ABSTRACT

Title of dissertation: GEOMETRIC AND TOPOLOGICAL ELLIPTICITY IN COHOMOGENEITY TWO

Joseph E. Yeager,
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Dissertation directed by: Karsten Grove
Department of Mathematics
Notre Dame University

Joel Cohen
Department of Mathematics
University of Maryland

Let M be a compact, connected and simply-connected Riemannian manifold, and suppose that G is a compact, connected Lie group acting on M by isometries. The dimension of the space of orbits is called the *cohomogeneity* of the action. If the direct sum of the higher homotopy groups of M, tensored with the field of rational numbers, is a finite-dimensional vector space over the rationals, then M is said to be *rationally elliptic*. It is known that M is rationally elliptic if it supports an action of cohomogeneity zero or one. When the cohomogeneity is two, this general result is no longer true. However, we prove that M is rationally elliptic in the two-dimensional case under the added assumption that M has nonnegative sectional curvature.

GEOMETRIC AND TOPOLOGICAL ELLIPTICITY IN COHOMOGENEITY TWO

by

Joseph E. Yeager

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Advisory Committee:
Professor Joel Cohen, Chair/Advisor
Professor Karsten Grove, Co-Advisor
Professor Stephen Halperin
Professor James Schafer
Professor Sylvester Gates, Jr.

To my wife Norma

with love and gratitude for her support and encouragement

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Chapter 1

Introduction

A simply-connected manifold M is said to be rationally elliptic if

$$\sum_{n=2}^{\infty} \dim(\pi_n(M) \otimes \mathbf{Q}) < \infty.$$

In the interest of brevity, we shall use the abbreviation elliptic for such manifolds. This definition is equivalent to the condition that the rational Betti numbers of the loop space ΩM are polynomially bounded. The simplest example of such a space is the sphere S^n for $n \geq 2$. It was shown by Serre [24] that compact Lie groups are elliptic. It is a straightforward consequence of the homotopy exact sequence of the fiber bundle $H \to G \to G/H$ that the same is true for their homogeneous spaces.

A manifold on which a group G acts is called a G-manifold. In this thesis we will consider a compact, connected Riemannian manifold M on which a compact, connected Lie group G acts as a group of isometries. The space of orbits of this G-action can be made into a metric space in a natural way and we will denote this space by M_G.

The dimension of the orbit space is called the *cohomogeneity* of the action. Manifolds having actions of cohomogeneity zero or one are known to be elliptic. In the first case, the manifolds are just the homogeneous spaces mentioned above. The cohomogeneity one case is more difficult and was proved in a paper of Grove and Halperin [11].

There is an important conjecture called the Ellipticity Conjecture that states that every simply connected manifold of (almost) nonnegative curvature is rationally elliptic [6]. In our work we will prove this conjecture for nonnegatively curved manifolds supporting a cohomogeneity two action.

Not all manifolds of cohomogeneity greater than one are elliptic. An example is the connected sum $(S^n \times S^n)\#(S^n \times S^n)$ which supports a cohomogeneity two isometric action by $SO(n) \times SO(n)$, and is known not to be rationally elliptic.

When the manifold has nonnegative curvature, the orbit space will also have nonnegative curvature in the sense of Alexandrov. We will determine all 2-dimensional orbit spaces that can support a metric having nonnegative sectional curvature and show, in each case, that the manifold M is elliptic. Specifically, we shall prove the following theorem.

Main Theorem. *Let M be a compact, connected and simply-connected Riemannian manifold having nonnegative sectional curvature. Let G be a compact, connected Lie group acting effectively on M as isometries, with orbit space M_G. If $dim(M_G) = 2$, then M is rationally elliptic.*

The proof of the main theorem utilizes two somewhat different techniques. Most of the orbit spaces considered can be given a metric of constant sectional curvature, either 0 or 1. These generate tilings of either the plane $\mathbf{R}^2$ or the sphere S^2, respectively. For these cases we will use the Morse theory, and a theorem of S. Halperin [14], to show that the Betti numbers of the loop space are polynomially bounded.

For the remaining cases, which are not amenable to this treatment, the result will follow almost directly from a theorem of K. Grove and S. Halperin in the paper referenced above.

In all that follows, we shall assume that M is a compact manifold without boundary, and that M is connected and simply-connected. G will always denote a compact and connected Lie group.

Chapter 2

Preliminaries

In this chapter we introduce the notations that we will employ, and briefly describe the basic tools from topology and geometry needed to develop our thesis. We will always assume that our manifold M and Lie group G are compact and connected and that $\pi_1(M) = 0$. Throughout this paper *curvature* always signifies sectional curvature, whenever the noun is used without a modifier.

Homotopy groups $\pi_n(X, x_0)$ are always computed with respect to a basepoint. Usually the basepoint is omitted from the notation for typographical convenience and we simply write $\pi_n(X)$. Of course $\pi_n(X)$ is abelian for $n \geq 2$.

2.1 Topological Preliminaries

We will now define the basic concepts, establish notations, and gather together the various topological tools that will be necessary in order to prove our main theorem.

2.1.1 Fibrations and the Homotopy Fiber

The following material is taken from Hatcher [15]. A map $p : E \to B$ is said to have the homotopy lifting property with respect to X if, whenever there is a homotopy $h_t : X \to B$ and a map $\tilde{h}_0 : X \to E$ lifting h_0, then there is an extension $\tilde{h}_t : X \to E$ lifting h_t. A map $p : E \to B$ is called a *fibration* if it has the homotopy lifting property with respect to any space. Fiber bundles over paracompact spaces are fibrations.

We assume in all that follows that B is a pointed space with basepoint b_0, and that E is pointed with basepoint $e_0 \in F = p^{-1}(b_0)$. The basic result on the homotopy groups of a fibration is the following theorem.

Theorem 2.1. *If $p : E \to B$ is a fibration, there is an exact sequence of homotopy groups*

$$... \to \pi_{k+1}(B) \to \pi_k(F) \to \pi_k(E) \to \pi_k(B) \to ... \to \pi_0(F) \to \pi_0(E) \to \pi_0(B).$$

This follows directly from the exact homotopy sequence of the pair (E, F) and the fact that $\pi_n(E, F) \cong \pi_n(B)$.

The results that we will need for our work are mostly direct consequences of the concept of the homotopy fiber of a mapping. If $f : E \to B$ is a mapping of topological spaces, f can always be expressed as a composition of a homotopy equivalence and a fibration. We define the space

$$E_f = \{(x, \gamma) \mid x \in E, \gamma : [0,1] \to B, \gamma(0) = f(x)\}$$

and the mapping $p : E_f \to B$ by $p(x, \gamma) = \gamma(1)$.

We denote the space of mappings of the interval [0,1] into B by B^I, and give B^I the compact-open topology. Then E_f is given the relative topology as a subset of $E \times B^I$. It is easy to see that the mapping p has the homotopy lifting property and is, therefore, a fibration. There is an inclusion $\tilde{f} : E \to E_f$ defined by $\tilde{f}(x) = (x, c_x)$, where c_x is the constant curve at $f(x)$. Shortening all curves to the interval $[0, 1-t]$ gives a deformation retraction of E_f onto E. Thus $\tilde{f}$ is a homotopy equivalence and $f = p \circ \tilde{f}$. If $b \in B$, the homotopy fiber at b is the space

$$F_b = \{(x, \gamma) \mid x \in E,\ \gamma : [0,1] \to B,\ \gamma(0) = f(x),\ \gamma(1) = b\}.$$

2.1.2 Rational Homotopy Theory

It is well known that the computation of the homotopy groups of a topological space X is a difficult, often impossible, task. By tensoring the homotopy groups with the field of rationals $\mathbf{Q}$, we remove the torsion elements from these groups and

considerably simplify their computation. As mentioned earlier, if the space X is the n-sphere S^n, only finitely many (at most two) of the rationalized groups are non-trivial, and the rationalized homotopy is finite dimensional. Topological spaces sharing this property with the n-sphere are called *rationally elliptic* or, for brevity, elliptic.

We will make the usual assumptions that X is simply-connected and path-connected. Then the ellipticity of X is equivalent to the condition that the rational homology of the loop space is polynomially bounded; i.e., there is a constant $A > 0$ such that

$$\sum_{i=0}^{n} \dim(H_i(\Omega X; \mathbf{Q}) \leq An^r,$$

for $n \geq 1$ and $r = \dim(\pi_{odd}(X) \otimes \mathbf{Q})$.

A short but enlightening discussion of these ideas can be found in the book by Bott and Tu [2]. Félix, Halperin and Thomas present a more complete treatment in their book [9]. We will not attempt to give the details here since this would require extensive preparation. It is a deep fact that the rational homotopy of a simply-connected space is completely described by a certain differential graded algebra (minimal Sullivan model [26]), whose generators correspond in a one-to-one fashion to the generators of the rational homotopy groups.

2.1.3 Two Fundamental Theorems

The theorems in this section are the basis for the main results of this paper. They are rather technical and we will require the following definitions in order to state them.

A space S is *nilpotent* if $\pi_1(S)$ is a nilpotent group that acts nilpotently on each $\pi_i(S),\ i \geq 2$. In particular, a simply connected space is nilpotent. A nilpotent space is *rational* if each $H_i(S, \mathbf{Z}),\ i \geq 1$, is a vector space over the rationals. A nilpotent space S is *rationally of finite type* if each $H_i(S, \mathbf{Q})$ is finite dimensional. Lastly, the *Lusternik-Schnirelmann category* of a space S, cat S, is the least integer m such that S is the union of $m+1$ open sets, each contractible in S. Clearly, each smooth compact manifold M has cat $M < \infty$.

One of the main tools that we will employ in our thesis is a theorem of Grove and Halperin [11]. In their paper they show, among other things, that a compact Riemannian manifold M supporting a cohomogeneity one action of a Lie group G is necessarily elliptic. An orbit space of such an action must be either a circle or an interval. The case of the circle is of no importance for us since, in that case, M will not be simply-connected. Denote the endpoints of the interval by a and b. Choose some point e in the interior of the interval. Let A, B, E denote the orbits in M corresponding to these points. Since these are all homogeneous spaces of G, they are elliptic. To prove their result, they note the fact that there are disk bundles $D(A)$ and $D(B)$ in M over A and B, respectively, having E as their common boundary.

The idea then is to create a "double mapping cylinder" to represent M as a disjoint union

$$A \amalg (E \times I) \amalg B$$

with attaching maps $\phi_0 : E \times \{0\} \to A$ and $\phi_1 : E \times \{1\} \to B$, where I is the closed interval [0,1]. By analyzing this cylinder for the various possible cohomogeneity one actions, they prove that M is elliptic.

The proof they give in their paper is carried out in greater generality, and does not actually require that A, B and E be homogeneous spaces, but only that they be elliptic. The cohomogeneity one property was used merely to establish the ellipticity of these spaces and to show the existence of the disk bundles. Therefore, if we know that A, B and E are elliptic, and if we can establish the existence of the disk bundles $D(A)$ and $D(B)$ in some other way, their basic argument will apply without modification. In view of these remarks, we restate their results in a form more suited to our needs.

Grove-Halperin Theorem. *Let M be a compact, connected Riemannian manifold. Suppose that A and B are disjoint submanifolds of M and that $D(A)$ and $D(B)$ are disk bundles in M over A and B, respectively, having the submanifold E as their common boundary. Then M is elliptic if and only if E (or A or B) is elliptic.*

When O is an orbit of M, we have an inclusion $i : O \to M$ and, as described earlier, we can construct a fibration $O_i \to M$ with homotopy fiber F at a point p in M. We will show later that F has polynomially bounded homology. When the orbit O is simply connected, it is a straightforward consequence of the Serre spectral sequence that M also has polynomially bounded homology. Even if the fundamental group is non-zero, but finite, we may still use this method by considering instead the universal covering space $\tilde{O}$ of O. For i we use the composite mapping

$$\tilde{O} \to O \to M.$$

However, when the fundamental group of O is infinite, this method breaks down. Fortunately, a recent theorem due to S. Halperin [14] provides a solution to this problem.

Halperin Theorem. *Let $F \to X \to Y$ be a fibration where X is a rationally elliptic nilpotent space. Suppose that Y is simply connected, rationally of finite type, and that cat $Y < \infty$. If the Betti numbers dim $H_i(F, \boldsymbol{Q})$ grow at most polynomially in i, then Y is rationally elliptic.*

Specializing to our situation, $F \to O_i \to M$, it follows that, if the Betti numbers of F grow at most polynomially, then M is rationally elliptic. This result will play a central role in our argument. It is important to note that a homogeneous space of a compact Lie group is nilpotent, and that a compact manifold has finite L-S category.

2.2 Geometric Preliminaries

In this section we gather together a number of results from Riemannian geometry and the theory of isometric Lie group actions on Riemannian manifolds that we will require for our later work.

2.2.1 Alexandrov Spaces

The orbit spaces in which we are interested are examples of what are called Alexandrov spaces. Many of the results we shall need for orbit spaces are also true in this more general context. A brief but illuminating overview of Alexandrov spaces is [10]. More details can be found in the excellent note of K. Shiohama [25] and in the book by D. Burago et al. [5]. A discussion of some of the relationships between orbit spaces and Alexandrov spaces can be found in [6].

We begin with the definition of a length space. A length space L is a metric space in which the distance between any two points is the infimum of the lengths of all curves connecting them. Only length spaces which are finite dimensional in the sense of the Haudorff measure [5] will be considered here. We will be primarily interested in the case where this metric is complete so that the minimum distance is actually achieved by some curve, at least when L is locally compact. Then any two points can be connected by a minimizing geodesic.

Let $S(k)$ be the simply connected space having constant curvature k (space form).

Definition 2.2. A space X is said to have curvature $\geq k$ if for each $x \in X$ there is an open set $U \subset X$, containing x, such that every quadruple of points $\{a, b, c, d\}$ in U can be isometrically embedded in $S(n)$ for some $n \geq k$.

A length space L of curvature $\geq k$ is called an Alexandrov space. Alexandrov spaces have much in common with Riemannian manifolds. For example, since it is possible to talk about geodesics in an Alexandrov space, we can consider the closure of the set of geodesic directions at p, and the tangent space is the Euclidean cone on this space.

2.2.2 Group Actions and Orbit Spaces

We will study an action $G \times M \to M$ of a compact, connected Lie group on a compact, connected Riemannian manifold. We assume, without loss of generality, that the action is effective; i.e., that the set $\{g \in G \mid gp = p, \text{ for all } p \in M\}$ reduces to the identity element of G. The isotropy group G_p of a point p is the subgroup of G fixing the point p. The orbit of p is the homogeneous space $Gp = G/G_p$.We will assume that G acts by isometries in all that follows.

Points on the same orbit have conjugate isotropy groups. We denote the class of isotropy groups conjugate to some subgroup K of G by $[K]$. If $[K_1]$ and $[K_2]$ are two such classes, define $[K_1] \leq [K_2]$ if some conjugate of K_1 is contained in K_2. With

this definition, the set of conjugacy classes forms a lattice. There is always a smallest such class $[H]$. Orbits having isotropy class equal to $[H]$ are called *principal* orbits. Other orbits having the same dimension as a principal orbit are called *exceptional* orbits. The remaining orbits are called *singular* orbits. The set of all orbits is called the orbit space and denoted by M_G, or sometimes by M/G. We define the distance between two points in M_G to be the minimum distance between the corresponding orbits in M. Since G acts by isometries on M the orbits are parallel in the sense that, if $\mathcal{O}_1$ and $\mathcal{O}_2$ are any two orbits and if $p \in \mathcal{O}_1$, then the minimum distance from p to $\mathcal{O}_2$ is independent of p. With this metric, the orbit space is an Alexandrov space. We shall see in sec. 2.2.3 that, if $\mathrm{curv}(M) \geq 0$, then $\mathrm{curv}(M_G) \geq 0$ as well.

An action is called *polar* if there is a complete immersed submanifold S of M which meets all orbits of G orthogonally. S is called a section of the action and is always a totally geodesic submanifold. The action is called *hyperpolar* if, in addition, the section is flat as, for example, in the well-known adjoint action of a Lie group on itself.

There is a stratification of orbit types based on the lattice of conjugacy classes of isotropy groups [7]. For each isotropy group K, the subset of M_G having an isotropy group in $[K]$ is a Riemannian manifold called an orbit stratum. These strata are locally totally geodesic submanifolds of M_G. If $\mathcal{S}$ is an orbit stratum, the set $\tilde{\mathcal{S}} = \pi^{-1}(\mathcal{S})$ is a minimal submanifold of M and the restriction of π to $\tilde{\mathcal{S}}$ is a Riemannian submersion. The orbit stratum corresponding to $[H]$, the set of principal orbits, is open and dense in M_G and has dimension equal to $\dim(M_G)$. All

of the remaining boundary strata correspond to singular or exceptional orbits and are submanifolds of M_G. The boundary, if any, of an orbit stratum of dimension k consists of one or more orbit strata of dimension less than or equal to $k-1$. There is a similar topological stratification of a general Alexandrov space.

Denote the tangent space at the point $p \in M$ by T_p. Let T_{Gp} be the tangent space to the orbit Gp, and let ${T_p}^{\perp}$ be the orthogonal complement of T_{Gp} so that $T_p = T_{Gp} \oplus {T_p}^{\perp}$. The differential of the group action induces an action of G_p on the tangent space T_p. Since G acts by isometries on M and since the tangent space to Gp is G_p-invariant, the normal space ${T_p}^{\perp}$ is also invariant. More specifically, the subgroup G_p acts by orthogonal transformations on ${T_p}^{\perp}$. For two-dimensional orbit spaces, this action is always polar [18]. The space of directions at $\pi(p)$ is realized as the quotient ${S_p}^{\perp}/G_p$, where ${S_p}^{\perp}$ is the unit sphere in ${T_p}^{\perp}$. It should be noted, in particular, that this result allows us to speak of angles at an orbit.

Choose a disk D_p in ${T_p}^{\perp}$. To the principal bundle $G_p \to G \to G/G_p$ there is an associated bundle $G \times_{G_p} D_p$ and the group G acts on this bundle in the obvious way. The Slice Theorem [7] provides an especially nice representation of the normal bundle of the orbit Gp as a G-invariant submanifold of M.

Slice Theorem. *For any $p \in M$, a sufficiently small tubular neighborhood $D(Gp)$ of Gp is equivariantly diffeomorphic to $G \times_{G_p} D_p^{\perp}$.*

An important tool which describes the way that the boundary strata are embedded in M_G is the Soul Theorem [23]. This theorem guarantees the existence of a small convex subspace of M_G (the soul) which captures the essence of M_G. A set is convex if every segment joining any two points of the set lies entirely in the set.

Soul Theorem. *If $curv(M_G) \geq 0$ and $\partial M_G \neq \varnothing$, then there exists a convex compact subset $S \subset M_G$ with $\partial S = \varnothing$, which is a strong deformation retract of M_G. If $curv(M_G) > 0$, then S is a point p, and ∂M_G is homeomorphic to $S_p^{\perp}/G_p$.*

2.2.3 Riemannian Submersions

The basic facts about Riemannian submersions can be found in O'Neill's original paper [21], and in the book [8]. Let M and B be Riemannian manifolds and let $\pi : M \to B$ be a map. Define the vertical subspace $V_p = \{v \in T_p \mid \pi_* v = 0\}$. The orthogonal complement of V_p, denoted by H_p, is called the horizontal subspace. The map π is called a Riemannian submersion if π has maximal rank and π_* preserves the lengths of horizontal vectors.

If X is a vector field on M, denote by vX and hX the vertical and horizontal components of X, respectively. Define a tensor $A_X Y$ by

$$A_X Y = v\nabla_{hX} hY + h\nabla_{hX} vY$$

where ∇ is the Levi-Civita connection.

When X, Y are horizontal, it is easy to show that

$$A_X Y = \tfrac{1}{2} v[X, Y]$$

where $[X, Y]$ is the Lie bracket of X and Y. Thus $A_X Y$ is a measure of the non-integrability of the horizontal distribution. Let X, Y denote an orthonormal pair of horizontal vectors at the point p, and let K denote the sectional curvature in the plane of X and Y. If K' denotes the sectional curvature in B corresponding to $\pi_* X$ and $\pi_* Y$, then

$$K = K' - 3|A_X Y|^2.$$

It follows that $K' \geq K$.

2.2.4 Geodesics

In this section we assume that the Riemannian manifold M has been assigned the canonical Levi-Civita connection with covariant derivative ∇_X. Then a geodesic $\gamma : [0, 1] \to M$ is a curve for which the tangent vector field $\gamma'(t)$ is parallel along itself

$$\nabla_{\gamma'(t)} \gamma'(t) = 0.$$

Let p be a point of M and let T_p be the tangent space at p. Consider the subspace V_P tangent to the orbit at p and let H_p be the orthogonal complement of V_p. H_p is

called the horizontal subspace at p. A geodesic is said to be horizontal if its tangent vector at every point $\gamma(t)$ is a vector in $H_{\gamma(t)}$. If $\overline{\gamma}(t)$ is a geodesic in M_G beginning at $\pi(p)$, and if v_p is a horizontal lift of $\overline{\gamma'}(0)$, there is a unique horizontal geodesic $\gamma(t)$ beginning at p with $\gamma'(0) = v_p$. Actually, we have the following lemma.

Lemma 2.3. *Let $\gamma(t)$ be a geodesic which is horizontal at $p \in M$. Then $\gamma(t)$ is horizontal at every point.*

Let O_p denote the orbit through p and let O_p and O_q denote any two distinct orbits of M. Suppose that γ is a geodesic in M that minimizes the distance between O_p and O_q. Then, except for possibly its endpoints, γ lies entirely in a single orbit stratum. This is the content of the lemma of Kleiner.

Lemma 2.4 (Kleiner)**.** *Let $\gamma : [0,1] \to M$ be a minimal geodesic joining the orbits $O_{\gamma(0)}$ and $O_{\gamma(1)}$. Then $G_{\gamma(t)}$ is constant for all $t \in (0,1)$ and is a subgroup of both $G_{\gamma(0)}$ and $G_{\gamma(1)}$.*

2.2.5 Morse Theory

The best elementary introduction to this material is the book of John Milnor [19]. Let M be a Riemannian manifold and let p and q be two points of M, where we do not rule out the possibility that $p = q$. The set of all piecewise smooth paths in M from p to q will be denoted by $\Omega(M; p, q)$. We define the energy E on $\Omega(M; p, q)$ by

$$E(\omega) = \int_0^1 \left| \frac{d\omega}{dt} \right|^2 \, dt$$

As is well known from the calculus of variations, the first variation of E vanishes; i.e., λ is a critical point of E, precisely when λ is a horizontal geodesic. At a critical point of E, the Hessian E_{**} is well-defined. We will be interested in computing the dimension of its nullspace at γ, which is always finite. For this we need to look at a second variation of E. We find that the nullspace of E_{**} consists of certain vector fields defined along the critical geodesics. These are called Jacobi fields and satisfy the differential equation

$$\frac{D^2J}{dt^2} + R(\gamma'(t), J)\gamma'(t) = 0,$$

where $\frac{D}{dt} = \nabla_{\gamma'(t)}$ is the covariant derivative along γ and R is the Riemann curvature tensor. We define points p, q of M to be conjugate along a geodesic γ if there is a non-zero Jacobi field along γ vanishing at both p and q. The multiplicity of the point q along γ is the dimension of the space of such Jacobi fields.

By definition, the index of the Hessian E_{**} at a critical point γ is the maximum dimension on which E_{**} is negative definite. We have the following fundamental theorem due to Morse. We assume that this geodesic is defined on the interval [0,T].

Index Theorem. *The index λ at a geodesic γ is equal to the number of points $\gamma(t)$ with $t \in (0, T)$ such that $\gamma(t)$ is conjugate to $\gamma(0)$ along γ, each conjugate point counted with its multiplicity. The index λ is always finite.*

Corollary 2.5. *A geodesic segment $\gamma : [0, T] \to M$ can contain only finitely many points which are conjugate to $\gamma(0)$ along γ.*

The great importance of the index is expressed in the fundamental theorem of Morse Theory.

Morse's Theorem. *Let M be a complete Riemannian manifold and let p, $q \in M$ be two points which are not conjugate along any geodesic. Then $\Omega(M; p, q)$ has the homotopy type of a countable CW-complex which contains one cell of dimension λ for each geodesic from p to q of index λ.*

There are generalized forms of the Index Theorem and Morse's Theorem where one or both of the points p and q are replaced by submanifolds P and Q. The critical points of the energy functional are smooth geodesics that intersect the manifolds P and Q orthogonally.

Let P_p denote the tangent space to P at the point $p \in P$. Suppose that $\gamma(t)$ is a geodesic defined on [0,T] such that $\gamma(0) \perp P_p$. A P-Jacobi field is a Jacobi field which is orthogonal to γ with $J(0) \in P_{\gamma(0)}$ and $J'(0) - S_0 J(0) \perp P_{\gamma(0)}$, where S_t is the second fundamental form of P at $\gamma(t)$ with respect to $\gamma'(t)$. A P-focal point is a point $\gamma(t)$, $t \in (0, T]$, for which there is a non-zero P-Jacobi field which vanishes at $\gamma(t)$. The multiplicity of the the P-focal point is the dimension of such Jacobi fields. Let S_T be the second fundamental form of P at $\gamma(T)$. Define a symmetric bilinear map A on the space spanned by these Jacobi fields

$$A(J_1, J_2) \ = < J_1'(T) - S_T J_1(T), J_2(T) >$$

whose value at T is contained in $Q_{\gamma(T)}$. We have the following extensions of the Index Theorem [16] and the Morse Theorem [17].

Extended Index Theorem. *If $\gamma : [0,T] \to M$ is a critical geodesic and $\gamma(T)$ is not a focal point of P, then the index of γ is the number of P-focal points $\gamma(t)$, with $0 < t < T$, each counted with its multiplicity, plus the index of A at T.*

Extended Morse Theorem. *Let M be a complete Riemannian manifold and let P, Q be two submanifolds of M which do not contain focal points of each other along any geodesic. Then $\Omega(M; P, Q)$ has the homotopy type of a countable CW-complex which contains one cell of dimension λ for each geodesic from P to Q of index λ.*

2.2.6 Two-Dimensional Orbit Spaces

In this section we assume that M is simply-connected, $\dim(M_G) = 2$, $\text{curv}(M) \geq 0$, and that $\text{curv}(M_G) \geq 0$. We begin by showing that M_G is simply-connected, severely limiting the possibilities for the orbit space. We note that a closed curve in M_G can be uniformly approximated by a broken geodesic curve in the same homotopy class.

Lemma 2.6. *Let $\gamma(t)$ be a continuous, piecewise geodesic curve in M_G. Then γ has a continuous lift to M.*

Proof. On each smooth segment, there is a (not necessarily unique) horizontal lift beginning at the terminal point of the previous segment. □

Theorem 2.7. *M_G is simply-connected.*

Proof. Let $\gamma(t)$ be a closed curve in M_G with basepoint p. As noted above, we can assume that γ is piecewise geodesic, without changing its homotopy class. Then we can lift $\gamma(t)$ (not uniquely) to a curve $\tilde{\gamma}(t)$ with both initial and terminal points in $\pi^{-1}(p)$. Since all orbits are arcwise connected, we can join these points by a curve α lying entirely in their common orbit. Since M is simply-connected, there is a homotopy of $\tilde{\gamma}\alpha$ to a point. The projection of this homotopy is a homotopy of $\gamma(t)$ to a point. □

Since M is simply-connected, it is orientable. In particular, $H_1(M; \mathbf{Z}_2) = 0$. Since G is connected, so are the orbits of the action. Let C^* denote the union of all orbit strata of codimension greater than or equal to three in M_G. Let E be the set of exceptional orbits and let SE denote the special exceptional orbits; i.e., the set of exceptional orbits on the boundary of M_G. Let B denote the set of singular boundary orbits. We cite the following theorems from Bredon [4], which further restrict the structure of the orbits and the orbit space.

Theorem 2.8. *If $H_1(M; \boldsymbol{Z}_2) = 0$ and if all orbits are connected, then $M_G - C^*$ is orientable.*

Theorem 2.9. *If $H_1(M; \boldsymbol{Z}_2) = 0$ and if all orbits are connected, then every orbit of maximal dimension is orientable.*

In particular, the principal orbits and the exceptional orbits are orientable. Of course when $dim(M_G) = 2$, then $C^* = \varnothing$ so that M_G is orientable.

Theorem 2.10. *Suppose that G is connected and that the cohomogeneity of the action is two. Suppose also that M is compact, connected, $H_1(M; \boldsymbol{Z}) = 0$, and that a singular orbit exists. Then the set of exceptional orbits E is empty, and the orbit space M_G is a 2-disk with boundary the set of singular orbits B.*

Theorem 2.11. *If the cohomogeneity of the action is two, then the boundary of M_G consists exactly of $B \cup$ closure(SE).*

Theorem 2.12. *If $H_1(M; \boldsymbol{Z}_2) = 0$ and if a principal orbit is connected, then there are no special exceptional orbits.*

A straightforward consequence of these theorems is the following important corollary.

Corollary 2.13. *Suppose that M is simply-connected with a cohomogeneity two action. If M_G has a boundary, it consists entirely of singular orbits, and there are no exceptional orbits.*

Summarizing, if $\partial M_G = \varnothing$, all of the non-principal orbits are isolated exceptional orbits, and M_G is homeomorphic to a 2-sphere. If $\partial M_G \neq \varnothing$, there are no exceptional orbits, and the orbit space is homeomorphic to a 2-disk.

The map $\pi : M \to M_G$, restricted to principal orbits, is a Riemannian submersion. We consider the case where $\text{curv}(M_G) = 0$, $\partial M_G = \varnothing$, and the only non-principal orbits are exceptional. Since we are assuming that $\text{curv}(M) \geq 0$, this implies (sec. 2.2.3) that $\text{curv}(M) = 0$. If X and Y are horizontal vectors, it follows

that $A_X Y = 0$; i.e., that $[X, Y]$ is horizontal. Then the horizontal distribution is integrable, and there is an immersed submanifold S that is orthogonal to each of the vertical subspaces. It is known in this case that the action is polar [3], and we have the following theorem of Alexandrino and Töben [1].

Theorem 2.14. *Any non-trivial polar action on a simply-connected manifold has no exceptional orbits.*

Grove and Ziller [13] show that in this case, when there are no singular orbits, that

$$M = \mathbf{R}^2 \times G/H$$

where G/H is a principal orbit. But then M is not compact, a contradiction. Therefore, no flat orbit space can be homeomorphic to a sphere.

We will later need to consider triangles in M_G whose sides are geodesics. The basic theorem regarding such geodesic triangles follows.

Theorem 2.15. *Suppose that dim$(M_G) = 2$ and that α, β, γ are the angles of a geodesic triangle in M_G whose interior consists entirely of principal orbits. If curv$(M_G) > 0$, then $\alpha + \beta + \gamma > \pi$. If curv$(M_G) = 0$, then $\alpha + \beta + \gamma = \pi$.*

Proof. Since such a geodesic triangle in a two-dimensional orbit space is a manifold with boundary, and since the curvature is non-negative, the theorem is a straightforward consequence of the Gauss-Bonnet theorem for surfaces. □

As previously mentioned, the orbit space can be stratified by components of different dimensions, corresponding to conjugacy classes of the isotropy groups. There is a single 2-dimensional stratum corresponding to the principal isotropy group. These are the principal orbits of the action. The remaining orbit strata are either 0 or 1-dimensional. When $\dim(M_G) = 2$, no exceptional orbit can lie on a boundary stratum, and non-boundary exceptional orbits correspond to isolated interior points of M_G.

The boundary strata can be either 0 or 1-dimensional. Let $p \in M$ be a point on a k-dimensional orbit stratum. Clearly, the action of the isotropy group G_p on the normal space at p maps a small ball onto a subset of M_G of dimension $2 - k$. It follows that the action of G_p restricted to a small sphere is a cohomogeneity $1 - k$ action. In particular, at a 0-dimensional boundary stratum, the isotropy group acts on a sphere with cohomogeneity one and, at a point of a 1-dimensional boundary stratum, the action on the sphere is transitive.

The closures of two 1-dimensional orbit strata are either disjoint or meet in 0-dimensional strata. When the strata intersect, the angle between them is well defined and must be one of the four angles $\pi/2, \pi/3, \pi/4$, or $\pi/6$. This important fact follows from the general theory of isoparametric submanifolds of S^n and the fact that the orbits of a cohomogeneity one action on S^n are isoparametric. See [27] for a general description of these ideas and additional references.

The point $p \in M_G$ is called an *orbifold point* if some neighborhood of p is isometric to a quotient N/Γ, where Γ is a finite group and N is a convex Riemannian manifold diffeomorphic to $\mathbf{R}^k$. By a result of Lytchak and Thorbergsson [18], orbit strata of codimension less than or equal to 2 in M_G consist entirely of orbifold points. Of course, when $\dim(M_G) = 2$, all points are orbifold points. The group Γ will be $\mathbf{Z}_p$ for a non-boundary exceptional orbit, and the action will be orientable since there is no boundary. On the 1-dimensional orbit strata, the group $\mathbf{Z}_2$ acts by reflection. At the 0-dimensional boundary strata, the dihedral group D_{2p} acts with reflections to produce the boundary edges.

Chapter 3

Two-Dimensional Orbit Spaces of Non-Negative Curvature

We continue to assume that M is a compact, connected and simply-connected manifold of nonnegative curvature, that $\dim(M_G) = 2$, and that G is a compact, connected Lie group of isometries of M. In this case, the orbit space M_G, with the induced metric, is also a space of nonnegative curvature in the sense of Alexandrov. Since M is simply-connected and G is connected, the orbit space is orientable by theorem 2.9. In this chapter, we will classify all possible two dimensional orientable orbit spaces that can admit a metric of nonnegative curvature. The author is not aware if it is known at this time whether all of these spaces can actually arise as the orbit space of some group action.

It was noted earlier that the points of M_G corresponding to principal orbits form a smooth manifold, with boundary either empty or consisting of closed geodesic arcs and exceptional orbits. Recall that the angle between two boundary arcs must be one of the angles in the set $\{\pi/2, \pi/3, \pi/4, \pi/6\}$. The interior may also contain isolated singular points; i.e., the exceptional orbits. At an exceptional orbit, the tangent space is a cone subtending a solid angle of $2\pi/p$, for some integer $p \geq 2$.

By a triangle in an orbit space, we mean a set of three points, connected pairwise by minimizing geodesics. By Kleiner's lemma, these geodesic arcs can meet pairwise only at the three points. In what follows, by a singular point we mean either an isolated exceptional orbit, or a boundary point corresponding to a 0-dimensional stratum. A boundary arc is meant to indicate a 1-dimensional orbit stratum, and may or may not contain its endpoints. Note the important fact that the boundary arcs are geodesics. It is possible that some of the triangles we construct will have a side along a boundary arc that is not minimizing. By bisecting each non-minimizing boundary arc, repeatedly if necessary, and joining the new points to the interior vertex of the original triangle, we will obtain a new configuration, consisting entirely of minimizing geodesics and satisfying the same conclusion as the original one. In positive curvature, let S be the total angle sum of the original triangles, and suppose that there are k such triangles. If an additional boundary point is added, the number of triangles will be increased by 1 and the angle sum by π. Since $S + \pi > (k+1)\pi$ if and only if $S > k\pi$, adding a point preserves the condition $S > k\pi$. In zero curvature, adding extra points preserves the flatness property. Therefore, for simplicity of exposition, we will not add the extra points, and will use nonminimal arcs instead, whenever it is convenient to do so.

3.1 Positive Curvature

In order to determine the orbit spaces that can support a metric having positive sectional curvature, we will use the fact that any geodesic triangle on such a space will have an angle sum exceeding π. We will systematically eliminate orbit spaces that cannot carry a positive curvature metric, and see that those that we do not eliminate do in fact support such a metric. Of the surviving orbit spaces, many can support a metric of constant positive curvature and can generate tilings of the 2-sphere in a natural way.

Our general method of attack to show that an orbit space does not support positive curvature is quite simple. We triangulate the orbit space in some way, obtain the sum of all angles for all triangles, and divide by the number of triangles. If the average is less than or equal to π, we conclude that at least one of the triangles must have an angle sum less than or equal to π. This conclusion contradicts the assumption of positive curvature.

Our first step in the classification is to establish limitations on the number of singular points and boundary arcs.

Theorem 3.1. *Let M_G be an orientable 2-dimensional orbit space having positive curvature in the sense of Alexandrov. Then*

a) ∂M_G cannot consist of more than 3 boundary arcs.

b) If $\partial M_G = \varnothing$, then there can be at most 3 exceptional orbits.

c) If ∂M_G consists of 1 boundary arc, then there can be at most 1 singular boundary point. If ∂M_G consists of 2 or 3 boundary arcs, then there will be 2 or 3 singular boundary points, respectively. In any case, there are no exceptional orbits.

Proof. a): This is a special case of a theorem which states that a positively curved orbit space of dimension n can have at most $n+1$ faces. However, we will give a direct elementary proof of this fact when n is 2. Suppose that there are $k > 3$ boundary arcs and, therefore, k singular boundary points. Choose one of these points p and construct geodesic arcs to each of the other singular boundary points not adjacent to p. This construction creates $k-2$ triangles for which the total angle sum is at most $k\pi/2$. The average angle sum for a triangle is $\frac{\pi}{2} \cdot \frac{k}{k-2} \leq \pi$, if $k \geq 4$. Then some triangle must have an angle sum less than or equal to π.

b): Suppose that there are at least four exceptional orbits. Choose any three of them and connect the points pairwise by minimizing geodesics, thus forming a geodesic triangle. Connect the fourth point to each of the other three points by minimizing geodesics. This construction creates four triangles with a total angle

sum of at most 4π (the sum of the angles at the four conical points) so that some triangle will have an individual angle sum less than or equal to π.

c): If the boundary consists of a single arc, there can be at most one boundary singular point (if the ends of the arc intersect at an angle less than π). Otherwise, there are as many singular boundary points as the number of arcs. By the set of theorems 2.8 to 2.12, there are no exceptional orbits. □

It is convenient to divide the orbit classification problem into two cases, depending on whether the boundary of M_G is empty or not. For the remainder of this chapter we assume that α, β, γ are angles in the set $\{\pi/2, \pi/3, \pi/4, \pi/6\}$ and that p, q, r are integers ≥ 2.

3.1.1 Empty Boundary

When $\partial M_G = \varnothing$, M_G is topologically S^2 but may have interior singular points. We have seen in this case that there can be at most three singular points. These are necessarily conical points; i.e., orbifold points resulting from an orientable $\mathbf{Z}_p$ action, subtending an angle of $2\pi/p$, for $p \geq 2$.

No Singular Point

This is the case where $M_G = S^2$ and trivially supports a metric of constant positive curvature.

One Singular Point

When there is only one singular point, the angle has a value $2\pi/p$. This orbit space is sometimes called the $\mathbf{Z}_p$ teardrop and is known to arise from an S^1 action on S^3. While this space has positive curvature, it cannot be given a metric of constant positive curvature.

Two Singular Points

If there are two singular points, let the angles be $2\pi/p$ and $2\pi/q$. This orbit space is known as the $\mathbf{Z}_p$-$\mathbf{Z}_q$ football and, like the $\mathbf{Z}_p$ teardrop, supports positive curvature but not constant positive curvature, unless $p = q$.

Three Singular Points

Finally, we consider the case of three singular points with angles $2\pi/p$, $2\pi/q$ and $2\pi/r$. We can view this space as the double of the space (with boundary) having boundary angles π/p, π/q and π/r. There are several possibilities for p, q and r. One set of possibilities is $p = q = 2$ and $r \geq 2$. The others are $p = 2$, $q = 3$ and $r = 3,\ 4$ or 5. Each of these spaces can be realized as a spherical triangle in S^2 and, consequently, can be given a constant positive curvature metric. All other combinations produce triangles with angle sum less than or equal to π.

3.1.2 Non-Empty Boundary

When $\partial M_G \neq \varnothing$, the boundary will consist of from one to three arcs, and will not contain an interior singular point.

One Boundary Arc

Suppose first that the boundary has no singular points. Then the orbit space can be given the metric of a closed hemisphere of S^2, which clearly has constant positive curvature. Suppose now that the boundary has one singular point with angle α. Then the double of M_G is the $\mathbf{Z}_p$ teardrop with $p = \pi/\alpha$ and M_G supports positive, but not constant, curvature.

Two Boundary Arcs

Let the boundary angles be denoted by α and β. The double of M_G is the $\mathbf{Z}_p$-$\mathbf{Z}_q$ football where $p = \pi/\alpha$ and $q = \pi/\beta$. Therefore, M_G supports positive but not constant curvature (unless $\alpha = \beta$).

Three Boundary Arcs

Let the boundary angles be denoted by α, β and γ. Then $\alpha + \beta + \gamma > \pi$. If $\alpha = \beta = \pi/2$, then γ can be any allowable angle. If $\alpha = \pi/2$ and $\beta = \pi/3$, then $\gamma = \pi/3$ or $\pi/4$. No other combinations are possible. All of these cases can be realized as spherical triangles and, consequently, can be given a metric of constant positive curvature.

3.2 Zero Curvature

In this section we will determine those orbit spaces that can support a flat metric. We will find that these are all triangles and squares. We summarize the results in a general theorem. Recall that, if M_G has a boundary, there can be no exceptional orbits.

Theorem 3.2. *Let M_G be an orientable 2-dimensional orbit space having nonnegative but not strictly positive curvature in the sense of Alexandrov.*
a) If $\partial M_G \neq \varnothing$, then we may assume that ∂M_G consists of 3 or 4 boundary arcs.
b) If $\partial M_G = \varnothing$, then M_G contains a point of positive curvature and the orbit space can be given a metric of strictly positive curvature.

Proof. a): Suppose that there are $k > 4$ boundary arcs and, therefore, k singular boundary points. Choose one of these points p and construct geodesic arcs to each of the other singular boundary points not adjacent to p. This construction creates $k-2$ triangles for which the total angle sum is at most $k\pi/2$. The average angle sum for a triangle is $\frac{\pi}{2} \cdot \frac{k}{k-2} < \pi$, if $k > 4$. Then some triangle must have an angle sum less than π. Orbit spaces with one or two boundary arcs, if they exist, are essentially the same as similar orbit spaces described in the section on positive curvature, and a proof of the main theorem for all of these spaces will be given that does not depend on the curvature of the space.

b): By the results of section 2.2.3, the orbit space must have a point of positive curvature. It cannot contain four exceptional orbits since, in this case, all of the angles at the exceptional orbits must be π, and the orbit space must be a flat square. Under our assumptions, there can be only three exceptional orbits. Connect them with geodesics to create two triangles. At least one of these triangles encloses a point of positive curvature and, therefore, must have an angle sum exceeding π. Since all angle sums for a triangle must be at least π, the average sum is greater than π and, as in the section on positive curvature, the orbit space can be given a metric of constant positive curvature. □

It follows that, if the orbit space cannot be given a metric of positive curvature, it must be a planar triangle or a square. As before, the angles of the triangle must be in the set $\{\pi/2, \pi/3, \pi/4, \pi/6\}$. There are only three such triangles, $(\pi/2, \pi/4, \pi/4)$, $(\pi/2, \pi/3, \pi/6)$ and $(\pi/3, \pi/3, \pi/3)$.

3.3 Tabular Listing

The previous results are summarized in tables 3.1 and 3.2. The exterior and interior angles are listed for each orbit space. In case 8, where there is one boundary arc but no angle (a circle), the angle is entered as π to indicate that there is a boundary.

When the orbit space can be represented as a spherical triangle, it is interesting to determine whether this triangle can be used to produce a tiling of the whole sphere. This question will be studied in the next section. We summarize the results of that discussion in the first table. Column "Tiling?" specifies whether or not S^2 can be tiled by the orbit space (or by two copies of a tiling triangle which, when when glued along some or all of their edges, give the orbit space). When the entry for "Tiling?" is "Yes", the column "Tile" gives the basic tiling domain and "No. Tiles" is the total number of tiles in the tiling.

The second table gives a similar classification of orbit spaces that can tile the plane. In this case, the number of tiles will be infinite.

We continue to use the convention that Greek letters denote boundary angles in the set $\{\pi/2, \pi/3, \pi/4, \pi/6\}$, while latin letters represent integers ≥ 2.

#	Ext ∠'s	Int ∠'s	Tiling?	Tile	No. Tiles
1			Yes	S^2	1
2		$2\pi/p$	No		
3		$2\pi/p$, $2\pi/q$	No		
4		π, π, $2\pi/p$	Yes	$(\pi/2, \pi/2, \pi/p)$	$4p$
5		π, $2\pi/3$, $2\pi/3$	Yes	$(\pi/2, \pi/3, \pi/3)$	24
6		π, $2\pi/3$, $\pi/2$	Yes	$(\pi/2, \pi/3, \pi/4)$	48
7		π, $2\pi/3$, $2\pi/5$	Yes	$(\pi/2, \pi/3, \pi/5)$	120
8	π		Yes	Hemisphere	2
9	α		No		
10	α, β		No		
11	$\pi/2$, $\pi/2$, α		Yes	$(\pi/2, \pi/2, \alpha/2)$	$8\pi/\alpha$
12	$\pi/2$, $\pi/3$, $\pi/3$		Yes	$(\pi/2, \pi/3, \pi/3)$	24
13	$\pi/2$, $\pi/3$, $\pi/4$		Yes	$(\pi/2, \pi/3, \pi/4)$	48

Table 3.1: Orbit Spaces Admitting Positive Curvature

#	Ext ∠'s	Int ∠'s	Tiling?	Tile
14	$\pi/2$, $\pi/3$, $\pi/6$		Yes	$(\pi/2, \pi/3, \pi/6)$
15	$\pi/2$, $\pi/4$, $\pi/4$		Yes	$(\pi/2, \pi/4, \pi/4)$
16	$\pi/3$, $\pi/3$, $\pi/3$		Yes	$(\pi/3, \pi/3, \pi/3)$
17	$\pi/2$, $\pi/2$, $\pi/2$, $\pi/2$		Yes	$(\pi/2, \pi/2, \pi/2, \pi/2)$

Table 3.2: Orbit Spaces Admitting Zero Curvature

3.4 Spherical and Planar Tilings

In this section we will see that many of the orbit spaces, those that support constant curvature, can be made to tile either the sphere or the plane. This will have important consequences for our work. We have seen that, except for the trivial cases of the sphere and hemisphere, all of the orbit spaces M_G supporting constant positive curvature can be realized as spherical triangles, or by gluing along some or all of the edges of such triangles. These are the cases 1, 4-8 and 11-13 in table 3.1. We will show that each triangle also generates a tiling of the sphere.

The orbit spaces 14-17 obviously generate a tiling of the plane.

Cases 1, 8 (Sphere and hemisphere): These cases clearly tile the sphere.

Cases 4, 11 $(\pi/2, \pi/2, \theta)$: In these cases θ divides 2π evenly with quotient k. Such a triangle can be bounded by an equator and two longitude lines, making an angle of θ with each other. Therefore, $2k$ triangles will tile the whole sphere.

Cases 5, 12 $(\pi/2, \pi/3, \pi/3)$: Reflect over an edge joining angles $\pi/2$ and $\pi/3$. This produces a triangle with angles $2\pi/3$, $\pi/3$ and $\pi/3$. Reflecting twice around the angle $2\pi/3$ produces a triangle with all three angles equal to $2\pi/3$. This is a fundamental region for the action of the tetrahedral group on S^2, and so gives a tiling of the sphere with 24 tiles.

Cases 6, 13 $(\pi/2, \pi/3, \pi/4)$: Reflecting the triangle over the edge connecting the angles $\pi/2$ and $\pi/4$ produces the triangle of case 5. The tiling consists of 48 tiles.

Case 7 $(\pi/2, \pi/3, \pi/5)$: Reflect around the angle $\pi/5$ ten times to remove this angle. When a reflection involves the angle $\pi/2$, the vertex is removed. There remain five vertices, each with angle $2\pi/3$. This pentagon is a fundamental region for the action of the dodecahedral group on the 2-sphere, and the triangle tiles the sphere with 120 tiles.

Chapter 4

Main Theorem

In this chapter we will prove our main theorem. We restate the theorem here for the convenience of the reader.

Main Theorem. *Let M be a compact, connected and simply-connected Riemannian manifold having nonnegative sectional curvature. Let G be a compact, connected Lie group acting effectively on M as isometries, with orbit space M_G. If $dim(M_G) = 2$, then M is rationally elliptic.*

The proof will be accomplished in two parts. First we will prove the theorem as an application of the Grove-Halperin theorem for those orbit spaces which do not accept a metric of constant nonnegative curvature. These are the cases 2, 3, 9 and 10 in table 3.1. All of the remaining cases can be given a metric of constant nonnegative curvature and, as we have just seen, can be used to tile either $\mathbf{R}^2$ or S^2. We will use these tilings, together with the Morse theory and Halperin's theorem, to prove our theorem for these orbit spaces.

4.1 Proof for Non-Constant Curvature

Our approach in the cases which do not admit spherical tilings is the following. We judiciously choose two points a and b, typically the most singular orbits in M_G, and a simple smooth arc γ such that $M_G - |\gamma|$ consists of two components, C_a and C_b, containing a and b, respectively. Here $|\gamma|$ denotes the support of γ. Set

$$A = \pi^{-1}(a),\ B = \pi^{-1}(b),\ E = \pi^{-1}(|\gamma|)$$

$$D(A) = \pi^{-1}(C_a),\ D(B) = \pi^{-1}(C_b),$$

where $\pi : M \to M_G$ is the projection of M onto the orbit space M_G. Clearly E is the common boundary of $D(A)$ and $D(B)$. We must choose γ so that E is a C^∞-submanifold of M. Since the action of G restricted to any of A, B and E has at most cohomogeneity one, we know by the Grove-Halperin theorem that each is elliptic. To finish the proof, we need to show that $D(A)$ and $D(B)$ are disk bundles over A and B, respectively. Then, by the method of proof of that theorem, it will follow that M is elliptic.

By the Slice Theorem, the normal bundles of A and B can be expressed as equivariant tubular neighborhoods $N_A(\rho_A)$ and $N_B(\rho_B)$. The fibers are open disks of common radii ρ_A and ρ_B, respectively. If necessary, we will reduce the radii of the normal bundles so that their closures have disjoint neighborhoods, and continue to use ρ_A and ρ_B to denote them. To avoid repetition, we will concentrate on the orbit A in what follows. Of course, similar remarks will hold for B as well. We denote

the closure of a set S by $\overline{S}$ and the interior by S^o in what follows.

Our immediate task is to extend the bundle $N_A(\rho_A/2)$ to a larger disk bundle M - $\overline{N_B(\rho_B/2)}$ over A. We begin by defining a vector field R on $\pi(N_A(\rho_A))$ which is a radial vector field along the geodesics emanating from a. Set

$$T = M_G \text{ - } \pi(N_A(\rho_A/2)) \cup \pi(N_B(\rho_B/2))$$

where $\pi(N_A(\rho_A/2))$ and $\pi(N_B(\rho_B/2))$ are the disks of radius $\rho_A/2$ and $\rho_B/2$ centered at a and b, respectively. Now T is diffeomorphic to either $S^1 \times I$ or to $I \times I$. This diffeomorphism defines a C^∞ family of C^∞ curves $\gamma_s(t)$, parametrized by the first coordinate. Of course any such diffeomorphism takes the boundary of I into the boundary of M_G, when M_G has a boundary. Choose the diffeomorphism so that the radial curves on the annulus defined by the disks of radius $\rho_A/2$ and ρ_A agree in some manner with the curves $\gamma_s(t)$, say for $0 \leq t \leq 1/4$, and do the same for the corresponding annulus around b on the interval $3/4 \leq t \leq 1$. Define a unit vector field Y on T in such a way that Y_q is the unit tangent vector to the curve γ_s through q in the direction of b. Choosing a partition of unity $\{\rho_1, \rho_2\}$ subordinate to the open sets $\pi(N_A(\rho_A/2))$ and T^o, we obtain a vector field

$$X_p = \rho_1(p)R_p + \rho_2(p)Y_p$$

on M_G - $\overline{\pi(N_B(\rho_B/2))}$ which is non-singular, except at a. The vector field X has a unique C^∞ horizontal lift Z, even when $\partial M_G \neq \varnothing$ since, on the boundary, the lift of the vector is parallel to the orbit stratum. The integral curves of Z fill out the

space M - $\overline{N_B(\rho_B/2)}$. It follows that M - $\overline{N_B(\rho_B/2)}$ is a disk bundle over A. The restriction of this bundle to $\pi^{-1}(C_a)$ is the disk bundle $D(A)$. Similarly, we obtain the disk bundle $D(B)$ by reversing the direction of the tangent vectors on T.

To finish the proof, we need only specify the orbits a, b, and the point set $|\gamma|$ for each case. In case 2, take a to be the exceptional orbit and let b be any other point. For case 3 choose a and b to be the two exceptional orbits. For case 9, take a to be the vertex point and let b be any other point on the boundary. Finally, for case 10, choose a and b to be the two vertices.

It remains to specify the curve γ. For cases 2 and 3, it suffices to set $\gamma(s) = \gamma_s(1/2)$. For cases 9 and 10, choose the points $\gamma_0(1/2)$ and $\gamma_1(1/2)$ on the boundary of T. Let p be a point of the orbit $P = \pi^{-1}(\gamma_0(1/2))$ and consider the normal space N_p to the orbit P at p. Then, by the Slice Theorem, there is a disk D centered at 0 in N_p such that $G \times_{G_p} D_p^{\perp}$ is equivariantly diffeomorphic to a sufficiently small tubular neighborhood of P. Since P lies on an orbit stratum of codimension 1 in M_G, the image of this bundle is a smooth arc α_0 in M_G. Since α_0 is perpendicular to the orbit stratum it follows that, for sufficiently small $\epsilon_0 > 0$, $\alpha_0(t)$ is transversal to the curves γ_s for t in the interval $[0, \epsilon_0)$. Set $\epsilon_0' = min(\epsilon_0, s/4)$. A similar result holds for the point $\gamma_1(1/2)$, giving an arc $\alpha_1(t)$ which is transversal in $[0, \epsilon_1')$. Let α be any C^∞ curve, smoothly connecting $\alpha_0(\epsilon_0'/2)$ and $\alpha_1(\epsilon_1'/2)$ and crossing each $\gamma_s(t)$ transversally and lying entirely in the interval $1/4 < t < 3/4$. The resulting curve satisfies all of our requirements. This completes the proof for these four cases.

4.2 Proof for Constant Curvature

In the constant curvature cases, the orbit spaces generate tilings of $\mathbf{R}^2$ or S^2. We will show that the homotopy fiber F of an inclusion of a principal orbit Q into M has polynomially bounded homology. The Halperin theorem 2.1.3 will then guarantee ellipticity. In either case, we will need to lift this constant curvature metric to the space M before we begin the analysis of F.

4.2.1 Lifting a Constant Curvature Metric

We will need to show that when the orbit space can be given a constant curvature metric, then this metric can be lifted to M. We begin with a basic lemma.

Lemma 4.1. *Let $<,>$ denote the usual inner product on $\boldsymbol{R}^n$ and consider the projection $\pi : \boldsymbol{R}^k \times \boldsymbol{R}^{n-k} \to \boldsymbol{R}^k$ of $\boldsymbol{R}^n$ onto the first factor. Let P, S be positive operators on $\boldsymbol{R}^k$ and $\boldsymbol{R}^n$, respectively. Define new metrics $g_P(u,v) = < Pu, v >$ on $\boldsymbol{R}^k$ and $g_S = < Su, v >$ on $\boldsymbol{R}^n$. Then $\pi : (\boldsymbol{R}^n, g_S) \to (\boldsymbol{R}^k, g_P)$ is a Riemannian submersion if and only if S and P are related by*

$$S^{-1} = \begin{pmatrix} P^{-1} & * \\ * & * \end{pmatrix}$$

Proof. Suppose first that π is a Riemannian submersion with horizontal subspace

$$H = \{(x, Ax) \mid x \in \mathbf{R}^k \text{ and } A : \mathbf{R}^k \to \mathbf{R}^{n-k}\}$$

and vertical subspace $V = \{0\} \times \mathbf{R}^{n-k}$. Since $< SH, V > = \ g_S(H, V) \ = \ 0$, $S(x, Ax) \in \mathbf{R}^k \times \{0\}$. Now suppose that x and $y \in \mathbf{R}^k$ have horizontal lifts (x, Ax) and (y, Ay). Then

$$g_S((x, Ax), (y, Ay)) \ = < S(x, Ax), (y, Ay) > = < S(x, Ax), (y, 0) >$$

$$= < \pi S(x, Ax), y > = \ g_P(x, y) \ = < Px, y >.$$

Therefore, $\pi S(x, Ax) \ = \ Px$ or $S(x, Ax) \ = \ (Px, 0)$. Then we have

$$(x, Ax) \ = \ S^{-1}S(x, Ax) \ = \ S^{-1}(Px, 0).$$

Expressing this relation in block form, where U and P are $k \times k$ matrices,

$$\begin{pmatrix} id_{R^k} \\ A \end{pmatrix} = \begin{pmatrix} U & * \\ W & * \end{pmatrix} \begin{pmatrix} P \\ 0 \end{pmatrix}$$

we have $UP \ = \ id_{\mathbf{R}^k}$ and $WP \ = \ A$, so that $U \ = \ P^{-1}$.

Conversely, suppose that S^{-1} has the above form and that $U = P^{-1}$. Define $A = WP$. Then, if $x, y \in \mathbf{R}^k$,

$$g_S((x, Ax), (y, Ay)) = < S(x, Ax), (y, Ay) > =$$

$$< S(UPx, WPx), (y, Ay) > = < SS^{-1}(Px, 0), (y, Ay) > =$$

$$< (Px, 0), (y, Ay) > = < Px, y > = g_P(x, y).$$

It follows that π is a Riemannian submersion with horizontal subspace

$$H = \{(x, Ax) \mid x \in \mathbf{R}^k\}. \qquad \square$$

We will use this lemma only in the case where $P = I$. Then S^{-1} will have the form

$$\begin{pmatrix} I & * \\ * & * \end{pmatrix}.$$

Now let $p \in M$. We know that the action of the isotropy group G_p on the normal sphere is polar and, therefore, so is the action of G_p on the the normal space ${T_p}^{\perp}$. Then there is a 2-dimensional section S perpendicular to all orbits. We can lift the constant curvature metric on the orbit space to this section, and then along the (orthogonal) orbits to the full normal space. The Slice Theorem guarantees that, if D_p is a sufficiently small disk in ${T_p}^{\perp}$, then $G \times_{\mathsf{G}_p} D_p^{\perp}$ is equivariantly diffeomorphic to a tubular neighborhood of the G-orbit Gp, and we may use this fact to transfer the metric to this neighborhood. We have thus shown that around each point of

M_G, there is a neighborhood U where the constant curvature metric on M_G can be lifted to a metric on $\tilde{U} = \pi^{-1}(U)$. Denote the totality of such neighborhoods for all points of M_G by $\{\tilde{U}_\alpha\}$. Choose a partition of unity f_α subordinate to the covering $\{\tilde{U}_\alpha\}$ and let S_α^{-1} represent the inverse of the lifted metric on $\tilde{U}_\alpha$, relative to the original metric on M. Then consider the operator S on M defined by

$$S^{-1} = \Sigma f_\alpha \; S_\alpha^{-1}.$$

This S gives the desired lift, since S^{-1} is globally equal to the identity in the upper left hand block of the matrix.

It is important to note that the new metric on M given by $g(Su, v)$ has the constant curvature orbit space but may no longer have nonnegative curvature. However, this property will no longer be needed in the sequel, since its only use was to limit the geometry of the orbit space.

4.2.2 Morse Theory of the Homotopy Fiber

We have seen that the constant curvature metric on M_G can be lifted to M. Since M_G tiles the plane or the sphere, we can view the orbit space as multiply embedded in $\mathbf{R}^2$ or S^2, and we can regard each horizontal geodesic in M as a curve over a geodesic in that space.

Let Q be a principal orbit of M. Consider the homotopy fiber F of the inclusion $i : Q \to M$ at $p \in M$. F is the set of all paths $\gamma : [0,1] \to M$ with $\gamma(0) \in Q$ and $\gamma(1) = p$. It is convenient for our purposes to consider the paths as beginning at p and ending on Q. Then $\gamma(0) = p$ and $\gamma(1) \in Q$. We will show that F has polynomially bounded homology and the Halperin theorem 2.1.3 will allow us to conclude that M is elliptic.

Let $E(\gamma)$ be the energy functional on F as in section 2.2.5. The critical points are horizontal geodesics that connect the point p with the orbit Q.

The methods we use are similar for positive and zero curvature, but the details are somewhat different, and so we will discuss these cases separately.

4.2.3 Positive Curvature

In the positive curvature cases, the orbit space is a spherical triangle, a double of such a triangle, the whole sphere, or a hemisphere. All generate tilings of S^2 where the triangular cases are of the form $(\pi/2, \pi/2, \pi/r)$, where r is arbitrary, or $(\pi/2, \pi/3, \pi/r)$, where $3 \leq r \leq 5$.

Let Q be any principal orbit, and let C denote the union of all great circles in S^2 that contain two or more copies of Q. Choose P in M - $\pi^{-1}(C)$ to be a principal orbit such that, for some $p \in P$, p is not a focal point of Q along any geodesic. Let p' denote the projection of P in M_G. In what follows we denote the antipodal point of p' by $\overline{p'}$ and the number of tiles in the spherical tiling by c (from the No. Tiles column in table 3.1).

Since all horizontal geodesics emanating from p lie over great circles passing through $\overline{p'}$ and, since the dimension of the horizontal subspace is 2, the contribution to the index of a critical geodesic at each crossing of the antipodal point must be 1.

Under our assumptions, there are exactly $2c$ critical geodesics that meet the orbit Q only once (c when the orbit space is the double of a spherical triangle). These correspond to the great circles on S^2 from p' to a point equivalent to Q. We must count two critical geodesics for each copy of Q, one for each direction from p. Every other critical geodesic must begin as one of these. For each initial segment there is just one critical geodesic lying over that great circle and terminating between two successive crossings of the antipodal point of p.

Choose one of these initial segments. Suppose that, on the corresponding great circle, γ is a geodesic having index d, thereby producing a cell of dimension d. We must consider the contribution of the index of the operator A (sec. 2.2.5) associated to the second fundamental form of Q, which may be a negative number. Set m equal to the smallest possible value of index(A). If $m \geq 0$ then, after a single circuit, there will be no additional cells of dimension d, only a cell of higher dimension, since there is a nonzero contribution at the antipodal point. Therefore, no two extensions of the same initial segment have the same index. If β_d is the d-dimensional Betti number of F, it follows that

$$\beta_d \leq 2c.$$

If $m < 0$, we may require as many as $1 - m$ complete circuits to obtain a larger value for the index of γ. Then

$$\beta_d \leq 2(1 - m)c.$$

Setting $\lambda = max(1, 1 - m)$ and summing the Betti numbers, we have

$$\sum_{i=0}^{n} \beta_i \leq 2c\lambda(n + 1)$$

and F has polynomially, in fact linearly, bounded homology. This proves the main theorem for positive curvature.

4.2.4 Zero Curvature

In the zero curvature cases, the orbit space is either a Euclidean triangle with angles in the restricted set $\{\pi/2, \pi/3, \pi/4, \pi/6\}$, or a square. All generate tilings of $\mathbf{R}^2$.

Let Q be any principal orbit, and let C denote the union of all straight lines in the plane that contain two or more copies of Q, together with the lines that connect a vertex of the tiling with a copy of Q. Choose P in M - $\pi^{-1}(C)$ to be a principal orbit such that, for some $p \in P$, p is not a focal point of Q along any geodesic. We note the important fact that, in view of our assumptions on p, no critical geodesic will cross a tile boundary at a vertex.

In zero curvature, since there is no antipodal point, we must look for focal points at the tile boundaries. This will occur if the focal point is on an orbit that is more singular than a principal orbit. More precisely, we have the following lemma.

Lemma 4.2. *Let Q be a principal orbit with isotropy group H, and let S be any other orbit with isotropy group $K \supseteq H$. Suppose that $\gamma(t)$ is a horizontal geodesic joining the points $s \in S$ and $q \in Q$. Define $\gamma_k(t) = k\gamma(t)$ for $k \in K$. Then $\gamma_k(t)$ is a horizontal geodesic and $\gamma_k(t) \neq \gamma(t)$ if $k \notin H$. The set $\{\gamma_k\}$ is parametrized by the space K/H, and s is a focal point of Q of multiplicity $dim(K/H)$. In particular, if S is a singular orbit, $dim(K/H) \geq 1$.*

Proof. Clearly, $\gamma_k(s) = s$ and $\gamma_k(q) \in Kq$. Furthermore, $kq = q$ if and only if $k \in H$. Since K acts by isometries, the geodesics $\gamma_k(t)$ are horizontal. □

The lemma implies that, since the boundary cannot contain exceptional orbits, the contribution to the index of a critical geodesic must be positive at every tile crossing. This fact will allow us to obtain a bound on the Betti numbers of the homotopy fiber. Since any straight line beginning at p will meet a copy of Q at most once, these must represent all of the critical geodesics. We can easily compute a lower bound for their indices and, from this lower bound, compute a bound on the Betti numbers.

In all cases where the orbit space tiles the plane, the orbit space is either an equilateral triangle, a square, or a triangle with a reflection that gives one of these two. We will consider, therefore, only the two cases of tilings by equilateral triangles and squares. Since we assume that a critical geodesic cannot pass through a vertex, the following figure illustrates the number of crossings necessary to enter a given tile.

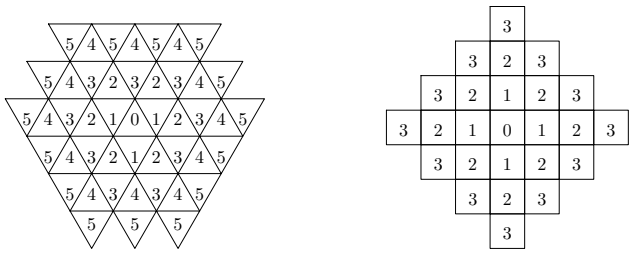

Figure 4.1: Number of Tile Crossings - Equilateral Triangle and Square

We will handle the case of the equilateral triangle first, since it is more involved. Let s_n denote the number of triangles in the nth ring with a single edge exposed, and let d_n denote the number with two edges exposed. Then it is easy to see that

$s_{n+1} = d_n - 3$ and $d_{n+1} = s_n + 6$. Adding these equations and setting $t_n = s_n + d_n$, we have

$$t_{n+1} = t_n + 3, \text{ and } t_1 = 3.$$

It follows that $t_n = 3n$. The total number of tiles in rings 0 through n is

$$1 + 3 + 6 + ... + 3n = 1 + 3n(n+1)/2.$$

Since each tile can contain either one or two copies of the orbit space, there is a maximum of $2 + 3n(n+1)$ copies of Q in rings 0 through n.

Each crossing contributes at least one to the index of a critical geodesic. We must also allow for the contribution of the index of the operator A (sec. 2.2.5) associated to the second fundamental form of Q. This may be a negative number. If we set m equal to its smallest possible value, we will require at most $n - m$ tile crossings to guarantee that all critical geodesics producing cells of dimension less than or equal to n have been obtained. It follows that

$$\sum_{i=0}^{n} \beta_i \leq 2 + 3(n-m)(n-m+1)$$

where β_i is the ith Betti number.

Similarly, for the square, we can see that

$$t_{n+1} = t_n + 4, \text{ and } t_1 = 4.$$

So $t_n = 4n$ and

$$\sum_{i=0}^{n} \beta_i \leq 2 + 4(n-m)(n-m+1).$$

In either case, the Betti numbers are quadratically bounded. This proves the main theorem for the case of zero curvature.

4.3 Remarks

1. The main theorem is really a theorem about orbit spaces. If M is a manifold satisfying all of the conditions of the theorem except for nonnegative curvature, and if the orbit space can be given a metric of nonnegative curvature, it still follows that M is elliptic.

2. The proof for constant curvature did not use the cohomogeneity 2 property except to show that each of these orbit spaces tiled the sphere. We required only that the action on the normal space was polar in order to lift the constant curvature metric. If we know that an orbit space can be given a constant curvature metric and can tile the k-plane or the k-sphere, then the orbit space must consist entirely of orbifold points. Lytchak and Thorbergsson have shown [18] that the action on the normal space is polar if and only if the point is an orbifold point. A straightforward argument will show, in this case, that M is elliptic.

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