ABSTRACT

Title of dissertation:	WEIL-ÉTALE COHOMOLOGY OVER LOCAL FIELDS
	David A. Karpuk, Doctor of Philosophy, 2012
Dissertation directed by:	Professor Niranjan Ramachandran Department of Mathematics

In [11], Lichtenbaum established the arithmetic utility of the Weil group of a finite field, by demonstrating a connection between certain Euler characteristics in Weil-étale cohomology and special values of zeta functions. In particular, the order of vanishing and leading coefficient of the zeta function of a smooth, projective variety over a finite field have a Weil-étale cohomological interpretation. These results rely on a duality theorem stated in terms of cup-product in Weil-étale cohomology.

With Lichtenbaum's paradigm in mind, we establish results for the cohomology of the Weil group of a local field, analogous to, but more general than, results from Galois cohomology. We prove a duality theorem for discrete Weil group modules, which implies the main theorem of Local Class Field Theory. We define Weil-smooth cohomology for varieties over local fields, and prove a duality theorem for the cohomology of $\mathbf{G}_m$ on a smooth, proper curve with a rational point. This last theorem is analogous to, and implies, a classical duality theorem for such curves.

WEIL-ÉTALE COHOMOLOGY
OVER LOCAL FIELDS

by

David A. Karpuk

Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2012

Advisory Committee:
Professor Niranjan Ramachandran, Chair/Advisor
Professor Lawrence Washington
Professor Jonathan Rosenberg
Professor Thomas Haines
Professor Alexander Barg

To my parents.

Acknowledgments

First and foremost, I would like to express my sincerest gratitude to my dissertation advisor, Niranjan Ramachandran, who originally suggested this topic as a thesis problem. He was always available to me whenever I had questions, and without his guidance, encouragement, mathematical insight, and consistently helpful suggestions, completing this research would not have been possible.

I would also like to thank Mathias Flach, Thomas Geisser, and Baptiste Morin for helpful questions and comments regarding the main results of my thesis. Thomas Haines and Lawrence Washington also deserve a great amount of my gratitude, for providing me with constant mathematical guidance while at the University of Maryland.

I would also like to thank the mathematics graduate student community at University of Maryland. Without a welcoming social community, graduate school would have been much more difficult. Sharing ideas with my peers has sharpened my dissertation results and general mathematical acumen a great deal. In particular, conversations with Jeff Frazier, Jon Huang, Anne Jorstad, Sean Kelly, Andy Sanders, and Sean Rostami have contributed greatly to my mathematical edification and development.

Table of Contents

Chapter 1

Introduction

1.1 Background and Motivation

Arithmetic applications of Weil groups have been a popular topic in recent years, starting with the article [11] of Lichtenbaum. In this article, Lichtenbaum defines a cohomology theory for varieties over finite fields, Weil-étale cohomology, wherein the Weil group plays the role that the Galois group plays in étale cohomology. The Weil-étale cohomology groups of smooth, projective varieties with $\mathbf{Z}$ coefficients are shown to be finitely generated abelian groups. The resulting Euler characteristics provide a cohomological interpretation of the order of vanishing and leading coefficient of the zeta function $Z(X, t)$ at $t = 1$.

More recent work by Lichtenbaum [12], Flach [4], Morin [16] and [17], and Flach and Morin [5] has been done towards a definition of Weil-étale cohomology for schemes of finite type over Spec $\mathbf{Z}$. These approaches have been partially successful in giving a Weil-étale cohomological interpretation of special values of zeta functions and L-functions of such schemes. In this thesis, we study Weil-étale cohomology over p-adic fields. Hopefully, this will provide an intermediate step between Weil-étale cohomology over finite fields and (the still largely conjectural) Weil-étale cohomology for arithmetic schemes.

If k is a finite field and we let $X = \text{Spec } k$, the étale cohomology and Weil-étale groups of X with $\mathbf{Z}$ coefficients are given, respectively, by

$$H^i(\hat{\mathbf{Z}}, \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } i = 0 \\ 0 & \text{if } i = 1 \\ \mathbf{Q}/\mathbf{Z} & \text{if } i = 2 \\ 0 & \text{if } i \geq 3 \end{cases} \quad \text{and} \quad H^i(\mathbf{Z}, \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } i = 0 \\ \mathbf{Z} & \text{if } i = 1 \\ 0 & \text{if } i \geq 2. \end{cases}$$

In general, taking Weil-étale cohomology of Spec k instead of étale cohomology has the effect of shifting the $\mathbf{Q}/\mathbf{Z}$'s that appear in the cohomology groups down a degree and turning them into $\mathbf{Z}$'s. In this sense, Weil-étale cohomology of Spec k determines the étale cohomology, as is made precise by Lemma 1.2 of [11].

Let K be a p-adic local field with absolute Galois group G and Weil group W. If we let $X = \text{Spec } K$, the étale cohomology and Weil-étale cohomology groups of X with $\mathbf{Z}$ coefficients are given, respectively, by

$$H^i(G, \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } i = 0 \\ 0 & \text{if } i = 1 \\ (K^\times)^* & \text{if } i = 2 \\ 0 & \text{if } i \geq 3 \end{cases} \quad \text{and} \quad H^i(W, \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } i = 0 \\ \mathbf{Z} & \text{if } i = 1 \\ U_K^* & \text{if } i = 2 \\ 0 & \text{if } i \geq 3. \end{cases}$$

Here U_K denotes the units of K, and $(-)^*$ denotes $\text{Hom}_{\text{cont}}(-, \mathbf{Q}/\mathbf{Z})$. Again, notice that the copy of $\mathbf{Q}/\mathbf{Z}$ in $H^2(G, \mathbf{Z})$, coming from dualizing the valuation map $K^\times \to \mathbf{Z}$, has shifted down a degree and become a copy of $\mathbf{Z}$. This behavior is generalized and made precise by our Theorem (4.1.3), which also implies that for any G-module

M, the groups $H^i(W, M)$ determine the groups $H^i(G, M)$ up to isomorphism. One can see that the converse fails by taking $M = \mathbf{Q}$ with trivial action.

Let L be the completion of the maximal unramified extension of K, and let $\bar{L}$ be a fixed algebraic closure of L. The group W acts naturally on $\bar{L}$, a fact first observed in the unpublished note [10] of Lichtenbaum. If A/K is a commutative algebraic group, we take this idea further by letting W act on $A(\bar{L})$ and studying the groups $H^i(W, A(\bar{L}))$. Taking $A = \mathbf{G}_m$ and computing the groups $H^i(G, A(\bar{K}))$ and $H^i(W, A(\bar{L}))$, we obtain

$$H^i(G, \bar{K}^\times) = \begin{cases} K^\times & \text{if } i = 0 \\ 0 & \text{if } i = 1 \\ \mathbf{Q}/\mathbf{Z} & \text{if } i = 2 \\ 0 & \text{if } i \geq 3 \end{cases} \qquad \text{and} \qquad H^i(W, \bar{L}^\times) = \begin{cases} K^\times & \text{if } i = 0 \\ \mathbf{Z} & \text{if } i = 1 \\ 0 & \text{if } i \geq 2, \end{cases}$$

which is familiar behavior. For any connected, commutative algebraic group A/K, the groups $H^i(W, A(\bar{L}))$ determine the groups $H^i(G, A(\bar{K}))$ up to isomorphism, which is made precise by our Theorem (4.1.5).

Lichtenbaum's computation of the groups $H^i(X_W, \mathbf{Z})$ for a curve X over a finite field, and therefore his interpretation of special values of zeta functions, relies on a duality theorem stated in terms of cup-product in Weil-étale cohomology. Our main theorems are analogous duality theorems for the Weil-étale cohomology of zero and one-dimensional schemes over p-adic fields. The former is the Weil analogue of Tate-Nakayama Duality, and the latter is the Weil analogue of Lichtenbaum Duality for curves over p-adic fields.

Our main theorem concerning the cohomology of W-modules is Theorem (3.3.1). Let $\Gamma_W : W\text{-Mod} \to \mathcal{A}b$ be the functor $M \mapsto M^W$, whose right derived functors define the groups $H^i(W, M)$. Let $R\Gamma_W : \mathcal{D}(W) \to \mathcal{D}(\mathbf{Z})$ be the derived functor of Γ_W. For a W-module M, let $M^D = \mathrm{Hom}(M, \bar{L}^\times)$. There is a natural map in $\mathcal{D}(\mathbf{Z})$,

$$\psi(M) : R\Gamma_W(M^D) \to R\mathrm{Hom}(R\Gamma_W(M), \mathbf{Z}[-1]),$$

induced by a cup-product pairing. Our theorem is the following:

Theorem 3.10. Suppose that M is finitely generated as an abelian group. Then $\psi(M)$ is an isomorphism in $\mathcal{D}(\mathbf{Z})$. □

This theorem implies Tate-Nakayama Duality, which in turn implies the main theorem of Local Class Field Theory. This theorem was originally proven by Jiang in [8] under the assumption that M be a G-module, and using Tate-Nakayama Duality. We have removed this assumption, and provided a proof which is independent of the main results of Galois cohomology.

Our second main theorem is a duality theorem for the Weil-étale cohomology of smooth, projective, geometrically connected curves X/K, which contain a rational point. Let Γ_X be the functor which takes a Weil-étale sheaf F on X to $H^0(X_W, F)$, whose derived functors define the Weil-étale cohomology groups $H^i(X_W, F)$. Let $R\Gamma_X$ be the derived functor of Γ_X. Our duality theorem for X is the following:

Theorem 5.13. Let X/K be a smooth, projective, geometrically connected curve over K, such that $X(K) \neq \emptyset$. There is a symmetric pairing

$$R\Gamma_X(\mathbf{G}_m) \otimes^L R\Gamma_X(\mathbf{G}_m) \to \mathbf{Z}[-2],$$

such that the induced map

$$R\Gamma_X(\mathbf{G}_m) \to R\mathrm{Hom}(R\Gamma_X(\mathbf{G}_m), \mathbf{Z}[-2])$$

is an isomorphism on cohomology in degree 0 and 1, and injective on cohomology in degree 2 and 3. The cohomology of both complexes vanishes outside of degrees 0 through 3. $\square$

Our theorem is stated and proved more naturally in the setting of Weil-smooth cohomology, which is defined in the last chapter. However, Weil-smooth and Weil-étale cohomology groups agree when the sheaf is given by a smooth, commutative group scheme, just as smooth and étale cohomology agree for such sheaves.

What prevents the map from being an isomorphism on the nose is that one essentially encounters the inclusion map $U_K^* \to \mathrm{Hom}(U_K, \mathbf{Q}/\mathbf{Z})$, which is not surjective as there exist discontinuous maps $U_K \to \mathbf{Q}/\mathbf{Z}$. This self-duality of $\mathbf{G}_m$ is the Weil analogue of the classical duality theorems of the article [9]. In fact, for curves with rational points, one can deduce the main result of [9], namely that there is a natural isomorphism $\mathrm{Br}(X) \to \mathrm{Pic}(X)^*$.

1.2 Notation

If R is a commutative ring, then $\mathcal{D}(R)$ denotes the bounded derived category of R-modules. If G is a discrete group, then $\mathcal{D}(G)$ denotes the bounded derived category of G-modules; if G is profinite, then we will always restrict our attention to discrete, continuous G-modules, unless stated otherwise. If $f : X \to Y$ is a map between two cochain complexes, we use f^i to denote the induced map in i^{th} cohomology. We will often write an exact triangle $X \to Y \to Z \to X[1]$ as simply $X \to Y \to Z$, with the $X[1]$ being implied.

If M is an abelian group and n is an integer, we use $M[n]$ and M/n to denote the kernel and cokernel, respectively, of the multiplication-by-n maps on M. If M and N are abelian groups with some possible extra structure (for example, M and N could be G-modules for some group G), then $\mathrm{Hom}(M, N)$ will always mean $\mathrm{Hom}_{\mathbf{Z}}(M, N)$, and similarly for Ext and $\otimes$. When working in the derived category of abelian groups, we will write $R\mathrm{Hom}$ for $R\mathrm{Hom}_{\mathcal{D}(\mathbf{Z})}$. If X and Y are objects in $\mathcal{D}(\mathbf{Z})$, we will write $\mathrm{Ext}^i(X, Y)$ for their i^{th} hyperext group, as defined in ([24], Definition 10.7.1). If X, Y are actually abelian groups, this coincides with the usual notion of Ext groups.

For a topological abelian group A, we define $A^* = \mathrm{Hom}_{\mathrm{cont}}(A, \mathbf{Q}/\mathbf{Z})$, where $\mathbf{Q}/\mathbf{Z}$ has the discrete topology. If A is the group of rational points of some commutative algebraic group scheme over a local field, it will be understood that A is endowed with the natural topology coming from the local field.

If G is a discrete group and M is a G-module, then by $H^*(G, M)$ we mean the traditional group cohomology of G with coefficients in M. If G is profinite, or an extension of a discrete group by a profinite group, and M is a discrete G-module, then by $H^*(G, M)$ we mean Galois cohomology in the sense of [21].

We briefly recall the notion of cohomological dimension. Let G be a discrete group, or an extension of a discrete group by a profinite group. The *cohomological dimension* of G is defined to be the smallest integer n such that for all $m > n$ and all torsion G-modules M, we have $H^m(G, M) = 0$ (provided such an n exists). We write $\mathrm{cd}(G)$ for the cohomological dimension of G. The *strict cohomological dimension* of G is defined to be the smallest integer n such that for all $m > n$ and all G-modules M, we have $H^m(G, M) = 0$ (provided such an n exists); we denote the strict cohomological dimension of G by $\mathrm{scd}(G)$.

If $\mathfrak{X}$ is any Grothendieck site, we denote by $\mathcal{S}(\mathfrak{X})$ the category of sheaves of abelian groups on $\mathfrak{X}$. We let $\mathcal{D}(\mathfrak{X})$ denote the bounded derived category of $\mathcal{S}(\mathfrak{X})$.

1.3 Setup

Let K be a local field of characteristic zero, with finite residue field k. Let K_{ur} be its maximal unramified extension, and let L be the completion of K_{ur}. Let $\bar{K}$ be an algebraic closure of K, and let $\bar{L}$ be an algebraic closure of L containing $\bar{K}$.

Let $\mathfrak{g} = \mathrm{Gal}(K_{ur}/K) = \mathrm{Gal}(\bar{k}/k) \simeq \hat{\mathbf{Z}}$, and let $\mathfrak{w} \simeq \mathbf{Z}$ be the subgroup of $\mathfrak{g}$ consisting of integral powers of the Frobenius morphism $\sigma : \bar{k} \to \bar{k}$. The action of $\mathfrak{w}$ on K_{ur} extends by continuity to L. We let $N = \mathrm{Gal}(\bar{K}/K_{ur})$ be the inertia subgroup of $G = \mathrm{Gal}(\bar{K}/K)$. It follows from Krasner's Lemma (see Lemma 8.1.6 of

[18]) applied to the extension $K_{ur} \subset L$ that $N = \mathrm{Gal}(\bar{L}/L)$.

We let W be the Weil group of K; it is the pullback of $\mathfrak{w}$ under the surjection $G \to \mathfrak{g}$. The diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & W & \longrightarrow & \mathfrak{w} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \end{array}$$

best summarizes the relationship between all of these groups. The topology on W is such that N is open, and translation by any preimage of σ is a homeomorphism.

By Chapter XIII, Lemma 1 of [20], the fixed points of $\mathfrak{w}$ acting on L are exactly K; it is immediate that the fixed points of W acting on $\bar{L}$ are also K. Similar remarks apply to $\mathfrak{w}$ and W acting on the L and $\bar{L}$-points, respectively, of some commutative algebraic group scheme defined over K.

Chapter 2

Weil Groups of Finite Fields

We begin by studying the cohomology of the Weil group $\mathfrak{w}$ of k, the residue field of our local field. As $\mathfrak{w} \simeq \mathbf{Z}$ as abstract groups, this is mostly an exercise in homological algebra. We prove a duality theorem for certain (complexes of) $\mathfrak{w}$-modules, and restate the theorem as a duality theorem in Weil-étale cohomology.

Since our goal is to prove duality theorems, we start by explicitly describing the $\mathbf{Z}$-dual of a complex in the bounded derived category of abelian groups.

Proposition 2.0.1. Let C be a bounded cochain complex of abelian groups, considered as an object in $\mathcal{D}(\mathbf{Z})$. Let $\mathbf{Z}[-n]$ be the cochain complex with $\mathbf{Z}$ in degree n and 0 everywhere else. Then for all i we have short exact sequences

$$0 \to \operatorname{Ext}(H^{n-i+1}(C), \mathbf{Z}) \to \operatorname{Ext}^i(C, \mathbf{Z}[-n]) \to \operatorname{Hom}(H^{n-i}(C), \mathbf{Z}) \to 0. \tag{2.1}$$

Proof. The exact sequence $0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0$ defines an exact triangle in $\mathcal{D}(\mathbf{Z})$. Applying $R\operatorname{Hom}(C, -)$ we obtain the exact triangle

$$R\operatorname{Hom}(C, \mathbf{Z}[-n]) \to R\operatorname{Hom}(C, \mathbf{Q}[-n]) \to R\operatorname{Hom}(C, \mathbf{Q}/\mathbf{Z}[-n])$$

Because $\mathbf{Q}$ is an injective abelian group, we have

$$\mathrm{Ext}^i(C, \mathbf{Q}[-n]) = \mathrm{Hom}(H^{n-i}(C), \mathbf{Q})$$

for all i, and similarly for $\mathbf{Q}/\mathbf{Z}$. By taking cohomology, we obtain the long exact sequence

$$\begin{aligned}\cdots \to &\mathrm{Hom}(H^{n-i+1}(C), \mathbf{Q}) \to \mathrm{Hom}(H^{n-i+1}(C), \mathbf{Q}/\mathbf{Z}) \to \mathrm{Ext}^i(C, \mathbf{Z}[-n]) \to \\ &\mathrm{Hom}(H^{n-i}(C), \mathbf{Q}) \to \mathrm{Hom}(H^{n-i}(C), \mathbf{Q}/\mathbf{Z}) \to \cdots .\end{aligned}$$

The result is now clear. □

2.1 Duality for Finitely Generated Modules

Proposition 2.1.1. Let R be a commutative ring with 1, and suppose that M is an $R[\mathfrak{w}]$-module. Let D be an injective R-module on which $\mathfrak{w}$ acts trivially, so that $H^1(\mathfrak{w}, D) = D$. Then the cup-product pairing

$$H^i(\mathfrak{w}, M) \otimes_R H^{1-i}(\mathfrak{w}, \mathrm{Hom}_R(M, D)) \to D$$

induces isomorphisms $H^{1-i}(\mathfrak{w}, \mathrm{Hom}_R(M, D)) \simeq \mathrm{Hom}_R(H^i(\mathfrak{w}, M), D)$ for $i = 0, 1$.

Proof. Applying the functor $\mathrm{Hom}_R(-, D)$ to the exact sequence $0 \to M^{\mathfrak{w}} \to M \overset{\sigma-1}{\to}$

$M \to M_{\mathfrak{w}} \to 0$ we obtain the exact sequence

$$0 \to \mathrm{Hom}_R(M_{\mathfrak{w}}, D) \to \mathrm{Hom}_R(M, D) \stackrel{\sigma-1}{\to} \mathrm{Hom}_R(M, D) \to \mathrm{Hom}_R(M^{\mathfrak{w}}, D) \to 0.$$

The proposition now follows from the fact that the groups $H^0(\mathfrak{w}, \mathrm{Hom}_R(M, D))$ and $H^1(\mathfrak{w}, \mathrm{Hom}_R(M, D))$ are, respectfully, the kernel and cokernel of the map $\sigma - 1$ from $\mathrm{Hom}_R(M, D)$ to itself. □

This proposition is particularly interesting in the following two cases:

Example 2.1.2. Let $R = \mathbf{Z}$, $D = \mathbf{Q}/\mathbf{Z}$, and suppose that M is finite. By the canonical isomorphism $M = M^{**}$, we obtain perfect pairings

$$H^i(\mathfrak{w}, M) \otimes H^{1-i}(\mathfrak{w}, M^*) \to \mathbf{Q}/\mathbf{Z} \tag{2.2}$$

of finite abelian groups. □

Example 2.1.3. Let R be a field F of characteristic zero, and suppose that $M = V$ is a finite-dimensional representation of $\mathfrak{w}$. Let $D = F$ be the trivial representation, and consider the dual representation $\mathrm{Hom}_F(V, D)$. By (2.1.1) we have perfect pairings

$$H^i(\mathfrak{w}, V) \otimes_F H^{1-i}(\mathfrak{w}, \mathrm{Hom}_F(V, F)) \to F \tag{2.3}$$

of vector spaces over F. □

Suppose that M is a bounded complex of $\mathfrak{w}$-modules, considered as an object in $\mathcal{D}(\mathfrak{w})$. Let us define its *dual complex* by $M^D := R\mathrm{Hom}(M, \mathbf{Z}) \in \mathcal{D}(\mathfrak{w})$. There is a canonical pairing

$$M \otimes^L M^D \to \mathbf{Z} \tag{2.4}$$

in $\mathcal{D}(\mathbf{Z})$. If the cohomology modules of M are finitely generated as abelian groups, then by Verdier Duality (see [11], Proposition 4.1) this pairing induces a canonical isomorphism $M = M^{DD}$ of objects of $\mathcal{D}(\mathfrak{w})$.

Consider the trivial $\mathfrak{w}$-module $\mathbf{Z}$. Because $H^1(\mathfrak{w}, \mathbf{Z}) = \mathbf{Z}$, there is a natural projection map $R\Gamma_{\mathfrak{w}}(\mathbf{Z}) \to \mathbf{Z}[-1]$ in $\mathcal{D}(\mathbf{Z})$. Thus if M is a bounded complex of $\mathfrak{w}$-modules, we have a cup-product pairing

$$R\Gamma_{\mathfrak{w}}(M) \otimes^L R\Gamma_{\mathfrak{w}}(M^D) \to \mathbf{Z}[-1] \tag{2.5}$$

which induces a map

$$\psi(M) : R\Gamma_{\mathfrak{w}}(M^D) \to R\mathrm{Hom}(R\Gamma_{\mathfrak{w}}(M), \mathbf{Z}[-1]) \tag{2.6}$$

in $\mathcal{D}(\mathbf{Z})$.

Proposition 2.1.4. Suppose that M is finite. Then $\psi(M)$ is an isomorphism.

Proof. Because M is finite, we have identifications $M^D = M^*[-1]$ and

$$\mathrm{Ext}^i(R\Gamma_{\mathfrak{w}}(M), \mathbf{Z}[-1]) = H^{2-i}(\mathfrak{w}, M)^*, \tag{2.7}$$

the second of which follows from (2.1.2). The map $\psi(M)^i$ is the map $H^{i-1}(\mathfrak{w}, M^*) \to H^{2-i}(\mathfrak{w}, M)^*$ induced by cup-product, which is an isomorphism by (2.1.2). □

Proposition 2.1.5. Suppose that M is free and finitely generated as an abelian group. Then $\psi(M)$ is an isomorphism.

Proof. Since the module M is fixed throughout, let us write ψ for $\psi(M)$ for simplicity. Because M is free, we have an identification $M^D = \mathrm{Hom}(M, \mathbf{Z})$. To explicitly describe the maps on cohomology, we apply (2.0.1) with $C = R\Gamma_{\mathfrak{w}}(M)$ and $n = 1$.

The map $\psi^0 : \mathrm{Hom}_{\mathfrak{w}}(M, \mathbf{Z}) \to \mathrm{Hom}(C, \mathbf{Z}[-1]) = \mathrm{Hom}(M_{\mathfrak{w}}, \mathbf{Z})$ is induced by the evaluation pairing $\mathrm{Hom}_{\mathfrak{w}}(M, \mathbf{Z}) \otimes M_{\mathfrak{w}} \to \mathbf{Z}$. An argument similar to the proof of (2.1.1) shows that ψ^0 is an isomorphism.

Now consider $\psi^1 : \mathrm{Hom}(M, \mathbf{Z})_{\mathfrak{w}} \to \mathrm{Ext}^1(C, \mathbf{Z}[-1])$. The short exact sequence of (2.0.1) which describes the latter group is

$$0 \to \mathrm{Ext}(M_{\mathfrak{w}}, \mathbf{Z}) \to \mathrm{Ext}^1(C, \mathbf{Z}[-1]) \to \mathrm{Hom}(M^{\mathfrak{w}}, \mathbf{Z}) \to 0.$$

We will prove that $\mathrm{Hom}(M, \mathbf{Z})_{\mathfrak{w}}$ fits into an isomorphic short exact sequence.

The inclusion $M^{\mathfrak{w}} \to M$ gives us a surjection $\mathrm{Hom}(M, \mathbf{Z})_{\mathfrak{w}} \to \mathrm{Hom}(M^{\mathfrak{w}}, \mathbf{Z})$, which is the map induced by the evaluation map $\mathrm{Hom}(M, \mathbf{Z})_{\mathfrak{w}} \otimes M^{\mathfrak{w}} \to \mathbf{Z}$. To identify the kernel of this surjection, consider the diagram

$$\begin{tikzcd}
 & & 0 & 0 \\
 & 0 \arrow[r] & \mathrm{Hom}(M^{\mathfrak{w}}, \mathbf{Z}) \arrow[r] \arrow[u] & \mathrm{Hom}(M^{\mathfrak{w}}, \mathbf{Q}) \arrow[u] \\
\mathrm{Hom}_{\mathfrak{w}}(M, \mathbf{Q}) \arrow[r] & \mathrm{Hom}_{\mathfrak{w}}(M, \mathbf{Q}/\mathbf{Z}) \arrow[r, "\delta"] & \mathrm{Hom}(M, \mathbf{Z})_{\mathfrak{w}} \arrow[r] \arrow[u] & \mathrm{Hom}(M, \mathbf{Q})_{\mathfrak{w}} \arrow[u, "\alpha_1"'] \\
\mathrm{Hom}(M_{\mathfrak{w}}, \mathbf{Q}) \arrow[r] \arrow[u, "\alpha_2"'] & \mathrm{Hom}(M_{\mathfrak{w}}, \mathbf{Q}/\mathbf{Z}) \arrow[u, "\alpha_3"'] & & 0. \arrow[u]
\end{tikzcd} \tag{2.8}$$

It follows from (2.1.1) that the α_i are all isomorphisms. Diagram chasing now shows that the kernel we are seeking to describe is equal to the cokernel of δ, which is $\mathrm{Ext}(M_{\mathfrak{w}}, \mathbf{Z})$.

By the previous paragraph, we have a commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(M_{\mathfrak{w}}, \mathbf{Z}) \arrow[r] & \mathrm{Ext}^1(C, \mathbf{Z}[-1]) \arrow[r] & \mathrm{Hom}(M^{\mathfrak{w}}, \mathbf{Z}) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}(M_{\mathfrak{w}}, \mathbf{Z}) \arrow[r] \arrow[u, "\mathrm{id}"] & \mathrm{Ext}^1_{\mathfrak{w}}(M, \mathbf{Z}) \arrow[r] \arrow[u, "\psi^1"] & \mathrm{Hom}(M^{\mathfrak{w}}, \mathbf{Z}) \arrow[r] \arrow[u, "\mathrm{id}"] & 0
\end{tikzcd}$$

and the Five Lemma now shows that ψ^1 is an isomorphism. For $i \neq 0, 1$ it is easy to see that the cohomology of both complexes vanishes. $\square$

Theorem 2.1.6. Let M be a bounded complex of $\mathfrak{w}$-modules, whose cohomology groups are finitely generated as abelian groups. Then the map $\psi(M)$ of (2.6) is an isomorphism.

Proof. Suppose that M is a $\mathfrak{w}$-module which is finitely generated as an abelian group. Write T for the torsion subgroup of M and F for the quotient M/T so that we have a short exact sequence $0 \to T \to M \to F \to 0$ of $\mathfrak{w}$-modules. We have a map of exact triangles

$$\begin{tikzcd}
R\Gamma_{\mathfrak{w}}(F^D) \arrow[r, "\psi(F)"] \arrow[d] & R\mathrm{Hom}(R\Gamma_{\mathfrak{w}}(F), \mathbf{Z}[-1]) \arrow[d] \\
R\Gamma_{\mathfrak{w}}(M^D) \arrow[r, "\psi(M)"] \arrow[d] & R\mathrm{Hom}(R\Gamma_{\mathfrak{w}}(M), \mathbf{Z}[-1]) \arrow[d] \\
R\Gamma_{\mathfrak{w}}(T^D) \arrow[r, "\psi(T)"] & R\mathrm{Hom}(R\Gamma_{\mathfrak{w}}(T), \mathbf{Z}[-1])
\end{tikzcd} \tag{2.9}$$

in $\mathcal{D}(\mathbf{Z})$. By the previous two propositions, $\psi(T)$ and $\psi(F)$ are isomorphisms. Therefore $\psi(M)$ is an isomorphism. The result for a general complex M follows by induction on the length of the complex. $\square$

2.2 Weil-étale Cohomology over Finite Fields

To paint a more complete picture, we restate the duality theorem of the last section in terms of Weil-étale cohomology. This rephrasing, together with Lichtenbaum's duality theorem for the Weil-étale cohomology of curves, give duality theorems for zero and one-dimensional schemes, which will later be paralleled in the local field setting.

For a scheme X which is finite type over k, let X_W denote X endowed with the Weil-étale topology, let $\Gamma_X : \mathcal{S}(X_W) \to \mathcal{A}b$ be the global sections functor, and let $R\Gamma_X : \mathcal{D}(X_W) \to \mathcal{D}(\mathbf{Z})$ be its derived functor.

Proposition 2.2 of [11] states that Weil-étale sheaves on X_W are equivalent to étale sheaves on $X_{\bar{k}}$ with a $\mathfrak{w}$-action. If we let $X = \operatorname{Spec} k$, then Weil-étale sheaves on X_W are simply $\mathfrak{w}$-modules. Theorem (2.1.6) has the following rephrasing in terms of Weil-étale cohomology:

Theorem 2.2.1. Let F be a bounded complex of Weil-étale sheaves on $X = \operatorname{Spec} k$, such that the cohomology sheaves $H^i(F)$ correspond to finitely generated $\mathfrak{w}$-modules. Let $F^D = R\underline{\operatorname{Hom}}_X(F, \mathbf{Z})$. Then the cup-product pairing

$$R\Gamma_X(F) \otimes^L R\Gamma_X(F^D) \to \mathbf{Z}[-1] \tag{2.10}$$

induces an isomorphism

$$R\Gamma_X(F^D) \to R\mathrm{Hom}(R\Gamma_X(F), \mathbf{Z}[-1]) \tag{2.11}$$

in the derived category of abelian groups. $\square$

We remind the reader of Lichtenbaum's duality theorem for the Weil-étale cohomology of curves, proved in [11]. Let X/k be a smooth, geometrically connected curve. For simplicity we assume X is projective, though Lichtenbaum does not make this assumption. One has $H^2(X_W, \mathbf{G}_m) = \mathbf{Z}$, and the higher cohomology groups of $\mathbf{G}_m$ on X vanish. The duality theorem is the following:

Theorem 2.2.2. ([11], Theorem 5.1) Let F be a locally constant Weil-étale sheaf on X, representable by a finitely generated abelian group, and let $F^D = R\underline{\mathrm{Hom}}_X(F, \mathbf{G}_m)$. The cup-product pairing

$$R\Gamma_X(F) \otimes^L R\Gamma_X(F^D) \to \mathbf{Z}[-2] \tag{2.12}$$

induces an isomorphism

$$R\Gamma_X(F^D) \to R\mathrm{Hom}(R\Gamma_X(F), \mathbf{Z}[-2]) \tag{2.13}$$

in the derived category of abelian groups. $\square$

2.3 Some Finiteness and Vanishing Lemmas

The following two lemmas can be thought of as extensions of the additive and multiplicative Hilbert Theorem 90 for finite fields, which will prove useful in the next chapter. It is convenient to collect in this section results on the cohomology of the types of $\mathfrak{w}$-modules that will appear when we study the cohomology of the Weil group of a local field.

Lemma 2.3.1. Let M be a $\mathfrak{w}$-module which is finitely generated and free as an abelian group. Then:

(i) $H^0(\mathfrak{w}, \mathrm{Hom}(M, \bar{k}^\times))$ is finite.

(ii) $H^0(\mathfrak{w}, \mathrm{Hom}(M, \bar{k}))$ is finite.

(iii) $H^0(\mathfrak{w}, \mathrm{Hom}(M, U_L))$ is profinite.

Proof. (i) Let $f : M \to \bar{k}^\times$ be $\mathfrak{w}$-equivariant. Then $f(\sigma m) = f(qm)$, so f factors through the finite group $M/(\sigma - q)M$. Since $\bar{k}^\times$ has only finitely many elements of any particular order, the image of f lands in a finite set which is independent of f. Therefore there are only finitely many possible f.

(ii) Choose a $\mathbf{Z}$-basis of M and a matrix A representing the action of σ on $M \cong \mathbf{Z}^r$. This $\mathbf{Z}$-basis provides us with an isomorphism $\mathrm{Hom}(M, \bar{k}) \simeq \bar{k}^r$. Under this isomorphism, elements of $H^0(\mathfrak{w}, \mathrm{Hom}(M, \bar{k}))$ correspond to solutions of the equation $Ax = x^q$, for $x \in \bar{k}^r$ (here raising to the q^{th} power is done component wise).

Writing the coordinates of x as $X_1, \dots, X_r$, we see that we must show that the affine variety V defined by the equations $\sum_{i=1}^r a_{ij} X_i - X_j^q = 0$ for $1 \leq j \leq r$ is finite, where $A = (a_{ij})$. Note that $\mathrm{Jac}(V) = A$ has full rank at every point of $\bar{k}^r$, by the invertibility of A.

Let $V_j \subset \bar{k}^r$ be the hypersurface defined by $\sum_{i=1}^r a_{ij} X_i - X_j^q$, so that $V = \bigcap_j V_j$. Let $v \in V$, and let T_v be the tangent space to V at v. Then

$$0 = \dim \ker(A) = \dim T_v \geq \dim V,$$

hence V has dimension zero and is therefore finite.

(iii) Since $H^0(\mathfrak{w}, -)$ commutes with inverse limits, we have

$$H^0(\mathfrak{w}, \mathrm{Hom}(M, U_L)) = \varprojlim_i H^0(\mathfrak{w}, \mathrm{Hom}(M, U_L/U_L^{(i)})).$$

Using induction on i on the sequences $0 \to U_L^{(i)}/U_L^{(i+1)} \to U_L/U_L^{(i+1)} \to U_L/U_L^{(i)} \to 0$, we see from parts (i) and (ii) that the terms appearing in the above inverse limit are all finite, hence the result. $\square$

Lemma 2.3.2. Let M be a $\mathfrak{w}$-module which is finitely generated and free as an abelian group. Then:

(i) $H^1(\mathfrak{w}, \mathrm{Hom}(M, \bar{k}^\times)) = 0$.

(ii) $H^1(\mathfrak{w}, \mathrm{Hom}(M, \bar{k})) = 0$.

(iii) $H^1(\mathfrak{w}, \mathrm{Hom}(M, U_L)) = 0$.

Proof. (i) Writing the group law on $\bar{k}^\times$ additively, we wish to show that the map $(\sigma - 1) : \mathrm{Hom}(M, \bar{k}^\times) \to \mathrm{Hom}(M, \bar{k}^\times)$ is surjective. Choosing a $\mathbf{Z}$-basis of M determines an isomorphism $\mathrm{Hom}(M, \bar{k}^\times) \simeq (\bar{k}^\times)^r$ and a matrix $A \in \mathrm{GL}_r(\mathbf{Z})$ representing the action of σ on M. Direct calculation shows that under our isomorphism $\mathrm{Hom}(M, \bar{k}^\times) \simeq (\bar{k}^\times)^r$, the map $\sigma - 1$ is transformed into $qA^{-1} - 1 : (\bar{k}^\times)^r \to (\bar{k}^\times)^r$. Multiplying by the automorphism $-A$, we conclude that it suffices to prove that $A - q$ is surjective, as a map from $(\bar{k}^\times)^r$ to itself.

It is clear that q is not an eigenvalue of A, since the product of the eigenvalues of A is 1, but all of these eigenvalues are algebraic integers. It follows that the map $A - q : \mathbf{Z}^r \to \mathbf{Z}^r$ is injective with finite cokernel C. Tensoring with $\bar{k}^\times$, we obtain an exact sequence

$$(\bar{k}^\times)^r \xrightarrow{A-q} (\bar{k}^\times)^r \to C \otimes \bar{k}^\times \to 0.$$

But $C \otimes \bar{k}^\times = 0$, because $\bar{k}^\times$ is divisible and C is finite.

(ii) As in part (i), we choose a $\mathbf{Z}$-basis for M and a matrix A representing the action of σ on M. On the group $\mathrm{Hom}(M, \bar{k})$, direct calculation shows that $((\sigma - 1)f)(m) = f(A^{-1}m)^q - f(m)$, and replacing m with Am and multiplying by -1, we are reduced to showing that the map $f \mapsto f \circ A - f^q$ is surjective. Under the isomorphism $\mathrm{Hom}(M, \bar{k}) \simeq \bar{k}^r$, this is the map $x \mapsto Ax - x^q$, where raising to the q^{th} power is done component-wise.

Let $y \in \bar{k}^r$; we must show that $Ax - x^q - y = 0$ has a solution in x. Replacing the components of x with formal variables $X_1, \ldots, X_r$ and projectivizing by adding in appropriate powers of X_{r+1}, we obtain the equations

$$\begin{aligned} L_1 X_{r+1}^{q-1} - X_1^q - y_1 X_{r+1}^q &= 0 \\ &\vdots \\ L_r X_{r+1}^{q-1} - X_r^q - y_r X_{r+1}^q &= 0 \end{aligned}$$

where L_i is a homogeneous linear polynomial in $X_1, \ldots, X_r$. By Chapter I, Theorem 7.2 of [7], the intersection of the corresponding hypersurfaces in $\mathbf{P}^r_{\bar{k}}$ is non-empty. It is clear from the equations that the intersection of the hypersurfaces with the hyperplane $X_{r+1} = 0$ is empty. Thus any point in the intersection of the hypersurfaces is also a solution to the original affine equations, proving that our map is surjective.

(iii) Recall that U_L comes equipped with a filtration

$$\cdots \subset U_L^{(i)} \subset \cdots \subset U_L^{(1)} \subset U_L$$

of $\mathfrak{w}$-modules, with $U_L/U_L^{(1)} \simeq \bar{k}^\times$, and higher successive quotients all isomorphic to $\bar{k}$. Let $f : M \to U_L$, and reduce modulo $U_L^{(1)}$. We obtain a map $\bar{f} : M \to \bar{k}^\times$, and by part (i) we can write $\bar{f} = (\sigma - 1)\bar{f}_0$ for some $\bar{f}_0 : M \to \bar{k}^\times$. As M is free, we can lift $\bar{f}_0$ to a map $f_0 : M \to U_L$, and we have $f = (\sigma - 1)f_0 + g_1$ for some $g_1 : M \to U_L^{(1)}$.

Reducing g_1 modulo $U_L^{(2)}$, we have a map $\bar{g}_1 : M \to \bar{k}$, which by part (ii) we can write as $\bar{g}_1 = (\sigma - 1)\bar{f}_1$ for some $\bar{f}_1 : M \to \bar{k}^\times$. Lifting $\bar{f}_1$ to some $f_1 : M \to U_L^{(1)}$, we obtain the equality $f = (\sigma - 1)f_0 + (\sigma - 1)f_1 + g_2$ where $g_2 : M \to U_L^{(2)}$. It is clear that by repeating this process, we can write $f = \sum_{i \geq 0}(\sigma - 1)f_i$, where $f_i : M \to U_L^{(i)}$. Because $\mathfrak{w}$ acts continuously on $\mathrm{Hom}(M, U_L)$, we have $f = (\sigma - 1)(\sum_{i \geq 0} f_i)$, and this is a well-defined homomorphism because U_L is complete. $\square$

Chapter 3

Weil Groups of Local Fields

We now turn our attention to local fields. The main theorem of this chapter is a duality theorem for the cohomology of the Weil group, which can be interpreted as a duality theorem for the Weil-étale cohomology of zero-dimensional schemes over Spec K.

The exact sequence $0 \to N \to W \to \mathfrak{w} \to 0$ of topological groups gives rise to a spectral sequence

$$H^i(\mathfrak{w}, H^j(N, M)) \Rightarrow H^{i+j}(W, M)$$

for any W-module M. Because $\mathrm{scd}(\mathfrak{w}) = 1$, this spectral sequence degenerates to a collection of short exact sequences

$$0 \to H^1(\mathfrak{w}, H^{i-1}(N, M)) \to H^i(W, M) \to H^0(\mathfrak{w}, H^i(N, M)) \to 0 \tag{3.1}$$

for all $i \geq 0$. We recall from Chapter II, §3.3(c) of [21] that $\mathrm{cd}(N) = 1$ and $\mathrm{scd}(N) = 2$; it follows that $H^i(W, M) = 0$ for all $i \geq 4$ and all M.

3.1 Cohomology of $\mathbf{G}_m$ and μ_n

Proposition 3.1.1. The cohomology groups $H^i(W, \bar{L}^\times)$ are given by

$$H^i(W, \bar{L}^\times) = \begin{cases} K^\times & \text{if } i = 0 \\ \mathbf{Z} & \text{if } i = 1 \\ 0 & \text{if } i \geq 2. \end{cases}$$

Proof. The field L is a C_1, hence the cohomology groups $H^i(N, \bar{L}^\times)$ all vanish for $i \geq 1$. It follows immediately from (3.1) that $H^i(W, \bar{L}^\times) = 0$ for $i \geq 2$.

For $i = 1$, (3.1) yields an isomorphism $H^1(\mathfrak{w}, L^\times) \simeq H^1(W, \bar{L}^\times)$. The long exact sequence of $\mathfrak{w}$-cohomology of $0 \to U_L \to L^\times \to \mathbf{Z} \to 0$ now gives us a canonical isomorphism $H^1(W, \bar{L}^\times) = H^1(\mathfrak{w}, \mathbf{Z}) = \mathbf{Z}$, since $H^1(\mathfrak{w}, U_L) = 0$ by (2.3.2). $\square$

Corollary 3.1.2. Let μ_n be the group of n^{th} roots of unity in $\bar{L}^\times$. Then

$$H^i(W, \mu_n) = \begin{cases} \mu_n(K) & \text{if } i = 0 \\ K^\times/(K^\times)^n & \text{if } i = 1 \\ \mathbf{Z}/n\mathbf{Z} & \text{if } i = 2 \\ 0 & \text{if } i \geq 3, \end{cases}$$

and thus $H^2(W, \mu) = \varinjlim_n H^2(W, \mu_n) = \mathbf{Q}/\mathbf{Z}$.

Proof. Consider the long exact sequence in cohomology of the Kummer sequence $0 \to \mu_n \to \bar{L}^\times \to \bar{L}^\times \to 0$. Using (3.1.1), we see that the long exact sequence reads

$$0 \to \mu_n(K) \to K^\times \xrightarrow{n} K^\times \to H^1(W, \mu_n) \to \mathbf{Z} \xrightarrow{n} \mathbf{Z} \to H^2(W, \mu_n) \to 0$$

from which the results follow immediately. $\square$

Corollary 3.1.3. Suppose that M is a finite W-module. Then the groups $H^i(W, M)$ are finite for all i, and vanish for $i \geq 3$.

Proof. If $M = \mu_n$ this is clear from (3.1.2). Otherwise, pick some finite Galois extension K' of K with Weil group W', such that as a W'-module, M is a direct sum of modules isomorphic to μ_n for various n. The quotient W/W' is naturally identified with $\mathrm{Gal}(K'/K)$. We thus have a spectral sequence

$$H^i(\mathrm{Gal}(K'/K), H^j(W', M)) \Rightarrow H^{i+j}(W, M) \tag{3.2}$$

where all the terms appearing on the second page are finite; it follows that all of the limit terms are finite as well. The vanishing of $H^i(W, M)$ for $i \geq 3$ is immediate from (3.1) and $\mathrm{cd}(N) = 1$. $\square$

Corollary 3.1.4. We have $\mathrm{cd}(W) = 2$.

Proof. From $\mathrm{scd}(\mathfrak{w}) = 1$ and $\mathrm{cd}(N) = 1$, and the short exact sequences of 3.1, we see that $\mathrm{cd}(W) \leq 2$. But $H^2(W, \mu_n) \neq 0$, hence the equality $\mathrm{cd}(W) = 2$. $\square$

Suppose that M is a finite W-module, and let $M' = \mathrm{Hom}(M, \mu)$; then M' is a W-module in the obvious way. There is a cup-product pairing

$$H^i(W, M) \otimes H^{2-i}(W, M') \to \mathbf{Q}/\mathbf{Z} \tag{3.3}$$

coming from the identification $H^2(W, \mu) = \mathbf{Q}/\mathbf{Z}$ of (3.1.2). The rest of this section is devoted to proving the following:

Theorem 3.1.5. (Weil-Tate Local Duality) The pairing of (3.3) is a perfect pairing of finite groups.

Proof. We will simply mimic the proof of Tate Local Duality in [21], checking along the way that the machinery used to prove Tate Local Duality still works in the setting of the cohomology of the Weil group.

Consider the functor $M \mapsto H^2(W, M)^*$ from finite W-modules to finite abelian groups. By Chapter I, §3.5, Lemma 6 of [21], this functor is representable by a W-module I which is a direct limit of finite modules. That is, for all finite W-moules M, there is a functorial isomorphism $H^0(W, \mathrm{Hom}(M, I)) \simeq H^2(W, M)^*$. Let us call I the *dualizing module* for W.

Let $V \subseteq W$ be an open, normal subgroup of finite index. If M is a finite V-module, we let $\mathrm{Ind}_V^W(M)$ be the set of continuous maps $f : W \to M$, such that $f(vw) = vf(w)$ for all $v \in V$ and all $w \in W$. It is easy to check that $\mathrm{Ind}_V^W(M)$ is finite, and carries an action of W by $(wf)(w_0) := f(w_0 w)$. The standard proof of Shapiro's Lemma (see Proposition 10 of Chapter I, §2.5 of [21]) applies, and shows that there are isomorphisms $H^i(W, \mathrm{Ind}_V^W(M)) \xrightarrow{\sim} H^i(V, M)$ for all i.

Shapiro's Lemma and the defining property of the dualizing module I of W

show together that we have functorial isomorphisms

$$\begin{aligned} H^0(V, \mathrm{Hom}(M, I)) &= H^0(W, \mathrm{Hom}(\mathrm{Ind}_V^W(M), I)) \\ &= H^2(W, \mathrm{Ind}_V^W(M))^* \\ &= H^2(V, M)^* \end{aligned}$$

for any finite V-module M. We conclude that I is also the dualizing module for any open, normal subgroup $V \subseteq W$ of finite index.

By the above paragraph and (3.1.2), we have

$$\mathrm{Hom}_V(\mu_n, I) = H^2(V, \mu_n)^* = (\mathbf{Z}/n\mathbf{Z})^* = \mathbf{Z}/n\mathbf{Z}$$

for any open, normal, finite index $V \subseteq W$. Thus we obtain an equality $\mathrm{Hom}_V(\mu_n, I) = \mathbf{Z}/n\mathbf{Z}$ which is independent of V, and we conclude that $\mathrm{Hom}(\mu_n, I) = \mathbf{Z}/n\mathbf{Z}$, as a W-module with trivial W-action. The canonical generator of $\mathbf{Z}/n\mathbf{Z}$ thus determines a W-module isomorphism $\mu_n \to I[n]$, and by passing to the limit over all n, we conclude that $\mu = \varinjlim_n \mu_n$ is the dualizing module for W.

The rest of the proof the theorem can be copied verbatim from the proof of Theorem 2 of Chapter II, §5.2 of [21]. □

3.2 Vanishing of $H^3(W, M)$ for Finitely Generated M

The main theorem in this section is the vanishing of $H^3(W, M)$, for M which are finitely generated as abelian groups. The vanishing theorem of this section is needed to prove the duality theorem of the next section, but it is interesting in its own right. As a consequence of this vanishing theorem, we deduce a theorem of Rajan (and offer a slight correction to his proof).

Theorem 3.2.1. Let M be a W-module which is finitely generated and free as an abelian group. Then $H^3(W, M) = 0$.

Proof. Let $\mathbf{Q}$ be the trivial W-module, and consider the exact sequence

$$0 \to M \to M \otimes \mathbf{Q} \to (M \otimes \mathbf{Q})/M \to 0$$

of W-modules. Let $M_{\mathbf{Q}} = M \otimes \mathbf{Q}$ and let $P = M_{\mathbf{Q}}/M$. The abelian group $M_{\mathbf{Q}}$ is a $\mathbf{Q}$-vector space, and hence uniquely divisible. Therefore $H^1(N, M_{\mathbf{Q}}) = H^2(N, M_{\mathbf{Q}}) = 0$, which implies by (3.1) that $H^2(W, M_{\mathbf{Q}}) = H^3(W, M_{\mathbf{Q}}) = 0$. Therefore we have an isomorphism $H^2(W, P) \simeq H^3(W, M)$.

Let P_n denote the n-torsion of P. Then P_n is finite, so we have by Weil-Tate Local Duality (3.1.5) that

$$\begin{aligned} H^2(W, P) &\simeq \varinjlim_n H^2(W, P_n) \\ &\simeq \varinjlim_n \mathrm{Hom}_W(P_n, \mu)^* \\ &\simeq (\varprojlim_n \mathrm{Hom}_W(P_n, \mu))^* \\ &\simeq \mathrm{Hom}_W(P, \mu)^*, \end{aligned}$$

where μ is the group of roots of unity endowed with its natural W-module structure, and $\varprojlim_n \mathrm{Hom}_W(P_n, \mu)$ has its profinite topology. From these isomorphisms, we see that it suffices to show $\mathrm{Hom}_W(P, \mu) = 0$.

Let $f : P \to \mu$ be W-equivariant, and let N' be an open, normal subgroup of N such that N' acts trivially on M. Then for all $\tau \in N'$ and $e \in P$, we have $f(e) = f(\tau \cdot e) = f(e)^\tau$. It follows that $f(P) \cap \mu_{p^\infty}$ is finite, where $p = \mathrm{char}(k)$. Since P has no finite quotients and we wish to prove that $f(P) = 1$, we may assume that $f(P) \subseteq \mu'$, the group of roots of unity of order prime to p. On μ', W acts via the natural action of σ, which is $\zeta \mapsto \zeta^q$, where $q = \#k$.

Let $a \in W$ be a preimage of σ. Applying the W-equivariance of f to a, we see that $f(a \cdot e) = f(e)^q = f(q \cdot e)$ for all $e \in P$. Therefore $(a - q)(P) \subseteq \ker(f)$. We claim that $(a - q) : P \to P$ is surjective, which implies $f = 1$. By the Snake Lemma, it suffices to prove that $a - q$ is surjective on $M_{\mathbf{Q}}$. Surjectivity of $a - q$ on $M_{\mathbf{Q}}$ is equivalent to injectivity, and injectivity will hold if and only if q is not an eigenvalue of a. But q cannot be an eigenvalue of a, because $\det(a) = \pm 1$ and the characteristic polynomial of a has integer coefficients. $\square$

Theorem 3.2.2. Let M be a W-module which is finitely generated as an abelian group. Then $H^3(W, M) = 0$.

Proof. This follows easily from (3.2.1) by considering the short exact sequence

$$0 \to M_{\mathrm{tors}} \to M \to M/M_{\mathrm{tors}} \to 0$$

and taking cohomology. $\square$

Suppose now that A is a *topological* W-module; that is A is a topological abelian group with a continuous action of W. For such A, we can define cohomology groups $H^i_{cc}(W, A)$ using complexes of continuous cochains. It follows from Corollary 2.4 of [12] and Corollary 2 of [4] that the groups $H^i_{cc}(W, A)$ agree with the topological group cohomology used in [12] and [4], which is defined using the classifying topos BW.

The non-discrete topological W-modules we are interested in are all complex manifolds, and hence Remark 2.2 of [12] shows that the groups $H^i_{cc}(W, A)$ also agree with the cohomology groups $H^i_M(W, A)$ defined by Moore in [15] and used by Rajan in [19]. If A is discrete, then $H^i_{cc}(W, A)$ can be identified with the Galois cohomology groups $H^i(W, A)$; this follows from the remarks preceding Lemma 1 of [19].

Theorem 3.2.3. Let T be an algebraic torus over $\mathbf{C}$, equipped with an action of W via algebraic automorphisms. Then $H^2_{cc}(W, T(\mathbf{C})) = 0$.

Proof. From Corollary 8 of [4], it follows that $H^i_{cc}(N, V) = 0$ for any finite-dimensional complex vector space V and any $i \geq 1$. Thus the spectral sequence of Moore (quoted by Rajan as Proposition 5 of [19]) gives isomorphisms

$$H^i(\mathfrak{w}, H^0_{cc}(N, V)) = H^i_{cc}(W, V),$$

and thus this latter group vanishes for $i \geq 2$.

Let $X_*(T)$ denote the cocharacter group of T; it is a finitely generated W-module. There is a short exact sequence of topological W-modules,

$$0 \to X_*(T) \to X_*(T) \otimes \mathbf{C} \to T(\mathbf{C}) \to 0$$

where $X_*(T)$ gets the discrete topology. By the previous paragraph, the groups $H^i_{cc}(W, X_*(T) \otimes \mathbf{C})$ vanish for $i \geq 2$. By (3.2.1), the groups $H^i(W, X_*(T))$ vanish for $i \geq 3$. The theorem now follows from taking the long exact cohomology sequence of the above short exact sequence. □

In [19], Rajan proves that $H^2_M(W, T(\mathbf{C})) = 0$, for those tori with an action of W that comes from an action of G. His proof seems to be slightly flawed, because his Proposition 6 asserts that $H^2(G, A) \to H^2(W, A)$ is an isomorphism for any G-module A, when in fact this map has a kernel for $A = \mathbf{Z}$. However, Rajan's proof of the vanishing of $H^2_M(W, T(\mathbf{C}))$ ultimately only relies on the fact that the restriction map $H^2(G, A) \to H^2(W, A)$ is surjective for all discrete G-modules A, which is true by (4.1.3).

Corollary 3.2.4. Consider $\mathbf{C}^\times$ as a trivial W-module with its natural Euclidean topology. Then $H^2_{cc}(W, \mathbf{C}^\times) = 0$.

Proof. Take $T = \mathbf{G}_m$ in (3.2.3), with trivial action. □

Corollary (3.2.4) implies that the map $H^1(W, \mathrm{GL}_n(\mathbf{C})) \to H^1(W, \mathrm{PGL}_n(\mathbf{C}))$ is surjective, which says exactly that every projective complex representation of W lifts to an affine representation.

3.3 Duality for Finitely Generated Modules

Our duality theorem is best stated using derived category language. The fixed points functor $\Gamma_W : W\text{-Mod} \to \mathcal{A}b$ has a derived functor $R\Gamma_W : \mathcal{D}(W) \to \mathcal{D}(\mathbf{Z})$, such that $H^i(R\Gamma_W(M)) = H^i(W, M)$ for any W-module M.

Consider now the W-module $\bar{L}^\times$. Because $H^1(W, \bar{L}^\times) = \mathbf{Z}$ and all higher cohomology groups vanish, there is a natural projection map $R\Gamma_W(\bar{L}^\times) \to \mathbf{Z}[-1]$. If M is any bounded complex of W-modules, we set $M^D := R\mathrm{Hom}(M, \bar{L}^\times)$. If M is concentrated in degree 0, then $M^D = \mathrm{Hom}(M, \bar{L}^\times)$. The cup-product pairing

$$R\Gamma_W(M) \otimes^L R\Gamma_W(M^D) \to \mathbf{Z}[-1] \tag{3.4}$$

induces a map

$$\psi(M) : R\Gamma_W(M^D) \to R\mathrm{Hom}(R\Gamma_W(M), \mathbf{Z}[-1]) \tag{3.5}$$

in $\mathcal{D}(\mathbf{Z})$.

Theorem 3.3.1. Suppose that M is a bounded complex of W-modules, whose cohomology groups are finitely generated as abelian groups. Then the map $\psi(M)$ of (3.5) is an isomorphism in $\mathcal{D}(\mathbf{Z})$.

By induction on the length of the complex, it is clear that it suffices to prove the result for M concentrated in degree 0; i.e. we may assume M is a W-module which is finitely generated as an abelian group.

The proof of this theorem relies on several lemmas. To begin, the discrete valuation $L^\times \to \mathbf{Z}$ induces a valuation $\bar{L}^\times \to \mathbf{Q}$. Let $U_{\bar{L}}$ be the kernel of this map, so that we have a short exact sequence

$$0 \to U_{\bar{L}} \to \bar{L}^\times \to \mathbf{Q} \to 0 \tag{3.6}$$

of W-modules. Taking W-cohomology, it follows easily that $H^2(W, U_{\bar{L}}) = \mathbf{Q}/\mathbf{Z}$, and $H^i(W, U_{\bar{L}}) = 0$ for $i \geq 3$. Cup-product therefore induces the map

$$\begin{tikzcd} R\Gamma_W(\mathrm{Hom}(M, U_{\bar{L}})) \arrow[r] \arrow[d] & R\mathrm{Hom}(R\Gamma_W(M), \mathbf{Q}/\mathbf{Z}[-2]) \arrow[d] \\ R\Gamma_W(M^D) \arrow[r, "\psi(M)"] \arrow[d] & R\mathrm{Hom}(R\Gamma_W(M), \mathbf{Z}[-1]) \arrow[d] \\ R\Gamma_W(\mathrm{Hom}(M, \mathbf{Q})) \arrow[r] & R\mathrm{Hom}(R\Gamma_W(M), \mathbf{Q}[-1]) \end{tikzcd} \tag{3.7}$$

of exact triangles in $\mathcal{D}(\mathbf{Z})$.

Lemma 3.3.2. To prove Theorem (3.3.1), it suffices to show that the maps

$$H^i(W, \mathrm{Hom}(M, U_{\bar{L}})) \to H^{2-i}(W, M)^* \tag{3.8}$$

and

$$H^j(W, \mathrm{Hom}(M, \mathbf{Q})) \to \mathrm{Hom}(H^{1-j}(W, M), \mathbf{Q}) \tag{3.9}$$

induced by cup-product are isomorphisms for all $i \geq 0$ and all $j \geq 0$.

Proof. By the Five Lemma, it to prove Theorem (3.3.1), it suffices to prove that the two outside vertical maps of (3.7) are isomorphisms. Because $\mathbf{Q}$ and $\mathbf{Q}/\mathbf{Z}$ are injective abelian groups, this is easily seen to be equivalent to the statement of the lemma. □

Lemma 3.3.3. Suppose that M is finite. Then the map $\psi(M)$ of Theorem (3.3.1) is an isomorphism.

Proof. This is a restatement of (3.1.5). □

By considering the short exact sequence $0 \to M_{\mathrm{tors}} \to M \to M/M_{\mathrm{tors}} \to 0$ of W-modules, we are now reduced to proving the maps of (3.8) and (3.9) are isomorphisms for finitely generated, free M. We will begin by showing that (3.9) is an isomorphism.

Proposition 3.3.4. Suppose that M is finitely generated as an abelian group. Then the map

$$H^j(W, \mathrm{Hom}(M, \mathbf{Q})) \to \mathrm{Hom}(H^{1-j}(W, M), \mathbf{Q})$$

induced by cup-product is an isomorphism for all $j \geq 0$.

Proof. Let $V = M \otimes \mathbf{Q}$. Because the quotient V/M has only trivial maps to $\mathbf{Q}$, our map can be identified with the map

$$H^j(W, \mathrm{Hom}_{\mathbf{Q}}(V, \mathbf{Q})) \to \mathrm{Hom}_{\mathbf{Q}}(H^{1-j}(W, V), \mathbf{Q}).$$

Since $H^i(N, V)$ vanishes for $i \geq 1$, proving the proposition is equivalent, by (3.1), to showing that the map

$$H^j(\mathfrak{w}, \mathrm{Hom}_{\mathbf{Q}}(V^N, \mathbf{Q})) \to \mathrm{Hom}_{\mathbf{Q}}(H^{1-j}(\mathfrak{w}, V^N), \mathbf{Q})$$

is an isomorphism for all $j \geq 0$. This follows immediately from our results on $\mathfrak{w}$-duality for vector space coefficients. $\square$

To prove that the maps $H^i(W, \mathrm{Hom}(M, U_{\bar{L}})) \to H^{2-i}(W, M)^*$ are all isomorphisms, we will use (3.1). In particular, for every $i \geq 0$, we have a map of short exact sequences

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
H^1(\mathfrak{w}, H^{i-1}(N, \mathrm{Hom}(M, U_{\bar{L}}))) \arrow[r] \arrow[d] & H^0(\mathfrak{w}, H^{2-i}(N, M))^* \arrow[d] \\
H^i(W, \mathrm{Hom}(M, U_{\bar{L}})) \arrow[r] \arrow[d] & H^{2-i}(W, M)^* \arrow[d] \\
H^0(\mathfrak{w}, H^i(N, \mathrm{Hom}(M, U_{\bar{L}})) \arrow[r] \arrow[d] & H^1(\mathfrak{w}, H^{1-i}(N, M))^* \arrow[d] \\
0 & 0
\end{tikzcd} \tag{3.10}$$

and to prove that the middle arrow is an isomorphism, we will prove, for $i \geq 1$, that the top and bottom horizontal arrows are isomorphisms. It should be noted that the top and bottom arrows are not isomorphisms for all $i \geq 0$: take $M = \mathbf{Z}$ and $i = 0$, then the bottom arrow is the zero map $U_K \to 0$.

Let L'/L be a finite Galois extension with group H and degree e. Consider the long exact sequence in H-cohomology of $0 \to U_{L'} \to L'^{\times} \to \mathbf{Z} \to 0$. Since the groups $H^i(H, L'^{\times})$ and $H^i(H, \mathbf{Q})$ vanish for $i \geq 1$, the long exact sequence reads

$$0 \to U_L \to L^{\times} \to \mathbf{Z} \to H^1(H, U_{L'}) \to 0 \tag{3.11}$$

which gives us a canonical identification $H^1(H, U_{L'}) = \mathbf{Z}/e\mathbf{Z}$. Taking the limit over all L'/L gives an identification $H^1(N, U_{\bar{L}}) = \mathbf{Q}/\mathbf{Z}$.

Throughout the proof of the next proposition, we will make use of the Tate cohomology groups $\hat{H}^*(H, -)$, for a finite group H. For definitions and basic properties of Tate cohomology groups, see Chapter VIII of [20].

Proposition 3.3.5. Let M be a W-module which is finitely generated and free as an abelian group. Then the maps

$$H^i(\mathfrak{w}, H^1(N, \mathrm{Hom}(M, U_{\bar{L}}))) \to H^{1-i}(\mathfrak{w}, H^0(N, M))^* \tag{3.12}$$

are isomorphisms for all $i \geq 0$.

Proof. Let M be an N-module which is finitely generated and free as an abelian group. We will show that

$$\alpha : H^1(N, \mathrm{Hom}(M, U_{\bar{L}})) \to H^0(N, M)^* \tag{3.13}$$

is an isomorphism. The proposition then follows easily from $\mathfrak{w}$-duality.

If $M = \mathbf{Z}$ with trivial N-action, then α is the canonical map $\mathbf{Q}/\mathbf{Z} \to \mathbf{Z}^*$. Since cohomology commutes with direct sums, this proves the result for M with trivial action.

Now let M be any finitely generated N-module, and choose a finite Galois extension L'/L with group H, such that the open subgroup $N' = \mathrm{Gal}(\bar{L}/L')$ of N acts trivially on M. We mimic the argument on page 128 of [9]. Consider the diagrams

$$\begin{tikzcd}
0 \arrow[r] & H^1(H, \mathrm{Hom}(M, U_{L'})) \arrow[r, "\mathrm{inf}"] \arrow[d, "\gamma"] & H^1(N, \mathrm{Hom}(M, U_{\bar{L}})) \arrow[r, "\mathrm{res}"] \arrow[d, "\alpha"] & H^1(N', \mathrm{Hom}(M, U_{\bar{L}})) \arrow[d, "\alpha'"] \\
0 \arrow[r] & \hat{H}^0(H, M)^* \arrow[r] & H^0(N, M)^* \arrow[r, "\mathrm{tr}^*"] & M^*
\end{tikzcd} \tag{3.14}$$

and

$$\begin{tikzcd}
H^1(N', \mathrm{Hom}(M, U_{\bar{L}})) \arrow[r, "\mathrm{tr}"] \arrow[d, "\alpha'"] & H^1(N, \mathrm{Hom}(M, U_{\bar{L}})) \arrow[d, "\alpha"] & \\
M^* \arrow[r] & H^0(N, M)^* \arrow[r] & 0
\end{tikzcd} \tag{3.15}$$

Where α' is the corresponding map for L'. By the previous paragraph, α' is an isomorphism. From (3.15) we see that α is surjective. To show that α is an isomorphism, we only need to show it is injective, and for this, it suffices to prove that γ is an isomorphism.

By ([20], Chapter X, §7, Proposition 11) $L'^{\times}$ is cohomologically trivial for H. By ([20], Chapter IX, §5, Theorem 9) it follows that $\mathrm{Hom}(M, L'^{\times})$ is also cohomologically trivial for H. Applying the exact functor $\mathrm{Hom}(M, -)$ to $0 \to U_{L'} \to L'^{\times} \to \mathbf{Z} \to 0$ and taking reduced H-cohomology gives us an isomorphism

$$\delta : \hat{H}^0(H, \mathrm{Hom}(M, \mathbf{Z})) \to H^1(H, \mathrm{Hom}(M, U_{L'})) \tag{3.16}$$

which commutes with cup-product in the sense that the diagram

$$\begin{tikzcd} \hat{H}^0(H, \mathrm{Hom}(M, \mathbf{Z})) \arrow[r] \arrow[d, "\delta"] & \hat{H}^0(H, M)^* \arrow[d, equal] \\ H^1(H, \mathrm{Hom}(M, U_{L'})) \arrow[r, "\gamma"] & \hat{H}^0(H, M)^* \end{tikzcd} \tag{3.17}$$

commutes. By ([18], Chapter III, Proposition 3.1.2) the top horizontal arrow is an isomorphism, and we conclude that γ is an isomorphism. □

Proposition 3.3.6. Suppose that M is a W-module which is finitely generated and free as an abelian group. Then the map

$$H^1(\mathfrak{w}, H^0(N, \mathrm{Hom}(M, U_{\bar{L}}))) \to H^0(\mathfrak{w}, H^1(N, M))^* \tag{3.18}$$

induced by cup-product is an isomorphism.

Proof. Let L'/L be a finite Galois extension with group H, such that $N' = \mathrm{Gal}(\bar{L}/L')$ acts trivially on M. Using the same argument as in the previous proposition and ([18], Chapter III, Proposition 3.1.2), we see that the map

$$\hat{H}^0(H, \mathrm{Hom}(M, U_{L'})) \to H^1(H, M)^* \tag{3.19}$$

is an isomorphism.

We claim that there is an isomorphism

$$H^1(\mathfrak{w}, H^0(N, \mathrm{Hom}(M, U_{\bar{L}}))) = H^1(\mathfrak{w}, \hat{H}^0(H, \mathrm{Hom}(M, U_{L'}))). \tag{3.20}$$

Together with the fact that the inflation map $H^1(H, M) \to H^1(N, M)$ is an isomorphism (since $H^1(N', M) = 0$), this will suffice to prove the proposition. It is clear that $H^0(N, \mathrm{Hom}(M, U_{\bar{L}})) = H^0(H, \mathrm{Hom}(M, U_{L'}))$, which reduces us to showing that the natural map

$$H^1(\mathfrak{w}, H^0(H, \mathrm{Hom}(M, U_{L'}))) \to H^1(\mathfrak{w}, \hat{H}^0(H, \mathrm{Hom}(M, U_{L'}))) \tag{3.21}$$

is an isomorphism.

For any H-module P, let $\mathrm{Nm}_H : P \to H^0(H, P)$ be the norm map, defined by $m \mapsto \sum_{s \in H} s \cdot m$. Consider the reduction map

$$\mathrm{Nm}_H(\mathrm{Hom}(M, U_{L'})) \to \mathrm{Nm}_H(\mathrm{Hom}(M, U_{L'}/U_{L'}^{(1)})) = \mathrm{Nm}_H(\mathrm{Hom}(M, \bar{k}^\times)).$$

The extension L'/L is totally ramified, hence H acts trivially on $\bar{k}^\times$. Thus for any $f : M \to \bar{k}^\times$, we have

$$\begin{aligned} \mathrm{Nm}_H(f)(m) &= \sum_{s \in H} s f(s^{-1} m) \\ &= \sum_{s \in H} f(s^{-1} m) \\ &= f(\mathrm{Nm}_H(m)) \end{aligned}$$

which proves that $\mathrm{Nm}_H(\mathrm{Hom}(M, \bar{k}^\times)) = \mathrm{Hom}(\mathrm{Nm}_H(M), \bar{k}^\times)$. As $\mathrm{Nm}_H(M) \subset M$ is a finitely generated and free abelian group, we see from (2.3.2) that $\sigma - 1$ maps $\mathrm{Hom}(\mathrm{Nm}_H(M), \bar{k}^\times)$ surjectively onto itself. The same argument shows that $\sigma - 1$

maps $\mathrm{Hom}(\mathrm{Nm}_H(M), \bar{k})$ surjectively onto itself. One now proceeds in the same fashion as in the proof of (2.3.2) to show that $\sigma - 1$ maps $\mathrm{Nm}_H(\mathrm{Hom}(M, U_{L'}))$ surjectively onto itself, hence $H^1(\mathfrak{w}, \mathrm{Nm}_H(\mathrm{Hom}(M, U_{L'}))) = 0$. □

Proposition 3.3.7. Let M be a W-module which is finitely generated and free as an abelian group. Then the maps

$$H^i(W, \mathrm{Hom}(M, U_{\bar{L}})) \to H^{2-i}(W, M)^* \tag{3.22}$$

are isomorphisms for all $i \geq 0$. For $i = 0$, this is a topological isomorphism of profinite groups.

Proof. For $i \geq 1$, this follows immediately from (3.10), (3.3.5), and (3.3.6). For $i = 0$, we consider the Kummer sequences

$$0 \to M \xrightarrow{n} M \to M/n \to 0 \tag{3.23}$$

and

$$0 \to \mathrm{Hom}(M, U_{\bar{L}})[n] \to \mathrm{Hom}(M, U_{\bar{L}}) \xrightarrow{n} \mathrm{Hom}(M, U_{\bar{L}}) \to 0. \tag{3.24}$$

These Kummer sequences induce maps of long exact sequences, the relevant part of which reads

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathrm{Hom}(M, U_{\bar{L}}))/n \arrow[r, "\delta"] \arrow[d] & H^1(\mathrm{Hom}(M, U_{\bar{L}})[n]) \arrow[r] \arrow[d, "\wr"] & H^1(\mathrm{Hom}(M, U_{\bar{L}}))[n] \arrow[d, "\wr"] \\
0 \arrow[r] & H^2(M)[n]^* \arrow[r, "\delta^*"] & H^1(M/n)^* \arrow[r] & (H^1(M)/n)^*
\end{tikzcd} \tag{3.25}$$

where we have written $H^i(-)$ for $H^i(W,-)$ for convenience. The middle arrow is Weil-Tate Local Duality applied to the finite modules $\mathrm{Hom}(M,U_{\bar{L}})[n] = \mathrm{Hom}(M/n,\mu_n)$ and M/n. The right-most arrow is the map of (3.22) when $i=1$, restricted to n-torsion; it is therefore an isomorphism. We conclude by the Five Lemma that

$$H^0(W,\mathrm{Hom}(M,U_{\bar{L}}))/n \to H^2(W,M)[n]^* \tag{3.26}$$

is an isomorphism for all n. Taking the inverse limit over all n, we conclude that

$$\varprojlim_n H^0(W,\mathrm{Hom}(M,U_{\bar{L}}))/n \to H^2(W,M)^* \tag{3.27}$$

is an isomorphism (recall that $H^2(W,M)$ is torsion).

It remains to show that the natural map

$$H^0(W,\mathrm{Hom}(M,U_{\bar{L}})) \to \varprojlim_n H^0(W,\mathrm{Hom}(M,U_{\bar{L}}))/n \tag{3.28}$$

is an isomorphism; in other words, to show that $H^0(W,\mathrm{Hom}(M,U_{\bar{L}}))$ is profinite. If N acts trivially on M, then this is contained in the statement of (2.3.1). Otherwise, pick an open normal subgroup N' of N acting trivially on M, corresponding to a finite Galois extension L'/L with group H. We have

$$H^0(W,\mathrm{Hom}(M,U_{\bar{L}})) = H^0(\mathfrak{w},H^0(H,\mathrm{Hom}(M,U_{L'})))$$

and the latter group is clearly profinite, hence we are done. $\square$

A natural question to ask is whether the other map induced by the cup-product pairing, namely

$$\eta(M) : R\Gamma_W(M) \to R\mathrm{Hom}(R\Gamma_W(M^D), \mathbf{Z}[-1]), \tag{3.29}$$

is an isomorphism. The next proposition shows that this map fails to be an isomorphism when $M = \mathbf{Z}$ with trivial action, due to the non-trivial natural topology on the cohomology groups of the complex $R\Gamma_W(M^D)$. However, elucidating this map will prove useful in later sections, so we now describe it explicitly.

Proposition 3.3.8. The map

$$\eta(\mathbf{Z}) : R\Gamma_W(\mathbf{Z}) \to R\mathrm{Hom}(R\Gamma_W(\bar{L}^\times), \mathbf{Z}[-1]) \tag{3.30}$$

has the following properties:

(i) $\eta(\mathbf{Z})^i$ is an isomorphism for $i \neq 2$.

(ii) $\eta(\mathbf{Z})^2$ induces an isomorphism of $H^2(W, \mathbf{Z})$ with the torsion subgroup U_K^* of $\mathrm{Ext}^2(R\Gamma_W(\bar{L}^\times), \mathbf{Z}[-1])$.

The cohomology of both complexes vanishes outside of degrees 0 through 2.

Proof. For $i = 0, 1$, explicit calculation using (2.0.1) shows that the cohomology of both sides is $\mathbf{Z}$, and the map between them is the identity map. For $i \geq 3$, using (2.0.1) one shows easily that the cohomology of both complexes vanishes.

The only assertion left to prove is that of (ii). Coming from the sequences

$0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/n\mathbf{Z} \to 0$ and $0 \to \mu_n \to U_{\bar{L}} \to U_{\bar{L}} \to 0$ we have a map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & H^1(W,\mathbf{Z})/n \arrow[r] \arrow[d, "\wr"] & H^1(W,\mathbf{Z}/n\mathbf{Z}) \arrow[r, "\delta"] \arrow[d, "\wr"] & H^2(W,\mathbf{Z})[n] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (H^1(W,U_{\bar{L}})^*)/n \arrow[r] & H^1(W,\mu_n)^* \arrow[r, "\delta^*"] & U_K^*[n] \arrow[r] & 0.
\end{tikzcd} \tag{3.31}$$

Here the left vertical arrow is the natural isomorphism $\mathbf{Z}/n\mathbf{Z} \to \hat{\mathbf{Z}}/n\hat{\mathbf{Z}}$, and the middle arrow is the isomorphism of Weil-Tate Local Duality. Therefore the right vertical arrow is an isomorphism, and by passing to the limit over all n, we see that $H^2(W,\mathbf{Z}) \to U_K^*$ is an isomorphism. A simple calculation using (2.0.1) shows that $\mathrm{Ext}^2(R\Gamma_W(\bar{L}^\times), \mathbf{Z}[-1]) = \mathrm{Ext}(U_K, \mathbf{Z})$, whose torsion subgroup is U_K^*. □

Corollary 3.3.9. The cohomology groups $H^i(W,\mathbf{Z})$ are given by

$$H^i(W,\mathbf{Z}) = \begin{cases} \mathbf{Z} & i = 0,1 \\ U_K^* & i = 2 \\ 0 & i \geq 3. \end{cases} \tag{3.32}$$

Proof. This is contained in the proof of the previous proposition. □

The main theorem of this chapter was proven by Jiang in his thesis (see Proposition 4.15 and Theorem 5.3 of [8]). However, Jiang assumes that M is a G-module, and uses the duality theorems of Local Class Field Theory in his proof. We have removed the condition that M be a G-module, and presented a proof which is independent of the main results of Local Class Field Theory.

Chapter 4

Local Class Field Theory via the Weil Group

In this chapter we show how to deduce the main theorems of Local Class Field Theory from the theorems of the previous chapter. We prove, in particular, that $\mathrm{Br}(K) = \mathbf{Q}/\mathbf{Z}$ and that there is a canonical isomorphism $K^\times \otimes \hat{\mathbf{Z}} \to G^{ab}$.

4.1 Comparison with Galois Cohomology

Any G-module can be given a W-module structure via the map $W \to G$, and this process induces restriction maps $H^i(G, M) \to H^i(W, M)$ on cohomology. The following comparison theorems describe these restriction maps.

Proposition 4.1.1. Let M be a torsion G-module. Then there are functorial isomorphisms $H^i(G, M) = H^i(W, M)$ for all $i \geq 0$.

Proof. Since Galois cohomology is always torsion, and $\mathfrak{g}$ and $\mathfrak{w}$ cohomology agree for torsion coefficients (see [20], Chapter XIII, Proposition 1), the restriction maps $H^i(\mathfrak{g}, H^j(N, M)) \to H^i(\mathfrak{w}, H^j(N, M))$ are isomorphisms for all $i, j \geq 0$. Thus the map of spectral sequences

$$\begin{array}{ccc} H^i(\mathfrak{g}, H^j(N, M)) & \Longrightarrow & H^{i+j}(G, M) \\ \downarrow & & \downarrow \\ H^i(\mathfrak{w}, H^j(N, M)) & \Longrightarrow & H^{i+j}(W, M) \end{array}$$

is an isomorphism on the second page and therefore in the limit. $\square$

Corollary 4.1.2. (Tate Local Duality) The cup-product pairing

$$H^i(G, M) \otimes H^{2-i}(G, M') \to \mathbf{Q}/\mathbf{Z} \tag{4.1}$$

is a perfect pairing of finite groups.

Proof. This follows from (3.1.5), and (4.1.1) applied to the finite module M. $\square$

Theorem 4.1.3. Let M be a discrete G-module. Then there are functorial isomorphisms

(i) $H^0(G, M) = H^0(W, M)$ and

(ii) $H^1(G, M) = H^1(W, M)_{\text{tors}}$,

(iii) there is a short exact sequence

$$0 \to H^1(W, M) \otimes \mathbf{Q}/\mathbf{Z} \to H^2(G, M) \to H^2(W, M) \to 0,$$

(iv) and there are isomorphisms $H^i(W, M) = H^i(G, M)$ for all $i \geq 3$.

Proof. Since W is dense in G it is clear that $H^0(G, M) = H^0(W, M)$. Suppose for now that M is torsion-free, and fix $n \in \mathbf{Z}$. Kummer sequences give rise to a diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(G, M)/n \arrow[r] \arrow[d, "\wr"] & H^0(G, M/nM) \arrow[r, "\delta"] \arrow[d, "\wr"] & H^1(G, M)[n] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(W, M)/n \arrow[r] & H^0(W, M/nM) \arrow[r, "\delta"] & H^1(W, M)[n] \arrow[r] & 0.
\end{tikzcd}$$

Since $H^1(G, M)$ is all torsion, we see by passing to the limit over all n that $H^1(G, M) = H^1(W, M)_{\mathrm{tors}}$.

Now consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(G, M)/n \arrow[r] \arrow[d, "\beta"] & H^1(G, M/n) \arrow[r, "\delta"] \arrow[d, "\wr"] & H^2(G, M)[n] \arrow[r] \arrow[d, "\kappa"] & 0 \\
0 \arrow[r] & H^1(W, M)/n \arrow[r] & H^1(W, M/n) \arrow[r, "\delta"] & H^2(W, M)[n] \arrow[r] & 0.
\end{tikzcd}$$

From the Snake Lemma it is clear that κ is surjective and that $\ker(\kappa) = \mathrm{coker}(\beta)$. Passing to the limit over all n, we have

$$\begin{aligned}
& \varinjlim_n \mathrm{coker}(H^1(G, M)/n \to H^1(W, M)/n) \\
= \ & \mathrm{coker}(H^1(G, M) \otimes \mathbf{Q}/\mathbf{Z} \to H^1(W, M) \otimes \mathbf{Q}/\mathbf{Z}) \\
= \ & H^1(W, M) \otimes \mathbf{Q}/\mathbf{Z}
\end{aligned}$$

because $H^1(G, M) \otimes \mathbf{Q}/\mathbf{Z} = 0$. The existence of our exact sequence is now clear.

For $i = 3$, we have the diagram

$$\begin{tikzcd}
0 \arrow[r] & H^2(G, M)/n \arrow[r] \arrow[d] & H^2(G, M/n) \arrow[r, "\delta"] \arrow[d, "\wr"] & H^3(G, M)[n] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^2(W, M)/n \arrow[r] & H^2(W, M/n) \arrow[r, "\delta"] & H^3(W, M)[n] \arrow[r] & 0.
\end{tikzcd}$$

and it follows from the exact sequence $0 \to H^1(W, M) \otimes \mathbf{Q}/\mathbf{Z} \to H^2(G, M) \to H^2(W, M) \to 0$ that the left arrow is an isomorphism. Therefore $H^3(G, M)[n] = H^3(W, M)[n]$, and passing to the limit over all n gives $H^3(G, M) = H^3(W, M)$. The result for $i \geq 4$ follows immediately by induction on i and considering Kummer sequences.

Consider now a general G-module M, and the exact sequence $0 \to M_{\text{tors}} \to M \to M/M_{\text{tors}} \to 0$. The map on long exact cohomology sequences, (4.1.1), and what we have shown above prove the results for M. $\square$

The above theorem implies that the groups $H^i(W,M)$ determine the groups $H^i(G,M)$ up to isomorphism, since $H^1(W,M) \otimes \mathbf{Q}/\mathbf{Z}$ is an injective abelian group. The converse fails: take for example $M = \mathbf{Q}$ with trivial action. Then $H^i(G,\mathbf{Q}) = 0$ for $i \geq 1$, but $H^1(W,\mathbf{Q}) = \operatorname{Hom}(\mathfrak{w},\mathbf{Q}) = \mathbf{Q}$. Thus the groups $H^i(W,M)$ contain more information than their Galois counterparts.

Corollary 4.1.4. The strict cohomological dimension of G is 2.

Proof. Let M be a G-module. Since the orbit under G of any $m \in M$ is finite, we can write M as a direct limit of G-modules M_n which are finitely generated as abelian groups. Then by the above comparison theorem and (3.2.2), we have $H^3(G,M) = \varinjlim_n H^3(W,M_n) = 0$ $\square$

Let $\mathcal{A}/K$ be a commutative algebraic group scheme defined over K. Then $\mathcal{A}(\bar{K})$ is naturally a G-module, and $\mathcal{A}(\bar{L})$ is naturally a W-module. The inclusion map $\mathcal{A}(\bar{K}) \to \mathcal{A}(\bar{L})$ induces restriction maps $H^i(G,\mathcal{A}(\bar{K})) \to H^i(W,\mathcal{A}(\bar{L}))$. We have the following comparison theorem:

Theorem 4.1.5. Let $\mathcal{A}$ be a connected commutative algebraic group scheme over K. There are functorial isomorphisms:

(i) $H^0(G, \mathcal{A}(\bar{K})) = H^0(W, \mathcal{A}(\bar{L}))$ and

(ii) $H^1(G, \mathcal{A}(\bar{K})) = H^1(W, \mathcal{A}(\bar{L}))_{\mathrm{tors}}$,

(iii) there is a short exact sequence

$$0 \to H^1(W, \mathcal{A}(\bar{L})) \otimes \mathbf{Q}/\mathbf{Z} \to H^2(G, \mathcal{A}(\bar{K})) \to H^2(W, \mathcal{A}(\bar{L})) \to 0,$$

(iv) and the higher cohomology groups $H^i(G, \mathcal{A}(\bar{K}))$ and $H^i(W, \mathcal{A}(\bar{L}))$ vanish for $i \geq 3$.

Proof. First we will show that the multiplication-by-n maps $\mathcal{A} \to \mathcal{A}$ are all surjective. Since $\mathcal{A}$ is connected, we have a short exact sequence

$$0 \to H \to \mathcal{A} \to B \to 0$$

of algebraic groups, where H is linear and B is an abelian variety (see Theorem 1.1 of [2]). As H is commutative, it is the product of a torus and a commutative unipotent group. It follows that the multiplication-by-n maps on H and B are surjective, hence the same is true of $\mathcal{A}$ by the Five Lemma.

The n-torsion of $\mathcal{A}(\bar{L})$ is contained in $\mathcal{A}(\bar{K})$, hence there is a map of Kummer sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{A}[n] \arrow[r] \arrow[d, equal] & \mathcal{A}(\bar{K}) \arrow[r, "n"] \arrow[d] & \mathcal{A}(\bar{K}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{A}[n] \arrow[r] & \mathcal{A}(\bar{L}) \arrow[r, "n"] & \mathcal{A}(\bar{L}) \arrow[r] & 0.
\end{tikzcd}$$

The rest of the proof is exactly as in (4.1.3). $\square$

Corollary 4.1.6. There is a natural isomorphism $H^1(W, \bar{L}^\times) \otimes \mathbf{Q}/\mathbf{Z} = \mathrm{Br}(K)$, and thus $\mathrm{Br}(K) = \mathbf{Q}/\mathbf{Z}$.

Proof. The isomorphism is the map in the short exact sequence of (4.1.5), and the second statement follows from (3.1.1). $\square$

From now on, we will denote the groups $H^i(G, \mathcal{A}(\bar{K}))$ by $H^i(G, \mathcal{A})$, and the groups $H^i(W, \mathcal{A}(\bar{L}))$ by $H^i(W, \mathcal{A})$.

4.2 The Reciprocity Isomorphism

In the previous chapter, we proved that for a W-module M which is finitely generated as an abelian group, cup-product gave a natural isomorphism

$$R\Gamma_W(M^D) \xrightarrow{\sim} R\mathrm{Hom}(R\Gamma_W(M), \mathbf{Z}[-1]) \tag{4.2}$$

in the derived category of abelian groups. Suppose now that T/K is a torus with character group $M = \underline{\mathrm{Hom}}(T, \mathbf{G}_m)$. Then $T(\bar{L})$ can be identified with M^D as a W-module, and the duality theorem reads

$$R\Gamma_W(T) \xrightarrow{\sim} R\mathrm{Hom}(R\Gamma_W(M), \mathbf{Z}[-1]). \tag{4.3}$$

We can use our duality theorem to prove the following:

Proposition 4.2.1. Let T/K be a torus. Then $H^2(W, T) = 0$.

Proof. By the duality theorem, this group is isomorphic to $\mathrm{Ext}^2(R\Gamma_W(M), \mathbf{Z}[-1])$.

By (2.0.1), there is an isomorphism

$$\mathrm{Ext}(H^0(W,M),\mathbf{Z}) \to \mathrm{Ext}^2(R\Gamma_W(M),\mathbf{Z}[-1]), \tag{4.4}$$

but the group $\mathrm{Ext}(H^0(W,M),\mathbf{Z})$ vanishes because $H^0(W,M) \subseteq M$ is finitely generated and free. $\square$

Corollary 4.2.2. Let T/K be a torus. Then there are natural isomorphisms

(i) $H^0(G,T) = H^0(W,T)$

(ii) $H^1(G,T) = H^1(W,T)_{\mathrm{tors}}$

(iii) $H^2(G,T) = H^1(W,T) \otimes \mathbf{Q}/\mathbf{Z}$

Proof. This follows immediately from (4.1.5) and (4.2.1). $\square$

Corollary 4.2.3. (Tate-Nakayama Duality) Let T/K be a torus with character group M. Then the map

$$H^i(G,T) \to H^{2-i}(G,M)^* \tag{4.5}$$

induced by cup-product is an isomorphism for $i = 1, 2$, and an isomorphism for $i = 0$ upon passing to the profinite completion of the left-hand side.

Proof. Using (3.1) it is easy to see that the group $H^1(W,M)$ is finitely generated, hence $\mathrm{Ext}(H^1(W,M),\mathbf{Z}) \otimes \mathbf{Q}/\mathbf{Z} = 0$. Our duality theorem therefore gives an isomorphism

$$H^1(W,T) \otimes \mathbf{Q}/\mathbf{Z} \xrightarrow{\sim} \mathrm{Hom}(H^0(W,M),\mathbf{Z}) \otimes \mathbf{Q}/\mathbf{Z} = H^0(W,M)^* \tag{4.6}$$

which, by the previous corollary, can be identified with the map $H^2(G,T) \to H^0(G,M)^*$.

Now consider the case where $i = 1$. Our duality theorem gives an isomorphism

$$H^1(W,T)_{\mathrm{tors}} \xrightarrow{\sim} \mathrm{Ext}(H^1(W,M),\mathbf{Z}) = (H^1(W,M)_{\mathrm{tors}})^* \tag{4.7}$$

which, again by the previous corollary, can be identified with the map $H^1(G,T) \to H^1(G,M)^*$.

Finally we treat the case $i = 0$. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H^2(W,M),\mathbf{Z}) \arrow[r] \arrow[d, "="] & H^0(G,T) \arrow[r] \arrow[d] & \mathrm{Hom}(H^1(W,M),\mathbf{Z}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^2(W,M)^* \arrow[r] & H^2(G,M)^* \arrow[r] & (H^1(W,M) \otimes \mathbf{Q}/\mathbf{Z})^* \arrow[r] & 0
\end{tikzcd} \tag{4.8}$$

where the top row comes from the identity $H^0(G,T) = H^0(W,T)$, our duality theorem, and (2.0.1). The bottom row is the dual of the exact sequence of (4.1.3). The left vertical arrow is the identity since $H^2(W,M)$ is all torsion, as can be seen easily from (3.1). The right vertical arrow is an isomorphism upon passing to the profinite completion of $\mathrm{Hom}(H^1(W,M),\mathbf{Z})$, hence the same is true of the middle vertical arrow. $\square$

Of course, for $T = \mathbf{G}_m$ and $i = 0$, one recovers from Tate-Nakayama Duality the reciprocity isomorphism $K^\times \otimes \hat{\mathbf{Z}} \xrightarrow{\sim} G^{ab}$ of Local Class Field Theory, where $- \otimes \hat{\mathbf{Z}}$ denotes profinite completion. Hence one can recover the main statements of Local Class Field Theory by studying the cohomology of the Weil group.

Chapter 5

The Weil-smooth Topology on Schemes over K

For any arbitrary scheme Y, let us recall the definition of the smooth site Y_{sm}, as stated in [23]. The underlying category is the category of schemes which are smooth and locally of finite type over Y, and the coverings are the surjective families. In [23], van Hamel illustrates the utility of the smooth site in the study of duality theorems; the cohomology groups coincide with those familiar from the étale site, but the internal hom functor is better suited to proving duality results.

This chapter is devoted to introducing a variant of the Weil-étale topology, the *Weil-smooth* topology. This definition is inspired the definition of the Weil-étale topology given by Jiang in [8], and is related to the smooth topology in the same way that the Weil-étale topology is related to the étale topology. As with the smooth site, the internal hom functor on the Weil-smooth site is more appropriate for a functorial approach to duality results.

5.1 Definitions and Basic Properties

Throughout this section we fix a scheme X which is smooth and finite type over K.

Definition 5.1.1. Let $\pi_1 : X_L \to X$ and $\pi_2 : X_L \to \text{Spec } L$ be the projections. We define the *Weil-smooth* topology $W(X)$ to be the following Grothendieck topology:

(i) The objects of $W(X)$ are the schemes which are smooth and locally of finite type over X_L. That is, they are the objects of the smooth site of X_L.

(ii) A morphism $(V \xrightarrow{f} X_L) \to (Z \xrightarrow{g} X_L)$ of objects in $W(X)$, for connected V, is a map $\phi : V \to Z$ of schemes, such that (a) $\pi_1 \circ g \circ \phi = \pi_1 \circ f$, and (b) there exists $n \in \mathbf{Z}$ such that $\sigma^n \circ \pi_2 \circ f = \pi_2 \circ g \circ \phi$, where σ is the Frobenius automorphism of L. If V is not connected, we impose these conditions component-wise.

(iii) The coverings in $W(X)$ are the surjective families.

We let X_W denote X endowed with the Weil-smooth topology.

We recall some basic results on groups acting on sheaves. If Y is a scheme, G is a discrete group of automorphisms of Y, and $F \in \mathcal{S}(Y_{sm})$, we say that G *acts on* F if there are morphisms $F \to \tau_* F$ of sheaves for all $\tau \in G$, compatible in the obvious sense with the multiplication in G. We denote the category of sheaves on Y_{sm} which carry a G-action by $\mathcal{S}(Y_{sm})_G$.

Proposition 5.1.2. The category $\mathcal{S}(X_W)$ is equivalent to the category $\mathcal{S}(X_{L,sm})_{\mathfrak{w}}$

Proof. This is the same proof as the analogous result for Weil-étale sheaves on schemes over finite fields; see ([11], Proposition 2.2). We restate it here for the sake of completeness.

Let $F \in \mathcal{S}(X_W)$; then F defines a sheaf $\upsilon(F)$ on $X_{L,sm}$ by restricting to the smooth site. We must show that $\upsilon(F)$ carries an action of $\mathfrak{w}$. Let $U \to X_L$ be smooth and locally of finite type, and consider $U_\tau = U \times_{X_L} X_L$, where the map from X_L to itself is the map induced by τ. Applying $\upsilon(F)$ to the projection map

$U_\tau \to U$, we obtain a map $\upsilon(F)(U) \to \tau_* \upsilon(F)(U)$ which is functorial in U, and therefore defines a map of sheaves.

Conversely, suppose that $H \in \mathcal{S}(X_{L,sm})$ has a $\mathfrak{w}$-action. Let $\tau \in \mathfrak{w}$. A map $V \to Z$ of objects in the underlying category of X_W gives rise to a diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & Z \arrow[d] \\ X_L \arrow[r, "\tau"] & X_L. \end{tikzcd}$$

The universal property of the fiber product now gives a unique map $V \to Z_\tau$, which commutes with the maps to X_L. Applying H to this map, we get a map $H(Z) \to H(Z_\tau) \to H(V)$, where the first map comes from the $\mathfrak{w}$-action on H. Therefore H defines an object of $\mathcal{S}(X_W)$. $\square$

Let $\bar{X} = X \times_K \bar{L}$, and let $F \in \mathcal{S}(X_W) = \mathcal{S}(X_{L,sm})_{\mathfrak{w}}$. The sheaf F defines a sheaf on $\bar{X}_{sm}$ by pulling back along the map $\phi : \bar{X} \to X_L$; let us denote this pullback by the standard notation $\phi^* F$.

Proposition 5.1.3. The abelian group $\phi^* F(\bar{X})$ is naturally a W-module, and we have $H^0(W, \phi^* F(\bar{X})) = H^0(\mathfrak{w}, F(X_L))$.

Proof. It is clear that $\phi^* F(\bar{X})$ is an N-module; we must show that this action extends to all of W. Let $a \in W$ be a preimage of the Frobenius element σ of $\mathfrak{w}$. We have $\phi^* F(\bar{X}) = \varinjlim_{L'/L} F(X_{L'})$ where the limit ranges over all of the finite extensions L' of L.

For a fixed finite extension L'/L, let $N' \subseteq N$ be the open subgroup corresponding to L', and let $a' \in W/N'$ be the image of a. There is a short exact sequence

$$1 \to \mathrm{Gal}(L'/L) \to W/N' \to \mathfrak{w} \to 1$$

of groups, and the choice of a' determines a splitting $W/N' \simeq \mathrm{Gal}(L'/L) \rtimes \mathfrak{w}$. Because F is endowed with an action of $\mathfrak{w}$, the groups $\mathrm{Gal}(L'/L)$ and $\mathfrak{w}$ both act on $F(X_{L'})$, and it is easy to check these actions are compatible with the decomposition of W/N' as a semi-direct product. Therefore W/N' acts on $F(X_{L'})$, and passing to the limit over all L' shows that $\phi^*F(\bar{X})$ is a W-module. One can verify easily that a different choice of a gives an isomorphic W-module.

The second statement follows from the description of the W-module structure, and the fact that $H^0(\mathrm{Gal}(L'/L), F(X_{L'})) = F(X_L)$ for any finite extension L'/L. □

Definition 5.1.4. Let $F \in \mathcal{S}(X_W)$. Define the i^{th} *Weil-smooth cohomology group* of X with coefficients in F by setting $\Gamma_X(F) = H^0(X_W, F) = H^0(\mathfrak{w}, F(X_L))$, and letting $H^i(X_W, F)$ be the i^{th} right derived functor of $H^0(X_W, -)$ applied to F. If F is a complex of sheaves in $\mathcal{D}(X_W)$, we let $R\Gamma_X(F)$ denote the derived functor of Γ_X applied to F.

Theorem 5.1.5. There are spectral sequences

(i) $H^p(\mathfrak{w}, H^q(X_{L,sm}, F)) \Rightarrow H^{p+q}(X_W, F)$, and

(ii) $H^p(W, H^q(\bar{X}_{sm}, \phi^*F)) \Rightarrow H^{p+q}(X_W, F)$

for any $F \in \mathcal{S}(X_W)$.

Proof. To establish the first spectral sequence, note that we can factor Γ_X as $F \mapsto F(X_L) \mapsto H^0(\mathfrak{w}, F(X_L))$. The functor $F \mapsto F(X_L)$ preserves injectives, since it has as exact left adjoint the functor $\mathfrak{w}$-Mod $\to \mathcal{S}(X_{L,sm})_{\mathfrak{w}}$ which takes a $\mathfrak{w}$-module to the corresponding locally constant sheaf. Part (i) is now just the spectral sequence of composite functors.

To see that the second spectral sequence holds, note that for any sheaf $F \in \mathcal{S}(X_{L,sm})_{\mathfrak{w}}$, we have a spectral sequence

$$H^p(N, H^q(\bar{X}_{sm}, \phi^* F)) \Rightarrow H^{p+q}(X_{L,sm}, F)$$

of $\mathfrak{w}$-modules; see Theorem III.2.20 and Remark III.2.21(a) of [13]. In derived category language, we have an isomorphism $R\Gamma_N \circ R\Gamma_{\bar{X}}(\phi^* F) \simeq R\Gamma_{X_L}(F)$. Applying $R\Gamma_{\mathfrak{w}}$ to both sides of this isomorphism, we obtain

$$\begin{aligned} R\Gamma_W \circ R\Gamma_{\bar{X}}(\phi^* F) &\simeq R\Gamma_{\mathfrak{w}} \circ R\Gamma_N \circ R\Gamma_{\bar{X}}(\phi^* F) \\ &\simeq R\Gamma_{\mathfrak{w}} \circ R\Gamma_{X_L}(F) \\ &\simeq R\Gamma_X(F). \end{aligned}$$

The first isomorphism is simply the spectral sequence coming from the group extension $1 \to N \to W \to \mathfrak{w} \to 1$, and the third isomorphism is from part (i). The isomorphism $R\Gamma_X(F) \simeq R\Gamma_W \circ R\Gamma_{\bar{X}}(\phi^* F)$ defines the desired spectral sequence, as in ([24], Corollary 10.8.3). $\square$

Corollary 5.1.6. Suppose that $F \in \mathcal{S}(K_W)$ is given by a smooth commutative group scheme defined over K. Then $H^p(K_W, F) = H^p(W, F(\bar{L}))$.

Proof. This is immediate from the second spectral sequence of the previous theorem, and the fact that $H^q(\bar{L}_{sm}, F) = 0$ for all $q \geq 1$ and all such F. This last statement follows, for example, from the fact that smooth and étale cohomology agree for sheaves given by smooth commutative group schemes; see §1.2 of [23]. □

5.2 Smooth and Weil-Smooth Sheaves

Let $\rho : X_L \to X$ be the natural map. Following ([11], Proposition 2.4), we use ρ to define a pair of adjoint functors. For any $F \in \mathcal{S}(X_{sm})$, the sheaf $\rho^*F \in \mathcal{S}(X_L)$ carries a natural $\mathfrak{w}$-action, and so we define $\rho^* : \mathcal{S}(X_{sm}) \to \mathcal{S}(X_W)$ to be the standard pullback functor. We define $\rho_*^{\mathfrak{w}} : \mathcal{S}(X_W) \to \mathcal{S}(X)$ by the rule $(\rho_*^{\mathfrak{w}}G)(U) = H^0(\mathfrak{w}, G(U_L))$ for any $G \in \mathcal{S}(X_W)$.

Proposition 5.2.1. Let ρ^* and $\rho_*^{\mathfrak{w}}$ be as above. We have the following:

(i) ρ^* is left adjoint to $\rho_*^{\mathfrak{w}}$.

(ii) ρ^* is exact, and therefore $\rho_*^{\mathfrak{w}}$ preserves injectives.

(iii) If G/X is a smooth commutative group scheme which is locally of finite type, then there is natural isomorphism $\rho^*G \xrightarrow{\sim} G_L$ in $\mathcal{S}(X_W)$.

(iv) For any $G \in \mathcal{S}(X_{sm})$, there is a canonical map $G \to \rho_*^{\mathfrak{w}}\rho^*G$, which is an isomorphism when G is representable by a smooth commutative group scheme while is locally of finite type.

(v) For any smooth sheaf $F \in \mathcal{S}(X_{sm})$, there is a map of spectral sequences from

$$H^i(G, H^j(X_{\bar{K},sm}, F)) \Rightarrow H^{i+j}(X_{sm}, F)$$

to

$$H^i(W, H^j(\bar{X}, \phi^*\rho^*F)) \Rightarrow H^{i+j}(X_W, \rho^*F).$$

Proof. Part (i) is proved in the usual manner, and part (ii) holds because pullback is always exact.

To see part (iii), we imitate the proof of ([13] Chapter II, Remark 3.1(d)). Let G/X be a smooth commutative group scheme, which we identify with the sheaf it defines on X_{sm}, and let $F \in \mathcal{S}(X_W)$ be any Weil-smooth sheaf. By definition of $\rho_*^{\mathfrak{w}}$ we have $H^0(\mathfrak{w}, F(G_L)) \xrightarrow{\sim} (\rho_*^{\mathfrak{w}}F)(G)$. By basic properties of representable sheaves, this implies that

$$\mathrm{Hom}_{X_W}(G_L, F) \xrightarrow{\sim} \mathrm{Hom}_{X_{sm}}(G, \rho_*^{\mathfrak{w}}F)$$

and the result follows by uniqueness of adjoints.

The map described in part (iv) is the map induced by the adjunction map $G \to \rho_*\rho^*G$, the image of which lands in the subsheaf of $\mathfrak{w}$-invariants, which is exactly $\rho_*^{\mathfrak{w}}\rho^*G$. When G is given by a smooth commutative group scheme, we have by part (iii) that $\rho^*G = G_L$, and thus the map is the natural map $G(U) \to H^0(\mathfrak{w}, G_L(U_L))$. It is easy to see that this induces an isomorphism of sheaves.

The map of spectral sequences in part (v) is simply the map induced by the inclusion $W \to G$ and the projection $\bar{X} \to X_{\bar{K}}$. □

The functor ρ^* extends naturally to a functor $\rho^* : \mathcal{D}(X_{sm}) \to \mathcal{D}(X_W)$ between the corresponding derived categories. If $F \in \mathcal{D}(X_{sm})$ is a complex such that $H^i(F)$ is representable by some smooth commutative group scheme G^i/X for all i, then by the exactness of ρ^* we have $H^i(\rho^* F) = \rho^* G^i = G^i_L$. If $F \in \mathcal{S}(X_{sm})$ is representable by a smooth commutative group scheme which is locally of finite type, then we will often simply write F instead of $\rho^* F$ for the corresponding Weil-smooth sheaf it defines.

5.3 Internal Hom and Pairings

Let X/K be a smooth scheme of finite type over K, and let $F, F' \in \mathcal{S}(X_W)$. We define the Weil-smooth sheaf hom by $\underline{\mathrm{Hom}}(F, F') = \underline{\mathrm{Hom}}_{X_W}(F, F') = \underline{\mathrm{Hom}}_{X_{L,sm}}(F, F')$, which carries a natural $\mathfrak{w}$-action. The functor $\underline{\mathrm{Hom}}(F, -)$ is left exact, and we denote by $R\underline{\mathrm{Hom}}(F, -)$ its derived functor.

Lemma 5.3.1. For any $F, G \in \mathcal{D}(K_{sm})$, there is a canonical map

$$\Phi(F, G) : \rho^* R\underline{\mathrm{Hom}}_{K_{sm}}(F, G) \to R\underline{\mathrm{Hom}}(\rho^* F, \rho^* G) \tag{5.1}$$

in $\mathcal{D}(K_W)$. If $G = \mathbf{G}_m$ and F is a torus, an abelian variety, or a free finitely generated group scheme, then $\Phi(F, G)$ is an isomorphism.

Proof. By standard adjointness properties (see [24], Chapter 10.7.1), we have identifications

$$\begin{aligned} & \mathrm{Hom}_{\mathcal{D}(K_W)}(\rho^* R\underline{\mathrm{Hom}}_{K_{sm}}(F, G), R\underline{\mathrm{Hom}}(\rho^* F, \rho^* G)) \\ = \ & \mathrm{Hom}_{\mathcal{D}(K_{sm})}(R\underline{\mathrm{Hom}}_{K_{sm}}(F, G), \rho^{\mathfrak{w}}_* R\underline{\mathrm{Hom}}(\rho^* F, \rho^* G)) \\ = \ & \mathrm{Hom}_{\mathcal{D}(K_{sm})}(R\underline{\mathrm{Hom}}_{K_{sm}}(F, G), R\underline{\mathrm{Hom}}_{K_{sm}}(F, \rho^{\mathfrak{w}}_* \rho^* G)). \end{aligned}$$

We define $\Phi(F, G)$ to be the map induced by the canonical map $G \to \rho_*^{\mathfrak{w}} \rho^* G$ of (5.2.1 (iv)), which is the identity map when $G = \mathbf{G}_m$.

Now set $G = \mathbf{G}_m$, and suppose that $F = M$ is a free finitely generated commutative group scheme. In this case $R\underline{\mathrm{Hom}}_{K_{sm}}(M, \mathbf{G}_m) = \underline{\mathrm{Hom}}_{K_{sm}}(M, \mathbf{G}_m) = T$ is a torus (see [23], Corollary 1.4). Thus $\Phi(M, \mathbf{G}_m)$ is the natural map

$$\Phi(M, \mathbf{G}_m) : T_L \to \underline{\mathrm{Hom}}_{K_W}(M_L, \mathbf{G}_{m,L})$$

which is clearly an isomorphism. For $F = T$ a torus, the same argument, with the roles of T and M reversed, shows that $\Phi(F, \mathbf{G}_m)$ is an isomorphism.

For $F = A$ an abelian variety we have $R\underline{\mathrm{Hom}}_{K_{sm}}(A, \mathbf{G}_m) = \underline{\mathrm{Ext}}^1_{K_{sm}}(A, \mathbf{G}_m) = A^t[-1]$ (see [23], Corollary 1.4), where A^t is the dual abelian variety of A. The map $\Phi(A, \mathbf{G}_m)$ now reads

$$\Phi(A, \mathbf{G}_m) : A^t_L[-1] \to \underline{\mathrm{Ext}}^1_{K_W}(A_L, \mathbf{G}_{m,L})$$

which is an isomorphism by the compatibility of the Barsotti-Weil formula with base change. $\square$

If $F \in \mathcal{D}(K_W)$ is any bounded complex of sheaves, we define its *Cartier Dual* by $F^D := R\underline{\mathrm{Hom}}(F, \mathbf{G}_m)$. If $G \in \mathcal{D}(K_{sm})$ we define $G^{D_{sm}} := R\underline{\mathrm{Hom}}_{K_{sm}}(G, \mathbf{G}_m)$. The previous proposition essentially says that if we restrict ourselves to tori and their cocharacter groups, and abelian varieties, we have $\rho^*((-)^{D_{sm}}) = (\rho^*(-))^D$.

Proposition 5.3.2. Let T be a torus over K with cocharacter group M, and A an abelian variety over K with dual abelian variety A^t. Then we have the following natural isomorphisms in $\mathcal{D}(K_W)$:

$$\begin{aligned} M^D &\simeq T && (5.2) \\ T^D &\simeq M && (5.3) \\ A^D &\simeq A^t[-1]. && (5.4) \end{aligned}$$

Proof. By Corollary 1.4 of [23], the isomorphisms we are trying to demonstrate hold in $\mathcal{D}(K_{sm})$. Applying ρ^* to van Hamel's isomorphisms and using (5.3.1), we arrive at the corresponding isomorphisms in $\mathcal{D}(K_W)$. □

Let us return now to an arbitrary smooth scheme X/K of finite type over K, and let $F \in \mathcal{D}(X_W)$. There is a Yoneda pairing

$$F' \otimes^L R\underline{\mathrm{Hom}}(F', F) \to F \qquad (5.5)$$

for any $F' \in \mathcal{D}(X_W)$. Suppose that $H^n(X_W, F) \neq 0$, but $H^m(X_W, F)$ vanishes for all $m > n$. Then by applying $R\Gamma_X(-)$ and projecting, we arrive at a pairing

$$R\Gamma_X(F') \otimes^L R\Gamma_X(R\underline{\mathrm{Hom}}(F', F)) \to H^n(X_W, F)[-n] \qquad (5.6)$$

in $\mathcal{D}(\mathbf{Z})$, which we will also call the Yoneda pairing. If $X = \mathrm{Spec}\ K$, $F' = T$ is a torus with cocharacter group M, and $F = \mathbf{G}_m$, then by the above proposition we arrive at the pairing $R\Gamma_K(T) \otimes^L R\Gamma_K(M) \to \mathbf{Z}[-1]$ of (3.3.1).

Proposition 5.3.3. Let $\pi : X \to K$ be a smooth, projective curve over a p-adic field K. Then in $\mathcal{D}(K_W)$, there is a canonical isomorphism

$$R\underline{\mathrm{Hom}}(R\pi_*\mathbf{G}_m, \mathbf{G}_m[-1]) \xrightarrow{\sim} R\pi_*\mathbf{G}_m \tag{5.7}$$

which induces a pairing

$$R\pi_*\mathbf{G}_m \otimes^L R\pi_*\mathbf{G}_m \to \mathbf{G}_m[-1]. \tag{5.8}$$

Proof. Quite generally, Let S be a scheme, and let $\pi : X \to S$ be a smooth, proper curve over S. Deligne, in [3], has constructed an isomorphism

$$\tau_{\leq 1}R\underline{\mathrm{Hom}}(\tau_{\leq 1}R\pi_*\mathbf{G}_m[1], \mathbf{G}_m) \to \tau_{\leq 1}R\pi_*\mathbf{G}_m \tag{5.9}$$

of sheaves on S_{fppf}. Let us make this isomorphism explicit when S is the spectrum of a field F of characteristic zero. As noted in [23], in this case (5.9) holds even on the smooth site of S.

Let $F(X)$ denote the function field of X. Then in $\mathcal{D}(X_{sm})$, the complex $\mathbf{G}_m[1]$ is isomorphic to the complex $F(X)^\times[1] \to \mathrm{Div}_X$ where the map takes a function to its divisor. Applying $R\pi_*$ to this complex, we see that $\pi_*\mathbf{G}_m[1] = \mathbf{G}_m[1]$ and $R^1\pi_*\mathbf{G}_m[1] = \mathrm{Pic}_X$, where by Pic_X we mean the sheaf on F_{sm} defined by the Picard scheme of X/F.

For U smooth over F, let $\mathbf{Z}[X(U)]$ be the free abelian group on the set of morphisms from U to X over F, and let $\mathbf{Z}^X$ be the sheaf on F_{sm} associated to

$U \mapsto \mathbf{Z}[X(U)]$. There is a map $\mathbf{Z}^X \to R\pi_*\mathbf{G}_m[1]$ in $\mathcal{D}(F_{sm})$, given by taking a morphism $U \to X$ to its divisor in $\mathrm{Div}(X \times_F U)$. Applying $R\underline{\mathrm{Hom}}_{F_{sm}}(-, \mathbf{G}_m)$ to this map, we arrive at Deligne's isomorphism

$$R\underline{\mathrm{Hom}}_{F_{sm}}(R\pi_*\mathbf{G}_m[1], \mathbf{G}_m) \xrightarrow{\sim} R\underline{\mathrm{Hom}}_{F_{sm}}(\mathbf{Z}^X, \mathbf{G}_m) = R\pi_*\mathbf{G}_m, \tag{5.10}$$

which encodes the auto-duality of the Jacobian of X, and the duality between the sheaves $\mathbf{Z}$ and $\mathbf{G}_m$. The identity $R\underline{\mathrm{Hom}}_{F_{sm}}(\mathbf{Z}^X, \mathbf{G}_m) = R\pi_*\mathbf{G}_m$ follows from Yoneda's Lemma (this is where we use the fact that X is smooth over S).

When $F = K$ Deligne's isomorphism reads $(R\pi_*\mathbf{G}_m[1])^{D_{sm}} \xrightarrow{\sim} R\pi_*\mathbf{G}_m$. The cohomology sheaves of the complex $R\pi_*\mathbf{G}_m$ are all free finitely generated group schemes, tori, or abelian varieties, or extensions of such sheaves. Thus (5.3.1) and (5.3.2) imply that

$$(\rho^* R\pi_*\mathbf{G}_m)^D \simeq \rho^*(R\pi_*\mathbf{G}_m[1])^{D_{sm}} \simeq \rho^* R\pi_*\mathbf{G}_m \tag{5.11}$$

which is the desired canonical isomorphism. The pairing $R\pi_*\mathbf{G}_m \otimes^L R\pi_*\mathbf{G}_m \to \mathbf{G}_m[-1]$ is induced by standard adjoint properties and a degree shift. $\square$

On applying $R\Gamma_K(-)$ to each term in the pairing (5.3.3) and composing with the map $R\Gamma_K(\mathbf{G}_m[-1]) \to \mathbf{Z}[-2]$, we arrive at a pairing

$$R\Gamma_X(\mathbf{G}_m) \otimes^L R\Gamma_X(\mathbf{G}_m) \to \mathbf{Z}[-2] \tag{5.12}$$

in the derived category of abelian groups. We let

$$\lambda(X) : R\Gamma_X(\mathbf{G}_m) \to R\mathrm{Hom}(R\Gamma_X(\mathbf{G}_m), \mathbf{Z}[-2]) \tag{5.13}$$

be the induced map (by the symmetry of Deligne's pairing, both induced maps are the same). The main theorem of this chapter will describe to what extent $\lambda(X)$ is an isomorphism.

5.4 Cohomology of K with Abelian Variety Coefficients

Studying the map $\lambda(X)$ will require us to understand the cohomology of W acting on the $\bar{L}$-points of the Jacobian of X. Therefore, it will be useful to establish a Weil group analogue of Tate's duality theorem for abelian varieties over local fields, found in [22].

To that end, we devote this section to the pairing (5.6) when $F = \mathbf{G}_m$ and $F' = A$ is an abelian variety over K. Using (5.2) and shifting, we see that the Yoneda pairing induces a pairing

$$R\Gamma_K(A) \otimes^L R\Gamma_K(A^t) \to \mathbf{Z} \tag{5.14}$$

which is equal to the pairing induced by the biextension map $A \otimes^L A^t \to \mathbf{G}_m[1]$. We let $\tau(A)$ denote the induced map,

$$\tau(A) : R\Gamma_K(A^t) \to R\mathrm{Hom}(R\Gamma_K(A), \mathbf{Z}). \tag{5.15}$$

Our duality theorem for abelian varieties will describe to what extent $\tau(A)$ is an isomorphism. For simplicity, we will write $H^i(W, A)$ for $H^i(W, A(\bar{L}))$ and $H^i(G, A)$ for $H^i(G, A(\bar{K}))$.

Lemma 5.4.1. $H^i(W, A) = 0$ for $i \neq 0, 1$.

Proof. (see [21], Chapter II.§5.3, Proposition 16) The only non-trivial assertion is that $H^2(W, A) = 0$. By considering Kummer sequences, we see that $H^2(W, A[n]) \to H^2(W, A)[n]$ is surjective. Thus it suffices to show that the group $\varinjlim_n H^2(W, A[n])$ is zero. The Weil pairing gives us an identification $A^t[n] = \mathrm{Hom}(A[n], \mu)$ of W-modules. Weil-Tate Local Duality gives us isomorphisms

$$\begin{aligned} \varinjlim_n H^2(W, A[n]) &\simeq \varinjlim_n H^0(W, A^t[n])^* \\ &\simeq (\varprojlim_n H^0(W, A^t[n]))^* \end{aligned}$$

but this last group vanishes, since $A^t(K)_{\mathrm{tors}}$ is finite. $\square$

Lemma 5.4.2. Let Y be any of the following groups: $\mathbf{Z}_p, \mathcal{O}_K, U_K, A(K)$. Then $\mathrm{Hom}(Y, \mathbf{Z})$ is zero.

Proof. First consider the case of $Y = \mathbf{Z}_p$. Let $f : \mathbf{Z}_p \to \mathbf{Z}$ be a non-zero homomorphism; since the only non-trivial subgroups of $\mathbf{Z}$ are isomorphic to $\mathbf{Z}$, we may assume f is surjective. Composing with the surjection $\mathbf{Z} \to \mathbf{Z}/p^n\mathbf{Z}$, we see that f induces a surjection $\mathbf{Z}_p \to \mathbf{Z}/p^n\mathbf{Z}$. This latter map must factor through $\mathbf{Z}_p/p^n\mathbf{Z}_p$, and therefore f induces a surjection $\mathbf{Z}_p/p^n\mathbf{Z}_p \to \mathbf{Z}/p^n\mathbf{Z}$. As these two finite groups have the same order, this is an isomorphism. It follows that $\ker(f) \subseteq \bigcap_n p^n\mathbf{Z}_p = 0$, and hence f is injective. Thus f is an isomorphism, which is a contradiction.

Now suppose that $Y = \mathcal{O}_K$. The vanishing of $\mathrm{Hom}(\mathcal{O}_K, \mathbf{Z})$ follows immediately from the fact that $\mathcal{O}_K$ is a free $\mathbf{Z}_p$-module of rank equal to $[K : \mathbf{Q}_p]$. The result for $Y = U_K$ follows from the fact that U_K contains a subgroup of finite index isomorphic to $\mathcal{O}_K$ as abstract abelian groups. Similarly, $A(K)$ contains a finite index subgroup isomorphic to $\dim A$ copies of $\mathcal{O}_K$. $\square$

Lemma 5.4.3. The restriction map $R\Gamma_G(A) \to R\Gamma_W(A)$ is an isomorphism in $\mathcal{D}(\mathbf{Z})$.

Proof. We must show that the maps $H^i(G, A) \to H^i(W, A)$ are isomorphisms for all $i \geq 0$. In light of (4.1.5), we only need to show that $H^1(W, A)$ is torsion. Recall from (3.1) that the group $H^1(W, A)$ fits into the exact sequence

$$0 \to H^1(\mathfrak{w}, A(L)) \to H^1(W, A) \to H^0(\mathfrak{w}, H^1(N, A)) \to 0.$$

The group on the right is torsion, because it is a subgroup of a Galois cohomology group. Thus we are reducing to showing that $H^1(\mathfrak{w}, A(L))$ is torsion.

Let $\mathcal{A}/\mathcal{O}_L$ be the Néron model for A over the ring of integers of L ($\mathcal{A}$ is the base change to $\mathcal{O}_L$ of the Néron model for A/K; see [1], Theorem 7.2.1 and Corollary 2.). Let $\mathcal{A}_0 \subseteq \mathcal{A}$ be the subscheme whose special fiber is the identity component of the special fiber of $\mathcal{A}$, and whose generic fiber is A_L. We have an exact sequence of $\mathfrak{w}$-modules,

$$0 \to \mathcal{A}_0(\mathcal{O}_L) \to \mathcal{A}(\mathcal{O}_L) \to \pi_0(\bar{k}) \to 0,$$

where $\pi_0(\bar{k})$ is the group of connected components of the special fiber of $\mathcal{A}$. This sequence is exact by Hensel's Lemma; see ([14], proof of Proposition I.3.8). Taking cohomology gives a short exact sequence

$$H^1(\mathfrak{w}, \mathcal{A}_0(\mathcal{O}_L)) \to H^1(\mathfrak{w}, \mathcal{A}(\mathcal{O}_L)) \to H^1(\mathfrak{w}, \pi_0(\bar{k})) \to 0,$$

and it follows from Proposition 3 of [6] that $H^1(\mathfrak{w}, \mathcal{A}_0(\mathcal{O}_L)) = 0$. Since $\mathcal{A}(\mathcal{O}_L) = A(L)$, we have that $H^1(\mathfrak{w}, A(L)) = H^1(\mathfrak{w}, \pi_0(\bar{k}))$, which is finite. $\square$

Theorem 5.4.4. The map $\tau(A)$ of (5.15) has the following properties:

(i) $\tau(A)^0 : A^t(K) \to \mathrm{Ext}(H^1(W, A), \mathbf{Z}) = H^1(W, A)^*$ is an isomorphism of profinite groups.

(ii) $\tau(A)^1 : H^1(W, A^t) \to \mathrm{Ext}(A(K), \mathbf{Z})$ induces an isomorphism of $H^1(W, A^t)$ with the torsion subgroup $A(K)^*$ of $\mathrm{Ext}(A(K), \mathbf{Z})$.

The cohomology of both complexes vanishes outside of degrees 0 and 1. In particular, $\tau(A)^i$ is injective for all i.

Proof. By (2.0.1) and (5.4.2), the maps $\tau(A)^i$ reduce to maps

$$\tau(A)^i : H^i(W, A^t) \to \mathrm{Ext}(H^{1-i}(W, A), \mathbf{Z}). \tag{5.16}$$

The group $H^1(W, A)$ is torsion, hence has no non-zero maps to $\mathbf{Q}$. Part (i) of the theorem now follows from (5.4.3) and Tate's duality theorem on abelian varieties over local fields (see the main theorem of [22]).

The profinite group $A(K)$ admits no continuous maps to $\mathbf{Q}$, so $A(K)^*$ injects into $\mathrm{Ext}(A(K), \mathbf{Z})$. Part (ii) of the theorem now follows again from (5.4.3) and Tate's theorem. $\square$

5.5 Duality for Weil-smooth Cohomology of Curves

Before stating our Weil-smooth duality theorem for curves, we would like to remind the reader of Lichtenbaum's duality theorem for curves over p-adic fields, and van Hamel's approach to its construction and proof of non-degeneracy. As always, let $\pi : X \to K$ be a smooth, projective, geometrically connected curve.

Theorem 5.5.1. (Lichtenbaum, [9]) There are natural pairings

$$H^i(X_{sm}, \mathbf{G}_m) \otimes H^{3-i}(X_{sm}, \mathbf{G}_m) \to \mathbf{Q}/\mathbf{Z}$$

which induce isomorphisms $H^i(X_{sm}, \mathbf{G}_m) \otimes \hat{\mathbf{Z}} \to H^{3-i}(X_{sm}, \mathbf{G}_m)^*$ for all i, where $H^i(X_{sm}, \mathbf{G}_m)$ has the natural topology coming from that on K.

Lichtenbaum defines his pairing by explicitly evaluating representatives of the Brauer group on divisor classes. Since our objects live in the derived category where the notion of "element" does not make sense, van Hamel's more functorial approach adapts better to the Weil-smooth situation.

Let $F = R\pi_* \mathbf{G}_m \in \mathcal{D}(X_{sm})$, so that $R\Gamma_{K_{sm}}(R\pi_* \mathbf{G}_m) = R\Gamma_{X_{sm}}(\mathbf{G}_m)$. In [23], van Hamel's approach to Lichtenbaum's duality theorem is to put an "ascending filtration" on F. That is, he defines complexes F_i and constructs a series of morphisms $0 \to F_0 \to F_1 \to F_2 = F$ in $\mathcal{D}(K_{sm})$. For each $i \geq 0$, van Hamel defines the i^{th} graded piece G_i to be the mapping cone of $F_{i-1} \to F_i$, yielding an exact triangle $F_{i-1} \to F_i \to G_i \to F_{i-1}[1]$ in $\mathcal{D}(K_{sm})$.

The sheaf F_0 is defined by $F_0 := H^0(F) = \mathbf{G}_m$, and F_1 is defined to be the mapping cone of the composite $F \to \mathrm{Pic}_X[-1] \xrightarrow{\deg} \mathbf{Z}[-1]$. Thus there are triangles

$$F_1 \to F \to \mathbf{Z}[-1] \to F_1[1] \quad \text{and} \quad \mathbf{G}_m \to F_1 \to \mathrm{Pic}^0_X[-1] \to \mathbf{G}_m[1]$$

in $\mathcal{D}(K_{sm})$. The graded pieces G_i are the given by

$$G_i = \begin{cases} \mathbf{G}_m & i = 0 \\ \mathrm{Pic}^0_X[-1] & i = 1 \\ \mathbf{Z}[-1] & i = 2 \\ 0 & i \geq 3. \end{cases} \tag{5.17}$$

The filtration on F induces a "descending filtration" $F^{D_{sm}} = F_2^{D_{sm}} \to F_1^{D_{sm}} \to F_0^{D_{sm}} \to 0$ on $F^{D_{sm}}$, and also induces triangles $G_i^{D_{sm}} \to F_i^{D_{sm}} \to F_{i-1}^{D_{sm}} \to G_i^{D_{sm}}[1]$ for all i. The sheaves $G_i^{D_{sm}}$ are given by

$$G_i^{D_{sm}} = \begin{cases} \mathbf{Z} & i = 0 \\ \mathrm{Alb}_X & i = 1 \\ \mathbf{G}_m[1] & i = 2 \\ 0 & i \geq 3. \end{cases} \tag{5.18}$$

To prove the non-degeneracy and perfectness results of the pairing, van Hamel then uses duality theorems for finitely generated group schemes, tori, and abelian varieties to analyze the pairings

$$R\Gamma_{K_{sm}}(G_i) \otimes^L R\Gamma_{K_{sm}}(G_i^{D_{sm}}) \to \mathbf{Q}/\mathbf{Z}[-2],$$

and pieces together a duality theorem for $R\Gamma_{K_{sm}}(F) = R\Gamma_{X_{sm}}(\mathbf{G}_m)$ using the Five Lemma. We will essentially copy this approach, by applying ρ^* to van Hamel's filtration and exact triangles.

We can now state and prove our duality theorem for the Weil-smooth cohomology of curves.

Theorem 5.5.2. Let X/K be a smooth, projective, geometrically connected curve over K, such that $X(K) \neq \emptyset$. The map

$$\lambda(X) : R\Gamma_X(\mathbf{G}_m) \to R\mathrm{Hom}(R\Gamma_X(\mathbf{G}_m), \mathbf{Z}[-2]) \tag{5.19}$$

induced by the pairing (5.12) has the following properties:

(i) $\lambda(X)^i$ is an isomorphism for $i = 0, 1$.

(ii) $\lambda(X)^i$ is injective for $i = 2, 3$.

The cohomology of both complexes vanishes outside of degrees 0 through 3.

Proof. As above, let F be the complex $R\pi_*\mathbf{G}_m$ considered on the smooth site of K, so that

$$R\Gamma_X(\mathbf{G}_m) = R\Gamma_{K_W}(\rho^* R\pi_*\mathbf{G}_m)$$

Applying ρ^* to van Hamel's filtration provides us with a filtration $0 \to \rho^* F_0 \to \rho^* F_1 \to \rho^* F_2 = \rho^* F$ on F which comes equipped with exact triangles $\rho^* F_{i-1} \to \rho^* F_i \to \rho^* G_i \to \rho^* F_{i-1}[1]$.

Now we apply ρ^* to the dual filtration, and note that by (5.3.2), ρ^* commutes with the dualizing functors in the sense that $\rho^*(G_i^{D_{sm}}) = (\rho^* G_i)^D$ for all i. Repeatedly applying the Five Lemma and (5.3.1) to van Hamel's exact triangles shows that $\rho^*(F_i^D) = (\rho^* F_i)^D$ for all i. Now the Yoneda pairing induces pairings

$$R\Gamma_K(\rho^* G_i) \otimes^L R\Gamma_K((\rho^* G_i)^D) \to \mathbf{Z}[-1] \tag{5.20}$$

$$R\Gamma_K(\rho^* F_i) \otimes^L R\Gamma_K((\rho^* F_i)^D) \to \mathbf{Z}[-1] \tag{5.21}$$

in $\mathcal{D}(\mathbf{Z})$ for all i. In a slight abuse of notation, we suppress ρ^* from now on.

Let

$$\gamma_i : R\Gamma_K(G_i^D) \to R\mathrm{Hom}(R\Gamma_K(G_i), \mathbf{Z}[-1]) \tag{5.22}$$

be the induced map in $\mathcal{D}(\mathbf{Z})$. We will describe the maps γ_i in terms of duality theorems we have already proven.

From (3.3.8) we see that $\gamma_0 = \eta(\mathbf{Z})$ has the following properties: γ_0^i is an isomorphism for $i \neq 2$, and γ_0^2 maps $H^2(W, \mathbf{Z})$ isomorphically onto the torsion subgroup U_K^* of $\mathrm{Ext}(K^\times, \mathbf{Z})$.

Let J_X be the Jacobian variety of X. Any rational point of X determines an embedding $X \hookrightarrow J_X$ defined over K, and thus a Weil-equivariant isomorphism $\mathrm{Pic}^0_X(\bar{L}) \to J_X(\bar{L})$. Hence we can identify $\mathrm{Pic}^0_X(\bar{L})$ with the $\bar{L}$-points of an abelian variety defined over K, and apply (5.4.4) to Pic^0_X and its dual abelian variety Alb_X.

From (5.4.4), we see that $\gamma_1 = \tau(\mathrm{Pic}^0_X)$ has the following properties: γ_1^0 is an isomorphism which maps $\mathrm{Alb}_X(K)$ isomorphically onto $H^1(W, \mathrm{Pic}^0_X)^*$, and γ_1^1 is an injective map which maps $H^1(W, \mathrm{Alb}_X)$ isomorphically onto the torsion subgroup $\mathrm{Pic}^0_X(K)^*$ of $\mathrm{Ext}(\mathrm{Pic}^0_X(K), \mathbf{Z})$. From (3.3.1) we see that $\gamma_2 = \psi(\mathbf{Z})[1]$ is an isomorphism.

Now consider the maps

$$\phi_i : R\Gamma_K(F_i^D) \to R\mathrm{Hom}(R\Gamma_K(F_i), \mathbf{Z}[-1]) \tag{5.23}$$

induced by the Yoneda pairing. We can determine to what extent the maps ϕ_i are isomorphisms, by using the triangles which defined G_i and G_i^D. When $i = 0$ one has $G_0 = F_0 = \mathbf{G}_m$, hence $\phi_0 = \gamma_0 = \eta(\mathbf{Z})$ is the map of (3.3.8). When $i = 1$ we have a diagram

$$\begin{tikzcd}
R\Gamma_K(\mathrm{Alb}_X) \arrow[r, "\gamma_1"] \arrow[d] & R\mathrm{Hom}(R\Gamma_K(\mathrm{Pic}^0_X[-1]), \mathbf{Z}[-1]) \arrow[d] \\
R\Gamma_K(F_1^D) \arrow[r, "\phi_1"] \arrow[d] & R\mathrm{Hom}(R\Gamma_K(F_1), \mathbf{Z}[-1]) \arrow[d] \\
R\Gamma_K(F_0^D) \arrow[r, "\phi_0"] & R\mathrm{Hom}(R\Gamma_K(F_0^D), \mathbf{Z}[-1]).
\end{tikzcd} \tag{5.24}$$

It follows that ϕ_1^0 is an isomorphism, ϕ_1^1 is injective, and ϕ_1^2 maps $H^2(K_W, F_1^D)$ isomorphically onto the torsion subgroup U_K^* of $\mathrm{Ext}^2(R\Gamma_K(F_1), \mathbf{Z}[-1])$. The cohomology of all of these complexes vanishes outside of degrees 0 through 2.

When $i = 2$ we have a diagram

$$\begin{tikzcd}
R\Gamma_K(\mathbf{G}_m[1]) \arrow[r, "\psi(\mathbf{Z})[1]", "\sim"'] \arrow[d] & R\mathrm{Hom}(R\Gamma_K(\mathbf{Z}[-1]), \mathbf{Z}[-1]) \arrow[d] \\
R\Gamma_K(F_2^D) \arrow[r, "\phi_2"] \arrow[d] & R\mathrm{Hom}(R\Gamma_K(F_2), \mathbf{Z}[-1]) \arrow[d] \\
R\Gamma_K(F_1^D) \arrow[r, "\phi_1"] & R\mathrm{Hom}(R\Gamma_K(F_1), \mathbf{Z}[-1]).
\end{tikzcd} \tag{5.25}$$

It follows that $\phi_2^{-1} = \psi(\mathbf{Z})^0$ is an isomorphism from $K^\times$ to $\mathrm{Hom}(R\Gamma_K(F_2), \mathbf{Z}[-1])$, that ϕ_2^0 is an isomorphism, that ϕ_2^1 is injective, and that ϕ_2^2 is an isomorphism from $H^2(K_W, F_2^D)$ to the torsion subgroup U_K^* of $\mathrm{Ext}^2(R\Gamma_K(F_2), \mathbf{Z}[-1])$.

The theorem is now clear, once we recall that $\lambda(X)$ is the map induced by the isomorphism $F_2[1]^D \to F_2$ of Weil-smooth sheaves, the map ϕ_2, and shifting degrees by one. □

One could speculate that there exists a natural way to "topologize" the complex $R\Gamma_X(\mathbf{G}_m)$ so that its cohomology groups inherit their natural p-adic topology, and a functor $R\mathrm{Hom}_{\mathrm{cont}}(-,-)$, such that the following "theorem" would be true: There exists a natural isomorphism

$$R\Gamma_X(\mathbf{G}_m) \xrightarrow{\sim} R\mathrm{Hom}_{\mathrm{cont}}(R\Gamma_X(\mathbf{G}_m), \mathbf{Z}[-2]), \tag{5.26}$$

inducing short exact sequences

$$0 \to \mathrm{Ext}^1_{\mathrm{cont}}(H^{3-i}(X_W, \mathbf{G}_m), \mathbf{Z}) \to H^i(X_W, \mathbf{G}_m) \to \mathrm{Hom}_{\mathrm{cont}}(H^{2-i}(X_W, \mathbf{G}_m), \mathbf{Z}) \to 0 \tag{5.27}$$

for all i.

While an isomorphism such as (5.26) seems out of reach, establishing (5.27) seems plausible using Yoneda-ext in the category of topological abelian groups, especially since $\mathrm{Hom}_{\mathrm{cont}}(H^i(X_W, \mathbf{G}_m), \mathbf{Z}) = \mathrm{Hom}(H^i(X_W, \mathbf{G}_m), \mathbf{Z})$ for all i. Furthermore, when $i = 0, 1$, $H^{3-i}(X_W, \mathbf{G}_m)$ has the discrete topology, and thus (5.27) holds.

5.6 Comparison with Smooth Cohomology

In this section we compare the duality theorem of the previous section with the main theorem of [9]. That the our pairing is compatible with the original pairing defined by Lichtenbaum follows from the results of §3.3 of [23].

Proposition 5.6.1. Suppose that X/K is a smooth, proper variety over K, and let F be a torsion sheaf in $\mathcal{S}(X_{sm})$. Then the restriction map $R\Gamma_{X_{sm}}(F) \to R\Gamma_X(F)$ is an isomorphism in $\mathcal{D}(\mathbf{Z})$.

Proof. By ([13], Chapter VI, Corollary 2.6), the map $R\Gamma_{X_{\bar{K},sm}}(F) \to R\Gamma_{\bar{X}_{sm}}(F)$ is an isomorphism. The result now follows from (4.1.1), since $R\Gamma_{X_{sm}} = R\Gamma_G \circ R\Gamma_{X_{\bar{K},sm}}$ and $R\Gamma_X = R\Gamma_W \circ R\Gamma_{\bar{X}_{sm}}$. □

By the previous proposition, smooth and Weil-smooth cohomology agree for the sheaf μ_n, hence we can use Kummer sequences to study the restriction maps

$H^i(X_{sm}, \mathbf{G}_m) \to H^i(X_W, \mathbf{G}_m)$. In particular there is a diagram

$$\begin{tikzcd}
0 \arrow[r] & H^i(X_{sm}, \mathbf{G}_m)/n \arrow[r, "\delta"] \arrow[d, "\mathrm{res}^i/n"] & H^{i+1}(X_{sm}, \mu_n) \arrow[r] \arrow[d, "\wr"] & H^{i+1}(X_{sm}, \mathbf{G}_m)[n] \arrow[r] \arrow[d, "\mathrm{res}^{i+1}[n]"] & 0 \\
0 \arrow[r] & H^i(X_W, \mathbf{G}_m)/n \arrow[r, "\delta"] & H^{i+1}(X_W, \mu_n) \arrow[r] & H^{i+1}(X_W, \mathbf{G}_m)[n] \arrow[r] & 0
\end{tikzcd} \tag{5.28}$$

from which we deduce an isomorphism $\delta_n^i : \ker(\mathrm{res}^{i+1}[n]) \to \mathrm{coker}(\mathrm{res}^i/n)$ for any pair of integers i, n. Passing to the limit over all n, we obtain a canonical isomorphism

$$\delta^i : \ker(\mathrm{res}^{i+1}|_{\mathrm{tors}}) \to \mathrm{coker}(\mathrm{res}^i \otimes 1) \tag{5.29}$$

where $\mathrm{res}^i \otimes 1$ is the obvious map $H^i(X_{sm}, \mathbf{G}_m) \otimes \mathbf{Q}/\mathbf{Z} \to H^i(X_W, \mathbf{G}_m) \otimes \mathbf{Q}/\mathbf{Z}$.

Proposition 5.6.2. Let X/K be a smooth, projective, geometrically connected curve over K such that $X(K) \neq \emptyset$. The restriction maps $\mathrm{res}^i : H^i(X_{sm}, \mathbf{G}_m) \to H^i(X_W, \mathbf{G}_m)$ are described by the following exact sequences:

$$0 \to H^1(X_{sm}, \mathbf{G}_m) \stackrel{\mathrm{res}^1}{\to} H^1(X_W, \mathbf{G}_m) \to H^1(W, \mathbf{G}_m) \to 0 \tag{5.30}$$

$$0 \to \mathrm{Br}(K) \to H^2(X_{sm}, \mathbf{G}_m) \stackrel{\mathrm{res}^2}{\to} H^1(W, \mathrm{Pic}^0_X) \to 0 \tag{5.31}$$

$$0 \to H^1(W, \mathbf{Z}) \otimes \mathbf{Q}/\mathbf{Z} \to H^3(X_{sm}, \mathbf{G}_m) \stackrel{\mathrm{res}^3}{\to} H^3(X_W, \mathbf{G}_m) \to 0. \tag{5.32}$$

Proof. The map of Hochschild-Serre spectral sequences computing smooth and Weil-smooth cohomology gives us a map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & H^1(X_{sm}, \mathbf{G}_m) \arrow[r, "\sim"] \arrow[d, "\mathrm{res}^1"] & H^0(G, \mathrm{Pic}_X) \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & H^1(W, \mathbf{G}_m) \arrow[r] & H^1(X_W, \mathbf{G}_m) \arrow[r] & H^0(W, \mathrm{Pic}_X) \arrow[r] & 0
\end{tikzcd} \tag{5.33}$$

coming from the long exact sequences of low degree. That the non-zero map in the top row is an isomorphism follows from $X(K) \neq \emptyset$. The existence of (5.30) follows by applying the Snake Lemma.

To prove the second existence of the second exact sequence, note that $H^2(X_{sm}, \mathbf{G}_m)$ is a torsion group, so $\ker(\mathrm{res}^2_{\mathrm{tors}}) = \ker(\mathrm{res}^2)$. We will show that there is a natural identification $\ker(\mathrm{res}^2) = \mathrm{Br}(K)$. The long exact sequences of low degree from the Hochschild-Serre spectral sequences give us a map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Br}(K) \arrow[r] \arrow[d] & H^2(X_{sm}, \mathbf{G}_m) \arrow[r] \arrow[d, "\mathrm{res}^2"] & H^1(G, \mathrm{Pic}_X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] & H^2(X_W, \mathbf{G}_m) \arrow[r, "\sim"] & H^1(W, \mathrm{Pic}_X) \arrow[r] & 0.
\end{tikzcd} \tag{5.34}$$

It follows from (4.1.3) and (4.1.5) that $H^1(G, \mathrm{Pic}_X)$ can be identified with the torsion subgroup of $H^1(W, \mathrm{Pic}_X)$ via the restriction map.

On the other hand, consider the long exact sequence in W-cohomology of $0 \to \mathrm{Pic}^0_X \to \mathrm{Pic}_X \to \mathbf{Z} \to 0$. Since $X(K) \neq \emptyset$, any rational point determines a Weil-equivariant degree 1 divisor class on $\bar{X}$, hence the map $\deg : H^0(W, \mathrm{Pic}_X) \to \mathbf{Z}$

is surjective. The relevant part of the long exact sequence now reads

$$H^0(W, \mathrm{Pic}_X) \xrightarrow{\deg} \mathbf{Z} \xrightarrow{0} H^1(W, \mathrm{Pic}^0_X) \to H^1(W, \mathrm{Pic}_X) \to \mathbf{Z} \to 0, \tag{5.35}$$

and we have an identification $H^1(W, \mathrm{Pic}^0_X) = H^1(W, \mathrm{Pic}_X)_{\mathrm{tors}}$. That (5.30) and (5.31) are exact is now clear.

The only remaining task is to identify the kernel of res^3. By (5.29) we have an identification $\ker(\mathrm{res}^3) = \mathrm{coker}(\mathrm{res}^2 \otimes 1)$. But as $H^2(X_{sm}, \mathbf{G}_m)$ is torsion, this last cokernel can be identified with $H^2(X_W, \mathbf{G}_m) \otimes \mathbf{Q}/\mathbf{Z} = H^1(W, \mathrm{Pic}_X) \otimes \mathbf{Q}/\mathbf{Z}$. The map on cohomology induced by the degree map gives an isomorphism of this last group with $H^1(W, \mathbf{Z}) \otimes \mathbf{Q}/\mathbf{Z}$, since $H^1(W, \mathrm{Pic}^0_X) \otimes \mathbf{Q}/\mathbf{Z} = 0$. □

With the above comparison theorem, we can reprove the main result of [9] for curves X/K containing a rational point.

Theorem 5.6.3. Suppose that X/K is a smooth, projective, geometrically connected curve, such that $X(K) \neq \emptyset$. Then the Lichtenbaum pairing $H^2(X_{sm}, \mathbf{G}_m) \otimes H^1(X_{sm}, \mathbf{G}_m) \to \mathbf{Q}/\mathbf{Z}$ induces an isomorphism $H^2(X_{sm}, \mathbf{G}_m) \to H^1(X_{sm}, \mathbf{G}_m)^*$.

Proof. The map induced by the Lichtenbaum pairing fits into the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Br}(K) \arrow[r] \arrow[d, "\wr"] & H^2(X_{sm}, \mathbf{G}_m) \arrow[r] \arrow[d] & H^1(W, \mathrm{Pic}^0_X) \arrow[r] \arrow[d, "\wr"] & 0 \\ 0 \arrow[r] & \mathbf{Z}^* \arrow[r] & H^1(X_{sm}, \mathbf{G}_m)^* \arrow[r] & \mathrm{Pic}^0_X(K)^* \arrow[r] & 0 \end{tikzcd} \tag{5.36}$$

where the top row is the exact sequence of (5.31). The result follows by the Five Lemma. □

Bibliography

[1] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron Models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990.

[2] Brian Conrad. A Modern Proof of Chevalley's Theorem on Algebraic Groups. *J. Ramanujan Math. Soc.*, 17(1):1–18, 2002.

[3] Pierre Deligne. Exposé XVIII: La Formule de Dualité Globale. In SGA 4, volume 305 of *Lecture Notes in Math*, pages 481–587. Springer-Verlag, New York, 1973.

[4] Mathias Flach. Cohomology of Topological Groups with Applications to the Weil Group. *Compos. Math.*, 144(3):633–656, 2008.

[5] Mathias Flach and Baptiste Morin. On the Weil-étale Topos of Regular Arithmetic Schemes. `http://arxiv.org/abs/1010.3833`, 2010.

[6] Marvin J. Greenberg. Schemata over Local Rings. II. *Ann. of Math. (2)*, 78:256–266, 1963.

[7] Robin Hartshorne. *Algebraic Geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[8] Yongbin Jiang. *Weil-étale Topology over Local Rings*. ProQuest LLC, Ann Arbor, MI, May 2006. Thesis (Ph.D.) – Brown University.

[9] Stephen Lichtenbaum. Duality Theorems for Curves over p-adic Fields. *Invent. Math.*, 7:120–136, 1969.

[10] Stephen Lichtenbaum. Duality Theorems for the Cohomology of Weil Groups. 1999.

[11] Stephen Lichtenbaum. The Weil-étale Topology on Schemes over Finite Fields. *Compos. Math.*, 141(3):689–702, 2005.

[12] Stephen Lichtenbaum. The Weil-étale Topology for Number Rings. *Ann. of Math. (2)*, 170(2):657–683, 2009.

[13] J. S. Milne. *Etale Cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.

[14] J. S. Milne. *Arithmetic Duality Theorems*. BookSurge, LLC, Charleston, SC, second edition, 2006.

[15] Calvin C. Moore. Group Extensions and Cohomology for Locally Compact Groups. III. *Trans. Amer. Math. Soc.*, 221(1):1–33, 1976.

[16] Baptiste Morin. The Weil-étale Fundamental Group of a Number Field I. *Kyushu J. Math.*, 65(1):101–140, 2011.

[17] Baptiste Morin. The Weil-étale Fundamental Group of a Number Field II. *Selecta Math. (N.S.)*, 17(1):67–137, 2011.

[18] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of Number Fields*, volume 323 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2008.

[19] C. S. Rajan. On the Vanishing of the Measurable Schur Cohomology Groups of Weil Groups. *Compos. Math.*, 140(1):84–98, 2004.

[20] Jean-Pierre Serre. *Local Fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1979. Translated from the French by Marvin Jay Greenberg.

[21] Jean-Pierre Serre. *Galois Cohomology*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, english edition, 2002. Translated from the French by Patrick Ion and revised by the author.

[22] John Tate. WC-groups over $\mathfrak{p}$-adic Fields. In *Séminaire Bourbaki, Vol. 4*, pages Exp. No. 156, 265–277. Soc. Math. France, Paris, 1957.

[23] Joost van Hamel. Lichtenbaum-Tate Duality for Varieties over p-adic Fields. *J. Reine Angew. Math.*, 575:101–134, 2004.

[24] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge University Press, 1994.