ABSTRACT

Title of dissertation: THE STRUCTURE OF OFFSHELL 3D SUPERSPACE WITH ARBITRARY N, COUPLED TO A YANG MILLS FIELD

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We introduce the form of 3D Superspace as a Z_2-graded Lie Algebra. A set of Superspace Algebras can be defined in this way, characterized by the number of supercharges, N. Superspace will then be coupled to a Yang Mills field, to create a covariant derivative algebra. Various constraints must then be implemented, for consistency and in order to form an irreducible representation of this algebra. An in-depth view into the well-known example of N = 2 will be given, as well as a comparison with the corresponding compactification of 4D N=1. Then, the theory will be expanded to N = 3 and N = 4, before attempting to generalize to arbitrary N. However, beyond N = 4, the theory is discovered to only be known to be consistent at N = 8. The corresponding 2-point superfield and component actions are shown, as a basis for later theories

THE STRUCTURE OF OFFSHELL 3D SUPERSPACE WITH ARBITRARY N, COUPLED TO A YANG MILLS FIELD

by

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Chapter 1

Introduction

1.1 The Standard Model and Beyond

It has long been a goal of humanity to take a glimpse into the mind of God, to determine how the world and everything around it operates. Through numerous experiments and discoveries, humans have continually improved on their limited understanding of the world, from the religion-based force dieties of the early polytheistic world, to the geocentric solar system models and Euclid geometries of the Greeks, Newtonian mechanics and the heliocentric solar system, Electromagnetism and Maxwell's equations, and finally Einstein's theory of Relativity and the rise of Quantum Mechanics and Quantum Field Theory, as well as countless other improvements in between. To this point, we have condensed our current knowledge of the universe as a whole into two separate realms, Gravitation, led by Einstein's Relativity, describing the physics of the very large, and the Standard Model of Particle Physics, describing the physics of the very small.

The Standard Model describes in detail all of the known particles and their interactions, to the precision of the main experiments performed to date. It is split into two sectors, the Electroweak sector and the Strong sector, each of which combined describes three of the four forces of nature.

The Electroweak sector unites the Electromagnetic and Weak Nuclear forces.[1]

The Electromagnetic force is governed by Quantum Electrodynamics, a Yang Mills theory based on an abelian U(1) gauge group, and in fact Yang Mills theory itself is based upon expanding Quantum Electrodynamics to incorporate more general, non-abelian gauge groups into a framework. It describes the photon, a gauged vector boson. The weak force is governed by a massive SU(2) Yang Mills theory, complete with two charged weak vector bosons and a neutral weak vector boson. Electroweak Theory predicts that above a certain energy scale, the Electroweak Scale, the photon and the neutral weak boson mix, encasing both into a combined SU(2) x U(1) Yang Mills theory. In addition to these bosons, Electroweak theory also describes the leptons

The Strong sector contains Quantum Chromodynamics (QCD), another Yang Mills theory, with the gauge group SU(3). It describes the quarks and the gluons, as well as their interactions. One important property that QCD has is that of 'Asymptotic Freedom', stating that the force between two strongly-charged particles decreases at high energies, or alternatively, low distances. Inversely, this leads to confinement, since as the force increases with distance, it reaches a point where it becomes energetically more favorable to create a quark-antiquark pair than to continue separating two quarks.[2]

In addition, the Standard model postulates the existence of a Higgs boson. The Higgs would be responsible for the breaking of the Electroweak force into the Electromagnetic and Weak forces. However, this particle has not been observed so far. The Large Hadron Collider is hoped to provide the discovery of this final Standard model particle.

While the Standard model has been extremely successful as a theory, several issues still remain unresolved, suggesting that it cannot be the final theory. Recent results have suggested that neutrinos in fact do have mass and can oscillate between their three generations.[3] While it is likely this result will be incorporated into the Standard Model, there is still debate as to how to best do this. Solving this issue has thus become a key part of many theories beyond the Standard model.

The Standard Model itself also does not include the force of gravity, and to this date the complete theory of quantum gravity is not known. Still, the scale at which gravity is expected to unite with particle physics is very high, some 16 orders of magnitude greater than the Electroweak Scale. It would require extreme fine-tuning of the many renormalized coupling constants in order to produce the range of particles seen in nature. It is thus expected that new physics will be discovered past the Electroweak Scale.

Many theorists have attempted to expand and unify the Standard Model into a single Yang Mills theory with a simple gauge group. Two famous models are the Georgi-Glashow SU(5) model[4] and the SO10 grand unified theory. These models make use of the running of the Electroweak and Strong coupling constants, noting that at a particular scale, 10^{16} GeV, the constants become comparable with each other. One drawback, however, is that the symmetries allow for proton decay, which has not been observed in nature.

1.2 Supersymmetry

Over the last 30 years, supersymmetry has become one of the leading elements in many physical models beyond the Standard Model. The Minimal Supersymmetric Standard Model (MSSM) takes the Standard Model and its gauge group and couples it to an N=1 Supersymmetry, in effect doubling the particle content of the Standard Model.[5] This has had some success as a theory, but none of the new particles predicted by the theory have been observed in experiments as of yet

Supersymmetric models tend to propose that each particle in the Standard model is paired up with a complementary particle, the 'superpartner.'[6] These particles are frozen out below the Electroweak Scale, breaking supersymmetry, but above, they provide instant cancellation of problematic interactions, making the process of renormalization proceed more naturally and the bounds on the running of the particle masses stop at the Electroweak scale rather than continue up to the Strong-Electroweak unification scale or the Planck scale.

However, as promising as supersymmetry has been, all but a handful of supersymmetric theories are only known on-shell, in the presence of equations of motion, whereas a complete quantum description of the theory requires it to be known off-shell. Theories with 4 supercharges, such as 4D N=1 Super Yang Mills (SYM) and Supergravity theories have been well studied, and the resulting offshell multiplets are known in detail, however, proceeding to higher supercharges has proven much more difficult. Harmonic Superspace and Projective Superspace are two methods for producing some results with higher supercharges, but they have the undesirable property of producing an infinite-dimensional multiplets.[7, 8] In expanding 4D N=1 to 4D N=2, Harmonic Superspace deals with 'complexifying' the N=1 fields, expanding the gauge group of the supercharges from U(1) to SU(2) through an analytic continuation.[9] Projective Superspace seeks to perform this same kind of analytic continuation, but it differs from Harmonic Superspace in that it assigns a projective isospin coordinate to the fermionic coordinates.[10]. Determining the structure of maximally-supersymmetric theories, however, has been the quest of many. Two maximally-supersymmetric theories in particular, 4D N=4 Super Yang Mills and 4D N=8 Supergravity are both suggested to possess many desirable features, such as ultraviolet finiteness for numerous supersymmetry-breaking interactions.[11, 12] These theories are the direct dimensional-reduction of two superstring theories within M theory – the 10D N=1 open string (Type I string theory) and the 10D N=(1,1) closed string (Type IIA string theory), two of the primary components of M-theory.[13] However, the complete offshell structure for these theories is not known, and it is with this in mind that this manuscript has been written.

1.3 3D Chern-Simons Theories

Recently, there has been extensive research into the realm of 3D Chern-Simons theories with extended supersymmetry. In particular, much of the motivation for this explosion of research is the discovery of the conformally-invariant N=6 ABJM model,[14] and the Bagger-Lambert-Gustavsson theory of maximal N=8.[15, 16, 17] It is expected that the dual to these two theories using the AdS/CFT duality is within the AdS(4) x S^7 background of M-theory.[18] However, the full offshell supersymmetry is not realized in either of these two theories, or in any other extended 3D SYM or CS theory beyond N=2. Providing an offshell realization to these and other 3D theories, complete with supersymmetric kinetic terms, would go a long way toward the development of these theories.

1.4 Plan for this Thesis

The goal of this thesis is to provide the framework for which complete off-shell theories of 3D SYM and Chern-Simons theories can be constructed. It can also be shown that theories in 4 dimensions can be reduced to equivalent theories in 3 dimensions using the process of dimension-reduction, and this will be demonstrated for a pair of known theories, 3D N=2 SYM and 4D N=1 SYM. While no known process is available to return to 4 dimensions, the field content discovered in the 3 dimensional theories is known to be equivalent in certain cases to corresponding theories in 4 dimensions[19]. In particular, the maximal case of 3D N=8 will be examined, and this is known to be the direct compactification of the active research field of 4D N=4. It will be determined that such a 3D N=8 SYM offshell multiplet does exist, containing 128 bosonic and fermionic field components, in agreement with the Roček-Siegel Theorem[20], after application of a self-duality condition, one that also ensures the equivalence of 3D N=3 and 3D N=4.

Chapter 2

An Illustrative Example: 3D N=2

2.1 Introduction

We begin by studying in detail a well-known test case, the theory of 3D N=2 Superspace coupled to a U(1) gauge field. The process used throughout this example can be used in larger, more general theories. This procedure begins by establishing the covariant derivative algebra of the theory, evaluating the various constraints required to promote this algebra into an irreducible representation, as well as preserve its integrity. After the field content and its relations are established, a prepotential superfield can be determined, containing the field content as components of the superfield. This prepotential comes with a natural 2-point action, as well as the possibility for additional actions to be built from it. The component-form of this action can be determined by integrating out the fermionic directions.

2.2 Defining the Algebra

The standard derivative algebra for a $3D$ $\mathcal{N}=2$ theory can look like:

$$\begin{aligned} [D_\alpha, D_\beta\} &= 0 \\ [D_\alpha, \bar{D}_\beta\} &= 2i(\gamma^a)_{\alpha\beta}\partial_a \\ [D_\alpha, \partial_a\} &= 0 \end{aligned}$$

$$[\partial_a, \partial_b\} \quad = 0 \tag{2.1}$$

where the $\mathcal{N} = 2$ is displayed through the use of a complex fermionic derivative, $D_\alpha = D^1_\alpha + iD^2_\alpha$.

In order to couple this algebra to a $\mathcal{U}(1)$ gauge field, one needs to create covariant derivatives with respect to the $\mathcal{U}(1)$.

$$\begin{aligned} \nabla_\alpha = &\quad D_\alpha + ig\Gamma_\alpha t \\ \nabla_a = &\quad \partial_a + ig\Gamma_a t \end{aligned} \tag{2.2}$$

where t is the $\mathcal{U}(1)$ generator and Γ is the connection. The generator t is pure imaginary and does not act on the covariant derivatives

$$[t, \nabla_A\} = 0 \tag{2.3}$$

By taking into account the inherent symmetries of the graded commutators, we can look for a theory of the following form:

$$\begin{aligned} [\nabla_\alpha, \nabla_\beta\} &= 2i((\gamma^a)_{\alpha\beta}A_a)(igt) \\ [\nabla_\alpha, \bar{\nabla}_\beta\} &= 2i(\gamma^a)_{\alpha\beta}(\nabla_a + B_a(igt)) + 2iC_{\alpha\beta}B)(igt) \\ [\nabla_\alpha, \nabla_a\} &= ((\gamma_a)^\beta_\alpha W_\beta + \tilde{C}_{\alpha a})(igt) \\ [\nabla_a, \nabla_b\} &= ig(\epsilon_{ab}{}^c F_c)t \end{aligned} \tag{2.4}$$

with all other terms related through complex conjugation,

$$\begin{aligned} [\bar{\nabla}_\alpha, \bar{\nabla}_\beta\} &= 2i((\gamma^a)_{\alpha\beta}\bar{A}_a)(igt) \\ [\bar{\nabla}_\alpha, \nabla_\beta\} &= 2i(\gamma^a)_{\alpha\beta}(\nabla_a + B_a(igt)) - 2iC_{\alpha\beta}B)(igt) \\ [\bar{\nabla}_\alpha, \nabla_a\} &= -((\gamma_a)^\beta_\alpha \bar{W}_\beta + \bar{\tilde{C}}_{\alpha a})(igt) \end{aligned} \tag{2.5}$$

2.3 Constraints on the Algebra

In order to reduce this algebra, there are several constraints that we can apply. As long as the constraints do not involve differential equations of motion, the theory will remain off-shell. There are three major kinds of constraints used to attempt to reduce an algebra like the one above into an irreducible form. The first deals with preserving the structure of fields, and is called a representation-preserving constraint. As this is a complex representation, in order to be able to define a chiral superfield in this theory, a certain representation-preserving constraint must be enforced.[21]

$$A_a = 0 \tag{2.6}$$

The second constraint type are the conventional constraints. Conventional constraints deal with evaluating the form of the covariant derivative algebra in terms of the superspace algebra and the corresponding connections. If there are redundancies in the degrees of freedom at the algebraic level, rather than the differential level, the redundancies can be set conventionally to get rid of unnecessary fields. In determining (4), there are two different terms that go into creating B_a. From a carefully-chosen relationship between Γ_a, Γ_α, and $\bar{\Gamma}_\alpha$, we can choose this constraint conventionally.

$$B_a = 0 \tag{2.7}$$

There is one minor consistency relation that should be taken into account, the fact that $\overline{[\nabla_\alpha, \bar{\nabla}_\beta\}} = -[\nabla_\beta, \bar{\nabla}_\alpha\}$. This forces one more constraint.

$$Im(B) = 0 \tag{2.8}$$

or in other words, B is real.

Finally, one more set of major constraints must be imposed, in order to ensure consistency of this algebra. The algebra must always satisfy the Jacobi identities. Jacobi identities with covariant derivatives replacing actual fields are called Bianchi Identities. These will provide most of the remaining constraints.[22]

$$(-1)^{AC}[\nabla_A,[\nabla_B,\nabla_C\}\}+(-1)^{BA}[\nabla_B,[\nabla_C,\nabla_A\}\}+(-1)^{CB}[\nabla_C,[\nabla_A,\nabla_B\}\}=0 \tag{2.9}$$

The convention for capital A here is that it must run over all indices, a and α, and the product $AB=1$ for A and B both spinor indices, $AB=0$ for all other products.

2.4 Solving the Bianchi Identities

In general, the double graded-commutator above will have the following form.

$$(-1)^{AC}[\nabla_A,[\nabla_B,\nabla_C\}\}=F_{ABC}{}^{D}\nabla_D+G_{ABC}t \tag{2.10}$$

So, we're left with the following set of equations.

$$\begin{aligned}&(F_{ABC}{}^{D}+F_{BCA}{}^{D}+F_{CAB}{}^{D})\nabla_D=0\\&(G_{ABC}+G_{BCA}+G_{CAB})t=0\end{aligned} \tag{2.11}$$

Separating out the indices from the equation above, there are several independent equations that must be satisfied. In order to keep track of whether or not I'm using ∇_α or $\bar{\nabla}_\alpha$, a bar will be placed on the spinor index in the latter. It does not mean a separate index. There are no equations of the top form, because the only way to obtain a F_{ABC} term is through the $[\nabla_\alpha,\bar{\nabla}_\beta\}$ graded commutator, and this piece always then ends in ∇_a, which does not allow any terms of the F_{ABC} form. The equations to examine thus are,

$$
\begin{aligned}
&(G_{\alpha\beta\bar{\gamma}} + G_{\beta\bar{\gamma}\alpha} + G_{\bar{\gamma}\alpha\beta})t = 0 \\
&(G_{\alpha\beta c} + G_{\beta c\alpha} + G_{c\alpha\beta})t = 0 \\
&(G_{\alpha\bar{\beta}c} + G_{\alpha\bar{\beta}c} + G_{c\alpha\bar{\beta}})t = 0 \\
&(G_{\alpha bc} + G_{bc\alpha} + G_{c\alpha b})t = 0 \\
&(G_{abc} + G_{bca} + G_{cab})t = 0 \qquad (2.12)
\end{aligned}
$$

Solving these equations gives the following conditions and variations.

$$
\begin{aligned}
\tilde{C}_{\alpha a} &= 0, (\nabla_\alpha B) = \bar{W}_\alpha \\
\nabla_\alpha \bar{W}_\beta &= 0 \\
\bar{\nabla}_\alpha \bar{W}_\beta &= (\gamma^a)_{\alpha\beta}(i\nabla_a B - F_a) + C_{\alpha\beta} d \\
\nabla_\alpha F_a &= -\epsilon_a{}^{bc}(\gamma_b)_\alpha^\beta(\nabla_c W_\beta) \\
\nabla^a F_a &= 0 \qquad (2.13)
\end{aligned}
$$

The last equation proves that F is not fundamental, and $\exists V_a$ s. t. $\epsilon^a{}_{bc} F_a = \nabla_b V_c - \nabla_c V_b$. V_a is thus a $U(1)$ gauge field.

The conjugate equation to the third identity also shows that F_a and $\nabla_a B$ are both purely real. However, there is one combination of variations that is not constrained by these equation. $C^{\alpha\beta}(\bar{\nabla}_\alpha \bar{W}_\beta + \nabla_\alpha W_\beta)$ can still be anything. I called this field d, so that $C^{\alpha\beta}(\bar{\nabla}_\alpha \bar{W}_\beta + \nabla_\alpha W_\beta) = +4d$. d also appears to be forced to be purely real.

One can then find the variation with respect to d by evaluating the following:

$$
C^{\beta\gamma}[\nabla_\alpha, \bar{\nabla}_\beta\} W_\gamma = 2\nabla_\alpha d
$$

$$
\begin{aligned}
&= -2i(\gamma^a)_\alpha^\gamma \nabla_a W_\gamma \\
\nabla_\alpha d &= -i(\gamma^a)_\alpha^\gamma \nabla_a W_\gamma
\end{aligned} \tag{2.14}
$$

2.5 Working with Prepotentials

Earlier, we saw that in order to couple the superspace to a $\mathcal{U}(1)$ gauge field, the introduction of christoffel symbols was required. These in particular reveal an underlying complex superfield, U.[21]

$$
\begin{aligned}
\Gamma_\alpha &= D_\alpha U \\
\bar{\Gamma}_\alpha &= \bar{D}_\alpha \bar{U} \\
\Gamma_a &= -\frac{i}{4}(\gamma_a)^{\alpha\beta}(D_\alpha(\bar{D}_\beta \bar{U}) + \bar{D}_\beta(D_\alpha U))
\end{aligned} \tag{2.15}
$$

Here, I also will define the following symbols:

$$
\begin{aligned}
[D_\alpha, D_\beta] &= 2C_{\alpha\beta} D^2 \\
[D_\alpha, \bar{D}_\beta] &= 2C_{\alpha\beta}\tilde{D}^2 + 2(\gamma^a)_{\alpha\beta}\Delta_a
\end{aligned} \tag{2.16}
$$

Note that, if U was real, $\Gamma_a = \partial_a U$. What can instead be said is, defining $V = Im(U)$

$$
\Gamma_a = \partial_a(Re(U)) - \Delta_a V \tag{2.17}
$$

We can then take a second look at the equation:

$$
[\nabla_\alpha, \bar{\nabla}_\beta\} = 2i(\gamma^a)_{\alpha\beta}\nabla_a + 2iC_{\alpha\beta}B(igt) \tag{2.18}
$$

and solve for B in terms of U or, alternatively, in terms of $Re(U)$ and V

$$
B = \tilde{D}^2 V \tag{2.19}
$$

Notice that B only depends on V. It does not depend on $Re(U)$. This procedure can be continued for the W and F fields as well.

$$\begin{aligned} W_\alpha &= D_\alpha \tilde{D}^2 V \\ F_c &= \epsilon^{ab}{}_c \partial_a (\Delta_b V) \end{aligned} \tag{2.20}$$

The auxiliary field, d is solved for in a slightly different way. Recall that

$$d = \frac{1}{2} C^{\alpha\beta} \bar{\nabla}_\alpha \bar{W}_\beta \tag{2.21}$$

The representation for W in terms of V can be conjugated and substituted in here, resulting in the following expression for d.

$$d = \tilde{D}^2 \tilde{D}^2 V \tag{2.22}$$

There are two possible real fermionic 4 forms that can be defined, $\tilde{D}^2\tilde{D}^2$ and $\{D^2, \bar{D}^2\}$. They are related by the following:

$$\tilde{D}^2 \tilde{D}^2 = -\frac{1}{4}\{D^2, \bar{D}^2\} - \Box V \tag{2.23}$$

For simplicity, we define $D^4 = \tilde{D}^2\tilde{D}^2$, so

$$d = D^4 V \tag{2.24}$$

2.6 The Wess-Zumino Gauge

The superfield, V, can be represented as a sum of components, each with its own product of grassmann numbers. Basically, it should take the following form:

$$V = V_0 + V_1^\alpha \theta_\alpha + V_2^\alpha \bar{\theta}_\alpha + V_3 \theta^2 + V_4^{\alpha\beta} \theta_\alpha \bar{\theta}_\beta + V_5 \bar{\theta}^2 + V_6^\alpha \theta^2 \bar{\theta}_\alpha + V_7^\alpha \theta_\alpha \bar{\theta}^2 + V_8 \theta^2 \bar{\theta}^2 \tag{2.25}$$

where θ represents an arbitrary grassmann variable.

Since V is real, $V_2 = -\bar{V}_1$, $V_3 = -\bar{V}_5$ and $V_7 = -\bar{V}_6$, and V_0, , V_4, and V_8 are all real. V_8 is the highest independent form, since the grassmann variables anticommute with each other,

The Wess-Zumino Gauge in this case sets several of these components, V_0, V_1, V_2, V_3, and V_5 all equal to 0.[23] The motivation for this is that each of these components do not appear to be necessary in describing the physical field degrees of freedom. This however is not quite enough, as we must also make sure that this choice of gauge does not introduce any on-shell equations in other ways, particularly with influencing the vector gauge field, $V_a = \frac{1}{2}\Delta_a V$, where $F_a = \epsilon_a{}^{bc}(\partial_b V_c - \partial_c V_b)$

Note back when we defined $\Gamma_\alpha = D_\alpha U$. If we make the following variation,

$$\delta U = \phi \tag{2.26}$$

where ϕ is a chiral superfield, Γ_α does not change. This represents a gauge degree of freedom in the superfield U, and consequently, in V. The same variation on V is

$$\delta V = -\frac{i}{2}(\bar{\phi} - \phi) \tag{2.27}$$

The idea of setting the superfield component V_0 is only consistent then if the variation of this component can be set to 0 offshell. We can check to see if this affects the gauge degree of freedom of V_a then by calculating its variation.

$$\delta V_a = \frac{1}{32}(\gamma_a)^{\alpha\beta}(D_\alpha \bar{D}_\beta \bar{\phi} - D_\alpha \bar{D}_\beta \phi - \bar{D}_\beta D_\alpha \bar{\phi} + \bar{D}_\beta D_\alpha \phi) \tag{2.28}$$

Because ϕ is a chiral superfield, and thus $\bar{\phi}$ is an antichiral superfield, $\bar{D}_\beta \phi = 0$ and $D_\alpha \bar{\phi} = 0$. Using this and the algebra, we can simplify the last equation to the following.

$$\delta V_a = \frac{i}{8}\partial_a(\bar{\phi} + \phi) \tag{2.29}$$

This expression thus relies on the real part of ϕ and is thus completely independent of setting the imaginary part of ϕ to 0.

The Wess-Zumino Gauge thus is a way to break supersymmetry explicitly, without breaking the bosonic gauge invariance.[22]

2.7 The Superspace Action

We are looking for an action of the form,[22]

$$S = a \int d^3x d^4\theta (Vd) \tag{2.30}$$

since d is the highest-ranking field in engineering dimension, and V is the superfield that appears to be connected with the fields.

For superfields, it turns out that integrating over grassmannian variables is necessarily equivalent to differentiating with respect to the variable, and evaluating in the limit the fermionic variables go to 0. This is due to the uniqueness theorem. Integration and differentiation must both be linear operations such that, if θ is a fermionic variable, $O(1) = 0$ and $O(\theta) = 1$. Since $\theta^2 = 0$, there can only be one operation that handles both.

Using the fact that integration over grassmannian variables is equivalent to differentiation, this reduces to the form of the component action, as expected

$$S_c = b \int d^3x [d^2 + B\Box B - \eta^{ab} F_a F_b + i(\gamma_a)^{\alpha\beta}(\bar{W}_\alpha \partial_a W_\beta + W_\alpha \partial_a \bar{W}_\beta)] \tag{2.31}$$

b is fixed by requiring the proper kinetic term coefficients.

2.8 Equations of Motion and Green's Functions

Once a Lagrangian is determined, the next step in solving the theory is to determine the physical equations of motion. Because we can redefine all the fields by scaling each one by a constant factor, we can, without loss of generality, choose the value of b. The choice $b = \frac{1}{2}$ gives the proper kinetic terms for each of the different fields

Auxiliary: $L = \frac{1}{2}d^2$

Scalar: $L = \frac{1}{2}B\Box B = -\frac{1}{2}\eta^{ab}\partial_a B\partial_b B$ mod total derivatives

Vector: $L = -\frac{1}{2}\eta^{ab}F_a F_b = \frac{1}{2}\eta^{ab}\eta cd(\partial_a V_c\partial_d V_b - \partial_a V_c\partial_b V_d)$

Spinor: $L = \frac{i}{2}(\gamma^a)^{\alpha\beta}(\bar{W}_\alpha\partial_a W_\beta + W_\alpha\partial_a\bar{W}_\beta)$

The idea next is to use Calculus of Variations on each Lagrangian to determine the equations of motion.

$$\frac{\delta L}{\delta X} - \partial_a\left(\frac{\delta L}{\delta(\partial_a X)}\right) = 0 \qquad (2.32)$$

Auxiliary: $d = 0$

Just as expected from an auxiliary field, the equations of motion for d cause it to drop out as an observable field. It only exists off-shell.

Scalar: $\Box B = 0$

Vector: $(\eta^{ab}\eta^{cd} - \eta^{ac}\eta^{bd})\partial_a\partial_b V_c = 0$

Spinor: $\frac{i}{2}(\gamma^a)^{\alpha\beta}\partial_a W_\beta = 0$

These equations of motion show how each of the fields act in free space, in the absence of sources of the fields. The Green's function of a field show how the given field reacts to a source. The equations of motion above are written for free space. If one was to add a source term at a point in space, x_0, each of the field equations can be seen to exist in the following form.

$$\mathfrak{D}(\partial_a)G(x) = J(x = x_0) \tag{2.33}$$

J is the source term, $\mathfrak{D}$ is a derivative operator, not necessarily linear, and G is the solution to the field with this source, called the Green's function. Indices were suppressed in this equation. The solution to this equation is shown below.

$$G(x) = \Delta(x - x_0)J(x_0) \tag{2.34}$$

Δ is called the Propagator, and in addition to determining the Green's function, it also is used in Quantum Field Theory to determine the probability of a particle traveling between the two points, x and x_0. So this is what needs to be calculated.

One of the most common methods of solving for the Propagator is the method of Fourier Transforms. By using a Fourier transformation, the differential equation in position space is transformed into an algebraic equation in momentum space. This can be solved, then transformed back into position space.

The Fourier transformation of a scalar field in one dimension, f(x), is defined by the integral,

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx f(x) exp(-ipx) \tag{2.35}$$

with inverse transformation

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp \tilde{f}(p) exp(ipx) \tag{2.36}$$

The constant factor, $1/\sqrt{2\pi}$, is determined by calculating a successive transformation and inverse transformation on a gaussian test function, then assuming a symmetric factor between the transformation and its inverse. This transformation can be extended into three dimensions by use of a triple integral.

$$\begin{aligned} \tilde{f}(\vec{p}) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3x f(x) exp(-ip_a x^a) \\ f(\vec{x}) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3p \tilde{f}(p) exp(ip_a x^a) \end{aligned} \tag{2.37}$$

So we finally have the tools necessary to compute the Propagator for each of the different fields. The auxiliary equation is trivial, simply $G(x) = J(x_0)$ if and only if $x = x_0$, or to put it another way, $\Delta(x - x_0) = \begin{cases} 1: & x - x_0 = 0 \\ 0: & x - x_0 \neq 0 \end{cases}$

The Fourier transformation has the ability to turn derivatives on position space into products of momentum, $\partial_a \to ip_a$

$$\begin{aligned} &\Box G(x) = J(x) \\ &\frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3p \tilde{G}(p) \eta^{ab}(ip_a)(ip_b) exp(ip_c x^c) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3p \tilde{J}(p) exp(ip_c x^c) \\ &-\tilde{G}p^2 = \tilde{J} \\ &\tilde{\Delta}(p) = -\frac{1}{p^2 + i\epsilon} \end{aligned} \tag{2.38}$$

Here, the addition of the term $i\epsilon$, for an infinitesimal ϵ, is used to offset a root that would otherwise be in the complex line of integration, when transforming back to position space.

$$G(x) = \frac{1}{(2\pi)^3} \int_{\Re^3} d^3p (exp(ip_a x^a)) \left(-\frac{1}{p^2 + i\epsilon} \right) \int d^3\tilde{x} J(\tilde{x}) exp(-ip_b \tilde{x}^b) \tag{2.39}$$

For the case $J(x) = \delta(x - x_0)$,

$$G(x) = \frac{1}{(2\pi)^3} \int_{\Re^3} d^3p(exp(ip_a(x^a - x_0^a))) \left(-\frac{1}{p^2 + i\epsilon} \right) \tag{2.40}$$

The other fields solve similarly, with a few differences. First, the Green's function and Propagator now have indices.

Vector: $G_a(x) = \Delta_{ab}(x - x_0)J^b(x_0)$

Spinor: $G_\alpha(x) = \Delta_{\alpha\beta}(x - x_0)J^\beta(x_0)$

Next, in the process of solving for the propagator in the Vector case, we arrive at the following equation.

$$\frac{1}{2}(\eta^{ab}\eta^{cd} - \eta^{ac}\eta^{bd})p_a p_c \tilde{G}_b = \tilde{J}^d \tag{2.41}$$

This is solved by taking the ansatz,

$$\tilde{G}_b = \frac{(a\eta_{ac}\eta_{bd} + b\eta_{ab}\eta_{cd})p^a p^c}{p^4}\tilde{J}^d \tag{2.42}$$

where a and b are constants to be determined. Solving for a and b, we get $a = 1$, but b is unconstrained. This is related to the fact that on-shell, the vector gauge field has not 2, but only 1 degree of freedom. The constant b acts like a Lagrange multiplier for gauge fixing this additional degree of freedom lost.[24] So this gives,

$$G_a(x) = \frac{1}{(2\pi)^3} \int_{\Re^3} d^3p(exp(ip_c x^c)) \left(-\frac{\eta_{ab}}{p^2 + i\epsilon} - b\frac{p_a p_b}{(p^2 + i\epsilon)^2} \right) \int d^3\tilde{x} J^b(\tilde{x}) exp(-ip_d \tilde{x}^d) \tag{2.43}$$

Finally, the Green's function for the Spinor field is,

$$G_\alpha(x) = \frac{1}{(2\pi)^3} \int_{\Re^3} d^3p(exp(ip_a x^a)) \left(-\frac{(\gamma^c)_{\alpha\beta} p_c}{p^2 + i\epsilon} \right) \int d^3\tilde{x} J(\tilde{x}) exp(-ip_b \tilde{x}^b) \tag{2.44}$$

Chapter 3

A Brief Look at 4 Dimensions

3.1 Introduction

In the previous chapter, we demonstrated in detail the derivation of a relatively simple example of an off–shell supersymmetric theory. While working in 3 dimensions is arguably simpler than working in 4, and most theories in 4D have not been discovered in depth, the case of 4D N=1 has been solved in detail.[21] Next we will demonstrate the similar theory of 4D N=1 over a U(1) gauge field, and compare this result with what is known for 3D N=2, using the process of dimensional reduction.

3.2 Defining the Algebra

We wish to follow the same procedure as in the previous chapter, to derive a covariant derivative algebra for a 4D N=1 supersymmetric theory, coupled to a U(1) field. First, using the conventions outlined in Appendix B, we can define a pair of fermionic covariant derivatives.

$$\begin{aligned} D_\alpha &= \partial_\alpha + i\frac{1}{2}\bar{\theta}^{\dot{\alpha}}\partial_{\underline{a}} \\ \bar{D}_{\dot{\alpha}} &= \partial_{\dot{\alpha}} + i\frac{1}{2}\theta^{\alpha}\partial_{\underline{a}} \end{aligned} \tag{3.1}$$

so that we have a complete superderivative, $D_A = (D_\alpha, \bar{D}_{\dot{\alpha}}, \partial_{\underline{a}})$

The 4D N=1 Superspace Algebra is thus given as:

$$[D_\alpha, D_{\dot\beta}\} = i\partial_{\underline{a}} \tag{3.2}$$

with all other commutation and anticommutation relations zero.

We then couple these derivatives to a U(1) field,

$$\nabla_A = D_A + \Gamma_A t \tag{3.3}$$

where t is the generator of U(1) transformations, and commutes with the superspace derivatives.

We can build this into a covariant derivative algebra by evaluating the commutation relations

$$[\nabla_A, \nabla_B\} = C_{AB}{}^C \partial_C - iF_{AB}t \tag{3.4}$$

C is called the anholonomity, and F is the field strength.

3.3 Constraints on the Algebra

Based upon expanding out the previous equation, we can see that clearly, $C_{\alpha\dot\alpha}{}^a = i$, while all other terms of the anholonomity are 0. We can set $F_{a\dot\beta}$ and $F_{\dot\alpha\beta}$ to 0 conventionally as well. $F_{\dot\alpha\dot\beta}$ and $F_{\alpha\beta}$ must both be symmetric with respect to their indices, due to the symmetry in the anticommutator itself. However, in order to be able to establish chiral and antichiral superfields respectively, we need to be able to set these to 0 as well. This would be the representation-preserving constraints. The algebra looks like:

$$[\nabla_\alpha, \bar\nabla_{\dot\alpha}\} \;=\; i\nabla_{\underline{a}}$$

$$
\begin{aligned}
[\nabla_\alpha, \nabla_b\} &= C_{\alpha\beta} \bar{W}_{\dot{\beta}} t \\
[\nabla_a, \nabla_b\} &= -i F_{ab} t
\end{aligned} \tag{3.5}
$$

$\bar{W}_{\dot{\alpha}}$ is an anti-chiral superfield, satisfying $D_\alpha \bar{W}_{\dot{\beta}} = 0$ Its conjugate, W_α is thus a chiral superfield.

The Bianchi Identities can then be used to establish relationships between these fields, and discover any additional fields. We find that

$$
\begin{aligned}
\nabla_\alpha W_\beta = C^{\dot{\alpha}\dot{\beta}} \frac{1}{2}(F_a b + \bar{F}_a b) + C_{\alpha\beta} \\
\nabla_\alpha F_{bc} = C_{\alpha\gamma} \nabla_b \bar{W}_{\dot{\gamma}} - C_{\alpha\beta} \nabla_c \bar{W}_{\dot{\beta}} \\
\bar{F}_{ab} = \frac{i}{2} \epsilon_{abcd} F^{cd}
\end{aligned} \tag{3.6}
$$

where we have introduced the auxiliary field, d. This field is required to have the variation below in order to close the algebra.

$$
\nabla_\alpha d = \frac{1}{2} \nabla_a C^{\dot{\alpha}\dot{\beta}} \bar{W}_{\dot{\beta}} \tag{3.7}
$$

3.4 The Prepotential Superfield

Just as in 2.15, we can define a complex superfield, U, with similar properties to the 3D N=2 case.

$$
\begin{aligned}
\Gamma_\alpha = &\quad D_\alpha U \\
\bar{\Gamma}_{\dot{\alpha}} = &\quad \bar{D}_\alpha \bar{U} \\
\Gamma_{\underline{a}} = &\quad -i(D_\alpha(\bar{D}_{\dot{\alpha}} \bar{U}) + \bar{D}_{\dot{\alpha}}(D_\alpha U))
\end{aligned} \tag{3.8}
$$

U can be written as the sum of a real part and an imaginary part, and the third equation can be rewritten in these terms

$$U = \tilde{U} + iV$$

$$\Gamma_{\underline{a}} = \partial_{\underline{a}}\tilde{U} + \Delta_{\underline{a}}V$$

$$\Delta_{\underline{a}} \equiv i(\bar{D}_{\dot{\alpha}}D_{\alpha} - D_{\alpha}\bar{D}_{\dot{\alpha}}) \tag{3.9}$$

The physical fields themselves can be represented completely in terms of V.

$$\begin{aligned} W_{\alpha} &= \bar{D}^2 D_{\alpha} V \\ F_{ab} &= (\partial_a \Delta_b - \partial_b \Delta_a) V \\ d &= D^4 V \\ D^2 &= 2C^{\alpha\beta} D_{\alpha} D_{\beta} \\ D^4 &= C^{\alpha\delta} C^{\dot{\beta}\dot{\gamma}} D_{\alpha} \bar{D}_{\beta} \bar{D}_{\gamma} D_{\delta} + \bar{D}_{\beta} D_{\alpha} D_{\delta} \bar{D}_{\gamma} \end{aligned} \tag{3.10}$$

This allows us to create a superfield action, and express it in component form as well. The other terms in the superfield are not true degrees of freedom and represent the gauge freedom of the superfield.

3.5 Superfield and Component Actions

Just as in the previous chapter, one can write down an action of the form

$$S = \int d^4x d^2\theta d^2\bar{q}(Vd) = \int d^4x d^2\theta d^2\bar{q}(VD^4V) \tag{3.11}$$

This action contains the form of every field within the algebra naturally. Evaluation of the component form of this action can proceed by integrating out the form of the grassmann derivatives with respect to the superfield action.

$$S = \int d^4x - \frac{1}{8} F^{\underline{ab}} F_{\underline{ab}} - \bar{W}^{\dot{\alpha}} \partial_{\underline{a}} W^{\alpha} + d^2 \tag{3.12}$$

3.6 Dimensional Reduction

Noting the similarities between the prepotential superfields between the 3D N=2 theory and the 4D N=1 theory, a good question is whether the two theories are actually related, and how to be sure. The main reason we have gone through this second example in detail is to show a procedure that will allow us to relate some theories in 4 dimensions to corresponding theories in 3 dimensions. The premise of dimensional reduction is that one or more dimensions is given a finite length, and the fields are then integrated out over that length. A limit can be taken where the size of the extra dimension becomes zero, in effect freezing out the fields in that direction.

Starting with equation 3.12, let us work with each field to reduce it into forms that can survive in 3 dimensions. First, the vector field, F_{ab} must have the extra dimension singled out from its indices.

$$F_{ab} = c \begin{bmatrix} 0 & F_{0j} \\ F_{i0} & F_{ij} \end{bmatrix} \tag{3.13}$$

c is a constant to be determined later. By dimensional analysis, c must be proportional to the square root of a mass. Also remember that $F_{ab} = \partial_a A_b - \partial_b A_a$, so $F_{i0} = c(\partial_i A_0 - \partial_0 A_a)$. In the limit of zero length in the 0th direction, the second term vanishes, and we find that this is only a function of a single scalar field, $A_0 \equiv \phi$. Also, since $i, j \in \{1, 2, 3\}$, we can also define F_i through

$$F_{ij} = \epsilon_{ijk} F^k \tag{3.14}$$

The action sees its terms come through $F^{ab}F_{ab}$, so let's decompose this product.

$$F^{ab}F_{ab} = c^2(F^{0j}F_{0j} + F^{i0}F_{i0} + F^{ij}F_{ij}) = c^2(2\partial^i\phi\partial_i\phi + 2F^kF_k) \tag{3.15}$$

Next, the W_α and $\bar{W}_{\dot{\alpha}}$ fields become two separate Majorana fermion fields, rather than two halves to a single Weyl fermion. But these two fermions can be recombined into one complex fermion. Piecewise, our fermion action term,

$$\bar{W}^{\dot{\alpha}}\partial_{\underline{a}}W^\alpha \rightarrow c^2\bar{W}^\alpha(\gamma^a)_\alpha^\beta\partial_a W_\alpha + \bar{W}^\alpha\delta_\alpha^\beta\partial_0 W_\alpha \tag{3.16}$$

In a theory with a specific length scale, this last term would become a mass term, signifying a winding number, but as the length of the 4th dimension goes to 0, this term vanishes as well.

The auxiliary field, $d(\vec{x}, y)$ transforms into $cd(\vec{x})$ and survives in tact. Thus, we can create a new, 3D component action:

$$S = \int d^3x(\frac{1}{4}\phi\Box\phi - \frac{1}{4}F^kF_k + \bar{W}^\alpha(\gamma^a)_\alpha^\beta\partial_a W_\alpha + d^2)\int dyc^2 \tag{3.17}$$

The y integral term only gives a constant which is used to renormalize the fields. As we can see, the field content between this action and 2.31 is identical. The slight difference in constants can be resolved by a more careful examination of the conventions between the two theories. Thus, what we can see is that many theories in 3D are directly related to theories in higher dimensions, and while no process exists to add an extra dimension rather than reduce by one, studying 3D theories can still provide insight into theories in higher dimensions.

Chapter 4

Generalizing the 3D Algebra: Non-Abelian Gauge Groups and Larger N

4.1 Motivation

While the proceeding chapters dealt with an example of some test theories, what we are really looking for is a more general theory framework that can be used as the basis or inspiration for theories to come. There are a number of ways we can generalize the 3D N=2 algebra dealt with in Chapter 2, and this discussion will deal with two of these. First, we can change the U(1) gauge field, allowing for larger, non-Abelian theories, and second, we can increase the number of supersymmetries, N, to arbitrary numbers, greater than 2.

4.2 Redefining the Superspace Algebra

As a true complex representation only exists for specific values of N, we need to start over with a group of real superspace generators. We start by defining a set of Grassmann numbers: $\theta^I_\alpha, \alpha \in \{1,2\}, I \in \{1,2,...,N\}$, and the spinor metric used to raise and lower the α index, $C^{\alpha\beta}$. Next, we can define the 3D gamma matrices, $(\gamma^a)_\alpha^{\ \beta}$ as solutions to the 3D Clifford Algebra,

$$(\gamma^a)_\alpha^{\ \beta}(\gamma_b)_\beta^{\ \chi} + (\gamma^b)_\alpha^{\ \beta}(\gamma_a)_\beta^{\ \gamma} = 2\eta^{ab}\delta_\alpha^{\ \gamma} \tag{4.1}$$

More details about the definitions and identities between the spinor metric and the gamma matrices can be found in Appendix C.

We can define derivatives over Grassmann numbers, and covariantize these derivatives with respect to the superspace.

$$
\begin{aligned}
\frac{\partial}{\partial\theta^I_\alpha}\theta^J_\beta &= \delta^\alpha_\beta\delta^{IJ} \\
D^I_\alpha &= \delta^{IJ}\frac{\partial}{\partial\theta^{\alpha J}} + i(\gamma^a)_\alpha^\beta\theta^I_\beta\partial_a \\
Q^I_\alpha &= \delta^{IJ}\frac{\partial}{\partial\theta^{\alpha J}} - i(\gamma^a)_\alpha^\beta\theta^I_\beta\partial_a
\end{aligned}
\tag{4.2}
$$

Either D^I_α, the generators of fermionic derivatives, or Q^I_α, the fermionic generators of Poincaré transformations can be used as a basis for superspace discussions, as both commute with each other and are independent. For a real representation of 3D superspace, the spacetime algebra takes the form,

$$
[D^I_\alpha, D^J_\beta\} = 2i(\gamma^a)_{\alpha\beta}\delta^{IJ}\partial_a \tag{4.3}
$$

Next, we wish to introduce a Yang Mills field, and couple this field to the superspace. We can do that by introducing a pair of covariant derivatives

$$
\begin{aligned}
\nabla^I_\alpha &= D^I_\alpha + ig\Gamma^{IA}_\alpha t^A \\
\nabla_a &= \partial_a + ig\Gamma^A_a t^A
\end{aligned}
\tag{4.4}
$$

where t^A are the generators of the Yang Mills algebra and Γ represent the connections between the covariant derivatives and the algebra

Thus, the most general decomposition of the covariant derivative algebra is,

$$
[\nabla^I_\alpha, \nabla^J_\beta\} = 2i(\gamma^a)_{\alpha\beta}\delta^{IJ}\nabla_a + 2i(\gamma^a)_{\alpha\beta}\tilde{A}^{IJA}_a t^A + 2i(\gamma^a)_{\alpha\beta}\delta^{IJ}A^A_a t^A + 2C_{\alpha\beta}B^{IJA}t^A
$$

$$
\begin{aligned}
[\nabla^I_\alpha, \nabla_a\} &= ((\gamma_a)_\alpha^\beta W^{IA}_\beta + \tilde{C}^{IA}_{\alpha a})(igt^A) \\
[\nabla_a, \nabla_b\} &= ig(\epsilon_{ab}{}^c F_c)^A t^A
\end{aligned}
\tag{4.5}
$$

where $\tilde{A}^{IJ}_a$ is symmetric traceless with respect to I and J, B^{IJ} is antisymmetric with respect to I and J, and $\tilde{C}^I_{\alpha a}$ is gamma-traceless, that is, $(\gamma^a)^{\alpha\beta}\tilde{C}^I_{\alpha a} = 0$.

The first thing to notice is that the covariant derivatives themselves do not alter the color of the gauge field. The colors factor out cleanly, and only are affected by the internal gauge symmetries. The interesting thing about this is that in this formulism, bosons and fermions alike exist in the same representation with respect to the internal gauge group. In order to have bosons and fermions exist in separate representations from each other, i.e. adjoint and fundamental, the gauge group in question needs to be part of the underlying supergroup, and is thus directly controlled by the number of supersymmetries. For instance, N=3 should include an underlying SO(3) symmetry just from relabeling the supersymmetric indices, so for 3D N=3, the fermions in W^{IA}_α would exist in the fundamental, 3 representation, of SO(3), while the bosons in B^{IJA} would exist in the adjoint, $\bar{3}$ representation.

Because of this clean factorization property, one can 'absorb' the Yang Mills generators into the fields, and make the following definitions:

$$
\begin{aligned}
\tilde{A}^{IJ}_a &\equiv \tilde{A}^{IJA}_a t^A \\
A_a &\equiv A^A_a t^A \\
B^{IJ} &\equiv B^{IJA} t^A \\
W^I_\beta &\equiv W^{IA}_\beta t^A \\
\tilde{C}^I_{\alpha a} &\equiv \tilde{C}^{IA}_{\alpha a} t^A
\end{aligned}
$$

$$F_c \equiv F_c^A t^A \tag{4.6}$$

This allows the procedure to determine the spinor covariant derivatives to continue in the same fashion as in the Abelian case. Interaction terms from the now non-vanishing of the various fields with each other may still exist, however.

4.3 Constraints on the Algebra

Once again, there are three major kinds of constraints used to attempt to reduce an algebra like the one above into an irreducible form: representation-preserving constraints, conventional constraints, and Jacobi/Bianchi identities. In this case, the constraint $\tilde{A}_a^{IJ} = 0$, which is suggested by the Adinkra theory of 1-dimensional supersymmetry,[25] holds as a representation-preserving constraint. While it is not clear a priori that it is necessary in the 3D case, setting this constraint will tend to reduce the number of auxiliary fields produced from an infinite number to a finite, manageable number. Thus, I will attempt to solve the covariant derivative algebra with this constraint. Next, solving for the terms in the decomposition in terms of the connections, $A_a = 0$ can be set conventionally. Then, we must solve the Bianchi and Jacobi Identities, to ensure a consistent algebra. These will also give the form of the covariant derivatives applied to the various fields, as well as suggest additional 'auxiliary' fields that need to be added.

4.4 Bianchi and Jacobi Identities for Arbitrary N

Finally, we arrive at possibly the most important set of constraints that must be imposed, the Bianchi and Jacobi Identities. The Bianchi Identities state that, the following equations must be satisfied, for all combinations of the following Lie triple system, G_{ABC}:

$$
\begin{aligned}
&G_{ABC} = (-1)^{AC}[[\nabla_A, \nabla_B\}\nabla_C\} \\
&G_{ABC} + G_{BCA} + G_{CAB} = 0
\end{aligned}
\tag{4.7}
$$

These equations are especially important because they give information about the covariant derivatives themselves, as well as suggest unconstrained fields that need to be added in as possibly auxiliary fields. The Bianchi Identities reduce to the following equations:

$$
\begin{aligned}
&2\nabla^K_\gamma B^{IJ} - \nabla^I_\gamma B^{JK} - \nabla^J_\gamma B^{KI} = 2(\delta^{JK} W^I_\gamma - \delta^{IK} W^J_\gamma) \\
&\tilde{C}^I_{\alpha a} = 0 \\
&(\gamma_a)^\gamma_\beta \nabla^I_\alpha W^J_\gamma + (\gamma_a)^\gamma_\alpha \nabla^J_\beta W^I_\gamma = 2i(\gamma^b)_{/a/b} \delta^{IJ} [\nabla_b, \nabla_a\} + 2C_{\alpha\beta} \nabla_a B^{IJ} \\
&\epsilon_{ab}{}^c \nabla^I_\alpha F_c = (\gamma_b)^\beta_\alpha \nabla_a W^I_\beta - (\gamma_a)^\beta_\alpha \nabla_b W^I_\beta \\
&\eta^{ab} \nabla_a F_b = 0
\end{aligned}
\tag{4.8}
$$

The last of these equations confirms that F^A_c is the dual to the Yang Mills field strength,

$$
\epsilon_{ab}{}^c F^A_c = -F^A_{ab} = \partial_b V^A_a - \partial_a V^A_b + i f^{ABC} [V^B_a, V^C_b] \tag{4.9}
$$

Solving these equations leads to the following fermionic covariant derivatives between the fields:

$$\begin{aligned}
\nabla^I_\alpha F_a &= \epsilon_a{}^{bc}\partial_b(\gamma_c)_\alpha^\beta W^I_\beta \\
\nabla^I_\alpha W^J_\beta &= (\gamma^a)_{\alpha\beta}(\delta^{IJ}F_a + i\partial_a B^{IJ}) + C_{\alpha\beta}X^{IJ} \\
\nabla^I_\alpha B^{JK} &= Y^{IJK}_\alpha + \delta^{IJ}W^K_\alpha - \delta^{IK}W^J_\alpha
\end{aligned} \tag{4.10}$$

X^{IJ} and Y^{IJK}_α are additional auxiliary fields that must exist in each multiplet for $N \geq 2, 3$ respectively. Each of the raised indices must be antisymmetric with each other. But the Bianchi identities do not give information on how X and Y transform under fermionic derivatives themselves. In order to get this information, one must apply the additional Jacobi Identities, replacing one of the covariant derivatives with a field. For instance, solving:

$$[[\nabla^I_\alpha, \nabla^J_\beta\}, W^K_\gamma\} + [[W^K_\gamma, \nabla^I_\alpha\}\nabla^J_\beta\} + [[\nabla^J_\beta, W^K_\gamma\}, \nabla^I_\alpha\} = 0 \tag{4.11}$$

leads to

$$\nabla^I_\alpha X^{JK} = -i(\gamma^a)_\alpha^\beta \partial_a Y^{IJK}_\beta + i(\gamma^a)_\alpha^\beta \partial_a(\delta^{IJ}W^K_\beta - \delta^{IK}W^J_\beta) \tag{4.12}$$

Additional Jacobi Identities can continue to be solved involving Y^{IJK}_α, potentially producing further auxiliary fields. But for the algebra to close, there must be a limit to this procedure. Let's see what happens with a couple of examples.

4.5 3D N=3 Algebra

For N=3, a very nice feature takes place, that suggests we're already just about done. Remember that Y^{IJK}_α must be antisymmetric with respect to the indices I, J, and K. But for N = 3, there is only one independent completely antisymmetric form. Thus, we can replace B, X, and Y with dual forms:

$$\begin{aligned} Y_\alpha^{IJK} &= \epsilon^{IJK} Y_\alpha \\ B^{IJ} &= \epsilon^{IJK} B^K \\ X^{IJ} &= \epsilon^{IJK} X^K \end{aligned} \tag{4.13}$$

The 3D N=3 Algebra is thus found to have the following form:

$$\begin{aligned} \nabla_\alpha^I F_a &= \epsilon_a{}^{bc} \partial_b (\gamma_c)_\alpha^\beta W_\beta^I \\ \nabla_\alpha^I W_\beta^J &= (\gamma^a)_{\alpha\beta} (\delta^{IJ} F_a + i\partial_a \epsilon^{IJK} B^K) + C_{\alpha\beta} \epsilon^{IJK} X^K \\ \nabla_\alpha^I B^J &= \delta^{IJ} Y_\beta + \epsilon^{IJK} W_\beta^K \\ \nabla_\alpha^I X^J &= -i(\gamma^a)_\alpha^\beta \partial_a (\delta^{IJ} Y_\beta - \epsilon^{IJK} W_\beta^K) \\ \nabla_\alpha^I Y_\beta &= i(\gamma^a)_{\alpha\beta} \partial_a B^I - C_{\alpha\beta} X^I \end{aligned} \tag{4.14}$$

Thus, no additional auxiliary fields are needed, as the derivatives of Y_α only depend on B^I and X^I. This also suggests a limit to the construction of auxiliary fields. If a field is proportional to the epsilon tensor, derivatives on that field should reduce the number of indices, resulting hopefully in only fields that have already been determined.

4.6 3D N=4 Algebra

The next case to examine then would be 3D N=4. The algebra can begin as in Eq's 4.10 and 4.12, but an additional field is needed, Z_a^{IJKL}, where

$$\begin{aligned} \nabla_\alpha^I Y_\beta^{JKL} &= (\gamma^a)_{\alpha\beta} (Z_a^{IJKL} + i\partial_a (\delta^{IJ} B^{KL} - \delta^{IK} B^{JL} + \delta^{IL} B^{JK})) \\ &\quad - C_{\alpha\beta} (\delta^{IJ} X^{KL} - \delta^{IK} X^{JL} + \delta^{IL} X^{JK}) \end{aligned}$$

$$\eta^{ab}\nabla_a Z_b^{IJKL} \;=\; 0 \tag{4.15}$$

Since Z_a^{IJKL} must be proportional to ϵ^{IJKL}, we can define the following dual forms:

$$\begin{aligned}
Z_a^{IJKL} &= \epsilon^{IJKL} Z_a \\
Y_\alpha^{IJK} &= \epsilon^{IJKL} Y_\alpha^L \\
\bar{B}^{IJ} &= \frac{1}{2}\epsilon^{IJKL} B^{KL} \\
\bar{X}^{IJ} &= \frac{1}{2}\epsilon^{IJKL} X^{KL}
\end{aligned} \tag{4.16}$$

The variation of Z_a^{IJKL} is best described in terms of its dual

$$D_\alpha^I Z_a = \epsilon_a{}^{bc}\partial_b(\gamma_c)_\alpha^\beta Y_\beta^I \tag{4.17}$$

and right away, we can see signs of an interesting property.

It turns out that this resulting algebra is in fact, further reducible, through a kind of mirror symmetry, into two pieces: $Z_a = F_a$ and $Z_a = -F_a$ Taking the first piece, the 3D N=4 Algebra is found to have the following form:

$$\begin{aligned}
\nabla_\alpha^I F_a &= \epsilon_a{}^{bc}\partial_b(\gamma_c)_\alpha^\beta W_\beta^I \\
\nabla_\alpha^I W_\beta^J &= (\gamma^a)_{\alpha\beta}(\delta^{IJ} F_a + i\partial_a B^{IJ}) + C_{\alpha\beta} X^{IJ} \\
\nabla_\alpha^I B^{JK} &= \epsilon^{IJKL} W_\alpha^L + \delta^{IJ} W_\alpha^K - \delta^{IK} W_\alpha^J \\
\nabla_\alpha^I X^{JK} &= -i(\gamma^a)_\alpha^\beta \partial_a \epsilon^{IJKL} W_\beta^L + i(\gamma^a)_\alpha^\beta \partial_a(\delta^{IJ} W_\beta^K - \delta^{IK} W_\beta^J)
\end{aligned} \tag{4.18}$$

with the following conditions:

$$\begin{aligned}
B^{IJ} &= \bar{B}^{IJ} \\
X^{IJ} &= \bar{X}^{IJ}
\end{aligned} \tag{4.19}$$

A similar algebra would also exist for the dual version.

This is the first instance of a particular symmetry: For N = 4k, the algebra 'folds' in on itself through the epsilon tensor. This "self-duality" will tend to halve the number of degrees of freedom in the algebra. In particular, the algebras for 3D N=3 and 3D N=4 are equivalent, with the degrees of freedom in Y in 3D N=3 absorbed into W in 3D N=4.

4.7 An attempt at an Arbitrary N Algebra: Index Theory

We want to be able to extend this algebra to an arbitrary value of N if possible. There are several features we can already notice about the theories. First, as there are no symmetric traceless forms, there must be a distinct ladder of terms, starting with the standard Yang Mills field strength pseudovector, F_a on the bottom rung. Each extra rung on this ladder adds an additional index I, so that if A is a field on the Mth rung, then it has to have M different indices, $I_1, I_2, ...I_M$. Furthermore, these indices must be antisymmetric with each other. Thus, taking a fermionic derivative D^J_α A must either move up the ladder, by inserting J with the M indices, or it must move down the ladder, by pulling an equivalent index to J out. This can be represented in mathematical terms, in the following way.

Define χ to be an equivalence class of ordered sets, concatenated with a boolean operator (+ or -), and let $x_1, x_2, ...x_n$ be indices $\in \{1, ..., N\}$, so that $\chi \ni \{+, x_1, x_2, ...x_n\}$ is a defining element of the class. Also define $\bar{\chi}$ to be the conjugate class defined by $\bar{\chi} \in \{-, x_1, x_2, ...x_n\}$. χ and $\bar{\chi}$ must have the the property of anticommutation: that is,

$$x_i \neq x_j \forall i, j$$
$$\forall i \in \{1, ..., n\}, \{+, ..., x_i, x_{i+1}, ...\} \in \chi \Rightarrow \{+, ..., x_{i+1}, x_i, ...\} \in \bar{\chi}$$
$$\forall i \in \{1, ..., n\}, \{-, ..., x_i, x_{i+1}, ...\} \in \chi \Rightarrow \{-, ..., x_{i+1}, x_i, ...\} \in \bar{\chi} \quad (4.20)$$

This will allow you to set a rung on the ladder. Next, we wish to be able to move up or down. Thus we need to define raising and lowering operations.

$$\begin{aligned} \chi &\ni \{+, x_1, x_2, ...x_n\}, I \in \{1, ...N\} \Rightarrow \\ I\chi &\ni \{+, I, x_1, x_2, ...x_n\} OR \\ \chi I &\ni \sum_{i=1}^{n} \delta^{I,x_i} \{(-1)^(i-1), x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_n\} \end{aligned} \quad (4.21)$$

Only one of $I\chi$ and χI can exist. It can be checked that the following relations apply:

$$(I\chi)J + I(\chi J) = \delta^{IJ}\chi$$
$$I(J\chi) + J(I\chi) = 0$$
$$(\chi J)I + (\chi I)J = 0 \quad (4.22)$$

These relations will provide a framework for our ladder, and moving 1 and 2 rungs along it. It is also clear that there is a maximum rung, $E, \bar{E}$:

$$\begin{aligned} E &\ni \{+, 1, 2, ...N\} \\ \bar{E} &\ni \{-, 1, 2, ..., N\} \\ \forall I &\in \{1, ..., N\}, \nexists IE, I\bar{E} \end{aligned} \quad (4.23)$$

Now, we can start to propose a form for the 3D Super Yang Mills algebra for arbitrary N:

$$
\begin{aligned}
\nabla^I_\alpha W^\chi_\beta &= (\gamma^a)_{\alpha\beta}(Z^{\chi I}_a + i\partial_a B^{I\chi}) + C_{\alpha\beta}X^{I\chi} \\
\nabla^I_\alpha B^\chi &= W^{\chi I}_\alpha + Y^{I\chi}_\alpha \\
\nabla^I_\alpha X^\chi &= i(\gamma^a)_\alpha^\beta \partial_a W^{\chi I}_\beta - i(\gamma^a)_\alpha^\beta \partial_a Y^{I\chi}_\beta \\
\nabla^I_\alpha Y^\chi_\beta &= (\gamma^a)_{\alpha\beta}(Z^{I\chi}_a + i\partial_a B^{\chi I}) - C_{\alpha\beta}X^{\chi I} \\
\nabla^I_\alpha Z^\chi_a &= \epsilon_a{}^{bc}\partial_b(\gamma_c)_\alpha^\beta(Y^{\chi I}_\beta + W^{I\chi}_\beta) \\
\eta^{ab}\partial_a Z^\chi_b &= 0
\end{aligned}
\tag{4.24}
$$

At first glance, this algebra seems very consistent. Z_a on the bottom rung is equated with F_a, the primary Yang-Mills field strength, and thus Z^χ_a is necessarily a group of gauge fields. A quick check of the degrees of freedom shows that, because of the binomial theorem, there are always going to be 2^N bosonic and 2^N fermionic degrees of freedom. For $N = 4k, k \in \mathbb{Z}$, there will be the self-dual property seen already in N=4, which can remove half of the degrees of freedom. But there is a catch. These fields must always be able to exist in superspace, and thus must satisfy the algebra $[D^I_\alpha, D^J_\beta\} = 2i(\gamma^a)_{\alpha\beta}\delta^{IJ}\partial_a$

Calculations for B^χ and X^χ confirm that these fields always satisfy the superspace algebra, as well as confirming the forms of the derivatives of W^χ_α and Y^χ_α with respect to B and X, but it can be shown that Z^χ_a in fact has an anomalous term as long as $Z^\chi_a \neq F_a$ and $\chi \neq E, \bar{E}$,

$$
[D^I_\alpha, D^J_\beta\}Z^\chi_a = 2i(\gamma^b)_{\alpha\beta}\delta^{IJ}\partial_b Z^\chi_a + C_{\alpha\beta}\epsilon_a{}^{bc}\partial_c(Z^{I(\chi J)}_b - Z^{(I\chi)J}_b - Z^{J(\chi I)}_b + Z^{(J\chi)I}_b) \tag{4.25}
$$

If Z is not the lowest or highest rung on the ladder, it does not close offshell. There are no obvious ways to repair this field, either to add additional terms to the fermionic derivatives or to add additional auxiliary fields that interact with Z. While this does not per se rule out that Z cannot be fixed, there are reasons to believe that it may be impossible for the general case. But while it would seem like the 3D N=4 case then may be the last one to follow this format, however, there is one more special case to check.

4.8 3D N=8 Algebra

While in the general case, this algebra does not appear to be able to close, there is a special case beyond N=4 that may be able to be solved. We saw that for the field Z_a^{χ}, there is an anomaly in the algebra, suggesting either equations of motion or additional auxiliary fields unless χ has no indices or maximum indices, but the same self-dual symmetry that existed for N=4 may work for N=8 to remove the anomaly. Due to this feature, an algebra of the following form can be tested, and does in fact close offshell.

$$
\begin{aligned}
\nabla_\alpha^I F_a &= \epsilon_a{}^{bc}\partial_b(\gamma_c)_\alpha^\beta W_\beta^I \\
\nabla_\alpha^I W_\beta^J &= (\gamma^a)_{\alpha\beta}(\delta^{IJ}F_a + i\partial_a B^{IJ}) + C_{\alpha\beta}X^{IJ} \\
\nabla_\alpha^I B^{JK} &= Y_\alpha^{IJK} + \delta^{IJ}W_\alpha^K - \delta^{IK}W_\alpha^J \\
\nabla_\alpha^I X^{JK} &= -i(\gamma^a)_\alpha^\beta\partial_a Y_\beta^{IJK} + i(\gamma^a)_\alpha^\beta\partial_a(\delta^{IJ}W_\beta^K - \delta^{IK}W_\beta^J) \\
\nabla_\alpha^I Y_\beta^{JKL} &= (\gamma^a)_{\alpha\beta}(Z_a^{IJKL} + i\partial_a(\delta^{IJ}B^{KL} - \delta^{IK}B^{JL} + \delta^{IL}B^{JK})) \\
&\quad -C_{\alpha\beta}(\delta^{IJ}X^{KL} - \delta^{IK}X^{JL} + \delta^{IL}X^{JK}) \\
\nabla_\alpha^I Z_a^{JKLM} &= \epsilon_a{}^{bc}\partial_b(\gamma_c)_\alpha^\beta((\delta^{IJ}Y_\beta^{KLM} - \delta^{IK}Y_\beta^{JLM} + \delta^{IL}Y_\beta^{JKM} - \delta^{IM}Y_\beta^{JKL}) \\
&\quad +\frac{1}{3!}\epsilon^{IJKLMNOP}Y_\beta^{NOP})
\end{aligned}
\tag{4.26}
$$

with the following condition:

$$Z_a^{IJKL} = \frac{1}{4!}\epsilon^{IJKLMNOP} Z_a^{MNOP} \tag{4.27}$$

This theory agrees with the well known result that in 4 dimensions, 4D N=4 is the maximally supersymmetric Yang Mills theory known to be consistent. 3D N=8 would thus be the direct compactification of this theory into 3 dimensions. It is also worthwhile to note that, as no further multiplets are known to exist at this time, 3D N=6 SYM in particular, though also N=5,7, naturally should be expanded to N=8 in order to complete its offshell structure. As 3D N=6 would be the direct compactification of 4D N=3, this agrees with the fact that 4D N=3 SYM is known on-shell to be equivalent to 4D N=4, and suggests that this is true offshell as well.[26]

Chapter 5

Solving for the 3D Actions

5.1 Prepotential Superfields and the Base Supersymmetric Action

While we have now finally derived the superalgebras involved in the 3D N=3, N=4, and N=8 theories, we are not done there. Just as was shown in the 3D N=2 case, the next step is to derive actions that relate to the theory. These are what display the kinds of fields used, as well as their interactions.

For every N, one can define a vector superfield, U_a. U_a has the gauge symmetry that

$$U'_a = U_a + \partial_a U \tag{5.1}$$

must give the same physics as U_a

The corresponding superfield strength pseudovector superfield, V_a then can be defined such that

$$\begin{aligned} V_a &= \epsilon_a{}^{bc}(\partial_b U_c - \partial_c U_b) \\ D^{2N} V_a &= F_a \end{aligned} \tag{5.2}$$

Note that V_a has the property that $\eta^{ab}\partial_a V_b = 0$

The benefit of using F_a as the highest derivative of our superfield is that a neat ladder is formed. One can check that if $D^\chi F_a \sim F$, then χ is not completely antisymmetric. This leads to the following supersymmetric action:

$$S = \int d^3x d^{2N}\theta(\eta^{ab}V_a D^{2N}V_b) \quad (5.3)$$

, where D^{2N} is the unique completely antisymmectric form of 2N spinor derivatives,

$$D^{2N} = \epsilon^{I_1 I_2 ... I_N}\epsilon^{J_1 J_2 ... J_N}C_{\alpha_1\beta_1}C_{\alpha_2\beta_2}...C_{\alpha_N\beta_N}D^{I_1}_{\alpha_1}D^{I_2}_{\alpha_2}...D^{I_N}_{\alpha_N}D^{J_1}_{\beta_1}D^{J_2}_{\beta_2}...D^{J_N}_{\beta_N} \quad (5.4)$$

While other actions may be able to be written, this is the general base action, and is likely to be included in any theory written containing the supersymmetric multiplets.

There is one caveat though. While this action can be written down, it does not include the self-dual symmetry found in 3D N=4 and 3D N=8. In order to obtain the proper action then, we must add in the dual action. This equates to solving for Z_a in terms of derivatives on V_a, then replacing $F_a = D^{2N}V_a$ with $\frac{1}{2}(F_a + Z_a)$, to ensure that $F_a - Z_a$ is not a true field.

For N=4, we can calculate that

$$\begin{aligned} Z_a &= K^{def}_{abc}(\gamma^b)^{\alpha\beta}(\gamma^c)^{\gamma\delta}\partial_d\partial_e D^I_\alpha D^J_\beta D^K_\gamma D^L_\delta V_f \\ K^{def}_{abc} &= \delta^f_b\eta_{ac}\eta^{de} + \delta^f_c\eta_{ab}\eta^{de} - \delta^f_a\eta_{bc}\eta^{de} + \delta^f_a\delta^d_b\delta^e_c \end{aligned} \quad (5.5)$$

A similar form can be defined for N=8.The base component action is straightforward. Due to the clean ladder of states, the component action takes the form

$$S = \int d^3x - \eta^{ab}F_aF_b + W^I_\alpha(\gamma^a)^{\alpha\beta}\partial_a W^I_\beta + X^{IJ}X^{IJ} + B^{IJ}\Box B^{IJ} + Y^{IJK}_\alpha(\gamma^a)^{\alpha\beta}\partial_a Y^{IJK}_\beta - \eta^{ab}Z^{IJKL}_a Z^{IJKL}_b \quad (5.6)$$

with some terms removed for N ¡ 8.

5.2 Additional Actions

Besides the base action, there are a number of higher-order, interacting actions that can be considered in our theory. For 3 dimensions in particular, Cherns-Simons terms and Skyrme terms can be added without any difficulties with renormalizability, corresponding to 3 point and 4 point interactions respectively.[27] In particular, as Chern-Simons theories have garnered a lot of attention in recent times, it would be Chern-Simons terms can arise naturally as surface-terms in 4D Super Yang Mills theories, and upon reduction to three dimensions these surface terms remain as massive states.[21] In addition, as 3-dimensional gravity does not have additional propagating fields, Supergravity theories can be formulated in terms of Chern-Simons actions.[28] Thus, the addition of Super Chern Simons terms would be a next step in the development of theories based upon this framework.

Chapter 6

Conclusions

In this paper, we have studied the algebra structure and field content for Off-shell 3-dimensional Super Yang Mills theory. Beginning with the known example of N=2, we demonstrated the tools used to study offshell theories. First, the 3D superspace algebra was defined and a coupling to a Yang Mills field was introduced. The covariant derivative algebra was derived, and conditions were placed for consistency and by convention in an attempt to reduce the algebra, including representation-preserving and conventional constraints, as well as constraints determined by solving the Bianchi and Jacobi Identities. Once the form of the algebra was set, and it was determined to close offshell, superfield prepotentials were introduced, and the field components of the algebra were neatly aligned within a superfield. A superfield action was proposed, and a component action was derived from the superfield action, showing to correspond with the action expected from the component fields themselves.

We demonstrated a similar theory in 4 dimensions, 4D N=1 and described the differences in the superspace structure between 3 and 4 dimensions, as well as demonstrated the process of dimensional reduction, showing how the two theories are in fact the same evaluated in different numbers of dimensions.

Using the tools developed previously, we extended our discussion to arbitrary numbers of supercharges. We derived the covariant derivative algebra for both N=3 and N=4, and in accordance with theory on Clifford Algebras, we found a duality in the N=4 theory that reduced it to the equivalent of 2 copies of N=3 theories, self-dual and anti-self-dual copies. We attempted to extend the 3D SYM theories beyond N=4, but only an algebra with N=8 was able to be found. This is expected to be the maximally supersymmetric theory, with 128 bosonic and 128 fermionic degrees of freedom. In addition to being studied as an interesting and useful theory in its own right, the 3D N=8 SYM theory is known to be the dimensional reduction of 4D N=4 SYM, a theory that is widely discussed and studied. It is expected then that knowing the offshell structure of 3D N=8 will aid in learning more about the field content and interactions of 4D N=4.

Each of the 3D SYM theories was determined to have a similar, standard vector prepotential superfield associated with it, but the cases of 3D N=4 and N=8 required additional insight in order to encompass their self-dual property. Thus, supersymmetric actions were able to be constructed, containing each of the fields in the corresponding algebras as component fields. These actions represent mainly 2-point functions, and further research into higher-order interaction terms can be done to extend these theories.

Appendix A

Spinor Conventions and Fierz Identities: 3D N=2 Complex Representation

While there are a few differences from the standard conventions, the conventions used for this paper were still designed to be consistent. The spinor metric and the corresponding gamma matrices are defined as follows.

$$
\begin{aligned}
(\gamma^a)^{\alpha\beta}(\gamma^b)_\beta^{\ \gamma} &= \eta^{ab}C^{\alpha\gamma} + i\epsilon^{abc}(\gamma_c)^{\alpha\gamma} \\
C^{\alpha\beta}C_{\gamma\delta} &= \delta_\gamma^\alpha\delta_\delta^\beta - \delta_\delta^\alpha\delta_\gamma^\beta \\
C^{\alpha\beta}C_{\alpha\beta} &= +2 \\
(\gamma^a)^{\alpha\beta}(\gamma_a)^{\gamma\delta} &= C^{\alpha\gamma}C^{\beta\delta} + C^{\alpha\delta}C^{\beta\gamma}
\end{aligned}
\tag{A.1}
$$

Conjugation is then defined by these two relations.

$$
\begin{aligned}
(A^\alpha)^* &= \bar{A}^\alpha \\
(A_\alpha)^* &= -\bar{A}_\alpha
\end{aligned}
\tag{A.2}
$$

Appendix B

4D N=1 Notations, Identities, and Spinor metrics

In 4 dimensions, the Clifford Algebra can be solved by a set of 4x4 matrices.

The 4-component spinor can be decomposed into a pair of 2 component spinors:

$$\theta_\mu = \begin{bmatrix} \theta_\alpha \\ \bar{\theta}_{\dot{\alpha}} \end{bmatrix} \tag{B.1}$$

where $\mu \in 1, 2, 3, 4$, and $\alpha, \dot{\alpha} \in 1, 2$

The metric, $C_{\mu\nu}$, does not mix the two sets of indices.

$$C_{\mu\nu} = \begin{bmatrix} C_{\alpha\beta} & 0 \\ 0 & C_{\dot{\alpha}\dot{\beta}} \end{bmatrix} \tag{B.2}$$

Next, there exists a representation of $(\gamma^a)_{\mu\nu}$ where μ and ν must always come from separate representations. For instance:

$$(\gamma)_{\mu\nu} = \begin{bmatrix} 0 & C_{\alpha\beta} \\ C_{\alpha\beta} & 0 \end{bmatrix}, \qquad (\gamma^1)_{\mu\nu} = \begin{bmatrix} 0 & (\sigma^1)_{\alpha\dot{\beta}} \\ (\sigma^1)_{\dot{\alpha}\beta} & 0 \end{bmatrix},$$

$$(\gamma^2)_{\mu\nu} = \begin{bmatrix} 0 & (\sigma^2)_{\alpha\dot{\beta}} \\ (\sigma^2)_{\dot{\alpha}\beta} & 0 \end{bmatrix}, \quad (\gamma^3)_{\mu\nu} = \begin{bmatrix} 0 & (\sigma^3)_{\alpha\dot{\beta}} \\ (\sigma^3)_{\dot{\alpha}\beta} & 0 \end{bmatrix} \tag{B.3}$$

This allows us to make the following decomposition of an object, $A_{\mu\nu}$

$$\begin{aligned} A_{\mu\nu} &= \begin{bmatrix} A_{\alpha\beta} & A_{\alpha\dot{\beta}} \\ A_{\dot{\alpha}\beta} & A_{\dot{\alpha}\dot{\beta}} \end{bmatrix} \\ A_{\alpha\beta} &= C_{\alpha\beta}A + \frac{1}{2}C_{\dot{\alpha}\dot{\beta}}(\gamma^a)_{\dot{\alpha}\alpha}(\gamma^b)_{\dot{\beta}\beta}(A_{ab} - i\frac{1}{2}\epsilon_{abcd}A^{cd}) \end{aligned}$$

$$
\begin{aligned}
A_{\alpha\dot{\beta}} &= \delta^{\dot{\alpha}}_{\dot{\beta}}(\gamma^a)_{\alpha\dot{\alpha}}A_a \\
A_{\dot{\alpha}\beta} &= \delta^{\alpha}_{\beta}(\gamma^a)_{\dot{\alpha}\alpha}A_a \\
A_{\dot{\alpha}\dot{\beta}} &= C_{\dot{\alpha}\dot{\beta}}A + \frac{1}{2}C^{\alpha\beta}(\gamma^a)_{\alpha\dot{\alpha}}(\gamma^b)_{\beta\dot{\beta}}(A_{ab} + i\frac{1}{2}\epsilon_{abcd}A^{cd})
\end{aligned}
\tag{B.4}
$$

Also note the projection operator that allows for this representation to exist:

$$
\begin{aligned}
(\gamma^5)^{\nu}_{\mu} &= i\frac{1}{4}\epsilon^{abcd}(\gamma^a)^{\rho}_{\mu}(\gamma^b)^{\sigma}_{\rho}(\gamma^c)^{\tau}_{\sigma}(\gamma^d)^{\nu}_{\tau} \\
(P_{\pm})^{\nu}_{\mu} &\equiv \frac{1}{2}(\delta^{\nu}_{\mu} \pm (\gamma^5)^{\nu}_{\mu})
\end{aligned}
\tag{B.5}
$$

and it can be verified that $P_{\pm}$ satisfies the properties of a projection operator

We then make the definition:

$$
x_{\underline{a}} \equiv x_{\alpha\dot{\alpha}} \tag{B.6}
$$

This definition with the previous decomposition allows us to forgo the gamma matrices in our discussion of 4D N=1

Appendix C

Spinor Conventions and Fierz Identities: 3D Real Representation

For the discussion of 3D Superspace with arbitrary N, a slightly different set of conventions were used from the 3D N=2 case. By building a representation for the gamma functions and spinor metric, I set a number of constants and Fierz Identities.

The gamma matrices are defined through the algebra,

$$(\gamma^a)_\alpha^\beta(\gamma^b)_\beta^\gamma + (\gamma^b)_\alpha^\beta(\gamma^a)_\beta^\gamma = 2\delta_\alpha^\gamma\eta^{ab} \tag{C.1}$$

In three bosonic dimensions, this can be satisfied by the following,

$$(\gamma^a)_\alpha^\beta(\gamma^b)_\beta^\gamma = \delta_\alpha^\gamma\eta^{ab} + i\epsilon^{ab}{}_c(\gamma^c)_\alpha^\gamma \tag{C.2}$$

After defining the spinor metric through the raising or lowering of one of the spinor indices,

$$(\gamma^a)^{\alpha\beta}(\gamma^b)_\beta^\gamma = C^{\alpha\gamma}\eta^{ab} + i\epsilon^{ab}{}_c(\gamma^c)^{\alpha\gamma} \tag{C.3}$$

, we can calculate the Fierz Identities from our representation.

$$\begin{aligned} C^{\alpha\beta}C_{\gamma\delta} &= \delta_\delta^\alpha\delta_\gamma^\beta - \delta_\gamma^\alpha\delta_\delta^\beta \\ C^{\alpha\beta}C_{\alpha\beta} &= -2 \\ (\gamma^a)_\alpha^\beta(\gamma_a)_\gamma^\delta &= C_{\alpha\gamma}C^{\beta\delta} + \delta_\alpha^\delta\delta_\gamma^\beta \end{aligned} \tag{C.4}$$

Other useful quantities are as follows:

$$\begin{aligned} (\gamma^a)_\alpha^\beta(\gamma_a)_\gamma^\delta + \delta_\alpha^\beta\delta_\gamma^\delta &= 2\delta_\alpha^\delta\delta_\gamma^\beta \\ (\gamma^a)_{\beta\gamma}\epsilon_a{}^{bc}(\gamma_c)_\alpha^\delta &= i(\gamma^b)_{\beta\gamma}\delta_\alpha^\delta + i(\gamma^b)_\gamma^\delta C_{\alpha\beta} + i(\gamma^b)_\beta^\delta C_{\alpha\gamma} \end{aligned} \tag{C.5}$$

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