ABSTRACT

Title of dissertation: BALAYAGE OF FOURIER TRANSFORMS AND THE THEORY OF FRAMES

Enrico Au-Yeung, Doctor of Philosophy, 2011

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Every separable Hilbert space has an orthogonal basis. This allows every element in the Hilbert space to be expressed as an infinite linear combination of the basis elements. The structure of a basis can be too rigid in some situations. Frames gives us greater flexibility than bases. A frame in Hilbert space is a spanning set with the reconstruction property.

A frame must satisfy both an upper frame bound and a lower frame bound. The requirement of an upper bound is rather modest. Most of the mathematical difficulty lies in showing the lower bound exists.

We examine the theory of Beurling on Balayage of Fourier transforms and the role of spectral synthesis in this theory. Beurling showed that if the condition of Balayage holds, then the lower frame bound for a Fourier frame exists under suitable hypothesis. We extend this theory to obtain lower bound inequalities for other types of frames. We prove that lower bounds exist for generalized Fourier frames and two types of semi-discrete Gabor frames.

BALAYAGE OF FOURIER TRANSFORMS AND THE THEORY OF FRAMES

by

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Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2011

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Dedicated to the 88th birthday of my grandmother

Acknowledgments

I am still alive.

First and foremost I'd like to thank my advisor, Professor John Benedetto, for his guidance throughout graduate school and for introducing me to the subject.

My parents have instilled in me a strong work ethics in an early age. During one summer when I was ten years old, my father patiently taught me how to solve equations with one variable. This lasted every day for the whole summer, for several hours each day, and he never gave up on me until I finally learned it.

There are many professors in graduate school that I wish to thank. These wonderful professors have introduced me to serious mathematics and many informal discussions have helped me a lot.

Survival in graduate school also depends on having friends and classmates to help me. In this regard, I am grateful to have a friend like Eric Hamilton. If anyone is going to graduate school and is so desperate that he will ask me for advice, then my advice would be to find a friend like Eric Hamilton.

I have many mathematical heroes, too many to name here. They are constantly a source of inspiration to me.

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Notations

$\mathbb{R}$	the set of real numbers
$\widehat{\mathbb{R}}$	the set of real numbers in the domain of Fourier transform
$\mathbb{Z}$	the set of integers
$\hat{f} = \mathcal{F}$	the Fourier transform of a function
$\check{f}$	the inverse Fourier transform of a function
$\stackrel{s}{=}$	equal, by switching two limits, such as an integral and a sum
E^o	the interior of a set E
$\mathcal{C}(\Lambda)$	the set of all bounded continuous functions ϕ on $\mathbb{R}^d$ with supp $\hat{\phi} \in \Lambda$
C_c	the set of all continuous functions with compact support
$\mathcal{S}(\mathbb{R}^d)$	the Schwartz space, rapidly decreasing infinitely differentiable functions
$M_b(E)$	the set of all bounded Radon measures with support on a set E
$M_b(\mathbb{R}^d)$	the set of all bounded Radon measures on $\mathbb{R}^d$
$L^1(\mathbb{R})$	the set of integrable functions on $\mathbb{R}$
$L^2(E)$	the set of square integrable functions that are zero outside set E
$L^2(\Lambda)$	the set of square integrable functions that are zero outside set Λ
$\int_E$	integration over the set E
$V_g f$	short-time Fourier transform of f with respect to g

Chapter 1

Introduction and Background

A central concept in linear algebra is that every finite dimensional vector space has a basis. This allows every element in the vector space to be written as a linear combination of the vectors in the basis. There is a natural generalization to infinite dimensional vector spaces. Every separable Hilbert space has an orthogonal basis. This allows every element in the Hilbert space to be expressed as an infinite linear combination of the basis elements.

The elements of the basis elements must be linearly independent. That means the structure of a basis can be too rigid in some situations. Suppose we want to express every element of a three-dimensional subspace of a five-dimensional space as a linear combination of the five basis vectors. This can certainly be done even though the five basis vectors do not form a basis for the three-dimensional subspace. Thus in this case, insisting on working with a basis would be a too rigid requirement. Another situation is that in a Hilbert space of functions, if we want each basis element to be a function that is smooth (infinitely differentiable) and vanish outside a bounded interval, then it may be difficult or impossible to find a basis with such criterion.

Frames give us greater flexibility than bases while still allow us to expand each function in a Hilbert space in terms of the elements of a frame.

Frames were introduced by Duffin and Schaeffer in 1952 to study some deep problems in nonharmonic Fourier series. For more than 30 years, their ideas did not seem to generate much interest outside of nonharmonic Fourier series. Motivated by the study of heat diffusion, Fourier tried to expand an arbitary function in terms of trigonometric series. Over the next century and half, many mathematicians have tried to put a rigourous foundation in classical Fourier analysis. Instead of a Fourier expansion of a function, Dennis Gabor considered in 1946 a method to represent a one-dimensional signal in two-dimensions, with time and frequency as coordinates. Eugene Wigner suggested in 1932 to represent a one-dimensional wave function in two dimensions, with position and momentum as coordinates.

Finally, in 1986, Daubechies, Grossman and Meyer in their groundbreaking paper observed that frames can be used for painless nonorthogonal expansions for functions. This was probably the time when many mathematicians and engineers began to see the potential of frames. Since then, frames have been used in signal processing, image processing, data compression. Frames have also been studied for the deep mathematical aspects arising in harmonic analysis, operator theory, group representation theory, and function spaces (Hardy, Sobolev, and Besov spaces).

The theory of frames is also connected to one of the most famous open problems in operator algebras, the Kadison-Singer Conjecture in C*-algebras. The conjecture remains open after more than 50 years. In the theory of frames, the Feichtinger Conjecture states that every bounded frame can be written as a finite union of Riesz basic sequences. It is known that the Kadison-Singer Conjecture implies the Feichtinger Conjecture.

An attractive aspect about the theory of frames is that much remains unknown. A frame must satisfy both an upper frame bound and a lower frame bound. It is usually not hard to verify the upper frame bound, but the lower frame bound often presents a serious mathematical challenge. It is a bit mysterious as to when or why a lower frame bound exists.

The theory of Balayage on Fourier transforms, as developed by the Swedish mathematician Beurling, is a promising tool in this direction of research. Balayage originated in potential theory and was introduced by Christoffel in the late 1870's. Poincaré used the Balayage method in 1890 to solve the Dirichlet problem for the Laplace equation.

Beurling showed that if a condition, which he called Balayage, exists between a pair of closed sets, then under some hypothesis, this implies that the lower frame bound for a Fourier frame exists. We want to extend this theory to other type of frames in order to obtain lower bound inequalities for these frames.

1.1 Riesz basis and complete sequence

Definition 1.1. In a normed space X, two sequences $\{e_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ are topologically isomorphic if there is a bounded and invertible linear operator T on X such that $f_n = Te_n$ for each n.

Definition 1.2. (Riesz basis) In a separable Hilbert space H, a sequence $\{f_n\}_{n=1}^{\infty}$ of elements is a Riesz basis if it is topologically isomorphic to an orthonormal basis.

Definition 1.3. A sequence $\{x_n\}_{n=1}^{\infty}$ of elements in a Hilbert space is ω-independent if $\sum_{n=1}^{\infty} c_n x_n = 0$ implies $c_n = 0$ for all n.

Theorem 1.1. *(Bari, 1951 [You01]) Let H be a separable Hilbert space.*
Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for H.
Let $\{f_n\}_{n=1}^{\infty}$ be an ω-independent sequence such that $\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < \infty$.
Then $\{f_n\}_{n=1}^{\infty}$ is a Riesz basis for H.

The Fourier series $\{e^{2\pi int} : n \in \mathbb{Z}\}$ is an othonormal basis for the Hilbert space $L^2[-\frac{1}{2}, \frac{1}{2}]$. The above theorem tells us that if we perturb each term in the series slightly, the resulting series will remain a Riesz basis. The following theorem quantifies how much we can perturb the terms in the series.

Theorem 1.2. *(Kadec's-$\frac{1}{4}$-Theorem) If $\{\lambda_n : n \in \mathbb{Z}\}$ is a sequence in $\mathbb{R}$ such that $|\lambda_n - n| \le L < \frac{1}{4}$ for $n \in \mathbb{Z}$, then $\{e^{2\pi i\lambda_n t}\}$ forms a Riesz basis for $L^2[-\frac{1}{2}, \frac{1}{2}]$.*

Definition 1.4. (Complete sequence) In a Hilbert space H, a sequence $\{f_n\}_{n=1}^{\infty}$ of elements is complete if its linear span is dense in H. That means, for each $f \in H$ and for every $\epsilon > 0$, there is a finite linear combination $\sum_{n=1}^{M} c_n f_n$ such that

$$\|f - \sum_{n=1}^{M} c_n f_n\| < \epsilon.$$

Unlike a Riesz basis, the elements of a complete sequence do not need to be linearly independent. For example, if we start with a Riesz basis, and add a finite number of elements to it, then the resulting sequence of elements is a complete sequence. Suppose the sequence $\{\lambda_n\}_{n=1}^{\infty}$ is spread out far apart in $\mathbb{R}$, must the sequence $\{e^{2\pi i \lambda_n t}\}_{n=1}^{\infty}$ be complete in $L^2[-A, A]$ for sufficiently small A?

Theorem 1.3. *(Schwartz [Sch59]) Let* $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ *be a sequence in* $\mathbb{R}$ *such that* $\sum_{n=1}^{\infty} \frac{1}{|\lambda_n|} < \infty$*. Then* $\{e^{2\pi i \lambda_n t}\}_{n=1}^{\infty}$ *is not complete in* $L^2[-A, A]$ *for any* $A > 0$*. i.e.* $R(\Lambda) = 0$*, where* $R(\Lambda) = \sup\{A : \{e^{2\pi i \lambda_n t}\}$ *is complete in* $L^2[-A, A]$*.*$\}$

There are situations when a basis is too rigid. For example, if $B = \{e_1, ..., e_d\}$ is an orthogonal basis in $\mathbb{C}^d$, then this set of vectors will not be a basis for any proper subspace of $\mathbb{C}^d$ but it is a spanning set of vectors, so it is a complete sequence. However, a complete sequence has its drawback. In an infinite dimensional Hilbert space H, if $\{x_n\}_{n=1}^{\infty}$ is a complete sequence for H, reconsruction of an element $f \in H$ from the coefficients $\langle f, x_n \rangle$ is generally not possible.

A Riesz basis is too rigid. A complete sequence does not have enough structure. What we need is something less rigid than a Riesz basis, but with more structure than a complete sequence.

1.2 Frames

Definition 1.5. (Frames) Let H be a separable Hilbert space. A sequence $\{x_n\}_{n=1}^{\infty}$ of elements in H is a frame if there are positive constants A and B such that,

$$\forall\ f \in H, \quad A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, x_n\rangle|^2 \leq B\|f\|^2.$$

If a sequence satisfies the right hand inequality, then it is a Bessel sequence.

Definition 1.6. Let $f(z)$ be an entire function such that for all values of z, we have $|f(z)| \leq Ae^{2\pi\Omega|z|}$, for positive constants A and Ω. We say that f is an entire function of exponential type.

Theorem 1.4. *(Paley-Wiener) Suppose f is an entire function of exponential type and $\int_{-\infty}^{\infty} |f(x)|^2\ dx < \infty$. Then there exists a function $\hat{f}$ in $L^2[-\Omega, \Omega]$ such that $f(z) = \int_{-\Omega}^{\Omega} \hat{f}(\lambda)e^{2\pi iz\lambda}d\lambda$.*

Definition 1.7. The Paley-Wiener space PW_Ω is the set of functions $f(x) \in L^2(-\infty, \infty)$ such that the Fourier transform $\hat{f}(\lambda)$ is zero outside the interval $[-\Omega, \Omega]$.

A sequence $\{t_n\}$ of real numbers is a sampling sequence if there are positive constants A, B such that

$$\forall f \in PW_\Omega,\ A\int_{-\infty}^{\infty} |f(x)|^2dx \leq \sum_{n=-\infty}^{\infty} |f(t_n)|^2 \leq B\int_{-\infty}^{\infty} |f(x)|^2dx.$$

By the Paley-Wiener Theorem, this is equivalent to

$$A\int_{-\Omega}^{\Omega} |\hat{f}(\gamma)|^2d\gamma \leq \sum_{n=-\infty}^{\infty} |\int_{-\Omega}^{\Omega} \hat{f}(\gamma)e^{2\pi it_n\gamma}d\gamma|^2 \leq B\int_{-\Omega}^{\Omega} |\hat{f}(\gamma)|^2d\gamma.$$

That means $\{e^{-2\pi it_n} : n \in \mathbb{Z}\}$ is a Fourier frame for $L^2[-\Omega, \Omega]$.

To understand why Fourier frames are important, we now briefly discuss the sampling problem. We can formulate the sampling problem as follows: Is it possible to reconstruct a function $f : \mathbb{R} \to \mathbb{C}$ if we only know a countable set of function values $\{f(t_n)\}$? The problem is ambiguous when formulated this way, since there are infinitely many functions that take the same values $\{f(t_n)\}$ on the set $\{t_n\}$, so we need to restrict our function to a certain class.

One way to do this is to restrict our attention to those functions in the Paley-Wiener space PW_Ω. They are also known as band-limited functions, i.e. these are precisely the functions in $L^2(-\infty, \infty)$ whose Fourier transforms vanish outside the interval $[-\Omega, \Omega]$. If we think of a piece of music, then in theory all frequencies can appear, but humans can only hear frequencies within a certain range. Therefore if we are trying to store a piece of music, we can discard those frequencies outside the human hearing range and treat the resulting function as band-limited.

When we discuss the theory of frames in Chapter 2, we will see that for any function f in a separable complex Hilbert space, if $\{x_n\}_{n=1}^{\infty}$ is a frame for H, then by using only the countable set of values $\{\langle f, x_n\rangle\}_{n=1}^{\infty}$, it is possible to resonstruct the original function from these sampled values. Furthermore, the reconstruction is robust if a small amount of noise is added to the sampled values. This robustness to noise is an important advantage over the use of an orthogonal basis.

A frame is a Bessel sequence that also satisfies the lower frame bound.

A frame is a spanning set in a Hilbert space with the reconstruction property.

To verify that a sequence is a frame in a Hilbert space, we have to show that both the upper and lower frame bounds inequalities are satisfied. The requirement of the upper bound is rather modest. For example, if we start with a Bessel sequence, then the removal of a finite number of elements will still leave us with a Bessel sequence, but the lower frame bound may no longer be satisfied.

It should be emphasized that most of the mathematical difficulty is to establish the lower frame bound.

We want to study the theory of Beurling on Balayage of Fourier transforms and the role of spectral synthesis played in this theory. Beurling has shown that if the condition of Balayage holds, then under some hypothesis the lower frame bound for a Fourier frame exists. One of our goals is to extend the theory of Beurling in order to obtain lower bound inequalities for other types of frames.

1.3 Summary of new results

All new results are contained in Chapter 4.

In the remaining of Chapter 1, we review some basic theory from real analysis and set the notations. We briefly discuss the theory of distributions that is necessary to justify the assertion that the Fourier transform of a bounded function exists in the sense of distributions. We introduce the short-time Fourier transform and prove the corresponding inversion formula.

In Chapter 2, we discuss the theory of frames. This chapter is not only a guided tour on the theory of frames, but it also contains less elementary results. For example, we include the proofs of two non-trivial theorems on Gabor expansions, a proof on a translation-invariant system that is a Bessel sequence, and a detailed constructive proof on the square root of a positive operator. We also give the proof of the reconstructive formula for frames (one of the most important result in the theory of frames) and illustrate how this formula works with a throughly worked out example.

In Chapter 3, we define the concept of Balayage. We follow Beurling's proof that Balayage implies Fourier frames (Theorem 3.6). Theorem 3.4 roughly says that if Balayage is possible for a pair of sets E and Λ, then the set Λ can be enlarged slightly and Balayage will remain possible. This thoerem is used to prove Lemma 3.9, which is the key tool for subsequent proofs in the next chapter. The proof of Theorem 3.4 is an arduous task. (It is not necessary to understand its proof in order to use Lemma 3.9 or to go through any proofs in Chapter 4).

An extremely important section in Chapter 3 is section 3.4, where we give the 4-line proof that Balayage implies Fourier frames. The 4-line proof is a summary of the full proof and its purpose is to give us the insight of why the proof works. The insight gained in this proof serves as an inspiration for the proofs in chapter 4.

In Chapter 4, we prove lower bound inequalities for different types of frames (under suitable hypothesis). These include generalized Fourier frames, Fourier frames on a weighted Hilbert space, and Semi-discrete Gabor frames. Two types of Semi-discrete Gabor frames are considered. We also consider a bilinear frame operator that is constructed from convolutions. This type of operator is not new. The novelty here lies in recognizing that it can be used to construct a bilinear frame operator. Balayage originated in potential theory, so it is fitting that we end this study with a result related to the Poisson kernel.

The new results are Theorem 4.1, 4.2, 4.3, 4.4, 4.6, and 4.7.

For the sake of convenience, we highlight three of these theorems in the next page.

Theorem (Generalized Fourier Frames)

Let $E = \{t_n\}$ be a separated sequence in $\mathbb{R}^d$. Let $\Lambda \subseteq \mathbb{R}^d$ be a set of spectral synthesis and symmetric about the origin. Assume Λ is a convex set. Assume Balayage is possible for (E, Λ).

Then there exists $A > 0$, such that for each $F \in L^2(\Lambda)$,

$$A\left(\int_\Lambda |F(\zeta)|^2\, d\zeta\right)^{1/2} \le \left(\sum_{n\in\mathbb{Z}^d} |\widehat{F}(t_n)|^2\right)^{1/2} + \left(\sum_{n\in\mathbb{Z}^d} |\widehat{F}(\frac{1}{2}t_n)|^2\right)^{1/2} + \left(\sum_{n\in\mathbb{Z}^d} |\widehat{F}(\frac{1}{3}t_n)|^2\right)^{1/2}.$$

Theorem (Fourier frames on a weighted Hilbert space)

Let $E = \{t_n\}$ be a separated sequence in $\mathbb{R}^d$. Let Λ be a set of spectral synthesis, symmetric about the origin. Assume Balayage is possible for (E, Λ). Let G be any positive bounded function defined on Λ. Define

$$L^2_G(\Lambda) = \{F : \int_\Lambda |F(\zeta)|^2\, G(\zeta)\, d\zeta < \infty.\}$$

Then the lower bound inequality holds, i.e. $\exists\ A > 0$, such that $\forall F \in L^2_G(\Lambda)$,

$$A \cdot \frac{\int_\Lambda |F(\zeta)|^2\, G(\zeta)\, d\zeta}{\left(\int_\Lambda |F(\zeta)|^2\, d\zeta\right)^{1/2}} \le \sum_{n\in\mathbb{Z}^d} |\,(F(\zeta)\cdot G(\zeta))\widehat{\ }(t_n)|^2.$$

Theorem (Semi-Discrete Gabor frames)

Let $g \in L^2(\mathbb{R}^d)$ be real-valued, where $\|g\|_2 = 1$.

Let $\Lambda \subseteq \mathbb{R}^d$ be a set of spectral synthesis and symmetric about 0.

Let $E = \{t_n\}$ be a separated sequence in $\mathbb{R}^d$. Assume Balayage holds for (E, Λ).

Then $\exists\ A > 0$, such that $\forall F \in L^2(\Lambda)$,

$$A\int_\Lambda |F(\zeta)|^2\, d\zeta \le \int_{\mathbb{R}^d} \sum_{n\in\mathbb{Z}^d} |V_gF(y, t_n)|^2\, dy.$$

1.4 Some theory and notations

In $\mathbb{R}^d$, we write x for $(x_1, x_2, ..., x_d)$.

Definition 1.8. The space of integrable functions on $\mathbb{R}^d$ is

$$L^1(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{C} : \|f\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |f(x)|\ dx < \infty\}.$$

The integral over $\mathbb{R}^d$ is $\int_{\mathbb{R}^d}$ and sometimes we write $\int$ when there is no ambiguity.

The Fourier transform plays a major role in this subject. The domain of Fourier transform is $\widehat{\mathbb{R}}^d = \mathbb{R}^d$. The Fourier transform $\hat{f}$ of $f \in L^1(\mathbb{R}^d)$ is

$$\hat{f}(\zeta) = \mathcal{F}(\zeta) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\zeta}dx, \quad \zeta \in \widehat{\mathbb{R}}^d.$$

The inversion formula for Fourier transform is

$$f(x) = \int_{\widehat{\mathbb{R}}^d} \hat{f}(\zeta)e^{2\pi i x\cdot\zeta}\ d\zeta.$$

Definition 1.9. Let $f, g \in L^1(\mathbb{R}^d)$. The convolution of f and g, denoted by $f * g$ is

$$(f * g)(x) = \int f(x-y)g(y)\ dy = \int g(x-y)f(y)\ dy.$$

Definition 1.10. The space of square integrable functions on $\mathbb{R}^d$ is

$$L^2(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{C} : \int_{\mathbb{R}^d} |f(x)|^2\ dx < \infty\}.$$

When $f \in L^2(\mathbb{R}^d)$, the norm of f is given by

$$\|f\|_{L^2(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |f(x)|^2\ dx\right)^{1/2}.$$

If there is no ambiguity, we sometimes write $\|f\|_2$ for $\|f\|_{L^2(\mathbb{R}^d)}$.

Definition 1.11. The collection of all square summable sequences is denoted by

$$l^2(\mathbb{Z}) = \{c = (..., c_1, c_2, c_3, ...) : \|c\|_{l^2(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} |c_n|^2 < \infty\}.$$

Definition 1.12. When E is a closed subset of $\mathbb{R}^d$,

$$L^2(E) = \{f : f \in L^2(\mathbb{R}^d), f = 0 \text{ outside set } E\}.$$

When $f \in L^2(E)$, the norm of f is given by

$$\|f\|_{L^2(E)} = \left(\int_E |f(x)|^2 \, dx\right)^{1/2}.$$

Definition 1.13. When Λ is a closed subset of $\widehat{\mathbb{R}}^d$,

$$L^2(\Lambda) = \{F : F \in L^2(\widehat{\mathbb{R}}^d), F = 0 \text{ outside set } \Lambda\}.$$

When $F \in L^2(\Lambda)$, the norm of F is given by

$$\|F\|_{L^2(\Lambda)} = \left(\int_\Lambda |F(\zeta)|^2 \, d\zeta\right)^{1/2}.$$

Definition 1.14. If $f : \mathbb{R}^d \to \mathbb{C}$ is any function on $\mathbb{R}^d$, then the support of f, denoted by supp f, is the smallest closed set outside of which f is zero.

If μ is a measure on $\mathbb{R}^d$, then the support of μ, denoted by supp μ, is the smallest closed set outside of which μ is zero.

Definition 1.15. When E is a closed subset of $\mathbb{R}^d$, the set of all bounded Radon measures on $\mathbb{R}^d$ with support in the set E is denoted by $M_b(E)$. The set of all bounded Radon measures on $\mathbb{R}^d$ is denoted by $M_b(\mathbb{R}^d)$.

The short-time Fourier transform of a function f with respect to g is given by

$$V_g f(y,\omega) = \int_{\mathbb{R}^d} f(x)\overline{g(x-y)}\; e^{-2\pi i x\cdot\omega}\; dx, \qquad \text{for } y,\omega \in \mathbb{R}^d.$$

In terms of the Fourier transform with respect to the x variable, we can write the above equation as

$$V_g f(y,\omega) = \mathcal{F}_x(f(x)\overline{g(x-y)})(\omega).$$

Lemma 1.1. *If $f \in L^2(\mathbb{R}^d)$ and $g \in L^2(\mathbb{R}^d)$, then*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |V_g f(y,\omega)|^2\; d\omega\; dy = \|f\|_2 \cdot \|g\|_2.$$

Proof.

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |V_g f(y,\omega)|^2\; d\omega\; dy \\
=\;& \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V_g f(y,\omega)\;\overline{V_g f(y,\omega)}\; d\omega\; dy \\
=\;& \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \mathcal{F}_x(f(x)\;\overline{g(x-y)})(\omega)\;\overline{\mathcal{F}_x(f(x)\;\overline{g(x-y)})(\omega)}\; d\omega\right)\; dy \\
=\;& \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f(x)\;\overline{g(x-y)}\;\overline{f(x)}\; g(x-y)\; dx\right)\; dy; \qquad \text{by Plancherel Theorem} \\
=\;& \int_{\mathbb{R}^d} f(x)\;\overline{f(x)}\left(\int_{\mathbb{R}^d} \overline{g(x-y)}\; g(x-y)\; dx\right)\; dy; \qquad |f(x)|^2 \in L^1(\mathbb{R}^d) \\
=\;& \int_{\mathbb{R}^d} f(x)\;\overline{f(x)}\left(\int_{\mathbb{R}^d} \overline{g(x-y)}\; g(x-y)\; dy\right)\; dx; \qquad \text{by Fubini Theorem} \\
=\;& \int_{\mathbb{R}^d} |f(x)|^2\left(\int_{\mathbb{R}^d} |g(y)|^2\; dy\right)\; dx \\
=\;& \|f\|_2 \cdot \|g\|_2
\end{aligned}$$

□

We can recover the original function from its short-time Fourier transform using the formula

$$f(x) = \frac{1}{\|g\|_2^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V_g f(y,\omega)\; g(x-y)\; e^{2\pi i y\cdot\omega} d\omega\; dy.$$

Theorem 1.5. *Let $g \in L^2(\mathbb{R}^d)$. Let $K_n = [-n,n]^{2d}$ or the closed ball $\bar{B}(0,n)$. Let*

$$f_n(x) = \frac{1}{\|g\|_2} \int\int_{K_n} V_g f(y,\omega)\ g(x-y)\ e^{2\pi i\omega\cdot x}\ dy\ d\omega.$$

Then

$$\lim_{n\to\infty} \|f - f_n\|_2 = 0.$$

Proof. Let $h \in L^2(\mathbb{R}^d)$.

$$\begin{aligned}
|\langle f_n, h\rangle| &= \frac{1}{\|g\|_2} \left| \int\int_{K_n} V_g f(y,\omega) \int_{\mathbb{R}^d} \overline{h(x)}\ g(x-y)\ e^{2\pi i\omega\cdot x}\ dx\ dy\ d\omega \right| \\
&= \frac{1}{\|g\|_2} \left| \int\int_{K_n} V_g f(y,\omega)\ \overline{V_g h(y,\omega)}\ dy\ d\omega \right| \\
&\leq \frac{1}{\|g\|_2^2} \|V_g f\|_{L^2(\mathbb{R}^{2d})}\ \|V_g h\|_{L^2(\mathbb{R}^{2d})} \\
&= \frac{1}{\|g\|_2^2} \|g\|_2 \cdot \|f\|_2 \cdot \|g\|_2 \cdot \|h\|_2 \\
\implies \|f_n\|_2 &\leq \|f\|_2
\end{aligned}$$

In particular, for each n, we have $f_n \in L^2(\mathbb{R}^d)$.

$$\begin{aligned}
|\langle f - f_n, h\rangle| &= \frac{1}{\|g\|_2^2} \left| \left(\int\int_{\mathbb{R}^{2d}} - \int\int_{K_n}\right) V_g f(y,\omega)\ \overline{V_g h(y,\omega)}\ dy\ d\omega \right| \\
&= \frac{1}{\|g\|_2^2} \left| \left(\int\int_{K_n^c} V_g f(y,\omega)\ \overline{V_g h(y,\omega)}\ dy\ d\omega\right) \right. \\
&\leq \frac{1}{\|g\|_2^2} \|V_g h\|_2 \cdot \left(\int\int_{K_n^c} |V_g f(y,\omega)|^2\ dy\ d\omega\right)^{1/2} \\
&= \frac{1}{\|g\|_2^2} \cdot \|g\|_2^2 \cdot \|h\|_2^2 \left(\int\int_{K_n^c} |V_g f(y,\omega)|^2\ dy\ d\omega\right)^{1/2}
\end{aligned}$$

This is true for all $h \in L^2(\mathbb{R}^d)$, so

$$\|f - f_n\|_2 = \sup_{\|h\|_2 = 1} \{|\langle f - f_n, h\rangle|\} \leq \frac{1}{\|g\|_2^2} \cdot \|h\|_2 \left(\int\int_{K_n^c} |V_g f(y,\omega)|^2\ dy\ d\omega\right)^{1/2}.$$

Since $V_g f \in L^2(\mathbb{R}^d)$, and $|K_n^c| \to 0$ as $n \to \infty$, hence $\|f - f_n\|_2 = 0$ as $n \to \infty$.

□

The Fourier transform is initially defined for an integrable function. It is significant that we can extend the Fourier transform to a larger class of objects to include measures and any bounded functions. If f is a bounded function on $\mathbb{R}$, the Fourier transform of f exists in the sense of distributions. We now discuss some basic facts about the theory of distributions. A more comprehensive treatment can be found in Benedetto [Ben97], Hörmander [Hor90], or Strichartz [Str94].

Definition 1.16. The space of infinitely differentiable complex-valued functions on $\mathbb{R}$ is denoted by $C^\infty(\mathbb{R})$.

$$C_c^\infty(\mathbb{R}) = \{\phi : \phi \in C^\infty(\mathbb{R}) \text{ and supp } \phi \text{ is compact.}\}$$

$C_c^\infty(\mathbb{R})$ is a vector space.

Definition 1.17. (Distribution) A linear mapping,

$$\begin{aligned} T : C_c^\infty(\mathbb{R}) &\rightarrow \mathbb{C} \\ \phi &\rightarrow T(\phi) \end{aligned}$$

is a distribution if $\lim_{n\rightarrow\infty} T(\phi_n) = 0$ for every sequence $\{\phi_n\} \subseteq C_c^\infty(\mathbb{R})$ satisfying the following properties

- $\exists\ K \subseteq \mathbb{R}$, a compact set, such that $\forall\ n, \text{supp } \phi_n \subseteq K$
- $\forall\ j \geq 0, \lim_{n\rightarrow\infty} \|\phi_n^{(j)}\|_{L^\infty(\mathbb{R})} = 0.$

The space of all distributions on $\mathbb{R}$ is denoted by $D'(\mathbb{R})$.

Let $f, \phi, \in L^2(\mathbb{R})$. By Plancherel Theorem, we have

$$\int_{\mathbb{R}} f(x)\overline{\phi(x)}\, dx = \int_{\widehat{\mathbb{R}}} \hat{f}(\zeta)\overline{\hat{\phi}(\zeta)}\, d\zeta.$$

If we treat the function ϕ above as the test function, we can rewrite this as

$$\begin{aligned} \hat{f}(\overline{\hat{\phi}}) &= f(\overline{\phi}) \\ \text{or} \quad T_{\hat{f}}(\overline{\hat{\phi}}) &= T_f(\overline{\phi}) \\ \text{or} \quad \langle T_{\hat{f}}, \hat{\phi}\rangle &= \langle T_f, \phi\rangle. \end{aligned}$$

Thus we are motivated to define formally the Fourier transform of a distribution T so that the Plancherel Theorem will hold. We can formally define the Fourier transform $\hat{T}$ of a distribution T by the equation

$$\hat{T}(\overline{\hat{\phi}}) = T(\bar{\phi})$$

or equivalently

$$\langle \hat{T}, \hat{\phi}\rangle = \langle T, \phi\rangle.$$

In order for this definition to make sense, we have to specify the class of test functions ϕ. In particular, we have to know that $\hat{\phi}$ is defined. This leads to the Schwartz space of test functions and the space of tempered distributions.

Definition 1.18. (Schwartz space) An infinitely differentiable function $\phi : \mathbb{R} \to \mathbb{C}$ is an element of the Schwartz space $\mathcal{S}(\mathbb{R})$ if

$$\forall\ n = 0, 1, 2, \ldots \qquad \sup_{0\le j\le n}\ \sup_{t\in\mathbb{R}} (1 + |t|^2)^n |\phi^{(j)}(t)| < \infty.$$

An important property of the Schwartz space is that the Fourier transform is a bijection from the Schwartz space to itself.

Definition 1.19. (Tempered distribution) A linear functional,

$$\begin{aligned} T : \mathcal{S}(\mathbb{R}) &\to \mathbb{C} \\ \phi &\to T(\phi) \end{aligned}$$

is a tempered distribution if $\lim_{n\to\infty} T(\phi_n) = 0$ for every sequence $\{\phi_n\} \subseteq \mathcal{S}(\mathbb{R})$ whenever

$$\forall\ j, m \geq 0, \quad \lim_{n\to\infty} \|t^m \phi_n^{(j)}\|_{L^\infty(\mathbb{R})} = 0.$$

The space of all tempered distributions on $\mathbb{R}$ is denoted by $\mathcal{S}'(\mathbb{R})$.

Definition 1.20. The Fourier transform $\hat{T}$ of $T \in \mathcal{S}'(\mathbb{R})$ is defined by

$$\forall\ \phi \in \mathcal{S}(\mathbb{R}), \quad \langle \hat{T}, \phi \rangle = \langle T, \hat{\phi} \rangle.$$

Let us verify that if $f \in L^1(\mathbb{R})$, then the Fourier transform of f is consistent with the new definition above, when we view f as a distribution. Let $f \in L^1(\mathbb{R})$, and let $\phi \in \mathcal{S}(\mathbb{R})$. Then $\hat{f}$ is a continuous function, and in the sense of distributions we have

$$\begin{aligned} \langle \hat{f}, \phi \rangle &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x) e^{-2\pi i x \zeta}\ dx \right) \phi(\zeta)\ d\zeta = \int_{\mathbb{R}} f(x) \int_{\mathbb{R}} \phi(\zeta) e^{-2\pi i x \zeta}\ d\zeta\ dx \\ &= \int_{\mathbb{R}} f(x) \hat{\phi}(x)\ dx = \langle f, \hat{\phi} \rangle. \end{aligned}$$

We have shown the Fourier transform of f exists in the sense of distributions.

□

Chapter 2

Theory of Frames

The theory of frames were introduced by Duffin and Schaeffer [DS52] in their seminal paper to deal with problems in nonharmonic Fourier series. Other excellent sources of references include the monograph of Dauberchies [Dau92], the fundamental paper "Painless nonorthogonal expansions" by Daubechies , Grossmann and Meyer [IDM86], the book on nonharmonic Fourier series by Robert Young [You01], the book on time-frequency analysis by Gröchenig [Gro00], the book on frames by Christensen [Chr03], the exposition by Benedetto and Walnut [BW94]. The monograph on frames and group representation by D. Han and David Larson [HL00] gives a slightly more advanced treatment. We pay tribute to all these authors here, otherwise our subsequent citations will be too repetitious.

2.1 The adjoint operator

Definition 2.1. Suppose H_1 and H_2 are Hilbert spaces. We say that b is a conjugate-bilinear functional on $H_1 \times H_2$, if b is a complex-valued functional on $H_1 \times H_2$ that is linear in the first variable and conjugate-linear in the second variable. We say that such a function b is bounded if there exists a real number c such that $|b(x, y)| \leq c\, \|x\|\|y\|$ for all x in H_1, for all y in H_2.

Theorem 2.1. *[KR97] Let H_1 and H_2 be Hilbert spaces. Let $T : H_1 \to H_2$ be a bounded linear operator. Then the equation*

$$b_T(x, y) = \langle Tx, y \rangle \quad x \in H_1, y \in H_2$$

defines a bounded conjugate-linear functional b_T on $H_1 \times H_2$, and $\|b_T\| = \|T\|$. Each bounded conjugate-linear functional on $H_1 \times H_2$ arises in this way, which means there is a unique bounded linear operator $T : H_1 \to H_2$ such that $b_T(x, y) = \langle Tx, y \rangle$.

Theorem 2.2. *If S and T are bounded linear operators acting on a complex Hilbert space H and $\langle Sx, x \rangle = \langle Tx, x \rangle$ for each $x \in H$, then $S = T$.*

Proof. Since $\langle Sx, x \rangle = \langle Tx, x \rangle$ for each $x \in H$, the polarization identity

$$\begin{aligned} 4\,\langle Tx, y \rangle &= \langle T(x+y), x+y \rangle - \langle T(x-y), x-y \rangle \\ &+ i\langle T(x+iy), x+iy \rangle - i\langle T(x-iy), x-iy \rangle \end{aligned}$$

implies that $\langle Sx, y \rangle = \langle Tx, y \rangle$ for all $x \in H_1, y \in H_2$.

Hence S and T give rise to the same conjugate-bilinear functional on H, and the uniqueness clause in Theorem 2.1 implies $S = T$.

□

Theorem 2.3. *Let H_1, H_2 be Hilbert spaces. If $T : H_1 \to H_2$ is a bounded linear operator, then there exists a unique bounded linear operator $T^* : H_2 \to H_1$ such that*

$$\forall\ x \in H_1,\ \forall\ y \in H_2, \qquad \langle Tx, y\rangle = \langle x, T^* y\rangle.$$

The operator T^* is called the adjoint operator of T. It has the following properties:

1. $(\lambda T)^* = \bar{\lambda} T^*, \quad \forall\ \lambda \in \mathbb{C}$

2. $(T^*)^* = T$

3. $\|T^* T\| = \|T\|^2$

4. $\|T^*\| = \|T\|$

Theorem 2.4. *Let H_1 and H_2 be Hilbert spaces. Let $T : H_1 \to H_2$ be a bounded linear operator. If T is surjective, then TT^* is bijective.*

Proof. We write Ker T for the kernel of T and Im T for the image of T.

Im $T = H_2$ is closed, since T is surjective.

$$H_1 = \text{Ker } T \oplus \text{Im } T^*$$

$$H_2 = \text{Ker } T^* \oplus \text{Im } T$$

We want to prove that TT^* is bijective.

If $TT^* x = 0$ for some x in H_1, then $T^* x \in \text{Ker } T \in \text{Im } T^* = \{0\}$, hence $T^* x = 0$.

Now, $x \in \text{Ker } T^* = (\text{Im } T)^{\perp} = H_2^{\perp} = \{0\}$, so $x = 0$. This proves TT^* is injective.

Let $z \in H_2$. T is surjective, so $z = Ty$ for some $y \in H_1$.

There exist $y_1 \in \text{Ker } T$ and $z_1 \in H_2$ so that $y = y_1 \oplus T^* z_1$.

Then $z = Ty = T\left(y_1 \oplus T^* z_1\right) = TT^* z_1$. This proves TT^* is surjective.

□

Definition 2.2. A bounded linear operator T on a Hilbert space is a self-adjoint operator if $T^* = T$.

Definition 2.3. We say that T is a positive operator if for each x in H, $\langle Tx, x\rangle \geq 0$. We write $T \geq 0$ or $0 \leq T$ if T is a positve operator. If A and B are positive operators, we write $A \geq B$ if $A - B \geq 0$.

Notation: $\mathcal{B}(H)$ is the set of all bounded linear operators on Hilbert space H. We summarize some useful properties of positive operators. Their proofs can be found in [Sch02].

Lemma 2.1. *Let H be a complex Hilbert space.*

1. *All positive operators on H are self-adjoint.*

2. *Let λ be a positive number. Let I be the identity operator on H.*

   If $-\lambda I \leq T \leq \lambda I$, then $\|T\| \leq \lambda$.

3. *If $\{S_n\}$ is a sequence of operators in $\mathcal{B}(H)$ satisfying*

$$0 \leq S_n \leq S_{n+1} \leq I \quad \text{for } n = 0, 1, 2, ...$$

   then there is an operator S in $\mathcal{B}(H)$ such that

$$\forall\ x \in H, \quad S_n x \to Sx.$$

4. *If $S \geq 0, T \geq 0$ and $ST = TS$, then $ST \geq 0$.*

2.2 The frame operator

We now come to frames, the central character of our story.

Definition 2.4. (Frames) Let H be a separable Hilbert space. A sequence $\{x_n\}_{n=1}^{\infty}$ of elements in H is a frame if there are positive constants A and B such that,

$$\forall\ f \in H, \quad A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, x_n\rangle|^2 \leq B\|f\|^2.$$

The numbers A and B are called frame bounds. They are not unique. The optimal upper frame bound is the infimum over all upper frame bounds. The optimal lower frame bound is the supremum over all lower frame bounds. When $A = B$, we say that the frame is a tight frame or an A-tight frame.

Definition 2.5. A sequence $\{x_n\}_{n=1}^{\infty}$ of elements in H is a Bessel sequence with Bessel bound B if there exists a positive constant B such that

$$\forall\ f \in H, \quad \sum_{n=1}^{\infty} |\langle f, x_n\rangle|^2 \leq B\|f\|^2.$$

We define two operators that associate to each frame. These two operators are crucial in the theory of frames.

The synthesis operator $T : H \to l^2$ is defined by

$$\forall\ f \in H, \quad Tf = \{\langle f, x_n\rangle\}_{n=1}^{\infty}.$$

The adjoint of the synthesis operator is the analysis operator.

The analysis operator $T^* : l^2 \to H$ is given by

$$\forall\ c = \{c_n\} \in l^2, \quad T^*c = \sum_{n=1}^{\infty} c_n x_n.$$

Since a frame is a Bessel sequence, the synthesis operator T is a bounded operator,

$$\forall\ f \in H, \quad \sum_{n=1}^{\infty} |\langle f, x_n \rangle|^2 \leq B\|f\|^2, \qquad \text{so } \|T\| \leq B.$$

By Theorem 2.3, since $\|T^*\| = \|T\|$, the analysis operator is also a bounded operator and has the same operator norm as the synthesis operator.

Composing these two operators T^* and T, we obtain the frame operator

$$S : H \to H, \quad Sf = T^*Tf = \sum_{n=1}^{\infty} \langle f, x_n \rangle x_n.$$

The frame operator $S = T^*T$ is a postive and self-adjoint operator.

$$\forall\ f \in H, \quad \langle Sf, f \rangle = \sum_{n=1}^{\infty} |\langle f, x_n \rangle|^2.$$

It follows from the definition of a frame that

$$\forall\ f \in H, \quad A\|f\|^2 \leq \langle Sf, f \rangle \leq B\|f\|^2.$$

or

$$AI \leq S \leq BI.$$

Remark: The Bessel condition remains the same, regardless of how the terms in the sum are ordered. The series converges unconditionally, i.e. reordering the terms of the sum will not change the sum. We can therefore choose an arbitary indexing of the elements $\{x_n\}$. We have chosen to index the elements by positive integers for convenience only.

Recall that if a linear operator $L : H \to H$ is bounded and $\|I - L\| < 1$, then L is invertible, and

$$L^{-1} = \sum_{n=0}^{\infty} (I - L)^n, \quad \text{with} \quad \|L^{-1}\| \leq \frac{1}{1 - \|I - L\|}.$$

Lemma 2.2. *The frame operator S is invertible.*

Proof. Since $AI \leq S \leq BI$, we can write

$$0 \leq I - B^{-1}S \leq \frac{B-A}{B} I,$$

and therefore

$$\|I - B^{-1}S\| \;=\; \sup_{\|f\|=1} |\langle (I - B^{-1}S)f, f\rangle| \;\leq\; \frac{B-A}{B} \;< 1.$$

This shows that S is invertible.

□

Since S is a self-adjoint operator, S^{-1} is also a self-adjoint operator.

Theorem 2.5. *(Reconstruction formula)*

Let $\{x_n\}_{n=1}^{\infty}$ be a frame for H with frame operator S. Then

$$\forall\ f \in H, \quad f = \sum_{n=1}^{\infty} \langle f, x_n\rangle S^{-1}x_n = \sum_{n=1}^{\infty} \langle f, S^{-1}x_n\rangle x_n$$

Proof.

$$\forall\ f \in H, \quad f = S^{-1}Sf = S^{-1} \sum_{n=1}^{\infty} \langle f, x_n\rangle x_n = \sum_{n=1}^{\infty} \langle f, x_n\rangle S^{-1}x_n,$$

and

$$\forall\ f \in H, \quad f = SS^{-1}f = \sum_{n=1}^{\infty} \langle S^{-1}f, x_n\rangle x_n = \sum_{n=1}^{\infty} \langle f, S^{-1}x_n\rangle x_n.$$

□

Definition 2.6. If $\{x_n\}_{n=1}^{\infty}$ is a frame and S is the frame operator, than $\{S^{-1}x_n\}_{n=1}^{\infty}$ is called the canonical dual frame of $\{x_n\}_{n=1}^{\infty}$.

Example 1. In $\mathbb{C}^3$, we have 4 vectors written in row vectors as

$e_1 = [1, 0, 0],\ e_2 = [0, 1, 0],\ e_3 = [0, 0, 1],\ e_4 = [1, 1, 1].$

To determine the frame operator, let $f = [a, b, c]$ be an arbitrary vector in C^3.

Then $\langle f, e_1\rangle = a,\ \langle f, e_2\rangle = b,\ \langle f, e_3\rangle = c,\ \langle f, e_4\rangle = a + b + c.$

$$Sf = \sum_{n=1}^{4} \langle f, e_n\rangle e_n = [a, b, c] + (a+b+c)\cdot[1, 1, 1] = [2a+b+c, a+2b+c, a+b+2c].$$

The frame operator S maps $[a, b, c]$ to $[2a + b + c, a + 2b + c, a + b + 2c]$.

$$Sf = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a+b+c \\ a+2b+c \\ a+b+2c \end{bmatrix}$$

With the frame operator as a 3×3 matrix, we can compute its inverse.

$$S^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

With the inverse S^{-1} as a 3×3 matrix, we can calculate $S^{-1}e_1$.

$$S^{-1}\ e_1 = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \end{bmatrix}$$

Similarly, we can calculate $S^{-1}e_2, S^{-1}e_3$, and $S^{-1}e_4$.

$$S^{-1}\ e_2 = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ \frac{3}{4} \\ -\frac{1}{4} \end{bmatrix}$$

$$S^{-1}\ e_3 = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ \frac{3}{4} \end{bmatrix}$$

$$S^{-1}\ e_4 = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \end{bmatrix}$$

Finally, we can use the reconstruction formula to recover the original vector f from the values of $\{\langle f, e_n\rangle : n = 1, 2, 3, 4\}$.

$$f = S^{-1}Sf = S^{-1}\left(\sum_{n=1}^{4}\langle f, e_n\rangle e_n\right) = \sum_{n=1}^{4}\langle f, e_n\rangle S^{-1}e_n.$$

$$\sum_{n=1}^{4}\langle f, e_n\rangle S^{-1}e_n = a\begin{bmatrix} \frac{3}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \end{bmatrix} + b\begin{bmatrix} -\frac{1}{4} \\ \frac{3}{4} \\ -\frac{1}{4} \end{bmatrix} + c\begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ \frac{3}{4} \end{bmatrix} + (a+b+c)\begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

The resulting factor is equal to $[a, b, c]$ as to be expected, since $f = [a, b, c]$.

Conclusion: The reconstruction formula works.

$\square$

In the last example, the 4 vectors form a frame for $\mathbb{C}^3$. An orthonormal basis with 3 vectors can span the same space.

What have we gained from the redundancy? Suppose Alice wanted to send a message to Bob. This message was encoded as a vector in $\mathbb{C}^3$ as $f = [a, b, c]$. Alice wanted to sent the 4 values of $\{\langle f, e_n \rangle\}_{n=1}^4$. During the transmission of the message, one of values was corrupted.

Instead of $\{a, b, c, a + b + c\}$, Bob received $\{a + \epsilon, b, c, a + b + c\}$. Using the same calculation and the reconstruction algorithm, Bob calculated

$$g = (a+\epsilon)\begin{bmatrix} \frac{3}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \end{bmatrix} + b\begin{bmatrix} -\frac{1}{4} \\ \frac{3}{4} \\ -\frac{1}{4} \end{bmatrix} + c\begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ \frac{3}{4} \end{bmatrix} + (a+b+c)\begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

The message that Bob received is g, which is not the same as f. The difference between f and g is the vector $[\frac{3}{4}\epsilon, \frac{-1}{4}\epsilon, \frac{-1}{4}\epsilon]$.

We calculate the reconstruction error by $\|f - g\|_2^2$.

$$\|f - g\|_2^2 = \epsilon^2 \times \left((\frac{3}{4})^2 + (\frac{1}{4})^2 + (\frac{1}{4})^2 \right) = \frac{11}{16}\,\epsilon^2.$$

This is strictly less than ϵ^2.

Example 2. Let $\{f_n\}$ and $\{g_n\}$ be a pair of sequenes on the Hilbert space H. We say that $\{f_n\}$ has the reconstruction property if for each f in H,

$$f = \sum_{n=1}^{\infty} \langle f, g_n \rangle f_n.$$

If $\{e_n\}$ is a frame for H and $\widetilde{e}_n = S^{-1}e_n$ then by the reconstruction formula,

$$f = S(S^{-1}f) = \sum_{n=1}^{\infty} \langle S^{-1}f, e_n \rangle e_n = \sum_{n=1}^{\infty} \langle f, S^{-1}e_n \rangle e_n = \sum_{n=1}^{\infty} \langle f, \widetilde{e}_n \rangle e_n.$$

So a frame has the reconstruction property. What about the converse: If $\{f_n\}$ is a sequence with the reconstruction property, must $\{f_n\}$ be a frame? This example illustrates the answer is no.

Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for H. Define $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ by

$$f_{2n} = e_n, \quad f_{2n-1} = e_1, \quad g_{2n} = e_n, \quad g_{2n+1} = e_{n+1} - e_n, \quad g_1 = e_1.$$

Let $f \in H$,

$$\sum_{n=1}^{\infty} \langle f, g_{2n} \rangle f_{2n} = \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n = f$$

$$\sum_{n=1}^{N} \langle f, g_{2n+1} \rangle f_{2n+1} = \langle f, e_1 \rangle e_1 + \sum_{n=1}^{N} \langle e_{n+1} - e_n, f \rangle e_1 = \langle e_{N+1}, f \rangle e_1.$$

Since $\lim_{N\to\infty} \langle e_{N+1}, f \rangle = 0$, we have $\sum_{n=1}^{\infty} \langle f, g_{2n+1} \rangle f_{2n+1} = 0$.

Hence, for all $f \in H, f = \sum_{n=1}^{\infty} \langle f, g_n \rangle f_n$. The reconstruction property holds.

But $\{f_n\}$ is not a frame. It is not even a Bessel sequence. Let $f = e_1$. Then

$$\sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 = \sum_{j=1}^{\infty} |\langle e_1, e_1 \rangle|^2 = \infty.$$

□

The elegant proof of the next theorem is due to Toda [Tod11].

Theorem 2.6. *Let $A, B : H \to H$ be bijective, self-adjoint and positive operators. If $A \geq B$, then $B^{-1} \geq A^{-1}$.*

Proof. Since $B \geq 0$, so for all $x, y \in H$, we have

$$\begin{aligned} 0 &\leq \langle y - B^{-1}x, B(y - B^{-1}x)\rangle \\ &= \langle y, By\rangle - \langle y, x\rangle - \langle B^{-1}x, By\rangle + \langle B^{-1}x, x\rangle \\ &= \langle y, By\rangle - 2\ \mathrm{Re}\langle x, y\rangle + \langle x, B^{-1}x\rangle \end{aligned}$$

Hence, $2\ \mathrm{Re}\langle x, y\rangle - \langle y, By\rangle \leq \langle x, B^{-1}x\rangle$.

Since $A \geq B$, this implies that

$$2\ \mathrm{Re}\langle x, y\rangle - \langle y, Ay\rangle \leq 2\ \mathrm{Re}\langle x, y\rangle - \langle y, By\rangle \leq \langle x, B^{-1}x\rangle.$$

Let $y = A^{-1}x$ in the leftmost expression of the above equation. We have

$$\begin{aligned} & 2\ \mathrm{Re}\langle x, A^{-1}x\rangle - \langle A^{-1}x, AA^{-1}x\rangle \\ =\ & 2\ \mathrm{Re}\langle x, A^{-1}x\rangle - \langle A^{-1}x, x\rangle \\ =\ & 2\ \langle x, A^{-1}x\rangle - \langle x, A^{-1}x\rangle, \quad A^{-1} \text{ is self-adjoint.} \end{aligned}$$

Therefore

$$\langle x, A^{-1}x\rangle \leq \langle x, B^{-1}x\rangle \quad \text{where } A^{-1} \text{ is self-adjoint.}$$

Since $x \in H$ is arbitary, we obtain $B^{-1} \geq A^{-1}$.

□

The canonical dual frame of a frame is also a frame.

Theorem 2.7. *Let* $\{x_n\}_{n=1}^{\infty}$ *be a frame for* H *with frame operator* S *and frame bounds* A *and* B*. Then* $\{S^{-1}x_n\}_{n=1}^{\infty}$ *is a frame with frame operator* S^{-1} *and frame bounds* B^{-1} *and* A^{-1}*.*

Proof. The frame operator S is invertible by Lemma 2.2. Since the operator S is self-adjoint, S^{-1} is also self-adjoint. For each $f \in H$,

$$\begin{aligned}\sum_{n=1}^{\infty} |\langle f, S^{-1}x_n\rangle|^2 &= \sum_{n=1}^{\infty} |\langle S^{-1}f, x_n\rangle|^2 \\ &\leq B\,\|S^{-1}f\|^2 \leq B\,\|S^{-1}\|_{op}^2\,\|f\|^2\end{aligned}$$

That means $\{S^{-1}x_n\}_{n=1}^{\infty}$ is a Bessel sequence. Hence the frame operator for the sequence $\{S^{-1}x_n\}_{n=1}^{\infty}$ is a bounded operator. This operator acts on $f \in H$ by

$$\begin{aligned}\sum_{n=1}^{\infty}\langle f, S^{-1}x_n\rangle S^{-1}x_n &= S^{-1}\sum_{n=1}^{\infty}\langle S^{-1}f, x_n\rangle x_n \\ &= S^{-1}S\left(S^{-1}f\right) = S^{-1}f. \qquad (*)\end{aligned}$$

We have shown that the frame operator for $\{S^{-1}x_n\}$ is S^{-1}. Since the frame operator S is invertible, it is bijective. We can apply Theorem 2.6 and obtain:

$$B^{-1}I \leq S^{-1} \leq A^{-1}I.$$

This means

$$B^{-1}\|f\|^2 \leq \langle S^{-1}f, f\rangle \leq A^{-1}\|f\|^2$$

By $(*)$, we obtain

$$B^{-1}\|f\|^2 \leq \sum_{n=1}^{\infty}|\langle f, S^{-1}x_n\rangle|^2 \leq A^{-1}\|f\|^2.$$

□

2.3 Fourier frames and density conditions

Definition 2.7. A sequence $E = \{t_n\} \subseteq \mathbb{R}^d$ is separated if

$$\inf\{|t_m - t_n| : m \neq n\} = \delta > 0.$$

The constant δ is called the separation constant.

Definition 2.8. Let E be a separated sequence. It has a uniform density D if there exists a number L such that, for all integers n,

$$|t_n - \frac{n}{D}| \leq L.$$

.

Duffin and Schaeffer proved the following result in their seminal paper.

Theorem 2.8. *Let $E = \{t_n : n \in \mathbb{Z}\}$ be a separated sequence of uniform density D, then $\{e^{-2\pi i t_n x} : n \in \mathbb{Z}\}$ is a frame for $L^2([\Omega/2, \Omega/2])$, where $0 < \Omega < D$.*

Definition 2.9. Let $n^+(r), n^-(r)$ denote respectively the largest and the smallest number of elements of E to be found in any interval of length r. Define

$$D^+(E) = \limsup_{r\to\infty} \frac{n^+(r)}{r},$$

and

$$D^-(E) = \liminf_{r\to\infty} \frac{n^+(r)}{r}.$$

We call $D^+(E)$ and $D^-(E)$ the Beurling upper and lower densities of E, respectively.

If a sequence is separated with separation constant δ, then there are at most $1 + \lfloor \delta \rfloor$ elements in any interval of length one. If a sequence is a finite union of separated sequences, then by similar reasoning, the number of elements of in any interval of length one must be uniformly bounded. The next theorem of Jaffard [Jaf91] tells us that sometimes the lim inf in the definition of the Beurling lower density can be replaced by a limit.

Theorem 2.9. *(Jaffard) If E is a finite union of separated sequences, then the limit*

$$D^-(E) = \lim_{r\to\infty} \frac{n^-(r)}{r}$$

exists. The limit is the Beurling lower density of E.

Proof. Define the function $Q(r) = n^-(r)/r$, for $r \geq 1$. Since E is a finite union of separated sequences, the number of elements of E in any interval of length 1 is uniformly bounded. There exists a constant C such that $n^-(r) \leq Cr$. Thus Q is a bounded function.

Let p be an integer. Let I be an interval of length pr. Write I as a disjoint union of p intervals $I_1, ..., I_p$, where each length has length r. For each k, the number of elements of $(E \cap I_k)$ is at least $n^-(r)$, so that the number of elements of $(E \cap I)$ is at least $pn^-(r)$. Thus

$$Q(pr) \geq Q(r).$$

Let $\alpha > 1$. Since $n^-(\alpha r) \geq n^-(r)$, so

$$Q(\alpha r) \geq \frac{1}{\alpha} Q(r).$$

Let

$$\bar{Q} = \sup_{r\geq 1} Q(r).$$

Given ϵ in $(0, \frac{1}{2})$, choose $a > 0$ and a positive integer n so that $Q(a) \geq \bar{Q} - \epsilon$ and

$$\frac{n+1}{n} \geq \frac{1}{1-\epsilon}.$$

Let $x \geq na$. There exists an integer p at least equal to n such that

$$pa \leq x < (p+1)a.$$

Then

$$\begin{aligned} Q(x) &= Q(\frac{x}{pa}pa) \geq \frac{pa}{x}Q(pa) \\ &\geq \frac{pa}{x}(\bar{Q} - \epsilon) \geq \frac{pa}{(p+1)a}(\bar{Q} - \epsilon) \\ &\geq \frac{n}{n+1}(\bar{Q} - \epsilon) \geq (1-\epsilon)(\bar{Q} - \epsilon). \end{aligned}$$

Since this is true for all $x \geq na$, hence the limit of $Q(x)$ as $|x| \to \infty$ exists.

□

The following characterization of Fourier frames in $\mathbb{R}$ is also due to Jaffard. Recall that for a closed set Λ, the Paley-Wiener space $PW(\Lambda)$ is the set of all functions in $L^2(\mathbb{R})$ whose Fourier transforms vanish outside the set Λ.

Theorem 2.10. *(Jaffard) The set $\{e^{-2\pi i t_n x} : n \in \mathbb{Z}\}$ is a Fourier frame for $PW([-\Omega, \Omega])$ for some $\Omega > 0$, if and only if $E = \{t_n : n \in \mathbb{Z}\}$ is a finite union of separated sequences and at least one of the sequences is uniformly dense.*

This is the qualitative version. A more quantitative version of the theorem will be stated shortly. But first, two more definitions.

Definition 2.10. Let $U(E)$ be the collection of all the subsequences of E with a uniform density. The the frame density of E is defined by

$$D^f(E) = \sup_{\Theta \in U(E)} D(\Theta).$$

This definition calls for some explanation. Each element of the set $U(E)$ is a subsequence of E. Among all the subsequences of E that are uniformly dense, we determine the uniform density for each of them, then take the supremum of these numbers. This supremum is the frame density.

Definition 2.11. Let E be a sequence of distinct real numbers. The frame radius of E is the upper bound of all numbers R such that $\{e^{it_n x}\}$ is a frame for $L^2[-R, R]$.

Theorem 2.11. *(Jaffard) Let E be a sequence of distinct real numbers. If there is at least one subsequene that is uniformly dense, and if the number of t_n in any interval of length 1 is uniformly bounded by a constant, then the frame radius is equal to $\pi D^f(E)$. Otherwise, $\{e^{it_n x}\}$ is not a Fourier frame for $L^2([-R, R])$ for any R.*

It is worth emphasizing that in the following theorem of Landau, the set Λ can be a union of several intervals, or Λ can be a Cantor set of positive measure.

Theorem 2.12. *(Landau) Let Λ be any closed set in $\mathbb{R}$. Let $E = \{t_n\}$ be a separated sequence in $\mathbb{R}$. Then a necessary condition for $\{e^{-2\pi i t_n x} : n \in \mathbb{Z}\}$ to be a Fourier frame for $PW(\Lambda)$ is that $D^-(E) \geq m(\Lambda)$. Here, m is the Lebesgue measure.*

Jaffard's theorem is in $\mathbb{R}$ and Landau's theorem remains valid in $\mathbb{R}^d$. Before we state the higher dimensional analogue, we need to extend the definition of Beurling's upper and lower densities to $\mathbb{R}^d$. In $\mathbb{R}$, we can measure the density of a separated sequence in terms of the functions $n^+(rI)$ and $n^-(rI)$, with I the unit interval. In $\mathbb{R}^d$, there is an additional degree of freedom: the unit interval I is replaced by a set of Lebesgue measure 1.

Definition 2.12. Let $n^+(rI), n^-(rI)$ denote respectively the largest and the smallest number of elements of E to be found in any translate of rI, where I is a set of Lebesgue measure 1 . Define

$$D^+(E) = \limsup_{r\to\infty} \frac{n^+(rI)}{r^d},$$

and

$$D^-(E) = \liminf_{r\to\infty} \frac{n^+(rI)}{r^d}.$$

Lemma 2.3. *(Landau) Let $E = \{t_n : n \in \mathbb{Z}\}$ be a separated sequence in $\mathbb{R}^d$. Let Q be the unit cube with sides parallel to the coordinate axis and with center at the origin. If I is a set of measure 1 whose boundary has measure 0, then in the definition of Beurling lower and upper densities, either I or Q can be used (and the definitions will remain the same).*

We can now state the fundamental result of Landau for Fourier frames in $\mathbb{R}^d$.

Theorem 2.13. *(Landau) Let Λ be any closed set in $\mathbb{R}^d$. Let $E = \{t_n\}$ be a separated sequence in $\mathbb{R}^d$. Then a necessary condition for $\{e^{-2\pi i t_n x} : n \in \mathbb{Z}\}$ to be a Fourier frame for $PW(\Lambda)$ is that $D^-(E) \geq m(\Lambda)$. Here, m is the Lebesgue measure.*

The proof of the following theorem is essentially due to Landau. It does not depend on the Phragmén-Lindelöff Theorem.

Theorem 2.14. *(Polya-Plancherel) Let E be a separated sequence in $\mathbb{R}$. That means $\exists\ \delta > 0,$ such that $\inf\{|t_m - t_n| : m \neq n\} = \delta$. Then there exists $B > 0$ such that $\forall f \in PW(\Omega)$; where $\Omega = [-\tau, \tau]$, we have*

$$\sum_k |f(t_k)|^2 \leq B \int_{\mathbb{R}} |f(x)|^2 dx.$$

Here, B depends on δ and τ, but not on f.

Proof. Let $h \in L^2(\mathbb{R})$ so that h vanishes outside $B(0, \frac{\delta}{2})$ and $|\hat{h}(\zeta)| \geq 1$ for all $\zeta \in \Omega$.

Given $f \in PW(\Omega)$, construct $g \in L^2(\mathbb{R})$ such that $\hat{g}(\zeta) = \hat{f}(\zeta)/\hat{h}(\zeta)$.

Since $\hat{f}(\zeta) = 0$ when $\zeta \notin \Omega$, so $\hat{g}(\zeta) = 0$ when $\zeta \notin \Omega$, so $g \in PW(\Omega)$.

$\hat{f} \ = \ \hat{g} \cdot \hat{h} \quad \Longrightarrow f = g * h.$

$$\begin{aligned}
\Longrightarrow\ f(x) \ &= \ \int_{\mathbb{R}} g(y)\ h(x-y)\ dy \\
&= \ \int_{|x-y|<\delta/2} g(y)\ h(x-y)\ dy. \\
\Longrightarrow\ |f(x)|^2 \ &\leq \ \left(\int_{|x-y|<\delta/2} |g(y)| \cdot |h(x-y)|\ dy\right)^2 \\
&\leq \ \left(\int_{|x-y|<\delta/2} |g(y)|^2\ dy\right) \cdot \|h\|_2^2 \qquad \text{by Cauchy-Schwarz} \\
\Longrightarrow |f(t_k)|^2 \ &\leq \ \|h\|_2^2 \cdot \int_{|y-t_k|<\delta/2} |g(y)|^2\ dy.
\end{aligned}$$

Since $|t_j - t_k| \geq \delta$ for all $j \neq k,$ we get $\quad \sum_k |f(t_k)|^2 \leq \|h\|_2^2 \cdot \int_{\mathbb{R}} |g(y)|^2\ dy.$

But $|\hat{h}(\zeta)| \geq 1 \ \Rightarrow |\hat{g}(\zeta)| \leq |\hat{f}(\zeta)| \ \Rightarrow \|\hat{g}\|_2 \ \leq \|\hat{f}\|_2 \ \Rightarrow \|g\|_2 \ \leq \|f\|_2$. Hence,

$$\sum_k |f(t_k)|^2 \leq \|h\|_2^2 \int_{\mathbb{R}} |f(x)|^2\ dx.$$

$\square$

It is natural to consider the set $\mathcal{E}(E) = \{e^{it_n x}\}$ and try to determine whether it is a frame based on its Beurling upper and lower densities. The situation is elegantly summarized by Seip [Sei95] in the following two theorems.

Definition 2.13. We say that a sequence E is relatively separated if it is a finite union of separated sequences.

Recall that a sequence $\{f_n\}$ is a Riesz sequence if it is a Riesz basis for the closure of the space spanned by $\{f_n\}$.

Theorem 2.15. *For the system $\{e^{it_n x}\}$ to be a frame in $L^2(-\pi, \pi)$, it is necessary that E be relatively separated and $D^-(E) \geq 1$, and it is sufficient that E be relatively separated and $D^-(E) > 1$.*

Theorem 2.16. *For the system $\{e^{it_n x}\}$ to be a Riesz sequence in $L^2(-\pi, \pi)$, it is necessary that E be relatively separated and $D^+(E) \leq 1$, and it is sufficient that E be relatively separated and $D^+(E) < 1$.*

2.4 Frames of translates

Translates of a single function play an important role in the theory of frames, in time-frequency analysis, and in sampling theory. Consider the periodic function

$$G(\zeta) \equiv \sum_{n \in \mathbb{Z}^d} |\hat{g}(\zeta - n)|^2.$$

The following theorem of Benedetto and Li [BL98] uses the periodic function G to characterize when a sequence based on the translations of a function generates a frame. A significant feature of this result is the function G is allowed to have a zero set of positive Lebesgue measure.

Theorem 2.17. *(Benedetto and Li) Let $g \in L^2(\mathbb{R}^d)$.*
Let $V = \overline{Span}\{g(x-n) : n \in \mathbb{Z}^d\}$ be a closed subspace of $L^2(\mathbb{R}^d)$. Then the sequence $\{g(x-n) : n \in \mathbb{Z}^d\}$ is a frame for V with frame bounds A and B if and only if there are positive constants A and B such that

$$A \leq G(\zeta) \leq B \quad \text{a.e. on } [0,1]^d \setminus N, \text{ where } N \equiv \{\zeta \in [0,1]^d : G(\zeta) = 0\}.$$

The sequence $\{g(x-n) : n \in \mathbb{Z}^d\}$ is an orthonormal sequence if and only if

$$G(\zeta) = 1 \quad \text{a.e.}$$

.

If we can use a single function and its translations to generate a frame, then it is natural to consider a finite or countable collection of functions $\{g_m\}$ and consider the translation-invariant system $\{g_m(x - na) : m, n \in \mathbb{Z}\}$. If such a system is a Bessel sequence, the following result of Janssen [Jan98] shows us how to estimate the upper frame bound.

Theorem 2.18. *Let $\{g_m : n \in \mathbb{Z}\}$ be a collection of functions in $L^2(\mathbb{R})$. Let $a > 0$. Consider the collection of functions $\{g_m(x - na) : m, n \in \mathbb{Z}\}$. Let $\{g_{mn}\} = \{T_{na} g_m\}$, where $(T_a f)(x) = f(x - a)$. Suppose $\{g_{mn}\}$ is a Bessel sequence with bound B. Then*

$$\sum_{m\in\mathbb{Z}} |\hat{g}_m(\zeta)|^2 \leq aB. \quad a.e.\ \zeta \in \mathbb{R}.$$

Proof. Let $f \in L^2(\mathbb{R})$. Let

$$\begin{aligned} G(x) &= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |\langle T_x f, g_{mn}\rangle|^2. \\ \forall\ x \in \mathbb{R}, \quad G(x) &\leq B\ \|T_x f\|^2 = B\ \|f\|^2. \end{aligned}$$

G is periodic, so G has a Fourier expansion on $L^2[0, a)$, with Fourier coefficient

$$c_k = \frac{1}{a}\int_0^a G(x)\ e^{2\pi i k x}\ dx.$$

In particular, the Fourier coefficient c_0 of G is:

$$\begin{aligned} c_0 &= \frac{1}{a}\int_0^a \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |\langle T_x f, g_{mn}\rangle|^2\ dx \\ &= \frac{1}{a}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\int_0^a |\int_{\mathbb{R}} f(t-x)\ \overline{g_m(t-na)}\ dt|^2\ dx \\ &= \frac{1}{a}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\int_0^a |\int_{\mathbb{R}} f(t-(x-na))\ \overline{g_m(t)}\ dt|^2\ dx \end{aligned}$$

Define $G_m(x)$ by the following formula,

$$\forall\ m \in \mathbb{Z}, \quad G_m(x) = |\int_{\mathbb{R}} f(t-x)\ \overline{g_m(t)}\ dt|^2 = |\langle T_x f, g_m\rangle|^2.$$

Then

$$\begin{aligned} c_0 &= \frac{1}{a}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\int_0^a G_m(x-na)\ dx \\ &= \frac{1}{a}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}} G_m(x)\ dx && \text{periodization} \\ &= \frac{1}{a}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}} |\langle T_x f, g_m\rangle|^2\ dx. \end{aligned}$$

Since $f, g_m \in L^2(\mathbb{R})$,

$$
\begin{aligned}
\langle T_x f, g_m \rangle &= \langle \mathcal{F} T_x f, \mathcal{F} g_m \rangle \\
&= \langle M_{-x} \hat{f}, \hat{g}_m \rangle; \qquad (M_b f)(x) = e^{2\pi i b x} f(x) \\
&= \int_{\mathbb{R}} \hat{f}(\zeta)\ \overline{\hat{g}_m(\zeta)}\ e^{-2\pi i x \zeta}\ d\zeta \\
&= \mathcal{F}(\hat{f}(\zeta)\ \overline{\hat{g}_m(\zeta)})(x).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
c_0 &= \frac{1}{a} \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} |\mathcal{F}(\hat{f}(\zeta)\ \overline{\hat{g}_m(\zeta)})(x)|^2\ dx \\
&= \frac{1}{a} \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} |\hat{f}(\zeta)\ \overline{\hat{g}_m(\zeta)}|^2\ d\zeta \\
&= \frac{1}{a} \int_{\mathbb{R}} |\hat{f}(\zeta)|^2 \sum_{m\in\mathbb{Z}} |\hat{g}_m(\zeta)|^2\ d\zeta.
\end{aligned}
$$

But

$$
\begin{aligned}
c_0 &= \frac{1}{a} \int_0^a \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |\langle T_x f, g_{mn} \rangle|^2\ dx = \frac{1}{a} \int_0^a G(x)\ dx \\
&\leq B\ \|f\|^2 = B \int_{\mathbb{R}} |\hat{f}(\zeta)|^2\ d\zeta.
\end{aligned}
$$

Therefore,

$$
\frac{1}{a} \int_{\mathbb{R}} |\hat{f}(\zeta)|^2 \sum_{m\in\mathbb{Z}} |\hat{g}_m(\zeta)|^2\ d\zeta \leq B \int_{\mathbb{R}} |\hat{f}(\zeta)|^2\ d\zeta.
$$

Since the inequality holds for each $f \in L^2(\mathbb{R})$, this implies

$$
\sum_{m\in\mathbb{Z}} |\hat{g}_m(\zeta)|^2 \leq aB. \quad \text{a.e. } \zeta \in \mathbb{R}.
$$

.

□

2.5 Gabor frames

Fix a function $g \neq 0$. Let $a, b > 0$. A collection of functions of the form

$$G(g, a, b) = \{g(x - ma)e^{2\pi inbx} : m, n \in \mathbb{Z}^d\}$$

is called a Gabor system.

If a Gabor system is a frame for $L^2(\mathbb{R}^d)$, then it is a Gabor frame. It is also called a Weyl-Heisenberg frame. The Gabor expansion of $f \in L^2(\mathbb{R}^d)$ is

$$f(x) = \sum_{m\in\mathbb{Z}^d} \sum_{n\in\mathbb{Z}^d} c_{mn}\ g(x - ma)e^{2\pi inbx}.$$

One of the goals of time-frequency analysis is to represent each function $f \in L^2(\mathbb{R}^d)$ as a Gabor expansion. An excellent reference for time-frequency analysis is the book by Gröchenig [Gro00].

The short-time Fourier transform of a function f with respect to g is given by

$$V_gF(y, \omega) = \int_{\mathbb{R}^d} f(x)\overline{g(x - y)}\ e^{-2\pi ix\cdot\omega}\ dx, \quad \text{for } y, \omega \in \mathbb{R}^d.$$

If $f \in L^2(\mathbb{R}^d)$ and $g \in L^2(\mathbb{R}^d)$, then

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |V_gf(y, \omega)|^2\ d\omega\ dy = \|f\|_2 \cdot \|g\|_2.$$

The inversion formula for short-time Fourier transform is

$$f(x) = \frac{1}{\|g\|_2^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} V_gF(y, \omega)\ g(x - y)\ e^{2\pi iy\cdot\omega} d\omega\ dy.$$

Definition 2.14. A function $g \in L^\infty(\mathbb{R}^d)$ belongs to the Wiener amalgam space $W(\mathbb{R}^d)$ if

$$\|g\|_W = \sum_n \operatorname{ess}\sup_{x\in[0,1]^d} |g(x + n)| < \infty.$$

The Wiener amalgam space was introduced by Wiener to study Tauberian theorems. The following theorem indicates one reason why the space is important.

Theorem 2.19. *(Gröchenig, Chapter 6)*

Let $g \in W(\mathbb{R}^d)$ *and* $a, b > 0$. *Let* $\{c_{mn}\} \in l^2(\mathbb{Z}^{2d})$.

Let

$$f(x) = \sum_{m \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} c_{mn}\ g(x - ma) e^{2\pi i n b x}$$

Then

$$\int_{\mathbb{R}^d} |f(x)|^2\ dx \leq (\frac{1}{a} + 1)^d\ (\frac{1}{b} + 1)^d\ \|g\|_W^2\ \left(\sum_{m \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} |c_{mn}|^2 \right).$$

We now want to discuss why Theorem 2.19 is true.

$$\begin{aligned} f(x) &= \sum_m \sum_n c_{mn}\ g(x - ma) e^{2\pi i n\ bx} \qquad\qquad g \in W(\mathbb{R}^d) \\ &= \sum_m \left(\sum_n c_{mn}\ e^{2\pi i n\ bx} \right) g(x - ma) \\ &= \sum_m f_m(x)\ g(x - ma) \end{aligned}$$

where $f_m(x) = \sum_n c_{mn}\ e^{2\pi i n\ bx}$ is periodic on $Q_{1/b} = [0, 1/b]^d$.

$$\|f_m\|^2_{L^2[0,1/b]^d} = \int_{Q_{1/b}} |f_m(x)|^2\ dx = \frac{1}{b^d} \sum_n |c_{mn}|^2.$$

$$\begin{aligned} &\|f_m\|^2_{L^2[0,1/b]^d} \\ &= \sum_{j \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \overline{f_j(x)} f_k(x) \cdot g(x - ka) \overline{g(x - ja)}\ dx \\ &= \sum_{j \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} \int_{Q_{1/b}} f_k(x) \overline{f_j(x)} \cdot \sum_{n \in \mathbb{Z}^d} g(x - ka - \frac{n}{b}) \overline{g(x - ja - \frac{n}{b})}\ dx \end{aligned}$$

For each $x \in \mathbb{R}^d$, define the matrix $\Gamma(x) = (\Gamma(x)_{jk})$ by

$$\Gamma_{jk}(x) = \sum_n \overline{g(x - ja - \frac{n}{b})}\ g(x - ka - \frac{n}{b}).$$

The operators $\Gamma(x)$ act on finite sequences $a = (a_k)_{k\in\mathbb{Z}^d}$ by

$$(\Gamma(x)\ a)_j = \sum_k \Gamma_{jk}(x)\ a_k.$$

Suppose we know that for each $x \in \mathbb{R}^d$, $\Gamma(x)$ is a bounded linear operator on $l_p(\mathbb{Z}^d)$, $1 \le p$, and

$$\|\Gamma(x)\|_{op} \le (\frac{1}{a}+1)^d\ (\frac{1}{b}+1)^d\ \|g\|_W^2 \stackrel{def}{=} \|\Gamma\|.$$

Then, for each $x \in \mathbb{R}^d$,

$$\begin{aligned} 0 &\le \sum_{j\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d} f_k(x)\overline{f_j(x)} \cdot \Gamma_{jk}(x) \le \|\Gamma\| \cdot \sum_{k\in\mathbb{Z}^d} |f_k(x)|^2 \\ &= \sum_{k\in\mathbb{Z}^d} f_k(x) \left(\sum_{j\in\mathbb{Z}^d} \Gamma_{jk}(x) \cdot \overline{f_j(x)} \right) \end{aligned}$$

Intergrate over $Q_{1/b}$ to get:

$$\begin{aligned} \|f\|_2^2 &= \int_{Q_{1/b}} \sum_{j\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d} f_k(x)\overline{f_j(x)} \cdot \Gamma_{jk}(x)\ dx \\ &\le \|\Gamma\| \cdot \sum_{k\in\mathbb{Z}^d} \int_{Q_{1/b}} |f_k(x)|^2\ dx \\ &= (\frac{1}{a}+1)^d\ (\frac{1}{b}+1)^d\ \|g\|_W^2 \cdot \left(\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d} |c_{mn}|^2 \right) \end{aligned}$$

□

Fix a number ω, with $0 < \omega < 1$.

Let ϕ and σ be complex-valued Schwartz functions on $\mathbb{R}^d$. We define a pair of Gabor families $\{\phi_{m,l} : m, l \in \mathbb{Z}^d\}$ and $\{\sigma_{m,l} : m, l \in \mathbb{Z}^d\}$ by:

$$\phi_{m,l}(x) = \phi(x-m)\ e^{2\pi i\omega x\cdot l}, \quad \sigma_{m,l}(x) = \sigma(x-m)\ e^{2\pi i\omega x\cdot l}$$

The following theorem is due to Grafakos and Lennard[GL01].

Theorem 2.20. *[GL01] Let ϕ and σ be complex-valued Schwartz functions on $\mathbb{R}^d$. Let $1 \le p < \infty$. Then there exists a constant $C_p > 0$ such that for all f in the Schwartz class of $\mathbb{R}^d$, we have*

$$\| \sum_{m\in\mathbb{Z}^d} | \sum_{l\in\mathbb{Z}^d} \langle f, \phi_{m,l}\rangle\ \sigma_{m,l}|\ \|_{L^p} \le C_p\ \|f\|_{L^p}.$$

The constant C_p depends only on $p, d,$ and the functions ϕ and σ, but not on f.

Proof. Let $f \in \mathcal{S}(\mathbb{R}^d)$. $\qquad \phi_{m,l}(x) = \phi(x-m)\ e^{2\pi i\omega x\cdot l}$

$$\begin{aligned}
& \sum_{l\in\mathbb{Z}^d} \langle f, \phi_{m,l}\rangle\ \sigma_{m,l}(x) \\
= \ & \sigma(x-m) \sum_{l\in\mathbb{Z}^d} e^{2\pi i\omega x\cdot l} \int_{\mathbb{R}^d} f(y)\ \overline{\phi(y-m)}\ e^{-2\pi i\omega y\cdot l}\ dy \\
= \ & \frac{\sigma(x-m)}{\omega^d} \sum_{l\in\mathbb{Z}^d} e^{2\pi i\omega x\cdot l} \int_{\mathbb{R}^d} f(\frac{y}{\omega})\ \overline{\phi(\frac{y}{\omega}-m)}\ e^{-2\pi i\omega y\cdot l}\ dy \\
= \ & \frac{\sigma(x-m)}{\omega^d} \sum_{l\in\mathbb{Z}^d} e^{2\pi i\omega x\cdot l} \left(f(\frac{\cdot}{\omega})\ \overline{\phi(\frac{\cdot}{\omega}-m)} \right)\widehat{\ }(l) \\
\overset{PSF}{=} \ & \frac{\sigma(x-m)}{\omega^d} \sum_{r\in\mathbb{Z}^d} f(x-\frac{1}{\omega}r)\ \overline{\phi(x-\frac{1}{\omega}r-m)}
\end{aligned}$$

In the last line, we applied Poisson summation formula (PSF):

If $|g(x)| \le \frac{C}{(1+|x|)^{d+\delta}}$, $|\hat{g}(\zeta)| \le \frac{C}{(1+|\zeta|)^{d+\delta}}$ for some $C, \delta > 0$, then

$$\sum_{l\in\mathbb{Z}^d} \hat{g}(l)\ e^{2\pi ix\cdot l} = \sum_{r\in\mathbb{Z}^d} g(x-r).$$

Let $T_m(f) = \sum_{l\in\mathbb{Z}^d}\langle f, \phi_{m,l}\rangle\ \sigma_{m,l}$.

$$\forall x \in \mathbb{R}^d, \qquad \sum_{m\in\mathbb{Z}^d} |T_m(f)(x)|$$
$$\leq \sum_{m\in\mathbb{Z}^d} |\frac{\sigma(x-m)}{\omega^d}| \sum_{r\in\mathbb{Z}^d} |f(x-\frac{1}{\omega}r)| \cdot \overline{|\phi(x-\frac{1}{\omega}r-m)|}$$
$$\leq \frac{C_n^2}{\omega^d} \sum_{r\in\mathbb{Z}^d} |f(x-\frac{1}{\omega}r)| \sum_{m\in\mathbb{Z}^d} (1+|x-m|)^{-n} \left(1+|x-\frac{1}{\omega}r-m|\right)^{-n} \qquad (*)$$

In the last line, we used the fact that ϕ and σ are Schwartz functions, so

$|\sigma(x-m)| \leq \frac{C_n}{(1+|x-m|)^n}$ and $|\phi(x-\frac{r}{\omega}-m)| \leq \frac{C_n}{(1+|x-(r/\omega)-m|)^n}$.

We will show that $(1+|x-m|)(1+|x-\frac{1}{\omega}r| \ -m) \geq 1+|\frac{1}{\omega}r|$. Assuming this for now, we use this to estimate the expression $(*)$ by

$$\frac{C_n^2}{\omega^d} \sum_{r\in\mathbb{Z}^d} \frac{f(x-\frac{r}{\omega})}{(1+|\frac{r}{\omega}|)^{n/2}} \sum_{m\in\mathbb{Z}^d} \frac{1}{(1+|x-m|)^{n/2}} \leq \left(\frac{C_n^2\ C'_{n/2}}{\omega^d}\right) \cdot \left(\sum_{r\in\mathbb{Z}^d} \frac{|f(x-\frac{r}{\omega})|}{(1+|\frac{r}{\omega}|)^{n/2}}\right).$$

The mapping $f \to \sum_{r\in\mathbb{Z}^d} \frac{|f(x-\frac{r}{\omega})|}{(1+|\frac{r}{\omega}|)^{n/2}}$ is bounded on $L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$ by Minkowski's inequality.

We have proved:

$$\| \sum_{m\in\mathbb{Z}^d} | \sum_{l\in\mathbb{Z}^d} \langle f, \phi_{m,l}\rangle\ \sigma_{m,l}|\ \|_{L^p} \leq C_p\ \|f\|_{L^p}.$$

It remains to show that $(1+|x-m|)(1+|x-\frac{1}{\omega}r| \ -m) \geq 1+|\frac{1}{\omega}r|$.

$$\begin{aligned}
1+|a| &\leq 1+|a-x|+|x| = 1+|x-a|+|x| \\
(1+|x|)(1+|x-a|) &= 1+|x-a|+|x|+|x||x-a| \\
&\geq 1+|x-a|+|x| \\
\Longrightarrow 1+|a| &\leq 1+|x-a|+|x| \leq (1+|x|)(1+|x-a|).
\end{aligned}$$

Then replace x by $x-m$ and replace a by r/ω.

□

Before continuing, it is convenient to do some straightforward calculations and summarize them in the following lemma.

Lemma 2.4.

$$\begin{aligned}
\langle \hat{f}, \hat{\phi}_{m,l} \rangle \psi_{m,l}(x) &= \int_{\mathbb{R}^d} \hat{f}(\zeta)\ \overline{\hat{\phi}(\zeta - \omega l)}\ e^{2\pi i m \cdot (\zeta - \omega l)}\ d\zeta\ \cdot \psi_{m,l}(x) \\
&= \int_{\mathbb{R}^d} \hat{f}(\zeta)\ \overline{\hat{\phi}(\zeta - \omega l)}\ \beta_{\zeta,l}(x-m)\ e^{2\pi i x \cdot \zeta}\ d\zeta \\
&\qquad \textit{where } \beta_{\zeta,l}(x) = \psi(x)\ e^{-2\pi i x \cdot (\zeta - \omega l)}.
\end{aligned}$$

Proof.

$$\begin{aligned}
\hat{\phi}_{m,l}(x) &= \int \phi(x-m)\ e^{2\pi i \omega x \cdot l}\ e^{-2\pi i x \cdot \zeta}\ dx \\
&= \int \phi(x-m)\ e^{-2\pi i x \cdot (\zeta - \omega l)}\ dx \\
&= \int \phi(x)\ e^{-2\pi i (x+m) \cdot (\zeta - \omega l)}\ dx \\
&= \hat{\phi}(\zeta - \omega l)\ e^{-2\pi i m \cdot (\zeta - \omega l)}
\end{aligned}$$

Hence,

$$\langle \hat{f}, \hat{\phi}_{m,l} \rangle \psi_{m,l}(x) = \int \hat{f}(\zeta)\ \overline{\hat{\phi}(\zeta - \omega l)}\ e^{2\pi i m \cdot (\zeta - \omega l)}\ d\zeta \cdot \psi_{m,l}(x).$$

Compute

$$\begin{aligned}
&\psi_{m,l}(x) \cdot e^{2\pi i m \cdot (\zeta - \omega l)} \\
= \ &\psi(x-m)\ e^{2\pi i \omega x \cdot l} \cdot e^{2\pi i m (\zeta - \omega l)} \\
= \ &\psi(x-m)\ e^{-2\pi i x \cdot (\zeta - \omega l)} \cdot e^{2\pi i x \cdot \zeta} \cdot e^{2\pi i m (\zeta - \omega l)} \\
= \ &\psi(x-m)\ e^{-2\pi i (x-m) \cdot (\zeta - \omega l)} \cdot e^{2\pi i x \cdot \zeta} \\
= \ &\beta_{\zeta,l}(x-m) \cdot e^{2\pi i x \zeta} \qquad \text{where } \beta_{\zeta,l}(x) = \psi(x)\ e^{-2\pi i x \cdot (\zeta - \omega l)}.
\end{aligned}$$

This completes the proof. □

One of the reason of using frames instead of basis is that it gives us greater flexibility. Consider a pair of Gabor families $\{\phi_{m,l} : m, l \in \mathbb{Z}^d\}$ and $\{\sigma_{m,l} : m, l \in \mathbb{Z}^d\}$ defined by:

$$\phi_{m,l}(x) = \phi(x - m)\ e^{2\pi i \omega x \cdot l}, \quad \sigma_{m,l}(x) = \sigma(x - m)\ e^{2\pi i \omega x \cdot l}$$

Can we choose the pair of functions ϕ and ψ so that resulting Gabor expansion

$$f = \sum_{m \in \mathbb{Z}^d} \sum_{l \in \mathbb{Z}^d} \langle f, \phi_{m,l} \rangle \psi_{m,l}$$

gives pointwise convergence to the function f?

The next theorem, due to Grafakos and Lennard, shows the answer is: Yes. Its proof uses the calculations we performed in Lemma 2.4.

Theorem 2.21. *Let ϕ and ψ be complex-valued Schwartz functions on $\mathbb{R}^d$.*

Assume $\widehat{\phi}$ and $\widehat{\psi}$ are supported on $Q = [0, 1]^d$.

Assume the following structural relationship holds.

$$\forall \zeta \in \mathbb{R}^d, \quad \sum_{l \in \mathbb{Z}^d} \widehat{\phi}(\zeta - \omega l)\ \overline{\widehat{\psi}(\zeta - \omega l)} = A;$$

for some fixed ω, where $0 < \omega < 1$ and $0 < A < \infty$.

Then each Schwartz function f can be represented as the pointwise limit of a Gabor expansion as:

$$\forall x \in \mathbb{R}^d, \quad f(x) = \frac{1}{A} \sum_{m \in \mathbb{Z}^d} \sum_{l \in \mathbb{Z}^d} \langle f, \phi_{m,l} \rangle \psi_{m,l}(x).$$

Proof. Let $f \in \mathcal{S}(\mathbb{R}^d)$. $\phi_{m,l}(x) = \phi(x-m)\ e^{2\pi i \omega x \cdot l}$

$$
\begin{aligned}
& \sum_{m\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d}\langle f, \phi_{m,l}\rangle \psi_{m,l}(x) \\
= \quad & \sum_{m\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d}\langle \widehat{f}, \widehat{\phi}_{m,l}\rangle \psi_{m,l}(x) \\
= \quad & \sum_{m\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d}\int_{\mathbb{R}^d} \widehat{f}(\zeta)\ \widehat{\phi}(\zeta - \omega\cdot l)\cdot e^{2\pi i m\cdot(\zeta-\omega\cdot l)}\ d\zeta \cdot \psi_{m,l}(x) \quad \text{by Lemma 2.4}
\end{aligned}
$$

$$\text{Let } \beta_{\zeta,l}(x) = \psi(x)\ e^{-2\pi i x\cdot(\zeta-\omega l)}$$

$$\text{Then } \widehat{\beta}_{\zeta,l}(k) = \widehat{\psi}(k+\zeta-\omega l)$$

$$
\begin{aligned}
= \quad & \sum_{m\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d}\int_{\mathbb{R}^d} \widehat{f}(\zeta)\ \overline{\widehat{\phi}(\zeta - \omega\cdot l)}\cdot e^{2\pi i x\cdot\zeta}\cdot \beta_{\zeta,l}(x-m)\ d\zeta \quad \text{by Lemma 2.4} \\
\overset{s}{=} \quad & \int_{\mathbb{R}^d} \widehat{f}(\zeta)\sum_{l\in\mathbb{Z}^d}\overline{\widehat{\phi}(\zeta - \omega\cdot l)}\cdot e^{2\pi i x\cdot\zeta}\cdot\left(\sum_{m\in\mathbb{Z}^d}\beta_{\zeta,l}(x-m)\right)\ d\zeta \\
\overset{PSF}{=} \quad & \int_{\mathbb{R}^d} \widehat{f}(\zeta)\sum_{l\in\mathbb{Z}^d}\overline{\widehat{\phi}(\zeta - \omega\cdot l)}\cdot e^{2\pi i x\cdot\zeta}\cdot\left(\sum_{k\in\mathbb{Z}^d}\widehat{\beta}_{\zeta,l}(k)\ e^{2\pi i x\cdot k}\right)\ d\zeta \\
= \quad & \int_{\mathbb{R}^d} \widehat{f}(\zeta)\sum_{l\in\mathbb{Z}^d}\overline{\widehat{\phi}(\zeta - \omega\cdot l)}\cdot e^{2\pi i x\cdot\zeta}\cdot\sum_{k\in\mathbb{Z}^d}\widehat{\psi}(k+\zeta-\omega\cdot l)\ e^{2\pi i x\cdot k}\ d\zeta \\
= \quad & \int_{\mathbb{R}^d} \widehat{f}(\zeta)\sum_{l\in\mathbb{Z}^d}\left(\sum_{k\in\mathbb{Z}^d}\overline{\widehat{\phi}(\zeta - \omega\cdot l)}\cdot\widehat{\psi}(k+\zeta-\omega\cdot l)\ e^{2\pi i x\cdot k}\right)\ e^{2\pi i x\cdot\zeta}\ d\zeta \\
= \quad & \int_{\mathbb{R}^d} \widehat{f}(\zeta)\sum_{l\in\mathbb{Z}^d}\overline{\widehat{\phi}(\zeta - \omega\cdot l)}\cdot\widehat{\psi}(\zeta-\omega\cdot l)\ e^{2\pi i x\cdot\zeta}\ d\zeta,
\end{aligned}
$$

$$\text{since supp } \widehat{\phi} \subseteq [0,1]^d,\ \text{supp } \widehat{\psi} \subseteq [0,1]^d$$

$$= \quad \int_{\mathbb{R}^d} \widehat{f}(\zeta)\ A\ e^{2\pi i x\cdot\zeta}\ d\zeta$$

$$\text{since } \sum_{l\in\mathbb{Z}^d}\overline{\widehat{\phi}(\zeta - \omega\cdot l)}\cdot\widehat{\psi}(\zeta-\omega\cdot l) = A \quad \text{by hypothesis}$$

$$= \quad A\ f(x).$$

The interchange of integration and summation, and the use of Possion summation formula are justified since f, ψ, ϕ are all Schwartz functions.

□

2.6 Tight frames

We now consider the following question: Given a finite set of vectors in a finite dimensional vector space, how do we construct a tight frame from the given set of vectors? Casazza and Leonhard [CL08] provided several methods to construct a tight frame from a finite set of vectors. We will discuss one such method.

Definition 2.15. Let l_2^N be the N-dimensional vector space consisting of all vectors $(a_1, a_2, a_3, ..., a_N)$ and the norm of such a vector is $\left(\sum_{j=1}^{N} |a_j|^2\right)^{1/2} < \infty$.

Theorem 2.22. *Let $\{f_i : 1 \leq i \leq M\}$ be a set of vectors in l_2^N and at least one of them is not the zero vector. We can add $(N-1)$ vectors $\{h_j : 2 \leq j \leq N\}$ to the family so that $\{f_i : 1 \leq i \leq M\} \cup \{h_j : 2 \leq j \leq N\}$ is a tight frame for l_2^N.*

Proof. Let $\{g_j : 1 \leq j \leq N\}$ be the basis corresponding to the eigenvectors for the frame operator of $\{f_i : 1 \leq i \leq M\}$ with respective eigenvalues $\{\lambda_j : 1 \leq j \leq N\}$. Some of these eigenvalues can be zero, but at least one of them must be non-zero. We may assume these eigenvalues are arranged so that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_N \geq 0$. For $2 \leq j \leq N$, let

$$h_j = \sqrt{\lambda_1 - \lambda_j}\; g_j.$$

Let S_1 be the frame operator for $\{f_i : 1 \leq i \leq M\} \cup \{h_j : 2 \leq j \leq N\}$.

Then for each $f \in l_2^N$,

$$\begin{aligned}
S_1 f &= \sum_{i=1}^{M} \langle f, f_i \rangle f_i + \sum_{j=2}^{N} \langle f, h_j \rangle h_j \\
&= \sum_{j=1}^{N} \lambda_j \langle f, g_j \rangle g_j + \sum_{j=2}^{N} \sqrt{\lambda_1 - \lambda_j}\ \langle f, g_j \rangle\ \sqrt{\lambda_1 - \lambda_j} g_j \\
&= \sum_{j=1}^{N} \lambda_j \langle f, g_j \rangle g_j + \sum_{j=2}^{N} (\lambda_1 - \lambda_j)\ \langle f, g_j \rangle\ g_j \\
&= \lambda_1 \langle f, g_1 \rangle g_1 + \sum_{j=2}^{N} \lambda_1 \langle f, g_j \rangle g_j \\
&= \lambda_1 \sum_{j=1}^{N} \langle f, g_j \rangle g_j \\
&= \lambda_1 f.
\end{aligned}$$

Therefore $\{f_i : 1 \leq i \leq M\} \cup \{h_j : 2 \leq j \leq N\}$ is a tight frame for l_2^N.

□

The resulting tight frame has a tight frame bound equal to λ_1*, which is the largest eigenvalue of the frame operator for the original set of vectors given. This method gives us a tight frame but the norm of the vectors may not be all equal.*

2.7 Other topics

Recall that a sequence is a Riesz basis if it is topologically isomorphic to an orthonormal sequence. That means a Riesz basis is precisely the image of an orthonormal basis under a bounded invertible operator. The following observation of D. Han and Larson is a generalization to frames. It is a generalization because a Riesz basis is always a frame, and an orthonormal basis is always a 1-tight frame.

Theorem 2.23. *A frame is precisely the image of a 1-tight frame under a bounded invertible operator. If T is the bounded invertible operator, the upper and lower frame bounds are $\|T\|^2$ and $\|T^{-1}\|^2$, respectively.*

With the observation of the above theorem, we can say more about frames. The following characterization of a frame in relation to the canonical dual frame, also due to D. Han and Larson.

Theorem 2.24. *Let $\{x_n\}$ be a frame for a Hilbert space H. Then there exists a unique operator $B \in \mathcal{B}(H)$ such that*

$$x = \sum_n \langle x, Bx_n \rangle x_n \quad \text{for all } x \in H. \qquad (*)$$

An explicit formula for B is given by $B = A^*A$ where A is any invertible operator in $\mathcal{B}(H, K)$ for some Hilbert space K with the property that $\{Ax_n\}$ is a 1-tight frame. In particular, B is an invertible positive operator.

Proof. Let $A \in \mathcal{B}(H, K)$ be any invertible from H to K for some Hilbert space K so that $\{Ax_n\}$ is a 1-tight frame. Such an operator exists by Theorem 2.23. Let

$f_n = Ax_n$. Let $B = A^*A \in \mathcal{B}(H)$. Then

$$\begin{aligned} & \sum_n \langle x, A^*Ax_n\rangle x_n = \sum_n \langle Ax_n, f_n\rangle x_n \\ = & \sum_n \langle Ax, f_n\rangle A^{-1} f_n \\ = & A^{-1} \sum_n \langle Ax, f_n\rangle f_n \\ = & A^{-1}Ax = x. \end{aligned}$$

Therefore, $B = A^*A$ satisfies the equation $(*)$.

For uniqueness, suppose that $T \in \mathcal{B}(H)$ satisfies $x = \sum_n \langle x, Tx_n\rangle x_n, \forall x \in H$.

Then

$$\begin{aligned} x &= \sum_n \langle x, Tx_n\rangle x_n = \sum_n \langle x, TA^{-1}f_n\rangle A^{-1} f_n \\ &= A^{-1} \sum_n \langle (A^*)^{-1}T^*x, f_n\rangle f_n \\ &= A^{-1}((A^*)^{-1}T^*x). \end{aligned}$$

This implies $A^{-1}(A^*)^{-1}T^* = I$, hence $T = A^*A$.

$\square$

We next turn to discuss the square root of a positive operator. The frame operator S is a positive operator, and in fact, if $\{f_n\}$ is a frame, then $\{S^{-1/2}f_n\}$ is a tight frame. The next theorem establishes the existence and uniqueness of the square root of a positive operator. The proof is long but elementary. It avoids using the heavy machinery of Banach algebra techniques.

Theorem 2.25. *If A is a positive operator in $\mathcal{B}(H)$, then there is a unique operator $B \geq 0$ such that $B^2 = A$. Furthermore, B commutes with any operator $C \in \mathcal{B}(H)$ that commutes with A.*

Proof. Without loss of generality, we may assume $0 \leq A \leq I$. To see this, note that the operator $A_1 = A/\|A\|$ satisfies $0 \leq A_1 \leq I$, and if we can find an operator G such that $G^2 = A_1$, then $B = \|A\|^{1/2}\, G$ satisfies $B^2 = A$.

Let $R = I - A$. Then $0 \leq R \leq I$.

Let $S = I - B$. Then $(I - S)^2 = I - R$, since $A = I - R, B = I - S$.

From $(I - S)^2 = I - R \Rightarrow I - 2S + S^2 = I - R$, we obtain

$$S = \frac{1}{2}(R + S^2).$$

To solve for S, define $S_0 = 0$ and define

$$S_{n+1} = \frac{1}{2}(R + S_n^2), \quad n = 0, 1, 2, ...$$

Then $0 \leq S_n \leq I$ for $n \geq 0$.

To see this, we use induction. This is true for $n = 0$. Assume it is true for n. Then

$$\langle S_{n+1}u, u\rangle = \frac{1}{2}\langle Ru, u\rangle + \frac{1}{2}\|S_n u\|^2,$$

and so $S_{n+1} \geq 0$.

$$\langle u, u\rangle - \langle Su, u\rangle = \|u\| - \frac{1}{2}\|S_n u\|^2 - \frac{1}{2}\langle Ru, u\rangle,$$

and so $S_{n+1} \leq I$.

Now, we note that S_n is a polynomial in R with nonnegative coefficients. To see this, we use induction again. This is true for $n = 0$. Assume it is true for n. Then the equation $S_{n+1} = \frac{1}{2}(R + S_n^2)$ immediately shows that it is true for $n+1$.

We can show that $S_{n+1} - S_n$ is a polynomial in R with nonnegative coefficients.

$$S_{n+1} - S_n = \frac{1}{2}(R + S_n^2) - \frac{1}{2}(R + S_{n-1}^2) = \frac{1}{2}(S_n + S_{n-1})(S_n - S_{n-1}).$$

(Note: Here, we used the fact that S_n commutes with S_{n-1}.) So if $S_n - S_{n-1}$ is a polynomial in R with nonnegative coefficients, then $S_{n+1} - S_n$ is also a polynomial in R with nonnegative coefficients.

Next, we can show that $R^k \geq 0$ for $k = 0, 1, 2, ...$

If $k = 2j$, then $\langle R^k u, u \rangle = \|R^j u\| \geq 0$,

and if $k = 2j + 1$, then $\langle R^k u, u \rangle = \langle RR^j u, R^j u \rangle \geq 0$.

Using the fact that $R^k \geq 0$ and the fact that each $S_{n+1} - S_n$ is a polynomial in R with nonnegative coefficients, we see that $S_{n+1} \geq S_n$ for $n \geq 0$. (This is because we can write $S_{n+1} - S_n = \sum_k c_k R^k$, where each $c_k \geq 0$ and each R^k is positive.)

We have shown that the sequence of operators $\{S_n\}$ satisfies:

$$0 \leq S_n \leq S_{n+1} \leq I \quad \text{ for } n = 0, 1, 2, ...$$

By Lemma 2.1 , this shows that S_n converges to the operator S.

We now want to prove that the operator B commutes with any operator that commutes with A. Let $C \in \mathcal{B}(H)$ be any operator that commutes with A. Then C commutes with $R = I - A$. Since each S_n is a polynomial in R, C must also commute with each S_n.

Then $CS_n u = S_n C u, \ \forall\ u \in H$.

Take the limit as $n \to \infty$, and we get $CSu = SCu, \ \forall\ u \in H$.

Hence C commutes with S, and therefore also with $B = I - S$.

For uniqueness, suppose $T \geq 0$ is another square root of A. Then

$$TA = TT^2 = T^2T = AT.$$

Since T commutes with A, by the above argument, T must also commute with B.

Hence, $(B+T)(B-T) = B^2 - T^2$ because $TB = BT$.

Let $u \in H$, and let $v = (B-T)u$. Then

$$\begin{aligned}\langle (B+T)v, v\rangle &= \langle (B^2 - T^2)u, v\rangle \\ &= \langle (A-A)u, v\rangle = 0.\end{aligned}$$

Since $B \geq 0$ and $T \geq 0$, so

$$\langle Bv, v\rangle = \langle Tv, v\rangle = 0.$$

Since $B \geq 0$, there is an operator $D \in \mathcal{B}(H)$ such that $D^2 = B$. Hence,

$$\|Dv\|^2 = \langle Dv, Dv\rangle = \langle D^2v, v\rangle = \langle Bv, v\rangle = 0.$$

So $Dv = 0$. Therefore, $Bv = D^2v = D(Dv) = 0$.

Similarly, $Tv = 0$. This implies,

$$\begin{aligned}\|(B-T)u\|^2 &= \langle (B-T)u, (B-T)u\rangle = \langle (B-T)^2u, u\rangle \\ &= \langle (B-T)v, u\rangle = 0. \\ \Longrightarrow (B-T)u &= 0 \quad \Rightarrow Bu = Tu.\end{aligned}$$

This holds for all $u \in H$. Hence B is the unique sqaure root of A.

$\square$

Chapter 3

The theory of Beurling on Balayage

3.1 Balayage

Let E be a subset of $\mathbb{R}^d$ and let Λ be a subset of $\widehat{\mathbb{R}}^d$.

Definition 3.1. We say that Balayage is possible for (E, Λ) if for each $\nu \in M_b(\mathbb{R}^d)$, there exists $\mu \in M_b(E)$ such that $\hat{\mu}(\zeta) = \hat{\nu}(\zeta),\ \forall\ \zeta \in \Lambda$.

In words, this says that for every bounded Radon measure whose support is on $\mathbb{R}^d$, there exists a bounded Radon measure whose support is on a subset $E \subset \mathbb{R}^d$, such that the Fourier transforms of these two measures agree on a set $\Lambda \subseteq \widehat{\mathbb{R}}^d$.

Let ϕ be a bounded continous function on $\mathbb{R}^d$. Then the Fourier transform of ϕ exists in the sense of distribution. We write $\hat{\phi}$ for the Fourier transform of ϕ.

Definition 3.2. The function space

$$\mathcal{C}(\Lambda) \stackrel{def}{=} \{\phi : \phi \text{ is bounded and continous on } \mathbb{R}^d,\ \text{supp}\ \hat{\phi} \subseteq \Lambda\}$$

plays a central role in this theory.

Definition 3.3. $J(E, \Lambda)$ is the smallest number J such that

$$\forall\ \phi \in \mathcal{C}(\Lambda), \quad \sup_{x \in \mathbb{R}^d} |\phi(x)| \leq J\ \sup_{x \in E} |\phi(x)|$$

We set $J(E, \Lambda) = \infty$ if no such positive number exists.

Lemma 3.1. *Suppose Balayage is possible for* (E, Λ).

Then $\exists\ K = K(E, \Lambda)$ *such that*

$$\inf\{\int_E |d\beta| : supp\ \beta \subseteq E,\ \hat{\beta}(\zeta) = \hat{\alpha}(\zeta)\ \forall\ \zeta \in \Lambda\} \leq K\ \int_{\mathbb{R}^d} |d\alpha|.$$

Proof. Let $\mathcal{B}(\Lambda) = \{\phi : \phi = \hat{\nu} \text{ restricted on } \Lambda, \text{ for some } \nu \in M_b(\mathbb{R}^d)\}$.

So, $\phi \in \mathcal{B}(\Lambda)$ means that for all $\zeta \in \Lambda$,

$\phi(\zeta) = \int e^{-2\pi i \zeta \cdot x}\ d\nu(x); \quad \nu \in M_b(\mathbb{R}^d)$.

Let $\|\phi\|_{\mathcal{B}(\Lambda)} = \inf\{\|\nu\| : \nu \in M_b(\mathbb{R}^d),\ \hat{\nu} \text{ restricted on } \Lambda = \phi\}$.

Define the linear map $L : M_b(E) \to \mathcal{B}(\Lambda)$ by $L\nu = \hat{\nu}|_{\Lambda}$.

L is surjective. Consider $L^{-1} : \mathcal{B}(\Lambda) \to M_b(E)$.

By the Open Mapping Theorem, L^{-1} is bounded, so

$\forall\ \phi \in \mathcal{B}(\Lambda), \quad \exists\ \beta \in M_B(E),$ such that $L^{-1}(\phi) = \beta$.

There can be more than one β in $M_b(E)$ such that $L^{-1}(\phi) = \beta$. Pick one of them.

L^{-1} is bounded, so $\exists\ M > 0$ such that $\|L^{-1}(\phi)\|_{M(E)}\ \leq\ M \cdot \|\phi\|_{\mathcal{B}(\Lambda)}$

$$\Longrightarrow \quad \int_E |d\beta|\ \leq\ M \cdot \|\phi\|_{\mathcal{B}} \quad \text{for all } \beta \in M_b(E), \text{ s.t. } \hat{\beta} = \phi \text{ on } \Lambda.$$

Now, given $\alpha \in M_b(\mathbb{R}^d),\ \exists\ \phi \in \mathcal{B}(\Lambda)$, such that $\hat{\alpha} = \phi$ on Λ.

By definition of $\|\phi\|_{\mathcal{B}(\Lambda)}$ and inf, $\quad \int_{\mathbb{R}^d} |d\alpha| \geq \|\phi\|_{\mathcal{B}(\Lambda)}$.

The above argument shows $\int_{\mathbb{R}^d} |d\alpha| \geq \|\phi\|_{\mathcal{B}(\Lambda)} \geq \frac{1}{M} \int_E |d\beta|$,

for all $\beta \in M_b(E)$, s.t. $\hat{\beta} = \phi$ on Λ, $\hat{\alpha} = \phi$ on Λ.

So the conclusion of the theorem follows if we take $K = \frac{1}{M}$, .

$\square$

Definition 3.4. $K(E,\Lambda)$ is the smallest number K such that

$$\inf\{\int_E |d\beta| : \text{supp } \beta \subseteq E,\ \hat{\beta}(\zeta) = \hat{\alpha}(\zeta)\ \forall\ \zeta \in \Lambda\} \leq K \int_{\mathbb{R}^d} |d\alpha|.$$

We set $K(E,\Lambda) = \infty$ if Balayage is not possible.

We need two conditions on the set Λ.

Definition 3.5. (NTF condition) A set Λ satisfies the not-too-thin (NTF) condition if for each $\zeta_1 \in \Lambda$, and for each $\epsilon > 0$, there exists a probability measure μ_ϵ with support in $\{\zeta : \zeta \in \Lambda, |\zeta - \zeta_1| \leq \epsilon\}$ so that $\check{\mu}_\epsilon(x) \equiv \int e^{2\pi i x\cdot\zeta}\ d\mu_\epsilon \to 0$ as $|x| \to \infty$.

Definition 3.6. (Spectral synthesis) A set Λ is a set of spectral synthesis if

$$\forall\ \phi \in \mathcal{C}(\Lambda),\ \forall\ \mu \in M_b(\mathbb{R}^d), \quad \hat{\mu} = 0 \text{ on } \Lambda \Rightarrow \int \phi\ d\mu = 0.$$

Lemma 3.2. *Assume the set Λ satisfies the NTF condition.*

Then $\quad K(E,\Lambda) \leq J(E,\Lambda)$.

Proof. Assume $J(E,\Lambda) < \infty$ and Λ satisfies the NTF condition.

Let $\mathcal{C}_0(\Lambda) = \{\phi \in \mathcal{C}(\Lambda),\ \lim_{|x|\to\infty}\ \phi(x) = 0.\}$

Given $\alpha \in M(\mathbb{R}^d)$, define $L : \mathcal{C}_0(\Lambda) \to \mathbb{R}$ by:

$$L(\phi) = \int_{\mathbb{R}^d} \phi(x)\ d\alpha(x).$$

By Riesz-Markov Representation Theorem, L is a linear functional on $C_0(\mathbb{R}^d)$, and $\|L\| = \int |d\alpha| \equiv \|\alpha\|$.

Let $\mathcal{A} = \{\{\phi(x)\}_{x\in E} : \phi \in \mathcal{C}_0(\Lambda)\}$.

Then $\mathcal{A} \subseteq c_0(E)$, where $c_0(E) \equiv \{\{a_x\}_{x\in E} : \lim_{|x|\to\infty}\ a_x \to 0\}$, i.e. the set of bounded sequences defined on E whose elements converge to zero.

Let $\{a_x\}_{x\in E} \in \mathcal{A}$. Let $\phi_1, \phi_2 \in \mathcal{C}_0(\Lambda)$, with $\phi_1(x) = \phi_2(x) = a_x$, $\forall x \in E$.

Since $\phi_1 - \phi_2 \in \mathcal{C}_0(\Lambda)$, and since $J(E,\Lambda) < \infty$, therefore

$$\sup_{x\in\mathbb{R}^d} |(\phi_1 - \phi_2)(x)| \ \leq \ J(E,\Lambda) \cdot \sup_{x\in E} |(\phi_1 - \phi_2)(x)| = 0.$$

$\Longrightarrow \ \phi_1 = \phi_2$.

$\Longrightarrow$ Given $\{a_x\}_{x\in E} \in \mathcal{A}$, $\exists$ unique $\phi \in \mathcal{C}_0(\Lambda)$, such that $\phi(x) = a_x$, $\forall x \in E$.

Define a linear function M on $\mathcal{A}$ by: $\{\phi(x)\}_{x\in E} \ \rightarrow \ L(\phi)$.

Given $\{\phi(x)\}_{x\in E} \in \mathcal{A}$, we have:

$$\begin{aligned}
|M(\ \{\phi(x)\}_{x\in E}\ )| \ &= \ |L(\phi)| \\
&\leq \ \|L\| \cdot \sup_{x\in\mathbb{R}^d} |\phi(x)| \\
&\leq \ \|L\| \cdot J(E,\Lambda) \cdot \sup_{x\in E} |\phi(x)| \\
&= \ \|L\| \cdot J(E,\Lambda) \cdot \|\{\phi(x)\}_{x\in E}\|_{\mathcal{A}}
\end{aligned}$$

$\Longrightarrow \|M\| \leq J(E,\Lambda) \cdot \|L\|$.

$\Longrightarrow M$ is a bounded linear functional on $\mathcal{A}$. $M : \mathcal{A} \rightarrow \mathbb{R}$.

By Riesz-Markov Representation Theorem, $\exists$ measure β, supp $\beta \in E$, with

$$M(\ \{\phi(x)\}_{x\in E}\ ) = \int_E \phi \ d\beta,$$

$$\text{and } \|M\| = \int_E |d\beta| \equiv \|\beta\| \leq J(E,\Lambda) \cdot \|L\| = J(E,\Lambda) \cdot \int_{\mathbb{R}^d} |d\alpha|.$$

Claim: $\hat{\alpha} = \hat{\beta}$, for all $\zeta \in \Lambda$.

Fix $\zeta_1 \in \Lambda$. Pick a measure ν_ϵ as in the NTF condition. Let $\phi = \hat{\nu}_\epsilon$.

$$\int_{\mathbb{R}^d} \hat{\nu}_\epsilon \ d\alpha = L(\phi) = M(\ \{\phi(x)\}_{x\in E}\ ) = \int_E \hat{\nu}_\epsilon \ d\beta$$

Since $\epsilon > 0$ is arbitary, this implies by a limiting process that $\hat{\alpha}(\zeta_1) = \hat{\beta}(\zeta_1)$.

By definition, $K(E, \Lambda)$ is the smallest number K such that $\int_E |d\beta| \leq K \int_{\mathbb{R}^d} |d\alpha|$.

Hence $K(E, \Lambda) \leq J(E, \Lambda)$.

□

We will now prove the reverse inequality.

Lemma 3.3. *Assume that* Λ *is a set of spectral synthesis.*

Then $J(E, \Lambda) \leq K(E, \Lambda)$.

Proof. Assume $K(E, \Lambda) < \infty$ and Λ is a set of spectral synthesis.

By hypothesis, if $\phi \in \mathcal{C}(\Lambda)$ and $\nu \in M_b(\mathbb{R}^d)$, $\hat{\nu} = 0$ on Λ, then $\int_\Lambda \phi(x)\ d\nu(x) = 0$.

Fix $y \in \mathbb{R}^d$. Balayage implies $\exists\ \mu_y \in M_b(E)$ such that

$$(\hat{\delta_y})(\zeta) = e^{-2\pi i y\cdot\zeta} = \int_E e^{-2\pi i x\cdot\zeta}\ d\mu_y(x),\ \forall\ \zeta \in \Lambda,\ \text{and}\ \int_E |d\mu_y|\ \leq\ K(E, \Lambda).$$

Spectral synthesis implies that if $\phi \in \mathcal{C}(\Lambda)$, then

$$\int \phi(d\delta_y - d\mu_y) = 0, \quad \text{i.e.}\quad \phi(y) = \int_E \phi\ d\mu_y.$$

$$\Longrightarrow\ |\phi(y)| = |\int_E \phi\ d\mu_y| \leq \|\phi\| \cdot \|\mu\| \leq K(E, \Lambda) \cdot \sup_{x\in E} |\phi(x)|.$$

By definition $J(E, \Lambda)$ is the smallest number J such that $|\phi(y)| \leq J \sup_{x\in E} |\phi(x)|$.

Hence $J(E, \Lambda) \leq K(E, \Lambda)$.

□

We are immediately rewarded by the following short and sweet result.

Lemma 3.4. *Assume that* Λ *is a set of spectral synthesis and* $K(E, \Lambda) < \infty$.

If $\phi \in \mathcal{C}(\Lambda)$ *and* $\phi = 0$ *on* E*, then* $\phi \equiv 0$.

Proof. Since Λ is a set of spectral synthesis, $J(E,\Lambda) \leq K(E,\Lambda)$. So

$$\sup_{x\in\mathbb{R}^d} |\phi(x)| \leq K(E,\Lambda) \ \sup_{x\in E} |\phi(x)|.$$

Therefore, if $\phi = 0$ on E, then $\phi \equiv 0$.

□

Recall that a sequence $\{t_n\}$ in $\mathbb{R}^d$ is separated if there exists $\delta > 0$ such that

$$\inf\{|t_m - t_n| : m \neq n\} \geq \delta.$$

Definition 3.7. For a given closed set Q and for $t > 0$, let $Q(t)$ be the set of points with distance $\leq t$ from Q. The Fréchet distance $[Q,R]$ between two closed sets Q and R is the smallest number t so that $Q \subset R(t)$ and $R \subset Q(t)$.

Theorem 3.1. *Let E_1, E_2 be two closed sets.*

Assume Λ is a set of spectral synthesis and satisfies the NTF condition.

Then $\quad |K(E_1,\Lambda)^{-1} - K(E_2,\Lambda)^{-1}| \ \leq$ *diam*(Λ) $[E_1,E_2]$.

Proof. Assume $\Lambda \subset \{x : |x| \leq r.\}$

Fix $\phi \in \mathcal{C}(\Lambda)$ with $\sup_{x\in\mathbb{R}^d} |\phi(x)| = 1$.

By lemma 3.3, Λ is a set of spectral synthesis implies $J(E_1,\Lambda) < K(E_1,\Lambda)$.

Then $\quad K(E_1,\Lambda)^{-1} \leq J(E_1,\Lambda)^{-1} \leq \sup_{x\in E_1} |\phi(x)|$.

Pick an $x_1 \in E_1$, such that $|\phi(x_1)| > K_1^{-1} - \epsilon$. $\quad$ We write K_1 for $K(E_1,\Lambda)$.

If $|x - x_1| \leq t$, then by Bernstein's Theorem, $|\text{grad } \phi(x_1)| \leq r \sup_{x\in\mathbb{R}^d} |\phi(x)|$.

So $\ K_1^{-1} - \epsilon - rt < |\phi(x)|$.

Let $t = [E_1, E_2]$. $\exists\ x \in E_2$ with $|x - x_1| \leq t$.

Hence $K_1^{-1} - \epsilon - r\ [E_1, E_2]\ <\ |\phi(x)|$.

Since $\epsilon > 0$ is arbitary, $\quad K_1^{-1} - r\ [E_1, E_2]\ \leq\ \sup_{x \in E_2} |\phi(x)|$.

$\implies J(E_2, \Lambda) \leq \left(K_1^{-1} - r[E_1, E_2]\right)^{-1}$. $\quad$ Recall: $\sup_{x \in \mathbb{R}^d} |\phi(x)| = 1$.

By lemma 3.2, Λ satifies NTF condition implies $J(E_2, \Lambda) \geq K(E_2, \Lambda)$.

Then $K_1^{-1} - r\ [E_1, E_2] \leq K(E_2, \Lambda)^{-1}$.

$\implies K(E_1, \Lambda)^{-1} - K(E_2, \Lambda)^{-1} \leq r\ [E_1, E_2]$.

Switch $K(E_1, \Lambda)$ and $K(E_2, \Lambda)$ to repeat the same argument, and we get

$K(E_2, \Lambda)^{-1} - K(E_1, \Lambda)^{-1} \leq r\ [E_1, E_2]$.

Therefore, $\quad |K(E_1, \Lambda)^{-1} - K(E_2, \Lambda)^{-1}|\ \leq \text{diam}(\Lambda)\ [E_1, E_2]$.

□

Corollary 3.1. *Suppose $K(E, \Lambda) < \infty$. Let Λ be a set of spectral synthesis and satisfies the NTF condition. Then given $\epsilon > 0$, there is a separated sequence $E_1 \subseteq E$ such that $K(E_1, \Lambda) < K(E, \Lambda) + \epsilon$.*

Definition 3.8. Let S be a closed set. Let $\{Q_n\}$ be a sequence of closed sets.

We say Q_n converges strongly to S, if $[Q_n, S] \to 0$.

Q_n converges weakly to S, if for every compact set $K, Q_n \cap K \to S \cap K$.

If Q_n converges strongly to S, we write $Q_n \to S$.

If Q_n converges weakly to S, we write $Q_n \xrightarrow{w} S$.

Example 3. $S = \{1, 2, 3\} \subset \mathbb{R}$. $\quad Q_n = \{1, 2, 3, \frac{1}{n}, n\}$.

Then the sequence of sets Q_n does not converge strongly to S.

But for every compact set K, $Q_n \cap K = \{1, 2, 3, \frac{1}{n}\}$ for n large enough,

so $Q_n \cap K \to S \cap K$. So this is weak convergence but not strong convergence.

Lemma 3.5. *Let Q_n be a collection of finite sets such that*

(1) $\forall Q_n, \quad \exists\ \delta > 0$ *such that* $\inf\{|a-b| : a \neq b;\ a, b \in Q_n\} \geq \delta$,

(2) $\forall Q_n, \quad \exists\ L > 0$ *such that* $Q_n \subseteq [-L, L]$.

Suppose $Q_n \to Q$.

(1) Then Card(Q_n) = Card(Q) for n sufficiently large.

Let a_m^n be the m^{th} element of Q_n. Let a_m be the m^{th} element of Q.

(2) Then $a_m^n \to a_m$.

Proof. $Q_n \to Q$. Fix $\epsilon > 0$.

$\exists\ N = N_\epsilon > 0$ such that $[Q_n, Q] < \epsilon$, for all $n > N$.

$\implies Q \subseteq Q_n(\epsilon)$ and $Q_n \subseteq Q(\epsilon)$, for all $n > N$.

$\implies Q_m \subseteq Q(\epsilon) \subseteq Q_n(2\epsilon)$, for all $n > N$.

Pick $\epsilon = \frac{1}{5}\delta$. Fix $b \in Q_n$.

Notice that there is no other point of Q_n inside the ball $B(b, 2\epsilon)$, since any other $x \in Q_n$ must have $|x - b| > \delta$.

There is at most one point a of Q_m, with $a \in B(b, 2\epsilon)$.

This is true for each $b \in Q_n$.

$\implies$ Card(Q_m) $\leq$ Card(Q_n), for all $m, n > N$.

To see why, we note that $Q_m \subseteq Q_n(2\epsilon)$. $|x - y| \geq \delta, \quad \forall x, y, \in Q_n$.

$\implies$ For n large enough, Card(Q_n) are all equal, i.e. $\exists N$, $\forall\, n > N$, Card(Q_n) $= M$.

Index the elements of Q_n by order. $\{a_j^n\} \subseteq [-L, L]$.

By compactness, $\exists$ subsequence $\{n_k\}$ of $\{n\}$ such that $a_j^{n_k} \to a_j;\ j = 1, 2, ..., M$.

Then $Q_n \to S$. Note that $S = Q$, since by hypothesis, $Q_n \to Q$. □

Lemma 3.6. *Let $E = \{t_n\}$ be a separated sequence in $\mathbb{R}^d$. Let $\{x_n\} \subset \mathbb{R}^d$. Then $\exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that the sets $E + x_{n_k}$ converges weakly. i.e. $E_{n_k} = E + x_{n_k}$, and the sets $E_{n_k} \xrightarrow{w} S$, for some set S (possibly empty).*

Proof. Consider the cube $Q = [-\frac{1}{2}, \frac{1}{2})^d$. Let $E_n = E + x_n$.

$$E_n = \{..., \ t_1 + x_n, \ t_2 + x_n, \ t_3 + x_n, \ t_4 + x_n, \ t_5 + x_n, \ ...\}.$$

Since E is a separated sequence, so the number of t_j inside any translate of $[-\frac{1}{2}, \frac{1}{2})^d$ is uniformly bounded.

So, the number of elements of E_n inside $[-\frac{1}{2}, \frac{1}{2})^d$ is uniformly bounded. (*)

Pick a subsequence of E_1, call this subsequence $\widetilde{E_1}$.

Pick a subsequence of E_2, call this subsequence $\widetilde{E_2}$.

Pick a subsequence of E_3, call this subsequence $\widetilde{E_3}$.

For each n, pick a subsequence of E_n, call this subsequence $\widetilde{E_n}$.

Then (*) implies that in each $\widetilde{E_n}$, we find the same number of elements in $[-\frac{1}{2}, \frac{1}{2})^d$.

Case (1). If this number is not zero, apply the previous Lemma 3.5, so that there is a finite set A_1 in $[-\frac{1}{2}, \frac{1}{2})^d$, and there is a subset of E_n which converges strongly to the set A_1.

Case (2). If the number is 0, then A_1 is the empty set.

Then split $\mathbb{R}^d$ into a collection of cubes, $\mathbb{R}^d = \bigcup_{n \in \mathbb{Z}^d} \left([-\frac{1}{2}, \frac{1}{2})^d + n\right)$.

For each $n \in \mathbb{Z}^d$, apply the above argument to the cube $[-\frac{1}{2}, \frac{1}{2})^d + n$.

□

Lemma 3.7. *(Compactness Property) Each ball* $\{\phi \in \mathcal{C}(\Lambda) : \sup|\phi(x)| \leq M\}$ *is compact. That means for every sequence* $\{\phi_n\}$ *in the ball, there exists a subsequence* $\{\phi_{n_k}\}$ *that converges pointwise and uniformly on every compact set to some function* ϕ *belonging to the ball.*

Proof. Let $\phi_n \in \mathcal{C}(\Lambda)$ with $\|\phi_n\|_\infty \leq M$. By Bernstein's Theorem, $\|\phi_n^{'}\|_\infty \leq rM$, so $\{\phi_n\}$ is a family of equicontinous functions. For any fixed $x, \{\phi_n(x)\}$ is uniformly bounded. By the Arzela-Ascoli Theorem, there exists a subsequence ϕ_{n_k} that converges to a function ϕ and this convergence is uniform over any compact set.

□

Theorem 3.2. *If the set* Λ *satisfies the NTF condition, then*

$$E_n \xrightarrow{w} E \ \textit{implies} \ \ K(E,\Lambda) \leq \ \underline{\lim} \ K(E_n,\Lambda).$$

Proof. Without loss of generality, we may assume $\underline{\lim} \ K(E_n,\Lambda) < \infty$.

Let $\phi \in \mathcal{C}(\Lambda)$. Let $\nu_n \in M_b(E_n)$ such that

$$\hat{\nu}_n(\zeta) = \phi(\zeta), \text{ for all } \zeta \in \Lambda, \quad \text{ and } \quad \int |d\nu_n| = K(E,\Lambda) \cdot \|\phi\|_\Lambda.$$

By definition of $\underline{\lim}$, there exists a subsequence ν_{n_k} of ν_n, such that

$$\int |d\nu_{n_k}| \ \rightarrow \ K(E_n,\Lambda) \cdot \|\phi\|_\Lambda < \infty.$$

By Banach-Alagolu Theorem, without loss of generality, $\nu_{n_k} \rightarrow \nu$ in weak*-topology,

$$\text{so} \quad \hat{\nu}(\zeta) = \phi(\zeta), \ \forall \ \zeta \in \Lambda, \text{ and } \int |d\nu| = \ \underline{\lim} \ K(E_n,\Lambda) \cdot \|\phi\|_\Lambda < \infty.$$

Since $E_n \xrightarrow{w} E$, so supp $\nu = E$, i.e. $\nu \in M_b(E)$.

Hence, $\forall\ \mu \in M_b(\mathbb{R}^d),\ \exists\ \phi \in \mathcal{C}(\Lambda)$, such that

$$\hat{\mu}|_\Lambda = \phi \quad \text{and} \quad \|\phi\|_\Lambda \le \int_{\mathbb{R}^d} |d\mu|.$$

For this ϕ, we find a measure $\nu \in M_b(E)$, such that

$$\int_E |d\nu| = \ \underline{\lim}\ K(E_n, \Lambda) \cdot \|\phi\|_\Lambda \ \le\ \underline{\lim}\ K(E_n, \Lambda) \cdot \int_{\mathbb{R}^d} |d\mu|.$$

$\Longrightarrow \qquad K(E, \Lambda) \le\ \underline{\lim}\ K(E_n, \Lambda)$

since $K(E, \Lambda)$ is the smallest number such that $\int_E |d\nu| \le K \int_{\mathbb{R}^d} |d\mu|$.

□

Definition 3.9. For a closed set E, let $W(E)$ be the collection of weak limits of translates $E_x = E + x$. Thus $E_1 \in W(E)$ means $\exists\ x_n$ with $E_{x_n} \to E_1$ as $n \to \infty$.

Theorem 3.3. *Let Λ be a set of spectral synthesis and satsifies the NTF condition. Then $K(E, \Lambda) < \infty$ iff for every $E_0 \in W(E)$,*

$$\phi \in \mathcal{C}(\Lambda) \ \ \textit{and} \ \ \phi(x) = 0 \ \ \textit{on } E_0 \ \ \textit{imply} \ \ \phi \ \ \textit{is identically 0.}$$

Proof. (1) Assume $K(E, \Lambda) < \infty$ and $E_0 \in W(E)$,

i.e. there is a sequence $\{a_n\}$ such that $\{E + a_n\} \xrightarrow{w} E_0$.

Theorem 3.2 says:

If the NTF condition holds, then $E_n \xrightarrow{w} E_0$ implies $K(E_0, \Lambda) \le\ \underline{\lim}\ K(E_n, \Lambda)$.

So by Theorem 3.2 and by the NTF condition,

$K(E_0, \Lambda) \le\ \underline{\lim}\ K(E + a_n, \Lambda) = K(E, \Lambda) < \infty$.

We used the fact: $K(E, \Lambda) = K(E + a, \Lambda)$ for any $a \in \mathbb{R}^d$.

Recall that lemma 3.4 says: If Λ is a set of spectral synthesis and $K(E, \Lambda) < \infty$, then $\phi \in \mathcal{C}(\Lambda)$ and $\phi = 0$ on E implies ϕ is identically 0.

So by lemma 3.4, and by the NTF condition,

if $\phi \in \mathcal{C}(\Lambda)$ and $\phi(x) = 0$ on E_0, then ϕ is identically 0.

(2) Assume for every $E_0 \in W(E)$, if $\phi \in \mathcal{C}(\Lambda)$ and $\phi = 0$ on E_0, then that implies ϕ is identically 0.

If $\phi \notin W(E)$, then the sentence:

"for every $E_0 \in W(E)$, if $\phi \in \mathcal{C}(\Lambda)$ and $\phi(x) = 0$ on E_0, then ϕ is identically 0" is not true. So we can assume ϕ is NOT in $W(E)$.

Suppose $K(E, \Lambda) = \infty$.

Since $E_1 \subset E_2$ implies $K(E_1, \Lambda) \geq K(E_2, \Lambda)$, without loss of generality we can assume E is a separated set.

Since the NTF condition holds, $\infty = K(E, \Lambda) \leq J(E, \Lambda)$. $\leftarrow$ (by Lemma 3.2)

So there exists a sequence $\{\phi_n\} \in \mathcal{C}(\Lambda)$ with

$$\sup_{x \in \mathbb{R}^d} |\phi_n(x)| = 1 \quad \text{and} \quad \sup_{x \in E} |\phi_n(x)| < \frac{1}{n}$$

Choose x_n such that $|\phi_n(x)| = \frac{1}{2}$, and define

$$\Psi_n(x) = \phi_n(x + x_n).$$

Then $|\Psi_n(0)| = \frac{1}{2}$. Let $E_n = E - x_n$. We have:

$$\sup\{|\Psi_n(x)| : x \in E_n\} < \frac{1}{n}.$$

Since E_n is separated, by Lemma 3.6, there exists a subsequence of x_n, such that E_{n_k} converges weakly. Without loss of generality, $E_n \xrightarrow{w} E_0$. By Compactness Property of $\mathcal{C}(\Lambda)$, there is a subsequence of $\Psi_n \to \Psi \in \mathcal{C}(\Lambda)$.

This implies $\Psi = 0$ on E_0. But $|\Psi(0)| = \frac{1}{2}$. This gives a contradiction.

□

Theorem 3.4. *Let Λ be a set of spectral synthesis and satsifies the NTF condition.*

Let $\Lambda_\epsilon = \{x : dist(x, \Lambda) \leq \epsilon\}$. Assume $K(E, \Lambda) < \infty$.

Then $\exists\ \epsilon_0$ such that $K(E, \Lambda_\epsilon) < \infty$ for all $\epsilon < \epsilon_0$.

Proof. By Corollary 3.1, we can assume E is separated.

Suppose the conclusion of theorem is not true.

Then $\exists\ \epsilon_n \searrow 0$ such that $K(E, \Lambda_n) \geq n$.

Note that $\Lambda_\epsilon^o = \{\zeta : \text{dist}(\zeta, \Lambda) < \epsilon\}, \quad \Lambda_\epsilon = \text{closure of } \Lambda_\epsilon^o$.

$\Longrightarrow$ NTF condition holds for Λ_{ϵ_n}.

Lemma 3.2 implies that $J(E, \Lambda_{\epsilon_n}) \geq K(E, \Lambda_{\epsilon_n}) \geq n$.

$\Longrightarrow \quad \exists\ \phi_n \in \mathcal{C}(\Lambda_{\epsilon_n}),$ s.t. $\sup_{x\in\mathbb{R}^d} |\phi_n(x)| = 1$ and $|\phi_n(x)| < \frac{1}{n}$ for $x \in E$.

Since $\sup_{x\in\mathbb{R}^d} |\phi_n(x)| = 1$ and ϕ_n is continuous, $\exists\ x_n$ s.t. $|\phi_n(x_n)| = \frac{1}{2}$.

Define $\Psi_n(x) = \phi_n(x + x_n)$. Then $|\Psi_n(0)| = \frac{1}{2}$.

Without loss of generality, $\{E - x_n\} \to E_1$, and $\Psi_n \to \Psi$, where $\Psi \in \mathcal{C}(\Lambda)$.

So, $|\Psi(0)| = \frac{1}{2}$, and $\Psi = 0$ on E_1. Note: Ψ is not identically 0.

By Theorem 3.3, since Λ is a set of spectral synthesis and satifies the NTF condition, this implies $K(E, \Lambda) = \infty$. But by hypotheis, $K(E, \Lambda) < \infty$. Hence we arrive at a contradiction if the conclusion of the theorem is not true.

□

The importance of Theorem 3.4 is that it will allow us to prove that: Balayage and Spectral Synthesis implies that for any fixed $y \in \mathbb{R}^d$,

$$e^{-2\pi i\zeta\cdot y} = \sum_n \alpha_n(y)\ h(t_n - y)\ e^{-2\pi i\zeta\cdot t_n}, \quad \forall\zeta\in\Lambda, \ \text{ where } \sum_n |\alpha_n| < \infty.$$

Here, $h \in L^2(\mathbb{R}^d)$, and there exists $\epsilon > 0$ such that supp $\hat{f} \subseteq \bar{B}(0,\epsilon)$.

We take up the proof in the next section. Before doing so, let us consider the question: What is a sufficient condition so that Balayage is possible for (E,Λ)? The following theorem of Beurling answers this question.

Theorem 3.5. *(Beurling) Let E be a sepearated sequence in $\mathbb{R}^d$. Define*

$$D = D(E) = \sup_{x\in\mathbb{R}^d} dist\ (x, E).$$

If $rD < 1/4$, then Balayage is possible for $(E, \bar{B}(0,r))$.

Note that the constant $1/4$ is the best possible. In $\mathbb{R}$, if we let $E = \mathbb{Z}$, then $D = 1/2$, but Balayage is not possible for $(E, [-1/2, 1/2])$.

3.2 Construction of a measure

We continue to assume that E is a separated sequence in $\mathbb{R}^d$, Λ is a set of spectral synthesis that is symmetric around the origin, Balayage holds for (E, Λ).

We will show that Balayage together with Spectral Synthesis implies that for any fixed $y \in \mathbb{R}^d$,

$$e^{-2\pi i \zeta \cdot y} = \sum_n \alpha_n(y)\ h(t_n - y)\ e^{-2\pi i \zeta \cdot t_n}, \quad \forall \zeta \in \Lambda, \ \text{ where } \sum_n |\alpha_n| < \infty.$$

Here, $h \in L^2(\mathbb{R}^d)$, $h(0) = 1$, and there exists $\epsilon > 0$ such that supp $\hat{f} \subseteq \bar{B}(0, \epsilon)$.

We will construct a measure $\nu \in M_b(\mathbb{R}^d)$ such that

$$\forall \zeta \in \Lambda, \quad \hat{\nu}(\zeta) = 0.$$

We begin by constructing a function $h \in L^2(\mathbb{R}^d)$, as mentioned above.

Lemma 3.8. *Let $\Omega(r)$ be a continous function that is increasing and positive. Assume Ω also satisfies the following properties.*

$$\frac{\Omega(r)}{r} \ \textit{is decreasing for } r > 0,$$

$$\textit{and} \quad \int_0^\infty \frac{\Omega(r)}{r^2}\, dr < \infty.$$

Let $\epsilon > 0$. Then there exists an even function g in $\mathbb{R}$ such that

$$g(0) = 1, \quad |g(x)| \le C\ e^{-\Omega(|x|)}, \quad \textit{and} \ \ \textit{supp}\ \hat{g} \subseteq [-\epsilon, \epsilon].$$

Proof. By hypothesis,

$$\frac{\Omega(r)}{r^2} \searrow 0 \quad \text{and} \quad \int_0^\infty \frac{\Omega(r)}{r^2}\, dr < \infty,$$

so $\sum_n \frac{\Omega(n)}{n^2} < \infty$. We may therefore pick a sequence $\{\beta_n\}$ with $\beta_n \searrow 0$, and $\beta_n \geq e\ \Omega(n)/n^2$ for n sufficiently large, with $\sum_n \beta_n \leq \epsilon$. Define

$$g(y) = \prod_{n=1}^{\infty} \frac{\sin 2\pi\beta_n y}{2\pi\beta_n y}.$$

Let $\gamma = [\Omega(y)]$. Note that since

$$\int_0^\infty \frac{\Omega(r)}{r^2}\ dr = \int_0^\infty \frac{\Omega(r)}{r}\frac{1}{r} < \infty,$$

and $\frac{\Omega(r)}{r}$ is decreasing, so $\frac{\Omega(r)}{r} \leq 1$, for all r.

Therefore

$$\frac{\gamma^2}{\Omega(\gamma)y} \leq \frac{\gamma[\Omega(y)]}{\Omega(\gamma)y} \leq \frac{\gamma}{\Omega(\gamma)}\frac{\Omega(y)}{y} \leq 1.$$

Since $|\frac{\sin x}{x}| \leq 1$ and $\beta_n \searrow 0$, we have

$$|g(y)| \leq \prod_{n=1}^{\gamma} \frac{1}{2\pi\beta_n y} \leq \left(\frac{1}{\beta_\gamma y}\right)^\gamma \leq \left(\frac{\gamma^2}{e\Omega(\gamma)y}\right)^\gamma \leq e^{-\gamma} < e^{-\Omega(y)+1}.$$

Since $\sum_n \beta_n \leq \epsilon$, supp $\hat{g} \subseteq [-\epsilon, \epsilon]$.

□

We have constructed an entire function g that is even, so $g(\zeta) = \sum_{n\geq 0} c_n \zeta^{2n}$.

Define

$$h(\zeta) = \sum_{n\geq 0} c_n \left(\sum_{\gamma=1}^{d} \zeta_\gamma^2\right)^n.$$

Then h is an entire function in $\mathbb{R}^d$ with the following properties

$$h(0) = 1, \quad |h(\zeta)| \leq Ce^{-\Omega(|\zeta|)},\ \forall\ \zeta \in \mathbb{R}^d, \quad \text{and supp } \hat{h} \subseteq \bar{B}(0,\epsilon).$$

Choose Ω so that $\int e^{-2\Omega(|\zeta|)}\ d\zeta < \infty$. Then $h \in L^2(\mathbb{R}^d)$.

With the function $h \in \mathbb{R}^d$, we can prove the following lemma.

Lemma 3.9. *Balayage and Spectral Synthesis implies that for any fixed* $y \in \mathbb{R}^d$,

$$e^{-2\pi i\zeta\cdot y} = \sum_n \alpha_n(y)\ h(t_n - y)\ e^{-2\pi i\zeta\cdot t_n}, \quad \forall\zeta \in \Lambda, \quad where \sum_n |\alpha_n| < \infty.$$

Here, $h \in L^2(\mathbb{R}^d),\ h(0) = 1,$ *and there exists* $\epsilon > 0$ *such that supp* $\hat{f} \subseteq \bar{B}(0, \epsilon)$.

Proof. Fix $y \in \mathbb{R}^d$. Let $\mu(x) = \delta_y$. By Theorem 3.4, there exists $\epsilon > 0$, such that Balayage holds for (E, Λ_ϵ). Since Balayage holds for (E, Λ_ϵ), there exists $\{\alpha_n\}$ depending on y, such that for ζ in Λ_ϵ, ,

$$(\delta_y)\hat{}(\zeta) = \sum_n \alpha_n(y)\ (\delta_{t_n})\hat{}(\zeta) \quad \text{where} \sum_n |\alpha_n(y)| \le K(E, \Lambda_\epsilon) < \infty.$$

Let $h \in L^2(\mathbb{R}^d), h(0) = 1,$ supp $\hat{h} \in \bar{B}(0, \epsilon)$. Define measure $\nu \in M_b(\mathbb{R}^d)$ by

$$\begin{aligned}
\nu(x) &= h(x - y)\ \left(\delta_y - \sum_n \alpha_n(y)\ \delta_{t_n}\right)(x). \\
\Longrightarrow \widehat{\nu}(\zeta) &= (h_y)\hat{} * \left(\delta_y - \sum_n \alpha_n(y)\ \delta_{t_n}\right)\hat{}(\zeta), \quad h_y(x) = h(x - y) \\
&= \int \widehat{h}(\zeta - \eta)\ e^{-2\pi i y\cdot(\zeta-\eta)}\ \left(\delta_y - \sum_n \alpha_n(y)\ \delta_{t_n}\right)\hat{}(\eta)\ d\eta \\
&= \int_{\Lambda_\epsilon^c} \widehat{h}(\zeta - \eta)\ e^{-2\pi i y\cdot(\zeta-\eta)}\ \left(\delta_y - \sum_n \alpha_n(y)\ \delta_{t_n}\right)\hat{}(\eta)\ d\eta
\end{aligned}$$

Since the support of $\widehat{h}$ is in $\bar{B}(0, \epsilon)$, $\widehat{h}(\zeta - \eta) = 0$ if $|\zeta - \eta| > \epsilon$.

That means when $\zeta \in \Lambda$ and $\eta \in \Lambda_\epsilon^c$, we have $\widehat{h}(\zeta - \eta) = 0$.

Hence, for each $\zeta \in \Lambda$, $\widehat{\nu}(\zeta) = 0$.

Fix $\zeta \in \Lambda$. Let $\phi(x) = e^{-2\pi i\zeta\cdot x}$. Then $\phi \in \mathcal{C}(\Lambda)$.

By Spectral synthesis, $\forall\phi \in \mathcal{C}(\Lambda)$, $\hat{\nu}(\zeta) = 0$ on Λ implies $\int \phi\ d\nu = 0$. Hence

$$e^{-2\pi i\zeta\cdot y} = \sum_n \alpha_n(y)\ h(t_n - y)e^{-2\pi i\zeta\cdot t_n}$$

□

3.3 Balayage implies Fourier frames

We now go through Beurling's proof that Balayage implies Fourier frames.

Theorem 3.6. *(Beurling [Beu89]) Let $E = \{t_n\}$ be a separated sequence in $\mathbb{R}^d$.*

Let Λ be a set of spectral synthesis, symmetric about the origin.

Assume Balayage is possible for (E, Λ).

Then the lower frame bound inequality holds, i.e. $\exists\ A > 0$, such that $\forall F \in L^2(\Lambda)$,

$$A \int_\Lambda |F(\zeta)|^2 \, d\zeta \leq \sum_{n \in \mathbb{Z}^d} |\widehat{F}(t_n)|^2.$$

Proof. Balayage and Spectral Synthesis implies that for any fixed $y \in \mathbb{R}^d$,

$$\forall \zeta \in \Lambda, \quad e^{-2\pi i \zeta \cdot y} = \sum_n \alpha_n(y)\ h(t_n - y)\ e^{-2\pi i \zeta \cdot t_n}. \qquad \text{(by Lemma 3.9)}$$

Let $f \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$.

$$\begin{aligned}
\hat{f}(\zeta) &= \int_{\mathbb{R}^d} f(y)\ e^{-2\pi i \zeta \cdot y}\ dy & \zeta \in \Lambda \\
&= \int f(y) \left(\sum_n \alpha_n(y)\ h(t_n - y)\ e^{-2\pi i \zeta \cdot t_n} \right) dy \\
&= \sum_n \left(\int \alpha_n(y)\ h(t_n - y) f(y)\ dy \right) e^{-2\pi i \zeta \cdot t_n} & |\sum_n \alpha_n(y)| < \infty \\
&= \sum_n \widetilde{f}(t_n)\ e^{-2\pi i \zeta \cdot t_n} \qquad \text{where } \widetilde{f}(t_n) = \int \alpha_n(y)\ h(t_n - y) f(y)\ dy
\end{aligned}$$

We can swith the integral and the sum above because

$|\sum_{|n| \leq k}\ \alpha_n(y)\ h(t_n - y)\ e^{-2\pi i \zeta \cdot t_n}\ f(y)|$

$\leq \sum_n\ |\alpha_n(y)\ h(t_n - y)|\ |f(y)|$ and $|\sum_n \alpha_n(y)| < \infty$, with $f \in L^1(\mathbb{R}^d)$.

Define an operator $T : L^2(\mathbb{R}^d) \to l^2(\{t_n\})$ by

$$T(f) = \int \alpha_n(y)\ h(t_n - y) f(y)\ dy = \widetilde{f}(t_n)$$

$$
\begin{aligned}
\|\widetilde{f}\|^2_{l^2(E)} &= \sum_n |\int \alpha_n(y)\ h(t_n-y)f(y)\ dy|^2 && \text{norm in } l^2(E)\\
&\le \sum_n \left(\int |\alpha_n(y)|\ |h(t_n-y)|^2\ dy\right)\ \left(\int |\alpha_n(y)|\ |f(y)|^2\ dy\right)\\
&\le \sum_n \left(C_1\ \int |h(t_n-y)|^2\ dy\right)\left(\int |\alpha_n(y)|\ |f(y)|^2\ dy\right) && \text{since } \{\alpha_n\} \text{ is bounded}\\
&\le C_1\ C_2\ \int |\alpha_n(y)|\ |f(y)|^2\ dy && \text{since } h\in L^2(\mathbb{R}^d)\\
&= M\ \int \left(\sum_n |\alpha_n(y)|\right)|f(y)|^2\ dy\\
& && \text{use the fact that } \sum_n |\alpha_n(y)| \le K(\{t_n\},\Lambda)\\
&\le a^2\ \|f\|^2_{L^2(\mathbb{R}^d)})
\end{aligned}
$$

So T is a bounded linear operator and $\|T\| \le a$.

$$
\begin{aligned}
\text{Then}\ \ \|\widetilde{f}\|_{l^1(E)} &= \sum_n |\int \alpha_n(y)\ h(t_n-y)f(y)\ dy| && f\in L^2(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)\\
&\le \sum_n \int |\alpha_n(y)|\ |h(t_n-y)|\ |f(y)|\ dy\\
&\le \int \left(\sum_n |\alpha_n(y)\ h(t_n-y)|\ |f(y)|\right)\ dy && \text{by Fatou's Lemma}\\
&\le \int\ C\ |\alpha_n(y)|\ |f(y)|\ dy\\
&< \infty. && f\in L^1(\mathbb{R}^d)
\end{aligned}
$$

Let $f, g \in L^2(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$. Note if $f\in C_c^\infty(\mathbb{R}^d)$, then $f\in L^2(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$.

$$
\begin{aligned}
\langle f,g\rangle_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} f(y)\overline{g(y)}\ dy\\
&= \int_\Lambda \hat{f}(\zeta)\ \overline{\hat{g}(\zeta)}\ d\zeta\\
&= \int \hat{f}(\zeta)\ \left(\sum_n\ \overline{\widetilde{g}(t_n)}\ e^{-2\pi i\zeta\cdot t_n}\right)\ d\zeta\\
&= \sum_n\ \overline{\widetilde{g}(t_n)}\ \int \hat{f}(\zeta)\ e^{-2\pi i\zeta\cdot t_n}\ d\zeta\\
&= \sum_n\ \overline{\widetilde{g}(t_n)}\ f(t_n)
\end{aligned}
$$

$$\text{Since } \left|\sum_{|n|\le k} \hat{f}(\zeta)\, \overline{\widetilde{g}(t_n)}\, e^{-2\pi i\cdot\zeta t_n}\right| \;\le\; \sum |\overline{\widetilde{g}(t_n)}|\, |\hat{f}(\zeta)| \;\le\; C\int |g(y)|\, dy\, \hat{f}(\zeta)$$

we can apply LDCT, so $\int$ and $\sum$ can be switched in the steps above.

$$\Longrightarrow \|f\|^2_{L^2(\mathbb{R}^d)} = \sum_n f(t_n)\, \overline{\widetilde{f}(t_n)} \le \left(\sum_n |f(t_n)|^2\right)^{1/2} \left(\sum_n |\widetilde{f}(t_n)|^2\right)^{1/2}$$

$$\Longrightarrow \|f\|^2_{L^2(\mathbb{R}^d)} \le \left(\sum_n |f(t_n)|^2\right)^{1/2} \|\widetilde{f}\|_{l^2_E}$$

and we have shown that $\|\widetilde{f}\|_{l^2_E} \le a\, \|f\|_{L^2(\mathbb{R}^d)}$

$$\Longrightarrow \left(\sum_n |f(t_n)|^2\right)^{1/2} \ge \frac{1}{a}\, \|f\|_{L^2(\mathbb{R}^d)}$$

$$\sum_n |f(t_n)|^2 \ge \frac{1}{a^2} \int_{\mathbb{R}^d} |f(y)|^2\, dy.$$

Remark: In the steps above, we have used LDCT to show that:

$$\hat{f}(\zeta) \sum_{|n|\le k} \widetilde{g}(t_n)\, e^{-2\pi i\zeta\cdot t_n} \;\to\; \hat{f}(\zeta) \sum_{-\infty}^{\infty} \widetilde{g}(t_n)\, e^{-2\pi i\zeta\cdot t_n}$$

implies that $\displaystyle\int \left(\hat{f}(\zeta) \sum_{|n|\le k} \widetilde{g}(t_n)\, e^{-2\pi i\zeta\cdot t_n}\right) d\zeta \;\to\; \int \left(\hat{f}(\zeta) \sum_{-\infty}^{\infty} \widetilde{g}(t_n)\, e^{-2\pi i\zeta\cdot t_n}\right) d\zeta.$

We have shown that the theorem holds for functions $f \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. We now continue the proof with the case when $f \in L^2(\mathbb{R}^d)$.

For $f, g \in L^2(\mathbb{R}^d)$, since $L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$; indeed C_c^∞ is dense in $L^2(\mathbb{R}^d)$, so $\exists\, f_m, g_m \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ such that

$\|f - f_m\|_{L^2(\mathbb{R}^d)} \to 0$ as $m \to \infty$, and $\|g - g_m\|_{L^2(\mathbb{R}^d)} \to 0$ as $m \to \infty$.

$$\text{So}\quad \langle f, g\rangle_{L^2(\mathbb{R}^d)} = \lim_{m\to\infty} \langle f_m, g_m\rangle_{L^2(\mathbb{R}^d)} = \lim_{m\to\infty} \sum_n f_m(t_n)\, \overline{\widetilde{g}_m(t_n)}.$$

Then

$$|\sum_n f(t_n)\ \overline{\widetilde{g}(t_n)} - \sum_n f_m(t_n)\ \overline{\widetilde{g}_m(t_n)}|$$

$$\leq\ \ |\sum_n f(t_n)\ \overline{\widetilde{g}(t_n)} - \sum_n f(t_n)\ \overline{\widetilde{g}_m(t_n)}|\ +\ |\sum_n f(t_n)\ \overline{\widetilde{g}_m(t_n)} - \sum_n f_m(t_n)\ \overline{\widetilde{g}_m(t_n)}|$$

$$\leq\ \ \left(\sum_n |f(t_n)|^2\right)^{1/2} \cdot \left(|\overline{\widetilde{g}(t_n)} - \overline{\widetilde{g}_m(t_n)}|^2\right)^{1/2}\ +\ \left(\sum_n |g_m(t_n)|^2\right)^{1/2} \cdot \left(|f(t_n) - f_m(t_n)|^2\right)^{1/2}$$

$$\leq\ \ \|f\|_{l^2_E} \cdot \|\widetilde{g} - \widetilde{g}_m\|_{l^2_E}\ +\ \|\widetilde{g}_m\|_{l^2_E} \cdot \|f - f_m\|_{l^2_E}$$

$$\leq\ \ \|f\| \cdot \|T\| \cdot \|g - g_m\|_{L^2(\mathbb{R}^d)}\ +\ \|f - f_m\| \cdot \|T\| \cdot \|g_m\|_{L^2(\mathbb{R}^d)} \quad \to 0 \text{ as } m \to \infty.$$

Note that $\|g - g_m\|_{L^2(\mathbb{R}^d)} \ \to 0$,

and $\|f - f_m\|_{l^2_E} \ \leq B \cdot \|f - f_m\|_{L^2(\mathbb{R}^d)} \ \to 0$,

where we used Polya-Plancherel Theorem: if $\{t_n\}$ is separated,

then $\exists\ B > 0$ such that $(\sum |f(t_n)|^2)^{1/2} \leq B \cdot \int |f(x)|^2\ dx$.

We have shown that

$$\langle f, g\rangle_{L^2(\mathbb{R}^d)}\ =\ \sum f(t_n)\ \overline{\widetilde{g}(t_n)}; \quad \forall\ f, g, \in L^2(\mathbb{R}^d).$$

Therefore, $\displaystyle\sum_n |f(t_n)|^2 \geq \frac{1}{a^2} \int_{\mathbb{R}^d} |f(y)|^2\ dy,$ as in previous page.

Hence the theorem holds for all $f \in L^2(\mathbb{R}^d)$. This completes the proof.

□

3.4 Insight behind the proof

In the last section, we saw that if Balayage is possible for (E, Λ), where Λ is a set of spectral synthesis, symmetric about the origin, and E is a separated sequence, then the lower bound inequality is obtained for Fourier frames.

Why does the proof work? To gain some insight, we present a summary of the proof with just 4 lines of equations, in order to extract the essence of the proof.

The 4-line proof of Beurling's Theorem.

$$\begin{aligned}
& \int_\Lambda F(\zeta)\ \overline{F(\zeta)}\ d\zeta \qquad\qquad\qquad\qquad F \in L^2(\Lambda) \\
= \ & \int_\Lambda F(\zeta) \left(\int_{\mathbb{R}^d} \overline{\widehat{F}(y)}\ e^{-2\pi i \zeta \cdot y}\ dy \right)\ d\zeta \\
= \ & \int_\Lambda F(\zeta) \left(\int_{\mathbb{R}^d} \overline{\widehat{F}(y)}\ \sum_n \alpha_n(y)\ h(t_n - y)\ e^{-2\pi i \zeta \cdot t_n}\ dy \right) d\zeta \\
\overset{s}{=} \ & \sum_n \left(\int_\Lambda F(\zeta)\ e^{-2\pi i \zeta \cdot t_n}\ d\zeta \right) \cdot \left(\int \overline{\widehat{F}(y)} \cdot \alpha_n(y)\ h(t_n - y)\ dy \right) \\
= \ & \sum_n \widehat{F}(t_n) \cdot \widetilde{F}(t_n) \le \left(\sum_n |\widehat{F}(t_n)|^2 \right)^{1/2} \cdot \left(\sum_n |\widetilde{F}(t_n)|^2 \right)^{1/2}
\end{aligned}$$

We can show that: $\sum_n |\widetilde{F}(t_n)|^2 \le a^2 \ \left(\int_\Lambda |F(\zeta)|^2\ d\zeta \right)$

$$\Longrightarrow \qquad \frac{1}{a} \cdot \frac{\int_\Lambda |F(\zeta)|^2\ d\zeta}{\left(\int_\Lambda |F(\zeta)|^2\ d\zeta \right)^{1/2}} \ \le \ \left(\sum_n |\widehat{F}(t_n)|^2 \right)^{1/2}$$

□

One key ingredient in the proof is this:

$$e^{-2\pi i \zeta \cdot y} = \sum_n \alpha_n(y)\ h(t_n - y)\ e^{-2\pi i \zeta \cdot t_n}, \quad \zeta \in \Lambda, y \in \mathbb{R}^d.$$

This equation is the result of Beurling's theory on Balayage of Fourier transforms and we saw that the proof is at times long and technical.

Remark: There are 2 other important reasons why the 4-line proof works.

1. $\sum_n |\int_{\mathbb{R}^d} \overline{\widehat{F}(y)} \; \sum_n \alpha_n(y) \; h(t_n - y) \; dy|^2 \; \leq a^2 \; \int_\Lambda |F(\zeta)|^2 \; d\zeta.$

   To prove this inequality, we had to explicitly use the fact that $h \in L^2(\mathbb{R}^d)$. Integration over the space of real numbers, which is a locally compact group, and not just a semi-group, has the property of translation invariance:

$$\int_G |h(x-t)|^2 \; dx = \int_G |h(x)|^2 \; dx, \quad \forall t \in G.$$

2. In order to prove the theorem is true for all $f \in L^2(\mathbb{R}^d)$, (i.e. the lower frame bound exists), it is sufficient to prove it is true for all $f \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ or any other dense subspace of $L^2(\mathbb{R}^d)$. To justify this point, we had to use the Polya-Plencherel Theorem towards the end of the proof in the last section and the fact that the set $E = \{t_n\}$ is a separated sequence. Another example of a dense subspace of $L^2(\mathbb{R}^d)$ is $C_0^\infty(\mathbb{R}^d)$, the set of all continous functions on $\mathbb{R}^d$ that vanish at infinity.

We extract Remark 2 above as a principle, since it is so important.

Principle of Density in proving lower bound exists

In order to prove a lower frame bound exists for all $f \in L^2(\mathbb{R}^d)$, it is sufficient to prove it exists for all f in a dense subspace of $L^2(\mathbb{R}^d)$, such as $L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$.

3.5 Spectral synthesis

We have seen that sets of spectral synthesis are important in our theory. In this section, we discuss the spectral synthesis problem in harmonic analysis [Ben75]. Let G be a locally compact abelian group and let $\widehat{G}$ be its dual group.

Definition 3.10. (Wiener, Beurling) A closed set $\Lambda \subseteq \widehat{G}$ is a set of spectral synthesis (S-set) if for each μ in $M(G)$, for each bounded continuous function f on G,

$$\text{supp}(\widehat{f}) \subseteq \Lambda \text{ and } \hat{\mu} = 0 \text{ on } \Lambda \Longrightarrow \int_G f \, d\mu = 0.$$

Here are some examples and counter-examples.

1. The surface of the unit ball in $\widehat{\mathbb{R}}^3$ is not an S-set (L. Schwartz).

2. Polyhedra are S-sets.

3. The $\frac{1}{3}$-Cantor set is an S-set with non-S-subsets.

We now follow the excellent explanation by Jean-Pierre Kahane.

In harmonic analysis, synthesis refers to the reconstruction of a function or some other quantity from its harmonics. For example, a periodic function can be reconstructed from its Fourier series. More generally, such reconstruction is possible for functions that are almost periodic in the sense of Harold Bohr, functions that are quasi-periodic in the sense of Paley and Wiener, and those that are mean-periodic in the sense of Laurent Schwartz. In all these cases, we associate with an element f of a specified function space the closed subspace $\tau(f)$ generated by the harmonics of f, and the harmonics of f are the generators of the simplest subspaces contained in $\tau(f)$.

The synthesis problem, in terms of these subspaces, is to determine whether the harmonics contained in $\tau(f)$ generate $\tau(f)$. This is a very general question, but suppose that we restrict our attention to spaces of bounded functions. For example, consider the space $L^\infty(\mathbb{R})$ with the weak topology it has as the dual of $L^1(\mathbb{R})$. The same question can be asked about $L^\infty(G)$ with the weak topology when G is a locally compact Abelian group such as $\mathbb{R}^d$, or $\mathbb{Z}^d$. When G is a compact Abelian group, synthesis always holds, but Laurent Schwartz showed that it fails when $G = \mathbb{R}^d$, for $d \geq 3$. The answer remained unknown in other cases, until Paul Malliavin proved that synthesis fails for $L^\infty(G)$ whenever G is a locally compact Abelian group which is not compact.

The problem has many equivalent forms. By duality, it can be viewed as a question about the structure of the closed ideals in the convolution algbera $L^1(G)$, and the question is whether such an ideal is the intersection of the maximal ideals containing it. Alternatively, if $\widehat{G}$ is the dual group of G and $A(\widehat{G})$ is Wiener's algebra, whose elements are the Fourier transforms of elements of $L^1(G)$, then the synthesis problem is the same as that of determining whether every closed ideal of $A(\widehat{G})$ is the ideal of functions in $A(\widehat{G})$ that vanish on some closed subset of G.

If instead of $A(\widehat{G})$, we look at the space of continuous functions on G that, if G is not compact, vanish at infinity, the analogous question has a positive answer. In fact, in that setting, the problem reduces to showing that if $f \in C(\widehat{G})$ vanishes on a closed set E and μ is a Radon measure on $\widehat{G}$ that is supported on E, then $\langle \mu, f \rangle = \int f \, d\mu = 0$. For $A(\widehat{G})$, the problem can be expressed in an analogous way, except the space of Radon measure is replaced by the space of pseudo-measures.

That means, we want to know whether if $f \in A(\widehat{G})$ vanished on a closed set $E \subseteq \widehat{G}$ and if T is a pseudo-measure that is supported on E, then it is necessarily true that $\langle T, f \rangle = 0$.

When G is a Euclidean space, the space of pseudo-measures can be identified with the space of tempered distributions with bounded Fourier transform. In his 1948 couterexample for $\mathbb{R}^3$, Schwartz took E to be the surface of the closed unit ball and T to be the derivative in the radial direction of the surface measure σ of the unit ball. Since as $|u| \to \infty, \widehat{\sigma}(u)$ is in the order of $\frac{1}{|u|}$, we deduce that $\widehat{T}(u)$ is in the order of 1 and hence T is a pseudo-measure. Thus the problem is reduced to the task of finding a test function that vanishes on the surface of the unit ball and has nonvanishing derivative in the direction in which σ was differentiated to get T.

When E is the line $\mathbb{R}$ or the circle $\mathbb{T}$, Malliavin's idea was to start with f instead of E and to choose f so that the formal composition $\delta'(f)$ of the derivative of Dirac delta function δ with f can be interpreted as a pseudo-measure whose support is the zero set of f. The successful implementaton of this idea is seriously challenging. Malliavin's idea applied equally well to $\mathbb{R}$ and to G when $\widehat{G}$ is not discrete.

Chapter 4

Main Results

4.1 Generalized Fourier Frames

4.1.1 Existence of a lower frame bound

Theorem 4.1. *Let $E = \{t_n\}$ be a separated sequence in $\mathbb{R}^d$.*

Let $\Lambda \subseteq \mathbb{R}^d$ be a set of spectral synthesis and symmetric about 0.

Assume Λ is a convex set.

Assume Balayage is possible for (E, Λ).

Then $\exists\ A > 0,$ *such that* $\forall F \in L^2(\Lambda),$

$$A\left(\int_\Lambda |F(\zeta)|^2\ d\zeta\right)^{1/2} \leq \left(\sum_{n\in\mathbb{Z}^d} |\widehat{F}(t_n)|^2\right)^{1/2} + \left(\sum_{n\in\mathbb{Z}^d} |\widehat{F}(\frac{1}{2}t_n)|^2\right)^{1/2} + \left(\sum_{n\in\mathbb{Z}^d} |\widehat{F}(\frac{1}{3}t_n)|^2\right)^{1/2}.$$

Proof. We choose bounded continuous functions $\phi_1, \phi_2, \phi_3 \in \mathcal{C}(\Lambda)$ such that:

$$\int \phi_1\ d\nu_1 = 0,\ \int \phi_2\ d\nu_2 = 0,\ \int \phi_3\ d\nu_3 = 0. \qquad \text{(Spectral Synthesis)}$$

Fix $y \in \mathbb{R}^d$. Consider these measures:

$$\mu_1 = \delta_y,\ \mu_2 = \delta_{2y},\ \mu_3 = \delta_{3y}.$$

Since Balayage holds for (E, Λ), we have:

$$\begin{aligned}
\hat{\delta}_y\ (\zeta) &= (\sum_n \alpha_n(y)\ \delta_{t_n})\hat{}\ (\zeta), \quad \zeta \in \Lambda. \\
\hat{\delta}_{2y}\ (\zeta) &= (\sum_n \alpha_n(2y)\ \delta_{t_n})\hat{}\ (\zeta), \\
\hat{\delta}_{3y}\ (\zeta) &= (\sum_n \alpha_n(3y)\ \delta_{t_n})\hat{}\ (\zeta).
\end{aligned}$$

We do not assume $\alpha_n(y)$ is related to $\alpha_n(2y)$ or $\alpha_n(3y)$.

$$\sum_n |\alpha_n(y)| \leq K_1(E, \Lambda_{\epsilon_1}) \quad \sum_n |\alpha_n(2y)| \leq K_2(E, \Lambda_{\epsilon_2}) \quad \sum_n |\alpha_n(3y)| \leq K_3(E, \Lambda_{\epsilon_3}).$$

Pick $\epsilon = \min(\epsilon_1, \epsilon_2, \epsilon_3)$. We proceed now as in the proof of Lemma 3.9.

Let $h \in L^2(\mathbb{R}^d),\ h(0) = 1,\ \text{supp}\ \hat{h} \subseteq \bar{B}(0, \epsilon)$.

For j = 1, 2, 3: define $\nu_j(x) = h(x - j\ y)\ (\delta_{jy} - \sum_n \alpha_n(jy)\ \delta_{t_n})\,(x)$.

$$\begin{aligned}
\nu_1(x) &= h(x - y)\ \left(\delta_y - \sum_n \alpha_n(y)\ \delta_{t_n}\right)(x). \\
\nu_2(x) &= h(x - 2\ y)\ \left(\delta_{2y} - \sum_n \alpha_n(2y)\ \delta_{t_n}\right)(x). \\
\nu_3(x) &= h(x - 3\ y)\ \left(\delta_{3y} - \sum_n \alpha_n(3y)\ \delta_{t_n}\right)(x).
\end{aligned}$$

Spectral synthesis implies that, for $j = 1, 2, 3$, we have:

$$\int \phi_j(x)\ h(x - jy)\ d\delta_{jy} = \int \left(\sum_n \alpha_n(jy)\ h(t_n - jy)\right)\ \phi_j(x)\ d\delta_{t_n}$$

$$\Rightarrow \quad \int \phi_j(x)\ h(x - jy)\ d\delta_{jy} = \sum_n \alpha_n(jy)\ h(t_n - jy)\ \phi_j(t_n)$$

$$\Rightarrow \quad \phi_j(jy) = \sum_n \alpha_n(jy)\ h(t_n - jy)\ \phi_j(t_n)$$

$$\begin{aligned}\phi_1(y) &= \sum_n \alpha_n(y)\ h(t_n - \ y)\ \phi_1(t_n)\\ \phi_2(2y) &= \sum_n \alpha_n(2y)\ h(t_n - \ 2y)\ \phi_2(t_n)\\ \phi_3(3y) &= \sum_n \alpha_n(3y)\ h(t_n - \ 3y)\ \phi_3(t_n).\end{aligned}$$

Fix $\zeta \in \Lambda$. Define the three functions

$$\begin{aligned}\phi_1(x) &= e^{-2\pi i\zeta\cdot x}\\ \phi_2(x) &= e^{-2\pi i(\zeta/2)\cdot x}\\ \phi_3(x) &= e^{-2\pi i(\zeta/3)\cdot x}.\end{aligned}$$

In the above, note that supp $\hat{\phi}_j \in \Lambda$.

Note: We assume that Λ is convex. So $\zeta \in \Lambda$ implies $\zeta/2 \in \Lambda$, and $\zeta/3 \in \Lambda$.

Let $F \in L^2(\Lambda)$.

$$\begin{aligned}&\overline{F(\zeta)}\\ &= \int_{\mathbb{R}^d} \overline{\widehat{F}(y)}\ e^{-2\pi i\zeta\cdot y}\ dy\\ &= \frac{1}{3}\int_{\mathbb{R}^d} \overline{\widehat{F}(y)}\ e^{-2\pi i\zeta\cdot y}\ dy + \ \frac{1}{3}\int_{\mathbb{R}^d} \overline{\widehat{F}(y)}\ e^{-2\pi i(\zeta/2)\cdot 2y}\ dy + \frac{1}{3}\int_{\mathbb{R}^d} \overline{\widehat{F}(y)}\ e^{-2\pi i(\zeta/3)\cdot 3y} dy\\ &= \frac{1}{3}\int \overline{\widehat{F}(y)}\ \left(\sum_n \alpha_n(y)\ h(t_n - y)\ e^{-2\pi i\zeta\cdot t_n}\right)\ dy\\ &\quad + \frac{1}{3}\int \overline{\widehat{F}(y)}\ \left(\sum_n \alpha_n(2y)\ h(t_n - 2y)\ e^{-2\pi i(\zeta/2)\cdot t_n}\right)\ dy\\ &\quad + \frac{1}{3}\int \overline{\widehat{F}(y)} \left(\sum_n \alpha_n(3y)\ h(t_n - 3y)\ e^{-2\pi i(\zeta/3)\cdot t_n}\right)\ dy\end{aligned}$$

This implies that:

$$
\begin{aligned}
& \int_\Lambda F(\zeta)\ \overline{F(\zeta)}\ d\zeta \\
= \ & \frac{1}{3}\int_\Lambda F(\zeta) \int_{\mathbb{R}^d} \overline{\widehat{F}(y)}\ \left(\sum_n \alpha_n(y)\ h(t_n-y)\ e^{-2\pi i\zeta\cdot t_n}\right)\ dy\ d\zeta \\
& + \frac{1}{3}\int_\Lambda F(\zeta) \int_{\mathbb{R}^d} \overline{\widehat{F}(y)}\ \left(\sum_n \alpha_n(2y)\ h(t_n-2y)\ e^{-2\pi i(\zeta/2)\cdot t_n}\right)\ dy\ d\zeta \\
& + \frac{1}{3}\int_\Lambda F(\zeta) \int_{\mathbb{R}^d} \overline{\widehat{F}(y)}\ \left(\sum_n \alpha_n(3y)\ h(t_n-3y)\ e^{-2\pi i(\zeta/3)\cdot t_n}\right)\ dy\ d\zeta \\
= \ & \frac{1}{3}\int_{\mathbb{R}^d} \sum_n \alpha_n(y)\ h(t_n-y)\ \left(\int_\Lambda F(\zeta)\ e^{-2\pi i\zeta\cdot t_n}\ d\zeta\right) \overline{\widehat{F}(y)}\ dy \\
& + \frac{1}{3}\int_{\mathbb{R}^d} \sum_n \alpha_n(2y)\ h(t_n-2y)\ \left(\int_\Lambda F(\zeta)\ e^{-2\pi i(\zeta/2)\cdot t_n}\ d\zeta\right) \overline{\widehat{F}(y)}\ dy \\
& + \frac{1}{3}\int_{\mathbb{R}^d} \sum_n \alpha_n(3y)\ h(t_n-3y)\ \left(\int_\Lambda F(\zeta)\ e^{-2\pi i(\zeta/3)\cdot t_n}\ d\zeta\right) \overline{\widehat{F}(y)}\ dy \\
= \ & \frac{1}{3}\int_{\mathbb{R}^d} \sum_n \alpha_n(y)\ h(t_n-y)\ \widehat{F}(t_n)\ \overline{\widehat{F}(y)}\ dy \\
& + \frac{1}{3}\int_{\mathbb{R}^d} \sum_n \alpha_n(2y)\ h(t_n-2y)\ \widehat{F}(\frac{1}{2}t_n)\ \overline{\widehat{F}(y)}\ dy \\
& + \frac{1}{3}\int_{\mathbb{R}^d} \sum_n \alpha_n(3y)\ h(t_n-3y)\ \widehat{F}(\frac{1}{3}t_n)\ \overline{\widehat{F}(y)}\ dy
\end{aligned}
$$

Now, by the Cauchy-Schwarz inequality,

$$
\begin{aligned}
& \int \sum_n \alpha_n(y)\ h(t_n-y)\ \widehat{F}(t_n)\ \overline{\widehat{F}(y)}\ dy \\
\leq \ & \left(\sum_n |\widehat{F}(t_n)|^2\right)^{1/2}\ \left(\sum_n |\int \alpha_n(y)\ h(t_n-y)\ \widehat{F}(y)\ dy|^2\right)^{1/2}
\end{aligned}
$$

Similarily for the other 2 integrals, and so we obtain:

$$\Longrightarrow \quad \int_\Lambda F(\zeta)\ \overline{F(\zeta)}\ d\zeta$$

$$= \frac{1}{3}\left(\sum_n |\widehat{F}(t_n)|^2\right)^{1/2} \left(\sum_n |\int \alpha_n(y)\ h(t_n - y)\ \widehat{F}(y)\ dy|^2\right)^{1/2}$$

$$+ \frac{1}{3}\left(\sum_n |\widehat{F}(\frac{1}{2}t_n)|^2\right)^{1/2} \left(\sum_n |\int \alpha_n(2y)\ h(t_n - 2y)\ \widehat{F}(y)\ dy|^2\right)^{1/2}$$

$$+ \frac{1}{3}\left(\sum_n |\widehat{F}(\frac{1}{3}t_n)|^2\right)^{1/2} \left(\sum_n |\int \alpha_n(3y)\ h(t_n - 3y)\ \widehat{F}(y)\ dy|^2\right)^{1/2}$$

Now,
$$\sum_n |\int \alpha_n(3y)\ h(t_n - 3y)\ \widehat{F}(y)\ dy|^2$$

$$\leq \sum_n \left(\int |\alpha_n(3y)|\ |h(t_n - 3y)|^2\ dy \quad \int |\alpha_n(3y)||\widehat{F}(y)|^2\ dy\right)$$

$$\leq \sum_n C_1\ \frac{1}{3}\ \|h\|^2 \int \left(\sum_n |\alpha_n(3y)|\right)\ |\widehat{F}(y)|^2\ dy,$$

since $|\alpha_n(3y)|$ is bounded, $h \in L^2(\mathbb{R}^d)$, $\sum_n |\alpha_n(3y)| \leq K_3(\Lambda, \epsilon_3)$.

Use the fact that: $\int_{\mathbb{R}^d} |\hat{F}(y)|^2\ dy = \int_\Lambda |F(\zeta)|^2\ d\zeta$.

$$\Longrightarrow \quad \left(\sum_n |\widehat{F}(t_n)|^2\right)^{1/2} + \left(\sum_n |\widehat{F}(\frac{1}{2}t_n)|^2\right)^{1/2} + \left(\sum_n |\widehat{F}(\frac{1}{3}t_n)|^2\right)^{1/2}$$

$$\geq A\left(\int_\Lambda |F(\zeta)|^2\ d\zeta\right)^{1/2}$$

Note that A depends on $K_1(\Lambda, \epsilon_1), K_2(\Lambda, \epsilon_2), K_3(\Lambda, \epsilon_3)$, and $\|h\|_2^2$.

$\square$

4.1.2 Construction of the frame operator

Recall that if a sequence of elements $\{e_n\}$ is a frame for a Hilbert space H, then the frame operator $S : H \to H$ is given by

$$\forall\ f \in H, \quad S(f) = \sum_n \langle f, e_n \rangle e_n.$$

Our goal now is to explicity write down the S operator for the generalized Fourier frames. We start with some elementary calculations.

$$2ab \le a^2 + b^2, \quad 2bc \le b^2 + c^2, \quad 2ac \le a^2 + c^2.$$

$$\begin{aligned}(a+b+c)^2 &= a^2 + b^2 + c^2 + 2ab + 2bc + 2ac \\ &\le a^2 + b^2 + c^2 + (a^2 + b^2) + (b^2 + c^2) + (a^2 + c^2) \\ &= 3(a^2 + b^2 + c^2).\end{aligned}$$

So, if $\quad A_1 \left(\int_\Lambda |F(\zeta)|^2\, d\zeta\right)^{1/2} \le a^2 + b^2 + c^2,$

then $\quad A_1^2 \int_\Lambda |F(\zeta)|^2\, d\zeta \le (a^{1/2} + b^{1/2} + c^{1/2})^2 \le 3(a+b+c).$

Set $\quad A = \frac{1}{3} A_1^2,\ a = \sum_n |\widehat{F}(t_n)|,\ b = \sum_n |\widehat{F}(\frac{1}{2} t_n)|,\ c = \sum_n |\widehat{F}(\frac{1}{3} t_n)|$, and we obtain:

$$A\ \|F\|^2_{L^2(\Lambda)} \le \sum_n |\widehat{F}(t_n)|^2 + \sum_n |\widehat{F}(\frac{1}{2} t_n)|^2 + \sum_n |\widehat{F}(\frac{1}{3} t_n)|^2$$

We can construct the S operator (with details over the next 2 pages)

$$\forall\ f \in H, \quad Sf = \sum_j \langle f, e_{1j} \rangle e_{1j}\ +\ \sum_j \langle f, e_{2j} \rangle e_{2j}\ +\ \sum_j \langle f, e_{3j} \rangle e_{3j},$$

Here $f = F(\zeta) \in L^2(\Lambda)$,

and $\forall n,\ e_{1n}(\zeta) = e^{-2\pi i \zeta \cdot t_n},\ e_{2n}(\zeta) = e^{-2\pi i \zeta \cdot (1/2) t_n},\ e_{3n}(\zeta) = e^{-2\pi i \zeta \cdot (1/3) t_n},\ \langle f, e_{1n} \rangle = F(\hat{t_n})$

From the S operator, we obtain:

$$\begin{aligned}\langle Sf, f\rangle &= \sum_j |\langle f, e_{1j}\rangle|^2 + \sum_j |\langle f, e_{2j}\rangle|^2 + \sum_j |\langle f, e_{3j}\rangle|^2 \\ &\geq A\,\|F\|^2_{L^2\Lambda} \qquad\qquad \text{i.e. } S^{-1} \text{ exists.}\end{aligned}$$

We now provide more details about constructing the S operator.

We use direct sum of two copies of a Hilbert space l_2 for the sake of clarity. This construction can easily be extended to direct sum of three or more copies of a Hilbert space.

We consider a linear operator L and the adjoint operator L^*.

$$L : H \to l_2^{(1)} \oplus l_2^{(2)}$$

$$L^* : l_2^{(1)} \oplus l_2^{(2)} \to H$$

Let $\{e_{1j}\} \in H, \{e_{2j}\} \in H$.

Let $c^{(1)} = (c_{11}, c_{12}, c_{13}, ...) \in l_2^{(1)}$

and $c^{(2)} = (c_{21}, c_{22}, c_{23}, ...) \in l_2^{(2)}$.

Then $c^{(1)} \oplus c^2 \in l_2^{(1)} \oplus l_2^{(2)}$.

$$\Longrightarrow \|c^{(1)} \oplus c^{(2)}\|^2_{l_2^{(1)} \oplus l_2^{(2)}} = \|c^{(1)}\|^2_{l_2^{(1)}} + \|c^{(2)}\|^2_{l_2^{(2)}}$$

$S = L^*L : H \to H$

$$\forall\ f \in H, \quad Lf = (\langle f, e_{11}\rangle, \langle f, e_{12}\rangle, \langle f, e_{13}\rangle, ...\ ) \oplus (\langle f, e_{21}\rangle, \langle f, e_{22}\rangle, \langle f, e_{23}\rangle, ...\ )$$

We calculate the adjoint operator L^* and it is given by:

$$L^*(c^{(1)} \;\oplus\; c^{(2)}) = \sum_j c_{1j} e_{1j} \;+\; \sum_j c_{2j} e_{2j}$$

Then $\forall f \in H$, we have:

$$\begin{aligned} Sf &= \langle f, e_{11}\rangle e_{11} + \langle f, e_{12}\rangle e_{12} + \langle f, e_{13}\rangle e_{13} + ... \\ &+ \langle f, e_{21}\rangle e_{21} + \langle f, e_{22}\rangle e_{22} + \langle f, e_{23}\rangle e_{23} + ... \end{aligned}$$

i.e. $Sf = \sum_j \langle f, e_{1j}\rangle e_{1j} \;+\; \sum_j \langle f, e_{2j}\rangle e_{2j}$

Remark: Given $E = \{t_n\}$, let $e_{1n} = e^{-2\pi i\zeta \cdot t_n}, e_{2n} = e^{-2\pi i\zeta \cdot (1/2)t_n}$.

4.1.3 Discussion

With minor changes, we can modify the proof of the last theorem to obtain the following version of the theorem.

Theorem 4.1 (**revised version**) Let $E = \{t_n\}$ be a separated sequence in $\mathbb{R}^d$. Let $\Lambda \subseteq \mathbb{R}^d$ be a set of spectral synthesis and symmetric about 0. Assume Λ is a convex set. Assume Balayage is possible for (E, Λ). Then $\exists\ A > 0,$ such that $\forall F \in L^2(\Lambda)$,

$$A\ \frac{\int_\Lambda |F(\zeta) + F(2\zeta) + F(3\zeta)|^2\ d\zeta}{\left(\int_\Lambda |F(\zeta)|^2\ d\zeta\right)^{1/2}}$$

$$\leq \quad \left(\sum_{n\in\mathbb{Z}^d} |\widehat{F}(t_n)|^2\right)^{1/2} + \frac{1}{2}\left(\sum_{n\in\mathbb{Z}^d} |\widehat{F}(\frac{1}{2}t_n)|^2\right)^{1/2} + \frac{1}{3}\left(\sum_{n\in\mathbb{Z}^d} |\widehat{F}(\frac{1}{3}t_n)|^2\right)^{1/2}.$$

Proof. We follow the proof of theorem 4.1, with minor changes.

Fix $y \in \mathbb{R}^d$. Consider these measures:

$$\mu_1 = \delta_y,\ \mu_2 = \delta_{2y},\ \mu_3 = \delta_{3y}.$$

For j = 1, 2, 3: define $\quad \nu_j(x) = h(x - j\ y)\ \left(\delta_{jy} - \sum_n \alpha_n(jy)\ \delta_{t_n}\right)(x).$

$$\begin{aligned} \nu_1(x) &= h(x - \ y)\ \left(\delta_y - \sum_n \alpha_n(y)\ \delta_{t_n}\right)(x). \\ \nu_2(x) &= h(x - 2\ y)\ \left(\delta_{2y} - \sum_n \alpha_n(2y)\ \delta_{t_n}\right)(x). \\ \nu_3(x) &= h(x - 3\ y)\ \left(\delta_{3y} - \sum_n \alpha_n(3y)\ \delta_{t_n}\right)(x). \end{aligned}$$

We do not assume $\alpha_n(y)$ is related to $\alpha_n(2y)$ or $\alpha_n(3y)$.

$$\sum_n |\alpha_n(y)| \leq K_1(E, \Lambda_{\epsilon_1}) \quad \sum_n |\alpha_n(2y)| \leq K_2(E, \Lambda_{\epsilon_2}) \quad \sum_n |\alpha_n(3y)| \leq K_3(E, \Lambda_{\epsilon_3}).$$

Pick $\epsilon = \min(\epsilon_1, \epsilon_2, \epsilon_3)$. Let $h \in L^2(\mathbb{R}^d),\ h(0) = 1,\ \text{supp } \hat{h} \subseteq \bar{B}(0, \epsilon)$.

Define $\nu(x) = \nu_1(x) + \nu_2(x) + \nu_3(x)$.

Then $\hat{\nu}_1(\zeta) + \hat{\nu}_2(\zeta) + \hat{\nu}_3(\zeta) = \hat{\nu}(\zeta) = 0,\ \text{ for } \zeta \in \Lambda$.

Let $\phi \in \mathcal{C}(\Lambda)$. Then we have

$$\int \phi\ d\nu = 0. \quad \text{(Spectral synthesis)}$$

$$\begin{aligned}
\Longrightarrow \qquad & \int \phi(x)\ h(x-y)\ d\delta_y + \int \phi(x)\ h(x-2y)\ d\delta_{2y} + \int \phi(x)\ h(x-3y)\ d\delta_{3y} \\
= \quad & \int \phi(x) \sum_n \alpha_n(y)\ h(x-y) d\delta_{t_n} \\
+ \quad & \int \phi(x) \sum_n \alpha_n(2y)\ h(x-2y) d\delta_{t_n} \\
+ \quad & \int \phi(x) \sum_n \alpha_n(3y)\ h(x-3y) d\delta_{t_n}. \\
\Longrightarrow \qquad & \phi(y) + \phi(2y) + \phi(3y) \\
= \quad & \sum_n \left(\alpha_n(y)\ h(t_n - y) + \alpha_n(2y)\ h(t_n - 2y) + \alpha_n(3y)\ h(t_n - 3y)\right)\ \phi(t_n).
\end{aligned}$$

Fix $\zeta \in \Lambda$. Let $\phi(x) = e^{-2\pi i \zeta \cdot x}$.

$$\begin{aligned}
\Longrightarrow \qquad & e^{-2\pi i \zeta \cdot y} + e^{-2\pi i \zeta \cdot 2y} + e^{-2\pi i \zeta \cdot 3y} \\
= \quad & \sum_n \left(\alpha_n(y)\ h(t_n - y) + \alpha_n(2y)\ h(t_n - 2y) + \alpha_n(3y)\ h(t_n - 3y)\right)\ e^{-2\pi i \zeta \cdot t_n} \\
\overset{def}{=} \quad & \sum_n b_n(y) \cdot e^{-2\pi i \zeta \cdot t_n}.
\end{aligned}$$

Let $F \in L^2(\Lambda)$ and $\hat{F} \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

$$\begin{aligned}
& \sum_n \int b_n(y) \cdot \hat{F}(y)\ dy \cdot e^{2\pi i \zeta \cdot t_n} \\
\overset{s}{=} \quad & \int \sum_n b_n(y) e^{2\pi i \zeta \cdot t_n} \cdot \hat{F}(y)\ dy \\
= \quad & \left(e^{2\pi i \zeta \cdot y} + e^{2\pi i \zeta \cdot 2y} + e^{2\pi i \zeta \cdot 3y}\right)\ \hat{F}(y)\ dy \\
= \quad & F(\zeta) + F(2\zeta) + F(3\zeta).
\end{aligned}$$

Let $J_F(\zeta) = F(\zeta) + F(2\zeta) + F(3\zeta)$.

Then $J_F(\zeta) = \sum_n \widetilde{F}(t_n) \cdot e^{2\pi i \zeta \cdot t_n}$; $\quad \widetilde{F}(t_n) = \int b_n(y) \cdot \hat{F}(y)\, dy$.

We compute the inner product $\langle J_F, J_F \rangle_\Lambda$.

$$
\begin{aligned}
& \int_\Lambda J_F(\zeta)\overline{J_F(\zeta)}\, d\zeta \\
= \quad & \int_\Lambda J_F(\zeta) \cdot \sum_n \overline{\widetilde{F}(t_n)} e^{-2\pi i \zeta \cdot t_n}\, d\zeta \\
\overset{s}{=} \quad & \sum_n \int_\Lambda J_F(\zeta) e^{-2\pi i \zeta \cdot t_n}\, d\zeta \cdot \overline{\widetilde{F}(t_n)} \\
= \quad & \sum_n \left(\hat{F}(t_n) + \frac{1}{2}\hat{F}(\frac{1}{2}t_n) + \frac{1}{3}\hat{F}(\frac{1}{3}t_n) \right) \cdot \overline{\widetilde{F}(t_n)} \\
\leq \quad & \left(\left(\sum_n |\hat{F}(t_n)|^2 \right)^{1/2} + \frac{1}{2} \left(\sum_n |\hat{F}(\frac{1}{2}t_n)|^2 \right)^{1/2} + \frac{1}{3} \left(\sum_n |\hat{F}(\frac{1}{3}t_n)|^2 \right)^{1/2} \right)^{1/2} \\
& \cdot \left(\sum_n |\widetilde{F}(t_n)|^2 \right)^{1/2}
\end{aligned}
$$

Now, $\widetilde{F}(t_n) = \int b_n(y)\hat{F}(y)\, dy$. As in the proof of theorem 4.1, we have

$$
\sum_n |\widetilde{F}(t_n)|^2 \leq a \cdot \int_\Lambda |F(\zeta)|^2\, d\zeta
$$

Therefore, we obtain

$$
\begin{aligned}
& A\ \frac{\int_\Lambda |F(\zeta) + F(2\zeta) + F(3\zeta)|^2\, d\zeta}{\left(\int_\Lambda |F(\zeta)|^2\, d\zeta\right)^{1/2}} \\
\leq \quad & \left(\sum_{n \in \mathbb{Z}^d} |\widehat{F}(t_n)|^2 \right)^{1/2} + \frac{1}{2} \left(\sum_{n \in \mathbb{Z}^d} |\widehat{F}(\frac{1}{2}t_n)|^2 \right)^{1/2} + \frac{1}{3} \left(\sum_{n \in \mathbb{Z}^d} |\widehat{F}(\frac{1}{3}t_n)|^2 \right)^{1/2}.
\end{aligned}
$$

□

4.2 Fourier frames on a weighted Hilbert space

Theorem 4.2. *Let $E = \{t_n\}$ be a separated sequence in $\mathbb{R}^d$. Let Λ be a set of spectral synthesis, symmetric about the origin. Assume Balayage is possible for (E, Λ). Let G be any positive bounded function defined on Λ. Define*

$$L^2_G(\Lambda) = \{F : \int_\Lambda |F(\zeta)|^2 \; G(\zeta) \; d\zeta < \infty.\}$$

Then the lower bound inequality holds, i.e. $\exists \; A > 0$, such that $\forall F \in L^2_G(\Lambda)$,

$$A \cdot \frac{\int_\Lambda |F(\zeta)|^2 \; G(\zeta) \; d\zeta}{\left(\int_\Lambda |F(\zeta)|^2 \; d\zeta\right)^{1/2}} \leq \sum_{n \in \mathbb{Z}^d} |\,(F(\zeta) \cdot G(\zeta))\check{}(t_n)|^2.$$

Proof. Fix $y \in \mathbb{R}^d$. Let $\mu(x) = \delta_y$. Let $h \in L^2(\mathbb{R}^d), \text{supp } \hat{h} \in \bar{B}(0, \epsilon)$. Since Balayage holds for (E, Λ), there exists $\{\alpha_n\}$ depending on y, such that for ζ in Λ,

$$(\delta_y)\check{}(\zeta) = \sum_n \alpha_n(y) \; (\delta_{t_n})\check{}(\zeta) \quad \text{where } \sum_n |\alpha_n(y)| \leq K < \infty.$$

As in the proof of Theorem 4.1 we can use μ and h to construct a measure $\nu \in M_b(\mathbb{R}^d)$ such that $\hat{\nu}(\zeta) = 0$ for all ζ in Λ. Define this measure ν by

$$\nu(x) = h(x - y) \; \left(\delta_y - \sum_n \alpha_n(y) \; \delta_{t_n}\right)(x).$$

Fix $\zeta \in \Lambda$. Let $\phi(x) = e^{-2\pi i \zeta \cdot x}$. Then $\phi \in \mathcal{C}(\Lambda)$.

By Spectral synthesis, $\forall \phi \in \mathcal{C}(\Lambda)$, $\hat{\nu}(\zeta) = 0$ on Λ implies $\int \phi \; d\nu = 0$. Hence

$$e^{-2\pi i \zeta \cdot y} = \sum_n \alpha_n(y) \; h(t_n - y) e^{-2\pi i \zeta \cdot t_n}$$

$$
\begin{aligned}
& \int_\Lambda F(\zeta)\cdot\overline{F(\zeta)}\cdot G(\zeta)\ d\zeta \qquad\qquad\qquad\qquad F\in L^2_G(\Lambda) \\
= \ & \int_\Lambda F(\zeta)\cdot G(\zeta)\left(\int_{\mathbb{R}^d}\overline{\widehat{F}(y)}\ e^{-2\pi i\zeta\cdot y}\ dy\right)\ d\zeta \\
= \ & \int_\Lambda F(\zeta)\cdot G(\zeta)\left(\int_{\mathbb{R}^d}\overline{\widehat{F}(y)}\ \sum_n \alpha_n(y)\ h(t_n-y)\ e^{-2\pi i\zeta\cdot t_n}\ dy\right)d\zeta \\
\stackrel{s}{=} \ & \sum_n\left(\int_\Lambda F(\zeta)\cdot G(\zeta)\ e^{-2\pi i\zeta\cdot t_n}\ d\zeta\right)\cdot\left(\int\overline{\widehat{F}(y)}\cdot\alpha_n(y)\ h(t_n-y)\ dy\right) \\
= \ & \sum_n\left(\int_\Lambda F(\zeta)\cdot G(\zeta)\ e^{-2\pi i\zeta\cdot t_n}\ d\zeta\right)\cdot\overline{\widetilde{F}(t_n)} \\
\leq \ & \left(\sum_n |\,(F(\zeta)\cdot G(\zeta))\widehat{\ }(t_n)|^2\right)^{1/2}\cdot\left(\sum_n|\widetilde{F}(t_n)|^2\right)^{1/2}
\end{aligned}
$$

We have shown that: $\sum_n|\widetilde{F}(t_n)|^2\leq a^2\ \left(\int_\Lambda|F(\zeta)|^2\ d\zeta\right)$

$$
\Longrightarrow \qquad \frac{1}{a}\cdot\frac{\int_\Lambda|F(\zeta)|^2\ G(\zeta)\ d\zeta}{\left(\int_\Lambda|F(\zeta)|^2\ d\zeta\right)^{1/2}}\ \leq\ \left(\sum_n|\,(F(\zeta)\cdot G(\zeta))\widehat{\ }(t_n)|^2\right)^{1/2}
$$

$\square$

4.3 Semi-Discrete Gabor frames

Fix a real-valued function $g \in L^2(\mathbb{R}^d)$, where $\|g\|_2 = 1$.

The short-time Fourier transform of a function $F \in L^2(\widehat{\mathbb{R}}^d)$ is given by

$$V_g F(y,\omega) = \int_{\widehat{\mathbb{R}}^d} F(\zeta) g(\zeta - y)\ e^{-2\pi i \zeta\cdot\omega}\ d\zeta.$$

We can recover the original function from its short-time Fourier transform using

$$F(\zeta) = \int_{\mathbb{R}^d} \int_{\widehat{\mathbb{R}}^d} V_g F(y,\omega)\ e^{2\pi i \zeta\cdot\omega}\ g(\zeta - y)\ d\omega\ dy.$$

Theorem 4.3. *Let $g \in L^2(\mathbb{R}^d)$ be real-valued, where $\|g\|_2 = 1$.*

Let $\Lambda \subseteq \mathbb{R}^d$ be a set of spectral synthesis and symmetric about 0.

Let $E = \{t_n : n \in \mathbb{Z}^d\}$ be a separated sequence in $\mathbb{R}^d$.

Assume Balayage is possible for (E, Λ).

Then $\exists\ A > 0$, such that $\forall F \in L^2(\Lambda)$,

$$A \int_\Lambda |F(\zeta)|^2\ d\zeta \leq \int_{\mathbb{R}^d} \sum_{n\in\mathbb{Z}^d} |V_g F(y, t_n)|^2\ dy.$$

Proof.

$$\begin{aligned}
& \int F(\zeta)\ \overline{F(\zeta)}\ d\zeta \\
= & \int F(\zeta)\ \left(\int\int \overline{V_g F(y,\omega)}\ e^{-2\pi i\zeta\cdot\omega}\ g(\zeta-y)\ d\omega\ dy\right)\ d\zeta \\
= & \int F(\zeta)\ \left(\int\int \overline{V_g F(y,\omega)}\ \left((\sum_n \alpha_n(\omega)\ h(t_n-\omega)\ e^{-2\pi i\zeta\cdot t_n})\ g(\zeta-y)\ d\omega\ dy\right)\right)\ d\zeta \\
\overset{s}{=} & \int\int \overline{V_g F(y,\omega)}\ \sum_n \alpha_n(\omega)\ h(t_n-\omega)\ \left(\int F(\zeta)\ e^{-2\pi i\zeta\cdot t_n}\ g(\zeta-y)\ d\zeta\right)\ dy\ d\omega \\
= & \int\int \overline{V_g F(y,\omega)}\ \sum_n \alpha_n(\omega)\ h(t_n-\omega)\ V_g F(y,t_n)\ dy\ d\omega
\end{aligned}$$

This implies that

$$
\begin{aligned}
& \int F(\zeta)\ \overline{F(\zeta)}\ d\zeta \\
= & \int\int \sum_n \alpha_n(\omega)\ h(t_n-\omega)\ \overline{V_gF(y,\omega)} \quad V_gF(y,t_n)\ dy\ d\omega \\
\overset{s}{=} & \int \sum_n \left(\int \alpha_n(\omega)\ h(t_n-\omega)\ \overline{V_gF(y,\omega)}\ d\omega \quad V_gF(y,t_n) \right)\ dy \\
\leq & \int \left(\sum_n \left| \int \alpha_n(\omega)\ h(t_n-\omega)\ \overline{V_gF(y,\omega)}\ d\omega \right|^2 \right)^{1/2} \left(\sum_n |V_gF(y,t_n)|^2 \right)^{1/2}\ dy
\end{aligned}
$$

by Cauchy-Schwarz inequality

We will show on the next page that there exists a constant $a > 0$ such that:

$$\sum_n \left| \int \alpha_n(\omega)\ h(t_n-\omega)\ \overline{V_gF(y,\omega)}\ d\omega \right|^2 \leq a^2 \int |V_gF(y,\omega)|^2\ d\omega.$$

Continuing the proof, this implies that:

$$
\begin{aligned}
& \int F(\zeta)\ \overline{F(\zeta)}\ d\zeta \\
\leq & \int a \left(\int |V_gF(y,\omega)|^2\ d\omega \right)^{1/2} \left(\sum_n |V_gF(y,t_n)|^2 \right)^{1/2}\ dy \\
\leq & \ a \left(\int\int |V_gF(y,\omega)|^2\ d\omega\ dy \right)^{1/2} \left(\int \sum_n |V_gF(y,t_n)|^2\ dy \right)^{1/2}
\end{aligned}
$$

by Cauchy-Schwarz inequality

$$= a \left(\int_\Lambda |F(\zeta)|^2\ d\zeta \right)^{1/2} \left(\int \sum_n |V_gF(y,t_n)|^2\ dy \right)^{1/2}$$

where we used the fact that $\int\int |V_gF(y,\omega)|^2\ d\omega\ dy = \int_\Lambda |F(\zeta)|^2\ d\zeta$

$$\implies \quad \frac{1}{a} \left(\int_\Lambda |F(\zeta)|^2\ d\zeta \right)^{1/2} \leq \left(\int_{\mathbb{R}^d} \sum_{n\in\mathbb{Z}^d} |V_gF(y,t_n)|^2\ dy \right)^{1/2}.$$

$$\implies \quad \frac{1}{a} \int_\Lambda |F(\zeta)|^2\ d\zeta \leq \int_{\mathbb{R}^d} \sum_{n\in\mathbb{Z}^d} |V_gF(y,t_n)|^2\ dy.$$

□

We now show that there exists a constant $a > 0$ such that:

$$\sum_n \left| \int \alpha_n(\omega)\ h(t_n - \omega)\ \overline{V_g F(y,\omega)}\ d\omega \right|^2 \le a^2 \int |V_g F(y,\omega)|^2\ d\omega.$$

Proof.

$$\begin{aligned}
& \sum_n \left| \int \alpha_n(\omega)\ h(t_n - \omega)\ \overline{V_g F(y,\omega)}\ d\omega \right|^2 \\
\le\ & \sum_n \left(\int |\alpha_n(\omega)|\ |h(t_n - \omega)|^2\ d\omega \right) \left(\int |\alpha_n(\omega)|\ |V_g F(y,\omega)|^2\ d\omega \right) \\
& \qquad\qquad\qquad\qquad \text{by Cauchy-Schwarz inequality} \\
\le\ & \sum_n \left(C_1 \int |h(t_n - \omega)|^2\ d\omega \right) \left(\int |\alpha_n(\omega)|\ |V_g F(y,\omega)|^2\ d\omega \right) \quad \text{; since } \sum_n |\alpha_n(\omega)| \le \infty \\
\le\ & \sum_n C_1\ C_2 \int |\alpha_n(\omega)|\ |V_g F(y,\omega)|^2\ d\omega \ \ \text{; since } h \in L^2(\mathbb{R}^d) \\
=\ & M \int \left(\sum_n |\alpha_n(\omega)| \right) |V_g F(y,\omega)|^2\ d\omega \\
\le\ & M\ K \int |V_g F(y,\omega)|^2\ d\omega.
\end{aligned}$$

□

Using a Gabor system for non-orthogonal expansion was suggested by Von Neumann in Mathematical Foundations of Quantum Mechanics. It is a curious that on page 407 [vN55], we can find the following sentence: "The proof of this fact leads to rather tedious calculations, which require no new concepts, and we shall omit them." From the context of the paragraph where the quote is taken, it is not clear what "this fact" refers to.

We have used the term semi-discrete frame to suggest that the sequence of elements $\{g(\zeta - y)\, e^{-2\pi i \zeta \cdot t_n} : y \in \mathbb{R}^d, t_n \in E\}$ in the Hilbert space $L^2(\Lambda)$ is a hybrid between a (fully discrete) frame and a fully continuous frame.

Analogous to a frame in a Hilbert space, we can define a fully continuous frame in a natural manner.

Definition 4.1. Let H be a separable complex Hilbert space. Let M be a measurable space with a (positive) measure μ. A sequence of elements $\{x_n : n \in M\}$ is a continuous frame for H with respect to the measure space (M, μ) if there are positive constants A and B such that

$$\forall\ f \in H, \quad A\|f\|^2 \leq \int_M |\langle f, x_n\rangle|^2\ d\mu(n) \leq B\|f\|^2.$$

For all f in H, the mapping $n \mapsto \langle f, x_n\rangle$ is a measurable function on M.

In particular, when the measure space (M, μ) is $(\mathbb{R}^d \times \widehat{\mathbb{R}}^d, dx \times d\omega)$, the sequence of functions $\{g_{y,\omega} \equiv g(x-y)e^{2\pi i x \cdot \omega} : y \in \mathbb{R}^d, \omega \in \widehat{\mathbb{R}}^d\}$ is a fully continuous frame for a Hilbert space H if there are positive constants A and B such that

$$\forall\ f \in H, \qquad A\|f\|^2 \leq \int_{\widehat{\mathbb{R}}^d} \int_{\mathbb{R}^d} |\langle f, g_{y,\omega}\rangle|^2\ dy\ d\omega \leq B\|f\|^2.$$

Analogously, the sequence of functions $\{g_{y,\omega} \equiv g(x-y)e^{2\pi i x \cdot t_n} : y \in \mathbb{R}^d, t_n \in E\}$ is a semi-discrete frame for a Hilbert space H if there are $A, B > 0$ such that

$$\forall\ f \in H, \qquad A\|f\|^2 \leq \int_{\mathbb{R}^d} \sum_{n \in \mathbb{Z}^d} |\langle f, g_{y,t_n}\rangle|^2\ dy \leq B\|f\|^2.$$

□

The next theorem is more difficult than the previous one.

Theorem 4.4. *Let $E = \{t_n\}$ be a separated sequence in $\mathbb{R}^d$.*

Let $\Lambda \subseteq \mathbb{R}^d$ be a set of spectral synthesis and symmetric about 0.

Assume Balayage is possible for (E, Λ).

Let $g \in L^2(\mathbb{R}^d)$ be real-valued, where $\|g\|_2 = 1$. We also assume

(i) g is a bounded continuous function,

(ii) supp $\hat{g} \subseteq \Lambda$.

Then $\exists\ A > 0,$ *such that* $\forall F \in L^2(\Lambda),$

$$A \int_\Lambda |F(\zeta)|^2\ d\zeta \leq \sum_{n \in \mathbb{Z}^d} \int_{\mathbb{R}^d} |V_g F(t_n, \omega)|^2\ dy.$$

Proof. Fix $y \in \mathbb{R}^d$. Let $\mu = \delta_y$.

Balayge holds for (E, Λ) implies there exists a sequence $\{\alpha_n(y)\}$ such that

$$\forall \zeta \in \Lambda_\epsilon, \quad (\delta_y)\check{}(\zeta) = \sum_n \alpha_n(y) e^{-2\pi i\ t_n \cdot \zeta}$$

Let $h \in L^2(\mathbb{R}^d)$, with support of $\hat{h} \subseteq \bar{B}(0, \epsilon)$.

Construct a measure ν such that $\forall\ \zeta \in \Lambda, \hat{\nu}(\zeta) = 0$ by setting

$$\nu(x) = h(x - y)(\delta_y - \sum_n \alpha_n(y)\ \delta_{t_n})(x).$$

Let $g \in \mathcal{C}(\Lambda)$. This implies for any fixed $\zeta \in \Lambda,$ $\text{supp}(g(\zeta - \cdot))\check{}(\gamma) \subseteq \Lambda.$

By Spectral synthesis, $\int g\ d\nu = 0$.

$$\implies \quad \forall \zeta \in \Lambda, \quad g(\zeta - y) = \sum \alpha_n(y)\ h(t_n - y)\ g(\zeta - t_n).$$

$$
\begin{aligned}
& \int F(\zeta)\ \overline{F(\zeta)}\ d\zeta \qquad\qquad\qquad\qquad F \in L^2(\Lambda) \\
= & \int\int |V_gF(y,\omega)|^2\ d\omega\ dy \\
= & \int\int V_gF(y,\omega) \cdot \overline{V_gF(y,\omega)}\ d\omega\ dy \\
= & \int\int V_gF(y,\omega)\ \left(\int \overline{F(\zeta)}\ g(\zeta - y)\ e^{2\pi i \zeta\cdot\omega}\ d\zeta\right) d\omega\ dy \\
= & \int\int V_gF(y,\omega) \left(\int \overline{F(\zeta)}\ \left(\sum_n \alpha_n(y)\ h(t(n) - y)\ g(\zeta - t_n)\right)\ d\zeta\right) d\omega\ dy \\
\stackrel{s}{=} & \int\int V_gF(y,\omega) \left(\sum_n \alpha_n(y)\ h(t_n - y)\ \int \overline{F(\zeta)}\ g(\zeta - t_n)\ e^{2\pi i \zeta\cdot\omega}\ d\zeta\right) d\omega\ dy \\
= & \int\int V_gF(y,\omega) \left(\sum_n \alpha_n(y)\ h(t_n - y)\ \overline{V_gF(t_n,\omega)}\right) d\omega\ dy \quad (*)
\end{aligned}
$$

Let us show that

$$
\sum_n |\int \alpha_n(y) h(t_n - y)\ V_gF(y,\omega)\ dy|^2 \le a^2 \int |V_gF(y,\omega)|^2\ dy. \quad (**)
$$

$$
\begin{aligned}
& \sum_n |\int \alpha_n(y) h(t_n - y)\ V_gF(y,\omega)\ dy|^2 \\
\le\ & \sum_n \left(\int |\alpha_n(y)\ h(t_n - y)|^2\ dy\right)\ \left(\int |\alpha_n(y)|\ |V_gF(y,\omega)|^2\ dy\right) \\
\le\ & \sum_n \left(C_1\ \int |h(t_n - y)|^2\ dy\right) \left(|\alpha_n(y)|\ |V_gF(y,\omega)|^2\ dy\right) \\
\le\ & \sum_n C_1\ C_2\ \int |\alpha_n(y)|\ |V_gF(y,\omega)|^2\ dy; \quad h \in L^2(\mathbb{R}^d) \\
=\ & M\ \int (\sum_n |\alpha_n(y)|)\ |V_gF(y,\omega)|^2\ dy; \quad \forall\ y, \sum_n |\alpha_n(y)| \le K \\
=\ & a^2\ \int |V_gF(y,\omega)|^2\ dy.
\end{aligned}
$$

Therefore, by $(*)$, we have

$$
\begin{aligned}
&\int F(\zeta)\ \overline{F(\zeta)}\ d\zeta = \int\int |V_gF(y,\omega)|^2\ d\omega\ dy \\
= \ & \int\int V_gF(y,\omega)\left(\sum_n \alpha_n(y)\ h(t_n-y)\ \overline{V_gF(t_n,\omega)}\right) d\omega\ dy \\
\overset{s}{=} \ & \sum_n \int\int V_gF(y,\omega)\left(\alpha_n(y)\ h(t_n-y)\ \overline{V_gF(t_n,\omega)}\right) d\omega\ dy \\
= \ & \sum_n \left[\int \overline{V_gF(t_n,\omega)}\ \left(\int \alpha_n(y)\ h(t_n-y)\ V_gF(y,\omega)\ dy\right)\ d\omega\right] \\
\leq \ & \sum_n \left[\left(\int |V_gF(t_n,\omega)|^2\ d\omega\right)^{1/2} \cdot \left(\int |\int \alpha_n(y)\ h(t_n-y)\ V_gF(y,\omega)\ dy|^2\ d\omega\right)^{1/2}\right] \quad \text{Cauchy-Schwarz} \\
\leq \ & \left(\sum_n \int |V_gF(t_n,\omega)|^2\ d\omega\right)^{1/2} \cdot \left(\sum_n \int |\int \alpha_n(y)\ h(t_n-y)\ V_gF(y,\omega)\ dy|^2\ d\omega\right)^{1/2}
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\sum_n \int |\int \alpha_n(y)\ h(t_n-y)\ V_gF(y,\omega)\ dy|^2\ d\omega \\
= \ & \int \left(\sum_n |\int \alpha_n(y) h(t_n-y)\ V_gF(y,\omega)\ dy|^2\right) d\omega \\
\leq \ & \int a^2 \left(\int |V_gF(y,\omega)|^2\ dy\right) d\omega \qquad \text{from } (**) \\
= \ & a^2 \int\int |V_gF(y,\omega)|^2\ dy\ d\omega \\
= \ & a^2\ \|F\|^2_{L^2(\Lambda)}.
\end{aligned}
$$

$$
\Longrightarrow \quad \|F\|^2_{L^2(\Lambda)} \leq \left[\sum_n \left(\int |V_gF(t_n,\omega)|^2 d\omega\right)\right]^{1/2} \cdot a \cdot \|F\|_{L^2(\Lambda)}
$$

$$
\Longrightarrow \quad A \cdot \|F\|^2_{L^2(\Lambda)} \ \leq \ \sum_n \int |V_gF(t_n,\omega)|^2 d\omega.
$$

□

4.4 Bilinear frame operator

We begin with a useful lemma that will simplify our calculations later.

Lemma 4.1. *(Convolution with a radial function is a self-adjoint operator)*

Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ *be a radial function, i.e.* $\forall\ x \in \mathbb{R}^d,\ \psi(x) = \overline{\psi(|x|)}$.

Define an operator $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ *by*

$$\forall\ f \in L^2(\mathbb{R}^d), \quad (Tf)(x) = \int_{\mathbb{R}^d} f(x-y)\psi(x)\ dy = (f * \psi)(x).$$

Then T *is a self-adjoint operator. That means,*

$$\forall\ f, g \in L^2(\mathbb{R}^d), \quad \langle Tf, g\rangle = \langle f, Tg\rangle.$$

Proof. Let $f, g \in L^2(\mathbb{R}^d)$.

We first note that $Tf \in L^2(\mathbb{R}^d)$, since $\|f * \psi\|_2 \leq \|f\|_2\ \|\psi\|_1$.

$$\begin{aligned}
\langle Tf, g\rangle &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(x-y)\psi(y)\ dy \right) \overline{g(x)}\ dx \\
&= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(y)\psi(x-y)\ dy \right) \overline{g(x)}\ dx \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y)\overline{\psi(y-x)}\ dy\ \overline{g(x)}\ dx; \qquad \psi \text{ is radial}, \psi(x) = \overline{\psi(|x|)} \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y)\ dy \cdot \overline{\psi(y-(x+y))}\ \overline{g(x+y)}\ dx; \qquad x \mapsto x+y \\
&= \int_{\mathbb{R}^d} f(y)\ dy \cdot \int_{\mathbb{R}^d} \overline{\psi(-x)}\ \overline{g(x+y)}\ dx \\
&= \int_{\mathbb{R}^d} f(y)\ dy \cdot \int_{\mathbb{R}^d} \overline{\psi(x)}\ \overline{g(y-x)}\ dx; \qquad x \mapsto -x \\
&= \int_{\mathbb{R}^d} f(y) \cdot \int_{\mathbb{R}^d} \overline{\psi(x)}\ \overline{g(y-x)}\ dx\ dy; \\
&= \int_{\mathbb{R}^d} f(y)\overline{(g * \psi)(y)}\ dy \\
&= \langle f, Tg\rangle.
\end{aligned}$$

□

Consider a bilinear operator $B : H \times H \to H$.

$\forall f, g \in H,\ B(f,g) \in H$, and B is linear in each of the two variables separately.

The Hilbert space H is either $L^2(\mathbb{R}^d)$ or $L^2(K)$, where K is a compact subset of $\mathbb{R}^d$.

Let $\psi, \phi \in \mathcal{S}(\mathbb{R}^d)$ with compact support, and

$$\int \psi\ dx = 0, \quad \int \phi\ dx = 1. \qquad \psi \text{ is radial, i.e. } \psi(x) = \overline{\psi(|x|)}.$$

We normalize ψ so that $\displaystyle\int_0^\infty |\hat{\psi}(t\zeta)|^2\ \frac{dt}{t} = 1.$ Write $\psi_t(x) = \dfrac{1}{t^d}\psi\left(\dfrac{x}{t}\right)$.

Define $B(f,g)$ by:

$$B(f,g) = \int_0^\infty \psi_t * \left((\psi_t * f) \cdot (\phi_t * g)\right)(x)\ \frac{dt}{t}.$$

This gives us a function of x in H.

Consider $\langle B(f,g), e_n \rangle = \displaystyle\int_{\mathbb{R}^d}\int_0^\infty \psi_t * ((\psi_t * f) \cdot (\phi_t * g)) \cdot e_n\ \frac{dt}{t}\ dx,$ where $e_n \in H$.

Note: If $g \equiv 1$, then $\langle B(f,1), e_n \rangle = \langle f, e_n \rangle$,

i.e. $\sum_n |\langle f, e_n \rangle|^2 = \sum_n |\langle B(f,1), e_n \rangle|^2$.

$$\begin{aligned}
\langle B(f,g), e_n \rangle &= \int_{\mathbb{R}^d}\int_0^\infty \psi_t * ((\psi_t * f) \cdot (\phi_t * g)) \cdot e_n\ \frac{dt}{t}\ dx \\
&= \int_{\mathbb{R}^d}\int_0^\infty (\psi_t * f) \cdot (\phi_t * g) \cdot (\psi_t * e_n)(x)\ \frac{dt}{t}\ dx \qquad \text{by Lemma 4.1} \\
&= \int_{\mathbb{R}^d}\int_0^\infty f(x) \cdot (\phi_t * g) \cdot (\psi_t * \psi_t * e_n)(x) \frac{dt}{t}\ dx \\
&= \int_{\mathbb{R}^d}\int_0^\infty (\psi_t * \psi_t * e_n)(x) \cdot (\phi_t * g)(x) \cdot f(x)\ \frac{dt}{t}\ dx
\end{aligned}$$

We are now ready to construct the S operator.

$\forall\ f, g \in H, B(f,g) \in H.\quad B : H \times H \to H.$

Fix $g \in H$. Define $M : H \to l_2$ by

$$\forall f \in H, \quad Mf = (\langle B(f,g),\ e_1 \rangle,\ \langle B(f,g),\ e_2 \rangle,\ \langle B(f,g),\ e_3 \rangle, \ldots).$$

Calculate the adjoint operator of M. $\quad M : H \to l_2. \quad M^* : l_2 \to H.$

Let $c = (c_1, c_2, c_3, ...)$ be a sequence in l_2.

$$\langle M^*c, f\rangle_H \;=\; \langle c, Mf\rangle_{l_2} \;=\; \sum_n c_n \; \langle B(f,g), e_n\rangle_H$$

$M : H \to l_2. \quad M^* : l_2 \to H. \quad S = M^*M : H \to H.$

$$Sf = M^*Mf = M^* \left(\langle B(f,g),\; e_1\rangle,\; \langle B(f,g),\; e_2\rangle,\; \langle B(f,g),\; e_3\rangle, ...\right)$$

By the calculations on $B(f,g)$ in the previous page, we can see that:

$$\begin{aligned}
\langle B(f,g),\; e_n\rangle \;&=\; \int_{\mathbb{R}^d}\int_0^\infty f(x)\cdot(\phi_t * g)\cdot(\psi_t * \psi_t * e_n)(x)\;\frac{dt}{t}\;dx \\
&=\; \int_{\mathbb{R}^d}\int_0^\infty (\psi_t * \psi_t * e_n)(x)\cdot(\phi_t * g)(x)\cdot f(x)\;\frac{dt}{t}\;dx \\
&=\; \int_{\mathbb{R}^d} Q(e_n, g)\; f(x)\; dx; \\
&\qquad\qquad \text{where } Q(e_n, g) = \int_{\mathbb{R}^d} (\psi_t * \psi_t * e_n)(x)\cdot(\phi_t * g)(x)\;\frac{dt}{t}
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow\; \langle M^*c,\; f\rangle_H = \langle c, Mf\rangle_{l_2} \;&=\; \sum_n c_n\cdot\langle Q(e_n,g), f\rangle \\
&=\; \langle \sum_n c_n\cdot Q(e_n,g), f\;\rangle;\quad \text{where } Q(e_n,g)\in H \\
\Longrightarrow\; M^*c \;&=\; \sum_n c_n\cdot Q(e_n,g);\quad c = (c_1,c_2,c_3,...),\quad M^*: l_2\to H. \\
\text{Hence,}\;\; Sf \;&=\; \sum_n \left(\langle B(f,g),\; e_n\rangle_H\cdot Q(e_n,g)\right);\quad S: H\to H
\end{aligned}$$

i.e. $\;S_g f = \sum_n \langle Q(e_n,g),\; f\rangle_H \cdot Q(e_n,g).\quad$ This is our bilinear frame operator.

We have constructed a bilinear frame operator. Let us summarize all our calculations in the following lemma.

Lemma 4.2. *(Bilinear frame operator) Let H be the Hilbert space $L^2(\mathbb{R}^d)$.*

Let $\psi, \phi \in \mathcal{S}(\mathbb{R}^d)$ with compact support, and

$$\int \psi \; dx = 0, \quad \int \phi \; dx = 1. \qquad \psi \text{ is radial, i.e. } \psi(x) = \overline{\psi(|x|)}.$$

$$\text{We normalize } \psi \text{ so that } \int_0^\infty |\hat{\psi}(t\zeta)|^2 \; \frac{dt}{t} = 1. \qquad \text{Write } \psi_t(x) = \frac{1}{t^d}\psi\left(\frac{x}{t}\right).$$

Consider a bilinear operator $B : H \times H \to H$.

$\forall f, g \in H,\ B(f, g) \in H$, and B is linear in each of the two variables separately.

Define $B(f, g)$ by:

$$B(f, g) = \int_0^\infty \psi_t * ((\psi_t * f) \cdot (\phi_t * g))\,(x) \; \frac{dt}{t}.$$

Let $\{e_n\}_{n=1}^\infty$ be a sequence in H. Fix $g \in H$. Define $M : H \to l_2$ by

$$\forall f \in H, \quad Mf = (\langle B(f, g),\ e_1\rangle,\ \langle B(f, g),\ e_2\rangle,\ \langle B(f, g),\ e_3\rangle, ...).$$

Let S be the operator obtained by composing M and M^*.

$M : H \to l_2$. $\quad M^* : l_2 \to H$. $\quad S = M^*M : H \to H$. Then

$$\begin{aligned} Sf &= \sum_n \langle B(f, g),\ e_n\rangle \cdot Q(e_n, g) \\ &= \sum_n \langle Q(e_n, g),\ f\rangle \cdot Q(e_n, g), \end{aligned}$$

$$\text{where } Q(e_n, g) = \int_{\mathbb{R}^d} (\psi_t * \psi_t * e_n)(x) \cdot (\phi_t * g)(x) \; \frac{dt}{t}$$

Proof. Our calculations in the previous pages established the lemma.

□

To prove that the bilinear frame operator is bounded, we need some preparation.

Lemma 4.3. *Let H be a separable Hilbert space (or a separable Banach space). Let M be a dense subspace of H. Let $B : M \times H \to H$ be a bilinear operator such that*

$$\forall\ f \in M,\ \forall\ g \in H, \quad \|B(f,g)\|_H \leq C \cdot \|f\|_H \|g\|_H. \qquad (*)$$

Then $(*)$ holds for all f in H, for all g in H, and B extends to a bounded bilinear operator from $H \times H$ to H.

Proof. For each f in H, there exists a sequence $\{f_n\}_{n=1}^{\infty} \in M$ such that

$\|f - f_n\|_H \to 0,$ as $n \to \infty$. Since for each $g \in H$,

$$\begin{aligned} & \|B(f_m, g) - B(f_n, g)\|_H \\ = & \ \|B(f_m - f_n, g)\|_H \\ \leq & \ C \cdot \|f_m - f_n\|_H \cdot \|g\|_H \quad \text{by } (*). \end{aligned}$$

So, for each $g \in H, \{B(f_n, g)\}_{n=1}^{\infty}$ is a Cauchy sequence in H.

Hence, $B(f_n, g)$ converges in H to an element in H, and we can define a bounded bilinear operator $B(f,g) : H \times H \to H$ by $B(f,g) = \lim_{n\to\infty} B(f_n, g)$.

□

Definition 4.2. (BMO) If f is a locally integrable function in $\mathbb{R}^d$, we say that $f \in BMO$ if for any cube $Q \subseteq \mathbb{R}^d$,

$$\frac{1}{Q} \int_Q |f(x) - f_Q| dx < \infty.$$

Here, f_Q is the average of f over the cube. The integration is over the cube.

Definition 4.3. If $f \in BMO \cap L^2(\mathbb{R}^d)$, then $f \in L^2_{BMO}(\mathbb{R}^d)$.

The following inequality is due to Fefferman and Stein [FS72].

Theorem 4.5. *Let $f \in BMO$. Let $\psi \in \mathcal{S}$ be such that $\int \psi \, dx = 0$. Then there exists a constant $C > 0$ such that for any cube Q,*

$$\frac{1}{|Q|} \int_Q \int_0^{l(Q)} |f * \psi_t(x)|^2 \, \frac{dt}{t} \, dx \leq C \, \|f\|_{BMO}^2.$$

Proof. Translation of a function does not change the BMO norm of a function, so we may assume without loss of generality that the cube Q is centered at the origin. Let Q^* be the cube with the same center as Q and whose side length is $2\sqrt{d}$ that of Q. Let f_Q be the average of f on f_Q and f_{Q^*} the average of f on Q^*.

Since $\int \psi \, dx = 0$, therefore $f * \psi_t(x) = (f - f_{Q^*}) \, * \, \psi_t(x)$.

Write $|\,(f - f_{Q^*}) * \psi_t(x)|^2 = |f_1 * \psi_t(x) + f_2 * \psi_t(x)|^2$, where $f_1 = (f - f_{Q^*})1_{Q^*}$ and $f_2 = (f - f_Q^*)1_{\{\mathbb{R}^d \setminus Q^*\}}$.

$$\begin{aligned} \Longrightarrow \quad \int_Q \int_0^{l(Q)} |f * \psi_t(x)|^2 \, \frac{dt}{t} \, dx \;&\leq\; 2 \int_Q \int_0^{l(Q)} |(f - f_Q^*)1_{Q^*} \, * \, \psi_t(x)|^2 \, \frac{dt}{t} \, dx \\ &+\; 2 \int_Q \int_0^{l(Q)} |(f - f_Q^*)1_{\{\mathbb{R}^d - Q^*\}} \, * \, \psi_t(x)|^2 \, \frac{dt}{t} \, dx \\ &=\; I_1 + I_2. \end{aligned}$$

To estimate I_1, we apply Plancherel Theorem,

$$I_1 \leq 2 \int_{\mathbb{R}^d} \int_0^{\infty} |\,(\,(f - f_{Q^*})1_{Q^*}\,)\widehat{\ }(\zeta)|^2 \, |\widehat{\psi}(t\zeta)|^2 \, \frac{dt}{t} \, d\zeta$$

By the estimate $\int_0^\infty |\widehat{\psi}(t\zeta)|^2 \, \frac{dt}{t} \leq C_1$ and by the result of John-Nirenberg inequality ($\forall \, p, 1 < p < \infty, \; \sup_Q \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p \, dx \right) \leq C_2 \; p \; \Gamma(p) \; \|f\|_{BMO}^p$), we obtain

$$I_1 \leq C \cdot \int_{Q^*} |f - f_{Q^*}|^2 \;\leq\; C \; |Q| \cdot \|f\|_{BMO}^2.$$

To estimtate I_2, let Q_n be the cube centered at the origin and whose side length is 2^n times that of Q^*. Then

$$I_2 \leq \int_Q \int_0^\infty |\sum_{n=0}^\infty \int_{Q_{n+1}-Q_n} |\,(f(y) - f_{Q^*})\ \psi_t(x-y)\ dy|^2\ \frac{dt}{t}\ dx.$$

Since $x \in Q$, if $y \notin Q_n$, then $2^{n-1}\ l(Q) \leq |x-y|$.

Since $\psi \in \mathcal{S}$, we also have the estimate $|\psi_t(x-y)| \leq C\ t^{-d}\ (t^{-d}\ 2^n \cdot l(Q))^{-d-1}$.

$$\begin{aligned} \text{Hence,} \quad I_2 &\leq C \int_Q \int_0^{l(Q)} |\sum_{n=0}^\infty \int_{Q_{n+1}} |f(y) - f_{Q^*}|\ dy \cdot \frac{t}{(2^n\ l(Q))}|^2\ \frac{dt}{t}\ dx \\ &\leq \int_Q \int_0^{l(Q)} \left(\sum_{n=0}^\infty \frac{t}{2^n\ l(Q)}\ \|f\|_{BMO} \right)^2\ \frac{dt}{t}\ dx \\ &\leq C \cdot |Q| \cdot \|f\|_{BMO}^2. \end{aligned}$$

□

Remark

The space $L^2_{BMO}(\mathbb{R}^d)$ is a dense subspace of $L^2(\mathbb{R}^d)$ in the topology of $L^2(\mathbb{R}^d)$.

Theorem 4.6. *(Boundedness of bilinear frame operator)*

Assume the hypothesis of Lemma 4.2. Define $B(f,g)$ by:

$$\forall\ f,g \in L^2(\mathbb{R}^d), \quad B(f,g) = \int_0^\infty \psi_t * ((\psi_t * f)\cdot(\phi_t * g))\,(x)\ \frac{dt}{t}.$$

Then there exists $C>0$ such that for each $e_n \in H$, with $\|e_n\|_2 = 1$,

$$\forall\ f \in L^2_{BMO}(\mathbb{R}^d), g \in L^2(\mathbb{R}^d), \quad |\langle B(f,g), e_n\rangle| \le C\cdot\|f\|_2\cdot\|g\|_2.$$

Proof. Let $f \in L^2_{BMO}(\mathbb{R}^d)$. Let $g \in L^2(\mathbb{R}^d)$. Let $e_n \in L^2(\mathbb{R}^d)$, with $\|e_n\|_2 = 1$.

$$\begin{aligned}
& \langle B(f,g), e_n\rangle \\
= & \int_{\mathbb{R}^d}\int_0^\infty \psi_t * ((\psi_t * f)\cdot(\phi_t * g))\,(x)\cdot e_n(x)\ \frac{dt}{t}\ dx \\
= & \int_{\mathbb{R}^d}\int_0^\infty (\psi_t * f)\cdot(\phi_t * g)(x)\cdot(\psi_t * e_n)(x)\ \frac{dt}{t}\ dx \\
\le & \left(\int_{\mathbb{R}^d}\int_0^\infty |\psi_t * f|^2(x)\cdot|\phi_t * g|^2(x)\ \frac{dt}{t}\ dx\right)^{1/2}\left(\int_{\mathbb{R}^d}\int_0^\infty |\psi_t * e_n|^2(x)\ \frac{dt}{t}\ dx\right)^{1/2} \\
\equiv & I_1\cdot I_2
\end{aligned}$$

Let $G(x) \overset{def}{=} \int_0^\infty |\phi_t * g|^2(x)\ \frac{dt}{t}$. Then $\|G\|_2^2 = C_1\cdot\|g\|_2^2$ by Plancherel Theorem.

$$\Longrightarrow\ I_1^2 = \int_{\mathbb{R}^d}\int_0^\infty |\psi_t * f|^2(x)\cdot|\phi_t * g|^2(x)\ \frac{dt}{t}\ dx \le C_1\cdot\|f\|^2_{BMO}\cdot\int_{\mathbb{R}^d}|g(x)|^2\ dx.$$

In the last inequality, we used Theorem 4.5 (the Fefferman-Stein inequality) .

Another application of Plancherel's Theorem gives us the following:

$$I_2^2 = \int_{\mathbb{R}^d}\int_0^\infty |\psi_t * e_n|^2(x)\ \frac{dt}{t}\ dx \le C_2\cdot\|e_n\|_2.$$

Hence,

$$|\langle B(f,g), e_n\rangle| \le C\cdot\|f\|_{BMO}\cdot\|g\|_2\cdot\|e_n\|_2.$$

Recall that the space $L^2_{BMO}(\mathbb{R}^d)$ is a dense subspace of $L^2(\mathbb{R}^d)$. Therefore B is a bounded bilinear operator on a dense subspace of $L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$. □

4.5 Poisson kernel

In this section, we give an application of Balayage of Fourier transforms.

Consider the following problem:

Let $\mathsf{S} = \{f : \text{supp } \widehat{f} \subseteq [-\Omega - \epsilon, \Omega + \epsilon]\}$.

Let $E = \{t_n\}$ be a separated set of sampling points in $\mathbb{R}$.

Given a function $f \in \mathsf{S}$, find a function u such that there exists a sequence $\{\alpha_n\}$,

$$u(x) = \sum_n \alpha_n \ f(x - t_n), \text{ and } \sum_n |\alpha_n| < \infty.$$

Theorem 4.7. *Fix $y > 0$. Define the Poisson kernel P_y by*

$$\forall \ x \in \mathbb{R}, \quad P_y(x) = \frac{1}{\pi} \frac{y}{y^2 + x^2}.$$

Then the function $u = f * P_y$ solves our problem. That means, there exists a function u such that $u = f * P_y$, and such that

$$u(x) = \sum_n \alpha_n \ f(x - t_n), \ \textit{and} \ \sum_n |\alpha_n| < \infty.$$

Proof. We fix $y > 0$. With the poission kernerl P_y defined as above, we have

$$\forall \ \zeta \in \Lambda, \quad \widehat{P_y}(\zeta) = e^{-2\pi y |\zeta|}.$$

Balayage holds for (E, Λ) implies (for fixed $y > 0$), there exists $\{\alpha_n\}$ depending on y, with $\sum_n |\alpha_n| < \infty$, such that

$$\forall \ \zeta \in \Lambda, \quad \widehat{P_y}(\zeta) = \sum_n \alpha_n(y) \ (\delta_{t_n})\widehat{\ }(\zeta).$$

$$\implies \quad e^{-2\pi y |\zeta|} = \sum_n \alpha_n(y) \ e^{-2\pi i \zeta \cdot t_n}$$

$$
\begin{aligned}
& \int \widehat{f}(\zeta)\ e^{-2\pi y|\zeta|}\ e^{2\pi i x\zeta} \qquad\qquad \text{supp } \widehat{f} \subseteq \Lambda \\
= \ & \int \widehat{f}(\zeta) \left(\sum_n \alpha_n(y)\ e^{-2\pi i\zeta\cdot t_n} \right) e^{2\pi i x\zeta}\ d\zeta \\
= \ & \int \widehat{f}(\zeta) \sum_n \alpha_n(y)\ e^{2\pi i\zeta\cdot(x-t_n)}\ d\zeta \\
\stackrel{s}{=} \ & \sum_n \alpha_n(y) \int \widehat{f}(\zeta)\ e^{2\pi i\zeta\cdot(x-t_n)}\ d\zeta; \qquad\qquad \sum_n |\alpha_n(y)| < \infty \\
= \ & \sum_n \alpha_n(y)\ f(x-t_n).
\end{aligned}
$$

$$
\Longrightarrow \quad \int f(x-t)\ P_y(t)\ dt = \sum_n \alpha_n(y)\ f(x-t_n); \quad y > 0
$$

But the left side is $u(x,y) \stackrel{def}{=} (f * P_y)(x)$. This completes the proof.

$\square$

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