ABSTRACT

Title of dissertation: ESSAYS ON AUCTION AND MATCHING THEORY

Terence Robert Johnson, Doctor of Philosophy, 2011

Dissertation directed by: Professors Lawrence Ausubel and Daniel Vincent

Department of Economics

This dissertation uses mechanism design theory to show how a matchmaker should design two-sided matching markets when agents are privately informed about their qualities or characteristics as a partner and can make monetary payments. Chapter Two uses a mechanism design approach to derive sufficient conditions for assortative matching to be profit-maximizing for the matchmaker or maximize social welfare, and then shows how to implement the optimal match and payments through two-sided position auctions as a Bayesian Nash equilibrium. Chapter Three broadens these results by showing how the implementation concept can be relaxed to ex post equilibrium through the use of market designs similar to the Vickrey-Clarke-Groves mechanism, as well as implemented through the use of dynamic games. Chapter Four shows how the ideas used in Chapters Two and Three can be extended to a multi-dimensional type framework, moving away from the supermodular paradigm that is the workhorse of models of matching with incomplete information.

ESSAYS ON AUCTION AND MATCHING THEORY

by

Terence Robert Johnson

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Advisory Committee:
Professor Lawrence Ausubel, Co-Chair/Advisor
Professor Daniel Vincent, Co-Chair/Advisor
Professor Peter Cramton
Professor John Rust
Professor Raghu Raghavan

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And God: "How precious to me are your thoughts, oh God! How vast is the sum of them! Were I to count them, they would outnumber the grains of sand."

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List of Abbreviations

$\mathbb{R}$	The real numbers
$\mathbf{E}[f(x)]$	The expectation of $f(x)$ with respect to x
$\mathbf{1}_{\{x:P(x)\}}$	The indicator function of $P(x)$

CHAPTER 2

VCG	Vickrey-Clarke-Groves
SRD	Serial Random Dictatorship
I, J	The two sides of the market
K_I, K_J	The number of agents on the I and J sides
K	The total number of possible matches
q_i, q_j	Privately known qualities of agents i and j
q_I, q_J	The vector of type realizations on the I and J side, respectively
$q_{I\setminus i}$	The vector of type realizations on the I side, with component q_i removed
$\mathbf{E}_i[h(q_i, q_j)]$	Agent i's expectation of $h(q_i, q_j)$, conditional on q_i
$m_{ij}(q_i, q_{I\setminus i}, q_j, q_{J\setminus j})$	The probability that i and j are matched
$t_i(q_i, q_{I\setminus i}, q_J)$	The payment made by agent i to the matchmaker
$s_I(q_i, q_j), s_J(q_j, q_i)$	Private surpluses for the I and J sides
$F_I(q_i), F_J(q_j), f_I(q_i), f_J(q_j)$	The probability distributions of q_i and q_j, and their respective probability distribution functions
$\psi_I(q_i, q_j), \psi_J(q_j, q_i)$	Virtual Valuation functions
$R_I(q_i)$	The reservation function
$\underline{q}_I$	The worst-off participating type on the I side
$\rho_X(x_i)$	The rank of component x_i in the vector x
$w_I(q_i, k)$	The probability that q_i is the k-th highest component of the vector q_I
$b_I^{AP}(q_i)$	The bidding function in the all-pay position auction
$b_I^{WP}(q_i)$	The bidding function in the winners-pay position auction

CHAPTERS 3 and 4

VCG	Vickrey-Clarke-Groves
I, J	The two sides of the market and the number of agents on each side
K	The total number of possible matches
s_i, s_j	Privately known qualities of agents i and j
s_I, s_J, s	The vector of type realizations on the I and J side, respectively
s	The realizations of all agents' types
$s_{I\setminus i}$	The vector of type realizations on the I side, with component s_i removed
$s_{\setminus i}$	The vector of type realizations of the whole market, with component s_i removed
$\mathbf{E}_{\setminus i}[h(s_i, s_j)]$	Agent i's expectation of $h(s_i, s_j)$, conditional on s_i
$m_{ij}(s)$	The probability that i and j are matched at realization s
$t_i(s)$	The payment made by agent i to the matchmaker
$v_I(s_i, s_j), v_J(s_j, s_i)$	Private surpluses for the I and J sides
$f(s)$	The joint distribution of all types in the market
σ_i	A type report
σ_I	The type reports of the I side
σ	A type report from all agents in the market
$\sigma_{\setminus i}$	A type report from all agents in the market except i
$\varnothing$	The null variable, mapping any input into the empty set
$x_i^*(s_i)$	The security bidder's value for an agent with type s_i
c_I	Clock prices on the I side
p_k^I	The price of the k-th match make on the I side
h_t	The history of a dynamic game
A_t	The set of active bidders at time t
P_t	The set of realized prices at time t
ϕ_i	A pure strategy in a dynamic game
τ_k^I	A drop-out strategy in a dynamic game
α_i	The announcement of agent i
$s_{I[k]}, \alpha_{I[k]}$	The k-th highest signal and announcement on the I side
δ_ℓ	The marginal value of the ℓ-th worker
W_i	An ordered package of workers assigned to firm i
2^X	The power set of the set X

Chapter 1

Introduction

1.1 Matching Markets

Matching markets are central to the modern economy, and intermediaries play an important role. For example, the employment relationships formed between workers and firms in labor markets are influenced by the supply of workers and jobs, the specific skills and needs of the firms, the behavior of intermediaries like Monster.com and headhunter firms like Korn/Ferry, and the idiosyncratic chemistry between the firm and pre-existing workers with any new hire. The financial sector of the economy involves a large web of interconnected obligations intermediated by brokers and market makers: When an agent forms a risk-sharing agreement with a particular partner, he is now exposed not just to that partner's characteristic risks and outcomes, but also those of his partners, and those partners' partners, and so on. Even the Internet, central to modern commerce and social life, is intermediated by search engines like Google or Yahoo! and Internet Service Providers like Comcast or Verizon. These intermediaries are constantly making deliberate choices to decide which links between consumers and content providers or merchants are allowed, and which receive priority for scarce bandwidth or are blocked altogether.

A *matching market* is one in which there are agents on two sides — workers and firms, for example — who seek to form partnerships. One-to-one matching, where each agent can only match to a single partner, has received extensive attention, but many-to-one matching markets — such as firms seeking packages of workers — or many-to-many markets — such as on the Internet, where content providers seek many visitors, and visitors patronize many content providers — are of great interest as well. Such a market exhibits *incomplete information* and suffers from *adverse selection* if the participants have private information about their characteristics as a partner or their tastes in partner attributes. In this case, it can be to an agent's advantage to misrepresent himself to ensure a better partner; for example, a worker might get a job at a better firm by overstating his qualifications. A market outcome is *efficient* if there is no alternative matching of partners where every agent weakly prefers his new partner to his old one, and some agents strictly prefer their new partners. A market outcome is *stable* if no set of agents have a mutual incentive to "cheat" on their current partners with each other.

The central goal of this dissertation is to study how the presence of incomplete information can prevent markets from achieving efficient or stable outcomes, and if possible, characterize market designs that mitigate or eliminate the negative effects of adverse selection. Studying whether a given market structure maximizes efficiency or the profits of the intermediaries then requires tools from two different fields: Mechanism design theory and matching theory. Mechanism design provides a formal way of studying what outcomes can be achieved in markets where agents have private information, and characterizing the non-cooperative games that implement those outcomes. Matching theory analyses various algorithms or processes that might be used to construct matches between markets that are split into two distinct groups —

worker and firms, for example — and investigate whether these matches are stable. By synthesizing and extending ideas from both these fields, this dissertation makes contributions to both by characterizing efficiency and stability in matching markets with private information, as well as providing practical market designs that produce outcomes that satisfy these criteria.

1.2 Literature Review

Since the work of Spence [63] on job market signaling and the marriage and labor market models of Becker [7], economists have carefully examined how information and complementarity can be important features in determining market outcomes. In matching environments, an analogy to competitive bidding is often used: If the best workers and best firms complement one another, matches between them should generate the most social value. If this is true, then the better the worker or firm, the more aggressively they should seek out other qualified partners by offering higher wages or competing more aggressively in attaining education or marketing themselves, since the return to searching for partners is higher. These classical studies jointly suggest that markets should separate vertically, with the best matches made, achieving efficiency and stability. Many subsequent papers follow this line of reasoning, including the recent contributions of Bulow and Levin [9], Damiano and Li [22], and Hoppe, Moldovanu, and Sela [35].

However, studies of markets with incomplete information suggest that inefficiency is likely to be widespread. The classic paper by Myerson and Satterthwaite [49] considers a situation where there is the possibility of trade between a buyer and seller, but the buyer's value for the good and the seller's cost of providing it are known only to them. In this context, they show that if the agents do not have common knowledge of gains from trade — both parties do not know for sure that the buyer values the good more than the seller — the buyer will typically try to understate his value while the seller tries to overstate his cost, resulting in inefficiency when trade does not occur due to the participants' misrepresentations of their private information. However, the seminal contributions by Myerson [48] and Vickrey [66] establish that efficiency can be achieved in similar situations in the presence of competition. For example, if the seller auctions the good to a number of buyers, full efficiency or profit maximization can be achieved. More recent papers try to incorporate matching elements into the classic mechanism design framework. Damiano and Li [22] consider a monopolist matchmaker that wants to organize "meeting places" with entry fees in a manner that induces agents to self-select. Hoppe, Moldovanu and Sela [35] show that in a model of matching through costly signaling where agents are matched on the rank of their signal leads to assortative matching, and provide conditions under which costly signaling can improve welfare over random matching. Hoppe et al., however, do not consider how an intermediary would design a market to achieve objectives like profit- or welfare-maximization, and their work focuses on establishing conditions under which signaling may be socially wasteful. Lastly, Gomes [29] considers a mechanism design framework similar to the problem faced by an Internet search engine when deciding which firms to present to consumers who are searching for trading opportunities. In his paper, a finite number of consumers receive signals about their preferences to shop at firms, and an intermediary designs a game that matches a single firm to all of the users who find it profitable to trade, once the identity of the firm is known. In this setting, the intermediary receives a noisy signal of the consumers' preferences, but never interacts with them directly, and only one firm is chosen to match to all consumers. In the work considered in my dissertation, the intermediary is free to interact with all participants, and other versions of one-to-one or many-to-one matching are considered.

Another, related literature follows David Gale and Lloyd Shapley [28], and considers how stable matches might be achieved through the use of dynamic "proposal games" called *deferred acceptance algorithms*; this is most often referred to as the "matching" literature. The majority of papers in this literature assume that agents know their preferences over potential partners but cannot make side-payments in the market (Gale and Shapley, [28], Roth and Sotomayor [56]), though some papers study the case where preferences over partners are private information (Roth [54], Niederle and Yariv [50]; Coles et al. [17]). Another part of this literature considers which dynamic games lead to stable outcomes when agents can make payments, including Crawford and Kelso [19], Crawford and Knoer [20], and Hatfield and Milgrom [34]. These papers primarily show that the set of stable matches with transfers is a lattice, and provide an algorithm that maps feasible matches into matches, guaranteed to terminate at a stable match. Both these literatures generally ignore the strategic incentives provided by the matching algorithm or price adjustment process, however, which allows the possibility of profitable manipulation by the agents. As

Roth [54] shows, stability, efficiency, and honest reporting can often be in conflict: There does not exist a stable matching mechanism in which honestly revealing one's preferences is a weakly dominant strategy. This negative finding is in contrast to positive results in the auction literature — particularly Vickrey [66], Myerson [48], Ausubel [2] and [1], and Edelman, Ostrovsky and Schwarz [25] — where the problem of selling goods to buyers with privately known values for the items being sold has been analyzed with considerable success.

1.3 Summary of Contributions

This dissertation studies the challenges to achieving stability or efficiency in matching markets that result from private information in matching markets by utilizing the techniques that were successful in the auction and mechanism design literature. A central result in the mechanism design literature is the *revelation principle*: Any Bayesian Nash equilibrium of a game of incomplete information can be supported as a Bayesian Nash equilibrium of an "announcement" game in which all agents tell a fictional "mechanism operator" their true private information, and he "plays" their equilibrium strategies for them. Such an announcement game is called through a *direct revelation mechanism.* This thought experiment allows the market designer to focus on characterizing the direct revelation mechanisms that are welfare- or profit-maximizing, then look for more practical games that satisfy the same characteristics, such as auctions or non-linear tariffs.

Chapter Two adds private information to the canonical complementary matching model of Becker [7], and investigates how an intermediary can design a market to achieve efficiency and stability, or maximize the revenue of the intermediary. This work clarifies that supermodular complementarity — a common assumption in the literature — it is not sufficient to guarantee that a profit-maximizing intermediary will choose the efficient and stable match. However, when the sufficient conditions are satisfied, a set of games are characterized that implement the optimal outcome as a Bayesian Nash equilibrium. These *two-sided position auctions* are similar to those studied by Damiano and Li [22], Edelman et al. [25], Bulow and Levin [9], Varian [65], and Hoppe, Moldovanu and Sela [35], but achieve the efficient or profit-maximizing outcome, where those papers are mainly using the competitive bidding framework as an analogy for the matching process, and do not provide an analysis of the ex post efficiency or profitability of the mechanism.

Chapter Three considers how to achieve efficient outcomes in matching markets through games similar to the work of Edelman et al. [25] and Ausubel [2] and [1]. This chapter shows that the static results of Chapter Two can largely be extended to dynamic games, using primarily ex post implementation rather than Bayesian implementation. The difference is that the results of Chapter Two make a number of strong assumptions about how information is distributed in the market, while ex post equilibrium only requires that if all players act on their private information in accordance with the equilibrium, no player can have a profitable deviation ex post. This concept is robust to whether agents are poorly informed about their potential partners, have accurate information, or this information is distribute asymmetrically throughout the market, while Bayesian equilibrium is not. In many matching markets, agents may have previously been matched or have differential access to information, making the weaker solution concept of ex post equilibrium a useful feature. The major contribution is a tractable model of many-to-one matching, which is new to the competitive matching literature with private information, which has previously focused on exclusively one-to-one matching.

Chapter Four extends these results to a multi-dimensional framework, allowing agents to have privately known types composed of many pieces of information. This allows, for example, heterogeneous firms that have business plans that differentially utilize workers with heterogeneous packages of skills. The paper first shows that the classic mechanisms for achieving honest revelation of private information — serial random dictatorship, Vickrey-Clarke-Groves mechanisms, and, for one side of the market, deferred acceptance algorithms — fail to implement truth-telling in this context. Then, a mechanism is proposed that achieves efficiency as an ex post equilibrium under a condition, called *reciprocity*, that ensures there is sufficient agreement between the preferences over partners of the two sides that the matchmaker can successfully intermediate with a market design that contains that of Chapter Three as a special case.

Chapter 2

Matching Through Position Auctions

2.1 Introduction

Intermediaries play a vital role in many markets, particularly where information is scarce or unreliable. By playing the role of matchmaker, a single agent can often improve market efficiency by providing incentives for agents to reveal what they know honestly. This paper explores this possibility by studying how a matchmaker can maximize profits or welfare in a two-sided matching model. Using a mechanism design approach, I show how the matchmaker balances the motive to maximize the welfare of the participants — and his cut of the surplus — with the monopolistic desire to restrict the supply of matches to increase his profits. The main results are sufficient conditions for assortative matching to be a solution to the matchmaker's problem, and a characterization of a class of bidding games (two-sided position auctions) that can be used in practice to implement the optimal outcome.

Consider a market that is split into two distinct sides, where each agent can produce surplus only by matching to a partner on the opposite side. All participants privately know their quality as a partner, which indexes agents from worst to best. If the match surplus for each agent is increasing in each agent's quality and exhibits supermodularity, an intermediary could maximize profits and efficiency simply by ranking the agents on both sides, matching the highest-ranked agents together, the second-highest, and so on, charging the matched agents up to their willingness to pay. This kind of matching is called *assortative*, and has long been recognized to be socially efficient in such an environment. However, if the agents' information is private and the matchmaker naively asks them to report their quality, the agents have an incentive to lie about their desirability as a partner to secure a better match, creating a case of two-sided adverse selection.

In such a situation, an intermediary can provide a valuable service by designing a game that induces them to reveal their characteristics honestly. A profit-maximizing matchmaker, however, is not seeking to maximize welfare but instead his own profits. This introduces a number of complications to the problem: Should matching still be assortative? What kind of price discrimination is profit-maximizing and what are the welfare costs? How can the matchmaker achieve this outcome using various market institutions, such as posted prices or auctions? Does the matchmaker ever find it profitable to arrange matches that reduce social welfare?

Using a mechanism design framework, this paper characterizes the set of implementable direct revelation mechanisms, and provides sufficient conditions for assortative matching to be profit-maximizing for an intermediary. The profit-maximizing matchmaker engages in price discrimination by withholding some socially beneficial matches in a manner similar to a seller setting a strictly positive reserve price in independent private value auctions regardless of whether he has any value for it. If the sufficient conditions for assortative matching are met, the optimal match can be implemented through a straightforward game where all agents simultaneously submit a single sealed bid, and then the matchmaker opens and ranks the bids, matching the highest-bidding agents together, the second-highest-bidding agents, and so on, until the remaining agents are denied a match or no eligible agents remain on one side. This *two-sided position auction* maximizes revenue and extracts the true quality rankings, rewarding the higher bidders by giving them more desirable partners. Surprisingly, winners-pay auctions in which agents only pay if they receive a partner are not always a successful means of implementing the optimal outcome, leaving all-pay auctions as the only sure method of implementation without making further assumptions. This occurs when markets are "unbalanced" in the sense that one side is much larger than the other and there is considerable uncertainty about the supply of partners. In particular, an agent might like to raise his bid, but the likelihood of winning may increase much faster than the expected quality of a partner, resulting in the potential for pooling in the winners-pay format but not the all-pay format. This provides a practical reason why a matchmaker may limit participation in a matching market, even though comparative statics on the optimal mechanism show that profits are always increasing in the size of the market, given that the allocation and expected payments are the same.

A common theme in many studies of matching is the analogy to competitive bidding for partners. This goes back at least to the ideas of Spence [63] on costly signaling, and the marriage market studied by Becker [7], and is developed in many subsequent papers, including the recent contributions of Bulow and Levin [9] and Hoppe, Moldovanu, and Sela [35]. This paper explains how such an analogy can be operationalized and exploited by an intermediary, and when assortative matching actually fails to be optimal. Alternatives to assortative matching such as *coarse* *matching*, where agents are sorted into sets wherein they are randomly matched, have been explored in Damiano and Li [22], McAfee [44], and Hoppe, Moldovanu and Ozdenoren [36], but these studies have focused on practical or institutional reasons why such a mechanism would be chosen. This paper shows that when conditions on the hazard rate and supermodularity of the surplus functions are violated, coarse matching (pooling) can turn out to be optimal. This is similar to the work by Shimer and Smith [61], which provides conditions for positive assortative matching to be a solution in repeated matching economies, and shows that under some conditions assortative matching may fail to prevail in equilibrium. This provides a useful link between matching with incomplete information and market design, since it clarifies why assortative or coarse matching arises, as well as the possibility of them coexisting in the same market.

An example of a market with this structure is the executive search industry, where firms seeking to fill vacant positions approach head-hunters, who represent a pool of candidates meeting the firms' criteria. Firms compete for qualified workers by reporting a wage offer for a suitable candidate, while potential employees compete either by reporting a minimum salary required for them to change jobs, or by acquiring costly signals of quality, such as executive degrees. The head-hunter has incentives to match profitable firms with talented executives, naturally leading to a larger surplus to be split among the parties. In practice, some head-hunters charge a "finder's fee" proportion to the wage, billing their clients up front for their services regardless of whether a match is found, while others charge an initial fee and collect the proportional payment after a match is found. This paper provides a theoretical explanation for why a head-hunter might choose between these two payment structures, and clarifies how the "wages" reported by the firms should be seen as bids, strategically influenced by the game designed by the head-hunter and intended to signal the firm's quality. A similar story can be told for many other intermediaries such as financial brokers, who match investors and entrepreneurs strategically to maximize the gains from trade, as well as taking a share of the surplus themselves.

In addition to labor and financial markets, the Internet provides many novel and exciting design problems. From general issues like Net Neutrality to specific questions such as how services like eBay might manipulate the allocation of buyers and sellers of goods, intermediaries and their incentives are at the core of many important debates about policy and welfare. One prominent example is the sale of advertising tied to keywords by search engines such as Google or Yahoo. For each keyword, such as "used cars" or "apartments", a search returns both "organic" links that are ranked by their observed use on the Internet, as well as "sponsored" links that are sold by the search engine to advertisers. In this setting, the search engine acts as an intermediary between consumers searching for goods or information, and firms or organizations willing to provide their services. Research on the mechanisms used in practice (Edelman et al. [25] and Varian [65]) has found that search engines have mostly adopted the *Generalized Second Price Auction*, where bidders submit a willingness to pay for a slot, the auctioneer ranks the bids and awards slots to the bidders according to the rankings, then charges the k-th ranked bidder the $k+1$-st ranked bid. The authors call these mechanisms *position auctions*, since they use a single bid to decide rankings, which then decide the allocation of goods. This paper shows how position auctions can be generalized not just to allocate goods, but to profitably match agents who have preferences over partners. Such a generalization is of practical use in the design of markets on the Internet, where information about other agents is difficult to obtain or verify or relationships are short-lived. Moreover, this paper clarifies the matchmaker's motives, allowing an analysis of the welfare loss that results from price discrimination.

The framework developed allows comparative statics analysis on a number of fundamentals in any profit-maximizing mechanism, and shows that increasing the size of one of the market's sides tends to erode the bargaining power of agents on that side, similar to the analysis of marriage markets of Crawford [18]. The matchmaker's profits are always increasing in the size of the market, but an "improvement" in the distribution of the types in the sense of hazard-rate dominance has two effects: The participants are likely to be higher quality, but there is more exclusion in the market. Since these effects go in opposite directions, it is ambiguous how these kinds of changes affect market welfare and profits.

This paper also complements the extensive literature on the "marriage problem", starting with David Gale and Lloyd Shapley [28], by considering mechanism design in a matching environment with adverse selection. This literature has improved real world matching markets, evidenced by the National Resident Matching Program and mechanisms used to allocate students to public schools in Boston and New York (Roth [55], Abdulkadiroglu and Somnez [4]). However, almost all these papers assume that agents know their preferences over potential partners (Gale and Shapley, [28], Roth and Sotomayor [56]), though these preferences may be private information (Niederle and Yariv [50]; Coles et al. [17]). In contrast, this paper assumes that the agents know they prefer higher quality partners, but cannot discern the desirability of any particular agent due to private information. In Niederle and Yariv [50], players are assumed to know their own preferences over partners and the existence of a unique stable match, but may be too impatient to wait for a decentralized market to identify their stable partners when players use proposal and acceptance strategies like those in a deferred acceptance algorithm. Many markets suffer from impatience or fixed deadlines and other imperfections which may tempt players to leave the market early or deviate from the behavior Niederle and Yariv consider, creating a scope for an intermediary like the one studied in this paper to improve performance.

This paper also contributes to the mechanism design literature by extending seminal work by Myerson and Satterthwaite [49], who studied whether and how to arrange a single transaction between two agents to which transactions to arrange among many possible partners with externalities within the match. The closest framework to the one in this paper is Damiano and Li [22], who consider a monopolist matchmaker that wants to organize "meeting places" with entry fees in a manner that induces agents to self-select. They conclude that adding more meeting places always increases revenue, and show that in the limit the monopolist would like to have an infinite number of meeting places and match agents assortatively, as well as efficiently. In this paper, the opposite conclusion holds: The matchmaker introduces significant inefficiency and, in general, has incentives to oppose changes that improve the ex ante quality of the participants. Moreover, they do not consider how the matchmaker could improve outcomes in the finite case by constructing a game in a more straightforward mechanism design framework where communication between the matchmaker and the agents is allowed, rather than exploring the limit of a model where posted prices are used. Hoppe, Moldovanu and Sela [35] show that in a model of matching through costly signaling where agents are matched on the rank of their signal leads to assortative matching, and provide conditions under which costly signaling can improve welfare over random matching. Hoppe et al., however, do not consider how an intermediary would design a market to achieve objectives like profit- or welfare-maximization, and their paper is mainly a positive analysis of costly signaling in the tradition of Spence [63]. The results in both Damiano and Li and Hoppe et al. make extensive use of the assumption that the matching surplus is Cobb-Douglas, which provides tractability for many results, but that assumption is quite restrictive. In particular, the analysis of Hoppe et al. relies on being able to convert agents' expected utilities as sums and differences of order statistics through an analogy to Vickrey-Clarke-Groves mechanisms, which is not possible in the framework in this paper. In the current paper, no assumptions are made about functional form, only the signs of derivatives and the relationship to the hazard rates of the players' type distributions. More importantly, this paper includes markets where some matches are not socially beneficial, which is not included in the Cobb-Douglas case. Lastly, Gomes [29] considers a mechanism design framework in which a finite number of agents on both sides of the market receive signals about their preferences for partners on the opposite side, and an intermediary platform constructs a revelation mechanism to match one firm to the users who find it profitable to pay an entry fee. This framework is modelled after Internet search, where firms are induced to reveal information honestly to the platform, who then sells one firm the right to match with users who participate. In my framework, the matchmaker is attempting to match users and firms one-to-one — rather than many-to-one — and receives messages from both sides, rather than just the firms.

2.2 Model

There are two disjoint sets of agents, I and J, with K_I agents on the I side and K_J agents on the J side; let $K = \min\{K_I, K_J\}$, which is the largest number of matches that can be arranged in this market. Each I-side agent would like to match to one agent on the J side, and each agent on the J side would like to match to one agent on the I side. Each I-side agent has a privately known quality drawn from an absolutely continuous probability distribution function $F_I(q)$, with support on $[0, \bar{q}_I]$, with q_j defined likewise. Let $q_I = (q_{I1}, q_{I2}, ..., q_{IK_I})$ be a vector of types for all the agents on the I side, and q_J defined similarly. To focus on a particular agent i on the I side, let $q_{I\setminus i} = (q_{I1}, q_{I2}, ..., q_{Ii-1}, q_{Ii+1}, ..., q_{IK_I})$, the vector of types from the I side where the i-th entry is removed, so that $q_I = (q_i, q_{I\setminus i})$.

Agents who are matched produce *pairwise private surpluses* $s_I(q_i, q_j)$ for the I-side agent and $s_J(q_j, q_i)$ for the J-side agent; this can be thought of as truly private surplus, in the case of a marriage between two individuals who derive value from each others' company, or as the equilibrium payoff from a perfect information, non-cooperative game played after the matching game that satisfies appropriate restrictions[1]. Surplus is a function of both agents' qualities, increasing in both arguments, and differentiable.

Agents have quasi-linear preferences, so an agent i paying t to match to agent j receives a payoff of

$$s_I(q_i, q_j) - t$$

Throughout, the rankings of components in a vector is important. For a vector $X = (x_1, x_2, ..., x_i, ..., x_K)$ with some component x_i, let $\rho_X(x_i)$ be the *rank* of x_i in X:

$$\rho_X(x_i) = |\{x_k \in X : x_k \geq x_i\}|$$

where $|A|$ is the number of elements in a set A. So $\rho_X(x_i) = 5$ implies that there are 5 elements in X greater than or equal to x_i (including itself), and if $\rho_X(x_i) = 1$, it is the largest element in X (since x_i is less than or equal to itself).

For an arbitrary function $h(q_I, q_J)$, let

$$\mathbf{E}_i[h(q_i, q_{I\setminus i}, q_J)] = \mathbf{E}_{q_{I\setminus i}, q_J}[h(q_i, q_{I\setminus i}, q_J)]$$

This is the expectation of $h(q_I, q_J)$ conditional on i's information; it can be read as "agent i's expectation of $h(q_I, q_J)$".

In addition, make the standard assumption that the monotone likelihood ratio

[1]For example, suppose the players' payoffs are a function $\pi_i(s_i, s_j, q_i)$ that exhibit strategic complementarities and $\partial^2\pi_i/\partial s_i \partial q_i \geq 0$. Then there is a lattice of equilibria for the players, with a "best" equilibria, and their strategies are increasing in their type and their opponent's type. (see Milgrom and Roberts [47])

property holds:

$$\frac{d}{dq}\frac{1-F_I(q)}{f_I(q)} \leq 0 \tag{2.1}$$

The monotone likelihood ratio property is often invoked in the mechanism design literature to rule out pooling, and many common distributions satisfy it, such as the Normal family, the exponential, the Pareto, and the uniform distributions (see Bagnoli and Bergstrom [6]).

Let the cost of arranging a match be c, independent of the realized qualities and reports. This can be interpreted as the bureaucratic cost to the matchmaker associated with arranging the consummation of the match or the cost of the legal burden of connecting agents, such as background checks, insurance against civil lawsuits, and investigating potential fraud.

A *direct revelation mechanism* is a set of functions

$$\{m_{ij}(q_I, q_J), t_i(q_I, q_J), t_j(q_I, q_J)\}_{i,j}$$

that take the type spaces of the agents as arguments, where $m_{ij}(q_I, q_J)$ is the probability that i is matched to j given types q_I and q_J, $t_i(q_I, q_J)$ is the amount paid by agent i given q_I and q_J, and $t_j(q_I, q_J)$ is the amount paid by agent j given q_I and q_J.

Consider the direct revelation mechanism as a non-cooperative game where agents each simultaneously announce a type — not necessarily truthfully — and get the payoff associated with their announced type, given the announcements of all the other players. Then the *revelation principle* asserts that given any game of incomplete information and an equilibrium for that game, there exists a direct revelation mechanism where truth-telling is a Bayesian Nash equilibrium and the outcomes are the same as the original equilibrium payoffs.

As in Myerson [48], we imagine that the matchmaker announces the mechanism to the agents, the agents then strategically report a type or decide not to participate, and then are matched and make payments according to the mechanism. To ensure that truth-telling is a Bayesian Nash equilibrium of the announcement game, incentive compatibility and individual rationality restrictions are imposed. Since this thought experiment characterizes all the implementable mechanisms, the profit- or welfare-maximizing mechanism can be found and characterized. With such a characterization available, any game can then be analyzed to see if it implements the same outcome, and is therefore an optimal *indirect implementation.* This paper characterized incentive compatible matching mechanisms, derives sufficient conditions for assortative matching to be a solution, then shows how two construct optimal two-sided position auctions.

Since agents have quasi-linear preferences, agent i has an *indirect utility function* from participating in a given mechanism of

$$U_I(q_i) = \max_{q'} \mathbf{E}_i \left[\sum_j m_{ij}(q', q_{I\setminus i}, q_J) s_I(q_i, q_j) - t_i(q', q_{I\setminus i}, q_J) \right]$$

Since no special distinctions are made between the I side and the J side of the market all analysis will apply equally to both sides, so discussion will focus on the I side, but all findings hold with the appropriate permutations of indices for the J side.

2.3 The Direct Revelation Mechanism

This section characterizes the profit-maximizing direct revelation mechanism, compares it with the welfare-maximizing direct revelation mechanism, and derives a number of comparative statics results showing how agents' payoffs are influenced by changes in the fundamentals of the market.

2.3.1 The Mechanism Design Problem

The matchmaker seeks to maximize expected profits:

$$\mathbf{E}\pi = \max_{t_i, t_j, m_{ij}} \mathbf{E}\left[\sum_i t_i(q_I, q_J) + \sum_j t_j(q_J, q_I) - \sum_i \sum_j m_{ij}(q_I, q_J)c\right] \tag{2.2}$$

subject to individual rationality constraints and incentive compatibility constraints: The mechanism is *individually rational* if, for all agents i on the I side with true quality q_i, and similarly for all agents j on the J side,

$$\mathbf{E}_i\left[\sum_j m_{ij}(q_i, q_{I\setminus i}, q_J)s_I(q_i, q_j) - t_i(q_i, q_{I\setminus i}, q_J)\right] \geq 0 \tag{2.3}$$

and the mechanism is *incentive compatible* if, for all agents i on the I side and similarly for all agents j on the J side, and for all q' not equal to the agent's true type q_i,

$$\begin{aligned}\mathbf{E}_i&\left[\sum_j m_{ij}(q_i, q_{I\setminus i}, q_J)s_I(q_i, q_j) - t_i(q_i, q_{I\setminus i}, q_J)\right] \\ &\geq \mathbf{E}_i\left[\sum_j m_{ij}(q', q_{I\setminus i}, q_J)s_I(q_i, q_j) - t_i(q', q_{I\setminus i}, q_J)\right]\end{aligned} \tag{2.4}$$

The incentive compatibility constraints require that no agent finds it in his best interests to lie about his type, while the individual rationality constraints require that no agent who participates receives a lower payoff than if he had refused to participate at all. In particular, they ensure that truthful reporting is a Bayesian Nash equilibrium of the announcement game.

2.3.2 Incentive Compatibility

Define $\underline{q}_I$ as the agent on the I side with the lowest q that chooses to participate, rather than withdraw from the market and receive a payoff of zero.

Lemma 2.3.1 (Incentive Compatibility) *The mechanism is incentive compatible if and only if the following conditions hold: (i)* the envelope condition

$$U_I(q_i) = \mathbf{E}_i\left[U_I(\underline{q}_I) + \int_{\underline{q}_I}^{q_i} \sum_j m_{ij}(z, q_{I\setminus i}, q_J)\frac{\partial s_I(z, q_j)}{\partial q_i}dz\right] \tag{2.5}$$

and (ii) the monotonicity condition *that for $q' \neq q$,*

$$\int_q^{q'} \mathbf{E}_i\left[\sum_j (m_{ij}(z, q_{I\setminus i}, q_J) - m_{ij}(q', q_{I\setminus i}, q_J))\frac{\partial s_I(z, q_j)}{\partial q_i}\right] dz \leq 0 \tag{2.6}$$

Since a profit-maximizing matchmaker is being considered, the individual rationality constraint for the worst-off participating type will bind, implying that $U_I(\underline{q}_I) = 0$. By equating the envelope condition and the indirect utility function, interim expected transfers can be isolated for any incentive compatible and individually rational mechanism:

$$\mathbf{E}_i[t_i(q_I, q_J)] = \mathbf{E}_i\left[\sum_j m_{ij}(q_i, q_{I\setminus i}, q_J)s_I(q_i, q_j) - \int_{\underline{q}_I}^{q} m_{ij}(z, q_{I\setminus i}, q_J)\frac{\partial s_I(z, q_j)}{\partial q_i}dz\right]$$

Then ex ante expected transfers can be found through an integration by parts:

$$\mathbf{E}\left[t_i(q_i, q_j)\right] = \mathbf{E}\left[\sum_j m_{ij}(q_i, q_{I\setminus i}, q_J)\left(s_I(q_i, q_j) - \frac{1 - F_I(q_i)}{f_I(q_i)}\frac{\partial s_I(q_i, q_j)}{\partial q_i}\right)\right]$$

Define *virtual surplus* as

$$\psi_I(q_i, q_j) = s_I(q_i, q_j) - \frac{1 - F_I(q_i)}{f_I(q_i)} \frac{\partial s_I(q_i, q_j)}{\partial q_i}$$

This can be interpreted as the marginal revenue accruing to the matchmaker from agent i when he is matched to agent j.

Then

$$\mathbf{E}\left[t_i(q_i, q_j)\right] = \mathbf{E}\left[\sum_j m_{ij}(q_I, q_J)\psi_I(q_i, q_j)\right]$$

Substituting this expression for expected transfers into the matchmaker's problem yields an equivalent program:

$$\mathbf{E}\pi = \max_{m_{ij}} \mathbf{E}\left[\sum_i \sum_j m_{ij}(q_I, q_J)(\psi_I(q_i, q_j) + \psi_J(q_j, q_i) - c)\right] \tag{2.7}$$

subject to the monotonicity condition.

So the matchmaker's problem is reduced to choosing a matching function on the basis of the reports, with the transfers eliminated. The monotonicity condition essentially requires that the matchmaker assign a more advantageous lottery over partners to agents who report higher types. Consider the *relaxed program* where the monotonicity condition is dropped:

$$\mathbf{E}\pi = \max_{m_{ij}} \mathbf{E}\left[\sum_i \sum_j m_{ij}(q_I, q_J)(\psi_I(q_i, q_j) + \psi_J(q_j, q_i) - c)\right]$$

To proceed, this relaxed optimization problem will be solved, and then it will be shown that the monotonicity constraint is satisfied at that solution (Theorem 2.3.6). Consequently, the unconstrained solution is a solution to the constrained problem.

2.3.3 Truncated Assortative Matching and the Reserve Function

By inspection of the objective function for the relaxed program, it is evident that the matchmaker refuses to put any agents together who satisfy $\psi_I(q_i, q_j) + \psi_J(q_j, q_i) < c$, since such a match generates negative profits. Consider the match function that pairs agents assortatively, putting the best I agent with the best J agent, the second-best I agent with the second-best J agent, and so on, but stops when a pair generates negative revenue for the matchmaker:

Definition *Truncated Assortative Matching (TAM)*

$$m_{ij}(q_I, q_J) = \begin{cases} 1 & , \rho_{q_I}(q_i) = \rho_{q_J}(q_j) \text{ and } \psi_I(q_i, q_j) + \psi_J(q_j, q_i) \geq c \\ 0 & , \text{ otherwise} \end{cases}$$

This mechanism has the notable property that for those agents who receive a match, the allocation is only dependent on rankings of the agents' qualities, not on the realization of qualities per se. This feature allows agents to contemplate whether they will attain a certain rank when formulating their strategies, rather that sorting through different possible realizations of all the other agents' qualities.

Lemma 2.3.2 (Assignment-Optimality of TAM) *If the surplus function is* supermodular

$$\frac{\partial^2 s_I(q_i, q_J)}{\partial q_i \partial q_j} \geq 0$$

and exhibits decreasing supermodularity

$$\frac{\partial^3 s_I(q_i, q_j)}{\partial q_i^2 \partial q_j} \leq 0 \tag{2.8}$$

then TAM is a solution to the relaxed matching problem.

Proof First, note that the cross-partial derivative of $\psi_I(q_i, q_j)$ is

$$\frac{\partial^2 \psi_I(q_i,q_j)}{\partial q_i \partial q_j} = \frac{\partial^2 s_I(q_i,q_j)}{\partial q_i \partial q_j} - \frac{1-F_I(q_i)}{f_I(q_i)} \frac{\partial^3 s_I(q_i,q_j)}{\partial q_i^2 \partial q_j} - \frac{d}{dq_i}\left[\frac{1-F_I(q_i)}{f_I(q_i)}\right] \frac{\partial^2 s_I(q_i,q_j)}{\partial q_i \partial q_j}$$

From the assumption that $s_I(q_i, q_j)$ is supermodular, the monotone likelihood ratio property of $F_I(q)$, and the added assumption of $\partial^3 s_I/\partial q_i^2 \partial q_j \leq 0$, the cross-partial of virtual surplus is positive, so virtual surplus is also supermodular.

It is obvious that leaving any two agents unmatched who satisfy $\psi_I(q_i, q_j) + \psi_J(q_j, q_i) \geq c$ is suboptimal, since these agents could be matched and revenue increased. Likewise, matching any pair of agents who satisfy who satisfy $\psi_I(q_i, q_j) + \psi_J(q_j, q_i) < c$ is suboptimal, since this reduces revenue for the matchmaker.

Now consider any match scheme other than truncated assortative matching, where all the current matches are generating positive revenue and no pair of unmatched agents could be profitably matched. It necessarily includes two pairs (q_{i1}, q_{j2}) and (q_{i2}, q_{j1}) with, say, $q_{i1} > q_{i2}$ but $q_{j1} > q_{j2}$. Consider the change in virtual surplus from switching to TAM, where both matches are still profitable after the switch:

$$[\psi_I(q_{i1}, q_{j1}) + \psi_I(q_{i2}, q_{j2}) + \psi_J(q_{j1}, q_{i1}) + \psi_J(q_{j2}, q_{i2})] -$$
$$[\psi_I(q_{i1}, q_{j2}) + \psi_I(q_{i2}, q_{j1}) + \psi_J(q_{j1}, q_{i2}) + \psi_J(q_{j2}, q_{i1})]$$

This equals

$$\int_{q_{i2}}^{q_{i1}} \int_{q_{j2}}^{q_{j1}} \frac{\partial^2 \psi_I(q_i,q_j)}{\partial q_i \partial q_j} dq_i dq_j + \int_{q_{i2}}^{q_{i1}} \int_{q_{j2}}^{q_{j1}} \frac{\partial^2 \psi_J(q_i,q_j)}{\partial q_i \partial q_j} dq_i dq_j$$

Since virtual surplus is supermodular, the integrands are positive, and this swap has increased revenue in equation (1.7). Therefore, any switch toward TAM raises the value of the objective function when no matches are destroyed.

Suppose, however, that such a switch results in the lower pair generating virtual surplus of $\psi_I(q_{i2}, q_{j2}) + \psi_J(q_{j2}, q_{i2}) < c$, so the choice is between one good match and two mediocre matches. By way of contradiction, suppose that choosing the one good match lowers profits, or

$$\psi_I(q_{i1}, q_{j1}) + \psi_I(q_{j1}, q_{i1}) - c \\ - [\psi_I(q_{i1}, q_{j2}) + \psi_I(q_{j2}, q_{i1}) - c + \psi_I(q_{i2}, q_{j1}) + \psi_I(q_{j1}, q_{i2}) - c] < 0$$

$$\int_{q_{i2}}^{q_{i1}} \int_{q_{j2}}^{q_{j1}} \frac{\partial^2 \psi_I(q_i, q_j)}{\partial q_i \partial q_j} dq_i dq_j + \int_{q_{i2}}^{q_{i1}} \int_{q_{j2}}^{q_{j1}} \frac{\partial^2 \psi_J(q_i, q_j)}{\partial q_i \partial q_j} dq_i dq_j \\ - [\psi_I(q_{i2}, q_{j2}) + \psi_J(q_{j2}, q_{i2}) - c] < 0$$

But this is a contradiction, since the integrands are positive, and $\psi_I(q_{i2}, q_{j2}) + \psi_J(q_{j2}, q_{i2}) < c$, so the one good match over two mediocre matches must be more profitable after all. This argument shows that any proposed match can be improved unless it is identically TAM. Therefore, TAM is profit-maximizing in the unconstrained problem. ∎

The key to the proof is that virtual surplus must also be supermodular to guarantee that assortative matching is profit-maximizing in this environment. To ensure this, three conditions are required: the monotone likelihood ratio property, supermodularity, and decreasing supermodularity. If these conditions are violated, the matchmaker might like to deviate from assortative matching, which can lead to *coarse matching*, or pooling. Previous work (Damiano and Li [22], McAfee [44], Hoppe, Moldovanu and Ozdenoren [36]) have focused on institutional or practical reasons why coarse matching might arise in practice, but this analysis shows that pooling can be the optimal choice for markets where one or more of conditions for virtual surplus to be supermodular are violated and the matchmaker would actually find it profitable to utilize non-assortative matching — which can, moreover, arise even if supermodularity holds.

Decreasing supermodularity appears elsewhere — see Fudenberg and Tirole [27] — and essentially brings stability to the market: If even the lowest-quality agents on one side found it profitable to bid aggressively in an attempt to get the best partner on the other side of the market due to quality externalities, the matchmaker may fail to be able to separate the types, forcing him — in the most extreme case— to adopt a lottery and pool all types. Under this condition, such a case is ruled out.

As previously noted, assortative matching is the socially optimal assignment in environments with supermodular surplus functions. However, due to matching costs and price discrimination motives, the matchmaker has incentives to restrict the supply of matches. The fact that the matchmaker refuses to put some agents together despite generating positive value is analogous to an auctioneer's decision to set a strictly positive reserve price, even if his value for the item for sale is zero. This is the source of inefficiency, and is a consequence of the profit-maximizing motives of the matchmaker. To warrant matching, the two agents must satisfy

$$\psi_I(q_i, q_j) + \psi_J(q_j, q_i) \geq c$$

Let the *reservation function* be

$$R_I(q_i) = \min_{q_j}\{q_j : \psi_I(q_i, q_j) + \psi_J(q_j, q_i) \geq c\}$$

This gives the lowest-quality partner that the matchmaker will allow to match to an agent with quality q_i. It is implicitly defined as the function satisfying

$$\psi_I(q_i, R_I(q_i)) + \psi_J(R_I(q_i), q_i) = c$$

Lemma 2.3.3 (Reserve Function) *Suppose*

$$\frac{f_I(q_i)}{1 - F_I(q_i)} \geq \frac{\partial}{\partial q_i} log\left(\frac{\partial s_I(q_i, q_j)}{\partial q_j}\right) \tag{2.9}$$

and likewise for the J *side. Then* $R_I(q_i)$ *is decreasing in* q_i*. If the worst-off agent who participates has a quality strictly greater than* 0*, his quality* $\underline{q}_I$ *satisfies*

$$\psi_I(\underline{q}_I, \bar{q}_J) + \psi_J(\bar{q}_J, \underline{q}_I) = c$$

The condition in Equation (2.1.9) states that the growth rate of marginal utility of partner quality in q_i is bounded by the hazard rate of q_i. If this condition were violated, it would be possible that getting a better partner would reduce an agent's virtual surplus. As a result, the matchmaker might like to block the match if joint virtual surplus falls below the match cost, but this could potentially lead to a violation in the monotonicity condition, resulting in pooling. Equation (2.1.9) provides a sufficient condition to rule this possibility out. Interestingly, the efficient matchmaking scheme in Section 2.3.6 does not require this condition, implying that it arises solely as a consequence of the matchmaker's efforts to price discriminate.

For some commonly used surplus functions, another approach to proving that $R_I(q_i)$ is decreasing is available:

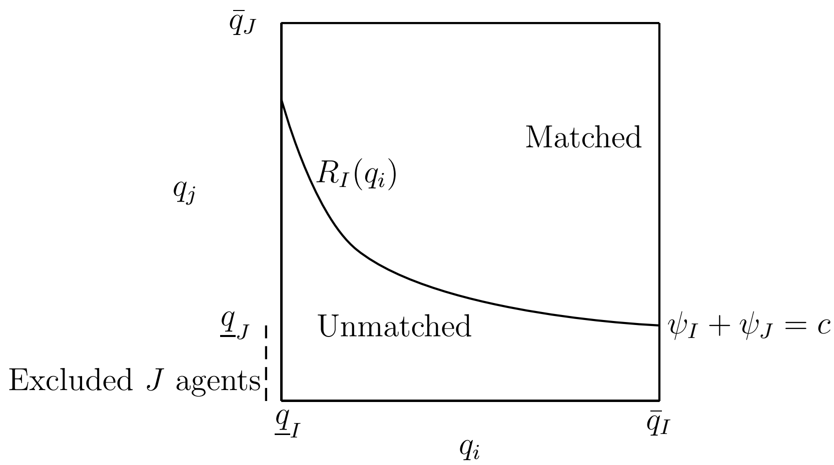

Figure 2.1: Downward-Sloping Reserve Function

Proposition 2.3.4 *Suppose that*

$$s_I(q_i, q_j) = \lambda_I a_I(q_i) a_J(q_j)$$

and

$$s_J(q_j, q_i) = \lambda_J a_I(q_i) a_J(q_j)$$

Assume $a_I(q)$ and $a_J(q)$ are positive, increasing and concave. Then $R_I(q_i)$ is downward sloping.

This includes any games with symmetric Cobb-Douglas or Stone-Geary matching surplus functions, or bargaining games with common surplus $a_I(q_i)a_J(q_j)$ where $\lambda_I + \lambda_J = 1$. Also, it can be adapted to include environments where $\lambda_I < 0$ but $\lambda_J > 0$, such as workers on the I side who incur the costs of labor effort, and firms on the J side who take in revenue from the efforts of the workers.

Figure 1.1 illustrates that there are two kinds of exclusion in the model. The mechanism exhibits *absolute exclusion* if there exists a set of types of strictly positive measure which prefer not to participate, in the sense that they expect non-positive surplus from the mechanism almost surely. *Relative exclusion* is realized after the type-announcements are made: Some pairs of agents are barred from matching, since their joint virtual surplus is too low, despite generating positive social surplus. Figure 1.1 shows an example where some agents on the J side face absolute exclusion and will never receive a partner by participating in the mechanism, while every agent on the I side has a non-empty set of types on the J side that he could be paired with.

It is remarkable that there is a single reserve function that depends only on joint virtual surplus and the matching cost: there is not a separate reserve function for the best match, the second-best match, and so on, and the reserve function does not depend on the number of agents in the market. This result is analogous to symmetric independent private value auctions where the optimal reserve price does not depend on the number of bidders, but surprising since the matchmaker optimally decides for only one criteria on which to allow or block a match, rather than a different one for each potential rank. If there was a cost schedule $\{c_1, c_2, ..., c_K\}$, there would be a different threshold for each match, creating a supply function for matches on the part of the intermediary and the potential for market congestion effects.

The following proposition provides a necessary and sufficient condition on the primitives of the model to determine if absolute exclusion occurs:

Proposition 2.3.5 *Absolute exclusion on the I side occurs if and only if*

$$s_I(0, \bar{q}_J) + s_J(\bar{q}_J, 0) - \frac{1}{f_I(0)} \frac{\partial s_I(0, \bar{q}_J)}{\partial q_i} < c \tag{2.10}$$

This shows that in matching environments, absolute exclusion may not occur on either side of the market, and all types have a positive probability of receiving a partner ex ante. It also provides a useful way of using only model fundamentals to check whether absolute exclusion occurs without solving for the optimal mechanism or equilibrium strategies.

2.3.4 Optimality of TAM in the Constrained Problem

With TAM sufficiently characterized, it can be verified that the monotonicity condition is satisfied under the sufficient conditions developed.

Let the *weighting function* be defined as

$$w_{I,k}(q) = \frac{(K_I - 1)!}{(K_I - k)!(k-1)!} \; F_I(q)^{K_I - k}(1 - F_I(q))^{k-1} \tag{2.11}$$

This is the probability of coming in rank k out of K_I draws from distribution F_I with value q (note that, in keeping with economic tradition, the highest value is ranked 1, and the lowest value is ranked K_I). The density of the k-th of K_J order statistics is:

$$f_{J,(k)}(x) = \frac{K_J!}{(K_J - k)!(k-1)!} \; F_J(x)^{K_J - k}(1 - F_J(x))^{k-1} f_J(x) \tag{2.12}$$

While Section 2.3.3 shows that TAM is optimal when the monotonicity condition is ignored and characterized reserve behavior of the matchmaker, it remains to show that the constraint is actually satisfied. Theorem 2.3.6 shows that the condition is satisfied under the set of assumptions previously developed. It also provides a revenue equivalence result: Under the assumptions of Theorem 2.3.6, any profit maximizing mechanism assigns matches according to TAM, and charges the agents the same expected payments.

Theorem 2.3.6 (Optimality and Revenue Equivalence) *Under the assumptions of supermodularity, decreasing supermodularity, and equation (1.9), TAM is a solution to the mechanism design problem; i.e., it satisfies the monotonicity condition. In any profit-maximizing mechanism, the interim expected transfers equal*

$$\begin{aligned}&\mathbf{E}_i[t_i(q_i, q_{I\setminus i}, q_J)] \\ &= \sum_k w_{I,k}(q_i) \int_{R_I(q_i)}^{\bar{q}_J} s_I(q_i, y) f_{J,(k)}(y) dy \\ &- \int_{\underline{q}_I}^{q_i} w_{I,k}(z) \int_{R_I(z)}^{\bar{q}_J} \frac{\partial s_I(z, y)}{\partial q_i} f_{J,(k)}(y) dy dz\end{aligned}$$

This theorem holds because the supermodularity property of the surplus functions and the first-order stochastic dominance properties of distributions of order statistics ensure that the mechanism assigns more favorable lotteries over partners to agents who submit higher reports, and this increases the matchmaker's profits. This aligns the matchmaker's incentives with those of the agents, and they have no profitable deviations from honestly reporting their quality if everyone else is honest.

The proof uses the fact that agents care about the quality of a partner rather than their "name", and TAM makes the assignment on the basis of ranking the agents' reports. Without this property, agents would have to worry about the realizations of all other agents' qualities, rather than just their ranks, making the problem much more complicated — this is what makes many-to-one matching with declining values for partners difficult to solve.

2.3.5 Welfare and Comparative Statics

In the framework developed above, a number of comparative statics predictions can be made concerning how exclusion and payoffs change under the profit-maximizing mechanism when the size or quality of the markets changes. An absolutely continuous distribution F *hazard rate dominates* an absolutely continuous distribution G if, for all x,

$$\frac{f(x)}{1-F(x)} \leq \frac{g(x)}{1-G(x)} \longleftrightarrow \frac{1-F(x)}{f(x)} \geq \frac{1-G(x)}{g(x)}$$

Hazard rate dominance implies the more commonly used *first-order stochastic dominance*, but is more closely related to the payoffs and strategies of the agents since the inverse of the hazard rate figures into the informational rents extracted by market participants.

Proposition 2.3.7 (Exclusion) *(i) If F_I^1 hazard-rate dominates F_I^2, then absolute exclusion is higher in the market under F_I^1 than the market under F_I^2; absolute exclusion does not depend on F_J. (ii) If F_J^1 hazard-rate dominates F_J^2, then there is more relative exclusion under F_J^1 than F_J^2 . (iii) If the cost schedule $c^1 \geq c^2$, then there is more absolute and relative exclusion under c^1 than c^2.*

Absolute and relative exclusion don't depend on K_I or K_J, similar to the result that reserve prices in auctions do not depend on the number of participants.

This result is surprising, since it might be anticipated that if one side were much larger than the other, payments might be depressed on the larger size. In fact, the matchmaker doesn't face this trade-off on exclusion, but on which kinds of bidding games implement the optimal outcome (see the discussion following Theorem 4.1).

Proposition 2.3.8 (Market Size) *(i) If K_J increases, the interim expected utility of the agents on the I side increase. (ii) If K_I increases, the interim expected utility of the agents on the I side decrease. (iii) If K_I or K_J increase, the profits of the matchmaker increase.*

This result mirrors Crawford [18], in that increasing the size of one side of the market generally reduces the interim expected payoffs of agents on that side, but increases the interim expected of agents on the other side.

Proposition 2.3.9 (Own-Side Effects) *Suppose F_{I1} hazard-rate dominates F_{I2}. Then the interim expected payoffs of I side agents are higher under F_{I2}.*

When the hazard rate decreases for all types, higher types become more likely. It might be expected that this would have unambiguous effects in increasing profit or the other side's payoffs, but that happens to fail. The reason is that the matchmaker responds to such a change in the market by increasing exclusion, thereby reducing the likelihood of a match. So while the matches made will be better in expectation, fewer are likely to be made. This provides an interesting insight into matchmaker behavior: Intermediaries will always be interested in fostering market size, but may be more reticent about changes that improve the quality of one side since that improves that side's bargaining power.

2.3.6 Efficiency

The above analysis is framed in terms of a monopolist matchmaker trying to maximize his profits. In many applications of a theory of matchmaking, the central authority may be organized by some organization or government, such as the NRMP or the junior academic recruiting process in economics. In electricity markets, for example, governments often empower a third-party regulator to design a market that matches electricity generators and distributors or consumers, and gives any economic profit back to the government. Suppose that the matchmaker has been mandated to maximize total surplus, the sum of transfers and the participants' utilities.

Since the necessary and sufficient conditions for individual rationality and incentive compatibility are derived independently of the objective function of the matchmaker, they also apply to this social planner's problem. Therefore, almost all the work done for the profit-maximizing matchmaker applies here, except the reserve function. Then the same transfer functions can be substituted into the objective function for the regulator, just as for the pure profit-maximizer. Then, summing the agents' welfare and the profits from matching them, the social planner faces the objective function:

$$\mathbf{E}W = \max_{m_{ij}^*} \mathbf{E}\left[\sum_i \sum_j m_{ij}^*(q_I, q_J)\left\{s_I(q_i, q_j) + s_J(q_j, q_i) - c\right\}\right]$$

Since the objective exhibits increasing differences, the optimal rule is:

$$m_{ij}^*(q_I, q_J) = \begin{cases} 1 & , \rho_{q_I}(q_i) = \rho_{q_J}(q_j) \text{ and } s_I(q_i, q_j) + s_J(q_j, q_i) \geq c \\ 0 & , \text{otherwise} \end{cases}$$

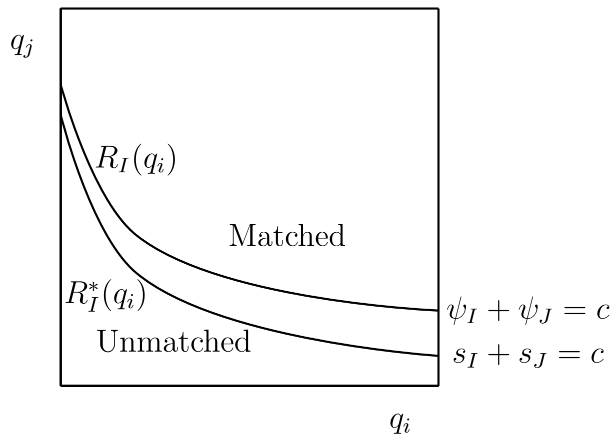

Figure 2.2: Surplus-Maximizing vs. Profit-Maximizing Matchmaking

The only difference between this case and the monopolist case is the reserve condition. Joint total surplus has to justify the matching cost, while the monopolist considers the joint virtual surplus. Just as with the monopolist matchmaker there is a reserve criterion that divides the set of matched and unmatched agents in equilibrium:

$$s_I(q_i, R_I^*(q_i)) + s_J(R_I^*(q_i), q_i) = c$$

The slope of $R_I^*(q_i)$ is

$$R_I^{*\prime}(q_i) = -\frac{\dfrac{\partial s_I(q_i, R_I^*(q_i))}{\partial q_i} + \dfrac{\partial s_J(R_I^*(q_i), q_i)}{\partial q_j}}{\dfrac{\partial s_I(q_i, R_I^*(q_i))}{\partial q_j} + \dfrac{\partial s_J(R_I^*(q_i), q_i)}{\partial q_i}} < 0$$

So there is a unique lowest-quality partner for every q_i that results in a successful match. The welfare-maximizing mechanism can be shown to be incentive compatible by showing that $m_{ij}^*(q_I, q_J)$ satisfies the monotonicity constraint.

In Figure 1.2, the pairs between R_I and R_I^* go unmatched when the monopolist decides the allocation, but are matched by the social planner. The quasi-linearity assumption on utilities insures that over the region in which both mechanisms match agents, the payments made by the agents to the matchmaker fall, and in the region where the efficient program matches the agents but the monopolist doesn't, the total surplus is completely lost.

Also interesting is the fact that $R^*(q_i)$ is decreasing without requiring any extra conditions such as the one in Lemma 2.3.3. This shows that the need for conditions such as Equation (2.2.9) stems from the matchmaker's attempts to price discriminate, rather than simply the presence of the externalities.

2.3.6.1 Comparison with the Double Auction

If $s_I(q_i, q_j) = q_i$ and $s_J(q_i, q_j) = -q_j$, then this framework corresponds to the same agent preferences as a double auction, similar to the models studied in Gresik and Satterthwaite [32], McAfee [43], Rustichini, Satterthwaite and Williams [57], Satterthwaite and Williams [59], and Cripps and Swinkels [21]. A common theme in that literature is that double auctions are only efficient as the number of traders becomes large, so that the probability that a given trader's bid affects the clearing price goes to zero. So it is surprising that this framework can achieve efficient allocation without passing to the limit.

The reason is that the double auction is a direct generalization of the Myerson and Satterthwaite bargaining game where a single trading price is maintained. The traders consider whether they will be the "marginal agent" who is setting the clearing price, and attempt to extract rents from their potential partner by bidding strategically, in the same way that agents in the Myerson and Satterthwaite framework try to over- or under-state their values to extract more rents for themselves. In short, agents are directing their informational advantage at both their opponents on their side of the market, but also at their potential partner. In the framework developed in this paper, the matchmaker utilizes essentially two separate auctions on each side of the market to direct the traders' informational advantage at opponents on their side. By setting the efficient level of exclusion, then, allocative efficiency can be achieved, even with the externalities present in a matching framework but not in a double auction.

2.4 Indirect Implementation

Any theory that provides a profit-maximizing direct revelation mechanism but does not investigate characterizations of games that implement the optimal outcomes is incomplete. The quality information held by the agents in the market might be abstract and difficult to communicate, requiring some other means of allowing them to report this information to the matchmaker. Similarly, this section provides a class of games that achieve the same outcomes as the direct revelation matching mechanism described in Section 2, but take the form of bidding games.

Since the optimal direct revelation matching mechanism depends on the ranks of the agents' reports, it turns out that position auctions are one appropriate tool to implement the profit-maximizing allocation. Broadly defined, a position auction is an indirect mechanism where agents submit bids for a series of goods that are clearly decreasing in value, the bids are ranked, and bidders receive the good associated with their rank.

Consider the *all-pay (winners-pay) position auction*:

1. The matchmaker announces a *bid-reservation schedule*, $(b_i, \underline{b}_J(b_i))$, giving the lowest bid a J-side agent can make and still be eligible to match to an I-side agent making a bid of b_i.

2. Agents submit a sealed bid b_i to the matchmaker.

3. The matchmaker opens all bids b_i and b_j and ranks them from greatest to least, tentatively matching the highest-bidding agent on the I side with the highest-bidding agent on the J side, the second-highest bidder on the I side with the second-highest bidder on the J side, and so on.

4. The matchmaker checks that the bids satisfy the bid-reservation schedule for all tentative matches. If so, he announces publicly that those agents are matched; otherwise, he reveals nothing. All agents pay their bids (All matched agents pay their bids).

Let $b_I = (b_1, b_2, ..., b_{K_I})$ be the list of bids from the I side of the market. Then

Theorem 2.4.1 (Profit-Maximizing Implementation) *The all-pay position auction has a Bayesian Nash equilibrium that implements the profit-maximizing matches* *and payments, with a symmetric bidding strategy given by*

$$\begin{aligned} b_I^{AP}(q) &= \sum_k^K w_{I,k}(q) \int_{R_I(q)}^{\bar{q}_J} s_I(q,y) f_{J,(k)}(y)dy \\ &\quad - \int_{\underline{q}_I}^{q} w_{I,k}(z) \int_{R_I(z)}^{\bar{q}_J} \frac{\partial s_I(z,y)}{\partial q_i} f_{J,(k)}(y)dydz \end{aligned}$$

and bid-reservation schedule $(b_i, \underline{b}_J(b_i))$ *schedule implicitly defined by*

$$\psi_I(b_I^{AP-1}(b_i), b_J^{AP-1}(\underline{b}_J)) + \psi_J(b_J^{AP-1}(\underline{b}_J)), b_I^{AP-1}(b_i)) - c = 0$$

Let

$$b_I^{WP}(q) = \frac{b_I^{AP}(q)}{\sum_k w_{I,k}(q)(1 - F_{J,(k)}(R_I(q))}$$

If $b_I^{WP}(q)$ *is increasing in* q*, then there is a symmetric equilibrium in the winner-pay position auction that implements the profit-maximizing matches and payments, with bid-reservation schedule* $(b_i, \underline{b}_J(b_i))$ *implicitly defined by*

$$\psi_I(b_I^{WP-1}(b_i), b_J^{WP-1}(\underline{b}_J)) + \psi_J(b_J^{WP-1}(\underline{b}_J)), b_I^{WP-1}(b_i)) - c = 0$$

While previous works have noted that supermodularity provides the right incentives for a separating equilibrium in matching markets, this result shows that position auctions can also be used in practice by intermediaries to maximize profits. It might have been necessary that a more complicated mechanism such as a menu auction in the form of bids for each rank of partner or a bid function expressing willingness to pay as a function of partner quality would be required to achieve profit maximization. But here, it turns out that this simple bidding procedure is optimal.

The winners-pay implementation, however, may fail to implement the profit-maximizing outcome if $b_I^{WP}(q)$ is non-monotonic. The numerator of $b_I^{WP}(q)$ is the expected payment in any profit-maximizing mechanism and the denominator is the probability that the agent receives a match; so conditional on receiving a match, this is the profit-maximizing payment to the matchmaker. However, since the probability of receiving a match is increasing in q, the numerator may not be increasing as fast as the denominator, and it follows these proposed bids do not form an equilibrium of the game (since they were assumed to be invertible in the derivation). Mathematically, $b_I^{WP}(q)$ is increasing if

$$\frac{\frac{d}{dq_i}\mathbf{E}_i[t_I(q_i, q_{I\setminus i}, q_J)]}{\mathbf{E}_i[t_I(q_i, q_{I\setminus i}, q_J)]} \geq \frac{\frac{d}{dq_i}\sum_k w_{I,k}(q_i)(1 - F_{J,(k)}(R_I(q_i))}{\sum_k w_{I,k}(q_i)(1 - F_{J,(k)}(R_I(q_i))} \tag{2.13}$$

The intuition for this is that if many agents are excluded on the other side of the market or the market is unbalanced — in the sense that K_I is much larger than K_J — the I-side agents are uncertain about whether or not they will receive a partner and an increase in their quality can increase the likelihood of matching dramatically. An increase in quality, however, also has a direct effect on their payoffs due to the match externalities, and agents may prefer to accept a slightly lower likelihood of receiving a partner as long as their payment is reduced. This means that if the numerator of the right-hand side in Equation (2.13) is strictly positive, the inequality may be violated. The winners-pay format will still have an equilibrium, but it will no longer be strictly monotonic, leading to unprofitable and inefficient pooling. This may explain why labor market intermediaries often focus on taking payments from one side of the market — firms, for example — and allow the other side to compete on costly but non-pecuniary signals like educational attainment.

Despite the drawbacks of an all-pay format, many matching environments have such characteristics: Some head-hunters collect fees up front, agents searching for opportunities on the Internet pay service providers by expending time looking at the screen or waiting in auction queues, students spend many years making costly payments to educational institutions to be matched to good employers, and researchers or entrepreneurs looking for grants often have to complete significant portions of the preliminary research to exhibit their project's fitness to investors. Additionally, in favor of an all-pay format is that if the bids made by the agents cannot be kept secret by the matchmaker, agents will have an incentive to renege ex post. If participants can walk away from the matchmaker and refuse to pay after learning the identity of their partner, not only are the matchmaker's profits reduced, but honest revelation is compromised: If all agents anticipate not paying, they have no incentives to be honest, and will simply submit a maximal bid.

2.5 A Simplified Mechanism and its Implementation

In many environments, however, the optimal implementation of Section 4 is not practical. First, if agents participate in the mechanism and can witness each other's bids or they receive some verified record of their participation, they can simply approach each other ex post to match, against the wishes of the matchmaker. In some situations, it may be possible for the matchmaker to keep records of participation and payments to himself, but for legal or practical reasons, many other environments will require him to release such information, undermining his ability to keep agents apart. Second, the information required to implement the optimal reserve function may be impossible to obtain in practice. While a standard auctioneer might experiment with the reserve price over time to find the optimal solution, matchmakers are essentially setting a reserve price for every q_i, resulting in an infinite number of such thresholds, $R_I(q_i)$. In this case, learning would be slow and incompetent use of relative exclusion may greatly reduce profits. On top of this, the matchmaker and participants would need to know the equilibrium strategies — including consistent beliefs about the distribution of types on both sides of the market and knowledge of both surplus functions — of the players to correctly infer their private information.

If these issues are binding on the matchmaker, the best he can do is prevent low-quality agents from transmitting their information though participation in the mechanism to the other agents. In terms of the direct revelation mechanism, he can set minimum types who are allowed to submit a report, and in terms of the bidding game, he can set minimum bid levels. Since virtual surplus is still supermodular, assortative matching will still be profit-maximizing. Consequently, the simplified match function then takes the form:

$$\tilde{m}_{ij}(q_I, q_J) = \begin{cases} 1 & , \rho_{q_I}(q_i) = \rho_{q_J}(q_j) \text{ and } q_i \geq \underline{q}_I, q_j \geq \underline{q}_J \\ 0 & , \text{ otherwise} \end{cases}$$

Matchmaker expected profits in this *simplified direct revelation mechanism* are then equal to

$$\max_{\underline{q}_I, \underline{q}_J} \sum_k \int_{\underline{q}_I}^{\bar{q}_I} \int_{\underline{q}_J}^{\bar{q}_J} \{\psi_I(q_i, q_j) + \psi_J(q_j, q_i) - c\} f_{J,(k)}(q_j) f_{I,(k)}(q_i) dq_j dq_i$$

Consider the *simplified all-pay (winners-pay) position auction*:

1. The matchmaker announces a *minimum bid* for each side, $(\underline{b}_I, \underline{b}_J)$, giving the lowest bids that agents can make and still be eligible to participate.

2. Agents submit a sealed bid b_i to the matchmaker.

3. The matchmaker opens all bids b_i and b_j and ranks them from greatest to least, matching the highest-bidding agent on the I side with the highest-bidding agent on the J side, the second-highest bidder on the I side with the second-highest bidder on the J side, and so on, until the supply of agents on one side is exhausted.

4. All agents are charged their bids (Any matched agent is charged his bid).

Then the following holds:

Theorem 2.5.1 (Simplified Implementation) *Absolute exclusion is higher in the simplified direct revelation mechanism than the profit-maximizing direct revelation mechanism and profits are lower. The identities of the worst-off types are determined by the system of equations:*

$$0 = \sum_k \int_{\underline{q}_J}^{\bar{q}_J} \left\{ \psi_I(\underline{q}_I, q_j) + \psi_J(q_j, \underline{q}_I) - c \right\} f_{J,(k)}(q_j) dq_j$$

$$0 = \sum_k \int_{\underline{q}_I}^{\bar{q}_I} \left\{ \psi_I(q_i, \underline{q}_J) + \psi_J(\underline{q}_J, q_i) - c \right\} f_{I,(k)}(q_j) dq_i$$

The simplified all-pay position auction has a symmetric equilibrium that implements the simplified direct revelation mechanism, with bidding strategies

$$b_I^{AP}(q) = \sum_k w_{I,k}(q) \int_{\underline{q}_J}^{\bar{q}_J} s_I(q,y) f_{J,(k)}(y) dy - \int_{\underline{q}_I}^{q_i} w_{I,k}(z) \int_{\underline{q}_J}^{\bar{q}_J} \frac{\partial s_I(z,y)}{\partial q_i} f_{J,(k)}(y) dy dz$$

with

$$b_I^{AP}(\underline{q}_I) = \sum_k w_{I,k}(\underline{q}_I) \int_{\underline{q}_J}^{\bar{q}_J} s_I(\underline{q}_I, y) f_{J,(k)}(y) dy$$

If

$$b_I^{WP}(q) = \frac{b_{I,AP}(q)}{\sum_k w_{I,k}(q)(1 - F_{J,(k)}(\underline{q}_J))}$$

is increasing, then the winners-pay format has a symmetric equilibrium that implements the simplified direct revelation mechanism where players use $b_I^{WP}(q)$. *A sufficient condition for* $b_I^{WP}(q)$ *to be increasing is*

$$\frac{\sum_k w'_{I,k}(q) \int_{\underline{q}_J}^{\bar{q}_J} s_I(q,y) dy}{\sum_k w'_{I,k}(q)(1 - F_{J,(k)}(\underline{q}_J)} \geq \frac{\sum_k w_{I,k}(q) \int_{\underline{q}_J}^{\bar{q}_J} s_I(q,y) f_{J,(k)}(y) dy}{\sum_k w_{I,k}(q)(1 - F_{J,(k)}(\underline{q}_J)}$$

The sufficient condition again shows how being on the long side of the market can lead to discouragement: If K_I is much larger than K_J, the sum in the denominator on the left-hand side $\sum_k^{K_J} w'_{I,k}(q)(1-F_{J,(k)}(\underline{q}_J))$ will be strictly positive, and the monotonicity of the strategies in the all-pay format may fail — note that if $K_I = K_J$ and $\underline{q}_I = \underline{q}_J = 0$, this condition is always satisfied. This is another advantage of simplified implementation: It is easier to check whether the winners-pay format will have strictly monotone bid functions, since only $\underline{q}_I$ and $\underline{q}_J$ need to be considered, rather than an entire schedule of exclusion, $R_I(q)$, and the equilibrium strategies.

Extensive use of distributional information about types is generally considered a weakness in mechanism design environments, since agents may not have such detailed information in real markets. The simplified mechanism reduces the problem of exclusion to choosing two minimum bids, thereby removing the need for the auctioneer to deduce each agent's type from his bid and the equilibrium strategies

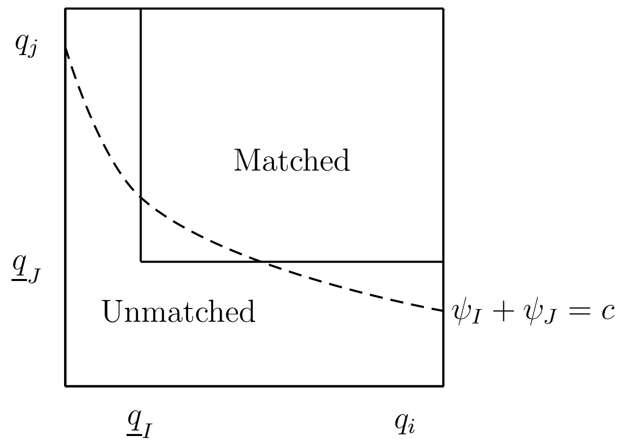

Figure 2.3: Exclusion in the Simplified Mechanism

before deciding whether a match should be allowed. Likewise, the process of mapping the bids into a match is much more transparent, reducing the likelihood that bidders will protest the outcome. This simplification, however, comes at the cost of more exclusion and lower profits.

2.6 Conclusion

This paper shows how matching in the presence of adverse selection can be analyzed with a mechanism design framework, and useful comparative statics and implementation results derived. While many papers have previously tackled these topics, they have often been in highly stylized models, such as Caillaud and Jullien [13], dealt with matching through costly signaling in markets without an intermediary, such as Hoppe et al. [35], or considered mechanisms in which the communication between the agents and the matchmaker is restricted in some way, as in Damiano and Li [22] or Gomes [29]. Using standard assumptions from the mechanism design and auction literature, the framework studied in this paper yields a number of useful conclusions about equilibrium matching with incomplete information, as well as illustrates how to implement the profit-maximizing and efficient mechanisms through practical bidding games. The analysis also uncovers some of practical limits of implementations such as the winners-pay and all-pay formats by illustrating how the winners-pay format can have non-monotone equilibria, providing some insight into why intermediaries adopt different kinds of payment schedules. Additionally, the results are encouraging for related research on matching with incomplete information: The optimal mechanisms are simple, comparative statics results are available, and the associated position auctions are straightforward and generate closed-form equilibrium bidding functions.

Similar to the auction literature, further progress on issues of ex ante heterogeneity among participants and more complicated demand structures can be technically difficult. Extensions to many-to-one matching will encounter the same challenges as the multi-unit and combinatorial auction literatures. The special feature of TAM that rankings decide the allocation of partners will break down in settings with complementarities or demand reduction, since the allocation of a given agent will be decided by the marginal values of the participants, which in turn depend on how all other agents are allocated.

Besides these technical issues, matching with two-sided adverse selection affords opportunities to study other issues that are difficult or impossible to capture in many current frameworks. Many jobs are found through informal networks, where agents are connected to one another through some structure that allows them to observe each others' characteristics. For example, a firm hires a number of workers who learn each others' abilities, then often disperse and get new jobs elsewhere. If someone is aware of two unmatched agents, they have incentives to try to arrange a match, provided the payoff to them is large enough. As the networks of relationships overlap, it becomes a competitive situation where intermediaries are forced to propose the best matches and reveal their information honestly. A framework similar to Kranton and Minehart [41] could be used to investigate when overlapping connections help or hinder agents connecting.

The presence of the matching cost c played a small role in the analysis, but the costs of matching in a decentralized market can be influential. To ensure the efficient match in a decentralized market, $K_I K_J$ communications would be necessary (on top of some form of costly signaling), while with the matchmaker, only $K_I + K_J$ are required. For large markets, this centralization could be a substantial reduction in costs. Moreover, if the cost of applying to a single firm is high enough, workers might be deterred from sending too many resumes, and would subsequently run the risk of receiving a worse partner than they might deserve. Constructing a model of decentralized communication and matching and comparing it to this centralized one can provide an explanation why some markets are highly organized and centralized, while others have no intermediation at all.

Finally, competition is a central issue to models of matchmaking. If a single monopolist is making large profits only on the strength of its position as a middleman between agents, new entry is inevitable. While mechanism design frameworks often struggle with competition, the fact that the "goods" in these markets are other agents suggests that stable and competitive markets with multiple matchmakers operating are possible. If the matchmaker offers differentiated services or is careful to set exclusion to deter entry, he might be able to maintain his position. Likewise, an entrant might target specific segments of the market, rather than try to capture all agents at once. Depending on how agents respond to different entry strategies, an entrant may be successful at getting a foothold in the market.

Chapter 3

Truthful Revelation in Complementary Matching Problems with Transfers

3.1 Introduction

The presence of intermediaries in today's markets is ubiquitous, including the network of traders and exchanges that form the global financial system and the Internet Service Providers connecting users to content producers. However, their position as a brokers of relationships raises a number of questions: How should the market be designed? To what extent do the insights from auction theory and Gale-Shapley-Roth-style matching theory carry over to markets with prices, externalities, and strategic participants? How can participants be incentivized to disclose information about themselves honestly, and what limitations does this place on the pursuit of efficiency or stability?

These issues are not purely of theoretical interest. During the recent financial crisis, the U.S. Treasury Department briefly considered using a two-sided market design to buy up troubled assets and improve banks' balance sheets. One component of the plan was to allow outsiders to compete for the banks' assets alongside the liquidity injection from the Treasury. Additionally, there are spot markets for labor organized online, where intermediaries like Yahoo! attempt to temporarily match workers to firms, for a fee. Further investigation of the mechanisms by which efficient or profit-maximizing matchings can be achieved is useful for illustrating the issues facing market designers and intermediaries.

This paper examines one-to-one and many-to-one matching in environments with incomplete information. Beginning with environments with supermodular matching surplus and one-dimensional types, it shows that the natural extension of the Vickrey-Clarke-Groves mechanism is not incentive compatible. An alternative mechanism is proposed that achieves truth-telling as an ex post equilibrium given some structure on agents' preferences over partners, called the *competitive externality mechanism*. This alternative mechanism can be implemented using an open format, and can be extended to a similar many-to-one framework, matching packages of workers to firms.

The main difference between this paper and works such as Shapley and Shubik [60], Crawford and Kelso [19], Crawford and Knoer [20], and Hatfield and Milgrom [34] — which explore dynamic adjustment processes that reach the core of matching games with transfers — is that incomplete information is explicitly introduced, and the agents are allowed to behave strategically. These papers primarily show that the set of stable matches is a lattice, and provide an algorithm that maps feasible matches into matches, guaranteed to terminate at a stable match. This approach follows the seminal paper by Gale and Shapley [28], but often ignores the strategic incentives provided by the algorithm. As Roth [54] shows, stability, efficiency, and honest reporting can often be in conflict. This is in stark contrast to results in the auction literature — particularly Vickrey [66], Myerson [48], Ausubel [2] and [1], and Edelman, Ostrovsky and Schwarz [25] — where the problem of selling goods to buyers with privately known values for the items being sold has been analyzed with considerable success.

A number of papers have attempted to bridge this gap, including Bulow and Levin [9], Damiano and Li [22], Hoppe, Moldovanu and Sela [35], Hoppe, Moldovanu and Ozdenoren [36], Gomes [29], Johnson [38], and McAfee [44]. These papers all analyze markets where agents derive a payoff from matching to partners — such as workers to firms in labor markets or spouses in marriage markets with dowries — and examine how competitive bidding might bring about favorable outcomes. In particular, Damiano and Li [22], Gomes [29] and Johnson [38] use Bayesian implementation methods from mechanism design to examine how a profit-maximizing matchmaker might deal with the problem of intermediation in the presence of adverse selection in various matching environments. The key theme underlying most of these papers is that by making the agents compete on each side for the right to a better partner, efficiency can be achieved, as long as the two sides have similar group preferences over which match should be chosen.

This paper contributes to the literature by applying this "competitive" reasoning, and studying some many-to-one and multi-dimensional type environments. Where possible, the solution concept used is ex post or ex post perfect. This is particularly useful for matching environments because it is often the case that agents have knowledge about potential partners or opponents, making Bayesian implementation inappropriate since information is distributed asymmetrically in the market. For example, workers might have previously been employed at the same firm, and in future periods, this gives that firm and set of workers better information about each other than other participants in the market. Analysing the market as if there was incomplete information may lead to misleading results, similar to how the equilibria in the first-price or all-pay auctions with and without private information can be very different.

3.2 Model

There are two disjoint sets of agents, I and J, with (abusing notation) I agents on the I side and J agents on the J side; let $K = \min\{I, J\}$ with $K > 1$, which is the largest number of matches that can be arranged in this market. Suppose that $I \geq J$, so that there is an excess supply of I-side agents. Each I-side agent would like to match to one agent on the J side, and each agent on the J side would like to match to one agent on the I side. Since the information and preferences considered here are symmetric considering both sides, only the I side will be described in detail, with the appropriate permutation of indices applying to the J side.

Each agent $i = I1, I2, ..., II$ draws a *type* s_i from the type space $[0, \bar{s}_I]$ that determines the preferences and abilities of that agent. Let a realization of types for the I side be written

$$s_I = (s_{I1}, s_{I2}, ..., s_{II})$$

and

$$s_{I\backslash k} = (s_{I1}, s_{I2}, ..., s_{I,k-1}, s_{I,k+1}, ..., s_{II})$$

Let $s = (s_I, s_J)$. A *report* $\sigma = (\sigma_I, \sigma_J)$ is any concatenation of types from the type spaces of all agents, not necessarily equal to the true types, and $s_{\setminus k} = (s_{I\setminus k}, s_J)$. Let the signals be distributed according to a joint density $f(s_I, s_J)$, and define $\mathbf{E}_{s_{\setminus k}}[h(s_k, s_{\setminus k})]$ as the expectation of $h(s_k, s_{\setminus k})$ conditional on s_k, with all other type realizations distributed according to the marginal density $f[s_{\setminus k}|s_k]$.

The match surplus for agent i from being matched to agent j is described by a *surplus function* $v_I(s_i, s_j)$ that maps the types of agent i and j into a real number. Since only s_i and s_j determine the value of the match, and no types from other agents enter the match surplus value, call this the *pairwise private values* case. If $v_I(s_i, s_j) \geq v_I(s_i, s_{j'})$, then agent i *prefers* agent j to j', or $j \geq_i j'$. These ordinal preferences correspond to those considered by Gale and Shapley [28], where the primitives are ordinal preferences rather than cardinal ones. Assume that for all s_i and s_j,

$$\frac{\partial v_I(s_i, s_j)}{\partial s_i} \geq 0 \ , \ \frac{\partial v_I(s_i, s_j)}{\partial s_j} \geq 0$$

and

$$\frac{\partial^2 v_I(s_i, s_j)}{\partial s_i \partial s_j} \geq 0$$

This implies that the players' payoffs are increasing in their own and each other's types, and there is complementarity between the types.

Agents have quasi-linear preferences, so an agent i paying t to match to agent j receives a payoff of

$$v_I(s_i, s_j) - t$$

If $v_I(s_i, y) \geq 0$ for all potential partner types y, then all partners are *acceptable for type* s_i. If $v_I(s_i, y) \geq 0$ for all partner types y and $v_J(s_j, y) \geq 0$ for all partner types

y, then say *all matches are acceptable.*

The game is modelled as a situation where a matchmaker is trying to construct a game to induce agents to honestly reveal their private information, so that the matchmaker can maximize ex post efficiency. From the revelation principle, any Bayesian game can be thought of as a "revelation mechanism", in which agents make a report to the matchmaker — not necessarily truthfully— and then the matchmaker acts as a "referee", playing the agent's equilibrium strategy for him. By studying revelation mechanisms generally, conditions for welfare- or profit-maximization can be developed subject to the constraint that agents find it in their best interests to reveal their private information honestly to the matchmaker.

A *direct revelation matching mechanism* (direct mechanism) is a set of functions

$$\{m_{ij}(\sigma), t_i(\sigma), t_j(\sigma)\}_{i,j}$$

where $m_{ij}(\sigma)$ maps any report σ into a probability distribution over matches in the set $I \times J$, and the payment functions $t_i(\sigma)$ and $t_j(\sigma)$ map any report σ into transfers to the matchmaker. In particular, $m_{ij}(\sigma)$ gives the probability that i and j are matched, given the report.

A direct mechanism is *Bayesian incentive compatible* for agents if

$$s_i \in \text{Argmax}_{\sigma_i} \mathbf{E}_{s_{I \setminus i}, s_J} \left[\sum_{j \in J} m_{ij}(\sigma_i, s_{\setminus i}) v_I(s_i, s_j) - t_i(\sigma_i, s_{\setminus}) \right]$$

If, for any profile of types $s'_{\setminus i}$ reported by i's opponents, it is true that

$$s_i \in \text{Argmax}_{\sigma_i} \sum_{j \in J} m_{ij}(\sigma_i, s'_{\setminus i}) v_I(s_i, s_j) - t_i(\sigma_i, s'_{\setminus i})$$

then honesty is a *weakly dominant strategy.*

Honesty is an *ex post equilibrium* if, for all realizations of the types,

$$s_i \in \operatorname{Argmax}_{\sigma_i} \sum_{j \in J} m_{ij}(\sigma_i, s_{\setminus i}) v_I(s_i, s_j) - t_i(\sigma_i, s_{\setminus i})$$

A match is *ex post individually rational* if all agents have positive payoffs ex post. Since there is quality uncertainty in the market, it is important to differentiate which market designs can guarantee ex post positive payoffs because agents may attempt to renege if they do worse by abiding by the results than opting out after the payments and qualities of partners are revealed.

Lastly, suppose there are two matches $(I1, J2)$ and $(I2, J1)$, where $I1$ prefers to match to $J1$ at the price paid by $I2$ to matching with his partner $J2$ at the price quoted for $J2$. If $J1$ prefers to match to $I1$ at the price paid by $J2$ rather than match to $I2$ at the price quoted for $I2$, the match is *unstable with transfers*; this includes the case where an agent prefers to be unmatched rather than match to his current partner. If a match is not unstable, it is *stable with transfers.*

3.3 One-to-One Matching

In the auction and public goods literatures, honest reporting in weakly dominant strategies can be achieved through the Vickrey-Clarke-Groves mechanism, in which an agent's impact on social welfare is internalized through his transfers so that the other participants receive the same payoff, regardless of the agent's report[1]. This section shows that, although matching markets have similarities to both auc-

[1]For a complete introduction to Vickrey-Clarke-Groves mechanisms, see Milgrom [46], p. 45-64

tions and public goods provision, the VCG mechanism fails to provide incentives for agents to report their types honestly, even in the special case of vertically differentiated supermodular preferences. Alternatively, the failure of the VCG mechanism to implement incentive compatible revelation can be seen as a parallel of Roth's [54] non-existence result in a transferable utility environment.

3.3.1 The VCG Mechanism

Suppose the matchmaker asks the agents to make a report about their types, not necessarily honestly, with the option of withdrawing from the market altogether by reporting the null variable $\varnothing$.

After receiving reports from the agents, the matchmaker can compute the welfare-maximizing allocation, given the reported types:

$$V(\sigma, I, J) \quad = \quad \max_{m_{ij}} \sum_{i \in I} \sum_{j \in J} m_{ij}(\sigma) \left\{ v_I(\sigma_i, \sigma_j) + v_J(\sigma_i, \sigma_j) \right\} \tag{3.1}$$

Then if $I\ell$ chooses to withdraw from the market, define

$$V(\sigma, I \backslash I\ell, J) \quad = \quad \max_{m_{ij}} \sum_{i \in I \backslash I\ell} \sum_{j \in J} m_{ij}(\varnothing, \sigma_{\backslash I\ell}) \left\{ v_I(\sigma_i, \sigma_j) + v_J(\sigma_j, \sigma_i) \right\} \tag{3.2}$$

to be market welfare when agent $I\ell$ refuses to participate. Construct the payment function of the k-th agent on the I side as

$$\begin{aligned} t_{I\ell}(\sigma) = \sum_{i \in I \backslash I\ell} \sum_{j \in J} m_{ij}(\varnothing, \sigma_{\backslash I\ell}) \left\{ v_I(\sigma_i, \sigma_j) + v_J(\sigma_j, \sigma_i) \right\} \\ - \sum_{i \in I \backslash I\ell} \sum_{j \in J} m_{ij}(\sigma_{I\ell}, \sigma_{\backslash I\ell}) \left\{ v_I(\sigma_i, \sigma_j) + v_J(\sigma_j, \sigma_i) \right\} + h_{I\ell}(\sigma_{\backslash I\ell}) \end{aligned}$$

In some cases — for example, auction environments — the VCG mechanism often satisfies all of a market designer's criteria: it implements the efficient outcome, honest reporting is an equilibrium in weakly dominant strategies, outcomes are individually rational, and it runs a budget surplus, but perhaps does not maximize revenue. Other environments however — for example, public goods — have the unfortunate property that outside funds may be required to implement the efficient outcome. Therefore, before proceeding to examine other mechanisms, the incentive properties of the VCG mechanism should be considered in this particular case.

Proposition 3.3.1 *Suppose all matches are acceptable. The VCG mechanism is not incentive compatible.*

Example Consider a simple supermodular market where $v_I(s_i, s_j) = v_J(s_j, s_i) = s_i s_j$, $(s_{I1}, s_{I2}) = (s_{J1}, s_{J2}) = (1, 2)$. If $I1$ honestly reports a 1 and the other agents do as well, $I1$ and $J1$ are matched, and $I2$ and $J2$ are matched. Then $I1$ gets a payoff from reporting honestly of

$$\begin{aligned} U_I(1,1) &= m_{I1,J1}(1, s_{-I1})(1*1+1*1) + m_{I2,J2}(1, s_{-I1})(2*2+2*2) \\ &\quad - m_{I2,J2}(\varnothing, s_{-I1})(2*2+2*2) \\ &= 2 \end{aligned}$$

But suppose he makes a report σ strictly higher than 2, he gets a better partner, as well as receives a higher payment based on his report:

$$\begin{aligned} U_I(1,\sigma) &= m_{I1,J2}(\sigma, s_{-I1})(1*2+2*\sigma) + m_{I2,J1}(\sigma, s_{-I1})(1*2+2*1) \\ &\quad - m_{I2,J2}(\varnothing, s_{-I1})(2*2+2*2) \\ &= 2*\sigma - 2 \\ &> U_I(1,1) \end{aligned}$$

So it is a profitable deviation to lie, and the VCG mechanism is not incentive compatible (truth-telling is not even a Nash equilibrium, let alone a dominant strategy).

$\blacksquare$

The failure of incentive compatibility arises because the VCG mechanism sets the payment of a given agent equal to his impact on social surplus *relative to the reports*, and agents can manipulate their perceived impact on social surplus through the valuation function of their partner. In an independent private values framework, each agent's valuation depends only on his own signal, so this motivation is not present. Here, however, partners who are compensated for the value they bring to their partner can always inflate their payoff by claiming they are a higher type. Consequently, the workhorse for designing efficient and incentive compatible mechanisms in other environments is not suited for *any* matchmaking purposes.

The problems arising with the VCG matching mechanism do not arise in other papers in the competitive matching literature, where markets can successfully be designed to implement the welfare-maximizing outcomes. In those papers, a key feature of the mechanisms studied is that agents on each side are forced to compete with each other, jockeying for position. Once their ranking is decided, it is straightforward for the matchmaker to pair the agents assortatively. This feature — considering the market as two auctions for a risky right to a partner on the other side — is what removes the cross-market payments that makes the VCG mechanism fail to implement honest reporting. However, if the cross-market payments are removed, the key feature of the VCG mechanism — that each player receives all the gains from trade of his or her participation, and subsequently has incentives to maximize social welfare — no longer holds. So this competitive approach will not, in general, implement truth-telling either. The next section shows that for the supermodular framework used in this paper, however, such a mechanism is incentive compatible.

3.3.2 Ex Post Implementation

Definition The *competitive externality mechanism* (CEM) satisfies the following construction:

- The matchmaker asks the agents to report their private types σ.
- On the I side, the matchmaker chooses a match function $m^I_{ij}(\sigma)$ to maximize

  $$V_I(\sigma, I) = \max_{m^I_{ij}} \sum_{i \in I} \sum_{j \in J} m^I_{ij}(\sigma) v_I(\sigma_i, \sigma_j)$$

  and on the J side, the matchmaker chooses a match function $m^J_{ij}(\sigma)$ to maximize

  $$V_J(\sigma, J) = \max_{m^J_{ij}} \sum_{j \in J} \sum_{i \in I} m^J_{ij}(\sigma) v_J(\sigma_j, \sigma_i)$$

- If the optimal allocations on the two sides agree, match the agents and charge agent Ik on the I side

  $$t_{Ik}(\sigma) = \sum_{i \in I \setminus Ik} \sum_{j \in J} m^I_{ij}(\varnothing, \sigma_{\setminus Ik}) v_I(\sigma_i, \sigma_j) - \sum_{I \setminus Ik} \sum_{J} m^I_{ij}(\sigma_{Ik}, \sigma_{\setminus Ik}) v_I(\sigma_i, \sigma_j)$$

  and the charge agent Jk on the J side

  $$t_{Jk}(\sigma) = \sum_{j \in J \setminus Jk} \sum_{i \in I} m^J_{ij}(\varnothing, \sigma_{\setminus Jk}) v_J(\sigma_j, \sigma_i) - \sum_{j \in J \setminus Jk} \sum_{i \in I} m^J_{ij}(\sigma_{Jk}, \sigma_{\setminus Jk}) v_J(\sigma_j, \sigma_i)$$

If the optimal allocations on the two sides do not match, no agents are matched and nothing is charged.

Say that CEM is *feasible* if, for all type reports σ, $m^I_{ij}(\sigma) = m^J_{ij}(\sigma)$.

Example Suppose there are three I agents with signals $s_I = (1, 2, 3)$, and two J agents with signals $s_J = (2, 3)$. The valuations of the agents are $v_I(s_i, s_j) = v_J(s_j, s_i) = s_i s_j$. The valuation functions can be summarized in a table:

	J2	J1
I3	$2, 2$	$3, 3$
I2	$4, 4$	$6, 6$
I1	$6, 6$	$9, 9$

So the agents report their types to the matchmaker, and he computes the efficient and stable match: $\{(I_3, J_2), (I_2, J_1), (I_3, \emptyset)\}$. Agent $I1$ has to pay for blocking both $I2$ and $I3$ from receiving better partners:

$$t_{I1} = -\{m_{I2,J2}(4) + m_{I3,\emptyset}(0)\} + \{m_{I2,J1}6 + m_{I3,J2}2\} = 4$$

And agent $I2$ has to pay for blocking $I3$ from receiving a better partner:

$$t_{I2} = -\{m_{I3,\emptyset}(0)\} + \{m_{I3,J2}2\} = 2$$

On the other side of the market, agent $J1$ blocks $J2$ yielding

$$t_{J1} = -\{m_{I2,J2}(4)\} + \{m_{I1,J2}(6)\} = 2$$

while $J2$ pays nothing, since he is not blocking anyone. ∎

The reason for constructing the mechanism in this way is that it ignores the positive welfare consequences of an agent's participation for the other side in computing payments and charges agents on the basis of the negative welfare consequences of their participation for their own side. It turns out that truth-telling is an ex post equilibrium.

Proposition 3.3.2 *Suppose all matches are acceptable. Then the competitive externality mechanism is feasible and truth-telling is an ex post equilibrium.*

Note that honesty is not a weakly dominant strategy, but only an ex post equilibrium. For example, suppose the best agent on the J side reported that he was actually the worst potential partner. In the competitive externality mechanism, some I side agent can now get the best partner and pay as if he were receiving the worst — clearly honesty is no longer a best response. The competitive externality mechanism includes no penalty for submitting reports that harm the opposite side of the market, so it requires restrictions on preferences such that there is sufficient pressure on each separate side of the market to induce honest reporting. Said another way, *if* the other side reported honestly, it *would* be a weakly dominant strategy to report honestly.

3.3.3 Protecting Agents from Bad Matches

One of the assumptions in the above analysis is that all matches are *acceptable*, or that matching with any partner is better than staying alone. Infeasibility of CEM may occur if there are matches which are acceptable for the partner on one side, but not the partner on the other. For example, if the agents enter into a partnership and split a surplus $v_I(s_i, s_j) = \lambda(r(s_i, s_j) - \gamma)$ and $v_J(s_j, s_i) = (1-\lambda)(r(s_i, s_j) - \gamma)$, no pair of agents from either side of the market will want to match if $r(s_i, s_j) - \gamma < 0$, and all the results will hold. However, if the two sides have significantly different private surplus there can easily be disagreement between the two sides about whether the low-value matches should be arranged, and the competitive externality mechanism is no longer feasible; for example, the I-side partner has valuation function $v_I(s_i, s_j) = r_I(s_i, s_j) - \gamma_I$ and the J-side partner has $v_J(s_j, s_i) = r_J(s_j, s_i) - \gamma_J$. Moreover, truth-telling will no longer be an ex post equilibrium if the matchmaker announces that he will not arrange matches that give either party a negative payoff ex post, since the agents then have an incentive to lie about their type to manipulate the operation of the mechanism on the *other* side of the market. This could lead to a mixed-strategy equilibrium and other strategic behaviors that undermine efficiency or stability.

To resolve this issue, introduce *security bidders*, who bid for only one agent and represent his interests on the other side of the market. In particular, the matchmaker takes i's report and instructs the security bidder for agent i bids to only on agent i, acting as a J side agent with a signal $x_i^*(s_i)$ implicitly defined by

$$v_I(s_i, x_i^*(s_i)) = 0$$

Proposition 3.3.3 *(i) Honest reporting is an ex post equilibrium in the competitive externality mechanism with security bidders. (ii) Revenue is weakly improved compared to the competitive externality mechanism without security bidding when all matches are acceptable. (iii) This is not necessarily efficient, but it is stable.*

Since submitting a security bid can only reduce the likelihood of receiving a partner without affecting their payment, agents have no incentive to make a false report. Revenue weakly increases because the lowest payment now increases from 0 (for the worst agent matched on the smaller side of the market) to the value of the security bid for the worst matched partner. As a result, the other payments on that side of the market weakly increase as well. So this small addition to the mechanism reduces the likelihood of unacceptable outcomes or infeasibility while improving revenue and incentives.

3.3.4 Dynamic Formats

This section investigates the properties of dynamic games that achieve the same allocation of partners as the static competitive externality mechanism, achieved through clock formats similar to the Generalized Second Price Auction studied in Edelman, Ostrovsky and Schwarz [25] and Varian [65]. Having a clock implementation in this environment is useful because the payments depend on the agents' valuation functions and types, and these may be difficult to describe in a matching environment where information can be a latent signal about one's own talent. For example, agents may find it difficult or disadvantageous to enumerate the mapping $v_I(s_i, s_j)$ for the matchmaker's use. Using a dynamic format can avoid problems in environments where the matchmaker is not sophisticated enough to compute the correct transfers or where there is no natural way to report private information.

Since this section considers dynamic games, a new notion of equilibrium is required. A strategy profile $\sigma^* = \{\sigma_i^*(s_i, h_t)\}_i$ of a dynamic game is an *ex post perfect equilibrium* if for every player i with private information s_i, every realization of private information $(s_i, s_{\backslash i})$, and at any time t following a history h_t, σ^* is a Nash equilibrium of the continuation game starting from time t. Alternatively, the strategy profile $\sigma^*(s_i, h_t)$ is an ex post perfect equilibrium if it is a Nash equilibrium in any subgame, for any realization of private information.

3.3.4.1 Achieving the Stable Match with a Dynamic Mechanism

Suppose the matchmaker uses the following *simple ascending competitive externality matching mechanism*: The *clock prices* c_I and c_J on each side of the market are set equal to zero. Initially, all agents are *active*, and indicate to the matchmaker at each moment in time whether they would like to remain active or *drop out*; agents who have dropped out cannot become active again. The matchmaker then uses the following procedure to determine the match and set prices:

- On the I side, the matchmaker begins the process in an *elimination stage* E by raising the clock price c_I until the number of active agents on each side of the market is the same (recall that $I \geq J$); the clock price at which the $J+1$-st agent on the I side dropped out is recorded as p_K^I, and $p_K^J = 0$.
- Suppose there are exactly k agents left on each side; call this the *k-th stage of the mechanism*. Then the matchmaker proceeds as follows for $k = K, K-1, ..., 2$:
  - Raise the clock price on the J side until an agent drops out; set p_{k-1}^J

equal to clock price at which this occurred. Raise the clock price on the I side until an agent drops out; set p^I_{k-1} equal to the clock price at which this occurred.

  – The matchmaker privately notes the identities of the two agents who most recently dropped out on each side — Ik and Jk— and the prices they should pay — p^I_k and p^J_k, respectively.

- Once all but two agents have dropped out, the matchmaker ends the auction, charges the agents the prices established by the game, and releases the identity of their partners.

During the process, only the drop-out behavior of agents on the I side is witnessed by agents on the I side, and only the drop-out behavior of agents on the J side is witnessed by agents on the J side. This turns the game into a dynamic game of incomplete information, since the agents on the I side cannot be sure what strategies the agents on the J side have played. If this were public information, agents on the I side could conceivably attempt to manipulate agents on the J side by staying in "too long" or dropping out early.

A strategy might be defined as being a function mapping all the information available to an agent at a given moment time into a probability of indicating to the matchmaker that the agent wishes to withdraw, 0, or remain active, 1. In particular, a *history* h_c is a I-dimensional vector A_I that contains a drop-out time for the k-th entry if at least k agents have dropped out, and the null variable $\varnothing$ otherwise. Then a *pure strategy* for a player on the I side is a function $\phi_i : (s_i, A_I, c_I) \rightarrow \Delta(\{0, 1\})$, where $\Delta(\{0,1\})$ is the set of all probability distributions over remaining active or dropping out. However, just as in the repeated games literature, it is preferable to use a simpler notion of a behavior or Markov perfect strategy — plans of action depending only on some aspects of the current state of play — rather than pure strategies that are functions of the entire history, especially if an equilibrium can be found that has a plausible behavioral interpretation. In particular, let the class of *drop-out strategies* be sequences of functions $\tau_k^I : (s_i, A_I) \to \mathbb{R}_+$ indexed by the stage $k = E, 1, 2, 3, ..., K$ that map the current state into a time at which the agent should withdraw, if no other withdrawals have yet occurred in that stage. In other words, the strategies τ_k^I propose an *optimal stopping time* for each stage and state, potentially using only some of the available information at that time.

Define the *ranking function*

$$\rho_{s_J}(j) = |\{\ell \in J : s_{J\ell} \leq s_j\}|$$

Then $\rho_{s_J}(j)$ gives that the cardinality of the agents on the J side whose signal is lower that s_j, or the *rank* of j in J.

Suppose the agents use the following drop out strategy along the equilibrium path to decide when to withdraw in the elimination phase:

$$\tau_E(s_i, s_{J,J}) = \mathbf{E}[v_I(s_i, s_j)|\rho_{s_J}(j) = J] \tag{3.3}$$

Here, agents are taking the expectation with respect to the J-th order statistic of a sample of size J. If agents follow this strategy, they stay in until their gains from matching to the worst potential partner in expectation are exhausted by the price of receiving a partner.

Once the number of active agents is the same on both sides, suppose the agents use the following drop-out strategy to decide when to withdraw from competition in stage k:

$$\tau_k^I(s_i, p_k^I) = \mathbf{E}[v_I(s_i, s_j)|\rho_{s_J}(j) = k-1] - \mathbf{E}[v_I(s_i, s_j)|\rho_{s_J}(j) = k] + p_k^I \qquad (3.4)$$

These strategies correspond to each agent deciding the highest clock price at which they are indifferent between getting the worst remaining active agent on the other side of the market in expectation and paying p_k^I, or waiting and matching to the next-best agent at a higher clock price:

$$\mathbf{E}[v_I(s_i, s_j)|\rho_{s_J}(j) = k-1] - \tau_k^I(s_i, p_k^I) = \mathbf{E}[v_I(s_i, s_j)|\rho_{s_J}(j) = k] - p_k^I$$

Edelman et al. [25] provide such an equilibrium for their model, showing that there is a set of drop-out strategies that implement the same allocation and payments as the VCG mechanism in the position auction they study.

Example Consider three agents on each side of the market, $I = \{a, b, c\}$ and $J = \{\alpha, \beta, \gamma\}$. Suppose the true realizations of their types are $(2, 5, 8)$ and $(2, 5, 6)$, all drawn independently from a uniform distribution over $[0, 10]$. Let $v_I(s_i, s_j) = v_J(s_j, s_i) = s_i s_j$.

The expected values of the the first, second, and third order statistics from a uniform variable on $[0, 10]$ are

$$s_{(1)} = \mathbf{E}[s_i|\rho_I(i) = 1] = \frac{15}{2} = 7\frac{1}{2}$$

$$s_{(2)} = \mathbf{E}[s_i|\rho_I(i) = 1] = 5$$

$$s_{(3)} = \mathbf{E}[s_i|\rho_I(i) = 1] = \frac{5}{2} = 2\frac{1}{2}$$

Consider the bidding on the I side. In the first stage of the game, the three agents are essentially bidding for the right to match to the third order statistic, $s_{(3)}$, at a price of $p_3^I = 0$, so their proposed drop-out strategies are

$$\begin{aligned}
&\tau_3^I(s_i, p_3^I)\\
\tau_3^I(2,0) &= \mathbf{E}[2 * s_j)|\rho_{s_J}(s_j) = 2] - \mathbf{E}[2 * s_j)|\rho_{s_J}(s_j) = 3]\\
&= 2 * s_{(2)} - 2 * s_{(3)} = 5\\
\tau_3^I(5,0) &= \mathbf{E}[5 * s_j)|\rho_{s_J}(s_j) = 2] - \mathbf{E}[5 * s_j)|\rho_{s_J}(s_j) = 3]\\
&= 5 * s_{(2)} - 5 * s_{(3)} = 12\frac{1}{2}\\
\tau_3^I(8,0) &= \mathbf{E}[8 * s_j)|\rho_{s_J}(s_j) = 2] - \mathbf{E}[8 * s_j)|\rho_{s_J}(s_j) = 3]\\
&= 8 * s_{(2)} - 8 * s_{(3)} = 20
\end{aligned}$$

So agent a drops out first at a price of 5, setting $p_2^I = 5$. The two remaining bidders, b and c, then adjust their drop out strategies for stage 2 of the auction:

$$\begin{aligned}
&\tau_2^I(s_i, p_2^I)\\
\tau_2^I(5,5) &= \mathbf{E}[5 * s_j)|\rho_{s_J}(s_j) = 1] - \mathbf{E}[5 * s_j)|\rho_{s_J}(s_j) = 2] + 5 =\\
&5 * s_{(1)} - 5 * s_{(2)} + 5 = 17\frac{1}{2}\\
\tau_2^I(8,5) &= \mathbf{E}[8 * s_j)|\rho_{s_J}(s_j) = 1] - \mathbf{E}[8 * s_j)|\rho_{s_J}(s_j) = 2] + 5\\
&= 8 * s_{(1)} - 8 * s_{(2)} + 5 = 25
\end{aligned}$$

So agent b drops out at a price of $17\frac{1}{2}$, and $p_1^I = 17\frac{1}{2}$. Then, in expectation, the players' payoffs are

$$\begin{aligned}
\mathbf{E}[2 * s_j | \rho_{s_J}(j) = 3] - 0 &= 5 \\
\mathbf{E}[5 * s_j | \rho_{s_J}(j) = 2] - 5 &= 20 \\
\mathbf{E}[8 * s_j | \rho_{s_J}(j) = 1] - 17\frac{1}{2} &= 42\frac{1}{2}
\end{aligned}$$

On the J side, the same process occurs, giving the set of prices (since α and β have the same signals as a and b): $p_3^I = 0$, $p_2^I = 5$, and $p_3^I = 17\frac{1}{2}$. Then the ex post payoffs are

$$\begin{aligned}
u_a = u_\alpha &= 2 * 2 - 0 = 4 \\
u_b = u_\beta &= 5 * 5 - 5 = 20 \\
u_c = u_\gamma &= 8 * 6 - 17\frac{1}{2} = 30\frac{1}{2}
\end{aligned}$$

However, it is possible to generate ex post *irrational* payoffs, despite the fact that agents expect positive payoffs at the interim stage. Suppose that $s_\alpha = 1$, $s_\beta = 1.5$, and $s_\gamma = 2$. Then the I side's ex post payoffs would have been

$$\begin{aligned}
u_a = u_\alpha &= 2 * 1 - 0 = 4 \\
u_b = u_\beta &= 5 * 1\frac{1}{2} - 5 = 5\frac{1}{2} \\
u_c = u_\gamma &= 8 * 2 - 17\frac{1}{2} = -1\frac{1}{2}
\end{aligned}$$

Therefore, even though the correct match is implemented, the lack of information about potential partners may be important to the participants. ∎

Even though this strategy completely ignores the drop-out behavior of the other players on the I side except for the realizations of the prices, this dynamic game does implement the stable match, where the proposed drop-out strategies are a Bayesian equilibrium.

Proposition 3.3.4 *Suppose all matches are acceptable, and the types of the agents are private information and independently and identically distributed on* $[0, \infty)$ *according to absolutely continuous densities* $F_I(s_i)$ *and* $F_J(s_j)$*. Then the drop-out strategies in Equations 3.3 and 3.4 form a perfect Bayesian equilibrium of the simple ascending competitive externality mechanism, which implements the stable match. Some players may receive negative payoffs ex post.*

This format has the useful feature of not requiring the matchmaker to know anything about the agents' information and preferences except that the valuation functions are supermodular and that all matches are acceptable. In particular, the matchmaker merely has to run the clocks and keep track of the payments. In a market where the signals or valuation functions are difficult to describe or communicate, this makes efficient intermediation possible. In some situations, however, the possibility of negative ex post payoffs may be a serious drawback. For that reason, an ascending mechanism which achieves both the same match and the same payments as an ex post perfect equilibrium is desirable.

3.3.4.2 Achieving the Static Outcome with an Announcement Phase

The matchmaker faces two difficulties in assisting agents to reveal more of their private information to potential partners: First, they may lie in the hopes of getting a better partner, and, second, the release of enough information may render the matchmaker irrelevant. If all information is released to the participants and becomes common knowledge they could presumably work out the unique stable match on their own, leading to dis-intermediation. However, without knowledge about the partners for which the agents are bidding, it is a challenge for the participants to decide when to drop out.

Suppose the matchmaker attempts to overcome the adverse selection by asking each agent on the I side to submit a report α_i that he will announce to the J side before using an ascending mechanism. This allows agents to replace the order statistics in their drop-out strategies from the previous mechanism with the announcements.

The new dynamic mechanism — the *announcement ascending competitive externality mechanism* — begins with an *announcement phase* where the matchmaker solicits a report α_i from each agent about his or her quality, and announces $\alpha_I = (\alpha_{I1}, \alpha_{I2}, ..., \alpha_{II})$ to the J side and α_J to the I side. The matchmaker then sets the *clock prices* c_I and c_J on each side of the market equal to zero. Initially, all agents are *active*, and indicate to the matchmaker at each moment in time whether they would like to remain active or *drop out*; agents who have dropped out cannot become active again. The matchmaker then uses the same dynamic process as for the simple ascending competitive externality mechanism to establish the match and payments.

Again, there are advantages to considering drop-out strategies that use only a limited amount of information at each stage of the game. Let $\alpha_{I[k]}$ be the commonly known value of the k-th highest announcement of an agent on the I side, and $\alpha_{J[k]}$

defined likewise. Suppose the agents use the following drop out strategies to decide when to withdraw in the elimination phase:

$$\tau_E^I(s_i, \alpha_{J[J]}) = v_I(s_i, \alpha_{J[J]}) \tag{3.5}$$

Once the number of active agents is the same on both sides, suppose the agents use the following drop-out strategy to decide when to withdraw from competition in stage k:

$$\tau_k^I(s_i, \alpha_{J[k-1]}, \alpha_{J[k]}, p_k^I) = v_I(s_i, \alpha_{J[k-1]}) - v_I(s_i, \alpha_{J[k]}) + p_k^I \tag{3.6}$$

To illustrate how the mechanism and drop-out strategies work, consider the following example:

Example Let $s_I = (1, 2, 3, 4)$ and $s_J = (1, 2, 3)$, so that $I = 4$ and $J = 3$, and assume that the valuation functions on both sides are $v(s_i, s_j) = s_i s_j$. Assume the agents announce their types honestly to the matchmaker.

The clocks are initially set to zero, and the matchmaker begins raising the one on the I side, since $I > J$. The drop-out strategies for the I-side agents in the elimination phase then are:

$$\tau_k^I(s_i, \alpha_{J[J]})$$

$$\begin{aligned}
\tau_E^I(1,1) &= 1 * 1 = 1 \\
\tau_E^I(2,1) &= 2 * 1 = 2 \\
\tau_E^I(3,1) &= 3 * 1 = 3 \\
\tau_E^I(4,1) &= 4 * 1 = 4
\end{aligned}$$

So the first drop-out on the I-side occurs at 1 when the first agent drops out, setting $p_3^I = 1$. Since no drop-outs were forced on the J side, $p_3^J = 0$.

Then at the beginning of the third stage, the agents on the J side anticipate getting a partner with a signal of 2 if they drop out or a partner with a signal of 3 if they drop out next round, which they use in their drop-out strategies:

$$\tau_3^J(s_j, \alpha_{I[2]}, \alpha_{I[3]}, p_3^I)$$

$$\begin{aligned}
\tau_3^J(1,3,2,0) &= 1*3-1*2+0=1 \\
\tau_3^J(2,3,2,0) &= 2*3-2*2+0=2 \\
\tau_3^J(3,3,2,0) &= 3*3-3*2+0=3
\end{aligned}$$

So the first agent drops out on the J side when the clock reaches 1. At the beginning of the third stage on the I side, the remaining I-side agents anticipate getting a partner with signal 1 if they drop out this stage at a price of $p_2^I = 1$ if they drop out in this stage, and a partner with a signal of 2 if they drop out in the next stage. Then their drop-out strategies are:

$$\tau_3^I(s_i, \alpha_{J[2]}, \alpha_{J[3]}, p_3^I)$$

$$\begin{aligned}
\tau_3^I(2,2,1,1) &= 2*2-2*1+1=3 \\
\tau_3^I(3,2,1,1) &= 3*2-3*1+1=4 \\
\tau_3^I(4,2,1,1) &= 4*2-4*1+1=5
\end{aligned}$$

So the second agent drops out on the I side at a clock price of 3. The matchmaker records that the second agent on the I side and the first agent on the J side should be partners, and they will pay $p_3^I = 1$ and $p_3^J = 0$, respectively. He sets $p_2^I = 3$ and

$p_2^J = 1$.

Now there are only two agents remaining on each side. At the beginning of the second stage on the J side, the participants anticipate that the worst remaining partner has a signal of 3, and matching in the next stage will yield a partner with a signal of 4. The matchmaker continues raising the clock on the J side, and the remaining J agents adopt the strategies:

$$\begin{aligned}
&\tau_2^J(s_i, \alpha_{I[1]}, \alpha_{I[2]}, p_2^I) \\
&\qquad \tau_2^J(2,4,3,1) \;=\; 2*4-2*3+1=3 \\
&\qquad \tau_2^J(3,4,3,1) \;=\; 3*4-3*3+1=4
\end{aligned}$$

So the second agent drops out on the J side at a clock price of 3. At the beginning of the second stage on the I side, the agents anticipate that the worst remaining partner has a signal of 2 and the best has a signal of 3. The matchmaker then begins raising the I-side clock, and the remaining I-side agents adopt the strategies:

$$\begin{aligned}
&\tau_2^I(s_i, \alpha_{J[1]}, \alpha_{J[2]}, p_2^I) \\
&\qquad \tau_2^I(3,3,2,3) \;=\; 3*3-3*2+3=6 \\
&\qquad \tau_2^I(4,3,2,3) \;=\; 4*3-4*2+3=7
\end{aligned}$$

So the third agent drops out on the I side at a clock price of 6. The matchmaker then records that the third agent on the I side and the second agent on the J side will be matched, and pay prices of $p_2^I = 3$ and $p_2^J = 1$, respectively. He sets $p_1^I = 6$ and $p_1^J = 4$.

Lastly, the matchmaker records that the fourth agent on the I side and the third agent on the J side should be matched, and should pay p_1^I and p_1^J, and the game concludes.

Now the matchmaker charges the matched agents the appropriate payment p_1^I, p_2^I, p_3^I or p_1^J, p_2^J, p_3^J, and releases the identities of the equilibrium partners. $\blacksquare$

This replicates the same match and payments as if the static competitive externality mechanism had been used, much the same as how the Generalized Second Price Auction replicates the same allocation and payments as the VCG mechanism in Edelman et al. [25]. It turns out that these strategies also form an ex post perfect equilibrium of the dynamic game:

Proposition 3.3.5 *Suppose all matches are acceptable. Then there exists an ex post perfect equilibrium of the announcement ascending competitive externality mechanism in which agents announce their types honestly to the matchmaker and use the drop-out strategies in Equations (3.5) and (3.6). This results in the same allocation and payments as the static competitive externality mechanism. There also exists an equilibrium in which the announcements are ignored, and the outcome is the same as the simple ascending competitive externality mechanism.*

Achieving the static outcome without the announcement phase is more complicated, because the static prices cannot be achieved if agents do not know what kind of partners they are bidding for. Like many cheap talk games, this game also has a "babbling equilibrium" where the announcements are false and ignored, and agents follow the equilibrium strategies of the simple ascending auction.

3.4 Many-to-One Matching

This section considers the case where the I side desires multiple partners on the J side, but the J side only wants one partner on the I side, analogous to firms hiring packages of workers.

Similar to specifying relationships between random variables, there are many ways to construct valuation functions for the agents that depend in non-trivial ways on their own type, the types of their co-workers or employer, and the package of workers a firm receives. In general, these potential dependencies could make honest revelation an intractable problem. In particular, workers will often have an incentive to misreport their type to manipulate the size of the firm, or because they have preferences over their "rank" at the firm.

Let a *package* of workers allocated to firm i be denoted $W_i = \{s_{j1}, s_{j2}, ..., s_{jL}\}$. A package is *ordered* if $s_{j1} \geq s_{j2} \geq ... \geq s_{jL}$; all packages are ordered in what follows.

The firms have *quantity-submodular* or *substitutes* preferences if for any two disjoint packages of workers W_{i1} and W_{i2},

$$v_I(s_i, W_{i1}) + v_I(s_j, W_{i2}) \geq v_I(s_i, W_{i1} \cup W_{i2})$$

Note that this class includes *additively submodular* firm values as a special case:

$$v_I(s_i, W_i) = \sum_{j\ell \in W_i = \{s_{j1}, s_{j2}, \ldots, s_{jL}\}} \pi_\ell(s_i, s_{j\ell})$$

where $\pi_\ell(s_i, s_j) \geq \pi_{\ell+k}(s_i, s_j)$, for all s_j and $k > 0$. This is a natural restriction and captures the idea of diminishing marginal returns to additional workers. If, for all

k and ℓ,

$$\frac{\partial v_I(s_i,\{...,s_{Jk},...\})}{\partial s_i \partial s_{Jk}} \geq 0\ ,\ \frac{\partial v_I(s_i,\{...,s_{Jk},...,s_{J\ell},...\})}{\partial s_{J\ell}\partial s_{Jk}} \geq 0$$

then *firm and worker signals are complementary* and *worker signals are complementary*, respectively. This is assumed throughout, and covers the natural case where worker and firm competence is mutually beneficial.

If workers get a constant share of their marginal contribution to firm profits, say that the workers are *paid their marginal product*: If worker j is the k-th worker at the firm,

$$v_J(s_j,s_i,\ell) = \alpha(v_I(s_i,\{s_{ji1},s_{ji2},....,s_{ji\ell-1},s_j\}) - v_I(s_i,\{s_{ji1},s_{ji2},....,s_{ji\ell-1}\}))$$

For the additively submodular case, this is equivalent to

$$v_J(s_j,s_i,\ell) = \alpha\pi_\ell(s_j,s_i)$$

and workers only care about their marginal contribution to the profitability of the firm.

This construction allows the worker's payoff to depend on the types of some of their co-workers. For example, the best worker at a given firm might be a manager, so that the payoffs of all the workers depend non-trivially on the firm's quality and the manager's talent, but not on any of the other worker's characteristics except their own — this is included in the additively submodular specification where workers are paid their marginal product.

The need for these definitions, rather than a general model, results from the fact that the two sides can now non-trivially disagree about how workers should be allocated to firms. For example, a worker might realize that reporting his type honestly will result in the addition of another employee who will lower his wages at his current firm. This might provoke him to understate his qualifications, thereby lowering the firm's expectations of productivity. This kind of manipulation is motivated by considerations about the *size* of the firm. A second motivation to lie is that the worker would rather be a "big fish in a small pond" and be the best worker at a bad firm rather than the worst worker at a better firm, especially if the difference in firm qualities is small. If the firms pay workers their marginal product, however, these motivations are mitigated.

Let $W = 2^J$ be the power set of J — all the possible packages of workers. Then the competitive externality mechanism can be extended to a *many-to-one competitive externality mechanism* in which an allocation is a probability distribution in the mixed extension over I, so that $m_{iW_i}(\sigma)$ gives the probability that a package of workers $W_i \in W$ will be assigned to firm i as a function of all the reports. Suppose the matchmaker separately chooses an allocation for both sides that maximizes the payoffs of the agents, and carries out the match only if the allocations match. Then the payments are decided for the firms as

$$t_{I\ell}(\sigma_{I\ell}, \sigma_{\setminus I\ell}) = -\sum_{i \in I \setminus I\ell} \sum_{W_i \in W} m_{iW_i}(\sigma_{I\ell}, \sigma_{\setminus I\ell}) v_I(\sigma_{I\ell}, W_i)$$

$$+ \sum_{i \in I \setminus I\ell} \sum_{W_i \in W} m_{iW_i}(\varnothing, \sigma_{\setminus I\ell}) v_I(\sigma_{I\ell}, W_i)$$

and for the workers as

$$t_{J\ell}(\sigma_{J\ell}, \sigma_{\backslash J\ell}) = -\sum_{j \in J \backslash J\ell} \sum_{W_i \in W} m_{iW_i}(\sigma_{J\ell}, \sigma_{\backslash J\ell}) v_J(\sigma_j, W_i) + \sum_{j \in J \backslash J\ell} \sum_{W_i \in W} m_{iW_i}(\varnothing, \sigma_{\backslash J\ell}) v_J(\sigma_j, W_i)$$

so the competitive externality of participation is internalized on all the agents.

Proposition 3.4.1 *Suppose all matches are acceptable. Then honest reporting is an ex post equilibrium of the many-to-one competitive externality mechanism if the firms pay workers their marginal product.*

So under these conditions, the matchmaker can intermediate in markets where there is many-to-one demand. Weakening these results will require assumptions about information, type distributions and payoffs that ensure agents are suitably ignorant of their opponents' and partners' characteristics so that they believe that reporting a higher type will get them a better package of workers or a better job in expectation. Since ex post equilibrium allows information to be distributed asymmetrically across the market, however, it applies more broadly in terms of achieving stability and efficiency in the presence of differential information, but with other payoff structures, there may be many additional incentive problems.

There are two difficulties in using a dynamic format to implement the same outcome as the static mechanism. First, the final payment of agent k is contingent on a hypothetical allocation of goods to the agents in market in which agent k has refused to participate. Since the marginal value of workers shifts whenever a firm acquires a worker, any straightforward extension of Proposition 3.4.1 fails to achieve the same outcome. One possibility is to employ a crediting/debiting mechanism, as studied in Ausubel [2]. However, the presence of private information on the part of the "goods" substantially complicates matters. It is not immediately clear whether the auction will release enough information for the agents to learn their value for all the potential packages of partners, and the notion of "sincere bidding" is complicated by the fact that information is being released about the value of potential partners continuously.

3.5 Conclusion

This paper provides a number of useful results on the dynamic implementation of ex post efficient matches in matching markets, and characterizes a set of assumptions that allow incentive compatible many-to-one matching, a case that has not before received attention in the competitive matching literature.

By combining ideas from the position auction literature — particularly Edelman et al. [25] — some useful dynamic games can be constructed to intermediate matching with supermodular complementarity, even in many-to-one matching contexts. While the assumptions in the many-to-one matching section — workers are paid their marginal product and firms have submodular preferences over package size — were strong, they are economically motivated and provide some insight into the challenges posed by many-to-one environments. Namely, workers have incentives to manipulate the size of the firm by over- or under-stating their complementarities with other workers, and a worker may have different preferences over firms about being the best worker at a better firm or the worst worker at a worse firm. Finding other mechanisms or sets of economically-meaningful assumptions that offer similar results might help distinguish important features of markets and shed light on why we see particular trading institutions emerge in the economy.

However, the case of single-dimensional types and supermodular valuation functions is quite restrictive. It would be preferable to allow firms to vary in their characteristics, allowing a rate of substitution between the worker types and their own. This would create other patterns of sorting different from the vertical type observed in the supermodular case. This introduces a number of new issues, however, because once the lattice of stable matches is no longer a singleton, the competitive framework used here breaks down: Agents are not only competing against the participants on their own side for partners, but groups of agents on each side are now competing against each other to insure their side's preferred match is chosen. Managing incentives in a situation like this poses many challenges because any combination of motives to compete, free ride, or mis-report private information can be in conflict or agreement throughout the type space.

Lastly, "matching" can be understood formally as choosing a weighted bipartite graph, where the edges represent the partner relationships and the weights represent the payments. There are many markets where relationships exist between and among many agents, such as in Kranton and Minehart [41] or Jackson [37]. For example, the entities who regulate power networks often have to procure electricity when there are limits on how much the various "sub-markets" can interact, and agents participate in many of these sub-markets — there are limits to how much electricity can be moved from Canada to Maine to Boston. Another example is the participation of outside commercial sellers on Amazon.com or eBay.com. How and why the matchmaker manipulates the structure of markets provides a new avenue for understanding commerce beyond the price mechanism of auction theory, or the partner-allocation algorithms of matching theory.

Chapter 4

One-to-One Matching with Multi-dimensional Types

4.1 Introduction

A common assumption in the matching literature is that agents know their preferences — whether it be cardinal utility or an ordered list — over partners. Incomplete information, when considered, then takes the form of uncertainty over the preferences of their potential partners or their opponents for mates. This paper shows that when there is uncertainty over the privately known characteristics of agents and their types are multi-dimensional, the common methods of achieving truthful reporting of private information as a weakly dominant strategy — serial random dictatorship, the Vickrey-Clarke-Groves mechanism, and, on one side of the market, deferred acceptance algorithms — fail to implement honest reporting. This occurs because when agents only know their own characteristics, they require knowledge of potential partners' characteristics to form a valuation, and this creates an avenue for manipulation that is assumed away when agents already know their preferences over partners. The second part of the paper constructs a mechanism with transfers that, with an appropriate restriction on preferences, does achieve honest revelation of private information as an ex post equilibrium. The restriction, called *reciprocity*, has a precedent in the linear assignment problem literature — see Shapley and Shubik [60] and Bikchandani and Ostroy [8] — and is not related to "vertical" or "horizontal" differentiation of market participants, nor the common assumption of supermodularity used throughout the literature.

In particular, this paper follows the approach taken in Damiano and Li [22], Hoppe, Moldovanu and Sela [35], Hoppe, Moldovanu and Ozdenoren [36], Bulow and Levin [9], Johnson [38], McAfee [44] and others following Becker [7]: When can competitive mechanisms in which agents' payments are based on competition for partners achieve stable or efficient outcomes? This literature will be referred to as the *competitive matching literature* because in contrast to other styles of inquiry such as Kelso and Crawford [19], Hatfield and Milgrom [34] and Chambers and Echenique [15], the emphasis is on how the market structure fosters competition for partners and leads to efficient matching, as opposed to studies where strategic behavior by the participants is ignored. All of the papers listed as part of the competitive matching literature, however, assume supermodularity and one-dimensional types. This paper extends those insights to multi-dimensional types and vertical or horizontal differentiation, but the results are not entirely positive: Restrictions must be placed on the relationship between the preferences for partners that the two sides of the market can have for each other, or the competition across the market undermines the competition within each side.

A recent paper by Chakraborty et al. [14] focuses on similar issues. They consider the problems facing a market designer who faces "students" with privately known multidimensional types and "universities" with publicly known types. Their focus is on how the amount of information released by the matching mechanism can undermine stability: By witnessing the final match, it may resolve some uncertainty on the part of the universities about the private information of the students that reveals the match is unstable. This is similar to the result in this paper that shows that serial random dictatorship, the classic deferred acceptance algorithm, and VCG do not implement honest revelation of private information in a multi-dimensional environment.

4.2 Model

There are two disjoint sets of agents, I and J, with (abusing notation) I agents on the I side and J agents on the J side; let $K = \min\{I, J\}$ with $K > 1$, which is the largest number of matches that can be arranged in this market. Suppose that $I \geq J$, so that there is an excess supply of I-side agents. Each I-side agent would like to match to one agent on the J side, and each agent on the J side would like to match to one agent on the I side. Since the information and preferences considered here are symmetric considering both sides, only the I side will be described in detail, with the appropriate permutation of indices applying to the J side.

Each agent receives a *type* $s_i \in S_I$ that determines the preferences and abilities of that agent. The type space for each agent on the I side, S_I, is a compact, convex subset of $\mathbb{R}^L$. A particular agent's type is a vector of *characteristics* $s_i = (s_i^1, s_i^2, ..., s_i^\ell, ...s_i^L)$, and a realization of types for the I side is an $L \times I$ $s_I = (s_{I1}, s_{I2}, ..., s_{II})$, including a column for each agent $I1, I2, ..., II$. Let the matrix with column Ik removed be

$$s_{I \backslash Ik} = (s_{I1}, s_{I2}, ..., s_{I,k-1}, s_{I,k+1}, ..., s_{II})$$

Let $s = (s_I, s_J)$. A *report* $\sigma = (\sigma_I, \sigma_J)$ is any collection of types from the type spaces of all agents, not necessarily equal to the truly realized types, and $s_{\backslash Ik} = (s_{I\backslash Ik}, s_J)$. Let the signals be distributed according to a joint density $f(s_I, s_J)$, and define $\mathbf{E}_{s_{\backslash i}}[h(s_i, s_{\backslash i})]$ as the expectation of $h(s_i, s_{\backslash i})$ conditional on s_i, with all other type realizations distributed according to the marginal density $f[s_{\backslash i}|s_i]$. Suppose that, given that some agent already exists with a type in the set $A \subset S_I$ where the measure of A is strictly positive, the probability that another agent draws a type in A is strictly positive for all A in the appropriately defined probability space. If this condition is satisfied, say that the type distribution is *rich*. This ensures that the agents' types are not deterministic in such a way that the sequence of realized types is carefully chosen to avoid incentive problems in an artificial fashion.

The *match surplus* accruing to agent i from being matched to agent j is given by a *surplus function* $v_I(s_i, s_j)$ that maps the types of agent i and j into a real number, with $v_J(s_j, s_i)$ defined similarly. Since only s_i and s_j determine the value of the match between agent i and j — and not any of the signals received by agents outside the i-j match — this the *pairwise private values* case. Agents have quasi-linear preferences, so an agent i paying t to match to agent j receives a payoff of

$$v_I(s_i, s_j) - t$$

A *direct revelation matching mechanism* (direct mechanism) is a set of functions

$$\{m_{ij}(\sigma), t_i(\sigma), t_j(\sigma)\}_{i,j}$$

where $m_{ij}(\sigma)$ maps any report into a lottery over matches in the set $I \times J$, and the payment functions $t_i(\sigma)$ and $t_j(\sigma)$ map a report σ into transfers to the matchmaker. The probability that i and j are matched at report σ is $m_{ij}(\sigma)$.

A direct mechanism is *Bayesian incentive compatible* for agents if

$$s_i \in \text{Argmax}_{\sigma_i} \mathbf{E}_{s_{I\setminus i}, s_J} \left[\sum_{j \in J} m_{ij}(\sigma_i, s_{\setminus i}) v_I(s_i, s_j) - t_i(\sigma_i, s_{\setminus i}) \right]$$

If, for any profile of types s'_{-i} reported by i's opponents, it is true that

$$s_i \in \text{Argmax}_{\sigma_i} \sum_{j \in J} m_{ij}(\sigma_i, s'_{\setminus i}) v_I(s_i, s_j) - t_i(\sigma_i, s'_{\setminus i})$$

then honesty is a *weakly dominant strategy*.

Honesty is an *ex post equilibrium* if, for all realizations of the types,

$$s_i \in \text{Argmax}_{\sigma_i} \sum_{j \in J} m_{ij}(\sigma_i, s_{\setminus i}) v_I(s_i, s_j) - t_i(\sigma_i, s_{\setminus i})$$

If $v_I(s_i, s_j) \geq v_I(s_i, s_{j'})$, then agent i *prefers* agent j to j', or $j \geq_i j'$. These ordinal preferences correspond to those considered by Gale and Shapley [28], where the primitives are ordinal preferences rather than cardinal ones. The allocation is *envy free* if there does not exist an agent on the I side, $I1$, who prefers a partner $J2$ matched to $I2$ at price t_{I2} to his own partner $J1$ at price t_{I1}:

$$v_I(s_{I1}, s_{J1}) - t_{I1} \geq v_I(s_{I1}, s_{J2}) - t_{I2}$$

This is an important aspect of the analysis in Edelman et al. [25], since it ensures that agents don't have an incentive to attempt to "steal" partners in an ascending auction for goods. Without it, there is the possibility that dynamic mechanisms can have equilibria where agents can profitably steal partners from one another.

There are two main examples used in examples and proofs in this paper:

- Agents have *increasing supermodular preferences* if the valuation functions are increasing in all arguments, and for all ℓ^1 and ℓ^2

$$\frac{\partial^2 v_I((s_i^1, s_i^2, ..., s_i^{\ell^1}, ...s_i^L), (s_j^1, s_j^2, ..., s_j^{\ell^2}, ...s_j^L))}{\partial s_i^{\ell^1} \partial s_j^{\ell^2}} \geq 0$$

  So there is complementarity between an agent's characteristics and his partner's characteristics. For example, $v_I(s_i, s_j) = s_i \cdot s_j = \sum_{\ell=1}^{L} s_i^\ell s_j^\ell$.

- Suppose $S_I = S_J$ and $\delta_I(s_i, s_j)$ is a distance metric, possibly specific to the I side. Then agents have *single-peaked preferences* if the valuation function takes the form

$$v_I(s_i, s_j) = f_I(\delta(s_i, s_j))$$

  where $f()$ is a decreasing function. For example, $v_I(s_i, s_j) = e^{-\sqrt{\sum_{\ell=1}^{L}(s_i^\ell - s_j^\ell)^2}}$.

These cases are intended to correspond to, respectively, a market in which agents always prefer a partner with a higher realization of signal type, and a market in which every agent has an "ideal" kind of partner, but these ideal vary on the characteristics of the agent. Of course, there are many other frameworks: one dimension could represent quality while the others are a location to nest the above preferences in one model, and so on. However, the set of stable matches can be non-trivial even these simple assumptions, as the next section shows.

4.3 Existence and Uniqueness of Stable Matches

Gale and Shapley [28] established the existence of a stable match in any marriage market, and Conway [40] proved that the set of stable matches is a lattice, where each side has a "group preference" that partially orders the set of stable matches, but that in the two-sided matching case, these orders run in opposite directions.

Theorem 4.3.1 (Gale and Shapley [28], Conway [40]) *The set of stable matches is a non-empty lattice. The I side agrees over a common preference ordering on the set of stable matches, and the J side agrees over a common preference ordering on the set of stable matches. These orders run in opposite directions.*

The single-dimensional matching cases with supermodular complementarity is studied in a number of papers, including Damiano and Li [22], Hoppe et al. [35], Johnson [38] [39], and McAfee [44]. However, this case has the convenient feature that the lattice of stable matches is a singleton: The best match for the I side is also the best match for the J side. In the multi-dimensional case, this can easily fail. This section presents a few simple examples in the increasing supermodular case of how the set of stable matches can be complicated by (i) the existence of an I-optimal stable match not equal to the J-optimal stable match, and (ii) existence of type realizations where each side does not have a uniform ranking of potential partners for some realization of partner quality.

4.3.0.3 Multiple Stable Matches

If the two sides have different preferences, the uniqueness of the stable match fails, even when supermodularity is assumed. This occurs whenever the two sides disagree on the marginal rate of substitution between two characteristics. For example, consider

$$v_I(s_i, s_j) = .75 s_i^1 s_j^1 + .25 s_i^2 s_j^2$$

$$v_J(s_j, s_i) = .25 s_i^1 s_j^1 + .75 s_i^2 s_j^2$$

If $s_{i1} = (1, 2)$, $s_{i2} = (2, 1)$, $s_{j1} = (1, 2)$ and $s_{j2} = (2, 1)$, then the values of the various matches are:

(v_I, v_J)	j_1	j_2
i_1	(1.75, 3.25)	(2,2)
i_2	(2, 2)	(3.25, 1.75)

So both the I-side agents prefer j_2 to j_1, but both the J-side agents prefer agent i_1 to i_2. So there are two stable matches[1]: $\mu_1 = \{(i_1, j_1), (i_2, j_2)\}$ and $\mu_2 = \{(i_1, j_2), (i_2, j_1)\}$.

4.3.0.4 Non-Unique Orderings of Partners

Another useful characteristic of the single-dimensional type environment is that all the agents on a given side agree on how to rank their potential partners. In the multi-dimensional type case, this no longer holds. Let

$$v_J(s_i, s_j) = .5 s_i^1 s_j^1 + .5 s_i^2 s_j^2$$

If $s_{i1} = (1, 2)$, $s_{i2} = (2, 1)$, $s_{j1} = (1, 2)$ and $s_{j2} = (2, 1)$, then the J side match values are

[1] To see this more formally, consider using the Gale-Shapley algorithm with the I side proposing, then the J side proposing. This yields two matches, so there is a non-trivial lattice of stable matches.

v_I, v_J	i_1	i_2
j_1	2.5	2
j_2	2	2.5

So the two J agents disagree about which I agent is better. Slightly perturbing the signals does not resolve this issue, either, since the cases above are not "knife-edge". So despite having supermodular surplus functions, there can be substantial disagreement over how the agents should be matched. This has substantial implications for designing the market. In particular, the approach used in Johnson [38] was to harness competition between the agents to compete for partners on the other side of the market. Now, however, the agents are not only competing against participants on their side, but necessarily against agents on the other side to determine which of the possible matchings is chosen by the matchmaker.

4.4 Deferred Acceptance Algorithms, Serial Random Dictatorship, and Vickrey-Clarke-Groves Mechanisms

In the matching literature, there are two main mechanisms used for matching problems: The first, serial random dictatorship (SRD), asks agents to report their preferences over partners to the matchmaker, who then draws agents in random order and matches them to the highest-ranked unmatched partner remaining on their lists. The second, deferred acceptance algorithms, asks agents to report their preferences as ordered lists to the matchmaker, who uses an appropriate version of the

Gale-Shapley algorithm to compute an allocation using the lists. These mechanisms are defended on strategic grounds by pointing out that it is a dominant strategy to report honestly in SRD because mis-reporting preferences does not provide any advantage: When chosen by the matchmaker to pick, it is better to have reported honestly than to have lied. For the Gale-Shapley algorithm, the defense is more subtle: Roth [54] has shown that it is never a dominant strategy for both sides to report honestly. However, in many practical applications, it is sufficient to guarantee that one side report honestly, since the other side of the market is often an institution such as a public school that has no way of manipulating its preference report. Roth has shown that the proposing side always has a dominant strategy to report honestly, so the mechanism can be arranged so that the side with the incentive to mis-report proposes, and incentive compatibility is established. Similarly, a common method of implementing honest reporting in the auction and public goods literatures is the Vickrey-Clarke-Groves mechanism. With incomplete information about types, however, these mechanisms no longer implement honest reporting. The reason is that by manipulating information revelation about himself, an agent can influence the perceived preferences of potential partners, thereby getting a better partner (and often a lower payment) merely by lying.

To show this, consider the following formal definitions of the mechanisms: The *incomplete information serial random dictatorship mechanism* is the direct revelation mechanism in which agents are asked to make a type report σ_i, the matchmaker computes the matrices of match values $V_I = [v_I(\sigma_i, \sigma_j)]_{ij}$ and $V_J = [v_J(\sigma_j, \sigma_i)]_{ji}$, and then proceeds by randomly picking one of the remaining agents and matching them to the remaining partner who gives them the highest valuation. The *incomplete information deferred acceptance algorithm* is the direct revelation mechanism in which agents are asked to make a type report σ_i, and then the matchmaker computes V_I and V_J , and uses the following matching algorithm:

- The matchmaker represents each agent's preferences over partners through an ordered list, where, for agent i, $j \succeq_i j'$ if $v_I(\sigma_i, \sigma_j) \geq v_I(\sigma_i, \sigma_{j'})$.
- Using the ordered lists, the matchmaker runs the Gale-Shapley algorithm "in virtual time":
  - All agents on the I side propose to their favorite potential partner on the J side. All J side agents tentatively accept their favorite proposal.
  - All the unmatched I-side agents propose to the next-highest potential partner on the J side, and the J side agents either retain their current partner or tentatively accept one of the new proposals.
  - The previous step is repeated until all the agents are matched or every unmatched I side agent has previously proposed to every acceptable J side partner.

Then the following is true:

Proposition 4.4.1 *Truth-telling is, in general, not an ex post equilibrium of (i) serial random dictatorship, (ii) deferred acceptance algorithms, even for the proposing side, and (iii) not an ex post equilibrium of the Vickrey-Clarke-Groves mechanism.*

The reason that Proposition 4.4.1 holds is that when agents' preferences are formed by the information revelation induced by the mechanism, there are possibilities for manipulation, and this is precisely the problem in matching markets with incomplete information: Agents are not sure about their partners' or opponents' types, and the market designer's goal is to help them through a process of mutual revelation. So the results of Roth [54] and others on the incentive compatibility properties of SRD and Gale-Shapley algorithms do not extend to this broader notion of uncertainty.

4.5 A Mechanism for "Reciprocal" Environments

As shown above, the Vickrey-Clarke-Groves mechanism is not well-suited to matching environments because agents' payments are based on their *anticipated* contribution to social welfare, and this can always be manipulated upwards by misrepresenting one's self as a better partner, thereby inflating the anticipated gain from matching. This makes constructing *any* matching mechanism that implements truth-telling a challenge.

In the competitive matching literature, the following pattern emerged for successful implementation: Make the agents on each side of the market compete for the rights to partners of better quality, and then match the two sides on the basis of the rankings. This logic immediately faces two problems when moving away from the single-dimensional supermodular case: What does better quality now mean, and what happens if the outcomes of the competition on the two sides of the market disagree?

4.5.1 Reciprocity and Admissible Transformations

A transformation $g : v_I(s_i, s_j) \rightarrow v_J(s_j, s_i)$ is *reciprocal* if, for any proposed match m^*_{ij} and any alternative allocation $m^{'}_{ij}$, if

$$\sum_i \sum_j m^*_{ij} v_I(s_i, s_j) \geq \sum_i \sum_j m'_{ij} v_I(s_i, s_j)$$

then

$$\sum_i \sum_j m^*_{ij} v_J(s_j, s_i) \geq \sum_i \sum_j m'_{ij} v_J(s_j, s_i)$$

This ensures that applying the transformation $g()$ does not change the relative ordering of matches on either side of the market with respect to social welfare. A related definition comes from Vo-Khac [68] and is developed further in Burkard, Hahn and Zimmerman [12], and reviewed in Burkard and Cela [11]. That literature considers an *assignment problem* of allocating jobs to workers or machines to minimize costs: For a set of n workers and n machines, with cost c_{ij} of allocating task i to worker j, the decision-maker seeks to minimize

$$\min_{x_{ij}} \sum_i \sum_j x_{ij} c_{ij}$$

where $x_{ij} \in \{0, 1\}$, $\sum_i x_{ij} = 1$, $\sum_j x_{ij} = 1$, and $c_{ij} > 0$. Clearly, this cost minimization problem is connected to the assignment problem of setting prices and allocations in a game theory setting, which is explored fruitfully in Shapley and Shubik [60], Bikchandani and Ostroy [8], and Vohra [67]. In the assignment setting, a transformation of cost matrix $C = (c_{ij})$ to $\tilde{C} = (\tilde{c}_{ij})$ is *admissible* if, for each j,

$$\sum_{i=1}^{n} c_{ij} = \sum_{i=1}^{n} \tilde{c}_{ij} + Z$$

where Z is a constant which depends on the transformation but not on anything else. In the assignment literature, a transformation of the costs is admissible if it leaves unchanged the relative order of objective function values of all feasible solutions. This ensures that the solution to an assignment problem is the same as that for any admissible transformation of it. This is exactly the condition needed for the two sides of the market to "agree" on a match in the incomplete information matching framework developed.

The set of reciprocal transformations is not empty. For example, let $g : v_I(s_i, s_j) \rightarrow \alpha v_I(s_i, s_j) + \beta$, with $\alpha > 0$ and $\beta \geq 0$, a positive affine transformation. Then

$$\begin{aligned}
\sum_i \sum_j m^*_{ij}(s) v_I(s_i, s_j) &\geq \sum_i \sum_j m'_{ij}(s) v_I(s_i, s_j) \\
\alpha \sum_i \sum_j m^*_{ij}(s) v_I(s_i, s_j) + J\beta &\geq \alpha \sum_i \sum_j m'_{ij}(s) v_I(s_i, s_j) + J\beta \\
\sum_i \sum_j m^*_{ij}(s)(\alpha v_I(s_i, s_j) + \beta) &\geq \sum_i \sum_j m'_{ij}(s)(\alpha v_I(s_i, s_j) + \beta) \\
\sum_i \sum_j m^*_{ij}(s) v_J(s_j, s_i) &\geq \sum_i \sum_j m'_{ij}(s) v_J(s_j, s_i)
\end{aligned}$$

implying that $\sum_i \sum_j m^*_{ij}(s) v_J(s_j, s_i) \geq \sum_i \sum_j m'_{ij}(s) v_J(s_j, s_i)$. So there is at least one class of admissible transformations. However, the definition potentially allows other kinds of transformations, and these transformations may depend on the valuation functions underlying the matching problem being investigated. For example, in the one-dimensional supermodular case, any transformation that maintains supermodularity is admissible.

4.5.2 Competitive Externality Pricing in Reciprocal Problems

Let $s_{I\ell}$ be agent $I\ell$'s true signal, $\sigma_{I\ell}$ a potentially false report to the matchmaker, and $\varnothing$ the null variable that indicates agent $I\ell$ prefers not to participate in the mechanism. Then the *competitive externality mechanism* is the direct revelation mechanism in which agents report possible types σ_i to the matchmaker, who then uses the following procedure to decide the match:

- The matchmaker finds the allocation $m^I_{ij}(\sigma)$ that solves

$$\max_{m^I_{ij}} \sum_i \sum_j m_{ij}(\sigma) v_I(\sigma_i, \sigma_j)$$

- The matchmaker finds the allocation $m^J_{ij}(\sigma)$ that solves

$$\max_{m^J_{ij}} \sum_i \sum_j m^J_{ij}(\sigma) v_J(\sigma_j, \sigma_i)$$

- If, for all i and j at σ, $m^I_{ij}(\sigma) = m^J_{ij}(\sigma)$, then the match is *feasible* and the matchmaker announces the identities of the partners. If, for some i and j at σ, $m^I_{ij}(\sigma) \neq m^J_{ij}(\sigma)$, the matchmaker does not match any agents or charge any payments. If agents are matched, the $I\ell$ agent pays

$$t_{I\ell}(\sigma_{I\ell}, \sigma_{\setminus I\ell}) = -\sum_{i \in I \setminus I\ell} \sum_{j \in J} m_{ij}(\sigma_{I\ell}, \sigma_{\setminus I\ell}) v_I(\sigma_i, \sigma_j) \\ + \sum_{i \in I \setminus I\ell} \sum_{j \in J} m_{ij}(\varnothing, \sigma_{\setminus I\ell}) v_I(\sigma_i, \sigma_j)$$

  and similarly for the J side.

This internalizes the welfare consequences of agent $I\ell$'s participation on his opponents on his side of the market, ignoring the welfare effects that impact the J side. In general, this need not lead to any kind of equilibrium — in particular, agent $I\ell$ might find it profitable to mis-represent himself to manipulate the behavior of the J side agents, and such manipulation might even benefit the other agents on his side of the market.

One obviously deficient feature of the competitive exernality mechanism is that if the welfare-maximizing match is different on the two sides, no action is taken by the matchmaker. A precedent for such a requirement is Gul and Stachhetti [33], where untruthful bidding in an auction can lead to a poorly defined excess demand vector, derailing the auction. In that paper and here, this lack of trade merely "closes" the model so that it is well-defined for all reports σ. In Johnson [39], ascending formats are used to implement the same outcome, given sufficient structure on the types and valuation functions of the participants, avoiding the inaction that results when the competitive externality mechanism is infeasible. Lastly, this feature actually highlights the key issue that arises in matching with incomplete information as opposed to package auctions, assignment problems, or other environments: As long as the two sides ex post agree on a unique efficient match, the problem is simply to provide incentives for agents on each side *separately* to reveal their information honestly. Once the two sides ex post disagree, each agent has a potential incentive to misrepresent himself to manipulate the other side's conception of an ex post efficient match.

Note that in Proposition 4.4.1, the preferences used to generate the counter-examples were $v(s_i, s_j) = s_i \cdot s_j$ for both sides. These preferences are reciprocal, and represent probably the simplest non-trivial case of complementarity between all the agents' characteristics. So the following proposition is a strict improvement over the failures there to generate incentive compatibility or efficient and stable matching:

Proposition 4.5.1 *Suppose the type distribution is rich. Then a competitive externality mechanism is feasible and truth-telling is an ex post equilibrium iff preferences are reciprocal.*

The added assumption that the type distribution is rich ensures that the probability distributions of types are not "degenerate" in such a way that any sequence of draws that can occur will not include disagreement between the two sides, even if the preferences and type spaces admit disagreement for other distributions of types. Proposition 4.5.1 shows that the reciprocity condition guarantees the CEM approach can be used, regardless of whether the agents have characteristics that are quality- or distance-based. Unfortunately, this is an unrealistic assumption in many situations. For example, in the special case of one-dimensional supermodular types, assortative matching is the unique efficient outcome, and any class of transformations that maintains supermodularity maintains the ranking of the assortative match as the best; this is the mathematical feature of that environment that makes it so tractable. However, Proposition 4.5.1 is useful because it illustrates how non-reciprocity complicates the demands of incentive compatibility in matching environments. As long as the incentive constraints can be handled separately on each side, the reasoning from the competitive matching literature extends to considerably more general environments.

To implement these outcomes dynamically, it would be helpful if none of the agents would like to "swap" partners and payments at the proposed allocation. This envy-free condition appears in Edelman et al. [25] and Johnson [38] [39]. Theorem 1 of Edelman et al., for example, requires envy-free prices to establish the result.

Proposition 4.5.2 *The allocation and payments of the competitive externality mechanism may not be envy-free.*

In Johnson, the match is ex post stable because of the supermodularity of the valuation functions and one-dimensionality of the types, which do not hold here. The failure of these features may have implications for efforts to design dynamic mechanisms, for example, since agents may be able to "intimidate" one another into dropping out early by over-bidding.

Lastly, it is possible for outcomes to be stable, given the payments, but unstable without them. This is not possible in the supermodular, one-dimensional type case, where the payments merely provide incentives for agents to report their information honestly. In the following example, there is a unique stable match without the payments, but welfare is maximized at a different matching and allocation. Consequently, the payments play two roles: The first is to provide incentives to report honestly, but the second is to compensate the agents for not moving to the stable match without the transfers. For example, consider a marriage market with dowries. Without the payments, one allocation might arise that is based only on the participants' preferences for their partners, but this is not efficient. With the introduction of the payments, however, the efficient match can be arranged.

The intuition of the following example is based on the idea that if agents prefer to be "closer" to one another and both sides of the market agree on the metric, then there will be a unique stable match as long as there are no indifferences among the agents. However, a social planner may prefer a different allocation that minimizes the total distance among partners. These solutions can be different, and making the efficient match stable will require carefully constructed payments.

Example Suppose $I = J = 2$ with types in $\mathbb{R}^2$ and agents have minimum distance preferences, $v_I(s_i, s_j) = v_J(s_j, s_i) = \bar{v} - ||s_i - s_j||$, where $||\cdot||$ is the Euclidean norm, and $\bar{v} > \sqrt{2}$. Then let the coordinates of $I1$'s type in the plane be $s_{I1} = (0, 1)$, the coordinates of $I2$'s type be $s_{I2} = (.25, .5)$, the coordinates of $J1$'s type be $s_{J1} = (0, 0)$, and the coordinates of $J2$'s type be $(1, 0)$. Then the surpluses associated with each of the matches are approximately

	$J1$	$J2$
$I1$	$\bar{v} - .559$	$\bar{v} - .901$
$I2$	$\bar{v} - 1$	$\bar{v} - \sqrt{2}$

So the unique stable match is $\{(I1, J1), (I2, J2)\}$, but welfare is maximized at $\{(I1, J2), (I2, J1)\}$ because

$$2(2\bar{v} - (.901 + .559)) = 2(2\bar{v} - 1.46) > 2(2\bar{v} - 1.901)$$

In the competitive externality matching mechanism with $\bar{v} = \sqrt{2}$, the payments are approximately

$$
\begin{aligned}
t_{I1} &= -.612 \\
t_{I2} &= .269 \\
t_{J1} &= -.414 \\
t_{J2} &= .368 \\
\sum_k t_k &= -.387
\end{aligned}
$$

so that agent $I1$ and $J1$ are compensated for not moving to block the efficient allocation, agents $I2$ and $J2$ make payments to the matchmaker. It turns out that the mechanism runs a deficit in this example, which is not necessarily the case when the efficient and stable matches are the same — in supermodular markets where there is a unique stable match, for example, there is a budget surplus. ∎

This shows that the following proposition is true:

Proposition 4.5.3 *A match generated by the competitive externality mechanism may be efficient, but not ex post stable without the payments.*

This might matter in situations where the payments are required to maintain stability, not just reveal information. For example, consider an infinitely repeated marriage market. Then agent i maximizes his discounted expected utility,

$$\sum_{\tau=0}^{\infty} \beta^{\tau} \mathbf{E}_i \left[\sum_j m_{ij\tau} v_I(s_i, s_j) - t_0 \right]$$

where $m_{ij\tau}$ is the probability that agent i and j are matched at time τ, and payments are only made at date 0, t_0. In a market where types are one-dimensional and surplus is supermodular, the unique stable match is identified in period 0 and the partners need no payments in subsequent periods for the efficient outcome to be supported. In the market considered in the previous example, however, the efficient outcome can only be supported in periods in which the mechanism designer is willing to make a payment into the market: Once information is revealed and the flow of payments stops, agents will be tempted to cheat. Consequently, honest revelation and efficiency can be achieved, but maintaining the efficient outcome will require outside funds in all future periods or the agents will revert to the stable, inefficient outcome. In these markets, then, stability and efficiency can be in conflict.

4.6 Conclusion

Further work on multi-dimensional types may provide many more satisfying answers to the implementation problem. Finding a mechanism that selects a particular element from the set of stable matches would provide useful insight into how to balance the competition for partners on each side of the market, versus the competition between the two sides to get the matchmaker to implement their preferred element of the set of stable matches. However, this paper provides a useful intermediate step by finding a mechanism that moves past the increasing-supermodular paradigm that is popular in the competitive matching literature.

Appendix A

Proofs for Chapter Two

Lemma A1 *If $q > q'$, the distribution $h(q', K) = \sum_{k=1}^{K} w_{I,k}(q')$ first-order stochastically dominates $h(q, K)$; having a higher q places more weight on the "high" rankings, corresponding to 1, 2, ...*

Proof First, note that $\sum_{k=1}^{K_I} w_{I,k}(q) = 1$, by the binomial theorem. Then

$$\sum_{k=1}^{K_I} w'_{I,k}(q) = 0$$

For $k = 2, 3, ..., K_I - 1$,

$$w'_{I,k}(q) = \frac{(K_I - 1)!}{(K_I - k)!(k-1)!} f_I(q) F_I(q)^{K_I - k - 1} (1 - F_I(q))^{k-2} \left[(K_I - k)(1 - F_I(q)) - (k-1)F_I(q)\right]$$

For $k = 1$ and $k = K_I$, the functions $w_{I,k}(q)$ are monotone increasing and decreasing, respectively. Let the sequence of points $\{\tilde{q}_k\}_{k=2}^{K_I - 1}$ be defined as

$$\tilde{q}_k = F_I^{-1}\left(\frac{K_I - k}{K_I - 1}\right)$$

This is a decreasing sequence in k. For a given q, find the interval $[\tilde{q}_k, \tilde{q}_{k+1}]$ and label the accompanying k as k^*. Now, for all the terms $w'_{I,k}(q)$ with $k < k^*$, $w'_{I,k}(q)$ is positive but for all terms $w'_{I,k}(q)$ with $k > k^*$, $w'_{I,k}(q)$ is negative. Then as we drop negative terms from the end of the sum, $K_I, K_I - 1, ..., K$, the value of $h'(q, K)$ must be positive. Then $h(q, K) = h(q', K) + \int_{q'}^{q} h'(z, K)dz \geq h(q', K)$. This shows that $h(q,K) \geq h(q',K)$ for all K, so $h(q',K)$ first-order stochastically dominates $h(q,K)$. ■

Lemma A2 *$F_{J,(\ell)}(q)$ first-order stochastically dominates $F_{J,(\ell+1)}(q)$.*

Proof I'll show that $f_{J,(\ell)}(q)$ likelihood ratio dominates $f_{J,(\ell+1)}(q)$, from which the conclusion follows. Note that

$$f_{J,(\ell)}(x) = \frac{K!}{(\ell-1)!(-+K-\ell)!}F_J(x)^{K-\ell}(1-F_J(x))^{\ell-1}f_J(x)$$

Take $x < y$. Then, since the (ℓ) distribution beats one more agent and the $(\ell+1)$ distribution loses to one more agent,

$$\frac{f_{J,(\ell)}(x)}{f_{J,(\ell+1)}(x)} = \frac{(K-\ell-1)!\ell!}{(K-\ell)!(\ell-1)!}\frac{F_J(x)}{1-F_J(x)}$$

Since cumulative density functions are non-decreasing, $F_J(y) \geq F_J(x)$, and

$$\frac{f_{J,(\ell)}(x)}{f_{J,(\ell+1)}(x)} = \frac{(K-\ell-1)!\ell!}{(K-\ell)!(\ell-1)!}\frac{F_J(x)}{1-F_J(x)} \leq \frac{(K-\ell-1)!\ell!}{(K-\ell)!(\ell-1)!}\frac{F_J(y)}{1-F_J(y)} = \frac{f_{J,(\ell)}(y)}{f_{J,(\ell+1)}(y)}$$

So $F_{J,(\ell)}(x)$ likelihood ratio dominates $F_{J,(\ell+1)}(x)$, which implies first-order stochastic dominance. ■

Lemma A3 *Consider*

$$\left\{\int_{R_I(z)}^{\bar{q}_J}\frac{\partial s_I(z,y)}{\partial q_i}f_{J,(k)}(y)dy\right\}_{k=1}^{K_J}$$

This sequence is non-increasing in k.

Proof By calculation,

$$\int_{R_I(z)}^{\bar{q}_J}\frac{\partial s_I(z,y)}{\partial q_i}f_{J,(k)}(y)dy - \int_{R_I(z)}^{\bar{q}_J}\frac{\partial s_I(z,y)}{\partial q_i}f_{J,(k+1)}(y)dy$$

$$= -\int_{R_I(z)}^{\bar{q}_J} \frac{\partial s_I(z,y)}{\partial q_i} d[1 - F_{J,(k)}(y)] + \int_{R_I(z)}^{\bar{q}_J} \frac{\partial s_I(z,y)}{\partial q_i} d[1 - F_{J,(k+1)}(y)]$$

Integrating by parts yields

$$= \left[F_{J,(k+1)}(R_I(z)) - F_{J,(k)}(R_I(z))\right] \frac{\partial s_I(z, R_I(z))}{\partial q_i} + \int_{R_I(z)}^{\bar{q}_J} \left[F_{J,(k+1)}(y) - F_{J,(k)}(y)\right] \frac{\partial^2 s_I(z,y)}{\partial q_i \partial q_j} dy$$

Because of the first-order stochastic dominance of $F_{J,(k)}$ over $F_{J,(k+1)}$ and supermodularity, this entire term is positive. ∎

Lemma A4 *Let $F_{J,(k),K_J}(q)$ be the distribution of the k-th order statistic, when there are K_J draws from $F_J(q)$. Then $F_{J,(k),K_J+1}(q)$ first-order stochastically dominates $F_{J,(k),K_J}(q)$.*

Proof Let $q > q'$. First, $f_{J,(k),K_J+1}(q)$ likelihood-ratio dominates $f_{J,(k),K_J}(q)$:

$$\frac{f_{J,(k),K_J+1}(q')}{f_{J,(k),K_J}(q')} = \frac{K_J + 1}{K_J + 1 - k} F_J(q') \leq \frac{K_J + 1}{K_J + 1 - k} F_J(q) = \frac{f_{J,(k),K_J+1}(q)}{f_{J,(k),K_J}(q)}$$

Since likelihood-ratio dominance implies first-order stochastic dominance, the result follows. ∎

Lemma A5 *Under TAM and for an integrable function $h(q_i, q_j)$,*

$$\mathbf{E}_i\left[\sum_j m_{ij}(q_i, q_{I\setminus i}, q_J) h(q_i, q_j)\right] =$$

$$\sum_k w_{I,k}(q)[1 - F_{J,(k)}(R_I(q_i))]\mathbf{E}_i[h(q_i, q_k) | \rho_{q_J}(q_k) = k, q_k \geq R_I(q_i)]$$

Proof Under TAM, the matching rule is:

$$m_{ij}(q_i, q_{I-i}, q_J) = \begin{cases} 1 \quad , \quad \rho_{q_I}(q_i) = \rho_{q_J}(q_j), \psi_I(q_i, q_j) + \psi_J(q_j, q_i) \geq c \\ 0 \qquad\qquad\qquad , \quad \text{Otherwise} \end{cases}$$

Then

$$\mathbf{E}_i\left[\sum_j m_{ij}(q_i, q_{I-i}, q_j, q_{J-j})h(q_i, q_J)\right] = \mathbf{E}_i\left[\sum_j \mathbf{1}_{\{\rho_{q_I}(q_i)=\rho_{q_J}(q_j), q_j \geq R_I(q_i)\}} h(q_i, q_j)\right]$$

Then the indicator function can be broken up into two events

$$\mathbf{1}_{\{\rho_{q_I}(q_i)=\rho_{q_J}(q_j), q_j \geq R_I(q_i)\}} = \mathbf{1}_{\{\rho_{q_I}(q_i)=\rho_{q_J}(q_j)\}}\mathbf{1}_{\{q_j \geq R_I(q_i)\}}$$

Then

$$\mathbf{E}_i\left[\sum_j \mathbf{1}_{\{\rho_{q_I}(q_i)=\rho_{q_J}(q_j) \cap q_j \geq R_I(q_i)\}} h(q_i, q_j)\right]$$
$$= \mathbf{E}_i\left[\sum_j \mathbf{1}_{\{\rho_{q_I}(q_i)=\rho_{q_J}(q_j)\}}\mathbf{1}_{\{q_j \geq R_I(q_i)\}} h(q_i, q_j)\right]$$

$$= \mathbf{E}_i\left[\mathbf{1}_{\{\rho_{q_I}(q_i)=1\}}\mathbf{1}_{\{q_j \geq R_I(q_i)\}} h(q_i, q_j) + ... + \mathbf{1}_{\{\rho_{q_I}(q_i)=K\}}\mathbf{1}_{\{q_j \geq R_I(q_i)\}} h(q_i, q_j)\right]$$

The above equation shows that for each agent on the other side, named j, there are $K = \min\{K_I, K_J\}$ ways to match to him: i can do it as the best ranked agent on the I side, as the second-best, and so on. Since the agents are symmetric, there are K_J terms for each of the ranks, and probability $1/K_J$ that each agent j achieves that rank, leading to

$$\mathbf{E}_i\left[\sum_j \sum_{k=1}^{K} \mathbf{1}_{\{\rho_{q_I}(q_i)=k \cap \rho_{q_J}(q_j)=k\}}\mathbf{1}_{\{q_j \geq R_I(q_i)\}} h(q_i, q_j)\right]$$

$$= \sum_{k=1}^{K} \mathbf{E}_{q_{I\setminus i}}\left[\mathbf{1}_{\{\rho_{q_I}(q_i)=k\}}\right]\left(\sum_j \frac{1}{K_J}\mathbf{E}_{q_J}\left[\mathbf{1}_{\{\rho_{q_J}(q_j)=k\}}\mathbf{1}_{\{q_j \geq R_I(q_i)\}} h(q_i, q_j)\right]\right)$$

Since the J-side agents are all symmetric and the expectation of an indicator function is a density,

$$= \sum_{k=1}^{K} w_{I,k}(q_i)\mathbf{E}_{q_J}\left[\mathbf{1}_{\{\rho_{q_J}(q_k)=k\}}\mathbf{1}_{\{q_k \geq R_I(q_i)\}}h(q_i, q_k)\right]$$

The remaining expectation can be written as

$$\int_{z_1} \cdots \int_{q_k} \cdots \int_{z_{K_J}} \mathbf{1}_{\{\rho_{q_J}(q_k)=k\}}\mathbf{1}_{\{q_k \geq R_I(q_i)\}}$$

$$h(q_i, q_k) f_J(z_1) \ldots f_J(q_k) \ldots f_J(z_{K_J}) dz_1 \ldots dq_k \ldots dz_{K_J}$$

The indicator function $\mathbf{1}_{\{\rho_{q_J}(q_j)=k\}}$ is activated whenever the particular component q_j takes the k-th slot less than K_J. The support of q_J can be divided into $K_J!$ disjoint sets that correspond to all the rankings of the components of the vector of qualities. These permutation transformations have a Jacobian of $|\pm 1|$, so that their joint distribution on the set on which the indicator function takes the value one, rather than zero, is $K_J! f_J(q_{(1)}) f_J(q_{(2)}) \ldots f_J(q_{(K_J)})$ which is the distribution of the order statistics. Taking this transformation and making a number of routine changes of order of integration leads to

$$= \sum_k w_{I,k}(q_i) \int_0^{\bar{q}_J} \mathbf{1}_{\{q_k \geq R_I(q_i)\}} h(q_i, q_k)$$

$$\frac{K_J!}{(k-1)!(K_J-k)!} F_J(q_k)^{K_J-k}(1-F_J(q_k))^{k-1} f_J(q_k) dq_k$$

$$= \sum_k w_{I,k}(q_i) \int_0^{\bar{q}_J} \mathbf{1}_{\{q_k \geq R_I(q_i)\}} h(q_i, q_k)) f_{J,(k)}(q_k) dq_k$$

Calculation yields

$$= \sum_k w_{I,k}(q_i) \int_0^{R_I(q_i)} 0 dq_k + \int_{R_I(q_i)}^{\bar{q}_J} h(q_i, q_k) f_{J,(k)}(q_k) dq_k$$

Dividing the remaining term by $(1 - F_{J,(k)}(R_I(q_i)))$ makes it into a conditional expectation. Substituting this into Equation 17 yields

$$\sum_k w_{I,k}(q_i) \left[1 - F_{J,(k)}(R_I(q_i))\right] \mathbf{E}_i \left[h(q_i, q_k) | \rho_{q_J}(q_k) = k, q_k \geq R_I(q_i)\right] \tag{A.1}$$

Therefore, the expectation can be written as claimed. ∎

Proof of Lemma 2.3.1 (Incentive Compatibility)

Proof *Only if:* Assume incentive compatibility holds; I now show (i) and (ii) hold.

Define q as the agent's true quality, and $\hat{q}$ as the report submitted to the mechanism operator. If the mechanism is locally incentive compatible, maximizing with respect to the report $\hat{q}$ in the indirect utility function yields

$$0 = \frac{\partial}{\partial \hat{q}} \left[\mathbf{E}_i \left[\sum_j m_{ij}(\hat{q}, q_{I \setminus i}, q_J) s_I(q, q_j) - t_i(\hat{q}, q_{I \setminus i}, q_J) \right] \right]_{\hat{q}=q}$$

Taking the total derivative of the indirect utility function then yields

$$\begin{aligned} U_I'(q) = \frac{\partial}{\partial \hat{q}} \left[\mathbf{E}_i \left[\sum_j m_{ij}(\hat{q}, q_{I \setminus i}, q_J) s_I(q, q_j) - t_i(\hat{q}, q_{I \setminus i}, q_J) \right] \right]_{\hat{q}=q} \\ + \mathbf{E}_i \left[\sum_j m_{ij}(q, q_{I \setminus i}, q_J) \frac{\partial s_I(q, q_j)}{\partial q_i} \right] \\ = \mathbf{E}_i \left[\sum_j m_{ij}(q, q_{I \setminus i}, q_J) \frac{\partial s_I(q, q_j)}{\partial q_i} \right] \end{aligned}$$

Integrating with respect to q yields the indirect utility function, (i):

$$U_I(q) = U_I(\underline{q}_I) + \mathbf{E}_i \left[\sum_j \int_{\underline{q}_I}^{q} m_{ij}(z, q_j) \frac{\partial s_I(z, q_j)}{\partial q_i} dz \right]$$

(ii) Using (i) and the indirect utility function, interim expected transfers can be written

$$\mathbf{E}_i[t_i(q, q_{I\setminus i}, q_J)] =$$
$$\mathbf{E}_i\left[\sum_j m_{ij}(q, q_{I\setminus i}, q_J)s_I(q, q_j) - \int_{\underline{q}_I}^{q} m_{ij}(z, q_{I\setminus i}, q_J)\frac{\partial s_I(z, q_j)}{\partial q_i}dz\right] - U_I(\underline{q}_I)$$

Then lying and submitting a report q' yields an expected payoff of

$$\begin{aligned}\tilde{U}(q, q') = \\ &\mathbf{E}_i\Big[\sum_j m_{ij}(q', q_{I\setminus i}, q_J)s_I(q, q_j) - \sum_j m_{ij}(q', q_{I\setminus i}, q_J)s_I(q', q_j) \\ &+ \int_{\underline{q}_I}^{q'} \sum_j m_{ij}(z, q_{I\setminus i}, q_J)\frac{\partial s_I(z, q_j)}{\partial q_i}dz\Big] - U_I(\underline{q}_I)\end{aligned} \tag{A.2}$$

Then lying is unprofitable if

$$\begin{aligned}\tilde{U}_I(q, q') - U_I(q) = \\ &\mathbf{E}_i\left[\sum_j m_{ij}(q', q_{I\setminus i}, q_J)(s_I(q, q_j) - s_I(q', q_j))\right. \\ &\left. + \int_q^{q'} \sum_j m_{ij}(z, q_{I\setminus i}, q_J)\frac{\partial s_I(z, q_j)}{\partial q_i}dz\right. \\ &\leq 0\end{aligned}$$

or

$$\tilde{U}_I(q, q') - U_I(q) = \int_q^{q'} \mathbf{E}_i\left[\sum_j \left\{m_{ij}(z, q_{I\setminus i}, q_J) - m_{ij}(q', q_{I\setminus i}, q_J)\right\} \frac{\partial s_I(z, q_j)}{\partial q_i}\right] dz \leq 0$$

This is condition (ii), the monotonicity condition.

If: Assume (i) and (ii) hold; I show incentive compatibility holds. Using condition (i) and the indirect utility function, transfers can be isolated and substituted into the indirect utility function to get Equation 14. Then deviating to q' rather than reporting q gives a change in payoff of

$\tilde{U}_I(q,q') - \tilde{U}_I(q,q) =$

$$
\begin{aligned}
&\mathbf{E}_i\Big[\sum_j m_{ij}(q', q_{I\setminus i}, q_J)(s_I(q,q_j) - s_I(q',q_j)) \\
&+ \int_q^{q'} \sum_j m_{ij}(z, q_{I\setminus i}, q_J)\frac{\partial s_I(z,q_j)}{\partial q_i} dz\Big] \\
&= \mathbf{E}_i\Big[-\sum_j m_{ij}(q', q_{I\setminus i}, q_J)\left\{\int_q^{q'} \frac{\partial s_I(z,q_j)}{\partial q_i} dz\right\} \\
&+ \int_q^{q'} \sum_j m_{ij}(z, q_{I\setminus i}, q_J)\frac{\partial s_I(z,q_j)}{\partial q_i} dz\Big] \\
&= \int_q^{q'} \mathbf{E}_i\left[\sum_j \left\{m_{ij}(z, q_{I\setminus i}, q_J) - m_{ij}(q', q_{I\setminus i}, q_J)\right\} \frac{\partial s_I(z,q_j)}{\partial q_i}\right] dz
\end{aligned}
$$

By (ii), the last line is non-positive, implying that reporting $q' \neq q$ isn't a profitable deviation. $\blacksquare$

Lemma 2.3.3 (Reserve Function)

Proof The reserve function is implicitly defined as:

$$\psi_I(q_i, R_I(q_i)) + \psi_J(R_I(q_i), q_i) = c$$

Totally differentiating with respect to q_i and rearranging yields

$$
\begin{aligned}
&\frac{\partial \psi_I(q_i, R_I(q_i))}{\partial q_i} + \frac{\partial \psi_I(q_i, R_I(q_i))}{\partial q_j} R_I'(q_i) + \\
&\qquad \frac{\partial \psi_J(R_I(q_i), q_i)}{\partial q_j} R_I'(q_i) + \frac{\partial \psi_J(R_I(q_i), q_i)}{\partial q_i} = 0
\end{aligned}
$$

$$R_I'(q_i) = -\frac{\partial \psi_I(q_i, R_I(q_i))/\partial q_i + \partial \psi_J(R_I(q_i), q_i)/\partial q_i}{\partial \psi_I(q_i, R_I(q_i))/\partial q_j + \partial \psi_J(R_I(q_i), q_i)/\partial q_j}$$

The terms $\partial \psi_I(q_i, q_j) \partial q_i$ is positive by assumption, but terms $\dfrac{\partial \psi_I(q_i, q_j)}{\partial q_j}$ and $\dfrac{\partial \psi_J(q_j, q_i)}{\partial q_i}$

are ambiguous; for instance,

$$\frac{\partial \psi_I(q_i, q_j)}{\partial q_j} = \underbrace{\frac{\partial s_I(q_i, q_j)}{\partial q_j}}_{+} - \underbrace{\frac{1 - F_I(q_i)}{f_I(q_i)} \frac{\partial^2 s_I(q_i, q_j)}{\partial q_j \partial q_i}}_{-}$$

To ensure the above term is positive, the above can be rearranged as

$$\frac{f_I(q_i)}{1 - F_I(q_i)} \geq \frac{\partial^2 s_I(q_i, q_j)}{\partial q_j \partial q_i} / \frac{\partial s_I(q_i, q_j)}{\partial q_j} = \frac{\partial}{\partial q_i} log\left(\frac{\partial s_I(q_i, q_j)}{\partial q_j}\right)$$

If this condition and its analog for the J side hold, $R_I(q_i)$ is decreasing, so it is almost everywhere differentiable. Then if $R_I(0) < \bar{q}_J$, 0 participates, and he is the worst-off type. Otherwise, the worst-off agent on the I-side can only match to the best agent on the other side of the market, namely $\bar{q}_J$. So this player must satisfy $\psi_I(\underline{q}_I, \bar{q}_J) + \psi_J(\bar{q}_J, \underline{q}_I) = c$. ∎

Proof of Proposition 2.3.4

Proof The value of $\underline{q}_I$ is determined by:

$$\begin{aligned} c &= \lambda_I a_I(\underline{q}_I) a_J(\bar{q}_J) - \frac{1 - F_I(\underline{q}_I)}{f_I(\underline{q}_I)} \lambda_I a_I'(\underline{q}_I) a_J(\bar{q}_J) + \lambda_J a_I(\underline{q}_I) a_J(\bar{q}_J) \\ \frac{c}{a_J(\bar{q}_J)} &= \left[\lambda_I a_I(\underline{q}_I) - \frac{1 - F_I(\underline{q}_I)}{f_I(\underline{q}_I)} \lambda_I a_I'(\underline{q}_I) + \lambda_J a_I(\underline{q}_I)\right] \end{aligned}$$

Then $R_I'(q) < 0$ if, for all q_i who participate, $\partial \psi_I(q_i, q_j)/\partial q_j + \partial \psi_J / \partial q_j \geq 0$, or

$$\begin{aligned} &\lambda_I a_I(q_i) a_J'(q_j) - \frac{1 - F_I(q_i)}{f_I(q_i)} \lambda_I a_I'(q_i) a_J'(q_j) + \lambda_J a_I(q_i) a_J'(q_j) \\ &\qquad - \frac{d}{dq_j}\left[\frac{1 - F_J(q_j)}{f_J(q_j)} \lambda_J a_I(q_i) a_J'(q_j)\right] > 0 \end{aligned}$$

Then the left-hand side of the above inequality is, for all $q_i > \underline{q}_I$,

$$\lambda_I a_I(q_i)a_J'(q_j) - \frac{1-F_I(q_i)}{f_I(q_i)}\lambda_I a_I'(q_i)a_J'(q_j) + \lambda_J a_I(q_i)a_J'(q_j)$$
$$-\frac{d}{dq_j}\left[\frac{1-F_J(q_j)}{f_J(q_j)}\lambda_J a_I(q_i)a_J'(q_j)\right]$$
$$> \lambda_I a_I(q_i)a_J'(q_j) - \frac{1-F_I(q_i)}{f_I(q_i)}\lambda_I a_I'(q_i)a_J'(q_j) + \lambda_J a_I(q_i)a_J'(q_j)$$
$$= a_J'(q_j)\left[\lambda_I a_I(q_i) - \frac{1-F_I(q_i)}{f_I(q_i)}\lambda_I a_I'(q_i) + \lambda_J a_I(q_i)\right]$$

The term in brackets is an increasing function of q_i, so using the condition defining $\underline{q}_I$,

$$> a_J'(q_j)\frac{c}{a_J(\bar{q}_J)}$$
$$> 0$$

Then for all $q_i \geq \underline{q}_I$, the reserve function is downward sloping. ∎

Proof of Proposition 2.3.5

Proof (Only If) Suppose there is a set of agents of positive measure who expect zero or negative surplus almost surely, so the mechanism exhibits absolute exclusion. Absolute exclusion is characterized by Lemma 1.3.3, by the condition

$$\psi_I(\underline{q}_I, \bar{q}_J) + \psi_J(\bar{q}_J, \underline{q}_I) = c$$

Since the set $[0, \underline{q}_I]$ has positive measure, and $\psi_I(q_i, q_j) + \psi_J(q_j, q_i)$ is increasing in both arguments, it must be true also that

$$\psi_I(0, \bar{q}_J) + \psi_J(\bar{q}_J, 0) < c$$

(If) The virtual surplus generated by an agent with $q_i = 0$ and the best match partner q_j is

$$s_I(0, \bar{q}_J) - \frac{1 - F_I(0)}{f_I(0)} + s_J(\bar{q}_J, 0) - \frac{1 - F_J(\bar{q}_J)}{f_J(\bar{q}_J)} = s_I(0, \bar{q}_J) - \frac{1}{f_I(0)} \frac{\partial s_I(0, \bar{q}_J)}{\partial q_i}$$

If this quantity is strictly negative, then the lowest quality agent on the I side produces negative virtual surplus a.s., since they produce negative surplus even when guaranteed the best partner. Since ψ_I is continuous, there are q_i' near $q_i = 0$ for which joint virtual surplus is strictly negative as well. Then there is a set of agents of positive measure who expect zero surplus a.s., so the mechanism exhibits absolute exclusion. ∎

Proof of Theorem 2.3.6 [Optimality and Revenue Equivalence]

Proof The proof relies on the lemmas:

- [Lemma A1] If $q > q'$, the distribution $h(q', k) = \sum_{j=1}^{k} w_{I,j}(q')$ first-order stochastically dominates $h(q, k)$; having a higher q places more weight on the "high" rankings, corresponding to 1, 2, ...

- [Lemma A2] $F_{J,(\ell)}(q)$ first-order stochastically dominates $F_{J,(\ell+1)}(q)$.

- [Lemma A3] Consider

  $$\left\{ \int_{R_I(z)}^{\bar{q}_J} \frac{\partial s_I(z, y)}{\partial q_i} f_{J,(k)}(y) dy \right\}_{k=1}^{K_J}$$

  This sequence is non-increasing in k.

and Lemma A5, which allows computation of expectations under TAM.

The strategy of the proof is to show that the integrand is negative for all values of z between q and q', implying that the entire integral is negative.

Case 1: Assume $q' > q$. Then it needs to be shown that

$$\int_q^{q'} \sum_j \Big\{ w_{I,j}(z) \int_{R_I(z)}^{\bar{q}_J} \frac{\partial s_I(z,y)}{\partial q_i} f_{J,(j)}(y)dy \\ - w_{I,j}(q')) \int_{R_I(q')}^{\bar{q}_J} \frac{\partial s_I(z,y)}{\partial q_i} f_{J,(j)}(y)dy \Big\} dz \leq 0$$

If the integrand is negative for all z, then the conclusion for this case follows. This is true if, for all $z > q$,

$$\sum_j w_{I,j}(z) \int_{R_I(z)}^{\bar{q}_J} \frac{\partial s_I(z,y)}{\partial q_i} f_{J,(j)}(y)dy \leq \sum_j w_{I,j}(q') \int_{R_I(q')}^{\bar{q}_J} \frac{\partial s_I(z,y)}{\partial q_i} f_{J,(j)}(y)dy$$

First note that since $q' > z$, $R_I(q') < R_I(z)$, and it suffices to show that

$$\sum_j w_{I,j}(z) \int_{R_I(z)}^{\bar{q}_J} \frac{\partial s_I(z,y)}{\partial q_i} f_{J,(j)}(y)dy \leq \sum_j w_{I,j}(q') \int_{R_I(z)}^{\bar{q}_J} \frac{\partial s_I(z,y)}{\partial q_i} f_{J,(j)}(y)dy$$

Since $q' > z$, the distribution over j on the left-hand side first-order stochastically dominates the distribution on the right-hand side (Lemma A1). The remaining terms on each side form a decreasing sequence in j (Lemma A3). The two sums can be considered as expectations over a decreasing sequence, with the right-hand side placing more weight on early terms in the sequence. Therefore, the inequality holds.

Case 2: Assume $q' < q$. Then it needs to be shown that

$$\int_{q'}^{q} \sum_j \Big\{ w_{I,j}(z) \int_{R_I(z)}^{\bar{q}_J} \frac{\partial s_I(z,y)}{\partial q_i} f_{J,(j)}(y)dy \\ - w_{I,j}(q')) \int_{R_I(q')}^{\bar{q}_J} \frac{\partial s_I(z,y)}{\partial q_i} f_{J,(j)}(y)dy \Big\} dz \geq 0$$

This would be true if for $z < q$ the integrand was positive, or

$$\sum_j w_{I,j}(z) \int_{R_I(z)}^{\bar{q}_J} \frac{\partial s_I(z,y)}{\partial q_i} f_{J,(j)}(y)dy \geq \sum_j w_{I,j}(q')) \int_{R_I(q')}^{\bar{q}_J} \frac{\partial s_I(z,y)}{\partial q_i} f_{J,(j)}(y)dy$$

Note that since $q' < z$, $R_I(q') > R_I(z)$, and it suffices to show

$$\sum_j w_{I,j}(z) \int_{R_I(z)}^{\bar{q}_J} \frac{\partial s_I(z,y)}{\partial q_i} f_{J,(j)}(y)dy \geq \sum_j w_{I,j}(q')) \int_{R_I(z)}^{\bar{q}_J} \frac{\partial s_I(z,y)}{\partial q_i} f_{J,(j)}(y)dy$$

From a similar argument to case 1, this is true, and the integrand is positive. Therefore, the monotonicity condidition is satisfied and TAM is a solution. ∎

Proof of Proposition 2.3.6 (Exclusion)

Proof Let $\underline{q}_{I1}$ be the exclusion associated with $F_I^1(q)$, and $\underline{q}_I^2$ the exclusion associated with $F_I^2(q)$. Hazard rate dominance implies

$$-\frac{1 - F_I^1(x)}{f_1^1(x)} \geq -\frac{1 - F_I^2(x)}{f_I^2(x)}$$

Then $\underline{q}_I^1$ satisfies

$$0 = s_I(\underline{q}_I^1, \bar{q}_J) - \frac{1 - F_I^1(\underline{q}_I^1)}{f_I^1(\underline{q}_I^1)} \frac{\partial s_I(\underline{q}_I^1, \bar{q}_J)}{\partial q_i} + s_J(\bar{q}_J, \underline{q}_I^1) - c$$

But then

$$0 < s_I(\underline{q}_I^1, \bar{q}_J) - \frac{1 - F_I^2(\underline{q}_I^1)}{f_I^2(\underline{q}_I^1)} \frac{\partial s_I(\underline{q}_I^1, \bar{q}_J)}{\partial q_i} + s_J(\bar{q}_J, \underline{q}_I^1) - c$$

Since the right-hand side is increasing in $\underline{q}_I$, exclusion must be higher under F_{I2} than F_{I1}, since the above inequality is zero when evaluated as $\underline{q}_{I2}$, but positive when evaluated at $\underline{q}_{I1}$.

Consider changing F_J^1 to F_J^2, where F_J^1 hazard-rate dominates F_J^2. Then relative exclusion function under F_J^1, $R_{Ik}^1(q)$, is defined by

$$0 = s_I(q, R_I(q)) - \frac{1-F_I(q)}{f_I(q)} \frac{\partial s_I(q, R_I^1(q))}{\partial q_i} + s_J(R_I^1(q), q) - \frac{1-F_J^1(R_I^1(q))}{f_J^1(R_I^1(q))} \frac{\partial s_J(R_I^1(q), q)}{\partial q_j} - c$$

But then

$$0 < s_I(q, (q)) - \frac{1-F_I(q)}{f_I(q)} \frac{\partial s_I(q, R_{Ik}^1(q))}{\partial q_i} + s_J(R_{Ik}^1(q), q) - \frac{1-F_J^2(R_{Ik}^1(q))}{f_J^2(R_{Ik}^1(q))} \frac{\partial s_J(R_I^1(q), q)}{\partial q_j} - c$$

because the second line equals 0 when evaluated at $R_{Ik}^2(q)$, and is increasing in $R_{Ik}^1(q)$. Since virtual surplus is increasing in both qualities, $R_{Ik}^1(q) > R_{Ik}^2(q)$.

Take the condition

$$s_I(\underline{q}_I, \bar{q}_J) - \frac{1-F_I(\underline{q}_I)}{f_I(\underline{q}_I)} \frac{\partial s_I(\underline{q}_I, \bar{q}_j)}{\partial q_i} + s_J(\bar{q}_J, \underline{q}_I) = c$$

Treat $\underline{q}_I(c)$ as a function of c_1 and totally differentiate to get:

$$\underline{q}_I'(c) = \frac{1}{\frac{\partial s_I(\underline{q}_I, \bar{q}_J)}{\partial q_i} - \frac{1-F_I(\underline{q}_I)}{f_I(\underline{q}_I)} \frac{\partial^2 s_I(\underline{q}_I, \bar{q}_j)}{\partial q_i^2} - \frac{d}{dq_i}\left[\frac{1-F_I(\underline{q}_I)}{f_I(\underline{q}_I)}\right] \frac{\partial s_I(\underline{q}_I, \bar{q}_J)}{\partial q_i} + \frac{\partial s_J(\bar{q}_J, \underline{q}_I)}{\partial q_i}} > 0$$

Similarly, take the condition

$$\psi_I(q, (q, c)) + \psi_J((q, c), q) = c$$

Then

$$\frac{\partial (q, c)}{\partial c} = \frac{1}{\frac{\psi_I(q, (q, c))}{\partial q_j} + \frac{\partial \psi_J((q, c), q)}{\partial q_j}} > 0$$

Verification that the denominator is positive comes from the conditions in Proposition 3.3. ∎

Proof of Proposition 2.3.7 (Market Size)

Proof (i) The interim indirect utility of agents on the I side is

$$U_I(q_i) = \int_{\underline{q}_I}^{q_i} \sum_k w_{I,k}(z) \int_{(z)}^{\bar{q}_J} \frac{\partial s_I(z, q_j)}{\partial q_i} f_{J,(k)}(y) dy dz$$

Since a change in K_J doesn't affect exclusion, only $f_{J,(k)}(y)$ is altered. The first-order stochastic dominance property (from Proposition A4) implies that probability is shifted onto higher realization of partner quality, and because of supermodularity, this function increases in K_J.

(ii) Let

$$w_{I,k:K_I}(q) = \frac{(K_I - 1)!}{(K_I - k)!(k-1)!} F_I(q)^{K_I - k}(1 - F_I(q))^{k-1}$$

Then

$$w_{I,k:K_I}(q) = w_I(q, k, K_I) \frac{K_I}{K_I + 1 - k} F_I(q)$$

Note that $\frac{K_I}{K_I + 1 - k} F_I(q) \leq 1$ only if

$$q \leq F^{-1}\left(\frac{K_I + 1 - k}{K_I}\right)$$

Let $\hat{q}_k$ be the value of q that satisfies the above inequality with an equality. That sequence of points is $\hat{q}_1 = \bar{q}_I, \hat{q}_2 = F^{-1}\left(\frac{K_I - 1}{K_I}\right), ..., \hat{q}_{K_I+1} = 0$. So for any q, the interval $q \in [\hat{q}_{\hat{k}}, \hat{q}_{\hat{k}+1}]$ can be found. For all terms $k < \hat{k}$, the probability of that agent attaining that rank has fallen due to an increase from K_I to $K_I + 1$; for all terms $k \geq \hat{k}$, that agent has become more likely to hold that rank. This entails a shift in probability from favorable, highly ranked partners to unfavorably ranked partners. Since

$$\sum_{k=1}^{K_I} w_{I,k,K_I}(q) = 1 = \sum_{k=1}^{K_I+1} w_{I,k:K_I+1}(q)$$

the sum on the left (associated with K_I) must place more weight on the better partners than the term on the right (associated with K_I+1). Note also that for this reason, cutting off the sums at any particular $K < K_I$ implies $\sum_{k=1}^{K} w_{I,k:K_I}(q) \geq \sum_{k=1}^{K} w_{I,k:K_I+1}(q)$. In other words, the lottery associated with K_I+1 first-order stochastically dominates the lottery associated with K_I over the terms $1, 2, ..., K_I$. Note also that $\int_{(q)}^{\bar{q}_J} \dfrac{\partial s_I(q,y)}{\partial q_i} f_{J,(k)}(y)dy$ is a decreasing sequence in k. Then let $U_I(q, K_I)$ be the interim utility of an agent on the I-side when there are K_I agents in the market: The agent has drawn his private information, but has not received any other information about his opponents.

Suppose that $K_I > K_J$, so that $\min\{K_I+1, K_J\} = K_J$. Then

$$\begin{aligned} U_I(q, K_{I+1}) &= \int_{\underline{q}_I}^{q} \sum_{k=1}^{K_J} w_{I,k:K_{I+1}}(z) \int_{(z)}^{\bar{q}_J} \frac{\partial s_I(z,y)}{\partial q_i} f_{J,(k)}(y)dydz \\ &\leq \int_{\underline{q}_I}^{q} \sum_{k=1}^{K_J} w_{I,k:K_I}(z) \int_{(z)}^{\bar{q}_J} \frac{\partial s_I(z,y)}{\partial q_i} f_{J,(k)}(y)dydz = U_I(q, K_I) \end{aligned}$$

since $\sum_1^\ell w_{I,\ell:K_I}(q)$ first-order stochastically dominates $\sum_1^\ell w_{I,\ell:K_{I+1}}(q)$ in ℓ. Now suppose $K_I \leq K_J + 1$, so that $\min\{K_{I+1}, K_J\} = K_I + 1$. Then adding the extra I-side agent leads to the creation of another slot. But since $\sum_1^{K_I} w_{I,k:K_I}(q) = 1 = \sum_1^{K_{I+1}} w_{I,k:K_I+1}(q)$, this shifts weight onto worse partners, since the sequence $\int \partial(s_I(q,y)/\partial q_i) f_{J(k)}(y)dy$ is decreasing, so $U_I(q, K_I) \geq U_I(q, K_I+1)$ in that case as well.

(iii) Matchmaker profits are

$$\sum_k \int_{\underline{q}_I}^{\bar{q}_I} \int_{(q_i)}^{\bar{q}_J} \{\psi_I(q_i, q_j) + \psi_J(q_j, q_i) - c\} f_{I,(k)}(q_i) f_{J,(k)}(q_j) dq_i dq_j$$

An increase in K_I or K_J leads implies first-order stochastic dominance by Proposition A4 without affecting exclusion, and since the integrand is supermodular, the value of all existing matches improves. If the small side of the market has increase, profits also increase because there are more matches made in expectation. ∎

Proof of Proposition 2.3.8 (Own-Side Effects)

Proof Note that from Proposition 3.6, absolute and relative exclusion is higher under F_I^1 than F_I^2. This reduces the welfare of newly excluded agents and raises the payments of the remaining agents. Additionally, consider

$$\frac{w_{I1,k}(q)}{w_{I2,k}(q)} = \left(\frac{F_{I1}(q)}{F_{I2}(q)}\right)^{K_I - k} \left(\frac{1 - F_{I1}(q)}{1 - F_{I2}(q)}\right)^{k-1}$$

Then $w_{I1,k}(q) \geq w_{I2,k}(q)$ if k is greater than

$$\tilde{k} = \frac{log\left(\dfrac{1 - F_{I1}(q)}{1 - F_{I2}(q)} \left(\dfrac{F_{I2}(q)}{F_{I1}(q)}\right)^{K_I}\right)}{log\left(\dfrac{1 - F_{I1}(q)}{1 - F_{I2}(q)} \dfrac{F_{I2}(q)}{F_{I1}(q)}\right)}$$

Since hazard rate dominance implies first-order stochastic dominance, $F_{I1}(q) < F_{I2}(q)$ but $1 - F_{I1}(q) > 1 - F_{I2}(q)$, and the above expression is positive. So for $k \leq \tilde{k}$, $w_{I2,k}(q)$ will be less than $w_{I1,k}(q)$, but for $k \geq \tilde{k}$, $w_{I1,k}(q)$ will be greater than $w_{I2,k}(q)$. This means that for all q, the lottery over partners is more favorable under $F_{I2}(q)$ than $F_{I1}(q)$, given that the agent participates under $F_{I2}(q)$. So the added exclusion doesn't benefit the remaining bidders, and the interim payoffs of the I-side agents fall. ∎

Proof of Theorem 2.4.1 (Profit-Maximizing Implementation)

Proof Consider first the all-pay position auction. Note that the minimum bid schedule is equivalent to the reservation function in the optimal mechanism, so we can work with $R_I(q)$ instead of $\underline{b}_J(b_i)$ in the agents' maximization problems. In any symmetric, increasing equilibrium, the bidders' common strategy $b_I^{AP}(q)$ is invertible, and agents face the problem:

$$\max_b \sum_{k=1}^{K} w_{I,k}(b_I^{AP-1}(b)) \int_{R_I(b_I^{AP-1}(b))}^{\bar{q}_J} s_I(q_i, y) f_{J,(k)}(y)dy - b$$

Then a necessary condition for maximization is that

$$b_I^{AP\prime}(q) =$$
$$\frac{d}{dq}\left[\sum_{k}^{K} w_{I,k}(q) \int_{R_I(q)}^{\bar{q}_J} s_I(q,y) f_{J,(k)}(y)dy\right] - \sum_{k}^{K} w_{I,k}(q) \int_{R_I(q)}^{\bar{q}_J} \frac{\partial s_I(q,y)}{\partial q_i} f_{J,(k)}(y)dy$$

Integration yields

$$b_I^{AP}(q) = \sum_{k}^{K} w_{I,k}(q) \int_{R_I(q)}^{\bar{q}_J} s_I(q,y) f_{J,(k)}(y)dy$$
$$- \int_{\underline{q}_I}^{q} w_{I,k}(z) \int_{R_I(z)}^{\bar{q}_J} \frac{\partial s_I(z,y)}{\partial q_i} f_{J,(k)}(y)dydz$$

Since the worst-off type receives zero surplus in the mechanism, there is a natural boundary condition to the differential equation of

$$b_I^{AP}(\underline{q}_I) = \sum_{k}^{K} w_{I,k}(\underline{q}_I) \int_{R_I(\underline{q}_I)}^{\bar{q}_J} s_I(q,y) f_{J,(k)}(y)dy$$

Since the right-hand side of the bid function is exactly the optimal payment in the direct revelation mechanism, this mechanism implements the optimal outcome. To ensure that this game is useful for implementation, we also have to check that

$b_{I,AP}(q)$ is monotonically increasing, so that the quality rankings of the agents can be correctly inferred from their bids (and the assumptions of invertibility is satisfied). Then

$$b_I^{AP\prime}(q) = \sum_k w'_{I,k}(q) \int_{R_I(q)}^{\bar{q}_J} s_I(q,y) f_{J,(k)}(y) dy$$
$$- w_{I,k}(q) s_I(q, R_I(q)) f_{J,(k)}(R_I(q)) R'_I(q)$$

Since $R'_I(q) < 0$, the second term is positive. Note that by the binomial theorem, $\sum_k^{K_I} w_{I,k}(q) = 1$, so $\sum_k^{K_I} w'_{I,k}(q) = 0$. Also, the sequence $\int_{R_I(q)}^{\bar{q}_J} s_I(q,y) f_{J,(k)}(y) dy$ is decreasing in k. Then for $k = 2, 3, ..., K_I - 1$,

$$w'_{I,k}(q) = \frac{(K_I - 1)!}{(K_I - k)!(k-1)!} f_I(q) F_I(q)^{K_I - k - 1} (1 - F_I(q))^{k-2}$$
$$[(K_I - k)(1 - F_I(q)) - (k-1) F_I(q)]$$

For $k = 1$ and $k = K_I$, the slopes of $w_{I,k}(q)$ are monotone increasing and decreasing, respectively. Let the sequence of points $\{\tilde{q}_k\}_{k=2}^{K_I - 1}$ be defined as

$$\tilde{q}_k = F_I^{-1}\left(\frac{K_I - k}{K_I - 1}\right)$$

This is a decreasing sequence in k. For a given q, find the interval $[\tilde{q}_k, \tilde{q}_{k+1}]$ and label the accompanying k as k^*. Now, for all the terms $w'_{I,k}(q)$ with $k < k^*$, $w'_{I,k}(q)$

is positive but for all terms $w'_{I,k}(q)$ with $k > k^*$, $w'_{I,k}(q)$ is negative. Then

$$\begin{aligned}
b_I^{AP\prime}(q) &= \sum_k^K w'_{I,k}(q) \int_{R_I(q)}^{\bar{q}_J} s_I(q,y) f_{J,(k)}(y) dy \\
&\quad - w_{I,k}(q) s_I(q, R_I(q)) f_{J,(k)}(R_I(q)) R'_I(q) \\
&> \sum_k^K w'_{I,k}(q) \int_{R_I(q)}^{\bar{q}_J} s_I(q,y) f_{J,(k^*)}(y) dy \\
&> \sum_k^{K_I} w'_{I,k}(q) \int_{R_I(q)}^{\bar{q}_J} s_I(q,y) f_{J,(k^*)}(y) dy \\
&= \int_{R_I(q)}^{\bar{q}_J} s_I(q,y) f_{J,(k^*)}(y) dy \sum_k^{K_I} w'_{I,k}(q) = 0
\end{aligned}$$

Where the second line follows since $\int_{R_I(q)}^{\bar{q}_J} s_I(q,y) f_{J,(k)}(y) dy$ is decreasing in k; decrease the terms above $k > k^*$ that appear with a negative $w'_{I,k}(q)$, and decrease the terms $k < k^*$ that appear with a positive $w'_{I,k}(q)$. So the bid function is monotone increasing.

If a winners-pay position auction is used instead, the agents maximize

$$\max_b \sum_k w_{I,k}(b_I^{WP-1}(b)) \left[\int_{R_I(b_I^{WP-1}(b)}^{\bar{q}_J} s_I(q,y) f_{J,(k)}(y) dy - (1 - F_{J,(k)}(R_I(b_I^{WP-1}(b)))) b \right]$$

with a potential solution (using the same approach as for the all-pay position auction)

$$b_I^{WP}(q) = \frac{\sum_k w_{I,k}(q) \int_{R_I(q)}^{\bar{q}_J} s_I(q,y) f_{J,(k)}(y) dy - \int_{\underline{q}_I}^{q} w_{I,k}(z) \int_{R_I(z)}^{\bar{q}_J} \frac{\partial s_I(z,y)}{\partial q_i} f_{J,(k)}(y) dy dz}{\sum_k w_{I,k}(q)[1 - F_{J,(k)}(R_I(q))]}$$

Then this equals

$$b_I^{WP}(q) = \frac{b_I^{AP}(q)}{\sum_k w_{I,k}(q)[1 - F_{J,(k)}(R_I(q))]}$$

The numerator and denominator are both increasing, so it is theoretically ambiguous whether the entire function is increasing or decreasing. ∎

Proof of Theorem 2.5.1 (Simplified Implementation)

Proof Matchmaker profits are

$$\sum_k \int_{\underline{q}_I}^{\bar{q}_I} \int_{\underline{q}_J}^{\bar{q}_J} \{\psi_I(q_i, q_j) + \psi_J(q_j, q_i) - c\} f_{J,(k)}(q_j) f_{I,(k)}(q_i) dq_j dq_i$$

Maximizing over $\underline{q}_I$ (and likewise for $\underline{q}_J$) yields a condition for maximization:

$$0 = \sum_k \int_{\underline{q}_J}^{\bar{q}_J} \left\{\psi_I(\underline{q}_I, q_j) + \psi_J(q_j, \underline{q}_I) - c\right\} f_{J,(k)}(q_j) dq_j$$

From an integration by parts and re-arranging,

$$\begin{aligned} &\psi_I(\underline{q}_I, \bar{q}_J) + \psi_J(\bar{q}_J, \underline{q}_I) - c = \\ &\qquad \frac{\sum_k \int_{\underline{q}_J}^{\bar{q}_J} \left\{\frac{\psi_I(\underline{q}_I, q_j) + \psi_J(q_j, \underline{q}_I) - c}{\partial q_j}\right\} [F_{J,(k)}(q_j) - F_{J,(k)}(\underline{q}_J)] dq_j}{\sum_k 1 - F_{J,(k)}(\underline{q}_J)} > 0 \end{aligned}$$

Absolute exclusion in the optimal mechanism comes from lemma 1.3.3, where the left-hand side of the above equation is set equal to zero. Since the left-hand side is increasing in $\underline{q}_I$ and the right-hand side is positive, exclusion here is higher.

The payments in the simplified mechanism can be computed similarly to the optimal mechanism to get

$$\begin{aligned} &\mathbf{E}_i[t_i(q_i, q_{I-i}, q_J)] \\ &= \sum_k w_{I,k}(q_i) \int_{\underline{q}_J}^{\bar{q}_J} s_I(q_i, y) f_{J,(k)}(y) dy - \int_{\underline{q}_I}^{q_i} w_{I,k}(z) \int_{\underline{q}_J}^{\bar{q}_J} \frac{\partial s_I(z, y)}{\partial q_i} f_{J,(k)}(y) dy dz \end{aligned}$$

Using this, the same arguments as in Section 1.3 show the monotonicity constraint fails to bind for the simplified direct mechanism, so it is incentive compatible. In the all-pay format, it is an equilibrium to bid the expected transfer exactly, and it is increasing by similar arguments to Theorem 2.4.1, so the all-pay format implements the same payments and allocations as the simplified direct mechanism. The minimum bid is derived as the boundary condition to the solution of the differential equation characterizing the bid function, using the condition that the worst-off type get a payoff of zero. Lastly, to provide a sufficient condition that the function is increasing, suppose $b_I^{WP}(q)$ is increasing and differentiate to get

$$\frac{\sum_k w'_{I,k}(q) \int_{\underline{q}_J}^{\bar{q}_J} s_I(q,y) f_{J,(k)}(y) dy}{\sum_k w_{I,k}(q) \int_{\underline{q}_J}^{\bar{q}_J} s_I(q,y) f_{J,(k)}(y) dy - \int_{\underline{q}_I}^{q} w_{I,k}(z) \int_{\underline{q}_J}^{\bar{q}_J} \frac{\partial s_I(z,y)}{\partial q_i} f_{J,(k)}(y) dy dz} \geq \frac{\sum_k w'_{I,k}(q)(1 - F_{J,(k)}(\underline{q}_J)}{\sum_k w_{I,k}(q)(1 - F_{J,(k)}(\underline{q}_J)}$$

Dropping the negative term in the denominator on the left-hand side (thereby reducing the magnitude of the left-hand side) and rearranging yields

$$\frac{\sum_k w'_{I,k}(q) \int_{\underline{q}_J}^{\bar{q}_J} s_I(q,y) dy}{\sum_k w'_{I,k}(q)(1 - F_{J,(k)}(\underline{q}_J)} \geq \frac{\sum_k w_{I,k}(q) \int_{\underline{q}_J}^{\bar{q}_J} s_I(q,y) f_{J,(k)}(y) dy}{\sum_k w_{I,k}(q)(1 - F_{J,(k)}(\underline{q}_J)}$$

The right-hand side is the expected value of a match, given that a match occurs. The left-hand side will be large only if $\sum_k w'_I(q,k)(1 - F_{J,(k)}(\underline{q}_J)$ is small, requiring that $1 - F_{J,(K)}(\underline{q}_J)$ is sufficiently close to 1 and K is close to K_I. ∎

Appendix B

Proofs for Chapter Three

Throughout the appendix, $s_{I\ell}$ refers to the signal received by agent $I\ell$ on the I side, while $s_{I[\ell]}$ refers to the ℓ-th highest signal received by any agent on the I side.

Proof of Proposition 3.3.1

Proof Let $U_I(\sigma_{Ik}, s_{\setminus Ik})$ be the indirect utility function for agent Ik when he reports σ_{Ik} while everyone else reports honestly, $s_{\setminus Ik}$. Note that k's payoff from reporting σ_{Ik} whenever everyone else reports their true type, $s_{\setminus Ik}$, is

$$
\begin{aligned}
U_I(\sigma_{Ik}, s_{\setminus Ik}) \quad = \quad & \sum_{j \in J} m_{kj}(\sigma_{Ik}, s_{\setminus Ik}) v_I(s_k, s_{\setminus Ik}) - t_k(\sigma_{Ik}, s_{\setminus Ik}) \\
& \sum_{j \in J} m_{kj}(\sigma_{Ik}, s_{\setminus Ik}) v_I(s_k, s_{\setminus Ik}) \\
& + \sum_{j \in J} m_{kj}(\sigma_{Ik}, s_{\setminus Ik}) v_J(s_{\setminus Ik}, \sigma_{Ik}) \\
& + \sum_{i \in I \setminus k} \sum_{j \in J} m_{ij}(\sigma_{Ik}, s_{\setminus Ik})(v_I(s_i, s_j) + v_J(s_j, s_i)) \\
& - \sum_{i \in I \setminus k} \sum_{j \in J} m_{ij}(\varnothing, s_{\setminus Ik})(v_I(s_i, s_j) + v_J(s_j, s_i)) - h_k(s_{\setminus Ik}))
\end{aligned}
$$

Then

$$
\begin{aligned}
U(s_k, s_{\setminus Ik}) - U(\sigma_{Ik}, s_{\setminus Ik}) \quad = & \\
& \sum_{j\in J} m_{kj}(s_k, s_{\setminus Ik}) v_I(s_k, s_{\setminus Ik}) \\
& - \sum_{j\in J} m_{kj}(\sigma_{Ik}, s_{\setminus Ik}) v_I(s_k, s_{\setminus Ik}) \\
& + \sum_{j\in J} m_{kj}(s_k, s_{\setminus Ik}) v_J(s_{\setminus Ik}, s_k) \\
& - \sum_{j\in J} m_{kj}(\sigma_{Ik}, s_{\setminus Ik}) v_J(s_{\setminus Ik}, \sigma_{Ik}) \\
& + \sum_{i\in I\setminus k} \sum_{j\in J} m_{ij}(s_k, s_{\setminus Ik})(v_I(s_i, s_j) + v_J(s_j, s_i)) \\
& - \sum_{i\in I\setminus k} \sum_{j\in J} m_{ij}(\sigma_{Ik}, s_{\setminus Ik})(v_I(s_i, s_j) + v_J(s_j, s_i))
\end{aligned}
$$

Let $s_{J[k]}$ be the k-th highest signal on the J side, and suppose that s_{Ik} is ranked k-th. Note that if the agent "goes after" the best partner on the other side, the term

$$
... + v_J(s_{J[k]}, s_{Ik)}) - v_J(s_{J[1])}, \sigma_{Ik}) + ...
$$

appears in Ik's payoff. This can be made arbitrarily negative by submitting a high-enough report σ_{Ik}, yielding $U(s_{Ik}, s_{\setminus Ik}) < U(\sigma_{Ik}, s_{\setminus Ik})$. Therefore, there is a profitable deviation, and truth-telling is not incentive compatible. ∎

Proof of Proposition 3.3.2

Proof (i) Since we can completely order the types according to either side's preferences, let $s_{I[k]}$ be the type of the agent with the k-th highest signal on the I side, so $s_{I[1]} \geq s_{I[2]} \geq ... \geq s_{I[I]}$. Likewise, we can completely order the types on the J side, $s_{J[1]} \geq s_{J[2]} \geq ... \geq s_{J[J]}$. By supermodularity, for all integer $\ell_1, \ell_2 \geq 1$

$$
v_I(s_{I[k]}, s_{J[k]}) + v_I(s_{I[k+\ell_1]}, s_{J[k+\ell_2]}) \geq v_I(s_{I[k+\ell_1]}, s_{J[k]}) + v_I(s_{I[k]}, s_{J[k+\ell_2]})
$$

and

$$v_J(s_{I[k]}, s_{J[k]}) + v_J(s_{I[k+\ell_1]}, s_{J[k+\ell_2]}) \geq v_J(s_{I[k+\ell_1]}, s_{J[k]}) + v_J(s_{I[k]}, s_{J[k+\ell_2]})$$

So the agents should be matched assortatively as

$$\{(I[1], J[1]), (I[2], J[2]), ..., (I[k], J[k]), ...(I[K], J[K])\}$$

This is the efficient and stable match.

(ii) Let $s_{I[k]}$ be the k-th highest signal received by an agent on the I side. This his payment is

$$\begin{aligned} t_{I[k]}(s) = \left\{v_I(s_{I[k+1]}, s_{J[k]}) - v_I(s_{I[k+1]}, s_{J[k+1]})\right\} \\ + \left\{v_I(s_{I[k+2]}, s_{J[k+1]}) - v_I(s_{I[k+2]}, s_{J[k+2]})\right\} \\ + ... + \left\{v_I(s_{I[K]}, s_{J[K-1]}) - v_I(s_{I[K]}, s_{J[K]})\right\} \end{aligned}$$

Note that because of the vertical nature of the market, the set of agents that k and $k+1$ are blocking overlap for all terms $k+2$ up to K

$$t_{I[k]}(s) = \left\{v_I(s_{I[k+1]}, s_{J[k]}) - v_I(s_{I[k+1]}, s_{J[k+1]})\right\} + t_{I[k+1]}(s)$$

Then the realized utility of an agent is:

$$\begin{aligned}
v_I(s_{I[k]}, s_{J[k]}) - t_{I[k]} &= \\
& v_I(s_{I[k]}, s_{J[k]}) - \left\{ v_I(s_{I[k+1]}, s_{J[k]}) - v_I(s_{I[k+1]}, s_{J[k+1]}) \right\} \\
& - t_{I[k+1]}(s) \\
&= \left\{ \int_{s_{J[k+1]}}^{s_{J[k]}} \frac{\partial v_I(s_{I[k]}, y)}{\partial s_j} dy + v_I(s_{I[k]}, s_{I[k+1]}) \right\} \\
& - \left\{ \int_{s_{J[k+1]}}^{s_{J[k]}} \frac{\partial v_I(s_{I[k+1]}, y)}{\partial s_j} \right\} - t_{I[k+1]}(s) \\
&= \int_{s_{J[k+1]}}^{s_{J[k]}} \int_{s_{I[k+1]}}^{s_{I[k]}} \frac{\partial^2 v_I(x, y)}{\partial s_i \partial s_j} dx dy + v_I(s_{I[k]}, s_{I[k+1]}) - t_{I[k+1]}(s)
\end{aligned}$$

We can continue to exploit the above pattern, yielding a final expression

$$\begin{aligned}
v_I(s_{I[k]}, s_{J[k]}) - t_{I[k]} &= \\
&= \int_{s_{J[k+1]}}^{s_{J[k]}} \int_{s_{I[k+1]}}^{s_{I[k]}} \frac{\partial^2 v(x, y)}{\partial s_i \partial s_j} dx dy \\
& + \int_{s_{J[k+2]}}^{s_{J[k+1]}} \int_{s_{I[k+2]}}^{s_{I[k]}} \frac{\partial^2 v(x, y)}{\partial s_i \partial s_j} dx dy \\
& + \ldots + \int_{s_{J[K]}}^{s_{J[K-1]}} \int_{s_{I[K]}}^{s_{I[k]}} \frac{\partial^2 v(x, y)}{\partial s_i \partial s_j} dx dy + v_I(s_{I[k]}, s_{J[K]})
\end{aligned}$$

Since an agent's payment doesn't depend on his type report, they can only deviate downward or upward in rank. Deviating down "throws away" some of the terms in the above sum, since the agent is no longer blocking those competitors. This is unprofitable, since the terms are all positive due to supermodularity. Likewise, deviating upward includes terms that are negative, because the order of integration is reversed when evaluated from a worse agent (the deviator) to a better one. So there are no profitable deviations for any realization of types.

So if the two halves of the competitive externality mechanism $\{m_{ij}^I(\sigma), t_i(\sigma)\}$

and $\{m_{ij}^J(\sigma), t_j(\sigma)\}$ agree on the allocation, it is an ex post equilibrium for agents to report honestly. It is not a dominant strategy, since if other agents lied about their types, it would no longer be a best-reply to be honest. For example, if the best J-side agent submitted a report that made her appear to be the worst partner on that side, a profitable deviation from truth-telling on the I side would be to pretend to be the worst partner who receives a match, and then making the lowest payment and getting the best partner. ∎

Proof of Proposition 3.3.3

Proof (i) Suppose all the types are common knowledge and an agent tries to lie about his type to exploit security bidding to his advantage. If $I\ell$ is ranked k-th, making a falsely high report that blocks all the matches from K to $k - \ell$ actually lowers the bar for receiving a partner, since

$$\frac{dx^*(s_i)}{ds_i} = -\frac{\partial v_I(s_i, x^*(s_i))}{\partial v_I(s_i, x^*(s_i))} < 0$$

So an upward deviation only exposes an agent to more risk of an unacceptable partner, given his true signal. The only way to get a better partner by falsely submitting a higher type is to achieve a better rank, which is already shown to be unprofitable in Proposition 3.2. Submitting a falsely low report, likewise, allows the possibility of matching to a low quality agent and getting a negative matching surplus in addition to potentially making a payment to the matchmaker. This, too, is obviously less profitable than reporting honestly and getting zero whenever the only available partners are unacceptable. So even if agent $I\ell$ knew the types of his opponents, lying about his type is unprofitable, so it cannot be profitable in expectation, and this is an ex post equilibrium strategy.

(ii) In the mechanism without security bidders, the lowest payment on the I side is $t_{IK} \geq 0$ whenever $I > J$, and t_{JK} is always equal to zero. Because of the security bidders, there are other bid-entries, weakly raising these lowest payments, and subsequently improving revenue.

(iii) Security bidding is not necessarily efficient because whenever a participant i on the I side "wins himself" through security bidding, a potential match — which would not have been stable — is not made. However, this match could have satisfied $v_I(s_i, s_j) < 0$ and $v_J(s_j, s_i) > 0$, but $v_I(s_i, s_j) + v_J(s_j, s_i) > 0$, so that blocking the match is not socially efficient. $\blacksquare$

Proof of Proposition 3.3.4

Proof Note that there is no way for agents on the I side to influence the behavior of the agents on the J side through their bidding behavior, since no information leaks across the two markets, so any deviation must be intended to improve an agent's behavior with respect to its implications for his opponents' payoffs.

First, the proposed drop-out strategies are monotone. Consider equation 2.4:

$$\tau_k^I(s_i, p_{k+1}^I) = \mathbf{E}[v_I(s_i, s_j)|\rho_{s_J}(j) = k] - \mathbf{E}[v_I(s_i, s_j)|\rho_{s_J}(j) = k+1] + p_{k+1}^I$$

The derivative with respect to s_i is

$$\frac{\partial \tau_k^I(s_i, p_{k+1}^I)}{\partial s_i} = \mathbf{E}\left[\frac{v_I(s_i, s_j)}{\partial s_i}|\rho_{s_J}(j) = k\right] - \mathbf{E}\left[\frac{\partial v_I(s_i, s_j)}{\partial s_i}|\rho_{s_J}(j) = k+1\right]$$

The support of s_i when ranked k-th or $k+1$-st is the same, since the only restriction is that $s_{J[k]} \geq s_{J[k+1]}$. Then let $f_{k:J}(s_j)$ be the distribution of the k-th of J order statistic.

$$\frac{\partial \tau_k^I(s_i, p_{k+1}^I)}{\partial s_i} = \int_{\underline{s}_J}^{\bar{s}_J} \frac{v_I(s_i, y)}{\partial s_i} f_{k:J}(y)dy - \int_{\underline{s}_J}^{\bar{s}_J} \frac{\partial v_I(s_i, y)}{\partial s_i} f_{k+1:J}(y)dy$$

From an integration by parts,

$$\begin{aligned}\frac{\partial \tau_k^I(s_i, p_{k+1}^I)}{\partial s_i} = (F_{k+1:J}(\underline{s}_J) - F_{k:J}(\underline{s}_J)) &\frac{v_I(s_i, \underline{s}_J)}{\partial s_i} \\ &+ \int_{\underline{s}_J}^{\bar{s}_J} (F_{k+1:J}(y) - F_{k:J}(y)) \frac{\partial^2 v_I(s_i, y)}{\partial s_j \partial s_i} dy\end{aligned}$$

This is positive, because $F_{k:J}(s_j)$ first-order stochastically dominates $F_{k+1:J}(s_j)$ and v_I is supermodular.

Now suppose agent $I[k]$ knew all of s_I, but the other agents on the I side only know their own information; it will be shown that there are no profitable deviations for $I[k]$, implying that there cannot be any profitable deviations in expectation, either. If the J side uses the proposed strategies, winning rank k entitles an I side agent to a draw from the distribution of the k-th order statistic, $F_{k:J}(s_j)$. Then the payoff for the k-th ranked agent on the I side from playing according to the proposed strategies is

$$\mathbf{E}[v_I(s_{I[k]}, s_j) | \rho_{s_J}(j) = k] - p_k^I$$

If the other players use the proposed strategies,

$$\begin{aligned} p_k^I &= \int v_I(s_{I[k+1]}, y) f_{k:J}(y)dy - \int v_I(s_{I[k+1]}, y) f_{k+1:J}(y)dy + p_{k+1}^I \\ &= \sum_{\ell=k+1}^{K} \left\{ \int v_I(s_{I[\ell]}, y) f_{\ell-1:J}(y)dy - \int v_I(s_{I[\ell]}, y) f_{\ell:J}(y)dy \right\} \\ &\quad + \int v_I(s_{I[K+1]}, y) f_{K:J}(y)dy \end{aligned}$$

Substituting this in and re-arranging yields

$$
\begin{aligned}
\mathbf{E}[v_I(s_{I[k]}, s_j) | \rho_{s_J}(j) = k] - p_k^I &= \int \left\{ v_I(s_{I[k]}, y) - v_I(s_{I[k+1]}, y) \right\} f_{k:J}(y) dy \\
&\quad + \int \left\{ v_I(s_{I[k+1]}, y) - v_I(s_{I[k+2]}, y) \right\} f_{k+1:J}(y) dy \\
&\quad + ... \\
&\quad + \int \left\{ v_I(s_{I,[K]}, y) - v_I(s_{I[K+1]}, y) \right\} f_{K:J}(y) dy \\
&= \int \int_{s_{I[k+1]}}^{s_{I[k]}} \frac{\partial v_I(x, y)}{\partial s_i} f_{k:J}(y) dx dy \\
&\quad + \int \int_{s_{I[k+2]}}^{s_{I[k+1]}} \frac{\partial v_I(x, y)}{\partial s_i} f_{k+1:J}(y) dx dy \\
&\quad + ... \\
&\quad + \int \int_{s_{I[K+1]}}^{s_{I[K]}} \frac{\partial v_I(x, y)}{\partial s_i} f_{K:J}(y) dx dy
\end{aligned}
$$

Since v_I is supermodular and $F_{k:J}(y)$ first-order stochastically dominates $F_{k+1:J}(y)$, all the terms above are positive if agent $I[k]$ uses the proposed strategies. Now, if agent $I[k]$ on the I side deviates by dropping out early, he foregoes one of the integral terms above. These are all positive, so his payoff falls. If agent $I[k]$ on the I side stays in and steals the position of an agent with a higher signal, the payoff will be negative, since the region of integration will flip (from $s_{I[k+\ell]}$ to $s_{I[k]}$ above becomes $s_{I[k]}$ to $s_{I[k-\ell]}$). Therefore, following the proposed strategies is an equilibrium: by dropping out earlier or later, for any realization of s_I, agent k's payoff will fall. Lastly, since each of $I[k]$'s opponents only knows his own type, none of them have a reason to "respond" to a deviation by $I[k]$ — to them, it simply appears that $I[k]$ has a higher value than he actually does. Subsequently, the agents who witness a drop-out occur at time $d_{I,k}$ can update their beliefs by inverting the drop-out strategy at stage k to get the probability that agent $I\ell$ withdrew:

$$pr[s_{I\ell} = s|d_{I,k}] = \begin{cases} 1 & , \tau_k^I(s, p_k^I) = d_{I,k} \\ 0 & , \text{otherwise} \end{cases}$$

and let beliefs at stage k at clock time c_I about an active agent $s_{I\ell}$ be

$$pr[s_{I\ell} = s|c_I] = \begin{cases} \frac{1}{I-(k-1)}\left(\sum_{n=1}^{k} f_{n:I}(s)\right) & , s \geq \left(\tau_k^I\right)^{-1}(c_I, s_{J[k-1]}, s_{J[k]}, p_k^I) \\ 0 & , \text{otherwise} \end{cases}$$

This provides beliefs that are consistent with the strategies and any observed history.

Since there are no profitable deviations for any realization of s_I, deviating cannot be profitable in expectation, either. Since this holds for the I side, it holds for the J side as well, and these strategies constitute a perfect Bayesian equilibrium.

■

Proof of Proposition 3.3.5

Proof First, if the players use the proposed drop-out strategies, the match and payments will be equivalent to the static CEM. Let that $\alpha_{I[k]}$ and $\alpha_{J[k]}$ be the announced values of the k-th highest signals on the I and J side, respectively.

Note that the drop-out strategies are monotone in the players' private information:

$$\begin{aligned} \frac{\partial \tau_k(s_i, \alpha_{J[k-1]}, \alpha_{J[k]}, p_k^I)}{\partial s_i} = \frac{\partial v_I(s_i, \alpha_{J[k-1]})}{\partial s_i} - \frac{\partial v_I(s_i, \alpha_{J[k]})}{\partial s_i} \\ = \int_{\alpha_{J[k]}}^{\alpha_{J[k-1]}} \frac{\partial^2 v_I(s_i, y)}{\partial s_j \partial s_i} dy \geq 0 \end{aligned}$$

Therefore, using this strategy, the worst agents drop out first, and the strategies implement the correct matching. This procedure also implements the same payments as the static competitive externality mechanism. Then the payment by the k-th highest ranked agent on the I side in the static CEM is

$$\begin{aligned} t_{I[k]}(s) = &\left\{v_I(s_{I[k+1]}, \alpha_{J[k]}) - v_I(s_{I[k+1]}, \alpha_{J[k+1]})\right\} \\ &+ \left\{v_I(s_{I[k+2]}, \alpha_{J[k+1]}) - v_I(s_{I[k+2]}, \alpha_{J[k+2]})\right\} \\ &+ ... + \left\{v_I(s_{I[K]}, \alpha_{J[K-1]}) - v_I(s_{I[K]}, \alpha_{J[K]})\right\} \end{aligned}$$

Then the payments in the proposed equilibrium of the dynamic competitive externality mechanism are

$$p_k^I = v_I(s_{I[k+1]}, \alpha_{J[k]}) - v_I(s_{I[k+1]}, \alpha_{J[k+1]}) + p_{k+1}^I$$

Iterating over k yields

$$p_K^I = v_I(s_{I[K+1]}, \alpha_{J[K]}) - 0$$

$$p_{K-1}^I = v_I(s_{I[K]}, \alpha_{J[K-1]}) - v_I(s_{I[K]}, \alpha_{J[K]}) + v_I(s_{I[K]}, \alpha_{J[K]})$$

$$\vdots$$

resulting in

$$\begin{aligned} p_k^I = &\left\{v_I(s_{I[k+1]}, \alpha_{J[k]}) - v_I(s_{I[k+1]}, \alpha_{J[k+1]})\right\} \\ &+ \left\{v_I(s_{I[k+2]}, \alpha_{J[k+1]}) - v_I(s_{I[k+2]}, \alpha_{J[k+2]})\right\} \\ &+ ... + \left\{v_I(s_{I[K]}, \alpha_{J[K-1]}) - v_I(s_{I[K]}, \alpha_{J[K]})\right\} \end{aligned}$$

Which are the same payments as the static version of the mechanism.

To show the proposed strategies form an equilibrium, first note that deviating at the announcement stage has no effect on bidding on the I side, because that information is not revealed to the agents on the I side. Also, lying misleads the J-side agents, but does not change the rankings, since the drop-out strategies are monotone. As a result, there are no profitable deviations at the announcement stage, since the matching will still be assortative.

Now we prove that if the agents on the J side adopt the proposed strategies, the agents on the I side have no profitable deviations (and since the two sides are symmetric, this is then an equilibrium). Since the agents on the J side cannot see what is happening on the I side, there is no way to manipulate their behavior by deviating in the I side. Subsequently, any profitable deviation must come by manipulating one's opponents on one's own side.

To prove this is an ex post perfect equilibrium, we need to show that for any history and at any subgame following the announcement phase, for any realization of private information, bidding according to the proposed drop-out strategies is a Nash equilibrium (i.e., the strategies are Nash in every subgame). This is shown by considering each stage as a subgame and using backwards induction.

- (Stage 1) Suppose there are only two agents left, $I1$ and $I2$, the clock price is any c_I, and the price for the second-best match is any p_2^I. Then the only payoff-relevant information left to be decided at this stage of the game is which agent gets the best partner and which gets the second-best partner, and what p_1^I will be. Suppose that $s_{I1} > s_{I2}$ (we are not assuming that $s_{I1} = s_{I[1]}$—

any pair of players may have made it into this final round, given the previous history of the game, so the "names" $I1$ and $I2$ are arbitrary labels).

First, suppose the clock price is higher than either agent's proposed drop-out strategy. Both agents should drop out immediately; since this is a tie, we can award the partners randomly. The agent who gets $\alpha_{J[2]}$ only has to pay p_2^I, and gets a payoff

$$v_I(s_i, \alpha_{J[2]}) - p_2^I$$

while the agent who gets $\alpha_{J[1]}$ has to pay the clock price and gets a payoff of

$$v_I(s_i, \alpha_{J[1]}) - c_I$$

but since c_I is greater than either agent's proposed drop-out price, so getting $\alpha_{J[2]}$ at p_2^I must be preferable to getting $\alpha_{J[1]}$ at c_I. Dropping out is better than staying in if

$$\frac{1}{2}\left\{v_I(s_i, \alpha_{J[1]}) - c_I + v_I(s_i, \alpha_{J[2]}) - p_2^I\right\} \geq v_I(s_i, \alpha_{J[1]}) - c_I$$

which is true for both players. Thus, they should follow the drop-out strategies and both withdraw immediately.

Alternatively, suppose the clock price is not higher than $I1$'s proposed drop-out time. If $I1$ adopts the proposed drop-out strategy, agent $I2$ can only win the best partner by outbidding $I1$. This entails staying in longer than

$$\tau_2^I(s_{I1}, \alpha_{J[1]}, \alpha_{J[2]}, p_2^I) = v_I(s_{I1}, \alpha_{J[1]}) - v_I(s_{I1}, \alpha_{J[2]}) + p_2^I$$

Then for agent $I2$, winning the best partner requires staying in until

$$c_I > v_I(s_{I1}, \alpha_{J[1]}) - v_I(s_{I1}, \alpha_{J[2]}) + p_2^I$$

yielding a payoff of

$$
\begin{aligned}
v_I(s_{I2}, \alpha_{J[1]}) - v_I(s_{I1}, \alpha_{J[1]}) + v_I(s_{I1}, \alpha_{J[2]}) - p_2^I \\
= -\int_{s_{I2}}^{s_{I1}} \int_{\alpha_{J[2]}}^{\alpha_{J[1]}} \frac{\partial^2 v_I(x,y)}{\partial s_i \partial s_j} dy dx + v_I(s_{I2}, \alpha_{J[2]}) - p_2^I \\
< v_I(s_{I2}, \alpha_{J[2]}) - p_2^I
\end{aligned}
$$

So usurping $I1$'s position is not a profitable strategy. Likewise, as long as player $I2$ drops out before the proposed drop-out time, player $I1$ prefers to wait. Therefore, in this final stage of the game and for any realizations of types and histories, the proposed drop-out strategies are subgame perfect.

- (Stage k) Now we extend the reasoning of the previous argument to k players at the k-th stage. Suppose there are a subset S of players greater than 2 for whom the clock price is greater than their proposed drop-out strategy. If the other players adopt the proposed drop-out strategies, the players in S whose partner on the equilibrium path at the expected prices — if all the agents in $I \backslash S$ use the proposed drop-out strategies in the continuation game — should all drop out immediately. Since this is a tie, we can match these agents randomly. Since the clock price was higher than their value for any achievable future partner, they prefer a lottery for the partner with announcement $\alpha_{J[k]}$ at price p_k^I to any partner they could achieve by staying in any longer. Therefore, they should follow the drop-out strategies.

  Suppose such a set S with $|S| > 1$ is not present. Let p_k^I be the price of the k-th match, and the remaining players have commonly known qualities $s_{I1} >$

$s_{I2} > ... > s_{Ik}$, and the relevant announcements are $\alpha_{J[1]} > \alpha_{J[2]} > ... > \alpha_{J[k]}$. If the agents $I1; I2; ...; I, k-2$ adopt the proposed drop-out strategies, they have no profitable deviations at this stage, since nothing happens at this point relevant to their payoffs and the induction hypothesis is that play is subgame perfect in the next stage. Then agent Ik can attempt to steal $I, k-1$'s position by outbidding him, but this results in a payoff of

$$
\begin{aligned}
& v_I(s_{Ik}, \alpha_{J[k-1]}) - v_I(s_{I,k-1}, \alpha_{J[k]}) + v_I(s_{I,k-1}, \alpha_{J[k]}) - p_k^I \\
& \qquad = -\int_{s_{Ik}}^{s_{I,k-1}} \int_{\alpha_{J[k]}}^{\alpha_{J[k-1]}} \frac{\partial^2 v_I(x,y)}{\partial s_i \partial s_j} dy dx + v_I(s_{Ik}, \alpha_{J[k]}) - p_k^I \\
& \qquad\qquad < v_I(s_{Ik}, \alpha_{J[k]}) - p_k^I
\end{aligned}
$$

So usurping the $k-1$-st slot is an unprofitable deviation. Likewise, the $I, k-1$ agent doesn't want to drop out any earlier for any realization of types and histories, since for any $\delta < \tau_k^I(s_{I,k-1}, \alpha_{J[k-1]}, \alpha_{J[k]}, p_k^I)$,

$$
v_I(s_{I,k-1}, \alpha_{J[k-1]}) - \tau_k^I(s_{I,k-1}, \alpha_{J[k-1]}, \alpha_{J[k]}, p_k^I) = v_I(s_{I,k-1}, \alpha_{J[k]}) - p_k^I
$$

so

$$
v_I(s_{I,k-1}, \alpha_{J[k-1]}) - \delta > v_I(s_{I,k-1}, \alpha_{J[k]}) - p_k^I
$$

So no players have a profitable deviation at any point in time in this stage, for all histories, prices and realizations of s_I, since any further deviations are already included in the two-player endgame considered in the stage 2 analysis, or the beginning paragraph of the stage k analysis.

Then players anticipate subgame perfect player in all future stages, and this anticipation makes current deviations unprofitable. Note that the above analysis includes scenarios where the $I[5]$ agent bids his way into the third stage, with, say, $I[1]$ and $I[2]$, since we took the set of currently active agents and the current price as given at the start of the stage.

So it is an ex post perfect equilibrium to bid according to the proposed drop-out strategies, since the drop-out strategies are a Nash equilibrium for every subgame.

$\blacksquare$

Proof of Proposition 3.4.1

Proof (i) Matching workers to firm slots sequentially and assortatively maximizes the value on both sides of the market: Consider the decision to allocate the k-th best worker on the J side. The firms have marginal values

$$\delta(s_i, s_j, W_i) = v_I(s_i, W_i \cup s_j) - v_I(s_i, W_i)$$

If firm a gets a higher marginal value from worker j than firm b, then

$$\begin{aligned} \delta(s_1, s_j, W_1) &\geq \delta(s_2, s_j, W_2) \\ \alpha\delta(s_1, s_j, W_1) &\geq \alpha\delta(s_2, s_j, W_2) \\ v_J(s_1, W_1 \cup s_j) &\geq v_J(s_2, W_2 \cup s_j) \end{aligned}$$

So worker j also prefers to work at firm 1 rather than firm 2. So both sides agree that on the margin, worker Jk belongs to the firm with the higher marginal value.

This assignment procedure is also socially optimal. Note that $\delta(s_i, s_j, W_i)$ only depends on the addition of the marginal worker, and the firm valuation function can be written

$$
\begin{aligned}
& v_I(s_i, \{s_{J1}, s_{J2}, ..., s_{JL}\}) \\
&\qquad = v_I(s_i, \{s_{J1}, s_{J2}, ..., s_{JL}\}) \\
&\qquad\quad - v_I(s_i, \{s_{J1}, s_{J2}, ..., s_{J,L-1}\}) \\
&\qquad\quad + v_I(s_i, \{s_{J1}, s_{J2}, ..., s_{JL-1}\}) \\
&\qquad = \delta_L(s_i, s_{JL}, \{s_{J1}, s_{J2}, ..., s_{J,L-1}\}) \\
&\qquad\quad + v_I(s_i, \{s_{J1}, s_{J2}, ..., s_{JL-1}\}) \\
&\qquad ... \\
&\qquad = \sum_{\ell=1}^{L} \delta_\ell(s_i, \{s_{J1}, s_{J2}, ..., s_{J\ell-1}\}, s_{J\ell})
\end{aligned}
$$

So at each step of the allocation process, the matchmaker never wants to revisit any earlier decisions about allocation because of complementarity between workers — if i has the highest value for j at the k-th stage, that will always be the most productive use of j, since it is better than any other open slot and raises the marginal value of all the subsequent slots at firm i.

(ii) It is a best-response for the firms to report honestly if all other agents do: Consider I, k's payoff for any report σ_{Ik}:

$$
\begin{aligned}
U(\sigma_{Ik}, s_{\setminus Ik}) = & \sum_{i \in I} \sum_{W_i \in W} m_{iW_i}(\sigma_{Ik}, s_{\setminus Ik}) v_I(s_i, W_i) \\
& - \sum_{i \in I} \sum_{W_i \in W} m_{iW_i}(\varnothing, s_{\setminus Ik}) v_I(s_i, W_i)
\end{aligned}
$$

Note that the right-hand side of the first line is the total welfare of the agents on the I side, and the second does not depend on Ik's report. Then from step (i), we know the whole expression is maximized at the sequentially assortative match, which coincides with making an honest report. So honest reporting is an ex post equilibrium for the I side. Likewise, on the J side, the agents (workers) have the same incentives as in Proposition 3.2 (the one-to-one case), so honesty is an ex post equilibrium for them as well. ∎

Appendix B

Proofs for Chapter Four

Proof of Proposition 4.4.1

Proof Suppose that $v_I(s_i, s_j) = v_J(s_j, s_i) = s_i \cdot s_j = \sum_\ell s_i^\ell s_j^\ell$.

(i) Let $s_{I1} = (3,3)$, $s_{I2} = (2,2)$, $s_{I3} = (1,1)$, $s_{J1} = (2,2)$, $s_{J2} = (1,1)$. If all agents report honestly, agent $I3$ only gets a partner if he is chosen first or second, for a final probability of getting *any* partner of $4\frac{1}{5}\frac{1}{3}$. By reporting $s_{I4} = (4,4)$, however, he improves the odds of getting a partner to $4\frac{1}{5}\frac{1}{3} + \frac{3}{5}$, since $J1$ and $J2$ now prefer him relative to the reported information and will be matched with him whenever the opportunity to do so arises. Therefore, truth-telling is not an ex post equilibrium.

(ii) Suppose $s_{I1} = (1,1)$, $s_{I2} = (2,2)$, $s_{J1} = (2,1)$ and $s_{J2} = (3,3)$. If agent $I1$ reports $(3,3)$, he moves to the top of both $J1$ and $J2$'s preference orderings. If the I side is proposing, $J2$ will clearly prefer an agent with $I1$'s report to agent $I2$, and the deviation is profitable for the proposing side. If the J side is proposing, $I1$ can report $(0,5)$, generating perceived values for the J side of

	$I1$	$I2$
$J1$	5	6
$J2$	15	12

Subsequently, if the J side reports honestly, agent $I1$ is matched to $I2$, and this is a profitable deviation.

(iii) (From Johnson [39]) Consider a simple supermodular market where $v_I(s_i, s_j) = v_J(s_j, s_i) = s_i s_j$ and

$$(s_{I1}, s_{I2}) = (s_{J1}, s_{J2}) = (1, 2)$$

If $I1$ honestly reports a 1 and the other agents do as well, $I1$ and $J1$ are matched, and $I2$ and $J2$ are matched. Then $I1$ gets a payoff from reporting honestly of

$$\begin{aligned} U_I(1,1) &= m_{I1,J1}(1, s_{-I1})(1*1+1*1) + m_{I2,J2}(1, s_{-I1})(2*2+2*2) \\ &\quad - m_{I2,J2}(\varnothing, s_{-I1})(2*2+2*2) \\ &= 2 \end{aligned}$$

But suppose he makes a report σ strictly higher than 3, he gets a better partner, as well as receives a higher payment based on his report:

$$\begin{aligned} U_I(1,\sigma) &= m_{I1,J2}(\sigma, s_{-I1})(1*2+2*\sigma) + m_{I2,J1}(\sigma, s_{-I1})(1*2+2*1) \\ &\quad - m_{I2,J2}(\varnothing, s_{-I1})(2*2+2*2) \\ &= 2*\sigma - 2 \\ &> U_I(1,1) \end{aligned}$$

So it is a profitable deviation to lie, and the VCG mechanism is not incentive compatible (truth-telling is not even a Nash equilibrium, let alone a dominant strategy).

∎

Proof of Proposition 4.5.1

Proof Proof (If preferences are reciprocal, then CEM is feasible and truth-telling is an ex post equilibrium:) Suppose the matchmaker solves the following linear assignment problems:

- $\max_{m_{ij}^I} \sum_i \sum_j m_{ij}^I v(s_i, s_j)$ subject to $\sum_i m_{ij}^I \leq 1$ and $\sum_j m_{ij}^I \leq 1$
- $\max_{m_{ij}^J} \sum_i \sum_j m_{ij}^J v(s_i, s_j)$ subject to $\sum_i m_{ij}^J \leq 1$ and $\sum_j m_{ij}^J \leq 1$

If $v_J(s_j, s_i)$ is reciprocal to $v_I(s_i, s_j)$, then the objective functions assign the same ordering to all the same matches; i.e., if $\sum_i \sum_j m_{ij} v_I(s_i, s_j) \geq \sum_i \sum_j \tilde{m}_{ij} v_I(s_i, s_j)$, then $\sum_i \sum_j m_{ij} v_J(s_j, s_i) \geq \sum_i \sum_j \tilde{m}_{ij} v_J(s_j, s_i)$. Therefore, CEM is feasible if preferences are reciprocal. The set of positive affine transformations $v_J(s_j, s_i) = \alpha v_I(s_i, s_j) + \beta$, where $\alpha, \beta > 0$, for example, is a class of reciprocal preferences for any $v_I(s_i, s_j)$.

To show that truth-telling is an ex post equilibrium of CEM, consider the transfers:

$$t_{I\ell} = -\left\{ \sum_{i \in I \setminus I\ell} \sum_{j \in J} m_{ij}(\sigma_{I\ell}, s_{\setminus I\ell}) v_I(s_i, s_j) \right\} + \left\{ \sum_{i \in I \setminus I\ell} \sum_{j \in J} m_{ij}(\varnothing, s_{\setminus i}) v_I(s_i, s_j) \right\}$$

Then $I\ell$'s payoff is

$$\begin{aligned} &\sum_j m_{ij}(\sigma_{I\ell}, s_{\setminus I\ell}) v_I(s_{I\ell}, s_j) - t_{I\ell}(\sigma_{I\ell}, s_{\setminus I\ell}) \\ &\qquad = \left\{ \sum_j m_{ij}(\sigma_{I\ell}, s_{\setminus I\ell}) v_I(s_{I\ell}, s_j) + \sum_{i \in I \setminus I\ell} \sum_{j \in J} m_{ij}(\sigma_{I\ell}, s_{\setminus I\ell}) v_I(s_i, s_j) \right\} \\ &\qquad\qquad - \left\{ \sum_{i \in I \setminus I\ell} \sum_{j \in J} m_{ij}(\varnothing, s_{\setminus i}) v_I(s_i, s_j) \right\} \end{aligned}$$

The second term in braces does not depend on $\sigma_{I\ell}$, so $I\ell$ cannot manipulate it. The first term in braces is the welfare of the I side, which is maximized at the true report, since then the matchmaker chooses the welfare maximizing allocation. So if all other agents report honestly, agent $I\ell$ should report honestly as well. Therefore, truth-telling is an ex post equilibrium of the CEM mechanism when preferences are reciprocal.

(If CEM is feasible and truth-telling is an ex post equilibrium, then preferences are reciprocal): This will be by way of contrapositive. Namely, if preferences are not reciprocal, then CEM is not feasible for all s, or truth-telling is not an ex post equilibrium for all s, or both. Since the match values are pairwise private value and preferences are not reciprocal, we can construct a situation with two agents on each side of the market where

$$v_I(s_{I1}, s_{J1}) > v_I(s_{I1}, s_{J2})$$

$$v_I(s_{I2}, s_{J2}) > v_I(s_{I2}, s_{J1})$$

$$v_J(s_{J1}, s_{I2}) > v_J(s_{I2}, s_{J1})$$

$$v_J(s_{J2}, s_{I1}) > v_J(s_{J2}, s_{I2})$$

or possibly

$$v_I(s_{I1}, s_{J1}) > v_I(s_{I1}, s_{J2})$$

$$v_I(s_{I2}, s_{J2}) > v_I(s_{I2}, s_{J1})$$

$$v_J(s_{J1}, s_{I2}) > v_J(s_{I2}, s_{J1})$$

$$v_J(s_{J2}, s_{I1}) < v_J(s_{J2}, s_{I2})$$

with $v_J(s_{J1}, s_{J2}) > v_J(s_{J2}, s_{I2})$. If this wasn't possible for any s, then we would have reciprocal preferences by definition because the social ordering of matches would be the same on both sides. The solution on the I side is $m^I_{11} = 1$, $m^I_{22} = 1$, $m^I_{12} = 0$, $m^I_{21} = 0$. The solution on the J side is $m^J_{11} = 0$, $m^J_{22} = 0$, $m^J_{12} = 1$, $m^J_{21} = 1$. So for $I = J = 2$, if the surplus functions aren't reciprocal, CEM is not feasible. By assumption, we can replicate the existing players arbitrary numbers of times — this does nothing to resolve the infeasibility problem, since the optimal matches now become lotteries where the two sides continue to disagree. Consequently, for any number of participants I and J, if preferences are not reciprocal, a type realization s can be constructed for which CEM is not feasible. $\blacksquare$

Proof of Proposition 4.5.2

Proof Player $I1$ is envious of player $I2$ if

$$v_I(s_{I2}, s_{J1}) - t_{I1} > v_I(s_{I2}, s_{J2}) - t_{I2}$$

This implies

$$\begin{aligned} v_I(s_{I2}, s_{J1}) + v_I(s_{I1}, s_{J1}) - (v_I(s_{I1}, s_{J1}) + t_{I1}) >& \\ \sum_{i \in I} m_{ij}(s) v_I(s_i, s_j) - \sum_{i \in I \setminus I2} m_{ij}(s) v_I(s_i, s_j)& \end{aligned}$$

$$\begin{aligned} \left(v_I(s_{I2}, s_{J1}) + \sum_{i \in I \setminus I2} m_{ij}(s) v_I(s_i, s_j) \right) + \left(v_I(s_{I1}, s_{J1}) + \sum_{i \in I \setminus I1} m_{ij}(s) v_I(s_i, s_j) \right)& \\ > 2 \sum_{i \in I} m_{ij}(s) v_I(s_i, s_j)& \end{aligned}$$

The two terms on the left-hand side of the inequality correspond to two proposed matches that are potentially different from the original one, but allowing agent $J1$ to be matched at most *twice*. The relaxation of this constraint introduces the possibility that the inequality holds, and that players are jealous of each other's resulting partner and payment. $\blacksquare$

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