend to infinity as ε' tends to zero. Choose $\varepsilon < \varepsilon'$ such that the circumference of $C_1^{(N)}$ is equal to ε for a sufficiently large value of N. Since the sequence $\{C_2^{(n)}\}_{n=0}^{\infty}$ converges to a cylinder of finite nonzero circumference, the circumferences of the cylinders $C_2^{(n)}$, denoted $w_2^{(n)}$ satisfy $0 < w_2^L \leq w_2^{(n)} \leq w_2^U < \infty$, for all n. The core curves of $C_1^{(n)}$ and $C_2^{(n)}$ are not parallel for all n, so for each n there exists a matrix $B_n \in \mathrm{SL}_2(\mathbb{R})$ that transforms the core curve of $C_1^{(n)}$ into a leaf of the vertical foliation and transforms the core curve of $C_2^{(n)}$ into a leaf of the horizontal foliation. For each N, consider the one parameter family of matrices, $G_t B_N \in \mathrm{SL}_2(\mathbb{R})$. Action by $G_t B_N$ on (X_N, ω_N) results in the core curve of $C_1^{(N)}$ expanding at the maximal rate e^t, while the core curve of $C_2^{(N)}$ contracts at the maximal rate e^{-t}. At time t, the circumference of $C_1^{(N)}$ is given by $e^t\varepsilon$, and the circumference of $C_2^{(N)}$ is given by $e^{-t}w_2^{(N)}$. Let T_N be the time satisfying the equation $e^{T_N}\varepsilon = e^{-T_N}w_2^{(N)}$. At time T_N, the circumference of each cylinder is given by $\sqrt{w_2^{(N)}\varepsilon}$. Define a sequence by

$$(X^{(N)}, \omega^{(N)}) := G_{T_N} B_N \cdot (X_N, \omega_N)$$

and consider $C_1^{(N)}, C_2^{(N)}$ to be cylinders in $X^{(N)}$. We claim the moduli of $C_1^{(N)}$ and $C_2^{(N)}$ diverge to infinity with N. Let h denote the height of a cylinder C, w its circumference, $A(C)$ its area, and $\mathrm{Mod}(C)$ its modulus. By the definition of the modulus,

$$\mathrm{Mod}(C) = \frac{h}{w} = \frac{A(C)}{w^2}.$$

In the case at hand, the areas of the cylinders $C_1^{(N)}$ and $C_2^{(N)}$ are bounded below for all N because $\mathrm{SL}_2(\mathbb{R})$ preserves area. Both cylinders have circumference $\sqrt{w_2^{(N)}\varepsilon}$, so their core curves pinch because

$$\lim_{\varepsilon' \to 0} \sqrt{w_2^{(N)}\varepsilon} \leq \lim_{\varepsilon' \to 0} \sqrt{w_2^U \varepsilon'} = 0.$$

Note that this argument can be applied to Case A above. Let $(X'^{(2)}, \omega'^{(2)})$ be the limit of the sequence $\{(X^{(N)}, \omega^{(N)})\}_{N=0}^{\infty}$. As N tends to infinity, the cylinders $C_1^{(N)}$ and $C_2^{(N)}$ degenerate to cylinders of equal circumference. If that circumference is non-zero, then $\omega'^{(2)}$ has two pairs of simple poles coming from non-homologous cylinders. By Lemma 2.1.4, $G^P(X', \omega')$ has a cycle with the pair of punctures represented by C_2' corresponding to an edge of G^P. Since cylinders with pinched core curves remain pinched under this procedure, $G^P(X'^{(2)}, \omega'^{(2)})$ must contain an edge e corresponding to C_1' in addition to the cycle of $G^P(X', \omega')$. It is impossible for e and the edges of $G^P(X', \omega')$ to be part of a larger cycle in $G^P(X'^{(2)}, \omega'^{(2)})$ because that would imply that e represents a cylinder whose core curve, a posteriori, must be parallel to the core curves of the cylinders represented by the edges of $G^P(X', \omega')$. This contradicts Lemma 2.1.4. However, it is still possible that the circumferences of both cylinders converge to zero in which case neither C_1' nor C_2' exist and $\omega'^{(2)}$ is holomorphic at both pairs of punctures. We address this possibility.

By Lemma 1.2.7, we can assume without loss of generality, that $\omega'^{(2)}$ has a pair of simple poles. We proceed by induction, where each step of the induction is to perform the argument of the preceding paragraph until we reach a contradiction. The first step is already done. We present the j^{th} step of the procedure. Let $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ denote the sequence of surfaces converging to a degenerate surface $(X'^{(j)}, \omega'^{(j)})$ such that (X_n, ω_n) has j pairwise non-homologous cylinders all of whose circumferences converge to zero while another sequence of cylinders $\{C_{j+1}^{(n)}\}_{n=0}^{\infty}$ converges to a pair of poles of $\omega'^{(j)}$. Let $\{C_k^{(n)}\}_{n=0}^{\infty}$, for $1 \leq k \leq j$, denote the j distinct sequences of cylinders whose circumferences converge to zero as n tends to infinity. Without loss of generality, let $\{C_1^{(n)}\}_{n=0}^{\infty}$ be a sequence of cylinders such that for infinitely many values of n and all $k \neq 1$, the circumference of $C_k^{(n)}$ is less than or equal to the circumference of $C_1^{(n)}$. This may require the sequences to be renamed. We pass to a subsequence such that this holds for all n. Recall that $\varepsilon' > 0$ was fixed in the preceding paragraph and an appropriate $\varepsilon > 0$ was chosen. It will become apparent that the circumference of the cylinder $C_1^{(n)}$ is $w_1^{(n)}\varepsilon^{1/(2^j)}$, where $w_1^{(n)}$ is a constant satisfying $0 < w_1^L \leq w_1^{(n)} \leq w_1^U < \infty$ for all n. Let $w_{j+1}^{(n)}$ denote the circumference of $C_{j+1}^{(n)}$, which also satisfies $0 < w_{j+1}^L \leq w_{j+1}^n \leq w_{j+1}^U < \infty$ for all n. We highlight the differences that arise in the course of repeating the argument of the preceding paragraph. Solving the equation $e^{T_N} w_1^{(N)} \varepsilon^{1/(2^j)} = e^{-T_N} w_{j+1}^{(N)}$ shows that at time T_N the lengths of the circumferences are $\sqrt{w_{j+1}^{(N)} w_1^{(N)}}\varepsilon^{1/(2^{j+1})}$. To see that the core curves of all $j+1$ cylinders still pinch as ε' tends to zero, note that, as before, the areas of all of the cylinders are fixed under the $\mathrm{SL}_2(\mathbb{R})$ action and thus their areas are bounded from below. Finally,

$$\lim_{\varepsilon' \to 0} \sqrt{w_{j+1}^{(N)} w_1^{(N)}}\varepsilon^{1/(2^{j+1})} \leq \lim_{\varepsilon' \to 0} \sqrt{w_{j+1}^{U} w_1^{U}}\varepsilon'^{1/(2^{j+1})} = 0.$$

Note that this induction procedure includes Case B that was left unaddressed above. Let $(X'^{(j+1)}, \omega'^{(j+1)})$ denote the degenerate surface formed by letting N tend to infinity in the sequence $\{G_{T_N} B_N \cdot (X_N, \omega_N)\}_{N=0}^{\infty}$. As above, the cylinders $C_1^{(N)}$ and $C_{j+1}^{(N)}$ degenerate to cylinders of equal circumference. If that circumference is non-zero, then $\omega'^{(j+1)}$ has at least two pairs of simple poles coming from non-homologous cylinders, namely C_1' and C_{j+1}'. By Lemma 2.1.4, $G^P(X'^{(j)}, \omega'^{(j)})$ has a cycle with the pair of punctures represented by C_{j+1}' corresponding to an edge of G^P. Since cylinders with pinched core curves remain pinched under this procedure, $G^P(X'^{(j+1)}, \omega'^{(j+1)})$ must contain an edge e corresponding to C_1' in addition to the cycle from $G^P(X'^{(j)}, \omega'^{(j)})$. As before, e and the edges of $G^P(X'^{(j)}, \omega'^{(j)})$ cannot be edges of a larger cycle. This contradicts Lemma 2.1.4. However, it is still possible that the circumferences of all $j+1$ cylinders converge to zero in which case $\omega'^{(j+1)}$ is holomorphic at $j+1$ pairs of punctures. In that case, repeat this argument.

This procedure must terminate at worst when $j = g$ because the core curves of the cylinders chosen at each step are pairwise non-homologous, and there are at most g such curves. Hence, performing this procedure at the $g-1$ iteration guarantees at least two poles from sequences of non-homologous cylinders and results in a contradiction. This contradiction demonstrates that $\tilde{X}$ must in fact be the empty set. In other words, the surface is filled by cylinders, and the vertical foliation of X by ω is periodic. Since this argument holds for all $\theta \in \mathbb{R}$ such that $(X, e^{i\theta}\omega)$ admits a cylinder in the vertical foliation, (X, ω) is completely periodic. $\square$

Theorem 2.1.5 is used implicitly in the following corollary to guarantee that it is not a vacuous statement. Compare this statement with [30][Lemma 5.3].

Corollary 2.1.6. *Let (X, ω) generate a Teichmüller disk $D \subset \mathcal{D}_g(1)$. For each $\theta \in \mathbb{R}$ such that the vertical foliation of $(X, e^{i\theta}\omega)$ is periodic, $(X, e^{i\theta}\omega)$ decomposes into a union of cylinders $C_1, \ldots, C_k$ such that all of the saddle connections on the top of C_i are identified to the saddle connections on the bottom of C_{i+1} and vice versa, for all $i \leq k-1$, and all of the saddle connections on the top of C_k are identified to the saddle connections on the bottom of C_1 and vice versa. Furthermore, the circumference of C_i equals the circumference of C_j, for all i, j.*

Proof. Without loss of generality, assume that the vertical foliation of (X, ω) is periodic. Consider a divergent sequence of times $\{t_n\}$ such that the sequence $G_{t_n} \cdot (X, \omega)$ converges to a degenerate surface (X', ω'). By [25][Theorem 3], the limit of this sequence is given by pinching the core curves of every cylinder in the cylinder decomposition of (X, ω). Furthermore, ω' has a pair of simple poles at all of the pairs of punctures of X'. Hence, $G(X', \omega') = G^P(X', \omega')$. Since $G(X', \omega')$ is a connected graph, $G(X', \omega')$ must be a cycle by Lemma 2.1.4. This implies that the cylinders must be arranged in exactly the configuration described in the statement of the corollary. Clearly this argument does not depend on θ, so the result follows. $\square$

Lemma 2.1.7. *Let (X, ω) generate a Teichmüller disk $D \subset \mathcal{D}_g(1)$. If (X', ω') is a degenerate surface in the closure of D and ω' is not holomorphic, then ω' has simple poles on every part of X'.*

Proof. Let $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ be a sequence in D converging to the degenerate surface (X', ω') as n tends to infinity. Since ω' is not holomorphic, there is a sequence of cylinders $\{C_1^{(n)}\}_{n=0}^{\infty}$, such that $C_1^{(n)} \subset X_n$ and the core curve of $C_1^{(n)}$ pinches to form a pair of simple poles of ω'. By Theorem 2.1.5, the foliation in which (X_n, ω_n) admits the cylinder $C_1^{(n)}$ is periodic. Therefore, there is a collection of cylinders $\{C_1^{(n)}, \ldots, C_k^{(n)}\}$ that fill X_n. Let $w_i^{(n)}$ denote the circumference of $C_i^{(n)}$. By Corollary 2.1.6, the ratios $w_i^{(n)}/w_1^{(n)} = 1$ for all $i \leq k$ and $n \geq 0$. Hence, if the core curve of $C_1^{(n)}$ pinches, then the core curve of every cylinder in that foliation pinches. Since the ratios between the circumferences are constant, every sequence of cylinders converges to an infinite cylinder on X', and ω' cannot be holomorphic on any part of X'. $\square$

Corollary 2.1.8. *Let (X, ω) generate a Teichmüller disk $D \subset \mathcal{D}_g(1)$. If (X', ω') is a degenerate surface in the closure of D and $\omega' \not\equiv 0$ on a part $S \subset X'$, then ω' is not identically zero on any part of X'.*

Proof. By contradiction, assume that there is a part S of (X', ω') such that $\omega' \equiv 0$ on S. By Lemma 2.1.7, ω' must be holomorphic on every part of X'. By Lemma 1.2.7 and [3][Proposition 11.1], we can find a sequence of surfaces $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ in the closure of D converging to the degenerate surface (X'', ω'') as n tends to infinity, such that ω'' is not holomorphic on a part of X''. Since the zero differential is obviously fixed by the $\mathrm{SL}_2(\mathbb{R})$ action, this contradicts Lemma 2.1.7 and completes the proof. $\square$

Definition. *An edge e of a connectivity graph $G(X')$ is called a* holomorphic edge *with respect to ω' if ω' is holomorphic at the pair of punctures corresponding to e.*

Lemma 2.1.9. *Let (X, ω) generate a Teichmüller disk $D \in \mathcal{D}_g(1)$ and let (X', ω') be a degenerate surface in the closure of D. If e is an edge in the connectivity graph $G(X')$ between two distinct vertices, then e is not a holomorphic edge with respect to ω'.*

Proof. By contradiction, assume there is a holomorphic edge e between two distinct vertices. First, we claim that ω' cannot be holomorphic on a surface with two or more parts. By Lemma 1.2.7, we can act by the $\mathrm{SL}_2(\mathbb{R})$ action on (X', ω') to reach a surface (X'', ω'') such that ω'' has a pair of simple poles. By Lemma 2.1.7, ω'' must have simple poles on every part of X''. However, for every pair of punctures (p, p') on X' where ω' is holomorphic, ω'' must also be holomorphic at the corresponding pair of punctures on X''. This forces $G^P(X'', \omega'')$ to be a disconnected graph where every vertex in G^P has degree at least two. This contradicts Lemma 2.1.4, so we assume that ω' is not holomorphic on every part of X'.

If ω' is not holomorphic, then Lemmas 2.1.4 and 2.1.7 imply that e is an edge between two vertices of the cycle $G^P(X', \omega')$. Let C_1 be a cylinder corresponding to an edge of $G^P(X', \omega')$. Let (X_1, ω_1) be a surface whose vertical foliation contains the core curve of C_1. The vertical foliation of (X_1, ω_1) is periodic by Theorem 2.1.5, and [25][Theorem 3] implies that the core curves of all of the cylinders parallel to

C_1 pinch under G_t. Let (X'', ω'') be the resulting degenerate surface. Note that ω'' has simple poles at every pair of punctures on X''. Moreover, since we pinched the core curve of every cylinder parallel to C_1, ω'' must have poles at all of the same punctures at which ω' has poles on X'. However, the edge e is no longer in the graph $G(X'', \omega'')$, which implies that the two vertices it joined are a single vertex in $G(X'', \omega'')$. This is impossible because it would imply that $\dim_C(G^P) \geq 2$. Therefore, $G(X')$ has no holomorphic edges with respect to ω'. $\square$

Lemma 2.1.10. *If (X', ω') is a degenerate surface in the closure of a Teichmüller disk $D \subset \mathcal{D}_g(1)$, then (X', ω') has one of the following three configurations:*

(1) (X', ω') has exactly one part with at most *two simple poles.*

(2) (X', ω') has exactly two parts that are joined by exactly two pairs of poles.

(3) $X' = S_1 \sqcup \cdots \sqcup S_n$ has $n \geq 3$ parts such that ω' has exactly one pair of poles joining S_j to S_{j+1}, for $1 \leq j \leq n-1$, and exactly one pair of poles joining S_n to S_1.

Furthermore, there are no pairs of punctures joining two distinct parts in the second and third configuration above such that ω' is holomorphic at those pairs of punctures.

Proof. By Lemma 2.1.2, if X' has one part, then ω' has at most one pair of poles. If X' has more than one part, then this lemma follows from Lemmas 2.1.4, 2.1.7, Corollary 2.1.8, and Lemma 2.1.9. $\square$

Remark. *Case (2) describes a cycle on two vertices that is simply a degenerate version of Case (3). We distinguished it from Case (3) for clarity.*

2.2 Applications of Complete Periodicity in $\mathcal{D}_g(1)$

The property of complete periodicity imposes very strong restrictions on a surface. With little effort we prove that there are no Teichmüller disks in $\mathcal{D}_g(1)$ in certain strata of Abelian differentials and apply this to genus two.

Lemma 2.2.1. *Given a completely periodic surface $(X, \omega) \in \mathcal{M}_g$, $g \geq 2$, there exists $\theta \in \mathbb{R}$ such that the cylinder decomposition of $(X, e^{i\theta}\omega)$ has at least two cylinders.*

Proof. Assume that (X, ω) is filled by a single cylinder C. We show that there exists a direction such that (X, ω) is not filled by a single cylinder. The top and bottom of C consist of a union of saddle connections. Choose one such saddle connection σ on the bottom of C joining zeros z_1 to z_2, which are not necessarily distinct. Let σ' be the saddle connection on the top of C to which σ is identified. Let σ' have endpoints z_1' and z_2' such that z_i is identified to z_i', for $i = 1, 2$. Consider the family of trajectories in C parallel to a trajectory from z_1 to z_1'. This determines a cylinder $C' \subset X$ with z_1 on its top and z_2 on its bottom formed by identifying σ to σ'. Since σ is a proper subset of the top of cylinder C, the cylinder C' does not fill (X, ω). Furthermore, (X, ω) is completely periodic, so the complement of C' must contain at least one cylinder. $\square$

Proposition 2.2.2. *There are no Teichmüller disks contained in $\mathcal{D}_g(1) \cap \mathcal{H}(2g-2)$.*

Proof. By contradiction, assume that there is a surface (X, ω) generating a Teichmüller disk in $\mathcal{D}_g(1) \cap \mathcal{H}(2g-2)$. By Lemma 2.2.1, choose a direction θ such that $(X, e^{i\theta}\omega)$ decomposes into two or more cylinders. Under the Teichmüller geodesic flow, $(X, e^{i\theta}\omega)$ degenerates to a surface (X', ω') with two or more parts by Lemma 2.1.10 and [25][Theorem 3]. Moreover, the zero of order $2g-2$ must lie on exactly one of the parts because [25][Theorem 3] implies that only the core curves of cylinders are pinched. This implies that there is a part of X' with two simple poles and no zeros, i.e. a twice punctured sphere. This is not admissible under the Deligne-Mumford compactification, thus we get a contradiction. □

Proposition 2.2.3. *Let n and m be odd numbers such that $n+m = 2g-2$. There are no Teichmüller disks contained in $\mathcal{D}_g(1) \cap \mathcal{H}(n,m)$.*

Proof. By contradiction, assume that there is a surface (X, ω) generating a Teichmüller disk in $\mathcal{D}_g(1) \cap \mathcal{H}(n,m)$. By Lemma 2.2.1, choose a direction θ such that $(X, e^{i\theta}\omega)$ decomposes into two or more cylinders. Under the Teichmüller geodesic flow, $(X, e^{i\theta}\omega)$ degenerates to a surface (X', ω') with two or more parts by Lemma 2.1.10 and [25][Theorem 3]. Moreover, the zeros must lie on one or two of the parts of X' because [25][Theorem 3] implies that only the core curves of cylinders were pinched. If they lie on the same part, then as before, every other part must be a twice punctured sphere, which is impossible. However, if they lie on different parts, then there is a part with two simple poles and a zero of order n. Since there does not exist an integer $g' \geq 0$ such that $n-2 = 2g'-2$, the Chern formula cannot be satisfied and we have a contradiction. □

Though Proposition 2.2.4 is well-known, we provide an original proof that there are no Teichmüller disks contained in $\mathcal{D}_2(1)$. The best possible result for the Lyapunov exponents of genus two surfaces was proven by Bainbridge [2], who used

McMullen's [29] classification of $\mathrm{SL}_2(\mathbb{R})$-invariant ergodic measures in genus two to calculate the Lyapunov exponents of the Kontsevich-Zorich cocycle explicitly. Bainbridge found $\lambda_2 = 1/2$, for all $\mathrm{SL}_2(\mathbb{R})$-invariant ergodic measures with support in $\mathcal{H}(1,1)$, and $\lambda_2 = 1/3$, for all $\mathrm{SL}_2(\mathbb{R})$-invariant ergodic measures with support in $\mathcal{H}(2)$.

Proposition 2.2.4. *There are no Teichmüller disks contained in* $\mathcal{D}_2(1)$.

Proof. This follows from Propositions 2.2.2 and 2.2.3 because $\mathcal{M}_2 = \mathcal{H}(2) \cup \mathcal{H}(1,1)$. □

Note that $\mathcal{D}_2(1)$ is the determinant locus in genus two. We remark that the author has another proof of Proposition 2.2.4 using more direct methods than those in this thesis and more elementary than those of [2].

Chapter 3

Relation to Veech Surfaces

3.1 Convergence to Veech Surfaces

The goal of this section is to prove Theorem 3.1.4, which will serve as the first step toward bridging the gap between the problem of classifying all Teichmüller disks in $\mathcal{D}_g(1)$ and Möller's [30] nearly complete classification of Teichmüller *curves* in $\mathcal{D}_g(1)$.

Lemma 3.1.1. *Given a surface (X, ω) generating a Teichmüller disk $D_1 \subset \mathcal{D}_g(1)$, let $\{(X_n, \omega_n)\}_{n=1}^{\infty}$ be a sequence of surfaces in D_1 converging to $(X', \omega') \in \overline{\mathcal{M}_g}$, where $(X', \omega') \notin D_1$ and ω' is holomorphic. If D_2 is the Teichmüller disk generated by (X', ω'), then $D_2 \subset \overline{\mathcal{D}_g(1)}$. Furthermore, $D_2 \subset \overline{D_1}$.*

Proof. We recall that the $\mathrm{SL}_2(\mathbb{R})$ action on $\overline{\mathcal{M}_g}$ is continuous by [3][Proposition 11.1]. Since $\overline{\mathcal{D}_g(1)}$ is closed, the closure of D_1 in $\overline{\mathcal{M}_g}$ is also contained in $\overline{\mathcal{D}_g(1)}$. Furthermore, every point in D_2 is the limit of a sequence of points in D_1. This can be seen by taking a sufficiently small neighborhood of (X', ω'), which contains points in D_1 by assumption. By the continuity of the $\mathrm{SL}_2(\mathbb{R})$ action on $\overline{\mathcal{M}_g}$, there is an arbitrarily small neighborhood of any point in D_2 that also contains points in D_1. Hence, $D_2 \subset \overline{D_1} \subset \overline{\mathcal{D}_g(1)}$. □

Definition. *A surface (X, ω) is called a* Veech surface *if its group $SL(X, \omega)$ of affine* *diffeomorphisms is a lattice in $SL_2(\mathbb{R})$. The Teichmüller disk generated by a Veech surface in the moduli space $\mathcal{M}_g$ is called a* Teichmüller curve.

The reason for the term Teichmüller curve follows from a result of Smillie, which states that the $\mathrm{SL}_2(\mathbb{R})$ orbit of a Veech surface projected into $\mathcal{R}_g$ is closed. This result was never published by John Smillie. However, it was communicated to William Veech, who outlined a proof of it in [35]. Moreover, when projected into $\mathcal{R}_g$, Teichmüller curves are algebraic curves. One striking property of Veech surfaces is the *Veech dichotomy.* The Veech dichotomy completely describes the dynamics of the trajectory of any point on the surface X [33]. It says that the geodesic flow on X with respect to the flat structure induced by ω is either periodic or uniquely ergodic. The following definition was introduced in [5].

Definition. *A completely periodic surface satisfies* topological dichotomy *if any direction that admits a saddle connection is periodic.*

Lemma 3.1.2. *Given a Teichmüller disk $D \subset \mathcal{D}_g(1)$ of a completely periodic surface $(X_0, \omega_0) \in \mathcal{M}_g$, which does not satisfy topological dichotomy, there exists a sequence of surfaces $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ in D converging to a surface $(X', \omega') \in \overline{\mathcal{D}_g(1)}$ such that X' has one part, ω' is holomorphic, and a saddle connection of (X_0, ω_0) contracts to a point on (X', ω').*

Proof. By assumption, there exists a saddle connection σ_0 lying in a nonperiodic foliation of the surface (X_0, ω_0). Without loss of generality, let σ_0 lie in the vertical foliation of (X_0, ω_0). Act by the Teichmüller geodesic flow G_t on (X_0, ω_0) so that σ_0 contracts by e^{-t} as t tends to infinity. We prove that we can choose a divergent sequences of times $\{t_n\}_{n=0}^{\infty}$ such that the corresponding sequence of surfaces $\{(X_n, \omega_n)\}_{n=0}^{\infty}$, defined by

$$(X_n, \omega_n) = G_{t_n} \cdot (X_0, \omega_0),$$

converges to a degenerate surface (X', ω'), where ω' is holomorphic. Let $t_0 = 0$.

Let σ_t be the saddle connection on $G_t \cdot (X_0, \omega_0)$ defined by contracting the saddle connection σ by e^{-t}. If ω' is not holomorphic, then, by Lemma 2.1.6, for all $\varepsilon > 0$, there exists an N and θ_N, such that the vertical foliation of $(X_N, e^{i\theta_N}\omega_N)$ determines a decomposition of $(X_N, e^{i\theta_N}\omega_N)$ into a union of cylinders $C_1, \ldots, C_p$, with waist lengths ε and heights $h_1, \ldots, h_p$, respectively, such that $\sum_k h_k = 1/\varepsilon$. This follows from the assumption that the area of every surface in the sequence is one. This sequence of surfaces defines a sequence of closed curves $\{\gamma_{n,t_n}\}_{n=0}^{\infty}$ whose lengths tend to zero as n tends to infinity, where γ_{n,t_n} is the waist curve of a cylinder on $(X_n, e^{i\theta_n}\omega_n)$. Furthermore, for each n, the curve γ_{n,t_n} corresponds to a closed curve γ_{n,t_0} on (X_0, ω_0) with the property that the image of γ_{n,t_0} under G_{t_n} is γ_{n,t_n}. Note that for all n and t_n, no curve γ_{n,t_n} is parallel to σ_{t_n} because σ_{t_n} does not lie in a periodic foliation while γ_{n,t_n} always lies in a periodic foliation.

We claim that we can pass to a subsequence such that γ_{n,t_n} is transverse to γ_{n+1,t_n}. Let $0 \leq \alpha_{n,t} < \pi$ denote the angle between $\gamma_{n,t}$ and $\sigma_{n,t}$. For all n and t_n, $\alpha_{n,t_n} \neq 0$ because γ_{n,t_n} is not parallel to σ_{t_n}. Fixing n and letting t tend to infinity, $|\alpha_{n,t}|$ tends to $\pi/2$ because γ_{n,t_n} has nontrivial length in the maximally expanding direction of G_t, so for sufficiently large t, $\gamma_{n,t}$ converges to the direction of maximum expansion, which is orthogonal to the direction of minimal expansion in which σ_0

lies. We prove that the set $\Gamma = \{\gamma_{n,0} | n \geq 0\}$ is infinite. If not, the previous comment would imply that given $\delta > 0$, there exists a time $T > 0$, such that for all n and $t > T$,

$$\sup_n ||\alpha_{n,t}| - \pi/2| < \delta.$$

This would contradict the fact that the lengths of the curves $\{\gamma_{n,t}\}_{n=0}^{\infty}$ tend to zero. Hence, the set Γ is infinite and we can pass to a subsequence such that γ_{n,t_n} is transverse to γ_{n+1,t_n}. Equivalently, $\gamma_{n,t_{n+1}}$ is transverse to $\gamma_{n+1,t_{n+1}}$.

Now we can construct a sequence of surfaces corresponding to a divergent sequence of times $\{t'_n\}_{n=0}^{\infty}$ such that the limit is holomorphic and the saddle connection σ_n degenerates to a point. Let $\varepsilon_N > 0$ be the infimum, taken over all cylinder decompositions of (X_N, ω_N), of the length of the waist curves of the cylinders at time t_N. By passing to a subsequence of times, we can assume $\gamma_{N+1,t_{N+1}}$ has length $\varepsilon_{N+1} < \varepsilon_N$. However, γ_{N,t_N} has length ε_N and γ_{N+1,t_N} is transverse to γ_{N,t_N}. For any surface (X, ω), whose Teichmüller disk is contained in $\mathcal{D}_g(1)$, let γ be the waist curve of a cylinder C_j which is an element of a cylinder decomposition $\mathcal{C}$ of (X, ω). It follows from Lemma 2.1.6 that every closed regular trajectory transverse to γ must pass through every cylinder in $\mathcal{C}$ at least once. Thus, in this case, γ_{N+1,t_N} has length at least $1/\varepsilon_N$. Since $\gamma_{N+1,t_{N+1}}$ has length $\varepsilon_{N+1} < \varepsilon_N$ and γ_{N+1,t_N} can be pinched under the Teichmüller geodesic flow so that the direction of σ_N contracts, then there is a time t'_{N+1} such that $t_N < t'_{N+1} < t_{N+1}$ and $\gamma_{N+1,t'_{N+1}}$ has length one. Furthermore, if $\gamma_{N+1,t'_{N+1}}$ has length one, then by the assumption that the area of (X_0, ω_0) is one, the fact that G_t preserves area, and the Teichmüller disk of (X_0, ω_0)

is contained in $\mathcal{D}_g(1)$, we have that the minimum length of any curve transverse to $\gamma_{N+1,t'_{N+1}}$ is also one. This implies that there are no short closed curves which are not unions of saddle connections. This defines a divergent sequence of times $\{t'_n\}_{n=0}^{\infty}$ such that the corresponding sequence of surfaces $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ converges to a degenerate surface (X', ω'), where ω' is holomorphic and σ_0 contracts to a point on X'. Finally, by Lemma 2.1.10, the only admissible boundary points of a Teichmüller disk contained in $\mathcal{D}_g(1)$, which carry holomorphic Abelian differentials, must have exactly one part. $\square$

The following definition was introduced by Vorobets [36]. In [31][Theorem 1.3, Parts (i) and (ii)], Smillie and Weiss prove that a surface is uniformly completely periodic if and only if it is a Veech surface.

Definition. *Let $\mathcal{S}_\theta$ denote the set of saddle connections of the vertical foliation of $(X, e^{i\theta}\omega)$. A surface is called* uniformly completely periodic *if it satisfies topological dichotomy and there exists a real number $s > 0$ such that for all θ, where $\mathcal{S}_\theta \neq \emptyset$, the ratio of the length of the longest saddle connection in $\mathcal{S}_\theta$ to the shortest saddle connection in $\mathcal{S}_\theta$ is bounded by s.*

Lemma 3.1.3. *Given a Teichmüller disk $D \subset \mathcal{D}_g(1)$ of a surface satisfying topological dichotomy $(X_0, \omega_0) \in \mathcal{M}_g$ that is not uniformly completely periodic, there exists a sequence of surfaces $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ in D converging to a surface $(X', \omega') \in \overline{\mathcal{D}_g(1)}$ such that X' has one part, ω' is holomorphic, and a saddle connection of (X_0, ω_0) contracts to a point on (X', ω').*

Proof. Since the surface (X_0, ω_0) is not uniformly completely periodic, given a divergent sequence of positive real numbers $\{s_j\}_{j=0}^{\infty}$, there exists a corresponding sequence of angles $\{\theta_j\}_{j=0}^{\infty}$ such that the ratio of the longest saddle connection to the shortest saddle connection on $(X_0, e^{i\theta_j}\omega_0)$ is greater than s_j, for all j. We show that there exists a sequence of times $\{t_n\}_{n=0}^{\infty}$ such that the sequence of surfaces $\{G_{t_n} \cdot (X_0, e^{i\theta_n}\omega_0)\}_{n=0}^{\infty}$ converges to a surface (X', ω'), where ω' is holomorphic. Moreover, there is a sequence of saddle connections on $G_{t_n} \cdot (X_0, e^{i\theta_n}\omega_0)$ converging to a point as n tends to infinity.

Pass to a subsequence of $\{\theta_j\}_{j=0}^{\infty}$ defined as follows. Since there is a finite number of zeros, there is a finite number of pairs of zeros. Choose a pair of zeros z_1 and z_2 that occur infinitely often in the sequence $\{(X_0, e^{i\theta_n}\omega_0)\}_{n=0}^{\infty}$ as the pairs of zeros which are joined by the shortest saddle connection. Pass to a subsequence $\{(X_0, e^{i\theta_n}\omega_0)\}_{n=0}^{\infty}$ such that a saddle connection between z_1 and z_2 represents the shortest saddle connection on $(X_1, e^{i\theta_n}\omega_1)$. By Lemma 2.1.6, all of the cylinders in the cylinder decomposition of a surface in $\mathcal{D}_g(1)$ have equal circumference and we can assume that (X_1, ω_1) has unit area and cylinders of unit circumference. For each angle θ_j, denote by $w_j > 1$ the length of the circumference of the cylinders in that direction. Then define the times t_j by $e^{-t_j} w_j = 1$, for all j. Then $G_{t_n} \cdot (X_0, e^{i\theta_n}\omega_0) = (X_n, \omega_n)$ is the action on the surface such that the waist curves of the cylinders of circumference w_j contract at the maximal rate. Furthermore, since the length of each saddle connection is bounded above by the circumference of the cylinders, the length of the shortest saddle connection on (X_n, ω_n) is bounded above by $1/s_n$. Note that $\lim_{n\to\infty} 1/s_n = 0$. The Teichmüller geodesic flow preserves area, so the surface (X_n, ω_n) also has unit area for all n. This implies that the sum of the heights of the cylinders is equal to one, as well. It follows from Lemma 2.1.6 that any curve transverse to a horizontal curve has length at least one because any such curve must travel the heights of every cylinder in the cylinder decomposition. Since the minimum length of a curve transverse to a vertical curve is the waist curve of a cylinder which has length one, there are no closed curves that can pinch that are not unions of saddle connections.

If a closed curve, which is a union of saddle connections, degenerates as n tends to infinity, then the limit is a degenerate surface carrying a holomorphic Abelian differential. By Lemma 2.1.10, the only such degenerate surfaces in the boundary of $\mathcal{D}_g(1)$ have one part. $\square$

Theorem 3.1.4. *If the Teichmüller disk D of (X, ω) is contained in $\mathcal{D}_g(1)$, then there is a Veech surface $(X', \omega') \in \overline{\mathcal{M}_g}$ such that the Teichmüller disk D' generated by (X', ω') is contained in $\overline{\mathcal{D}_g(1)}$. Furthermore, every surface in D' is the limit of a sequence of surfaces in D.*

Proof. If (X, ω) is a Veech surface, let $(X, \omega) = (X', \omega')$. Otherwise, assume that $(X, \omega) = (X_{0,1}, \omega_{0,1})$ is not a Veech surface and let D_1 be its Teichmüller disk. Since (X, ω) is not a Veech surface, but its Teichmüller disk is contained in $\overline{\mathcal{D}_g(1)}$, (X, ω) is completely periodic by Theorem 2.1.5. Furthermore, (X, ω) is not uniformly completely periodic by [31][Theorem 1.3, Parts (i) and (ii)]. By Lemmas 3.1.2 and 3.1.3, there exists a sequence $\{(X_{n,1}, \omega_{n,1})\}_{n=1}^{\infty}$ converging to a surface $(X_{0,2}, \omega_{0,2}) \in \overline{\mathcal{M}_g}$ with one part carrying a holomorphic Abelian differential with $(X_{0,2}, \omega_{0,2}) \in \mathcal{D}_g(1)$ and a saddle connection on $\omega_{0,1}$ degenerates to a point on $X_{0,2}$. A degenerate saddle connection implies either two or more zeros of $\omega_{0,1}$ converge to a single zero of $\omega_{0,2}$ or a closed curve of $X_{0,1}$ converges to a pair of punctures on $X_{0,2}$. Then $(X_{0,2}, \omega_{0,2})$ has Teichmüller disk D_2 and by Lemma 3.1.1, $D_2 \subset \overline{\mathcal{D}_g(1)}$. By Theorem 2.1.5, $(X_{0,2}, \omega_{0,2})$ is also completely periodic. If it is a Veech surface, then we are done. Otherwise, we proceed by induction using Lemmas 3.1.2 and 3.1.3 to create a sequence of surfaces $\{(X_{0,j}, \omega_{0,j})\}_{j=1}^{N}$ in $\overline{\mathcal{M}_g}$ such that each surface in the sequence carries a differential either with fewer distinct zeros or lower genus than the previous surface in the sequence. Since both the number of zeros as well as the genus are finite, this process will terminate at some step N resulting in a surface $(X_{0,N}, \omega_{0,N}) \in \overline{\mathcal{D}_g(1)}$ with Teichmüller disk D_N. By Lemma 3.1.1, $D_N \subset \overline{\mathcal{D}_g(1)}$. The surface $X_{0,N}$ cannot be a sphere because $\omega_{0,N}$ is holomorphic and $\omega_{0,N}$ is nonzero by Lemma 1.2.2. Hence, there are three possibilities. Either $\omega_{0,N}$ has a single zero, $X_{0,N}$ is a punctured torus, or $(X_{0,N}, \omega_{0,N})$ is a Veech surface. By Lemma 2.2.2, $\omega_{0,N}$ cannot have a single zero and Lemma 3.2.8 says $X_{0,N}$ cannot be a punctured torus. Thus, the only remaining possibility is that $(X_{0,N}, \omega_{0,N})$ is a Veech surface.

Let D' be the Teichmüller disk generated by $(X_{0,N}, \omega_{0,N})$. Lemma 3.1.1 implies that every surface in D' is the limit of a sequence of surfaces in D_1. □

3.2 Punctured Veech Surfaces

There are several key results that give a nearly complete picture of Teichmüller curves in $\mathcal{D}_g(1)$. We recall all of the results here for the sake of completeness and convenience of the reader. There are two similarly named, related concepts: a square-tiled covering and a square-tiled cyclic cover. A *square-tiled covering* is a specific type of Veech surface introduced by Thurston formed by gluing unit squares together to form a genus g surface. Naturally, such a surface comes with a covering of the unit square, i.e. the torus. A surface is a square-tiled covering if and only if it has affine group commensurable to $\mathrm{SL}_2(\mathbb{Z})$, by [15][Theorem 5.9].

We define a *square-tiled cyclic cover* using the exposition of [14]. A square-tiled cyclic cover is a specific type of square-tiled covering. Let $N > 1$ be an integer and $(a_1, a_2, a_3, a_4) \in \mathbb{Z}^4$ such that they satisfy

$$0 < a_i \leq N; \;\; \gcd(N, a_1, \ldots, a_4) = 1; \;\; \sum_{i=1}^{4} a_i \equiv 0(\mod N).$$

Then the algebraic equation

$$w^N = (z - z_1)^{a_1}(z - z_2)^{a_2}(z - z_3)^{a_3}(z - z_4)^{a_4}$$

defines a closed, connected and nonsingular Riemann surface denoted by $M_N(a_1, a_2, a_3, a_4)$. By construction, $M_N(a_1, a_2, a_3, a_4)$ is a ramified cover over the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ branched over the points $z_1, \ldots, z_4$. Consider the meromorphic quadratic differential

$$q_0 = \frac{dz^2}{(z - z_1)(z - z_2)(z - z_3)(z - z_4)}$$

on $\mathbb{P}^1(\mathbb{C})$. It has simple poles at $z_1, \ldots, z_4$ and no other zeros or poles. Then the canonical projection

$$p : M_N(a_1, a_2, a_3, a_4) \to \mathbb{P}^1(\mathbb{C})$$

induces a quadratic differential $q = p^*q_0$ by pull-back. Lemma 3.2.1 follows from [30][Cor. 3.3, Sect. 3.6].

Remark. *The name cyclic cover comes from the fact that the group of deck transformations of a cyclic cover is the cyclic group $\mathbb{Z}/N\mathbb{Z}$.*

Lemma 3.2.1 (Möller)**.** *If (X,ω) is a Veech surface whose Teichmüller disk is contained in $\mathcal{D}_g(1)$, then (X,ω) is a square-tiled covering.*

We recall the two known examples of surfaces that generate Teichmüller disks in $\mathcal{D}_g(1)$. The genus three example, denoted here by (M_3, ω_{M_3}), is commonly known as the Eierlegende Wollmilchsau for its numerous remarkable properties [17]. Forni [11] discovered that its Kontsevich-Zorich spectrum is indeed completely degenerate. The surface (M_3, ω_{M_3}) is a square-tiled surface given by the algebraic equation

$$w^4 = (z - z_1)(z - z_2)(z - z_3)(z - z_4).$$

Its differential, given in [28], can be written explicitly as

$$\omega_{M_3} = \frac{dz}{w^2}.$$

It is easy to see that this lies in the principal stratum of genus three, $\mathcal{H}(1,1,1,1)$. The surface is pictured in Figure 3.1 and the zeros lie at the corners of the squares and are denoted by $v_1, \ldots, v_4$. For completeness, note that the stratum $\mathcal{H}(1,1,1,1)$ is connected by [21].

Proposition 3.2.2 (Forni)**.** *The square-tiled surface (M_3, ω_{M_3}) generates a Teichmüller curve in $\mathcal{D}_3(1)$.*

The genus four example was discovered by Forni and Matheus [12] and we denote it by (M_4, ω_{M_4}). Recently, Vincent Delecroix and Barak Weiss have proposed

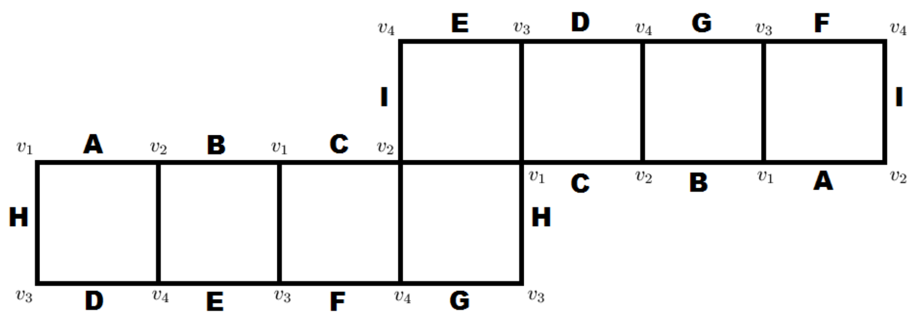

Figure 3.1: The Eierlegende Wollmilchsau (M_3, ω_{M_3})

to Carlos Matheus that (M_4, ω_{M_4}) be named "Platypus" translated into a romance language, e.g. Ornithorynque or Ornitorinco. The surface (M_4, ω_{M_4}) is a square-tiled surface given by the algebraic equation

$$w^6 = (z - z_1)(z - z_2)(z - z_3)(z - z_4)^3.$$

Its differential, see [28], can be written explicitly as

$$\omega_{M_4} = \frac{z\,dz}{w^2}.$$

It is easy to see that this lies in the stratum $\mathcal{H}(2,2,2)$. The surface is pictured in Figure 3.2 and the zeros, denoted by v_1, v_2, v_3, lie at the corners of the squares. For completeness, note that $\mathcal{H}(2,2,2)$ has two connected components by [21], and it was proven in [28] and again in [14] that (M_4, ω_{M_4}) lies in the connected component $\mathcal{H}^{even}(2,2,2)$ where the spin-structure has even parity.

Proposition 3.2.3 (Forni-Matheus)**.** *The square-tiled surface* (M_4, ω_{M_4}) *generates a Teichmüller curve in* $\mathcal{D}_4(1)$.

Möller [30] showed that Teichmüller curves in $\mathcal{D}_g(1)$ must also be Shimura curves. This allowed him to give a nearly complete classification of Teichmüller

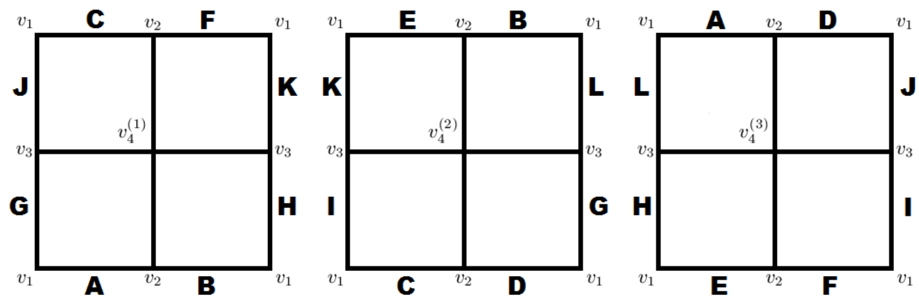

Figure 3.2: The Surface (M_4, ω_{M_4})

curves in $\mathcal{D}_g(1)$.

Theorem 3.2.4 (Möller)**.** *Other than possible examples in certain strata of $\mathcal{M}_5$, listed in Table 3.2, and the examples of Propositions 3.2.2 and 3.2.3, there are no other Teichmüller curves contained in $\mathcal{D}_g(1)$, for $g \geq 2$.*

Stratum
$(1, 1, 1, 1, 1, 1, 2)$
$(1, 1, 1, 1, 2, 2)$
$(1, 1, 2, 2, 2)$
$(2, 2, 2, 2)$
$(1, 1, 1, 1, 1, 3)$
$(1, 1, 3, 3)$
$(1, 1, 1, 1, 4)$

Table 3.1: Strata of $\mathcal{M}_5$ with a Possible Teichmüller Curve in $\mathcal{D}_5(1)$

These results are key to the remainder of the thesis. Theorem 3.1.4 implies that for any Teichmüller disk in $\mathcal{D}_g(1)$ there is a sequence of surfaces converging to a Veech surface. This Veech surface may arise from pinching curves to pairs of punctures thereby resulting in a punctured Veech surface. Moreover, Lemma 3.2.1 implies that this punctured Veech surface is, in fact, a punctured square-tiled surface. The strategy will be to proceed by contradiction and assume that there is such a sequence of surfaces converging to a punctured square-tiled surface. The theme of the remainder of this thesis is captured in the following question.

Question. *Given a sequence of surfaces in a Teichmüller disk contained in $\mathcal{D}_g(1)$ converging to a degenerate surface X', which is square-tiled and carries a holomorphic Abelian differential ω', at which points of X' can the pairs of punctures lie?*

Definition. *Let $(X, \omega) \in \mathcal{M}_g$ and $p \in X$. Let $\Gamma_p(X)$ denote the set of all closed regular trajectories γ passing through p with respect to $e^{i\theta}$, for all $\theta \in \mathbb{R}$. Define the set*

$$C_p(X) = \bigcap_{\gamma \in \Gamma_p(X)} \gamma.$$

It should be obvious to the reader that for any compact Riemann surface X and any $p \in X$, $C_p(X)$ is a finite set. Otherwise, it would have an accumulation point on X, which is impossible.

Theorem 3.2.5. *Let D be a Teichmüller disk in $\mathcal{D}_g(1)$. Let (X', ω') be a degenerate surface in the closure of D such that ω' is holomorphic and X' has exactly one part. If (p, p') is a pair of punctures on (X', ω'), then $p' \in C_p(X')$.*

Proof. We proceed by contradiction and assume $p' \notin C_p(X')$. By definition of $C_p(X')$, there exists a $\theta \in \mathbb{R}$ such that $(X', e^{i\theta}\omega')$ has a closed leaf γ passing through

p and not through p'. We act on $(X', e^{i\theta}\omega')$ by G_t and claim that we can find a divergent sequence of times $\{t_n\}_{n=1}^{\infty}$ such that $G_{t_n} \cdot (X', e^{i\theta}\omega')$ converges to a degenerate surface which cannot be a boundary point of a Teichmüller disk in $\overline{\mathcal{D}_g(1)}$. Since (X', ω') is completely periodic, by Theorem 2.1.5, all of the leaves of the vertical foliation of $(X', e^{i\theta}\omega')$ are closed. By Corollary 2.1.6, all of the leaves have the same length ℓ. After time t, they have length $e^{-t}\ell$. Furthermore, since p and p' do not both lie on γ, the distance between them tends to infinity exponentially with t. Let (X', ω') degenerate to (X'', ω'') under the action by G_t. This implies that (p, p') are a pair of holomorphic punctures paired between two distinct parts of (X'', ω''). However, Lemma 2.1.10 says that there cannot be a pair of holomorphic punctures on two distinct parts of a degenerate surface whose Teichmüller disk is contained in $\mathcal{D}_g(1)$. This contradiction implies that $p' \in C_p(X')$. $\square$

Lemma 3.2.6. *Let $\mathbb{T}^2$ denote the torus. For all $p \in \mathbb{T}^2$, $C_p(\mathbb{T}^2) = \{p\}$.*

Proof. Identify $\mathbb{T}^2$ with the unit square S. Consider the horizontal and vertical lines intersecting at $p \in S$. It is obvious that these two lines have no other intersection point. Hence, $C_p(\mathbb{T}^2) = \{p\}$. $\square$

Corollary 3.2.7. *Let D be a Teichmüller disk in $\mathcal{D}_g(1)$ such that (X', ω') is a degenerate surface carrying a holomorphic Abelian differential, and (X', ω') is a square-tiled surface with covering map $\pi : X \to \mathbb{T}^2$. If (p, p') is a pair of punctures on X', then $\pi(p) = \pi(p')$.*

Proof. Closed trajectories on X descend to closed trajectories on $\mathbb{T}^2$ under π. Hence, it follows from Theorem 3.2.5 and Lemma 3.2.6 that

$$\pi(p') \in \pi(C_p(X)) = C_{\pi(p)}(\mathbb{T}^2) = \{\pi(p)\}.$$

$\square$

Remark. *Corollary 3.2.7 is weaker than Theorem 3.2.5 because $\pi(p) = \pi(p')$ does* not *imply $p' \in C_p(X)$.*

Lemma 3.2.8. *Given a Teichmüller disk $D \subset \mathcal{D}_g(1)$, for $g \geq 2$, there is no degenerate surface in the closure of $D \subset \overline{\mathcal{D}_g(1)}$ of the form (S, ω), where S is a punctured torus and ω is holomorphic.*

Proof. By contradiction, assume there is a degenerate surface of the form (S, ω) in the boundary of D. By the assumption that S arises from pinching curves on a higher genus surface, S has an even, nonzero, number of punctures. Let (p, p') be a pair of punctures on S. By Lemma 3.2.6, there are two parallel curves on S, γ_1 and γ_2 passing through p and p', respectively. There are two more curves γ_1' and γ_2' parallel to γ_1 that do not pass through punctures of S such that γ_1' is not homotopic to γ_2' (because S is not a torus, but a *punctured* torus). Pinching the curves γ_1' and γ_2' degenerates the torus S into a union of two or more spheres S' such that p and p' do not lie on the same sphere. However, (p, p') represent a pinched curve, so if S' consists of exactly two parts, then there are three pairs of punctures between those two parts and if S' has more than two parts, then there are two parts of S' with two pairs of punctures between them. This directly contradicts Lemma 2.1.10 and proves that no surface can degenerate to a punctured torus carrying a holomorphic Abelian differential. $\square$

Theorem 3.2.9. *The Eierlegende Wollmilchsau* (M_3, ω_{M_3}) *generates the only Teichmüller disk in* $\mathcal{D}_3(1)$.

Proof. By [30] (restated in Theorem 3.2.4 above), the Eierlegende Wollmilchsau is the only Veech surface that generates a Teichmüller disk in $\mathcal{D}_3(1)$. By contradiction, assume that there is a genus three surface (X, ω) that generates a Teichmüller disk $D \subset \mathcal{D}_3(1)$. Then X is not a Veech surface, but it is completely periodic by Theorem 2.1.5. By Theorem 3.1.4, there is a sequence of surfaces in D converging to a Veech surface (X', ω') contained in $\overline{\mathcal{D}_3(1)}$. Since Theorem 3.1.4 guarantees that either the Abelian differential ω' has fewer zeros than ω, which implies (X', ω') cannot lie in the principal stratum of $\mathcal{M}_3$, or X' has lower genus than X. However, X' cannot have lower genus by Lemma 3.2.8 and Proposition 2.2.4 and the fact that the sphere carries no nonzero holomorphic differentials. Moreover, Theorem 3.2.4 implies that (X', ω') cannot be a Veech surface because (X', ω') does lie in the principal stratum. This contradiction implies that no other Teichmüller disk is contained in $\mathcal{D}_3(1)$. □

We complete this section by proving that pairs of punctures cannot lie at the zeros in the known examples of Veech surfaces generating Teichmüller disks in $\mathcal{D}_g(1)$.

Lemma 3.2.10. *Let* (M_3', ω_{M_3}') *denote* (M_3, ω_{M_3}) *with punctures. Let* (X, ω) *be a surface generating a Teichmüller disk in* $\mathcal{D}_g(1)$ *such that* (M_3', ω_{M_3}') *is a degenerate surface in the closure of* D*. If* (p, p') *is a pair of punctures on* (M_3', ω_{M_3}')*, then*

$$\{p, p'\} \cap \{v_1, v_2, v_3, v_4\} = \emptyset.$$

Proof. Let $V = \{v_1, v_2, v_3, v_4\}$ and $\pi : M_3 \to \mathbb{T}^2$ be a covering map of the torus. The points in V lie at the corners of each of the squares in the square tile decomposition of M_3, so $\pi^{-1} \circ \pi(v_1) = V$. By contradiction, assume $p \in V$. By Corollary 3.2.7, $p \in V$ implies $\{p, p'\} \subset V$. Hence, there are six pairs of points at which (p, p') can lie. For each pair of points, listed in the first column of Table 3.2, choose the closed trajectory on (M_3', ω_{M_3}') in the direction specified in the second column of Table 3.2, according to the orientation of Figure 3.1, and pinch the core curves of the cylinders in that direction. This results in p and p' lying on different parts of a degenerate surface. This contradicts Lemma 2.1.10, which states that there are no holomorphic punctures between parts of a degenerate surface in the boundary of $\mathcal{D}_g(1)$.

Pair of Points	Direction
$(v_1, v_3), (v_2, v_4), (v_2, v_3), (v_1, v_4)$	Horizontal
$(v_1, v_2), (v_3, v_4)$	Vertical

Table 3.2: Pinching Directions for Lemma 3.2.10

□

Lemma 3.2.11. *Let (M_4', ω_{M_4}') denote (M_4, ω_{M_4}) with punctures. Let (X, ω) be a surface generating a Teichmüller disk in $\mathcal{D}_g(1)$ such that (M_4', ω_{M_4}') is a degenerate surface in the closure of D. If (p, p') is a pair of punctures on (M_4', ω_{M_4}'), then*

$$\{p, p'\} \cap \{v_1, v_2, v_3\} = \emptyset.$$

Proof. Let $V = \{v_1, v_2, v_3, v_4^{(1)}, v_4^{(2)}, v_4^{(3)}\}$ and $\pi : M_4 \to \mathbb{T}^2$ be a covering map of the torus. As above, $\pi^{-1} \circ \pi(v_1) = V$. By contradiction, assume $p \in V$. Then Corollary 3.2.7 tells us that $\{p, p'\} \subset V$. There are 15 subsets of V of order two. We exclude the three subsets containing only points of the form $v_4^{(j)}$, for all j, because they are not relevant to the statement of this lemma. We proceed as in the proof of Lemma 3.2.10 using Table 3.3 to specify the direction in which to pinch M_4' given the orientation of Figure 3.2 to reach a contradiction with Lemma 2.1.10.

Pair of Points	Direction
$(v_1, v_3), (v_2, v_3), (v_2, v_4^{(j)}), (v_1, v_4^{(j)}), \forall j$	Horizontal
$(v_1, v_2), (v_3, v_4^{(j)}), \forall j$	Vertical

Table 3.3: Pinching Directions for Lemma 3.2.11

□

3.3 Directions for Future Research

The results of this thesis could lead to a complete classification of Teichmüller disks in $\mathcal{D}_g(1)$. In this section, we pose several conjectures that could lead to such a classification.

Conjecture. *Let D be a Teichmüller disk contained in $\mathcal{D}_g(1)$. Let (X', ω') be a degenerate (punctured) surface carrying a holomorphic Abelian differential in the closure of D and let (X', ω') be a Veech surface. If (p, p') are a pair of punctures on X, then ω' has a zero at p or p'.*

The main corollary of this conjecture would be a new proof that for sufficiently high genus, there are no $\mathrm{SL}_2(\mathbb{R})$-invariant ergodic measures with completely degenerate Kontsevich-Zorich spectrum. Though, the bound produced here is weaker than that of [7], where they prove that there are no regular $\mathrm{SL}_2(\mathbb{R})$-invariant suborbifolds supporting such a measure for $g \geq 7$, this result would be stronger for $g \geq 13$ because it would not rely on any open conjecture. Recall that it is only conjectured that the closure of every Teichmüller disk is regular.

Corollary 3.3.1. *For $g \geq 13$, there are no Teichmüller disks contained in $\mathcal{D}_g(1)$.*

Proof. By [30] (see Theorem 3.2.4), there are no Veech surfaces in $\mathcal{D}_g(1)$, for $g \geq 6$. In particular, Möller proves that a Veech surface generating a Teichmüller disk in $\mathcal{D}_g(1)$ has at most seven distinct zeros in genus five. By the conjecture above, at least one puncture in each pair of punctures must lie at a zero. In the worst case, there could be a surface in genus $5+7=12$, generating a Teichmüller disk in $\mathcal{D}_g(1)$, that degenerates to a Veech surface (X', ω') in genus five with exactly seven pairs of punctures, such that one puncture in each pair of punctures lies at a zero of ω'. $\square$

Next, we believe that the techniques used in this thesis, particularly those of Section 3.1, could lead to a proof of the following conjecture.

Conjecture. *The surface (M_4, ω_{M_4}) generates the only Teichmüller disk contained in $\mathcal{D}_4(1)$.*

Recall Möller's conjecture. We believe that resolving this conjecture could quickly lead to a proof of the final conjecture below.

Conjecture (Möller)**.** *There are no Teichmüller curves in $\mathcal{D}_5(1)$.*

Conjecture. *There are no Teichmüller disks contained in $\mathcal{D}_g(1)$, for $g \geq 5$.*

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