ABSTRACT

Title of Dissertation: APPLYING MATHEMATICS TO PHYSICS AND ENGINEERING: SYMBOLIC FORMS OF THE INTEGRAL

Steven Robert Jones, Doctor of Philosophy, 2010

Dissertation directed by: Professor Patricia F. Campbell
Department of Curriculum and Instruction

A perception exists that physics and engineering students experience difficulty in applying mathematics to physics and engineering coursework. While some curricular projects aim to improve calculus instruction for these students, it is important to specify where calculus curriculum and instructional practice could be enhanced by examining the knowledge and understanding that students do or do not access after instruction. This qualitative study is intended to shed light on students' knowledge about the integral and how that knowledge is applied to physics and engineering.

In this study, nine introductory-level physics and engineering students were interviewed about their understanding of the integral. They were interviewed twice, with one interview focused on and described as problems similar to those encountered in a mathematics class and the other focused on and described as problems similar to those found in a physics class.

These students provided evidence for several “symbolic forms” that may exist in their cognition. Some of these symbolic forms resembled the typical interpretations of the integral: an area, an addition over several pieces, and an anti-derivative process. However, unique features of the students’ interpretations help explain how this knowledge has been compiled. Furthermore, the way in which these symbolic forms were employed throughout the interviews shows a context-dependence on the activation of this knowledge. The symbolic forms related to area and anti-derivatives were more common and productive during the mathematics interview, while less common and less productive during the physics interview. By contrast, the symbolic form relating to an addition over several pieces was productive for both interview sessions, suggesting its general utility in understanding the integral in various contexts.

This study suggests that mathematics instruction may need to provide physics and engineering students with more opportunities to understand the integral as an addition over several pieces. Also, it suggests that physics and engineering instruction may need to reiterate the importance, in physics and engineering contexts, of the integral as an addition over several pieces in order to assist students in applying their knowledge about the integral.

APPLYING MATHEMATICS TO PHYSICS AND ENGINEERING: SYMBOLIC FORMS OF THE INTEGRAL

by

Steven Robert Jones

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Advisory Committee:

Professor Patricia F. Campbell, Chair
Professor Ann Edwards
Professor David Hammer
Professor Edward Redish
Professor Wesley Lawson

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Introduction

If one walks into a first or second semester calculus class what will one see? Most likely one will see a large number of students from varied disciplines funneled into a "generic" calculus course. From these courses the students are expected to learn the essential elements of mathematics that they need to bring back to their own fields of study for use (Terrell, 2007; Mustoe, 2002). One may encounter students from physics, engineering, chemistry, mathematics, or biology together in this classroom. This provides a ripe space for research where studies can seek to learn about how students understand the mathematics they need in their field of study.

This study specifically considers the mathematical understanding of physics and engineering majors. In order to establish a picture of previous research, the following literature review is organized along three main questions addressing the usage and understanding of mathematics in physics and engineering.

1. *What are possible difficulties in applying mathematics to physics and engineering courses?*
2. *What is known about the knowledge and understanding students activate when applying mathematics to physics and engineering problems?*
3. *What is the significance of studying student understanding of mathematics in physics and engineering contexts?*

1.1 What are possible difficulties in applying mathematics to physics and engineering courses?

In their 100- and 200-level calculus courses, mathematics departments have the challenge of educating students from various fields of study, many of whom need to apply mathematical content in their chosen disciplines. This is not unique to mathematics departments; one can also find students from various disciplines in a basic English or history course. However, mathematics departments must face the challenge that many students in calculus courses need to use mathematics regularly in their field of study and that their success depends in part on their mathematical ability (Cardella, 2004; Maloney, 1994). Therefore, many students enroll in mathematics not simply to gain appreciation for calculus, but because these courses are required and will be needed if students are to apply mathematical procedures and concepts in physics, engineering, chemistry, and biology.

This can potentially prove difficult for students who are attempting to apply mathematics to their field of study (Redish, 2005). There is growing evidence of a perception among physics and engineering educators that students are often unable to apply their mathematical knowledge successfully to their science courses (Baggi, 2007; Gainsburg, 2006; Hoffmann, 2004; Fuller 2002; Clement, Lochhead, & Monk, 1981). Some instructors "are often surprised by how little math [their] students seem to know, despite successful performances in their math classes" (Redish, 2005, p. 1). It is important to note from this quote that many of these students, those who are having difficulties accessing mathematical knowledge in their physics courses, are perceived to have done well in their mathematics classes. If there is merit to this perception, it seems that there is more at play than simply remembering the mathematics. Some educators claim that “students can often perform mathematical operations correctly in the context of a math problem, but are unable to perform the same operations in the context of a physics problem” (Redish, Steinberg, & Saul, 1996, p. 1). Is it the case then, that calculus courses are successfully providing students with one type of knowledge, the type needed to perform well in their mathematics classes, but not preparing them to use this knowledge successfully in their science classes?

Research has provided some examples of student knowledge and understanding of mathematical content that may not be successfully activated in science classrooms. One difficulty physics and engineering students may have in applying mathematical knowledge is in translating situations dealing with words, figures, and data tables into mathematical terms and equations (McDermott, 1984; Clement, Lochhead & Monk, 1981). In a physics or an engineering course, problems are often presented in real world contexts, using words, figures, and tables to organize and communicate the situation to be solved. Students are expected to take these situations and to create mathematical equations from which they can perform procedures. Students also need to dissect equations and to describe relationships between multiple variables.

By contrast, in a mathematics course students are often given bare equations and asked to perform routine procedures on them (Meel, 1998; Park & Travers, 1996; Clement, Lochhead, & Monk, 1981). While not true of all mathematics courses, there is a possibility that students may be taught in such a way that does not require them to understand the meaning of symbols (Clement, Lochhead, & Monk, 1981). This can lead, for example, to the common problem of directly mapping words to mathematical symbols without attention to their physical meaning. Clement, Lochhead, & Monk provide the well-known example of asking students to write an equation for the statement: “There are six times as many students as professors at this university” (p. 288). If the students are not attentive to the meaning of the symbols, they may directly map the words of the sentence into the equation “$6S = P$,” which resembles the word order of “six, times, students, professors.” This equation is, of course, a reversal of the original statement and actually claims that there are six times as many professors as students. This example suggests that when students deal with equations that have physical associations, they may need to activate specialized knowledge. This knowledge is something that is necessary to describe. Though prior research has investigated the cognitive features that will help a student work successfully with equations in science courses (Lee & Sherin, 2006; Sherin, 2001), it has only begun to describe potential knowledge that students may draw upon while applying mathematics to a physics or engineering context.

This leads to another issue in applying mathematics to science courses, which is the connection of mathematical symbols to physical meaning. There is a difference between the interpretation of certain symbols in a mathematics classroom and the interpretation of those same symbols in a science classroom (Torigoe & Gladding, 2007; Gainsburg, 2006; Glazebrook, 2001). The following example illustrates this difference. Consider the equation $T(x, y) = k(x^2 + y^2)$. The equation shows a function T with two variables x and y. Now imagine showing this equation to mathematicians and to physicists asking, “What is $T(r, \theta)$?” (Dray & Manogue, 2004b). With the variables x and y changed to r and θ, what impact does this have on the meaning of the function? While many physicists respond that $T(r, \theta) = kr^2$, mathematicians usually argue instead that $T(r,\theta) = k(r^2 + \theta^2)$. To a mathematician, *T*(*x*,*y*) is a function of two variables where the variables *x* and *y* are placeholders for numeric values. Changing their names to *r* and θ does not change their meaning as placeholders. Thus it is natural for a mathematician to assert that the equation remains the same but with different names for the symbols. That is, $k(x^2 + y^2)$ becomes $k(r^2 + \theta^2)$.

To a physicist, *T*(*x*,*y*) is also a function given in two variables. However, the key difference is that the variables *x* and *y* are not simply numbers, nor is *T*. They are physical quantities representing objects in the everyday physical world. *T* is often used to denote the temperature of something and *x* and *y* most likely denote a two-dimensional plane, measured in feet, meters, or inches. For example, one may see this equation as representing the temperature, in Celsius, of a hot plate in a factory. Thus changing the variables from *x* and *y* to *r* and θ change the meaning of the equation. Since the temperature, *T*, will not change at a given point just because the variables are changed, the *formula* required to calculate *T* must change to accommodate these new variables. If *r* and θ are taken to be polar coordinates, this will prompt the physicist to say that the formula is now kr^2. This example highlights the fact that mathematics is different depending on the context in which one considers it. “We speak different languages—but the basic vocabulary is the same!” (Dray & Manogue, 2004b, p. 13). There is a difference in the way that mathematics is understood in mathematics courses and the way the same mathematics is understood in physics and engineering courses.

Physics educators have attempted to propose reasons for these inherent differences regarding mathematics in physics courses in contrast to mathematics courses. Some have identified several tasks that students potentially need to be able to do in order to apply mathematical knowledge and understanding to problems in their science classes. For example, students must (1) see symbols as representing physical measurements rather than numbers, (2) parse equations to understand the various physical quantities at play, (3) coordinate time and space in problems, and (4) treat equations as representations of reality (Redish et al, 1996). It is also necessary in physics to use particular symbols that carry ancillary information not otherwise present in the mathematical structure of the equation (Redish, 2004).

These proposals help us know more about what students *should* know and understand concerning the connection between mathematical symbols and physical meaning in order to successfully apply mathematics to their science courses. However, they do not tell us *what* knowledge students activate when dealing with mathematical symbols applied to physical meaning. While some research has been done to look at how mathematical symbols are blended with physical meaning, especially in the context of the structure of equations (Lee & Sherin 2006; Sherin, 2001), there is still much work to do to describe the knowledge students bring to bear on physics and engineering problems with regard to many mathematical symbols systems, such as the symbols used in concepts from calculus. If students encounter a mathematical operation with associated physical meaning, such as considering force as the derivative of momentum, what knowledge or understanding do they activate to deal with it?

Many physics and engineering students may indeed be learning from their mathematics classes and may be successful at using mathematical concepts within the mathematics classroom. However, there is a growing perception among educators that students may not be applying this knowledge successfully in physics and engineering courses (Baggi, 2007; Hoffmann, 2004; Fuller 2002; Clement, Lochhead, & Monk, 1981). But it is not yet to fully understand what knowledge and understanding students *are* in possession of when they apply mathematics to science contexts. If a student who knows about derivatives and integrals from their mathematics course has trouble applying that knowledge in a physics or an engineering course, what knowledge *is* that student drawing on in those situations? If a student can take a bare function in a mathematics class and determine an integral, what does that student's understanding look like when the function has physical meaning and the integral carries a specific interpretation?

1.2 What is known about the knowledge and understanding students activate when applying mathematics to physics and engineering problems?

Fauconnier and Turner (2002) developed a theory of "cognitive blending" to describe the way two or more concepts are melded together. This theory has been applied to the cognitive aspects of blending mathematical knowledge with physical knowledge (Bing & Redish, 2007). In this theoretical model, "the mind combines two or more mental spaces to make sense of linguistic input in new, emergent ways" (Bing & Redish, 2007, p. 1). For example, a student might combine the "positive" and "negative" concepts from mathematics with an "up" and "down" conception from the real world to produce "positive means upward" and "negative means downward." Bing and Redish claim that "An important sign of physics students' progress is their combining the symbols and structures of mathematics with their physical knowledge and intuition, enhancing both" (p. 1). Thus students can apply mathematical concepts by blending them with the science concepts they are learning about in their classes. In order for the mathematics to be useful it needs to be tied to the physical quantities being studied.

Some studies have investigated the role that physics knowledge has in understanding mathematics (Marrongelle, 2004; 2001), though less attention has been given to the productive mathematical knowledge students have at their disposal when applying mathematics to physics and engineering. There is a need to know more about the activation of mathematical knowledge when applied to physics and engineering. For example, if students see an integral of force, what does their knowledge of this mathematical concept applied to this real-world concept consist of?

In order to discuss the knowledge students draw upon while applying mathematics to physics and engineering, it is necessary to establish the components of cognition that constitute student knowledge. It is possible to view the basic elements of knowledge through the constructs of "p-prims" (diSessa, 1993) or "cognitive resources" (Hammer, 2000). Knowledge defined in this way consists in part of abstractions from everyday experiences, including classroom experiences. In order to better understand how students apply mathematics to their science courses, it is necessary to know what sorts of p-prims or resources students possess that they could activate in the classroom. Thus the next step to build an understanding of the knowledge students activate while applying mathematical concepts in physics and engineering concepts is to begin to document what cognitive resources the students hold and draw upon when dealing with mathematics in physics and engineering contexts.

The theory of cognitive resources (Hammer, 2000), with symbolic forms in particular (Sherin, 1996), provides a useful lens for considering how students apply mathematics to their science courses. When presented with a problem to solve, there are various ways to approach that problem. There are many ideas, notions, strategies, beliefs, memorized facts, and/or intuitions that may, may not, or may incorrectly come into play when thinking about and solving a problem. These tools, abilities, and beliefs that are brought to bear on the problem are called “resources.” Hammer explains that the word “resource” is derived “loosely from the notion of a resource in computer science, a chunk of computer code that can be incorporated into programs to perform some function” (2000, p. 53). Given a problem, many resources might be activated simultaneously, but not every resource will be used and oft times the resources drawn upon might be used in a way that does not satisfactorily solve the problem.

Hammer is careful, however, to point out that this “differs from the notion of a ‘misconception,’ according to which a student’s incorrect reasoning results from a single cognitive unit, namely the ‘conception,’ which is either consistent or inconsistent with expert understanding” (2000, p. 53). Each “concept” is made up of a multitude of small-grained “resources” which are used to solve problems. Students will develop resources that help them to be successful in learning, and these resources are often tied to the problem solving situations in which students find themselves (Hammer & Elby, 2003). Thus it is quite possible that students will compile cognitive resources in a mathematics course that could be dependent on the mathematics context, or that could be devoid of the necessary blending with physical notions that would make them more readily available to students when they work on problems in science contexts.

These notions support the idea that there may be a difference between using mathematics in a mathematics course and using mathematics in a science course. There is a significant number of tasks required to apply mathematics to science courses (Redish, 2005; McDermott, Rosenquist, & van Zee, 1987; Clement, Lochhead, & Monk, 1981).

These tasks might require the students to draw upon cognitive resources that were compiled in a different context—a mathematics classroom. But little is known as to what resources students activate (and how) when applying mathematics in science contexts. In order to support the creation and appropriate activation of productive resources it is necessary to know what resources students hold in their cognition, as well as which resources they activate when dealing with mathematical symbols in physics and engineering problems.

The cognitive resources that physics and engineering students draw upon when applying mathematics to physics and engineering are at the heart of this dissertation study. Research has begun to look into the cognitive resources that students draw upon when dealing with mathematics in a science context. Specifically, some educators have looked into how physics students use equations in physical contexts (Sherin, 2006; Lee & Sherin, 2006; Tuminaro, 2004; Sherin, 2001; Sherin, 1996). According to these studies, students' cognitive resources pertaining to equations can be expressed in terms of "symbolic forms." A symbolic form is a cognitive resource that is comprised of two components: a "symbol template" and a "conceptual schema." These two are blended into a stable cognitive resource. The idea of symbolic forms is discussed further in Chapter two. At this point it is sufficient to state that the work of describing the symbolic forms that students hold in their cognition is primarily limited to the basic equation symbol structure. This research has described how students understand the relation "[] = []" as well as operations between symbols such as "[] + []", "[] – []", or "[] x []." Any number, variable, term, or complicated expression can populate the "boxes" in the symbol template. The symbolic form is then composed of the conceptual schema that is blended with this symbol structure.

While the work of revealing student cognitive resources pertaining to equations and basic operations is fundamental to understanding how students apply mathematics to science contexts, it is only the beginning of the manifold mathematical symbols used in physics and engineering. In a calculus course, students learn about the mathematical concepts of the limit, the derivative, and the integral. Each of these concepts carries with it notations and symbols. Thus this provides a rich environment within which to observe the symbolic forms students compile and hold with respect to these symbol templates. While research has looked at student understanding of various calculus concepts, there is no research specifically aimed at describing the symbolic forms that students hold regarding these symbol templates. Furthermore, all of the fields of mathematics comprise such a vast world of symbols that the fields of mathematics and physics education have barely begun the work of documenting and describing the symbolic forms students may hold in their cognition pertaining to these diverse symbol systems.

Thus, in summation, while research has begun to investigate the cognitive resources that students hold and draw upon, there is a significant amount of work left to do. Since the ideas from calculus, namely the limit, derivative, and integral, are so central to many physics and engineering concepts, it is important to know what cognitive resources (specifically, what symbolic forms) students hold about these concepts. This may help shed light on how students apply mathematics to science contexts and also clarifies the potential difficulties students may have in doing so.

1.3 What is the significance of studying student understanding of mathematics in physics and engineering contexts?

Recently there have been charges that mathematics education at the undergraduate level is not sufficiently equipping science majors for their field of study. Some educators warn of "distressing gaps in the breadth of students' education" between mathematics, physics, and engineering (Hoffmann, 2004, p. 191). Indeed, traditional mathematics instruction has been termed an "old-fashioned" part of the pillars of engineering education that should be revised to fit the current needs of today's world (Baggi, 2007). There is concern among businesses that this gap in education between mathematics, physics, and engineering may lead to an engineering force that lacks the fundamental skills to compete in today's competitive global market (Booth, 2008; Kennedy, 2006; Gainsburg, 2006).

There are also concerns that this gap in education contributes negatively to student success in college. There is a high rate of failure and withdrawal among science majors, which has been found to be linked to the process through which students learn mathematics (Borges, Do Carmo, Goncalves & Cunha, 2003). As a result, in some European universities engineering schools are reducing the number of required mathematics courses and are inserting the mathematics content relevant to engineering into their own classes (Fuller, 2002). This casts doubt on the effectiveness of mathematics service courses and causes concern about the mathematical opportunities given to physics and engineering majors. European colleges of science, in particular, are looking to find up-to-date methods for teaching mathematics to their students as they distance themselves from mathematics departments (Baggi, 2007).

The conclusions one draws from this literature call into question the mathematical opportunities given to students. If students were to learn mathematics in such a way as to be able to understand and apply mathematical content in future applied and theoretical contexts, what would their mathematical instruction consist of? This has prompted several researchers and institutions to begin work on redesigning curriculum in order to "bridge the gap" between mathematics and science (Dray & Manogue, 2003; Berry, DiPiazza & Sauer, 2003; More & Hill, 2002; Meredith & Black, 2001). The Vector Calculus Bridge Project at Oregon State University (Dray & Manogue, 2006) seeks to switch the emphasis of traditionally taught calculus concepts from the limit to the differential, from slope to rate of change, and from area to total amount. They argue that these changes in emphasis will have the effect of making generalizations of the mathematical concepts to physical situations easier. The Studio Calculus/Physics course designed at the University of New Hampshire (Meredith & Black, 2001) aims to help students "see the use of the calculus immediately" (p. 5) by mixing introductory calculus and introductory physics together into one class. These authors are attentive to the order in which the topics are taught so that as students learn about physics concepts, they also learn the requisite calculus concepts. The activities are explicitly a mixture of calculus and physics and students are required to solve several open-ended problems that make use of both curricular domains.

The mathematics education community is also engaged in improving curriculum for calculus courses (Hughes-Hallett, et al, 2005). The Harvard Calculus Project takes a different approach to teaching the basic calculus concepts. Those who collaborated on the textbook for this project "started with a clean slate" and came to decisions about what topics to emphasize "after discussions with mathematicians, engineers, physicists, chemists, biologists, and economists" (Hughes-Hallet, et al, 2005, p. v). Each idea is presented from a graphical, numerical, symbolic, and verbal perspective. Doing this creates opportunities for the students to construct a well-rounded understanding of the calculus concepts by learning about them from several angles. The section on integration begins with a discussion of how far an object has traveled given that its velocity is known. The idea of the Riemann sum is discussed which is then tied to the notion of area underneath the curve. It is important to understand the impacts that this approach would have on students' attempts to apply their knowledge about the integral to physics and engineering.

These curriculum projects are generating substantial interest in the need to identify what students know and understand about mathematical concepts in physics and engineering. In order to best design curriculum, teaching strategies, and support material, the research field needs to be familiar with student thinking and understanding in these contexts (NCTM, 2000). For example, how would a shift of emphasis better support the creation of productive knowledge? What cognitive resources can be supported by teaching calculus and science concepts simultaneously? If derivatives, integrals, differentials, graphs, and equations are intertwined with physics and engineering concepts, what will the knowledge that students have look like? At this point, the research field is still in the process of documenting and understanding the cognitive resources students draw on in these situations.

Several studies have been conducted about various aspects of learning calculus. The focus of many of these studies has been on the limit (Oehrtman, 2008; Oehrtman, 2004; Brown, 2004; Tall & Vinner, 1981), on the derivative (Zandieh, 2002; Carlson, et al, 2002; Marrongelle, 2001; Zandieh, 2000; Ortin, 1983b), and on the conception of the function (Oehrtman, Carlson, & Thompson, 2008; Carlson, et al, 2002). Some studies have also looked at student understanding of the integral (Sealey & Oehrtman, 2007; Sealey, 2006; Sealey & Oehrtman, 2005; Marrongelle, 2001; Ortin, 1983a). The work done by Sealey and Oehrtman has looked primarily at student understanding of the Riemann sum. The purpose of their research "was to examine student development of the concepts of the Riemann sum and definite integrals" (Sealey & Oehrtman, 2007, p. 78). Thus, this work focused on understanding how students compiled particular knowledge about one of the ways of interpreting the integral. This provides valuable insight into how students come to construct their understanding of the integral as an addition process. Also, Marrongelle studied the way in which students used their physics knowledge to inform their mathematical understanding of the calculus concepts of the limit, the derivative, and the integral.

The research that has been done around student understanding of the integral sheds light on how students come to understand the Riemann sum process and how their physics knowledge influences the creation of their ideas. This yields results that can affect instructional practice and curriculum design. However, there are important components to how students understand the integral that are missing from this research. First, there is not enough information about the knowledge of the integral that students draw upon in various contexts. Instead of focusing on one interpretation of the integral, it can also be asked what knowledge will students spontaneously draw on when working with problems involving integrals? Specifically, is there a difference between the knowledge about the integral that students will draw on in a mathematics setting as opposed to a physics setting? Second, while there is information about how students use physics knowledge to inform the creation of their mathematical knowledge, there is not enough information about how students activate and apply their mathematics knowledge of the integral to physics and engineering classrooms. This provides a place for continued research to contribute to an understanding of how students apply the knowledge they are intended to learn in a calculus course.

1.4 Conclusion and Research Questions

There is more to applying mathematics concepts to physics and engineering courses than simply remembering the concepts learned in the mathematics course. There is a call for improved undergraduate mathematics education for physics and engineering students. Calculus is fundamental to both of these disciplines and is prerequisite for advanced courses. Conceptual understanding of the derivative and the integral are important for a student to be successful in physics or engineering coursework. There is limited information about some difficulties students have in translating between mathematical symbols and real-world concepts found in physics and engineering courses.

There are useful theoretical tools for understanding student cognition during the application of mathematics to science. Theory suggests that when students use mathematics concepts in physics and engineering contexts, they may activate several cognitive resources that can either be helpful or unproductive in their attempts to solve problems. These cognitive resources can consist of symbol templates “cognitively blended" with conceptual schemas into symbolic forms. By knowing more about these resources, it may be possible to support the productive application of symbolic forms in science contexts. Prior research has provided information about how students compile their understanding of the Riemann sum. It has also provided information about how students use their physical understanding to assist in building mathematical knowledge.

However, at this point there is not enough information about the cognitive resources students spontaneously activate around calculus concepts when dealing with mathematical symbols in physics and engineering contexts. While there is some research on symbolic forms related to equations and simple operations, there is not sufficient documentation of the cognitive resources that students hold about the mathematical symbols and ideas from calculus. This includes a lack of documentation and description of the symbolic forms students hold and draw on regarding the symbols used in derivatives and integrals. When students make use of the integral in physics and engineering courses, there is little clarity with respect to what resources around this concept, including symbolic forms, students may activate. The next step is to discover and describe the symbolic forms from a broader range of symbol templates that students might hold and activate. Specifically, research may consider the symbolic forms students hold and activate around the integral symbol template.

Thus it is appropriate to continue the work of documenting and describing students' cognitive resources by finding evidence for the symbolic forms they activate in both mathematics and physics settings. Because the concepts in calculus are fundamental to coursework in physics and engineering, this study considers the symbolic forms pertaining to the integral that students may hold in their cognition. In order to shed light on the apparent disconnect between doing mathematics in a mathematics course and applying that mathematics to a physics or an engineering course, this research investigates the symbolic forms students draw on in both a mathematics and a physics context. In particular, this research addresses the following question:

- *What are the symbolic forms relating to the integral that physics and engineering students have and draw upon?*

Furthermore, this work compares the activation of symbolic forms relating to the integral in physics and engineering contexts as opposed to mathematics contexts. Specifically, this work will characterize the symbolic forms that are activated when the students are engaged in physics-framed problems in possible contrast to those that are activated in mathematics-framed problems. Thus, this study seeks to provide insight around each of these sub-questions:

- *What symbolic forms for the integral do students activate in mathematics-framed settings?*
- *What symbolic forms for the integral do students activate in physics-framed settings?*
- *What is the intersection and/or disjunction in symbolic form activation between these two settings?*

1.5 Implications and Limitations

This research study is not able to define the conditions that will activate certain symbolic forms. There are many factors that can contribute to the "choice" of symbolic forms and this study is limited to the specific interview items presented to the students. Thus the findings presented in this research are influenced by the particular items that were chosen for the interviews. However, this study does shed light on the relationships between certain problems or particular tasks and the activation of symbolic forms. It is also able to provide a description of the similarities and differences between symbolic form activation for interview items that are framed as “mathematics” problems or “physics” problems. What this study cannot clearly answer is the question of *why* symbolic forms are activated in the manner that they are. It can only express correlations between the items and the symbolic forms that are activated. Also, these interview items can serve as nothing more than very rough approximations to classroom settings where students learn about the integral and apply that knowledge to physics and engineering concepts. Thus the results of this study cannot directly imply a particular way that students think in mathematics classrooms or physics classrooms.

Despite these limitations, this study does suggest implications for both instruction and curriculum. It provides evidence for general ways in which the integral could be expressed in a mathematics course to better support the application of knowledge about the integral to physics and engineering. It also suggests particular ways of understanding the integral that appear to be most productive in physics contexts. This leads to implications for how the integral could be introduced and explicitly discussed in physics and engineering courses to support the application of knowledge about the integral.

CHAPTER 2: LITERATURE REVIEW

2.1 Domain and Definitions

Domain of the Study

It is beyond the scope of one project to include all of the academic disciplines that use mathematics. Therefore, it is necessary to limit the fields considered in order to focus more deeply on the particular issues of applying mathematics to those fields. Though it is difficult to say that some fields are more "important," it is possible to say that some fields depend more heavily on mathematics than others. Physics and engineering make particular use of mathematics (Baggi, 2007; Maloney, 1994) and thus this study and literature review are limited to these fields. For simplicity, in this review physics and engineering students are sometimes referred to as "science students." In this study they will be interchangeable.

Calculus is fundamental to the study of physics and engineering. As a result it has received much attention from curriculum developers who are attempting to close the perceived gap between mathematics and science learning (Dray & Manogue, 2003; More & Hill, 2002; Meredith & Black, 2001). Since there is much interest in calculus learning and usage, this study is focused within its boundaries. However, even within the concepts of calculus, there is much knowledge that students could construct relative to the concepts of the limit, the derivative, and the integral as well as their applications to physics and engineering. In order to delve more deeply into student knowledge around a single topic, the domain of this study is restricted to the integral, specifically the knowledge students have about the integral in a mathematics context and a science context. Consequently, the results of this study regarding "mathematical knowledge that is applied to physics and engineering" should be understood to mean "mathematical knowledge about the integral that is applied to physics and engineering." This study's methodology, which is described in Chapter three, makes use of items pertaining to the integral in both a mathematics context and a physics context.

Definitions

In common language, once students have learned mathematics, they can "apply" it to other contexts. By this is meant the ability of a student to perform mathematical procedures or to relate mathematical concepts outside of the class in which they were learned. This includes (1) using mathematics abilities after time has elapsed (Kwon, Rasmussen & Allen, 2005), (2) using mathematical procedures and concepts in non-symbolic problems (Clement, Lochhead, & Monk, 1981), and (3) performing mathematical actions in a science classroom (Maloney, 1994). Thus, within this report "applying" mathematics to physics and engineering refers to these three features.

Since this study focuses on students' application of mathematics in physics and engineering, it necessarily deals with student knowledge. The domain of "knowledge" is broad and complex, including several facets such as the quantity of information one has about a subject, the interpretation of symbols, familiarity with mathematical objects and procedures, and the ability to dissect the subject (Nickerson, 1985). In this study, the aspects of "knowledge" are those of "cognitive resources" (Hammer, 2000), which will be explained later in this chapter. This study has not exhausted the depth of student knowledge about the topics or contexts that are the settings for the problem-solving interviews, nor does it determine their ability to abstract mathematical or scientific knowledge. This investigation has not *directly* studied students' beliefs about mathematics or science (Elby & Hammer, 2001) though it is acknowledged that their beliefs impact the theoretical framework. Rather this study focuses on documenting the symbolic forms activated in a mathematics context and a physics context as well as comparing and contrasting the forms activated in each setting.

Additionally, this dissertation speaks about "mathematical knowledge" especially as it is applied to physics and engineering. This implies a distinction between mathematical knowledge and physics knowledge. This study looks at the knowledge students have about the integral, which will be considered a mathematical concept. Thus, the knowledge students have about the integral is by necessity connected to "mathematical knowledge." However, the integral has many uses in fields other than mathematics, such as in physics and engineering, meaning that the knowledge that students have about the integral may also connect with their knowledge about physics or engineering. By "mathematical knowledge" several things are meant. It is how students understand the syntax of formulas and equations, how students interpret and manipulate mathematical symbols, and the conceptual structures and representations of mathematical notations. That is, it deals both with performing procedures and with providing a conceptual structure for the mathematical object.

This study does not seek to explore "physics knowledge" in isolation, but instead considers mathematical knowledge that is connected to this physics knowledge. Thus, "mathematical knowledge connected to physics" is defined as that which combines mathematical objects, such as the derivative, the integral, or equations, with objects found in the everyday world, whether material or immaterial, such as mass, force, electrons, velocity, or density. Through these definitions, then, mathematical knowledge is that which does not require any knowledge about physical objects in order to be functional. If a student can provide an interpretation of the integral that is not dependent on their knowledge of physical objects, such as velocity and acceleration for example, then this could be called mathematical knowledge. When this mathematical knowledge is linked to particular knowledge about these physical objects from the everyday world, then it has become mathematical knowledge connected to physics. The idea of “symbolic forms” which is presented in this chapter provides one way to talk about mathematical knowledge of the integral as well as mathematical knowledge that is connected to physics knowledge.

In this dissertation both physics and engineering are grounded in “real-world contexts” and study physical phenomena. As a consequence, the explanatory text frequently includes the phrases “physical objects” and “physical world.” The word “physical” is intended to mean “real-world” in that it pertains to *both* physics and engineering, despite the close relationship between the words “physics” and “physical.” Thus physical objects refer to the things that both physics and engineering students might study. Also, “objects” are to be understood as both concrete objects (e.g. liquids, rigid bodies, etc) as well as conceptual objects (e.g. force, momentum, etc). Lastly, for simplicity the phrases “science students” or “science classes” mean physics and engineering students and classes. In this dissertation, the word “science” is limited to these two fields of study unless otherwise noted.

2.2 Mathematics in Physics and Engineering

Mathematics, Physics, and Engineering as Symbolic Systems

This section describes the interconnectedness of the domains for this study. The mathematics, physics, and engineering disciplines are steeped in symbolic notation, where entire concepts can be described using a single symbol. In a mathematics course, ideas are given names and symbols that enable them to be communicated; these names and symbols also are intended to aid in mental manipulation (Tall & Vinner, 1981). For instance, the symbol "sin(*x*)" has multiple meanings, representations, and uses in mathematics. It can be a "wave-like" graph, the relationship of an angle and sides of a triangle, or simply a numerical value. The development of understanding of mathematical symbols is a gateway between direct computations and more abstract thinking (Graham & Thomas, 2000). "[T]he total cognitive structure which colours the meaning of the concept is far greater than the evocation of a single symbol" (p. 151). For example, the idea of a "variable" is in one way or another fundamental to every advanced mathematical domain. Yet despite its simplistic appearance, one variable can take on multiple meanings, even simultaneously, which (ideally) allows for new connections to be made. Variables, equations, graphs, functions, tables, and manipulatives all serve to extract concepts into the theoretical realm and serve as symbols for real or constructed objects (Goldin & Shteingold, 2001). Undergraduate students encounter symbols every day in a mathematics classroom.

Physics and engineering also make extensive use of mathematical symbols. They borrow much from mathematics by way of formulas, theorems, operations, and notations. However, mathematical symbols in physics may take on different meanings than those intended in pure mathematics (Redish, 2005). In physics, symbols often represent physical measurements rather than simple numerical values. This is depicted by the example from Dray and Manogue, described in chapter one, regarding the way mathematicians and physicists interpret $T(r, \theta)$ differently. An important sign of progress in physics and engineering students is their ability to mesh the symbols and structure of mathematics with physical knowledge and intuition (Bing & Redish, 2007).

Mathematics is Fundamental to Physics and Engineering

As just expressed, there is a fundamental link between mathematics, physics, and engineering. Considering its history, several mathematical domains (including calculus) were created to satisfy understanding about measurement, construction, and motion. It is the language that physicists and engineers use to theorize about the nature of the world and universe. Because of their close connection, there is a call to pay attention to the relationship between teaching mathematics and teaching physics and engineering (More & Hill, 2002; Fuller, 2002). Recent efforts have included looking at how science students are using their mathematical knowledge in order to solve physical problems (Bing & Redish, 2007; Lee & Sherin, 2006; Sherin, 2001; Dray & Manogue, 2006). Mathematics, then, is not an isolated subject matter, nor can it be considered only in the environment of a mathematics classroom. Its applications warrant discussion in a broader context. When considering the connection between the domains, it is important to remember that applying mathematics to science involves representing physical meaning in addition to expressing abstract relationships (Redish, 2005).

The Problem to be Addressed: The Gap between Math and Physics/Engineering

There is a compelling perception that students do not readily apply their mathematical knowledge to physics and engineering (Booth, 2008; Baggi, 2007; Gainsburg, 2006; Redish, 2005; Hoffmann, 2004; Fuller 2002; Clement, Lochhead, & Monk, 1981). Is this because students are not learning the mathematics, or is there more to applying mathematics than just remembering? If they *are* learning from their mathematics courses, but then struggling to apply this knowledge to physics and engineering, what could cause this phenomenon? Some educators have suggested areas that may cause problems when trying to apply mathematics to physics, including relating symbols to measurement, parsing equations, understanding equations as relationships, coordinating time and space, and treating equations as representations of reality (Redish et al., 1996). Hence there are numerous places in the process of applying mathematical knowledge to science classes where students could potentially have difficulty.

In order to address this perceived issue, many curricular projects have sprung up to more closely relate the mathematics instruction to physics and engineering (Dray & Manogue, 2003; Berry, DiPiazza & Sauer, 2003; More & Hill, 2002; Meredith & Black, 2001). However, the curricular approaches should have a solid base of understanding of student knowledge to rely on. If it is not known what knowledge students have and are drawing on, how will instruction be tailored to support the application of knowledge to other domains? At this point there is limited understanding of the knowledge that students draw on when applying mathematics to science contexts, especially concerning the concepts within calculus. This dissertation attempts to describe more about the specific pieces of knowledge that students hold in their cognition pertaining to the concept of the integral. To analyze the problem, consider the idea of “cognitive resources.”

2.3 Theoretical Perspective: Cognitive Resources

The “Predecessors” to Resources

First, is it helpful to discuss the ideas that serve as a basis for the theoretical construct of “resources.” Suppose that a “concept” in a student’s cognition is not a single entity, but rather consists of smaller units. These smaller units together may create an overall concept, belief, or strategy. Thus it has been argued that calling a student’s “conception” correct or incorrect is too simplistic (Clement, Brown & Zietsman, 1989). In physics education, researchers began testing this idea by creating tasks that drew upon smaller, correct ideas that students employed in order to build up knowledge about a larger concept. For example, Minstrel (1982) noted that students had difficulties understanding how a table can exert an upward force on a book that rests upon it. He showed that this could be overcome by first discussing springs. Students were able to easily recognize the force the spring exerted back on the object compressing it. The students in his study were then more easily able to understand the notion of this “upward force” from the table. Thus this study had bypassed the idea that “force” is a unitary conception in a student’s cognition, either to be understood or misunderstood. Instead, the concept of “force” could be subdivided into smaller ideas, namely downward force, upward force, spring force, static force, etc. Some of the smaller ideas about force then proved useful in explaining the larger picture about the force exerted by the table.

DiSessa (1993) pushed this notion further by describing “phenomenological primitives” or “p-prims.” DiSessa defined these as the smallest pieces of cognitive structure that are accessible by a person. The p-prims were much smaller than the conceptual total, meaning that even within one idea there could be several p-prims. Additionally, one p-prim could contribute to many distinct concepts. Defining knowledge “in pieces” in this way was important in that it continued to push away from the idea of a single mental entity called a concept, and argued instead that understanding and knowledge are made up of lots of individual bits of cognitive structure. Several researchers have built on this work to produce the theory of cognitive and epistemological resources.

Definition and Explanation of Resources

Cognitive and epistemological resources are a theoretical construct that has been developed recently to explain the nature of knowledge (Gupta, Redish, & Hammer, 2008; Hammer, Elby, Scherr, & Redish, 2005; Louca, Elby, Hammer, & Kagey, 2004; Hammer & Elby, 2003; Elby & Hammer, 2001; Hammer, 2000; diSessa & Sherin, 1998; Hammer 1994). In order to present this construct, consider someone who is presented with a novel problem. This person may begin by searching their knowledge and experience in an attempt to use what they already know in order to solve it. This person might have employed memorized facts, informal notions, large concepts, or beliefs about the nature of the problem. There are various ways to consider a problem. There have many concepts, notions, strategies, beliefs, or intuitions that may, may not, or may incorrectly come into play when thinking about a given problem.

These tools, knowledge, concepts, and beliefs that are brought to bear on the problem are called “resources.” The word “resource” is derived “loosely from the notion of a resource in computer science, a chunk of computer code that can be incorporated into programs to perform some function" (Hammer, 2000, p. 53). A computer programmer does not create all of her or his code from scratch every time. There is a vast reservoir of previous work, which includes many self-contained subprograms that are retrieved and implemented by a programmer. The programmer connects several of these resources and combines them with intuition and logic of his or her own to create a new project. Note that an important characteristic of these resources is that they are used and re-used under many circumstances. Also, resources may be different sizes: some cognitive resources may be memorized facts about multiplication, while another may be the overall compiled view of the space R^n. A resource is simply whatever cognitive structure, large or small, that can be activated and implemented as a unit. Hence one resource can be said to be made up of other smaller resources. For example, the overall concept of R^n might be made up of ideas about vectors, dimensions, axes, and the origin.

It is important to remember that resources are not always used to a constructive end (Hammer, 2000). Given a problem, many resources may be activated simultaneously, but not every resource will be used and oft times the resources drawn upon might be used in a way that does not satisfactorily solve the problem. Consider a person who is in the process of working out an answer only to discover that their solution method was not providing a useful path and that they needed to start over. Hammer is careful, however, to point out that this "differs from the notion of a 'misconception,' according to which a student's incorrect reasoning results from a single cognitive unit, namely the 'conception,' which is either consistent or inconsistent with expert understanding" (2000, p. 53). Thus under the theory of resources, concepts do not come in the form of a single unit, but are rather built up of several components, meaning that a "misconception" might really just be a useful idea that was merely misapplied to the wrong situation. The piece of knowledge employed in understanding a situation might be completely valid in one context, but inappropriate for another. One example is to consider the answer to the question: "why is it warmer in the summer than the winter?" The correct answer to this question has to do with the angle of the Earth's tilt relative to the sun. As one hemisphere tilts toward the sun, the rays can penetrate to the surface more easily and for a longer period of time each day, causing the temperatures to increase. However, a student may think about this question and draw on the good, often useful resource "closer means more intense." This resource certainly is true when it comes to a fire, or a hot stove. The closer a person is to a fire, the warmer they are. Therefore, this usually productive resource may be misapplied to the question and the student might answer "because the Earth is closer to the sun during the summer." Therefore, it is not to say that this student has created a single cognitive unit about the seasons that is in disharmony with expert understanding. Rather, the student took a resource grounded in good experience and applied it to the wrong situation.

Epistemological Resources

An "epistemology" is essentially the idea of "how we know that we know something." A person's epistemology describes what they understand about what knowledge is, what it's made up of, or how it's created. In the research community there is an apparent consensus about what constitutes a correct epistemology about science—that science is tentative and evolving (Elby & Hammer, 2001). However, this consensus can be questioned on the grounds that epistemologies should be treated as finer grained. In essence, much like the cognitive structures that can be broken down into smaller components, epistemologies can be subdivided into epistemological resources (Hammer & Elby, 2002). Thus it would be important to understand that students may have epistemologies along many different dimensions, including beliefs about the nature of a subject, how knowledge is gained in that subject, or the certainty of that knowledge. The beliefs that students hold about the interview setting, the tasks given to them, and about mathematics and physics themselves affects how they understand and approach the task (i.e. how they "frame" the task). Additionally, their beliefs affect the perceived goal of the task, the kind of knowledge they activate, and how they attempt to solve the task. In short, epistemological resources affect which cognitive resources students activate.

Furthermore, the work done with epistemological resources provides a more detailed understanding of certain features of a resource. Resources can better conceptualized by giving attention to the "form" of the epistemological or cognitive resource (Louca, Elby, Hammer & Kagey, 2004). The "form" of a resource consists of the "grain size, stability, and context dependence of the relevant cognitive elements" (p. 57). The grain size refers to the relative size of the resource, whether tiny and primitive, or large and compiled of other resources. One could say that diSessa's "p-prims" might be generally considered as having a small grain size because of their primitive nature. However, Sherin's symbolic forms (to be discussed presently) might have a larger grain size because they are compiled from other cognitive resources. As previously discussed, the idea of a resource's "grain size" turns away from the theories that concepts and epistemologies exist only as developmental stages or unitary cognitive objects, stating instead that they can exist as a collection of finer-grained cognitive elements (Hammer & Elby, 2002).

Next, resources can be context dependent, meaning that a person will not necessarily apply the same resource to every situation—resulting in knowledge of an "object" being different things at different times. For example, it is possible to perceive that "*a* is a constant" in contexts such as a mathematics course, where *a* predominantly shows up in situations where there is no associated physical meaning, like in the equation $f(x) = ax^2 + b$. Yet if the equation deals with a physical situation such as $a = \frac{dv}{dt}$, like one would see in a physics course, this resource may become dormant while another resource "*a* is the acceleration and might not be constant" is activated and used. This may happen because of the context of the physics course, where *a* often has the meaning of acceleration and the acceleration may continuously change depending on the situation. Thus these resources show a dependence on the context. Finally, a resource may have a certain stability or instability associated with it. For instance, one potential resource that might be activated when dealing with integrals is that "the integral is the area under the curve." This resource might be so stable in a student's cognitive structure that the resource will be used every time an integral is encountered, even if there are other ways of considering the integral. Other times the resource might be so tentative that it is activated in one situation with an integral and then not activated in a nearly identical situation.

This study is not intended to document student epistemologies nor to explicitly analyze how they influence "choice" of cognitive resource activation. However, since this study does seek to document and analyze the symbolic forms that students activate, it must be acknowledged that these resources are playing a fundamental role in what is observable in the interviews. Epistemologies play a role in two different places: in the "framing" of the situation and in the "selection" of cognitive resources. (I put "selection" in quotes as it is most often a tacit cognitive function.) The following paragraphs describe more about the role of framing.

Framing

The concept of framing helps us understand how students interpret a given situation, problem, or goal. When a person encounters any situation, they automatically (and most often tacitly) make choices about what the situation means and how to construe it. It is the way a person answers the question "What is going on here?" where the interpretations happen as a continuous process (MacLachlan & Reid, 1994; Lunzer, 1989). As a working definition, framing in this study means "a set of expectations an individual has about the situation in which she finds herself that affect what she notices and how she thinks to act" (Hammer, Elby, Scherr, & Redish, 2005, p. 97). Thus, in a grocery store a person might use estimation to add up the prices of items in his or her their cart, where in a mathematics class that same person might use a written-out procedure or a calculator to find the exact sum. The "task" is identical (add up these numbers), but the person frames it differently depending on the situation (grocery store vs. mathematics class) and hence uses a different strategy for computing the numbers.

In the interview setting, there are a number of components that influence students' framing of the tasks. While it is not possible to account for nor control all of them in order to make the interview a perfectly authentic reproduction of classroom, homework, exam, or work settings, there are some practices implemented that may control some of the effects of framing during the interview. First, I interviewed students in pairs where I presented them with a problem and asked them to discuss it and solve it until they were both fully satisfied. Interviewing the students in pairs reduces the tendency of students to respond to interviewer probing as directional hints about what is correct and to guess at what I want them to say. Instead, they must explain their thinking to each other and respond to each other's questions. It is less likely that the other student is seen as an authority figure, though it may happen, and thus less likely that the student would look for cues as to what to say and what is correct. Second, student pairs came in for two interview sessions where I intentionally framed each interview as either a "mathematics day" or a "physics day." Thus by forcing a set of expectations regarding the interview, I more cleanly expected the students to "frame" the tasks in a particular way. During the first interview, I told the students that we were looking at problems from a mathematics class and provided them with tasks that closely resemble mathematics classroom notations and styles. In the second interview, I told the students that we would focus on problems from a physics class and gave them items more similar to those encountered in a physics course.

This study is also not intended to explicitly study the effects of framing, nor to provide an analysis of it in the results. Framing is only intended to be considered as an important component of what happened during the interview that may color the data that I collected. It is possible that some of the data could be interpreted as directly consequent from student framing.

Resources and Transfer

It is important to note why a theory of transfer is not used in this study. A key feature to the theory of cognitive and epistemological resources is the abandonment of the notion of "transfer" and the development of the notion of "activating resources"

(Hammer, Elby, Scherr & Redish, 2005). This hearkens to the rejection of conceptions and epistemology as unitary objects to be transferred to a new situation. More recently researchers have instead begun to argue for a "manifold" view of conception and epistemology. The idea is that knowledge "involves many simple elements whose origins are relatively unproblematic, as minimal abstractions of common events" (diSessa, 1993). If one reject conceptions and epistemology as unitary objects, one cannot expect to see an entire conception (which is built up by many fine-grained resources) to be summarily "transferred" to a new situation. Instead, individual components of knowledge and beliefs, or rather cognitive and epistemological resources, are activated upon encountering a new situation. Consequently, instead of interpreting errors as students simply having "incorrect" conceptions that they learned in another context and are applying to this new context, it is possible that a student has merely misapplied an otherwise "correct" resource. Thus instruction is not required to replace incorrect conceptions or beliefs with correct ones, but rather to support the activation of resources the student already possesses (Hammer, Elby, Scherr & Redish, 2005). This study is aligned with the theory of cognitive and epistemological resources.

Resources as a Way to Understand the Gap between Mathematics and Physics/Engineering

This section seeks to explain how the theory of cognitive and epistemological resources sheds light on applying mathematics to physics and engineering. This is done by demonstrating how the theory of resources implies a difference between learning mathematics in a mathematics classroom and learning mathematics in a science classroom. The focus for exploring such differences resides in the way that the two classroom settings create and encourage different sets of cognitive and epistemological resources.

Consider the set of curricular materials for an undergraduate calculus course. According to the literature, typical traditional curricular materials are described as "formal" (Meel, 1998) and "routine" (Park & Travers, 1996). Traditional calculus curricula introduce concepts through formal introductions and skill development (Meel, 1998). What cognitive and epistemological resources are developed and encouraged through such materials? Since students develop resources that help them to be successful in the environment they find themselves in (Hammer & Elby, 2003), they may build epistemological resources that "solving a problem" means executing a procedure. It is possible to come to believe that the variable should always be x from regularly seeing functions such as $f(x) = x^2+2x+1$ or $f(x) = \sin(x)$. The derivative might be understood to be a procedure that is done to a function, where the exponent is dropped down as a coefficient and then the exponent is reduced by one. Note that this does not imply that any of these resources are wrong or hurtful.

However, by contrast consider those students in a physics classroom. Suppose they were given the equation $M = \int_S \rho \, dV$. If the student tries to apply the resource "the variable should be x," which was productive in the mathematics course, the student may have difficulty understanding this integral equation because it lacks an x. Furthermore, if he or she understands the integral to be the "anti-derivative of the integrand with respect to the differential" (which again is a productive resource in a mathematics course), he or she may struggle to understand how ρ is a function of the variable V. This exemplifies how productive resources for a mathematics class may become unproductive when applied to very similar-looking situations in a physics course. The resources that students have at their disposal and which ones they activate will impact their understanding of physics concepts. Thus the theory of resources affords an important lens through which one may investigate the application of one subject to another, i.e. applying mathematics in physics and engineering courses.

2.4 Symbolic Forms

The cognitive resources that physics and engineering students draw upon when applying mathematics to physics and engineering is at the heart of this dissertation study. Thus it is important to review documented evidence of the resources that these students draw upon when dealing with mathematics. Research has looked into how physics students use equations in physical contexts (Sherin, 2006; Lee & Sherin, 2006; Tuminaro, 2004; Sherin, 2001; Sherin, 1996). According to these studies, there are certain types of cognitive resources that inhabit students' understanding of equations, which can be expressed in terms of "symbolic forms." A symbolic form is a cognitive resource that is comprised of two components: a "symbol template" and a "conceptual schema" blended with the symbol template. Let's take a moment to discuss this terminology.

In order to have a tangible example to work with, consider the following two equations from physics:

$$(1)\ v = v_o + at \quad \text{and} \quad (2)\ E = P + K\ .$$

The first equation describes the velocity of an object, v, that is subjected to constant acceleration, a. One takes the initial velocity of the object plus the additional velocity acquired from the acceleration in order to determine the object's velocity at a particular point in time. The second equation describes the relationship between total energy, E, and potential energy, *P*, and kinetic energy, *K*. By adding the components *P* and *K* one can determine the total energy.

Considering these examples, let us first explore the "symbol template." The template is simply the arrangement of the symbols in the equation. First, the right hand side of both equations bears the template "[] + []." That is, in each equation there are two terms separated by a plus sign. Also, each equation has another template, which is "[] = []." That is, there are two expressions separated by an equals sign. Together, both of these equations have the template "[] = [] + []." The template does not require any meaning to be associated with any of the parts. It is no more than the structure or arrangement of the symbols. Thus, as far as the symbol template is concerned, these two equations are identical.

The "conceptual schema" refers to the meaning underlying the arrangement of the symbols. Let us look at the conceptual schema that relate to these two equations. In the right hand side of the first equation (1), the two terms refer to a "base" and a "change" (Sherin, 2001). That is, the v_o is the *starting* point of the velocity and the *at* is the amount of *additional* velocity that the object receives after *t* amount of time. Thus, the *symbolic form* associated with the right hand side of the first equation is "[base] + [change]." Additionally, one invokes the symbolic form "same amount" ([] = []) to know that the *v* on the left hand side is equal to the resulting amount on the right hand side. For the second equation, the *P* and the *K* in the right hand side refer to two components of the total energy. They are each a "part" of a "whole" (Sherin, 2001). The *symbolic form* associated with the second equation is "[whole] = [part] + [part]." This is a different symbolic form than the one for the first equation. From this example it can be seen that two symbolic forms may share the exact same symbol template, but have different conceptual schemas. Therefore, two symbolic forms may differ in their symbol templates, their conceptual schemas, or both.

According to this idea, students understand mathematical expressions as a combination of the symbol arrangement as well as the concepts that underlie the symbols. Furthermore, physics and engineering students understand mathematical expressions by taking the symbolic form and attaching it to the physical meaning of a given situation (Sherin, 2001; 1996). For instance, in the velocity equation from the previous example, a student would associate the "base" with the velocity an object was already traveling at and the "change" with the additional velocity produced due to the acceleration that the object experienced. By blending the symbolic form with the physical situation of velocity and acceleration (Bing & Redish, 2007) the student can apply the mathematical structure of an equation to the movement of an object.

Sherin's work in this area (2006; 2001; 1996) provides a rich list of many symbolic forms that students draw on while working with equations in physics. He has organized the symbolic forms into clusters based on similarities in structure or conceptual schema. The following table lists the symbolic forms that Sherin identified. (Note that some symbolic forms have the same symbol template.)

Cluster	Name of Symbolic Form	Symbol Template of Form
Competing terms	Competing terms	[]± []± [] ...
	Opposition	[] – []
	Balancing	[] = []
	Canceling	0 = [] – []
Terms are amounts	Parts-of-a-whole	[[]± []± [] ...]
	Base ± change	[[]± Δ]
	Whole – part	[[] – []]
	Same amount	[] = []
Dependence	Dependence	[...*x*...]
	No dependence	[...]
	Sole dependence	[...*x*...]
Coefficient	Coefficient	[*x*[]]
	Scaling	[*n*[]]
Multiplication	Intensive-extensive	*x* x *y*
	Extensive-extensive	*x* x *y*
Proportionality	Prop+	$\left[\frac{...x...}{...}\right]$
	Prop–	$\left[\frac{...}{...x...}\right]$
	Ratio	$\left[\frac{x}{y}\right]$
	Canceling (b)	$\left[\frac{...x...}{...x...}\right]$
Other	Identity	*x* = ...
	Dying away	$[e^{-x\,...}]$

Table 2.3.1: Symbolic forms identified in previous research (Sherin 2006; 2001; 1996)

This work provides a rich vocabulary for talking about how students use equations within the context of physics and engineering. However, it is known that there are many mathematical symbols beyond equations that students must use in physics and engineering in order to apply mathematics successfully. For instance, this work has begun to be extended to graphs (Lee & Sherin, 2006). Yet, as students at the post-secondary level continue in their physics and engineering education, calculus concepts become increasingly more common and important. Students must understand and use derivatives and integrals in their coursework. At this point, there are not descriptions of the cognitive resources along the lines of symbolic forms that students have regarding the derivative or the integral. This research project attempts to extend the work of Sherin's symbolic forms to the integral. Specifically, given the integral symbol template (for example "$\int_{[]}^{[]}[]d[]$" or "$\int[]d[]$"), what conceptual meaning is blended with the symbols in order to produce symbolic forms pertaining to the integral? This dissertation reports on the symbolic forms for the integral that students showed evidence of possessing and drawing upon during a mathematics-framed setting and a physics-framed setting.

CHAPTER 3: METHODOLOGY

This study aims to provide insight into the following question:

- *What are the symbolic forms relating to the integral that physics and engineering students have and draw upon?*

This study compares the activation of symbolic forms relating to the integral in physics and engineering contexts to those activated in mathematics contexts. As such, it describes the forms that are activated when the students are engaged in physics-framed problems in possible contrast to those that are activated in mathematics-framed problems. Thus, this work also seeks to provide insight around each of these sub-questions:

- *What symbolic forms do students activate in mathematics-framed settings?*
- *What symbolic forms do students activate in physics-framed settings?*
- *What is the intersection and/or disjunction in symbolic form activation between these two settings?*

This chapter explains the methodology employed in order to shed light on these research questions. I interviewed students in pairs around mathematics and physics items. This chapter begins with a description of the student population as well as a description of the courses they had taken or were enrolled in. It then details the process of creating the framework developed around activating symbolic forms, and the method employed to prompt resource activation during the interviews. This chapter describes the interview and provides a rationale for the items included in it.

3.1 Background Information of Participants

Student Population

This dissertation is intended to study the cognitive resources students draw upon while applying mathematics to physics and engineering, in response to the perception that students are routinely struggling to apply mathematics in their undergraduate physics and engineering courses. One critical detail is the selection of representative undergraduate students, which would allow for the results to be at least somewhat applicable to the broader population of physics students (Becker, 1990). By demonstrating that the students interviewed are "typical" of those characterized in the literature (Schofield, 1990), it could be argued that the cognitive resources described in this study might be found amid students in a normal classroom.

The literature often refers to introductory students in physics and engineering. This student population was narrowed down to students who were intending on majoring in physics and engineering and who were at the introductory level for their major courses. Since I collected my data at the University of California, Davis (UC Davis), I reviewed the physics course curricula at UC Davis in order to determine which courses these students would be enrolled in. The courses at UC Davis are on a "quarter system" meaning that each course runs for a ten-week period. The Physics Department contains three tracks of physics courses as well as lower-level general courses. The top level includes PHY 9HA, 9HB, 9HC, 9HD, and 9HE, which are "intended primarily for first-year students with a strong interest in physics and with advanced placement in mathematics" (UCD, 2009a). The next level includes PHY 9A, 9B, 9C, and 9D, which are designed "primarily for students in the physical sciences and engineering" (though it appears that one can still become a physics major after taking these courses). And lastly, there is the PHY 7A, 7B, and 7C track for biological science majors, chemistry majors, and pre-medical and pre-dental students. There are also lower-level general education courses. In order to focus on physics and engineering majors at the introductory level, I recruited students who had nearly completed PHY 9A and PHY 9HA.

Trivially, if students are struggling to apply mathematics to their physics and engineering courses, they are at least at a point in their studies where they *should* be applying mathematical knowledge to physics and engineering courses. It is possible that part of the difficulties some students are having is that they either have not learned the mathematics sufficiently, or that they have learned it in such a way that it is not readily applicable to physics and engineering coursework. This study is not meant to describe student deficiencies nor depict students from a deficit perspective (NCSM, 2008). It is not meant to characterize students who have not adequately learned the requisite mathematics. Rather, it is more concerned with the ways in which "successful" mathematics students apply their mathematical knowledge to physics and engineering, or struggle in doing so. Thus, I did not want to recruit students who were too inexperienced with calculus, where the results of my study could be accounted for by their lack of exposure to mathematics. Instead, I intend to shed light on the cognitive resources students draw upon in hopes of illuminating the field around the subject of applying mathematics.

In order to recruit students, I asked permission to enter the introductory physics courses at UC Davis during Fall 2009 and briefly explained the study to the entire class. Sufficient quantities of students responded that I was able to select several pairs of students who had “successful” backgrounds in mathematics. There were no students who declined participation after initially expressing interest. Since I recruited students who had nearly completed PHY 9A or PHY 9HA, I was able to enlist students that had experience with calculus, including integrals. PHY 9A and PHY 9HA both have first quarter calculus (MATH 21A) as a prerequisite and second quarter calculus (MATH 21B) as either a prerequisite or a co-requisite. Furthermore, I only recruited students who had “successful” performances in their calculus courses. (For my definition of “successful” here, I used the requirement that they had either an A or a B, or that they had passed the AP calculus exam with a “5.”) This way I could ensure that the students (1) had completed a first quarter calculus course (covering limits and derivatives), (2) had taken or had been enrolled in a second quarter calculus course (covering integration), and (3) had experience in applying mathematics to science concepts throughout their physics 9A or 9HA course. Before a student was admitted to the study, I asked to know which of these classes they had taken as well as the grades the student had received in each class. This assured me that the student met the conditions as outlined here. The end of this section includes a table that displays the background of each student who participated in the study.

The students were interviewed in pairs and were asked to come to an agreement on mathematics and physics problems. For this study, I had a total of four pairs of students from UC Davis, plus another student recruited through the University of Maryland’s Physics Education research group. The pairs of students from UC Davis were interviewed together two times, though one of the students did not come to the second interview. The first session consisted of a “mathematics-day interview” where items resembled problems typically encountered in a mathematics classroom. The students were explicitly told to think about the interview from the perspective of being in a mathematics course. The second session consisted of a "physics-day interview" with items resembling those seen in a physics course. The students were again explicitly told this context. The ninth student was from the University of Maryland (UMD) and was interviewed only once. The items in his interview were drawn mostly from those in the "physics-day" interview for the UC Davis students.

In order to better capture the context that these students were in at the time of the interviews, I briefly describe here the following courses: PHY 9A, PHY 9HA, MATH 21A, and MATH 21B. All of the students were enrolled in either PHY 9A or PHY 9HA, all had either taken MATH 21A or passed the AP exam, and all either were taking or had completed MATH 21B. Describing these courses aids in depicting why the students I recruited are representative of those characterized in the literature. Students enrolled in PHY 9A or PHY 9HA are required to have completed MATH 21A and are also required to have completed or be concurrently enrolled in MATH 21B (UCD, 2009a).

MATH 21A: *Calculus I*

Calculus I is a prerequisite for any students taking PHY 9A or PHY 9HA. It is possible that students may have completed high school AP credit and have tested out of MATH 21A, but similar concepts would have been learned (and I required them to have scored a "5" on the exam). According to the math department's website, the course focuses on "differential calculus" including the topics of functions, limits, continuity, derivatives and applications of the derivative, optimization, related rates, and L'Hopital's rule (UCD, 2009b). "Successful" completion of this course, or a similar AP course in high school, ensures that the students had a working understanding of the important basic concepts of differential calculus.

MATH 21B: Calculus II

Calculus II is a pre-requisite or a co-requisite for PHY 9A or PHY 9HA. All of the students in this study either had taken this course or were enrolled in it. This means the students would have either completed or mostly completed this course by the time I interviewed them. The department website indicates that the course centers on anti-derivatives, techniques of integration, center of mass, surfaces of revolutions and volume, work, fluid pressure, exponential and logarithmic functions, partial fractions, numerical methods, and polar coordinates (UCD, 2009b). This suggests that students would not only have seen the basic concepts of calculus, but would have had the opportunity to extend their knowledge through a full semester of integration. Since the students had completed or nearly completed this course, it is less likely that their difficulties with mathematics in their science courses would lie purely in their inability to do the math itself. They have been exposed to the type of mathematics required of them in their introductory physics and engineering courses.

PHY 9A: Classical Physics

This course is a general physics course for engineering majors and physical sciences majors (UCD, 2009a). It is the first of a four-quarter track and it is explicitly calculus-based. According to the syllabi available on the physics department website, it covers the topics typically dealt with in this type of course, including the laws of motion, force and energy, principles of mechanics, collisions, linear momentum, rotation, and gravitation. The book used for this course in Spring 2009 (Young & Freedman, 2004) follows similar topics as other typically-used undergraduate physics texts (Serway & Jewett, 2008; Tipler & Mosca, 2008), though it does contain slight variations from others texts in the presentation of the material, as would be expected. The book uses rudimentary calculus explicitly and regularly, including both the derivative and the integral, and homework problems include calculus concepts (UCD, 2009c). This course exposes students to applying calculus to physics.

PHY 9HA: Honors Physics

This is the first of five courses designed specifically for those with a strong interest in physics, who also have a strong mathematical background (UCD, 2009a). It is calculus-based and covers a variety of topics, including kinematics, Newton's laws, energy and work, linear and angular momenta, temperature and pressure, statics, and oscillations. The book used for this course in Spring 2009 (Moore, 2003) follows similar topics as other books used at this level. As with PHY 9A, the book makes use of calculus and the students are required to use basic calculus to work through problems. Students will have had numerous occasions to see and use mathematics in this physics course.

Brief Student Profiles

For this study, I interviewed a total of nine students. Eight of the students were recruited at UC Davis and were interviewed in pairs. Each pair was interviewed twice, with the exception of one student who did not attend the second interview. Thus I interviewed the other student alone for the second interview. The ninth student was recruited from the University of Maryland and was interviewed once. In this paper, pseudonyms have been given to the students which indicate their gender; the first pair consists of Adam and Alice, the second pair Bill and Becky, the third Clay and

Christopher, the fourth Devon and David. David did not show up for the second interview. The ninth student is Ethan, from the University of Maryland, who was interviewed once by himself.

Each student was required to have a successful background in mathematics. I defined a "successful background" as having an A or a B in the appropriate courses (or a 5 on the calculus AP exam). The students had to have completed the first calculus course and have completed (or nearly completed) the second semester calculus course. The students would have already received a grade for these courses unless they were enrolled in one of them at the time of the interviews. Most of the students exceeded these preliminary requirements. The students were recruited from introductory physics courses for physics or engineering majors. Table 3.1.1 shows the mathematics and physics background for each student.

Student	Math 21A	Math 21B	Math 21C	Math 21D	PHY 9A or PHY 9HA*
Adam	AP exam	B	A-	enrolled	9A
Alice	A-	A	enrolled		9A
Bill	AP exam	A	enrolled		9HA
Becky	B	A-	(completed)	enrolled	9A
Clay	A	enrolled			9HA
Chris	AP exam	B	enrolled		9A
Devon	A	A	A	A	9HA
David	B-	B	enrolled		9A
Ethan**	(completed)	(completed)	(completed)	(enrolled)	(PHY 161)

Table 3.1.1: Mathematics and physics background for student participants
* 9HA is for intended for physics majors; 9A is for intended for engineering majors.
** Ethan completed comparable courses at UMD and was an engineering major.

3.2 Framework: Activating Symbolic Forms of the Integral

This sections describes the framework that was compiled for this dissertation, which represents applying mathematics to physics and engineering via the activation of cognitive resources in the form of symbolic forms. Symbolic forms have provided a useful language for describing the way students interact with the mathematics they see in their physics and engineering courses. These cognitive resources can be activated by students in order to apply mathematical knowledge to a physics-framed situation. While certainly not the only piece of the puzzle, symbolic forms may be manifest whenever a student is asked to make meaning out of mathematical symbols. Essentially, if the student has stably blended a conceptual schema with a particular symbol template, then by definition a symbolic form exists in the student's cognitive structure. Evidence of this might be seen in the way a student talks about and works with a physics or engineering problem.

There are manifold places in the application of mathematics to physics and engineering where students are confronted with a symbolic structure that is meant to convey meaning. As such, there are many places where the work of symbolic forms could be extended. Work centered on equations has already yielded a robust collection of symbolic forms, some of which have been backed up by other studies (Lee & Sherin, 2006; Sherin, 2006; Tuminaro, 2004; Sherin, 2001; Sherin, 1996). Additionally, some exploratory work has been done in looking at the symbolic forms that students have and activate when using graphs (Lee & Sherin, 2006). However, there are other topics that have yet to be explored with a symbolic forms lens, including the derivative and the integral. Both are symbolic structures with associated meaning. For this study, I explore the symbolic forms relating to the integral that students have and activate.

As discussed in the previous chapter, the framing students employ in the tasks they are given will influence the resources they activate (Hammer, Elby, Scherr, & Redish, 2005). Underlying the activation of symbolic forms are the students' epistemologies about mathematics, physics, and engineering. The epistemological resources that they have active in the interview influence their framing, which may in turn play a significant role in the choice of the cognitive resources they activate for solving the problem. Thus it can be expected that the cognitive resources that are employed during the interview are mediated by the framing done by the students. The framings are tied to the activated epistemological resources in the interview context. The way the students understand the interview setting, what is requested of them, and what constitutes mathematical or physics knowledge will effect how the students approach the problems in the interview. A student may hold a particular symbolic form, but not apply it because of the set-up of a problem or because of what they think they are being asked to do. These pieces are put together into the following framework that I use for this study, which is represented in Figure 3.2.1.

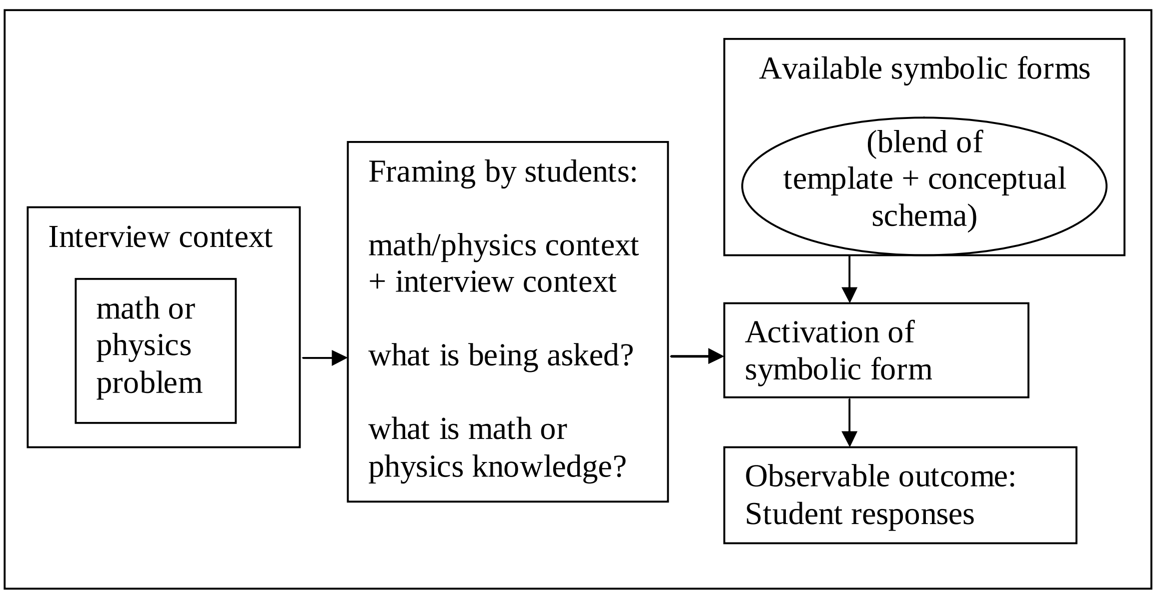


Figure 3.2.1: Framework around the activation of symbolic forms during the interview

This study focuses on the cognitive resources (as symbolic forms) that students activate in mathematics and physics contexts. Thus, while the important role that both framing and epistemologies play in the process of resource activation is acknowledged, and that they certainly interact with the cognitive processes of the student, this analysis is restricted to documenting and describing the symbolic forms that students gave evidence as having. Also, this research does not attempt to examine how or why symbolic forms are cognitively compiled. Thus this framework may be conceptualized according to Figure 3.2.2, depicting the pieces of the framework explicitly analyzed (bold font).

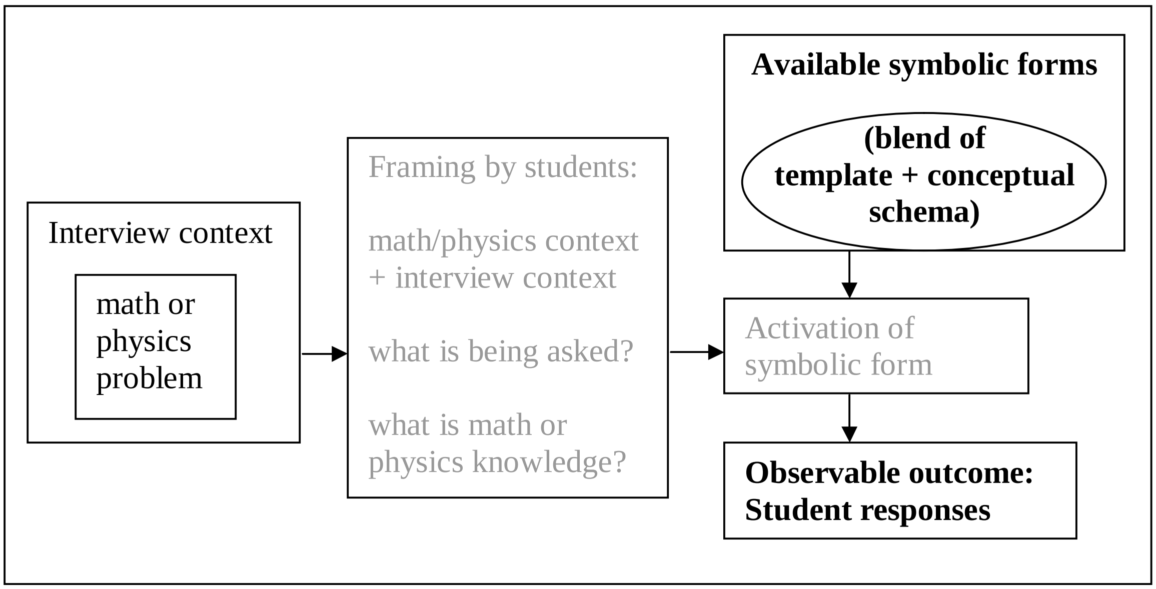


Figure 3.2.2: Framework showing pieces that are explicitly analyzed

3.3 Data Collection, Resource Activation, and Coding

Interview Data

The data used for this study consists of interviews with pairs of students, who were audio recorded and video recorded. I also took notes during the interview and kept student written work to support the other forms of data. Video was an important component of data collection as it allowed me (1) to capture non-verbal data (including and especially gestures), (2) to correlate the audio with written work, and (3) to observe where and when students make references to written materials. Student written work gives me the opportunity to represent drawings, equations, or work produced from the student during the interview. These can shed light into their thinking and understanding. Finally, notes taken during the interview provide a record of impressions, interesting moments, unique responses, or other instances that are important to keep track of. These sources of data together help create a more accurate description of student thinking during the interviews.

The students were interviewed in pairs in order to avoid some of the difficulties in accurately capturing student thinking in one-on-one interviews. Often in one-on-one interviews, the questions, reactions, and follow-up of the interviewer can lead student responses in such a way that detracts from natural student thinking. The student might assume that a follow-up question means that the interviewer values a particular way of doing the problem, which can correspondingly alter the student's course of action. Similarly, if the interviewer asks the student to explain a different way of doing the problem, the student might believe that his or her answer is wrong and will try to change it. On the other hand, if students are working in pairs, they must discuss their solutions with each other instead of the interviewer. The advantage is that if the first student disagrees with the second student, the second student cannot automatically assume that that means they are incorrect (as might be the case if the *interviewer* disagreed). Thus the student may still persist in their way of thinking. If they do change their mind, it is more likely an indication that their thinking has changed instead of an attempt to read into the interviewer's reactions.

Evidence of students' symbolic forms primarily came from what students said and what they wrote. It was important to encourage student talking as much as possible during the interviews. Students were encouraged to discuss their ideas with each other and to come to a consensus regarding the solution to the problem. In addition, probing questions were often used to follow up student explanations. If a student did not provide much explanation along an important line of thinking, I asked follow-up questions targeted at exploring their understanding more fully in that area. This helped ensure that I could make reasonable assertions about the cognitive resources they were drawing on during the interview. If a student had not discussed a certain type of understanding that a particular interview item was intended to cover, I explicitly brought it up at the end of their discussions in an effort to see what the student would express about it. In the next section I describe my interview items and planned questions.

Despite the utility of interviews (Bogdan & Biklen, 2006), I do note that interviews in general have certain limitations. First, analyzing student responses, spoken or written, only approximates the intended goal of capturing “knowledge” as something that can be observed and documented. The verbal and written data can only serve as a rough approximation to student thinking. Second, an interview occurs over a limited amount of time, and therefore only captures what the participant thinks about during that time frame. Thus I cannot claim that the results account for all of the students’ available cognitive resources, nor that the results would necessarily be reproducible across several interviews. This together with other issues, such as my own interpretation of their answers, will naturally bring up questions about the validity of the results (Wolcott, 1990). However, by being explicit about the intent of the interview to the students (Maxwell, 1996), by allowing the interviewees to explain freely their ideas, and by being as open and clear as I can be about my own interpretations in my results (Wolcott, 1990), this study can posit the existence of legitimate symbolic forms. However, this study cannot make any claims to the frequency of the symbolic forms within the overall student population.

Potential Symbolic Forms and Interview Items

In preparing to write items for the interview sessions, I brainstormed potential symbolic forms relating to integrals that might be present in student thinking. I consulted with faculty advisors as well as peer workgroups to obtain a decent collection of good "candidate symbolic forms" that students may actually possess and that would be documentable from the data I would collect. Some of these forms also came from preliminary pilot interviews done with engineering students. Keeping these possible forms in mind, I created interview items in order to maximize the possibility of capturing them (or other potential resources) during the interview sessions. I discuss the interview items in the next section. I now describe the symbolic templates relating to these "potential symbolic forms" that I used in the creation of the interview items. Note that these "potential symbolic forms" do *not* represent the results from the study, but are merely reported to indicate the method I used in creating the interview items. The symbolic *templates* associated with these potential forms are:

1. $\int [\,]\, d[\,]$
2. $\int d[\,]$
3. $\int_{[\,]}^{[\,]}$
4. $\int_{[\,]}$
5. $\int$
6. $[\,]\int$

Let me make some comments regarding these *templates*. First, note that what counts as the "template" depends on what is seen as the "symbolic placeholders." In

Sherin's work, this is analogous to a "+" or a "=" which count as symbolic placeholders separating two terms in a template. The first of these templates (1) looks solely at the interpretation of the symbols in the place of the integrand and differential. According to standard integral notation, these two symbols are separated by a *d*. This symbol template could then be further meshed with an integral sign with limits or one of the other templates to produce a larger, more complex symbolic template. The second template (2) looks at those integrals with "no" integrand, that is with an integrand equal to 1. Here there are no symbols in between the integral sign and the *d*. This template could also be meshed with an integral sign with limits or other symbol templates to produce a more complex template. The next symbol template (3) takes into account the meaning given to the limits of integration, as placed on the integral sign. The fourth template (4) is related, but different in that some limits of integration are represented generically with a symbol for the domain, such as *R* or *D*. These symbols could potentially be interpreted differently from the limits as presented in the third template. By contrast, the fifth template (5) looks at the meaning of an integral symbol *without* limits of integration. This relates to the difference between "definite" and "indefinite" integrals. Finally, the sixth template (6) looks at the meaning given to a "multiplier" on the front of the integral symbol. It is possible that this template might produce some forms that overlap with Sherin's work, which includes forms relating to multiplication. Some of these symbol templates listed above could be meshed together to give a more complex symbol template, such as "$[]\int_{[]}^{[]}[]d[]$."

These templates may provide the grounds for compiling some of the symbolic forms for the integral. If these templates are stably blended with a conceptual schema then a symbolic form could be said to exist in the student's cognition. The interview items were created with these templates in mind to see what meanings students would give to these symbol structures.

Looking for Symbolic Forms: Grounded Theory

In each interview item, the students had the chance to work with symbol templates for the integral. They were asked to think about various integrals in different contexts, at times simply calculating an integral, or at other times creating an integral that would match a particular scenario. In the data collected from each item I looked for any of the following clues, or others similar to these, that might serve as places in the data that could provide evidence for a symbolic form a student had activated.

1) Students point to written symbols
2) Students verbally mention symbols
3) Students use words such as "the integral…" or "this represents…"
4) Students make visual representations of the integral
5) Students say phrases such as "I know that…" or "I see this as…"

These phrases are obviously not an exhaustive list of clues for places in the data where I might discover evidence for a symbolic form. However, they served as a useful sieve at times for flagging places to investigate more deeply.

My analysis could be said to follow a "grounded theory" approach (Glaser & Strauss, 1967). In looking for symbolic forms, I let the data provide the basis for conjecturing about the existence of symbolic forms. By continuing to look for confirming or disconfirming evidence, I was able to refine the conjectures I had for symbolic forms until they took definite shape and were able to withstand continued scrutiny. These forms were shown to faculty and fellow students who challenged and debated their structure. My first step in coding the data consisted of going through each interview, creating an outline of what was said by the students. This outline contained the main ideas of each student explanation as well as brief descriptions of any written work done while the student was talking. Whenever I saw a place in the data where one of the aforementioned (or other) “clues” were present, I made a note in the outline to come back and investigate those pieces more carefully. Upon returning to these places in the data, I made a full transcription of the entire episode, along with insertions of the students’ corresponding written work in the transcription. By having the full transcription and written work together in one body, I was able to carefully consider the meaning the students were giving to the symbols in the template.

If it appeared that there was evidence of a symbolic form, I wrote out a conjecture of what the overall symbolic form consisted of and the conceptual schema applied to the symbols. I kept track of all of these “candidate symbolic forms” in a list. As I continued through the data I would look for confirming or disconfirming evidence of these symbolic forms in other places of the same interview, or in other interviews, or with other students. By doing so, I was able to reject inaccurate conjectures and refine misjudged notions until I had a reasonable argument for a symbolic form. As I continued to look through the data, I would use other episodes as verification of the proposed symbolic form.

3.4 Interview Items and Analysis

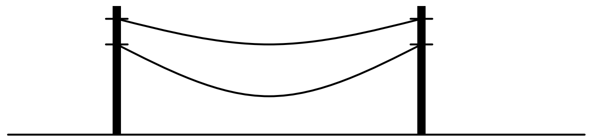
Previously, I listed several potential symbolic templates that might serve as the basis for a symbolic form. Based on the insight of this preliminary exercise, I created interview items that would likely allow me to investigate symbolic forms that students held around these particular templates. I will now explain the interview items I used in order to find evidence of symbolic forms in the students' work. As previously explained, most of the students participated in two interview sessions. The first session consisted of a mathematics-framed interview setting. Here the items were more similar to problems seen in mathematics courses. The second session consisted of a physics-framed interview setting. In this interview the items were more similar to what is seen in physics courses. Each interview session lasted for about 45 to 70 minutes. For each item, the students were given an initial problem to work on. After they had completely solved it and had finished all discussion between themselves, I prompted them with several follow-up questions. The follow-up questions were often targeted at making sure the students talked about each aspect of the symbol template. For instance, I often asked the students something like: "Does that '*dx*' have any meaning?" Then I prompted them to describe what its meaning was.

The mathematics-framed items gave insight into the symbolic forms students had and drew upon in mathematics contexts. The physics-framed items provided data for the symbolic forms students drew on in physics contexts and provided a contrast for the similarity or discrepancy in the symbolic forms that students activated in each situation. I limited myself solely to interview items based on integrals in order to maximize the time the students spent discussing the integral symbol templates. The mathematics-day interview items and physics-day interview items were as follows:

Interview Items for the Mathematics-Day Interview

ITEM Math1 (presented to all student pairs, excluding Ethan)

Two wires are attached to two telephone poles (see picture). Suppose we wanted to know the area between the two wires. How could you figure that out?



ITEM Math2 (presented to all student pairs, excluding Ethan)

$\int_1^2 \frac{2}{x^3} - x^2 \, dx$ Compute and then discuss this integral.

ITEM Math3 (presented to all student pairs, excluding Ethan)

I want you to look at each of the following integrals and talk about what they mean. Talk about each one individually.

$$\int \sin(x) \qquad \int_2^0 e^x \, dx$$

ITEM Math4 (presented to all student pairs, excluding Ethan)

I want you to look at each of the following integrals and talk about what they mean. Talk about each one individually.

$$\int dx \qquad \int \sqrt{t} \, dx$$

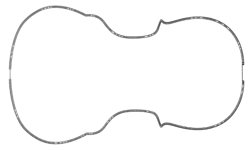
ITEM Math5 (presented to all student pairs, excluding Ethan)

Suppose we had a function *f(x)* with a domain *D*. What does this integral mean?

$$-2\int_D f(x) \, dx$$

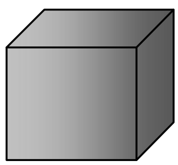
ITEM Math 6 (presented only to David and Devon to produce more data)

This picture shows the outline of a violin body. If you wanted to know the area of this shape, how could you figure that out?



Interview Items for the Physics-Day Interview

ITEM Physics1 (presented to all student pairs)



This shows a box with varying density. (dark = more dense, light = less dense) Suppose you wanted to know the box's mass. How could you figure that out?

ITEM Physics2 (presented to all student pairs)

The durability of a car motor is being tested. The engineers run the motor at *varying* levels of "revolutions per minute" over a 10 hour period. Denote "revolutions per minute" by *R*.

What is the *meaning* of the integral $\int_0^{600} R\, dt$?

ITEM Physics3 (presented to all student pairs)

A 2-dimensional surface (S) experiences a non-uniform pressure (P) and we want to know the total force exerted. We can use the surface's area (A) to compute this through the integral:

$$F = \int_S P\, dA \,.$$

Why does this integral calculate the total force exerted?

<u>ITEM Physics4</u> (presented to all student pairs, excluding Ethan)

We know from kinematics that acceleration and velocity are related by $a(t) = \frac{d\,v(t)}{dt}$. We can rearrange this equation and integrate to get the equation

$$\int a\,dt = \int dv$$

What does this equation mean? Why are these two terms equal to each other?

<u>ITEM Physics5</u> (presented to all student pairs, excluding Adam/Alice and Ethan)

F_y is used to denote the amount of a force in the y-direction. ΔU is used to denote the change in potential energy. These two concepts are related through this equation:

$$\Delta U = -\int_{y_i}^{y_f} F_y\,dy\,.$$

Explain this equation. What does each part of the equation/integral mean?

<u>ITEM Physics6</u> (presented to all student pairs, excluding Bill/Becky and Ethan)

This represents a metal bar with varying mass along its length
(lighter = less dense, less mass / darker = more dense, more mass)

How could you figure out the center of mass for the bar along its length?

I now discuss the interview items and the goal I had in presenting each item to the students during the interviews. The mathematics-day interview started off with an open-ended question regarding the area between two hanging wires. The problem is a mathematical one despite the use of the “real-world” objects, because the physical properties of the wires have no bearing on the problem. The item is devoid of any numbers, functions, or other clues as to how to proceed (other than the direction to find the area). In this way, the students were free to use any approach. If they came to use an integral, the construction of that integral was purely their own. The symbols that they wrote in the integral could all be “clues” about their thinking, which would lead to a rich place to analyze the conceptual schema they had meshed with those symbols. By opening the interview in such a way, I could expect to find evidence of the way they had compiled the meaning of an integral. Fortunately, in this set of interviews, every student pair naturally came to the idea of an integral and did not require any prompting about using an integral, which would have had the effect of influencing their framing of the problem.

Since the first interview item is open ended, it could be possible for the students to employ various symbolic forms with varying symbol templates. Thus in the next few items, I attempted to both involve other symbol templates as well as use non-area based problems. The next item uses the template “$\int_{[]}^{[]}[]d[]$” allowing for opportunities for the students to talk about the integrand, the differential and the limits. (Note: this item additionally has the template “$\int_{[]}^{[]}([1]-[2])d[]$” which ended up being significant.) The third item takes a look at this same symbol template where the limits are “reversed.” This created a place for discussing the limits more carefully. Also, this item uses an integral that lacks a *dx*. This omission was made on purpose in order to provide a place for potential discussion about the differential. By paying close attention to what interpretations the students gave to this expression, I could better analyze the meaning they gave to the differential. The fourth item then moves on to discuss the template with “no” integrand, that is the template “$\int d[]$.” This particular item does not have limits, which also allowed students to discuss the relevance of the limits of integration. In this same item is another integral that does "have" an integrand, but the variable in the integrand is different than the variable in the differential. This provided an opportunity for the students to discuss the relationship between the integrand and the differential. Finally, the fifth item presents a more generic integral that both has a multiplier in the front as well as a change in the way the *limits* are presented. Here there is a domain affixed to the integral symbol, in contrast to an upper and lower bound. By carefully considering how the students attended to the -2 in front and the *D* on the integral, I was able to learn about the understanding they attached to symbols in these locations. The last mathematics-day interview item was used as a back-up in case I felt like I needed more data from a particular pair of students. This was only the case with David and Devon who had been describing the integral often as a "mapping" to some "original function." I wanted to see if this thinking would persist even in the context of another "area"-framed item (which it did not).

The physics-day interview also opened up with an open-ended question about finding the mass of a box. Again, no information was given about the numbers involved, the functions involved, or anything else other than the fact that the density was varying. Like the opening mathematics item, this allowed me to see the students create their own integral. This afforded the opportunity to see more clearly how the students put the pieces of the integral together based on the idea of a box with varying density. By watching them do this, I was able to see the meaning they gave to the various symbols associated with the integral, such as the integrand, differential, and limits. It also provided insight into "the way the integral works."

Following the same train of thought as the mathematics-day items, I wanted to make sure that after giving the students an open-ended question related to integrals, I would be able to see the students work with a variety of symbol templates. Thus the subsequent items were designed to provide the students with problems using an integral with different templates. The second item looks at an integral with "regular" upper and lower limits, while the third item attaches a "domain" onto the integral. This provides a way of looking at the templates "$\int_{[]}^{[]}$ " and "$\int_{[]}$ " and seeing possible comparisons or contrasts between the meaning made out of these symbol structures. The fourth item looks at an integral equation involving acceleration and velocity. I want to note that in this equation, the integrals *should* (to be physically correct) have limits on them. They are only truly equal if the two integrations are happening over the same time interval. However, I wanted to provide the students items using integrals *without* limits on them in order to see how the students interpreted an integral without them. I decided that this "mistake" could potentially provide a place for discussion with the students. It would allow me to see them talk about the role of the limits of integration. It would also allow for a discussion on the difference between definite and indefinite integrals. In item five, I made an attempt to find an integral with a multiplier on the front of the integral. This equation shows a negative on the outside of the integral, which could provide a place to talk about the meaning of the multiplier. Lastly, as with the mathematics-day interview, I had an additional item that I could use for more data. This question is also open ended and directed the students to determine a way to calculate the center of mass of a bar. It turned out that I was able to use this item with most of the pairs of students. Unfortunately, Bill and Becky were experiencing a certain degree of tension between each other and began to dwindle in their efforts toward the end of the interview. Thus I decided it would not be useful to ask them to continue through another task.

The symbol templates that I listed in the previous section are certainly not all of the symbol templates possible for the integral. They represent, instead, the groundwork for creating interview items. During the analysis I also looked for other possible symbolic forms that made use of other symbol templates that were not included in my list. Additionally, it is possible that other templates (and associated forms) exist beyond what I captured in this dissertation project. However, based on this groundwork, I wrote interview items to provide opportunities for capturing forms relating to these templates. In order to see the relationship between the symbol templates and the interview items, I visually depict how these interview items overlap with the posited symbolic templates. The table below shows which symbol templates are covered by each interview item. An "x" in the table means that that interview item has a task that would likely provide data to support the documentation of symbolic forms with that particular symbolic template.

		Interview Items											
Symbol Templates		m1	m2	m3	m4	m5	m6	p1	p2	p3	p4	p5	p6
	1	x	x	x	x	x	x	x	x	x	x	x	x
	2				x						x		
	3	x	x	x			x	x	x			x	x
	4	x				x	x	x		x			x
	5	x		x	x		x	x			x		x
	6	x				x	x	x				x	x

Table 3.4.1: Intersection of interview items with symbolic templates
(m = math, p = physics)

Through this methodology, I was able to create interview items that enabled me to detect and document student cognitive resources in the form of symbolic forms. By conceiving of candidate symbolic forms before the interview process, I could be sure that my items would cover the breadth of these potential sources for data. By working with a grounded theory lens as I sifted through and coded my data, I was guided by the student responses in detailing the existence of symbolic forms in the students' cognition. Due to the "typical" nature of the students I interviewed, it is possible to claim that some of the symbolic forms presented in the next chapter might exist in any common university classroom.

CHAPTER 4: RESULTS

In this chapter I discuss the results from the interviews with the students. As discussed in chapter three, I interviewed a total of nine students, eight of which were interviewed in pairs, with one additional student interviewed alone. I discuss the symbolic forms that can be postulated as evidenced by the students' work. I use the descriptions that the students provided of their thinking in order to determine what meaning they gave to certain symbolic elements of the integral symbol template. While I acknowledge the limitations that verbal and written descriptions have on approximating the actual cognitive resources that students may hold, I provide support for the symbolic forms I ascribe to their cognition.

Throughout the interviews, there appeared four "major" symbolic forms that encompassed the entire integral symbol template, "$\int_{[]}^{[]} [] d[]$." By "major" I mean that they have the complete symbol template with different conceptual schemas. The meaning given to each of the parts of the symbol template were self-consistent within each of these four major symbolic forms. That is, the meaning ascribed to one part of the template corresponded tightly with the meaning ascribed to the other parts of the symbol template. Each of these symbolic forms can be supported through the work done by individual students, as well as laterally across the various students I interviewed. After discussing each of these symbolic forms independently, I then compare and contrast the four symbolic forms with each other, taking note of the differences in conceptual schemas given to the "[]d[]" symbols, the "$\int$ " symbol, and the limits, "$\int_{[]}^{[]}$."

Following the exhaustive description of these four major symbolic forms, I then turn my attention to other symbolic forms pertaining to the integral. These forms related to specific parts of the integral template, such as the differential, a multiplier preceding the integral sign, or the relationship between certain symbols and graphical representations. Lastly, I look briefly at some other interesting cognitive resources that students exhibited, but that do not necessarily constitute a "symbolic form."

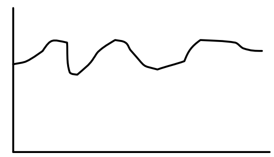
Following the description of these various symbolic forms and cognitive resources, in chapter five I launch into an analysis of the symbolic forms I discovered. I detail which symbolic forms were activated in the mathematics-day interviews and the physics-day interviews, and which interview items in particular elicited certain symbolic forms. I compare and contrast the symbolic forms activated in each of these two interviews and explore the conclusions that might be drawn from the differences in the way students drew on their cognitive resources in a mathematics setting versus a physics setting.

4.1 The "Area" Symbolic Form

The first symbolic form of the integral that I explore equates the integral to a fixed region in the *x*-*y* plane, whose area is the value of the integral. The students first ascribe meaning to the "[]*d*[]" part of the symbol template, proceeding to then interpret the "$\int$ " symbol itself as well as the limits "$\int_{[]}^{[]}$ " of the integral. I use an example from the mathematics-day interview with Clay and Chris. In the interview, I wrote the integral "$-2\int_D f(x)\,dx$" on the board and asked them to describe what the integral meant. Chris immediately began working on the problem.

> Chris: OK, so essentially I always like graphs. So if you want to draw a graph [draws axes], um, we have f of x [draws squiggly graph above *x*-axis. See Figure 4.1.1a].

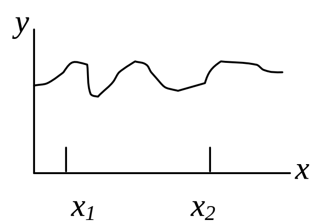
Figure 4.1.1a



We see that Chris' first action was to attend to the function *f(x)* and represent it graphically in the plane. This serves as a important step in creating a fixed region in the *x*-*y* plane that relates to the integral. Chris continued:

> Chris: And then since we're saying over the domain D, domain is usually when we're dealing with x, y axes. We assume it's with respect to the x axis and also the integral deals with x [points to *dx*] and we have a function of x, so we can assume D is a domain from some point x1 to some point x2 [labels x_1 and x_2 on the *x*-axis. See Figure 4.1.1b].

Figure 4.1.1b

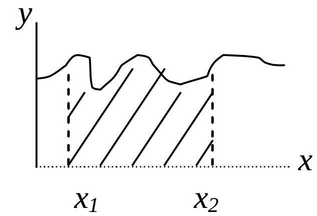


Chris explained the fact that he was dealing with an *x*-*y* plane and that "it" was with respect to the *x*-axis. Part of his reasoning for assuming an *x*-axis was the fact that "the integral deals with *x*." Chris said this as he pointed to the *dx* in the integral. This signifies that part of the role of the *dx* is to determine one of the sides of the region. The *dx* helped Chris make the decision about the axis that would serve as the "bottom side" of the fixed region. Next, Chris marked x_1 and x_2 on the *x*-axis in order to graphically represent the boundaries of the integration.

> Chris: So we have… [writes D:(x_1,x_2)]. And then so that would be equal to… [writes $-2\int_{x_1}^{x_2} f(x)\,dx$. Draws dotted vertical lines from x_1 and x_2]. And then

> we'd take the integral from x1 to x2 [said as he shades the area. See Figure 4.1.1c].

Figure 4.1.1c



Chris used the x_1 and x_2 to create the left and right sides of a fixed object in the *x*-*y* plane. He used vertical dashed lines to mark off this region and shaded the region in to clearly show the object that was represented by the integration. Thus the limits of integration are not merely numbers, but actual boundaries of a region in the plane. They become the "left and right sides" of the fixed region itself. Then *while* he shaded in the region he said, "And then we'd take the integral from x_1 to x_2." This clearly connects the actual area to the act of integration.

This episode shows Chris decomposing the integral and describing what each part means, activating what I call the "area" symbolic form. First, he chose to display the function *f(x)* as a randomly drawn graph in the *x*-*y* plane. Then the fact that "the integral deals with *x* [a reference to the *dx*]" meant to Chris that the *x*-axis was the bottom of the region and that the upper and lower limits for this variable *x* would provide the left and right sides for a spatial region in the plane. He marked off two vertical lines underneath the graph of *f(x)* and shaded in everything in between. Speaking the words "And then we'd take the integral from x_1 to x_2" while shading the region in shows that the integration itself does not take place until the fixed area has been determined. The key feature of this symbolic form is that the "area underneath the curve" (as we often say in mathematics) is taken to be one static whole. It is not subdivided into parts nor measured using successively more accurate approximations, as in a Riemann sum. The bounded region is seen as a fixed body whose fixed area is taken, as a whole, to be the value of the integral.

The *area* symbolic form essentially takes the integral and construes it as a region, usually in the *x*-*y* plane, whose area will be computed. This symbolic form may not be any great surprise to educators, since an area model is used extensively in mathematics courses, though there are interesting components to it. The limits of integration actually become the physical sides of the shape, as opposed to representing simple numbers. This imbues the limits with additional meaning, more than just the numbers that are plugged into the anti-derivative. This meaning contrasts with the meaning given to the limits in the other symbolic forms. The horizontal axis creates the bottom of the shape, since the differential is a *dx*. The integral symbol is then taken to mean the area of this fixed shape. This symbolic form is depicted visually in Figure 4.1.2.

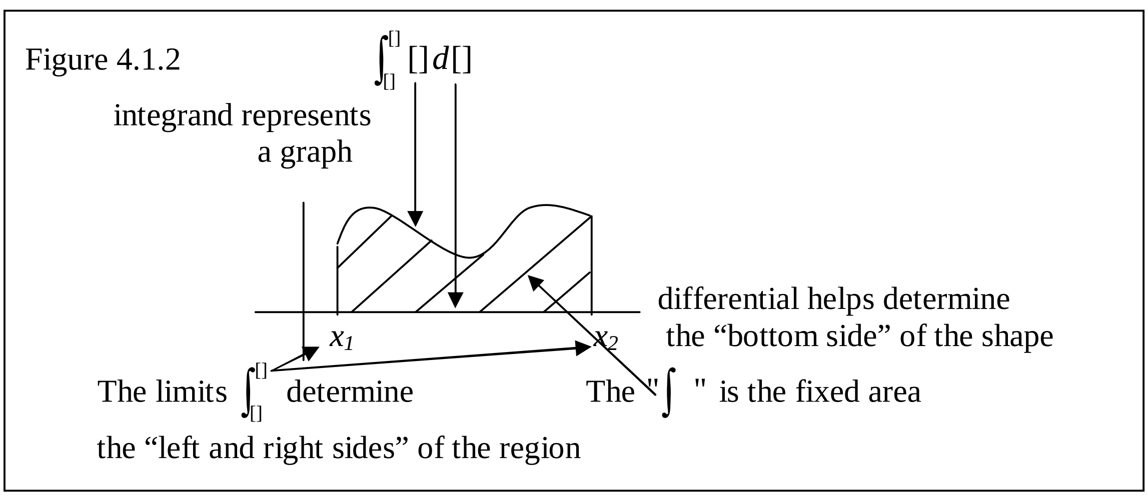
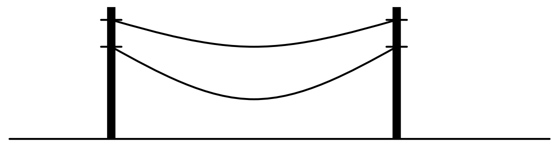


Figure 4.1.2: A visual representation of the *area* symbolic form

This symbolic form guided much of Clay's and Chris' work during other items in the mathematics-day interview. In one interview question, Clay and Chris were shown the following item:

> Two wires are attached to two telephone poles (see picture). Suppose we wanted to know the area between the two wires. How could you figure that out?



They set up an integral by labeling one curve f_1 and the other curve f_2 and writing down "$\int_{-x}^{x} f_1 - f_2 \, dx$" (the fact that the limits, -*x* and *x* are technically incorrect do not play an important role in this part of the discussion). Then Clay explained in more detail how he thought about figuring out the problem.

> Clay: Uh, well you could have probably solved it like… so using integrals, like finding the area of the whole thing, and minus it from the bottom [redraws figure and shades the two regions. See Figure 4.1.3a]. …That would be another approach.

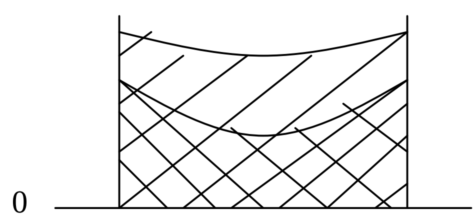


Figure 4.1.3a

For a moment, Chris did not understand exactly what Clay was doing. I prompted them to discuss it with each other until they came to an agreement. They began dividing up the region into various segments and adding and subtracting them from each other until they had the desired leftover area.

Chris: How are you adding them?

…

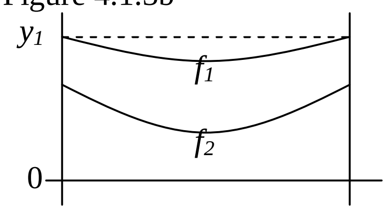
Chris: So, we're finding this whole rectangle's area, and then subtract it by this area?

Clay: It's probably more complicated actually.

Interviewer: But this is good. Go ahead and work through that.

Clay: Ok, I put 0 at the bottom of it. And then… [writes f_1 and f_2 on the two curves]. On this top part I could put y1, so we have this line y1 [adds a line across the top and labels it y_1. See Figure 4.1.3b].

Figure 4.1.3b



Clay: And we subtract out the function f1 [underneath the figure writes $y_1 - f_1$]. And then you get this value in between [points to small area in between y_1 and f_1]. Then you subtract that from the lower value of…

Chris: So basically you're finding this area [points to area between y_1 and f_1], this area [points to area below f_2], and this whole area [traces finger around the entire drawing], and then subtracting these from this.

Clay: Yeah. Basically. So we're going to have this value so far [points to area between y_1 and f_1]. And then we need to find this value [points to area underneath f_2].

Chris: …So, we could write this integral from the top area.

Clay: Wait, this is the top area, right [points to $y_1 - f_1$]?

Chris: Yeah.

Clay: This part [points to area underneath f_2]. And you have to subtract that from the total area. Like this whole thing [motions over whole region]… [writes $y_1 - (y_1 - f_1)$]. And we need to subtract this value from that.

Chris: So minus…f2 [writes $y_1 - (y_1 - f_1) - (f_2)$]

…

Clay: So these cancel out and this just becomes f1 minus f2 [writes $f_1 - f_2$].

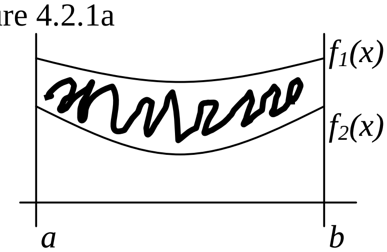
Chris: So [laughs] that's just where we were.

In this episode Clay and Chris again demonstrated activation of an *area* symbolic form. The fact that their sole purpose was to reduce the overall picture to the bounded region in between the f_1 and f_2 curves shows that they understood the integral template "$\int_{-x}^{x} f_1 - f_2 \, dx$" to be the area of the fixed region in between the f_1 and f_2 curves in the *x*-*y* plane. The symbol *dx*, in this particular discussion, served no purpose beyond determining the "bottom side" of the regions. The areas were not partitioned, but rather were thought of as static wholes. Consider again the symbol template "$\int_{[]}^{[]} [] d[]$." Their entire discussion revolved around what should be in the "integrand box" in the template, such that the extraneous regions would be disregarded. This interpretation of the integral as an "area" drove their whole method for determining how to proceed. This provides evidence that they had activated an *area* symbolic form.

4.2 The "Adding Up Pieces" Symbolic Form

The next symbolic form for the integral I discuss deals with thinking similar to that expressed in the Riemann sum. However, there may be minor differences between the way this form is compiled and the actual process that the Riemann sum follows. The integral is viewed as slicing an object or a region into tiny pieces that are added up to give the value of the integral. In the mathematics-day interview with Devon and David, I showed them the interview item with the two telephone wires (see the previous section), where they were asked to come up with a way to obtain the area in between them. After a minute's work they came up with the integral "$\int_a^b f_1(x) - f_2(x)\, dx$," based on a figure they had drawn (see Figure 4.2.1a).

Figure 4.2.1a



We can see that they also began with a concept of a fixed region in the plane, evidencing the activation of an *area* form, which this interview item admittedly lends itself to.

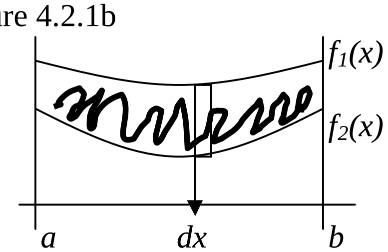
However, when I asked them to explain how they came up with their integral, they began drawing on other meanings for the integral symbols. Devon explained how he understood the integral that they had come up with.

> Devon: You can't just put area, you have to somehow divide it into, let's say the length, let's say you slice it this way [draws several vertical lines from top curve to bottom curve], and then you add up all the individual lengths [puts hand on left side of shaded region and sweeps hand across to the right side]. And then that means we have to find the difference between these two curves, that's why we label it [points to f_1 and f_2]. And by finding the curve and then integrating over them [again sweeps hand from left to right], that's how we find the area.

Devon started his explanation by stating that in order to understand their integral, he had to "divide" the region of interest. Then these "individual lengths" had to be systematically added up, which he demonstrated by sweeping his hand from the left of the shaded region to the right. We see a hint of what it means to him to "integrate" when he again swept his hand from left to right while saying "and then integrating over them." There appears to be some connection for Devon between "integrating" and going from the left to the right. I prompted him to continue describing what he meant.

> Devon: I would imagine it as, you slice it [draws a thin rectangle. See Figure 4.2.1b], like very small pieces and each of them is a dx [draws an arrow from the bottom and writes *dx*].

Figure 4.2.1b



> Devon: And this part [puts fingers along the height of the thin rectangle] is the, is this part right here, this term right here [points to $f_1 - f_2$ inside the integral].
>
> Interview: Which part is? Just to make sure.

> Devon: This part right here, the length here [underlines $f_1 - f_2$ inside the integral and draws an arrow over the height of the rectangle]. And then every little bit [uses finger and thumb to mark a small width], I call it a dx.

As Devon continued his explanation, we see that the first part of his thinking consisted of making a rectangle that served as a reference for what was happening in the integration. Furthermore, he described where the rectangle came from. The height of the rectangle came directly from the integrand, namely "$f_1 - f_2$." The width of the rectangle was referenced by the differential *dx*. I then asked Devon to say what the *a* and the *b* meant.

> Devon: When we label this a and b, it's kind of like a natural thing to us. Because we did it like that in class. But if you really think why we put it there, like I said, I slice it into little pieces. And all the pieces we're looking at is from here to here [motions with hand from left of the shaded region to the right], and it has to do with the values of it [says this as he moves his hand from left to right again]. It's more like an action thing I think.

Devon had previously described that the integral takes all of the individual pieces, like the rectangle he created, and adds them up. Now we can see how he saw this addition as taking place. The limits of the integration, *a* and *b*, served as a "starting" and "ending" place for the addition. The integration happens "from here to here." The integration is "more like an action." Under this conceptual schema, the integrand and differential divide the object or region of interest into small pieces. The integral itself then adds up all of these pieces and the limits of integration dictate where to start the addition and where to end the addition. This constitutes the "adding up pieces" symbolic form. This form is visually represented in Figure 4.2.2.

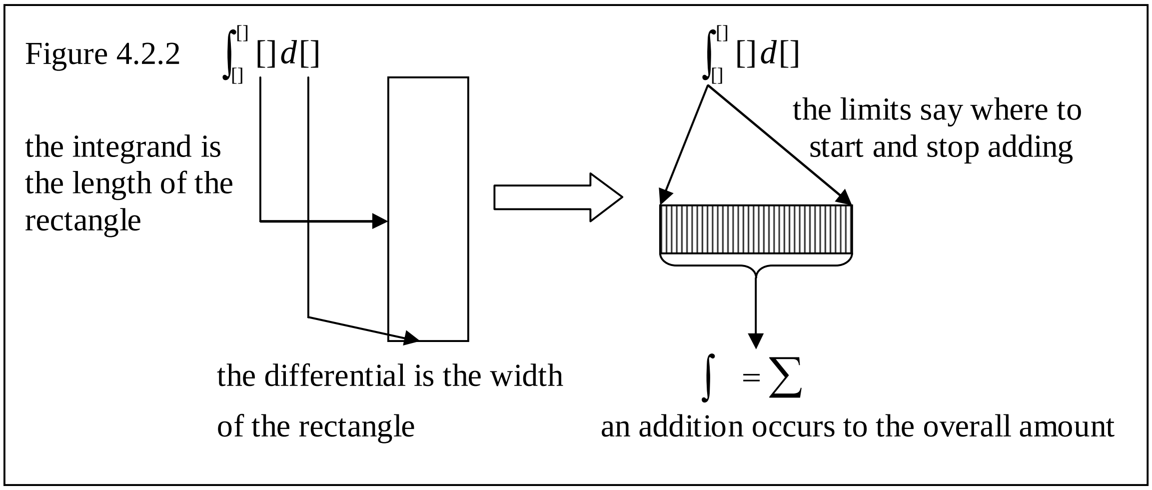
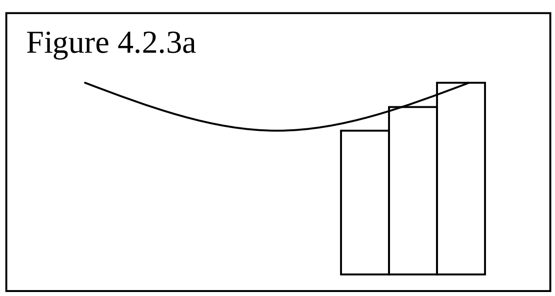


Figure 4.2.2: A visual representation of the *adding up pieces* symbolic form

The Infinite Addition

In most instances where the students had activated the *adding up pieces* symbolic form, there was strong evidence that they viewed the rectangles as "infinitely thin" and the addition as happening over "infinitely many" rectangles. When Chris and Clay were working on the problem of finding the area between the two wires, I asked them to explain why an integral gives the area. Chris, who had drawn on the *adding up pieces* symbolic form had been describing rectangles that would add to give the value of the integral. He then talked about how these rectangles would be added up.

> Chris: We want to find the area, so theoretically we could add up the value of a bunch of rectangles, and add them up. But we're going to constantly have little gaps [draws approximation rectangles. See Figure 4.2.3a].



Chris: So we're going to be missing this area [points to gaps in between curve and rectangles]. So we assume, by integrating we assume that dx is infinitesimally small.

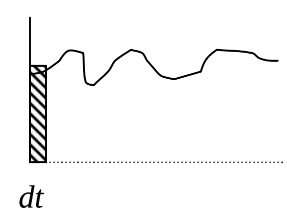
Note that when integrating "we assume that *dx* is infinitesimally small." Thus integration does not occur with a finite number of rectangles. Chris further clarified this thinking during the next interview. He and Clay had been given the following problem:

The durability of a car motor is being tested. The engineers run the motor at *varying* levels of "revolutions per minute" over a 10 hour period. Denote "revolutions per minute" by *R*.

What is the *meaning* of the integral $\int_0^{600} R\,dt$?

They correctly determined that the integral was calculating the total number of revolutions that occurred over the ten hour period. When prompted to discuss how they came up with this answer, Clay drew a squiggly graph to represent *R* and drew in a rectangle underneath the graph (see Figure 4.2.3b).

Figure 4.2.3b



Clay identified the height of the rectangle as "the revolutions per minute at that time," and the width of the rectangle as a small unit of time. Chris then interjected his thoughts:

Chris: Essentially the length times width of the infinitesimally small rectangle, that we're integrating over, is going to have units revs per minutes times minutes, equals revs.

Once again, Chris expressed the idea that these rectangles were infinitely small during the integration process. This type of language was common and consistent across the students. During instances of using the *adding up pieces* symbolic form, the students described the rectangles as “infinitesimal,” “infinitesimally small,” “infinite number,” “infinite amount,” “infinitesimal rectangles,” and “infinitely many.” It seems clear that for many students, the *adding up pieces* symbolic form also has embedded in it an inherent notion that the rectangles have already achieved the status of being infinitely thin and that the addition process requires an infinite summation over the infinitely many pieces.

The *adding up pieces* symbolic form compiled in this manner diverges in an important way from the traditional Riemann sum process. The Riemann sum takes an arbitrary finite partition, which yields a finite amount of rectangles whose areas are added together. The partition is then refined and the areas of the finitely many rectangles are again added. By systematically doing this, one constructs a sequence of numerical values. By a theoretical process of refining the partition infinitely many times, the numerical sequence may converge to a single number. If it converges in a particular, well-defined way, this number is the value of the integral. Many theorems help establish the value of the integral as being independent from the choice of partitions. By contrast, the students in the interviews commonly understood the integral to be a process of taking infinitely small pieces and adding up the infinitely many pieces to capture the total. That is, the limiting process seems to occur *before* the addition takes place.

This distinction is more than just linguistic. There is evidence that some students separate the finite “Riemann sum” process from the final, infinite “integral” process. Adam and Alice had been working on the problem regarding the area between two wires and they had drawn in thin rectangles between the two wires. I asked them to speak more about the thin rectangles they had drawn on their graph. Adam offered the following explanation.

> Adam: So, this goes back to a Riemann sum, where you take small portions of each graph [draws a "thin rectangle" with his finger]. Like what I have here which would be dx. In Riemann sum, you define what the width is, or how many sections per… graph you have. With the dx, when you're integrating, you're taking an infinite number of lengths of portions, so the dx gets really really small, making the area of each sliver more accurate to what it actually is.

Adam claimed that in a Riemann sum the graph is partitioned according to a defined width. However, he noted that "when you're integrating" the process changes a little bit, so that now there is "an infinite number of lengths of portions." This is how the integral is "more accurate" than the Riemann sum. These two ideas certainly appear to be connected in his thinking. That is, the integration is like having a Riemann sum when "the *dx* gets really really small." But again, this is conceptually distinct from the actual Riemann sum process. The students show evidence of thinking of the integration as happening over "an infinite number of lengths." At another point, during the physics-day interview, Adam was again drawing rectangles into his figure to talk about the integral he and Alice had come up with. When prompted to talk more about those rectangles he said,

> Adam: It's kind of like you're adding them all up. Going back to Riemann sums, it represents the infinite amount… or sums for Riemann, so… If you had the infinite amount of portions for a Riemann sum, that would represent this, the integral.

Hence, an integral is when one has "the infinite amount of portions for a Riemann sum." That is, the integral is the "infinite" case of a Riemann sum. Bill produced a similar explanation in his interview when he said,

> Bill: Well, I would say dx is just a… I think of an integral as just a way of expressing an infinite Riemann sum. And, as dx goes to 0. Well, as, as the length of each rectangle goes to 0, then it becomes a dx. That's how I think of it.

Again, an integral is "an infinite Riemann sum." This way of compiling the *adding up pieces* symbolic form exhibits a qualitative difference from the Riemann sum process. Here the integral itself could even be considered a "special case" of a Riemann sum, namely a Riemann sum with an "infinite amount of portions."

4.3 The "Function Mapping" Symbolic Form

The third main symbolic form I discuss here is somewhat different from the other two. While the others draw heavily on visual representations of the integral, this form views the integral as more of a "pairing of objects," which takes the integrand and matches it with an appropriately selected function. Like the other forms, the students imbue the integrand, the differential, and the limits of integration each with their customized conceptual meaning. Though this form resembles the "rote procedure" for calculating an integral, I argue that the students are not, in fact, simply producing rote calculations, but are giving conceptual meaning to the symbol template of the integral. This may be similar to the reification of "processes" into "objects" (Sfard, 1991). In the mathematics-day interview Devon and David were given the integral "$\int_1^2 \frac{2}{x^3} - x^2 \; dx$." I asked them to calculate it and then began asking them about the way they solved it. In their first step they had broken the integral into two parts, "$\int_1^2 2x^{-3} - \int_1^2 x^2 dx$." David recognized that they had left out the *dx* from the first of these two integrals and added it in. I then asked them to explain why it needed a *dx*.

Interviewer: Why does it need a dx?
David: Well, it's still an integration. So, in an integration the dx is always essential, because it shows that this entire thing [waves hand over $2x^{-3} - x^2$ inside the integral] is a derivative of x.

The first explanation we have from them revolves around the fact that *dx* signifies that the integrand is "a derivative of *x*." Hence it appears that the function may have "come" from somewhere else. The conversation then moved to talk about what is meant by the fact that $(2x^{-3} - x^2)$ and *dx* are placed next to each other in the integral symbol template.

David: The fact that this entire thing is sitting right next to each other, and dx outside, means that basically this entire function [motions hand over $2x^{-3} - x^2$] is the derivative of an original function.

David had conceptualized the integrand "$2x^{-3} - x^2$" as the "derivative of an original function." This means that there exists a function out there, or several functions potentially, that would map to this function via the derivative/integration process. The function in the integral is connected with this other function (or functions) in that it *came from* this other function. At this point, their explanations about *dx* are centered on the idea that it determines the variable that was used to take the derivative of this "original function." In the next interview item, I wrote "$\int \sin(x)$" and "$\int_2^0 e^x dx$" on the board and asked David and Devon to talk about them (note: the *dx* was left off of the first and the limits were written 2 to 0 on the second intentionally, as a means to generate conversation). They wrote below the first integral "-cos(x) + c." Then Devon turned to me to explain what he was thinking.

Devon: From my memory it's like finding the anti-derivative of, like, this function [points to sin(x)].

Here Devon stated that he was thinking of the object "sin(x)" as mapping to its antiderivative. The actual meaning of the integration, then, is to map the integrand appropriately with another function, by "finding the anti-derivative of this function." During the conversation, David added in a *dx* to the original expression, "$\int \sin(x)$," so I took the opportunity to ask him why he did so and what significance that *dx* had.

> David: Again, I guess it matters because if you don't have the dx, then it's just going to be like sine x [sin(x)]… but, is it the second derivative, or the first derivative or something like that? … So I think it's just for the sake of organization just to have the dx in there, to signify that this is the derivative of the original function.

David clearly laid out his thinking that the idea behind an integral is to take a function and map it to some "original function" from whence the integrand function came. The *dx* serves the exclusive purpose of indicating which variable was used to take "the derivative of the original function." It is the "link" between the two functions. The purpose of the integral, then, is to appropriately select this "original function" that the integrand maps to. In this case "$-\cos(x) + c$" maps with "sin(x)." David also explained the significance of the "+ c" in their solution.

> Interviewer: Why in this case is there a plus c?
> David: Because in the original equation, you could always have a constant. You could always have a function added with a constant [waves finger over $-\cos(x) + c$]. But the thing is when you derive the entire function the constant just goes away.

Again, David talked about an "original equation" (David often used the word "equation" when he meant "function") that could "always have a constant." In his explanations there clearly exists some other function that maps directly to the function inside the integral. The conceptual understanding of the integrand, then, is that the integrand exists in some kind of correspondence to this other "original function" that it came from. And that the meaning of the integral is to find the original function that matches with the function in the integral.

So the question still remains that if the integral is conceptualized as mapping an "original function" with the integrand and the *dx* serves as a referent for the variable of differentiation, what is the meaning of the limits of integration? I asked David and Devon why the integral of sin(*x*) ended up with a "+ *c*" when the previous integral they worked on did *not* have a "+ *c*."

> Interviewer: So why do we have a plus c in this one, where in the last example we didn't have a plus c anywhere?
>
> Devon: I think it's plus c, we didn't see, but it's there, but we cancel it out. We make this function [moves over to $\int_2^0 e^x dx$], we have the antiderivative of this [writes $e^x + c$], but when you're doing the [writes $e^x + c\,|_0^2$ (note: later he notices his mistake and changes it to $|_2^0$)], doing this, this c got cancelled out, so it's not shown.
>
> …
>
> David: [Referring back to $\int \sin(x)\, dx$] And we need the constant in order to come up with a concrete function.
>
> …
>
> Devon: But this one [moves back to $\int_2^0 e^x dx$], you are finding the difference between these two [points fingers to the 2 and 0]. So, regardless of the c, it would just be difference. So that's how I think of it, as difference. So it doesn't matter.

In their discussion, they revealed something about the meaning of the limits of integration when considering the integral as a map to another function. Devon explained that he was "finding the difference" and pointed at the 2 and the 0. Thus the limits of integration have to do with the "difference in values" of the original function. Devon stated that that was how he thought of it, "as difference." I want to be clear that this is not merely a rote procedure, but an understanding that 2 and 0 signify a difference between two values. There is nothing inherent in a 2 and a 0 that mean "difference," so we can see that they were providing these numbers with a layer of meaning. As a sidenote, we also see that David referenced the need for a constant in order to come up with a "concrete function" that corresponds with the function in the integral. Thus we can see strong evidence of a "mapping" idea in his thinking.

We can put these pieces together to describe what I call a "function mapping" symbolic form. The integrand is conceived of as having come from some "original function." The differential dictates the "link" between the original function and the integrand function. It is "how you know" how to pair the function in the integral with the original function. The meaning of the integral itself is to find this companion function that maps directly onto the function in the integral. Though of course, there may be more than one function, hence the need for a "+ *c*" in some cases. The limits then correspond with certain values of the "original function" and the difference between them is measured. This symbolic form is visually described in Figure 4.3.1.

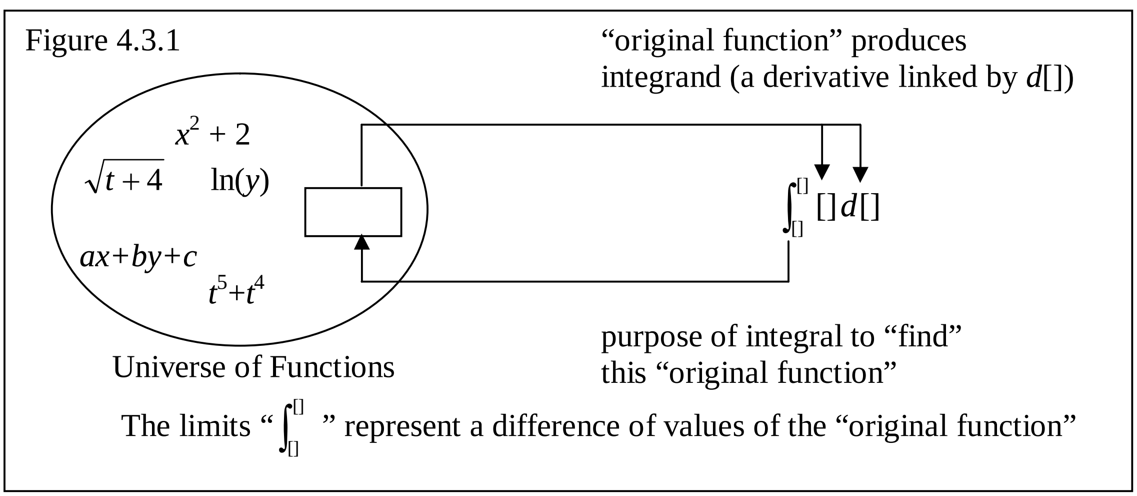
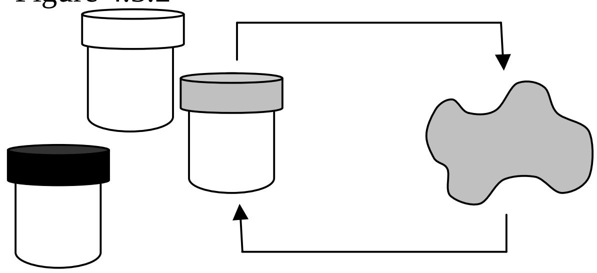


Figure 4.3.1: A visual representation of the *function mapping* symbolic form

There needs to be one important note made about the *function mapping* symbolic form. It could be easy to dismiss this form by saying that it does not meet the criteria of a symbolic form. However, it is possible that this symbolic form could be seen as something like the reification of the anti-derivative process into a conceptual object (Sfard, 1991). The procedure of reversing the derivative process has created the meaning of “finding” a companion function for the integrand. Here we have meaning being given to the symbols in the template “$\int_{[]}^{[]} [] d[]$.” I argue that the way in which the students are thinking about function mapping here does not fall under the category of “rote procedure,” though it is closely related to the regular procedure for calculating an integral. The students were able to articulate conceptually about the meaning of the symbols in the template. Devon and David consistently refer to an “original function” that pairs up with the integrand. The means by which they are paired is mediated through the differential. This thinking transcends mere rote procedure, as much as any of the other symbolic forms do, and gives the integrand and the differential an identity with real meaning.

In order to create an analogy to a more concrete situation to explain the conceptual merit of this symbolic form, consider a child who is playing with play-doh. As the child begins to play, he or she might pull clay from several different jars, where each jar has a lid that matches the color of the play-doh inside. When the child is done playing with the play-doh, she or he is able to “map” each color of play-doh with the jar that it came from. The child does this by matching the color of the play-doh with the color of the jar’s lid (see Figure 4.3.2). If there is more than one jar with a “red” lid, then it does not matter which of these jars the child returns the play-doh to. This scenario is analogous to the type of thinking David and Devon exhibit with the integration. The integrand comes from some original function, and the integral seeks to “return” the

Figure 4.3.2



integrand to this original function. The clues for how to do this are given by the differential, which indicates how the integrand and original function are related.

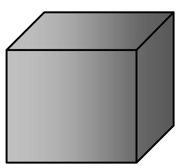
Some branches of mathematics have been devoted to the study of such “function mappings,” such as is the case with some Banach spaces or with some areas of set theory. Thus unless we wish to call those fields of study mere “rote procedure” then we should consider what these students are doing as conceptually meaningful. Just as a mathematician sees meaning in looking at mappings from one function space to another, the students appear to also be giving meaning to the pairing of the integrand with some other “concrete function.” The whole meaning of the integral through this lens *is* function mapping.

In fact, it is true that the essential meaning of the indefinite integral “$\int [\,]d[\,]$” is just that. It is a shorthand notation for the mapping of the function to a class of anti-derivatives. The conversation that David and Devon had regarding the “+ *c*” shows that they hold understanding regarding the mapping of the one function, the integrand, to a whole host of potential candidates. Thus we can say they are conceptualizing the integral in a similar way to how a mathematician might. They are demonstrating a certain type of conceptual understanding that can be placed on the integral template.

4.4 The Problematic "Add Up Then Multiply" Symbolic Form

The resources framework, and the "knowledge in pieces" ideas in general, push away from deficit-based explanations for student mistakes. In this vein, I do not want to wrongfully portray student misconceptions as the culprit to student difficulties. However, it is still possible that students may develop a piece of knowledge that is based in sound cognitive resources, but conflicts with accepted conventions. It is even possible for this "knowledge" to become a resource that the student uses. Here I describe a symbolic form that students may hold that is problematic, but also show that the form is based in other correct and useful resources.

First, I describe the work done by Ethan during his interview. I gave him the following interview item and asked him how he could determine the box's mass.



> This shows a box with varying density. (dark = more dense, light = less dense)
> Suppose you wanted to know the box's mass. How could you figure that out?

He decided that he needed to look at the density over the whole volume in order to the find the mass. He began to write down an integral of density with respect to volume ($\int D\, dV$), and I asked him to explain what he was writing down.

> Ethan: Density is varying. So this integral means that we're going to add up all the densities [points to *D*], infinitely small, so that you get the overall idea. You'll get the exact idea of its density once you do that, then you just multiply by volume [points do *dV*].

During his explanation he pointed to the *D* in his integral (representing density) and *dV* (representing the volume). Thus he appears to have conceptualized the pieces of the integral as being multiplied together. However, this multiplication seems to take place *after* the adding up of the integrand. Following this train of thought, the differential dV would act as an infinitesimally fine-grained partition over which the density D (not mass) is added up over each piece. At the conclusion of this summation, the resultant density would be multiplied by the volume to get mass. This view of the integral is different from the standard understanding in a significant way and it played an important role in Ethan's thinking about the integral. I now provide support that this understanding of the integral was not an isolated case, but was drawn on in other contexts during the interview as well.

In another interview item, I presented Ethan with the following problem.

> A 2-dimensional surface (S) experiences a non-uniform pressure (P) and we want to know the total force exerted. We can use the surface's area (A) to compute this through the integral:
>
> $$F = \int_S P \, dA .$$
>
> Why does this integral calculate the total force exerted?

I asked Ethan to talk about what was happening in this integral and what it meant to integrate pressure. Ethan's response again yielded evidence of a similar way of thinking.

> Ethan: Since it's a non-uniform pressure, you're adding up all the pressures that are at each point, each kind of location on the surface, and over all they would tell you the pressures—you can get the whole pressure. You can divide, after you integrate you can divide it by how much area... and it'll tell you the average area.

The first part of this quote makes it clear that Ethan was thinking about adding up pressures, not forces. This is similar to his idea of adding up densities (not masses) in the previous example. The second part of the quote, about dividing by area, left some questions about what he was thinking, so I asked him to say more about what he meant.

> Ethan: The higher the P's the more the F's...Force equals, so I'm adding up the P's. So if P is higher, you'll get more, a higher sum and that relates to F.

Here he explicitly stated a relationship between adding up the pressure and the resultant force. He understood the integral to be a summation of pressures, which if they are higher give you a larger total pressure. This summation of pressure is "related to *F*."

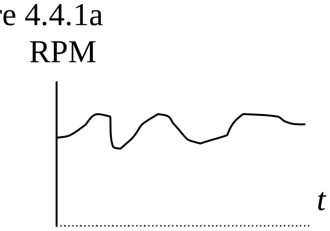
In a third instance, Ethan displayed thinking along these same lines. I provided Ethan with the interview item:

> The durability of a car motor is being tested. The engineers run the motor at *varying* levels of "revolutions per minute" over a 10 hour period. Denote "revolutions per minute" by *R*.
>
> What is the *meaning* of the integral $\int_0^{600} R\, dt$?

He first talked about how the "10-hour period" had been changed into minutes and that that was where the 0 and the 600 came from. I then prompted him to explain what the integral was computing.

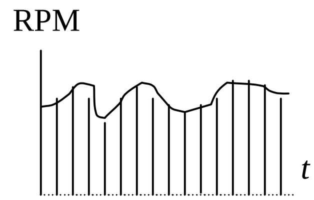
> Ethan: You're just adding up all of the, I guess in this context, you're just, over time, you just add up all the RPMs that happened from 0 minutes to 600 minutes. Because 10 times 60 is 600 minutes.
>
> Interviewer: What do you mean by add up all the RPMs?
>
> Ethan: The integral just adds up things, just keeps adding them up [places hand on table to his left and sweeps it across to the right]. I guess if you had it, if you just had a function of RPMs, something like [draws axes], this is time here [points to horizontal axis], maybe this is RPMs [points to vertical axis] and you have something like [draws squiggly graph. See Figure 4.4.1a].

Figure 4.4.1a



> This function will tell you the area here, which is just the way you're just adding it up [draws several vertical lines from his graph to horizontal axis. See Figure 4.4.1b].

Figure 4.4.1b



Interviewer: What does the 0 and what does the 600 mean?
Ethan: It just means… wait, I don't understand.
Interviewer: So, what does the 0 refer to and what does the 600 refer to?
Ethan: Oh. It's going to add up a whole bunch of RPMs [points to *R* in the integral] with respect to time from 0 minutes to 600 minutes [points to the limits on the integral]. In that region.

This third episode clearly shows Ethan stating that the quantity being added up is RPMs. The fact that he labeled his vertical axis RPM illustrated that it was not due to confusion about what the function was, and that he was thinking of RPMs as the values of the function. Thus, in his thinking it is the quantity represented in the *integrand* that is added up over the region dictated by the limits. His language and hand movements also clarified what he saw the limits of integration as representing. He used his hands to visually show that the addition was happening from left to right, from 0 to 600 minutes, showing a "from…to" type thinking. He also employed the words "*from* 0 minutes *to* 600 minutes," which is language similar to that found in the *adding up pieces* symbolic form.

I wish to draw attention to the fact that Ethan had a *stable* blend between the symbolic template and a conceptual schema for the symbols. He viewed the integral template as "$\int_{[]}^{[]} [] d[]$," where the first box (density, pressure, RPMs) was added up over the infinitesimally small pieces generated from the partition created by the differential (*dV, dA, dt*). The resultant summation was then multiplied by the quantity represented by the differential (volume, area, time) in order to get the value of the integral. I call this the "add up then multiply" symbolic form and give a visual representation of it in Figure 4.4.2.

Figure 4.4.2

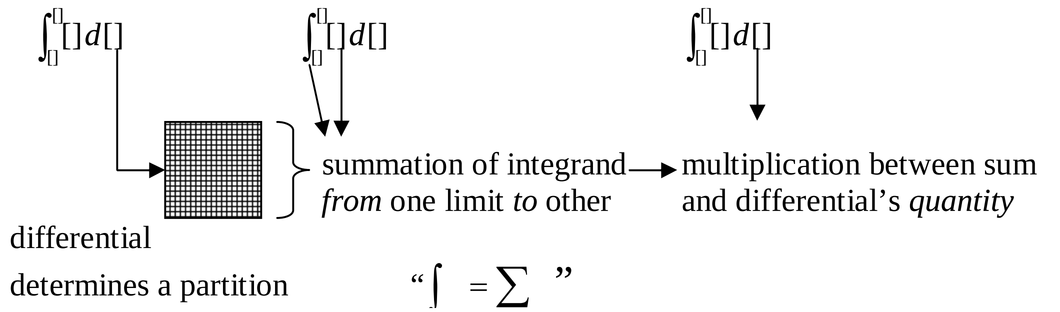


Figure 4.4.2: A visual representation of the *add up then multiply* symbolic form

This symbolic form, while problematic, seems rooted in productive resources. For example, it is reasonable to claim that Ethan understands a connection between integrals and multiplication. Whether this comes from working with Riemann sums or somewhere else is unclear, but it is certain that Ethan understood some multiplicative relationship between the integrand and the differential. This is similar to the multiplicative relationship in the *adding up pieces* symbolic form, except for *when* the multiplication occurs. While Ethan was working with the mass of the box, he drew on the fact that "mass is…density times volume." Thus he not only has understanding about the inherent multiplication in integration, but he drew on knowledge about the relationships between certain variables to come up with his integral.

Additionally, the differential "*d*[]" does have a connection with the "infinitesimally small." If the partition is at a finite level, usually a symbol such as Δx is used to represent the small width of the partition. Often we say that $\Delta x \to 0$ and becomes *dx*. Hence Ethan also may have been drawing on a relevant cognitive resource that *dx* means "infinitesimally small."

Lastly, the Riemann sum process (or similarly the *adding up pieces* symbolic form) has elements of a summation that approaches a limiting process of an "infinite summation." This feature of the Riemann sum has most likely been used by Ethan in constructing his notion of integration. The only place where Ethan's thinking diverged from correct understanding about the integral is that the multiplication between integrand and differential happened *after* the summation, instead of *before* the summation in each "little piece." We could say then that his understanding has most likely arisen from these correct underlying pieces of knowledge. However, what is important to note is that since this knowledge has been compiled into a stable cognitive unit, it should be considered a symbolic form in his cognition.

4.5 Contrasting the Four Major Symbolic Forms

In this chapter I have presented four major symbolic forms that take the entire integral symbol template "$\int_{[]}^{[]}[]d[]$" and associate a conceptual schema to it. All of the symbolic forms presented so far provide unique meaning to each of the template pieces: (1) the integrand and differential "[]*d*[]," (2) the "$\int$" sign, and (3) the limits of integration "$\int_{[]}^{[]}$." At the initiation of this research, there was an implicit presumption that students would associate meaning with each of these pieces of the symbol template separately. However, these data indicate that for these students a conceptual schema applied, for example, to the limits "$\int_{[]}^{[]}$" correlated with another specific conceptual schema applied to the integrand and differential "[]*d*[]." That is, the meaning that these students gave to one piece of the integral template automatically linked to specific meanings that were then given to the other parts of the template. That is to say, if a student saw “[]*d*[]” as a thin rectangle, then their interpretation of the “$\int$ ” sign and the limits “$\int_{[]}^{[]}$ ” automatically had specific meanings associated with a “thin rectangles” view of “[]*d*[].”

The relationship between the meanings given to each of these parts of the symbol template exactly correlated with the specific symbolic form the student was drawing on at that point. Thus, for these students, each of the parts of the symbol template was viewed in terms of how they fit into either the *area*, *adding up pieces*, *function mapping*, or *add up then multiply* symbolic forms (or others not yet conceived). What follows in the subsequent section is a consideration of each of the parts of the integral symbol template, recapping the part of conceptual schema given to them depending on which symbolic form is being activated.

Conceptual Schema Blended with “[]d[]”

First, consider the part of the integral symbol template “[]*d*[].” This component addresses the integrand and the differential that are involved in the integration. When students drew on an *area* symbolic form, the integrand was taken to mean a graph drawn on a set of axes and the differential had the role of determining the axis that counted as the “bottom side” of the fixed region. Bill and Becky, who drew on the *area* symbolic form extensively provide a nice example of this. After they had calculated (numerically) the integral “$\int_1^2 \frac{2}{x^3} - x^2 \; dx$,” I began asking them about different parts of the integral symbol template.

> Interviewer: When you see the 1 and the 2, what does that mean?
> Bill: It means to me that if we’re looking at this on a graph, this is 1 and this is 2. Let’s just say it looks something like that [draws in squiggly graph], it would be the area in between 1 and 2 [shades in region]

Becky: I totally agree with that. First I would look to see if it was a dx or a dy.

I then asked Becky to clarify how the meaning would change if it were a *dy* instead of a *dx*. She explained that the axis involved in forming the region would shift to the other axis.

> Becky: Picture-wise, it would mean the 1 and the 2 would be on the y axis [points to two places on the *y*-axis].

We can see from this episode that the integrand is made to be a graph in the plane. There seems to be no further purpose for, nor action taken on the integrand. The integrand simply defines the "top" part of the region being considered. The major purpose of the differential unit in this symbolic form is to dictate the axis over which the region would be bounded. Thus, in a sense, the differential is responsible for the "bottom side" of the region being considered. If there is a *dx*, the bottom part of the region lies on the *x*-axis, but if there is a *dy*, then the bottom part of the region lies on the *y*-axis. Later, Bill was discussing the meaning of the integral $\int_2^0 e^x dx$ and was trying to justify why the limits would end up yielding a negative area.

> Bill: Let's see, for this… if I drew e to the x… [draws e^x graph]. When I look at this, when I try to justify why it's negative area and not positive area, cause you normally... Like with the identity [a reference to a rule], it would be from 0 to 2, but it would be negative. So if it was redrawn, like that [draws in $-e^x$ graph].

Later, while trying to explain why $\int dx$ was just the same thing as $\int 1 \cdot dx$, he offered the following,

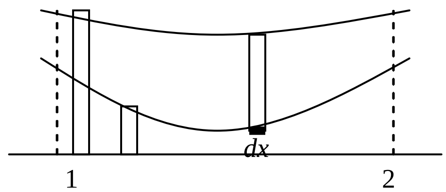
> Bill: If it's only just 1, times, the graph is just y equals 1. So it's 1. Straight line [draws "$y = 1$" graph, horizontal line].

Bill continued to use a "graph" approach many times throughout the interview, even causing Becky, his interview partner to remark, "You like graphs, huh?" The importance that these data have is that they show Bill regularly interpreting the "[]*d*[]" part of the integral template as a graph in the plane. The integrand only serves the purpose of creating the "top" part of the region whose area will be determined. The region under the graph was viewed as a static, fixed whole. The differential determines which axis the bounded region will be attached to. I call this the "parts of the region" interpretation of the integrand and differential.

This interpretation contrasts in important ways to the meaning of "[]*d*[]" when looking through the other symbolic forms. When the students were drawing on the *adding up pieces* symbolic form, the meaning that they gave to "[]*d*[]," though still related to a graphical, visual representation of the integral, included a representative "rectangle" that was used to analyze the integral. The integral includes a partition of tiny pieces, where a little bit of the overall quantity is ascribed to each of these tiny pieces. *One* of these "tiny pieces" is looked at as representative of *all* of the tiny pieces and its properties are used to understand what is being done in the integral. The integrand and differential are used to construct this "representative rectangle." Thus the integrand is not merely the "top" part of a region, but is involved in the creation of these "tiny pieces."

Chris and Clay drew on the *adding up pieces* symbolic form regularly during their interviews. During the mathematics-day interview, they were computing and explaining the integral "$\int_1^2 \frac{2}{x^3} - x^2 dx$." They had drawn a picture with two curves, one representing $\frac{2}{x^3}$ and another representing x^2. While they were talking about why they could split the integral up into two separate integrals, Chris drew a thin rectangle running up to the higher curve, another running up to the lower curve, and then a third one running between the two curves. He drew a dark line across the base of the thin rectangle and labeled it *dx* (see Figure 4.5.1).

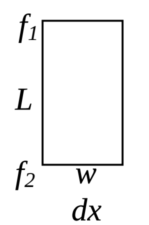
Figure 4.5.1



Chris used this "representative rectangle" to explain why the integral could either be broken up into two pieces or considered together as one integral. In doing this, he separated out the representative rectangle from his drawing and redrew it larger on the side.

> Chris: That's basically like, if you want to find the area of the rectangle [refers to the rectangle in between the two curves], you have length times width [draws another, larger rectangle to the side of his figure]. And so, we'll assume that length [points to the rectangle] is the difference between the two functions. I should draw it this way [vertically] and length [writes *L*] is the difference between the two functions [writes in f_1 and f_2]. And width is dx [writes *w* and *dx*]. [See Figure 4.5.2.]

Figure 4.5.2



Chris used the rectangle to dissect and make meaning of the integral. Notice that the single "representative rectangle" was redrawn larger off to the side of the figure. This shows that Chris was thinking of this one rectangle specifically, and that its properties had general implications for the overall integral. But in order to discover these general properties, attention had to be paid to one single representative rectangle. In Figure 4.5.2, f_1 and f_2 represent the functions that are being subtracted in the integral. The length of the rectangle, *L*, is given by their difference. The width of the rectangle is given by *dx*. We can see the connection, then, between the symbols "[]*d*[]" and this rectangle. The integrand represents the height of this representative rectangle, whereas the differential represents the width. Thus the symbols "[]*d*[]" are used to construct these rectangles. This contrasts with the *area* symbolic form, where the integrand and the differential served the purpose of creating the "top" and "bottom" parts of a fixed region in the plane.

Later in the interview, Clay and Chris were discussing the integrals "$\int dx$" and "$\int \sqrt{t}dx$." Chris approached the integrals in the same way as before, by drawing one representative rectangle to discuss in depth. He began by discussing the first of the two integrals.

> Chris: So here obviously, we don't have a function, so we can assume it's 1.
> Interviewer: Ok. Why can you do that?
> Chris: Cause when we're going back to the length times width back here [draws large rectangle next to integral], we have length and width. We have our width equaling dx [writes $w = dx$]. So if the, the area is going to be dx [writes $A = dx$], then that means our length has to be 1 [changes it to $A = 1 \cdot dx$. See Figure 4.5.3].

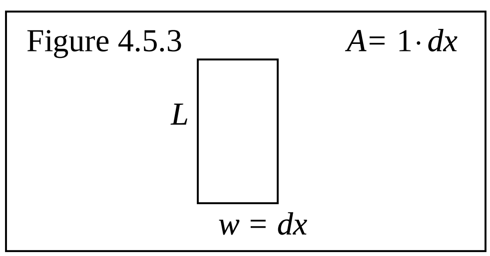


Figure 4.5.3

Chris used the rectangle to reason about the properties of the integral "$\int dx$." He did not even use a graph to situate this rectangle. He was able to take one representative rectangle completely out of context as a way of discussing the integral. According to this rectangle, whose area is supposed to equal *dx*, in order to match the integral, the length would have to be $L = 1$. That meant that "1" could be substituted back into the integral as the integrand. Through this Chris showed the close connection between the integrand

and the height of the rectangle. The rectangle was satisfactory for Chris in justifying this idea. Furthermore, as he moved on to discuss the second integral, "$\int \sqrt{t}dx$," he came back to the rectangle idea. Clay had just finished saying that if *t* did not have a dependence on *x*, then it could be treated as a constant and pulled out of the integral. When I prompted them to talk more about what they meant, Chris brought up the rectangle he had used for the previous discussion. He erased the *L* and put "$\sqrt{t}$" instead. He then said, "Yeah, so instead of having 1 being here, we would have root *t*." Thus the use of a single representative rectangle figured prominently in his thinking. I call this the "representative rectangle" view of the integrand-differential part of the symbol template.

These two graphical-visual interpretations of "[]*d*[]" contrast with the *function mapping* symbolic form where the integrand is seen as having come from some other function. From this "original" function, whose derivative was calculated with respect to the differential variable, we get the function represented by the integrand. David and Devon demonstrated how the integrand could be conceptualized within a *function mapping* symbolic form. At one point they were discussing the need for the *dx* in the integral "$\int_1^2 (2x^{-3} - x^2)\, dx$."

> Interviewer: Why does it need a dx?
> David: Well, it's still an integration. So, in an integration the dx is always essential, because it shows that this entire thing [waves hand over $2x^{-3} - x^2$] is a derivative of x.

A little bit later David added the following when asked to talk about the relationship between the integrand function and the differential.

> David: The fact that this entire thing is sitting right next to each other, and dx outside, means that basically this entire function [motions hand over $2x^{-3} - x^2$] is the derivative of an original function.

David had conceptualized the integrand "$2x^{-3} - x^2$" as the "derivative of an original function." The purpose of the differential was to indicate that the way the integrand was formed was by taking this "original function" and differentiating it with respect to *x*. The differentiation should happen with respect to *x* because *x* is the variable of the differential. The *dx* "shows that this entire thing is a derivative of *x*." Thus the "[]*d*[]" is seen as having "clues" about the origin of the integrand function. I call this the "function origins" view of the integrand-differential part of the symbolic template.

Lastly, the *add up then multiply* symbolic form, like the first two symbolic forms, views "[]*d*[]" in more graphical-visual terms. It is similar to the *adding up pieces* symbolic form in that the differential directs some kind of partition and each piece of the partition is considered. However, the difference is in the relationship between the integrand and the differential during this process. In the *adding up pieces* symbolic form, the integrand and differential interact within each individual rectangle to create a small bit of the quantity, as seen by the use of one representative rectangle to describe the integral. In the *add up then multiply* symbolic form, small quantities of the *integrand* are considered to exist within each little piece of the partition. Chris and Clay were working on the interview item in which they were to figure out how to determine the mass of a box with varying density. They had devised the integral "$\int_0^4 x\,dx$" and were explaining how their integral calculated the box's mass. They had arbitrarily decided to use a linearly increasing density function, *x*, and arbitrarily came up with the limits of integration.

> Chris: We have this density that's increasing. Along the one side it's linearly increasing. So dx would just be us trying to calculate density at each point along the x axis that's infinitesimally small. And then adding up all these infinitesimally small values.

Chris described that the purpose of the *dx* was to allow him "to calculate density at each point." These densities were thought to be distributed throughout the *dx*-related partition. Each small piece within the partition had some small amount of the density.

Ethan, in his interview, worked on the same problem and came to the exact same conclusion. He had written his integral "$\int D\,dV$" and was explaining why it worked.

> Ethan: Density is varying. So this integral means that we're going to add up all the densities [points to *D*], infinitely small, so that you get the overall idea. You'll get the exact idea of its density once you do that, then you just multiply by volume [points do *dV*].

Similarly to Chris, Ethan stated that each "infinitely small" piece held a little bit of "density." We can see from their descriptions that the meaning of "[]*d*[]" is that a small amount of the integrand "[]" is contained in each tiny piece created by the partition made by the differential "*d*[]." But there is no interaction between the two at this fine-grained level. I call this the "integrand in each piece" view of these symbols.

Conceptual Schema Applied to the "$\int$ " Symbol

Next, I examine the meaning given to the integral symbol itself, "$\int$." Once meaning is given to "[]*d*[]," then the integral symbol comes to mean different things. Each of the four major symbolic forms takes a different interpretation of the symbol. I want to note that the meaning given to the "$\int$ " sign stably correlates with the meaning of both "[]*d*[]" and the limits "$\int_{[]}^{[]}$." So while there are several interpretations of the "$\int$ " sign, they appear *not* to be independent of the symbolic form that is being activated.

First, I talk about the students who were drawing on the *area* symbolic form. Bill had made much use of this symbolic form in his thinking. He often immediately took the integrals of the interview items and construed them as graphs in the plane, taking a fixed, static view of the region underneath the graph. This pair of students had been given the integral "$\int_2^0 e^x dx$," and Bill was trying to explain why it gave negative area as opposed to positive area.

> Bill: Let's see, for this… if I drew e to the x… [draws e^x graph.] When I look at this, when I try to justify why it's negative area and not positive area, cause you normally... Like with the identity [a reference to a rule], it would be from 0 to 2, but it would be negative. So if it was redrawn, like that [draws in $-e^x$ graph].

Bill talked about the fact that when integrating from 2 to 0, he knew the area should come out to be negative. According to a "rule," he reconceptualized the problem as being an integral from 0 to 2 over the function "$-e^x$." When he drew in the $-e^x$ graph, it helped him explain why the result should be negative, since it was yielding a "negative area." We can see that his understanding of the integral seems tied to the area of a fixed, bounded region. He redrew his graph in order to provide an area that was "negative," since $-e^x$ is below the *x*-axis. I asked them to talk more about why having the limits as 2 and 0 would end up giving negative area for the integral.

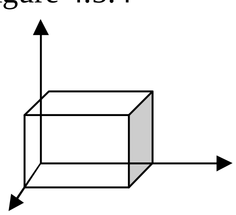
> Becky: It basically means you have your graph [references the e^x graph] and you have your two points, here's 0 and here's 2 [points to two places on the *x*-axis]. Instead of looking at it as if you were wanting the area from this point to this point [puts left hand on 0, then right hand on 2], you're just looking at it as the area from this point to this point [puts right hand on 2 then left hand on 0].
>
> Bill: The only problem I have with that is that it looks like positive area on the graph.

Here we can see that once the e^x graph had been drawn in the plane, Bill conceptualized the integral as the area of the fixed, bounded region above the graph.

Consequently he ended up having a "problem" with the fact that "it looks like positive area on the graph" when dealing with the function e^x. We can see that he was interpreting the integral symbol as meaning "the area of this bounded region." The area *looked* positive to him, so it seemed like it should have ended up being positive area. The main point we see in this episode is the fact that the integral actually *means* the area, which is the conceptual schema given to the "$\int$ " symbol. Thus this symbolic form is labeled the "area" symbolic form.

The *adding up pieces* symbolic form, while also based in visual, graphical representations, is quite distinct from the *area* symbolic form's conceptual meaning of the "$\int$ " symbol. The *adding up pieces* form starts by taking the "[]*d*[]" and construing it as several tiny rectangles, where one representative rectangle can be taken out of context in order to understand the properties of the integral. Hence, the "$\int$ " symbol here will necessarily have to do with these rectangles, which are often viewed as "infinite" in number by the students. In the physics-day interview, Devon was working on the problem of how to determine the mass of a box with varying density. He had drawn a box in a three-dimensional coordinate system and then written an integral that corresponded to it (see Figure 4.5.4). He had chosen $\rho(r)$ to represent his density function.

Figure 4.5.4



$$\int_V \rho(r)\, dV$$

He then drew several little cubes inside of his larger box and then talked about why his integral would compute the mass of the box.

> Devon: I think it's more the conceptual idea that when you find out a mass of every small piece [points to the little cubes inside his box], you find the mass of the whole thing [waves hand over the whole box]. And this would find the mass of every piece [underlines $p(r)dV$ inside of the integral], and then you just add them up together [waves hand over the integral symbol].

Devon provided a very clear picture of what he saw the integral as doing. The integral "just adds them up together." As discussed in the previous section, the fact that he was drawing on an *adding up pieces* symbolic form meant that the integrand and differential "[]*d*[]" interacted within each little cube. He stated that "this," meaning the function $p(r)$ and the differential dV, "would find the mass of every piece" for the little cubes he had inside of the larger box. Then, he showed his interpretation for the integral sign itself, saying "then you just add them up together" while he waved his hand over the integral sign. So the "$\int$ " means "add them up." The masses from each of the little cubes were added up in order to "find the mass of the whole thing." He then furthered this idea.

> Devon: It's basically, how many boxes there are, and each box has its density and then you calculate the volume of, I mean the mass of each box [points to little cubes inside of the box] and then you add them up.

Again, we can see that once the integrand and differential have "interacted" in each cube in order to determine the mass of each cube, the integral itself takes all of these masses and adds them up. Thus the meaning given to the "$\int$ " sign is to add up all of the little quantities and for this reason this form has been termed the "adding up pieces" symbolic form.

These two graphical-visual interpretations of "$\int$" again contrast with the *function mapping* symbolic form where the meaning of the integral has more to do with selecting an appropriate companion function that matches what is in the integral. In this symbolic form, the integral seeks to pair the function presented with a "companion function" from whence it came. Specifically the relationship between the integrand function and the companion function is a derivative with respect to the variable of the differential.

Adam and Alice, who drew often on the *function mapping* symbolic form demonstrated how the "$\int$" sign could be conceptualized from the standpoint of this form. They had been given the integral equation $\int a\, dt = \int dv$ and were talking about why they were equal to each other. (Note: no limits were placed on these integrals intentionally as a potential source of conversation.) The conversation was revolving around why these two integrals were equal and what the meaning of *dv* was.

> Interviewer: What does it mean that the integral on the right is dv?
> Adam: So I see these terms, a and dv as functions [writes "functions: *a*, *dv*"]. So they [*a* and *dv*] equal something like y…[writes *y* =]. And this [points to *y*] can also be denoted as f of x [writes *y* = *f(x)*] … This f is like a or dv [points to *a* and *dv*]. These represent functions…You could put t, or whatever, it's just another variable [writes *dv(x)* and then changes to *dv(t)*. Then changes *y* = *f*(*x*) to *y* = *f*(*t*)]. It's just denoting the function with something, so you can recognize it.

Adam talked about a third function, *y* or *f*(*t*), that could be paired with both *a* and *dv*. Hence the reason for the equality of these two integrals is the fact that they both map to the same function, *f*(*t*). The conceptual meaning of the integral itself was about finding the function that is paired with the integrand function, in this case with either *a* or *dv* (though he will clarify shortly that by "*dv*" he is actually thinking of *dv/dt*). The equality of the two integrals was due to the fact that they both map to the same companion function. It didn't matter whether the companion function was written as *y*, or *f(x)*, or *f(t)*, etc. All that mattered was that it was a match for *a* as well as a match for *dv*. And the variable *t* from *dt* is merely "just another variable," facilitating the choice of the companion function. A few moments later, Adam came back to this point.

> Adam: And what you were saying before about the dv. I think this is just like a name. You call someone a name, like Adam or something. Here you're just doing integral terms. dv equals acceleration. The derivative of velocity. But you're just making this the name of the function.
>
> …
>
> Adam: I think I wrote this wrong. It's not dv in terms of t, it's the derivative of v of t over dt [writes *dv(t)/dt*]. It's just another way to write a [points to *a*].

Again, the integrand is a function that represents the derivative of some other function; in this case *a* is the derivative of velocity. But *dv/dt* is *also* the derivative of velocity. Hence these two functions came from the same original function, namely the velocity function. Adam was able to use this to contend that the two integrals were, in fact, equal to each other since *a* and *dv/dt* are just two different "names" for the "derivative of velocity," where velocity counts as the "original function." Hence, "the derivative of *v(t)* is *a*" means that *v(t)* is the "original function" from whence *a* came. So it works as the companion function to the integrand function, *a(t)*. Similarly, *dv/dt* is the derivative of velocity, and so *v(t)* is also the "original function" for that integral. Thus the name "function mapping" is given to this symbolic form.

Finally, in the *add up then multiply* symbolic form the meaning of the "$\int$ " sign takes on a similar meaning to the *adding up pieces* symbolic form. It likewise is conceptualized as taking all of the tiny pieces and adding them up. The signature difference between the two symbolic forms resides in the meaning given to each tiny piece. In the *adding up pieces* form, the meaning given to "[]*d*[]" dictates that each tiny piece is a product of the quantity of the integrand and the quantity of the differential. That is, density and volume give mass. Or pressure over an area gives a force. However, in the *add up them multiply* symbolic form, the meaning of "[]*d*[]" dictates that the quantity of the integrand alone is added up over each piece of the partition.

When Ethan described the integrals he had come up with during his interview, it was clear that he was thinking of a summation over the quantity of the integrand. When he had the integral over density, "$\int D\,dV$," which was used by him as a way to calculate mass, he said,

> Ethan: Density is varying. So this integral means that we're going to add up all the densities [points to *D*], infinitely small, so that you get the overall idea.

Then when working with the integral equation "$F = \int_S P\,dA$," he explained how this integral computed the overall force.

> Ethan: Since it's a non-uniform pressure, you're adding up all the pressures that are at each point, each kind of location on the surface, and over all they would tell you the pressures—you can get the whole pressure.

Similarly, when working with "$\int_0^{600} R\,dt$," where *R* represented the revolutions per minute, he claimed that the integral symbol indicated an addition.

> Ethan: It's going to add up a whole bunch of RPMs with respect to time from 0 minutes to 600 minutes.

It is clear that this symbolic form views the "$\int$ " symbol as an addition, but that the addition happens over the quantity that stands in the place of the integrand. The density, the pressure, or the revolutions per minute is what is added up.

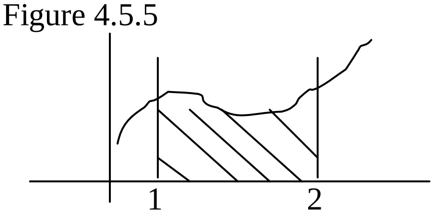
Conceptual Schema Applied to the Limits, "$\int_{[]}^{[]}$ "

Last, I consider the meaning given to the limits of integration "$\int_{[]}^{[]}$." As with the other parts of the symbol template, the meaning given to the limits of integration correlated with the meaning given to both the "[]*d*[]" and the "$\int$ " symbols. When the students were drawing on the *area* symbolic form, the limits of the integration came to mean the sides, or perimeter, of a fixed region in the plane. Much like the four lines of a trapezoid mark out the boundary of what is considered "the trapezoid," the limits of integration become actual vertical lines that help mark out the boundary of the object whose area is considered by the integral.

When the students were using the *area* symbolic form in their thinking, the limits often manifested themselves as vertical lines extending up (or down) from the horizontal axis in order to mark off the shape of the region. Bill and Becky were working with the integral "$\int_1^2 \frac{2}{x^3} - x^2 dx$" when I asked them specifically to talk about what the 1 and 2 meant.

> Interviewer: When you see the 1 and the 2, what does that mean?
> Bill: It means to me that if we're looking at this on a graph, this is 1 and this is 2 [draws vertical lines]. Let's just say it looks something like that [draws in squiggly graph. See Figure 4.5.5]. It would be the area in between 1 and 2 [shades in region].

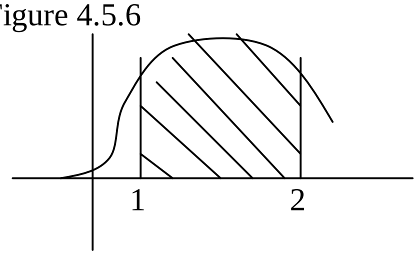
Figure 4.5.5



The 1 and the 2 appear to represent lines drawn straight up in such a way that there is a specific, closed off region in the plane. Therefore, the 1 and the 2 do much more than represent numbers on the x-axis. They are actually the two sides of the shape itself. I call this the "sides of the region" view of the limits. At another point, Bill was describing the integral "$\int \sin(x)\,dx$" and the difference between definite and indefinite integrals.

> Bill: So the definite integral, it would be [draws squiggly graph on *x*-*y* axes]. If it was, for example, from 1 to 2 [makes marks on *x*-axis at 1 and 2]. It would just be [draws vertical lines at $x = 1$ and $x = 2$] this area right here [shades in region. See Figure 4.5.6].

Figure 4.5.6



Again, we see Bill conceptualizing the limits of the integral (he used the example of 1 and 2 for the limits) as the two sides of a fixed region in the plane, where the integral was going to represent the area. The vertical lines going up from the numbers on the x-axis were a common hallmark in the figures students produced while drawing on the *area* symbolic form. Thus the limits of integration actually *are* the sides of the shape. They are more than simple numeric values, but they are part of the fixed region itself.

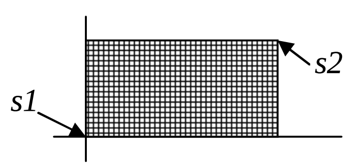
The limits of integration looked somewhat different when the students were drawing on the *adding up pieces* symbolic form. They did not have quite the same visual-graphical interpretation that they had in the *area* symbolic form. In *adding up pieces*, the integral involves some action, where the little pieces are all "added up." The limits in this case represented a sort of "starting" point and "ending" point for this addition. Clay and Chris were working on the following interview item and had activated an *adding up pieces* form.

> A 2-dimensional surface (*S*) experiences a non-uniform pressure (*P*) and we want to know the total force exerted. We can use the surface's area (*A*) to compute this through the integral:
>
> $$F = \int_S P\, dA.$$
>
> Why does this integral calculate the total force exerted?

They drew a two-dimensional rectangular surface and eventually "sliced it up" into a grid of smaller rectangles (see Figure 4.5.7). During their work, they changed the integral to be $\int_{s1}^{s2} P\, dA$. I asked them to discuss the limits *s1* and *s2* that they put on their integration.

> Interviewer: It was integral with S, and you translated that in an integral with an s1 and an s2. So what exactly are s1 and s2?
>
> Clay: s1 is the minimum, where you're starting from, I guess. And s2 would be the last one you're integrating, but that would be inclusive of all the ones in between.
>
> Chris: s1 would be the first square here [points to lower left-hand corner of the surface, see Figure 4.5.7], and then s2 would be this square here because it's where both x and y reach the maximum [points to upper right-hand corner of the surface].

Figure 4.5.7



As a note of clarification, in the last statement Chris meant by "reach the maximum" that the upper right-hand corner of their two-dimensional surface was the largest *x,y* pair on the surface. But more importantly, we can see that the lower limit represented "where you're starting from" and the upper limit represented "the last one" that was being added up. Thus, when the integral is given the meaning of "adding up" all of the small quantities, the limits come to mean "from where to where" the addition happens. I call this the "*from…to*" view of the limits.

In a separate problem, Chris was trying to explain why the integral "$\int dv$" was the same as the integral "$\int a\,dt$," where *a* and *v* were acceleration and velocity. (Note: In order to generate conversation, no limits of integration were specified in this problem.) Chris had noted that $\int dv$ did not have any time component in it, but that it was still equivalent to the other integral anyway. He explained this by discussing the quantities that were being integrated and had been drawing on an *adding up pieces* symbolic form during his work. The following is what he said about the limits of integration.

> Chris: So we don't really need to worry about time anymore here. We're just integrating from the first point of velocity [writes v_0 as lower limit: $\int_{vo} dv$], to the last point [writes *v1* as upper limit: $\int_{vo}^{v1} dv$].

Again this shows the nature of the limits of integration while drawing on the *adding up pieces* symbolic form. The lower limit was "the first point of velocity" and the upper limit was the "last point of velocity." We can see that a "*from…to*" way of looking at the limits was active in Chris' thinking while using this symbolic form.

As already discussed in more detail in section 4.3, the limits of integration, when looking through a *function mapping* symbolic form, mean the values of the original function and the difference between them. Devon, who was drawing on a *function mapping* form, was explaining why there was no "+ *c*" when computing the integral "$\int_2^0 e^x dx$."

> Devon: But this one [points to $\int_2^0 e^x dx$], you are finding the difference between these two [points fingers to the 2 and 0]. So, regardless of the c, it would just be difference. So that's how I think of it, as difference. So it doesn't matter.

Devon described that the way he thought about the limits, 2 and 0, was "as difference." The limits of integration come into play to find "the difference between these two" values. Through his work, it was clear that the difference he was talking about was the values of the "original function" that maps to the integrand. Note that 0 and 2 do not have any inherent meaning of "difference." Thus we can say that Devon was ascribing the meaning of a difference between two values to the limits of integration. I call this the "difference" view of the limits.

Finally, in the *add up then multiply* symbolic form, the limits of integration are given a similar meaning to that in the *adding up pieces* symbolic form. The integral is considered to be "adding up" small quantities in both of these symbolic forms. The only major difference is *what* is being added up. However, the fact that this addition takes place in both means that it would make sense to have similar interpretations of the limits of integration for both. Ethan was describing the details of the integral "$\int_0^{600} R\,dt$" and was drawing on the *add up then multiply* form. I asked him to describe exactly what the limits of integration meant.

> Interviewer: What does the 0 and what does the 600 mean?
> Ethan: It just means… wait, I don't understand.
> Interviewer: So, what does the 0 refer to and what does the 600 refer to?
> Ethan: Oh. It's going to add up a whole bunch of RPMs [points to *R* in the integral] with respect to time from 0 minutes to 600 minutes [points to the limits on the integral]. In that region.

Thus the limits 0 and 600 again refer to the "starting" point and "ending" point of the addition. The addition happens "*from* 0 minutes *to* 600 minutes." Ethan also used hand gestures while he was talking about this integral, where he swept his hand from the left side of his figure to the right side. The difference, then, between the *adding up pieces* and the *add up then multiply* symbolic forms in general is quite subtle. There is similar meaning given to the integral symbol "$\int$ " as well as the limits of integration "$\int_{[]}^{[]}$." They mean "*add* them up *from* here *to* here" in both forms. The difference comes mainly in the interpretation of the symbols "[]*d*[]." In the *adding up pieces* form, the integrand and differential create small rectangles, where inside each rectangle there is an interaction between the integrand and the differential to make a small quantity. However, in the *add up then multiply* form, there is no such interaction and it is only the quantity of the integrand that is added up.

4.6 Other Symbolic Forms Pertaining to the Integral Symbol Template

In the previous sections, I outlined four major symbolic forms that were detectable during the interviews with the students. By "major" I simply mean that they take into account the entire integral symbol template "$\int_{[]}^{[]}[]d[]$." However, there are other symbolic forms that pertain to the integral, but that are specific to either only one part of the template, or to the way the symbol template interacts with other symbols around it. First I look at the students conceptual schemas blended with just the "$\int$ " symbol (i.e. with no limits attached to it). Then I discuss a symbolic form that is actually a special case of the *area* form where the "area" is thought to be between two curves in the plane. Then I describe other forms that deal with the meaning of a number multiplied onto the outside of the integral.

Two Interpretations of the "$\int$ " Symbol with No Limits

During the interviews with the students, there emerged two different conceptual schemas that could be applied to the "$\int$ " symbol with no limits on it. The students, in general, discussed this symbol more on the mathematics-day interview, but rarely during the physics-day interview. Instead, they often translated those integrals into integrals with limits. However, during the mathematics-day interview, I asked the students what the symbol meant and how it differed from an integral symbol with limits on it. In the interview with Bill and Becky, they had been given the integral "$\int \sin(x)\,dx$" and were trying to explain why the answer needed a "+ *c*" on it. I asked them to summarize what the difference was between an integral like this one, without limits, and one that had limits on it.

> Becky: I would say the one that has numbers, you're asking for a specific area or a specific region of like whatever y it is. One that doesn't, you're just asking for it in general. I kind of like interpret that as later on, if you want to know it, what values it's between, you have a more broad range to put the values into. Whereas when you solve for a specific 0 and 2 you're giving the answer and you can't really, like, work on that.

Becky described that by not having limits on the integral, "you're just asking for it in general." She explained what she meant by saying that "later" you could attach numbers on to it to find out the integral for "specific" values.

> Becky: Later on, if you want to know it, what values it's between you have a more broad range to put the values into.

Thus, the integral without limits is like a "generic answer" that is waiting for more specific limits of integration in order to provide a more specific value. By contrast, the integral with limits already gives an "answer," so there is nothing else that can be done with it. When the "$\int$ " symbol is seen in this way, I call it the "generic answer" symbolic form.

The second way the students appeared to think of the integral symbol with no limits deals more with the kind of "object" that results after the computation. It differentiates the two types of integrals based on their outcome. This interpretation of the "$\int$ " symbol could almost be thought of as a special case of the *function mapping* symbolic form. It is similar in that the object of "$\int$ " is to find an appropriate companion function that fits the integrand. I asked David and Devon to describe what the difference was between "$\int \sin(x)\,dx$," which has no limits, and "$\int_2^0 e^x\,dx$," which does.

> David: I guess when it has no limits, no upper bound or lower bound, it just means you're trying to find the anti-derivative of the equation. The original equation to the derivative inside the integration.
>
> …
>
> David: So in this case [points to $\int_2^0 e^x\,dx$] you're just trying to find a number, in this one [points to $\int \sin(x)\,dx$] you're finding a function.

Since the integral "$\int \sin(x)\,dx$" has no limits, they were "finding a function," as opposed to "$\int_2^0 e^x\,dx$," where they were "trying to find a number." Thus the "$\int$ " symbol is taken to mean "finding a function." Note that David claimed that "$\int$ " meant that they were trying to find the "original equation" of the integrand function. (Note: as in other explanations, David often said "equation" when he meant "function.") Because of its similarity to the *function mapping* symbolic form for the entire symbol template, I simply call this a special case of that symbolic form for the "$\int$ " symbol, or the "function mapping with no limits" symbolic form. I want to make clear, however, that this is not equivalent to the *function mapping* symbolic form since the symbol template used is different. Because a form consists of a symbol template blended with a conceptual schema, these become two separate (though related) symbolic forms.

Alice further illustrated how this thinking is similar to that expressed in the *function mapping* form. She was explaining her computation of the integral "$\int_1^2 \frac{2}{x^3} - x^2\, dx$" and noted that if the limits 1 and 2 were not there, it would need a "+ *c*."

> Interviewer: So why, why would you need a plus c?
> Alice: Because, like we did some examples in class where you have a function and they might have the same end derivative, but there's some extra variable or constant to it. So it could be like 2 x plus 4 [writes $2x + 4$] or 2 x minus 2 [writes $2x$ -2], so they're completely different, but you might get the same, so that's why you need to add the constant. But because we have definite boundaries, we don't need it.

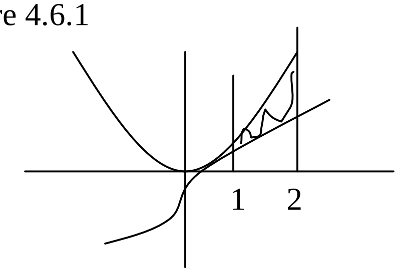
Alice talked about various functions that could all have the same "end derivative." Here we again see the ideas expressed in the *function mapping* symbolic form, where the integrand is conceptualized as having come from some other "original function." She noted that the functions "$2x + 4$" and "$2x - 2$" were both candidates for the "original function" of the same integral. Both of them would yield the same "end derivative" and hence would both work for the integral "$\int 2\, dx$."

Symbolic Form for the Template: $\int_{[]}^{[]} ([1] - [2])\, d[]$

The symbol template "$\int_{[]}^{[]} ([1] - [2])\, d[]$" is really just a special case of the regular integral template "$\int_{[]}^{[]} []\, d[]$." However, the extra layer of having the integrand represented as "[1] – [2]" provides the potential for additional meaning to be given to the integral. Several of the students took integrals in this form and equated it to finding the area in between two different curves in the plane. In one interview item, I wrote the integral "$\int_1^2 \frac{2}{x^3} - x^2 dx$" on the board and told them to compute it and talk about it while they worked on it. Bill was trying to explain what the "area" would be for this integral when Becky brought in the idea to think of it as two curves.

> Bill: Well I think it means it's just that, separately it would be, well they combine together to make one function, that in terms of… I'm trying to explain.
> Becky: I don't know if this is correct, but you could do it kind of like the f of x and g of x [*f(x)* and *g(x)*]. It's like f of x plus g of x [*f(x)* + *g(x)*].
> Bill: Yeah I guess you could do that, if you took 2 over x cubed [$\frac{2}{x^3}$], that's probably just … let's just say it's this [draws a random curve. See Figure 4.6.1]. And then you take x squared [draws the x^2 graph] and then you take all the area here… The area is just the difference between those two curves [draws in vertical lines at $x = 1$ and $x = 2$].

Figure 4.6.1



The integral in this scenario was only a single integral. However, Becky suggested looking at the integrand of this single integral as two functions instead of as one function. Bill caught onto this idea readily and began drawing out the two curves, one for the $\frac{2}{x^3}$ function and another for the x^2 function. Then he interpreted the integral, drawing on an *area* symbolic form, as the area in between these two functions. Thus Becky and Bill were able to see a special case in the integral symbol template and imbue a conceptual schema onto it.

Other students showed evidence of similar kinds of thinking. When given this same integral to work with, David said the following.

> David: The fact they're just sitting right next to each other, it's like the first problem [a reference to the "area between two wires" item], where you have two different equations, where you're trying to find the area between the two. I think what they're trying to do is find the area in between each other.

David had a consistent habit of using the word "equation" to mean "function," and I have no doubt that similarly he means "function" here. Thus David, like Bill and Becky, saw the special format of "[1] – [2]" in the integrand and gave it the meaning of finding "the area between the two" functions. To emphasize this point, he repeated himself, "I think what they're trying to do is find the area in between" the two curves. We have evidence that the students may contribute an extra layer of meaning to this special case of the integral. I call this the "area between [] and []" symbolic form.

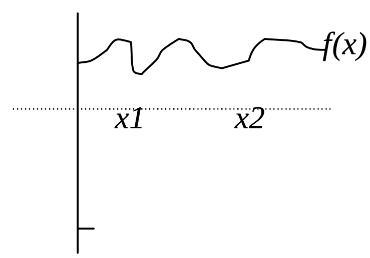
The Front Multiplier: "[] $\int$ "

As part of the interview items, I gave the students integrals that had a constant preceding the integral. For example, in one item, I gave the students the integral "$-2\int_D f(x)\,dx$" where *D* was stated to be the domain of the function *f*. There seemed to be three major ways of interpreting this "front multiplier" of the integral. One was rooted in a more visual-graphical conceptual schema, another was similar to more basic symbolic forms for multiplication (see chapter two), and the third dealt with the symmetry of the graph or function. First I discuss the more visual-graphical schema.

Clay and Chris were discussing this integral and had approached it by giving *D* a specific range of values, *D*=(*x1*,*x2*). Activating the *area* form, Chris visually represented the basic meaning of the integral by drawing a random, squiggly graph in the coordinate system, marking off *x1* and *x2*, and shading in the region. Then, without any prompting about the -2, he launched immediately into explaining the effects of the -2 in front of the integral sign.

Chris: If we multiply by negative 2, essentially that means that we're flipping this over negatively [points to graph] and we're multiplying it by a double magnitude. So this is what it would be [starts to make mark below *x*-axis. See Figure 4.6.2a].

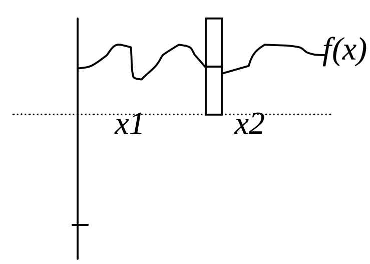
Figure 4.6.2a



Interviewer: So when you say that we're multiplying by double, what exactly are you multiplying by double?

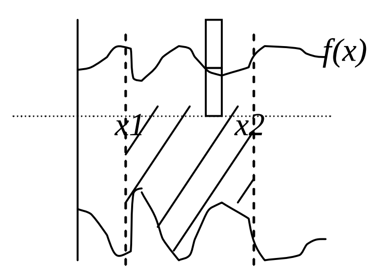
Chris: So each infinitesimal rectangle [draws in thin rectangle. See Figure 4.6.2b], we're doubling its y magnitude [doubles length of the rectangle] because… or we'd be doubling the area of it, and since dx, it's staying constant then that would mean we're doubling the y component.

Figure 4.6.2b



So we'd get [draws in "flipped over" graph. See figure 4.6.2c]. And so this would be our resultant value [says this while shading the region].

Figure 4.6.2c



Interviewer: Now let me ask you a question on that. So you're talking about doubling the area, and it's down below the x axis. Now that negative 2 is kind of sitting out in front of the integral, so why is it that that negative 2 is making that kind of a result?

Chris: So if we're dealing with an infinitesimal region, we could assume that this is area [writes *A* above the *f(x)dx* in the integral]. And either… we could think of it two different ways. We could think of it as being, um doubling the area and moving it negatively [makes hand motion like he's flipping something upside down]. Like I did here. Double that. Or you could just move it in and have it be doubling and multiplying by negative 2 on the function itself. So moving this up [puts chalk on the graph and moves chalk upward] and negatively [swings chalk down below *x*-axis] and then multiply by dx.

Chris was describing here a conceptual schema relating to the "front multiplier" of the integral. He explained in detail the fact that the -2 could interact with the function and influence the result of the integral by doing so. He conceptualized the -2 as "doubling the magnitude" of the function values (or "*y*-values," as he called them). Then these doubled values were "flipped over" to the region underneath the *x*-axis. This provided a new graph with a stretched out mirror-image of the first graph. There is good evidence that this is a well-compiled cognitive resource, owing to the fact that he was able to explain it both visually in the graph and algebraically through the function values. He called these "two different ways" to think about the -2. This significant conceptual schema assigned to the symbol template "[] $\int$ " provides another symbolic form that the students held in their cognition. It takes the multiplier in the box "[]" and blends it with the function in the integrand so that a new function is being used in the integral. This affects the size of the "area" being computed, or the height of the "rectangles" that are being analyzed. Note that this symbolic form is compatible with both the *area* and the *adding up pieces* symbolic forms. Chris was able to successfully explain the meaning of "[] $\int$ " from both an *area* perspective and a *representative rectangle* perspective. The key characteristic of this symbolic form is that the multiplier is taken as blended with the integrand before the integration takes place. I call this symbolic form the "melds with integrand" form.

Consider another episode where David and Devon were working with the same integral, "$-2\int_D f(x)\,dx$." In this episode, they had activated the *function mapping* symbolic form. During their work, David involved the -2 in the relationship between the integrand $f(x)$ and the "original function" $F(x)$.

> David: If we think that f of x equals f, x [writes $F'(x) = f(x)$], then I guess this would equal negative 2, f, x over the domain [writes -2 $F(x)$].

After a brief discussion of the domain D, I came back to the -2 and asked them to further describe what they meant by that. Devon then hinted at the idea that the -2 could be taken into the integral which would give a "new function."

> Devon: Either take the 2 in and then you can, like, take this as a new function, or you take this as some kind of value [points to -2] and this some kind of value [points to the integral]. It has nothing to do with the integration here.
>
> …
>
> Devon: That way the 2 is part of the function. So, it's one of the determined of the function. It's just that you can't take it out, you take that, it's the whole function.
>
> David: Or you could actually think of this also like, f of x [*f(x)*], like if you look at it, you could say f of x could have been some function, and like, let's say I had negative 6 x squared minus 4 x [writes $-6x^2 - 4x$], and this is a function, but you could have taken out the negative 2 and made it negative 2 times 3 x squared plus 2 x [writes $-2(3x^2+2x)$]. You know, and this constant would just come out and this would be the function [points to $(3x^2+2x)$].
>
> Interviewer: Alright. Anything else about…
>
> Devon: Yeah. If you take it as a whole function, the 2, like you would not see the 2, you don't even see the 2 here, it's just like a part of it.

David provided an example of how the negative 2 could have come from the function in the integral and that by removing it, he ended up with a "new function."

Devon furthered this by claiming that if you have the -2 on the inside, “you take it as a whole function” and that “you would not see the 2, you don’t even see the 2 here.” This language indicates that Devon saw the -2 as being able to completely blend with the function, providing a new function for the integrand. This is similar to the way that Chris earlier talked about using the -2 to double the function values and swing them over below the *x*-axis. In both cases, the student was conceptualizing the multiplier in front of the integral as “melding” with the function.

This episode with David and Devon provides evidence that the *melds with integrand* form is also compatible with the *function mapping* symbolic form. David had been discussing this integral in terms of *f(x)* being the derivative of *F(x)* and the effect the -2 would have on the relationship between these functions. While drawing on the *function mapping* form, David was able to draw on a *melds with integrand* symbolic form for the multiplier -2. His explanation was that the algebraic form of the function would change as the -2 was pulled out of the integral. That is, it would change from $-6x^2 - 4x$ to $3x^2 + 2x$. The key, however, is that David was still interpreting the -2 as something that blends with the integrand, or that can be taken out of the integral. Thus this symbolic form appears to be compatible with *all* of the major symbolic forms I have presented.

In this episode, Devon also talked about another way to view the multiplier, “$[]\int$.” In this other view, the multiplier is taken to just be “some kind of a value” that might have “nothing to do with the integration.” In this case, the value of the multiplier and the value of the integral are simply multiplied together like any other two values might be. Devon offered his ideas about how the multiplier would be conceptualized looking at it from this perspective.

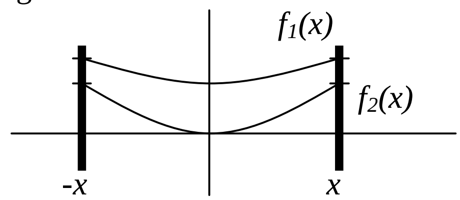
Interviewer: If you do it that way, you said take that as a value and take that as a value, then what's the relationship between this value and that value?
Devon: This and this [points to -2 and then the integral]? They don't have to have any kind of relation, it's just a random thing, times them together. Multiply together.

In this case, there is no special relationship between the -2 and the integral. Instead, it is merely a multiplication between two values. The integral would need to be computed to get its "value" and then that would be multiplied by -2, in this example. In this case, this reduces the symbol template "[] $\int$ " down to the symbolic forms for basic multiplication "[] $\times$ []." The symbols filling in the second box may look more complicated, being an integral, than other basic expressions and equations, but the conceptual schema is the same. The two are simply "multiplied together." I call this the "scales the result" symbolic form for the integral.

There was also evidence that this "front multiplier" could be interpreted as correlating with the graph of a function that was symmetric about a vertical line. There were few instances of this type of thinking during the interviews, but there was one important occurrence during the mathematics-day interview with Clay and Chris that helps explain this conceptual schema applied to the symbol.

Clay and Chris were working on the problem of finding the area between two curves in the plane. They had named the two curves $f_1(x)$ and $f_2(x)$ and had started to create an integral that would match the scenario. They drew a coordinate system where the y-axis split the curves directly in half, and used the *x*-values "$-x$" and "x" for the left and right endpoint (see Figure 4.6.3). (Note: their use of the limits $-x$ and x is technically incorrect, but they do not interfere with the present discussion.)

Figure 4.6.3



They had decided that based on the shapes of the curves, they could use the functions "$f_1(x)=\frac{x^4}{4}$" and "$f_2(x)=\frac{x^2}{2}$" (they later revise these—for our purposes here, the functions they use do not matter) and agreed that they would be able to calculate the area using the integral "$\int_{-x}^{x}\frac{x^2}{4}-\frac{x^2}{2}dx$." As they started to write this down, Chris noted,

> Chris: Now we could just say from negative x to x, take the two integrals, subtract the difference [he writes $\int_{-x}^{x}\frac{x^2}{4}-\frac{x^2}{2}dx$]. Or we could just double from 0 to x because it's simpler [erases –x, writes $2\int_{0}^{x}\frac{x^2}{4}-\frac{x^2}{2}dx$]. Because the right side is a mirror of the left and stuff.
>
> Clay: Right, 'cause it's an even function.

In the middle of writing the integral, Chris decided that they could take advantage of the fact that the curves were symmetric about the *y*-axis to come up with an "easier" integral. Instead of having an integral from –*x* to *x*, he could have an integral from 0 to *x*, which would just be doubled in order to recapture all of the area between the two curves. We can see evidence that this front multiplier was connected with the symmetric graph. Later I asked them to come back and explain in more detail why they could do that and what motivated them to do that.

> Interviewer: In your integral, you had started with a negative x up to x, and you decided, let's just change that, and you put a 2 in front and you changed the

bottom one to 0. So talk a little more about that, as far as why can you do that? What motivated you to say that's a good thing to go ahead and do?

Clay: Well, it's, like, an even function. So you know that whatever is on this side [points to the right half], it's the same as that [points to the left half].

Chris: Yeah, so if you take an integral, the difference between the right side [moves hands over right half of the picture] and you add it to the integral, the difference on the left side [moves hands over left half of the picture], they're going to be two equal values, so we might as well just make it a 0 and multiply by 2 for simplicity reasons.

It's clear from their explanations that they saw the symmetry as a motivator to dividing the integral up and multiplying the result by 2. They explained that the meaning of the 2 is tied to this symmetry, since the right half and the left half are equal to each other. They produce two equal parts of the area, so "for simplicity reasons" you could find out the integral for only part of the area, and then double it to recapture the whole area. I call this the "symmetric graph" symbolic form for the symbol template "[]$\int$."

The Dependence of the Differential "d[]"on either the Integrand or the Domain

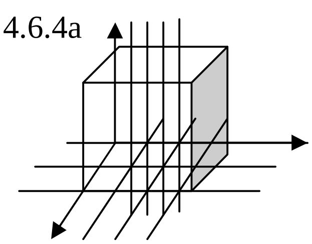
The next symbolic form is specific just to the interaction of the integrand and the differential, "[]*d*[]." While it is related to the *adding up pieces* and *add up then multiply* symbolic forms, it constitutes a separate piece of thinking that is not necessarily inherent in either of those symbolic forms. Thus I claim that it constitutes a separate symbolic form, attaching a meaning specifically to "[]*d*[]." Here the differential "*d*[]" is seen as dependent on the integrand. During the physics-day interview, Devon brought this idea up several times. In the first instance, Devon was working on an item in which he was trying to determine the mass of a box with varying density. He drew a figure of a box in a three-dimensional coordinate system and then created the integral "$\int_V \rho(r)\, dV$." I prompted him to explain how the *dV* corresponded to the picture he had drawn.

Interviewer: What would dV be over in your picture that you have over there?
Devon: In this case it really depends… It's just dV as corresponding to the rho r [$\rho(r)$]. Like I said, you could either integrate the horizontal and vertical way, or if…it depends on how the trend of the density is. Let's say, it depends on the distance from the origin to that point, then dV, maybe I'd use the coordinate system, the polar system, so dV would be a different shape. Like I would have a different way to slice it. Generally, I would say that it's the dV that's corresponding to the rho of r [$\rho(r)$].

Here Devon claimed that there was a "correspondence" between dV and $\rho(r)$. However, there is something unique about this "correspondence." Most students, while drawing on an *adding up pieces* symbolic form, used the differential as a starting place to talk about the representative rectangles that were created. They would draw in a rectangle, whose width was dependent on the differential. While it is true that Devon was connecting the differential dV with the way the domain would be "sliced up," it appears that the nature of these slices could not be determined without some previous knowledge of the integrand, in this case $\rho(r)$. Hence dV is dependent in some way on the integrand. He then continued to provide an example of how the partition and the differential dV were connected.

Interviewer: So maybe do a couple of examples, one where it is polar, or one where it's not polar or something. What are some ways you could represent dV over in your picture?
Devon: Let's say this is case 1, and the function rho r [$\rho(r)$] is something related just to a x plus b y plus z [writes $ax+by+z$]. And then I would, it would be easy for me to slice it, I'll show you a graph here, just vertical and horizontal [draws a box]. Just this way and then… [draws several lines going left to right, then back to front, then top to bottom. See Figure 4.6.4a].

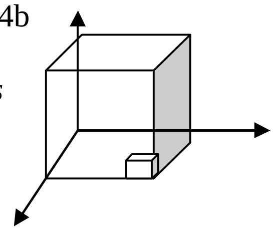
Figure 4.6.4a



> Devon: It would be like, every little dV would be a small piece of box, a small box [draws a small box inside the larger box. See Figure 4.6.4b]. And really I would guess it would be a triple integral, with like, it would be dx, dy, dz. That would be one case.

Figure 4.6.4b

(*note: lines removed for clarity*)

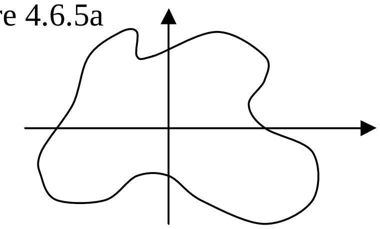


Devon's description showed that the integrand directly impacted what *dV* was. It was not until he instantiated a particular function for $\rho(r)$, namely "$ax+by+z$" that he was able to move forward on his partition. Thus the partition and the differential *dV* were partially dependent on $\rho(r)$. Because it was "easy" to slice it up along the directions of the three axes, the natural shape for *dV* ended up being a "box." Devon then tried to come up with another example of a way to slice the domain up, but had difficulty coming up with one because "this is a box" and for that reason "this is the easiest way to slice it." He explained his difficulty in choosing another way to partition.

> Devon: I can't really come up with a rho [$\rho(r)$] that I have to slice another way. But like the principle is, you have to be corresponding to the rho r [$\rho(r)$]. Let's say if this is not a box, but like a sphere, I can have multiple ways to slice it.

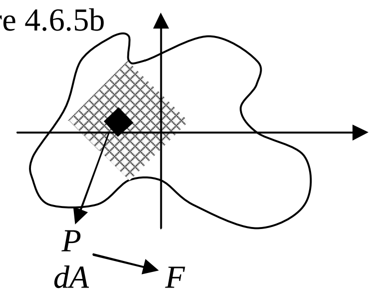
Devon stated that his difficulty in finding another way to partition the domain was based on the fact that he "can't really come up with a rho [$\rho(r)$]" that would require a different type of "slicing." The method of partitioning, and hence the shape of *dV*, is dependent on the integrand, $\rho(r)$. "You have to be corresponding to the rho r [$\rho(r)$]," he claimed. If a particular function or domain lent itself well to a spherical-type domain, *then* he could find another way to slice it. Later in the interview, Devon was working with the integral equation "$F = \int_S P\,dA$" and was trying to explain why the integral calculated the total force on a surface. After explaining the relationship $F = PA$, he drew a picture and represented the surface, S, by a randomly drawn shape in the plane (see Figure 4.6.5a).

Figure 4.6.5a



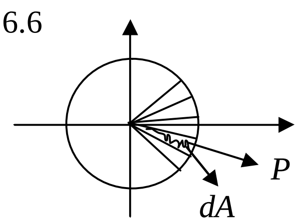
Devon: Every…let's say we slice it this way [slices region into a grid. See Figure 4.6.5b]. And this little piece would be dA [shades in one square]. At every little piece you can find the pressure at that point. You can find the P and find the area of that little piece [draws arrow, writes *P* and *dA*], and multiply them and you get the force at that point [draws arrow and writes *F*].

Figure 4.6.5b



Devon: And again, that's not the only way you could slice it. Like you could do it like a radian, do a radian thing or system. Let's say it's a circle [draws a circle and divides it. See Figure 4.6.6]. Like every piece of pizza can be dA as well [draws arrow, writes *dA*]. And then you find the P at that, of that slice [shades in a piece, writes *P*].

Figure 4.6.6



Devon again explained that the differential *dA* was dependent on the shape of the domain. Here we have an interesting problem, which is that Devon does not invoke the integrand, *P*, as the main factor in determining the partition, and hence in determining *dA*. Instead, he spoke only about the shape of the surface, *S*. Thus in contrast to the previous episode, he used the domain as the means of getting the shape of the *dA*. While these two episodes are at odds in terms of what really determines the way *dA* comes into the picture, they both agree in the fact that Devon saw the *dA* as being dependent on something else. This is a significant departure from the way that most other students approached the problem, where instead the differential was the thing *responsible* for the partition, not *dependent* on it. Devon's descriptions clearly show that he favored the latter.

This symbolic form for "[]*d*[]" is related in some ways to the *adding up pieces* or the *add up then multiply* symbolic forms because of the attention given to representative "slices" that explain what is being "added up." However, this form for "[]*d*[]" should not be confused as being an inherent or essential part of either of these broader symbolic forms. There is evidence that many students held the *adding up pieces* and *add up then multiply* symbolic forms without necessarily having this specific conceptual schema attached to the symbols "[]*d*[]." Most students discussed the differential as the major component in creating the representative rectangle, not as being dependent on it.

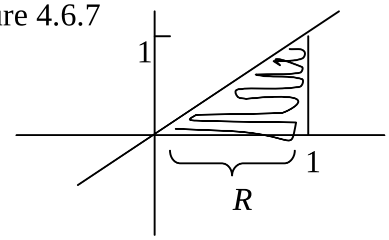
The exception to this is Devon, who used this thinking in several places during the physics-day interview. Hence it could be said that either he has integrated this into his *adding up pieces* symbolic form, creating a new "advanced" version of this form, or he has compiled a separate symbolic form specifically for the differential "*d*[]." All we can say for sure is that Devon drew on an understanding of the differential "*d*[]" as being dependent on either the integrand or on the shape of the domain. However, we know Devon activated the *adding up pieces* symbolic form during both the math- and physics-day interviews, but he only drew on this understanding of the differential during the physics-day interview. Consequently, it appears that Devon has a separate symbolic form for the differential since he seems able to draw on the *adding up pieces* form without necessarily activating this thinking. That is, to Devon this form for the differential "*d*[]" likely exists as a separate symbolic form, despite its close relationship to the *adding up pieces* symbolic form. I call this the "dependent differential" symbolic form.

The Symbol "$\int_{[]}$" as Representing a Region in Space

The symbol template "$\int_{[]}$" deals with an integral whose domain is summarized by one symbol, as in the integral $\int_S P\,dA$. There were two interpretations that the students placed on this symbol structure, though I would only consider one of them to be a "symbolic form." I discuss the other interpretation in the next section about other resources the students drew on. In this section, I describe this symbol as being interpreted as a region marked off in space. That is, the domain of the function consists of some two-dimensional object in the plane, or a three-dimensional object in three-dimensional space. In one example, Adam and Alice were discussing the integral "$-2\int_D f(x)\,dx$" and Adam had just explained to Alice that the *D* could represent any domain for the function *f(x)*. He used the examples "*D*:(1,2)" and "*D*:(2,3)" to explain the meaning of *D*. However, he then added a different thought about the meaning of that symbol.

Adam: I guess just as far as the domain goes, sometimes there's a double integral [writes $\iint_R$]. There would be an R right here. Which would be the region, like right here [draws a graph and shades in the region. See Figure 4.6.7].

Figure 4.6.7



Adam: So it would be like…dx…dy [writes $\iint_R \; dxdy$]. So the x would go from, the region for the x would be 1, 0, or something. 0 to 1, right here. But you could say, instead of writing these out, you could just put R [writes *R* underneath the graph] for the region. So I guess that's similar to the domain.

Adam takes the integral to potentially mean a "double integral." This led him to discuss a domain that would exist in a two-dimensional plane, as opposed to a one-dimensional line. He claimed that the *D*, which he renamed *R*, "would be the region." He then drew a graph to explicitly demonstrate what "the region" might look like. He marked off the boundaries of the region, and used the underlining bracket to show that the *R* represented the entire region. By rewriting the integral as "$\iint_R \; dxdy$" he provided evidence that he specifically connected the subscript *D* (or *R*) as the region in the plane. Thus the symbol means more than an interval of "*x*-values," but means an entire bounded region in the plane. The *R* stands in place for the shape of the region. I call this the "region in space" symbolic form.

4.7 Other Cognitive Resources of Interest

In addition to the number of symbolic forms that I have presented, there were two other significant cognitive resources that students drew on during the interviews. I do not give these cognitive resources the label of "symbolic form" because they do not necessarily mesh a specific symbol with a specific conceptual schema. However, they were stable elements of knowledge that the students drew on and worked with in their thinking.

The Symbol "$\int_{[]}$ " as Shorthand for "$\int_{[]}^{[]}$ "

The first of these deals again with the symbol template "$\int_{[]}$." Throughout the interview this notation was very consistently thought of as a shorthand notation for the limits "$\int_{[]}^{[]}$." This does not necessarily comprise a symbolic form, because there is not much by way of a "conceptual schema" that is being attached to the "$\int_{[]}$ " template. Rather, this symbol template is seen as equivalent to, and hence replaceable by, the similar symbol template "$\int_{[]}^{[]}$." It constitutes a stable piece of knowledge that students may hold and then activate when thinking about integrals.

I gave Adam and Alice the integral "$-2\int_D f(x)\,dx$" and asked them to talk about what it meant. Alice had some questions about what the *D* meant, because she said she had not seen that notation before. Adam offered an explanation of *D*.

> Adam: Well I see, well it says with the domain. I picture, there's no value, there's no boundaries, there's no values for the boundaries. You just put D to represent what boundaries there are going to be. So, in an assignment you could put, like, the domain would be like 1 to 2.
>
> …
>
> Adam: Let's say like 1, 2 or something like that [writes *D*:(1,2)]. Like I would imagine in a book, it would just, like, list a couple of different domains [writes *D*:(2,3)].
>
> Alice: Now, since it's at the bottom, does that mean the highest one goes here and the lowest one here, or does it matter? Like could you write it where you have the D up here [points to top of the integral sign]?
>
> Adam: No, this just represents, this isn't really a value for the bottom boundary. It's just saying between… for this domain. Like sometimes there's a

> question: what's the domain for the function? And it goes from 0 to infinity or 0 to 2. This is just saying the domain would be such and such, and it would give you something. Right here it's ambiguous to what the domain is.

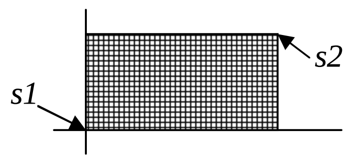
When I asked Adam to rewrite the integral using those domains he chose, he wrote "$-2\int_1^2 f(x)\,dx$" and "$-2\int_2^3 f(x)\,dx$." Adam saw the *D* as standing in place for the "regular" limits of integration, or two *x*-values. The *D* was there to represent the generic domain, and it was to be replaced with two numbers as soon as those numbers were discovered. For instance if "an assignment" told him to find the domain of the function and then integrate between those values, the *D* would be replaced by the interval that represented the domain for that function. If the domain of the function was found to be something like (1,2) or (2,3), then the symbol template "$\int_{[]}$" could be replaced with "$\int_1^2$" or "$\int_2^3$."

This cognitive resource seems innocuous and is, in fact, a correct way to interpret the symbol "$\int_{[]}$" much of the time. However, there is strong evidence that the students overgeneralized this resource to most situations where this symbol template was used. In the physics-day interview, the students were asked to interpret the integral equation "$F = \int_S P\,dA$." When Chris and Clay were given this interview item, they drew a two-dimensional rectangular surface and eventually "sliced it up" into a grid of smaller rectangles (see Figure 4.7.1). During their work, they changed the integral to be "$\int_{s1}^{s2} P\,dA$." I asked them to discuss the limits *s1* and *s2* that they put on their integration.

> Interviewer: It was integral with S, and you translated that into an integral with an s1 and an s2. So what exactly are s1 and s2?

> Clay: s1 is the minimum, where you're starting from, I guess. And s2 would be the last one you're integrating, but that would be inclusive of all the ones in between.
>
> Chris: s1 would be the first square here [points to lower left-hand corner of the surface, see Figure 4.7.1], and then s2 would be this square here because it's where both x and y reach the maximum [points to upper right-hand corner of the surface].

Figure 4.7.1

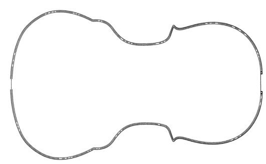


Like Adam, Chris and Clay viewed the symbol template "$\int_{[]}$," which in this case was "$\int_S$," as generic for the symbol template "$\int_{[]}^{[]}$." They changed the integral to have two limits, *s1* and *s2*, where the integration was to happen from *s1* up to *s2*. While there is certainly some merit in what they have said, as one could devise a summation that ranged from *s1* to *s2* in the plane using an appropriate ordering, it also does not regard "$\int_S$" as denoting a region or boundary in the plane. Instead, one symbol template is merely exchanged for the other. This way of thinking does not place a conceptual schema on the symbol "$\int_{[]}$," but rather views it as an equivalent symbolic expression to the limits of integration, "$\int_{[]}^{[]}$." Thus it does not attain the status of a "symbolic form" and I instead consider it simply, and more generically, to be a cognitive resource in the students' knowledge system. I call this cognitive resource the "shorthand for limits" resource. I discuss the implications of this resource more in chapter five.

Facing the Other Way and Negative Area

Some of the students demonstrated that they understood certain regions in the plane to contain “negative” area. Here I explain the connection “negative area” had with symmetry in their discussions. David and Devon were working on a problem and were trying to find the area of a region that was in the shape of a violin, as given in the following item.

> This picture shows the outline of a violin body. If you wanted to know the area of this shape, how could you figure that out?



They quickly agreed that an integration would be helpful, though David began his explanation with a warning that this integration needed to be done carefully. He said that otherwise they could just end up with 0 for the result.

> David: I think you could [use an integral], but you’d have to be careful because, say you graphed it like this [draws axis through middle. See Figure 4.7.2]. Then if you use, like, integration without being attentive you just get 0. Because this is positive and this is negative [writes + and – in two halves]. I would suppose that this would be like negative f of x [writes –*f(x)*] and this would be positive f of x [writes *f(x)*].

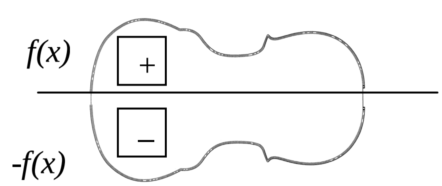


Figure 4.7.2

We can see that David had some idea of a positive area versus a negative area. However, an important feature in his explanation is that the symmetry of the shape seemed to prompt him to think of negative area. There was nothing about the interview item that said anything about negative area, nor a choice of axes. But David recognized that if this particular axis was chosen along the line of symmetry, there would be equal portions of "positive" and "negative" area. As they continued to work on this concept, David revealed more about the connection between symmetry and negative area. During their work they created the integrals "$\int_a^b f(x)dx + \int_a^b f(x)dx$." I then prompted David to go back and discuss the negative and positive areas in more detail.

> David: This [points to top half of figure] is pretty much the same as this one [points to bottom half of figure], just flipped around. So I would suppose that they're identical, the only thing is that this is negative, so it's facing the other way. It's like a complete opposite reflection… And let's say we were just to ignore the fact that… Y'know, let's say we did this [points to integral $\int_a^b f(x)dx + \int_a^b f(x)dx$], but instead we put a negative here [changes to $\int_a^b f(x)dx + \int_a^b - f(x)dx$]. We'd probably get a negative answer that is exactly the same as this one. So positive x minus x. And you get 0. They just cancel out. So I think that just paying close attention and knowing that this entire area is not 0, it's the combination of two halves.
>
> Devon: Yeah, we're doing that integration, it's actually we calculate the signed value of it. So this one would have a positive signed value, a signed area. And this one has a signed area too, and this, they're equal but opposite, so they would cancel out if you don't do it carefully.

There are two important ideas that come out of their discussion. David talked about the bottom half of the figure as being "flipped around," a "complete opposite reflection," and as "facing the other way." His consistent use of these phrases makes it clear that a salient feature in this item for David was the symmetry between the top and bottom halves of the figure. Then he took the region that "faces the other way" as the negative area. Thus David saw some kind of directionality in the assignment of positive or negative area. Devon called this "signed area," meaning that each area actually has a positive or a negative sign attached to it. The symmetry of the figure, which prompted the notion of directionality, specifically activated the concept of "negative area." There appears to be a link between graphical symmetry and negative area in the way David and Devon were thinking about this problem.

The second idea connects a curve that "faces the other way" to a function with negative values. David and Devon had assigned the top half of the violin body the function $f(x)$ and then given the bottom half the function $-f(x)$. The "negative area" that they got from the integral corresponds to this "negative function." By stating this, David and Devon showed that there is a cognitive connection between the following: (1) graphical symmetry, (2) curves that "face the other way," (3) negative area, and (4) negative function values. The symmetry of the graph activated knowledge that cast the bottom half of the object as a curve that "faces the other way," which in turn cast the area of the bottom half as negative area. Thus we see a connection between the symmetry and the idea of negative area. This is not a symbolic form, because these students are not attaching this meaning to symbols, but are rather ascribing meaning to a graphical feature. I claim that this is, generically, a cognitive resource instead. I call it the "facing the other way" resource.

4.8 Summary of Results

In chapter five I elaborate further on the activation of certain symbolic forms during the mathematics-day interview and the physics-day interview. I also elaborate on the similarities and differences between the symbolic forms activated during the two interviews. In this section I present a summary of the symbolic forms that were activated during each interview item for the mathematics-day and physics-day interviews. In order to do this, first I display a summary of the various symbolic forms that students drew upon during the interviews. I want to be clear that this does not represent an exhaustive list of all possible symbolic forms relating to the integral template. However, they are the ones that I had evidence for during the interviews. The symbolic forms I detected are listed in the following, Table 4.8.1.

Symbol Template	**Brief Description of Conceptual Schema**	**Name of Symbolic Form**
$\int_{[]}^{[]}[]d[]$	Integrand represents a graph and the differential determines the axis used. Limits and axis create the actual sides of a fixed, static region in the plane. The integral is the area of this region.	Area
$\int_{[]}^{[]}[]d[]$	Integrand and differential create rectangles, each with a small piece of the quantity. The integral is an addition over these pieces. The limits indicate the starting and stopping point of the addition.	Adding Up Pieces
$\int_{[]}^{[]}[]d[]$	Integrand originated from another function. The link between them rests in the differential. The limits represent values of this original function whose difference is measured.	Function Mapping
$\int_{[]}^{[]}[]d[]$	Differential determines a partition in which a small quantity of the integrand exists. These small quantities are added up and then the result is multiplied by the variable of the differential.	Add Up Then Multiply
$\int []d[]$	The integral yields a function, which is a generic version of a numerical value. This result is waiting for limits to be attached so a more specific numerical result can be calculated.	Generic Answer
$\int []d[]$	Integrand originated from another function. The link between them rests in the differential. The meaning of the integral is to search for this original function.	Function Mapping with No Limits
$\int_{[]}^{[]}([1]-[2])d[]$	Similar to the *area* form. However, the unique structure of the integrand suggests that the fixed area is situated in between two curves in the plane.	Area between [] and []
$[]\int$	The multiplier on the front can be combined with the integrand function. This alters the area, rectangles, or function involved in the integration.	Melds with Integrand
$[]\int$	The multiplier on the front is seen as a value to be multiplied with the value of the integral, thus scaling the result. This also may reduce to symbolic forms for multiplication.	Scales the Result
$[]\int$	The function is symmetric, yielding the possibility of finding a smaller part of the integral and then doubling it (or some other magnification) in order to recapture the entire integral.	Symmetric Graph

[]*d*[]	The differential is dependent either on (1) the function of the integrand or (2) the shape of the domain.	Dependent Differential
$\int_{[]}$	The symbol "$\int_{[]}$" is seen as representing a region in space.	Region in Space
$\int_{[]}$	The symbol "$\int_{[]}$" is seen as shorthand for "$\int_{[]}^{[]}$."	Shorthand for Limits*
Graphs	Certain parts of the figure are deemed to be "facing the other way" which corresponds to negative area.	Facing the Other Way*

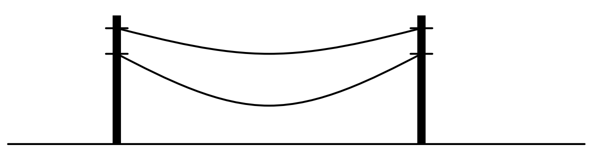
Table 4.8.1: Summary of the symbolic forms detected during the interviews
* These two do not represent symbolic forms, in that they do not constitute a conceptual schema blended with a symbol template. However, they appear to be stable cognitive resources.

To facilitate a summary of which symbolic forms the students drew on during each interview item, I now recap the interview items that were given to the students during each interview. Not every item was given to every pair of students, depending on the way the interviews progressed. See chapter three for a more detailed discussion on the interview items. Here is a listing of the interview items presented to the students for quick reference. Note that Ethan did not receive any interview items from the mathematics-day interview.

Interview Items for the Mathematics-Day Interview

<u>ITEM Math1</u> (presented to all student pairs, excluding Ethan)

Two wires are attached to two telephone poles (see picture). Suppose we wanted to know the area between the two wires. How could you figure that out?



<u>ITEM Math2</u> (presented to all student pairs, excluding Ethan)

$\int_1^2 \frac{2}{x^3} - x^2 \, dx$ Compute and then discuss this integral.

<u>ITEM Math3</u> (presented to all student pairs, excluding Ethan)

I want you to look at each of the following integrals and talk about what they mean. Talk about each one individually.

$$\int \sin(x) \qquad\qquad \int_2^0 e^x \, dx$$

<u>ITEM Math4</u> (presented to all student pairs, excluding Ethan)

I want you to look at each of the following integrals and talk about what they mean. Talk about each one individually.

$$\int dx \qquad\qquad \int \sqrt{t} \, dx$$

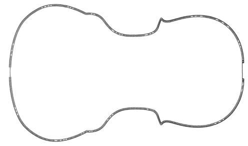
<u>ITEM Math5</u> (presented to all student pairs, excluding Ethan)

Suppose we had a function *f(x)* with a domain D. What does this integral mean?

$$-2\int_D f(x) \, dx$$

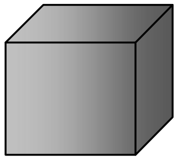
<u>ITEM Math 6</u> (presented only to David and Devon to produce more data)

This picture shows the outline of a violin body. If you wanted to know the area of this shape, how could you figure that out?



Interview Items for the Physics-Day Interview

ITEM Physics1 (presented to all student pairs)



This shows a box with varying density. (dark = more dense, light = less dense) Suppose you wanted to know the box's mass. How could you figure that out?

ITEM Physics2 (presented to all student pairs)

The durability of a car motor is being tested. The engineers run the motor at *varying* levels of "revolutions per minute" over a 10 hour period. Denote "revolutions per minute" by R.

What is the *meaning* of the integral $\int_0^{600} R\,dt$?

ITEM Physics3 (presented to all student pairs)

A 2-dimensional surface (S) experiences a non-uniform pressure (P) and we want to know the total force exerted. We can use the surface's area (A) to compute this through the integral:

$$F = \int_S P\,dA\,.$$

Why does this integral calculate the total force exerted?

ITEM Physics4 (presented to all student pairs, excluding Ethan)

We know from kinematics that acceleration and velocity are related by $a(t) = \frac{d\,v(t)}{dt}$. We can rearrange this equation and integrate to get the equation

$$\int a\,dt = \int dv$$

What does this equation mean? Why are these two terms equal to each other?

ITEM Physics5 (presented to all student pairs, excluding Adam/Alice and Ethan)

F_y is used to denote the amount of a force in the y-direction. ΔU is used to denote the change in potential energy. These two concepts are related through this equation:

$$\Delta U = -\int_{y_i}^{y_f} F_y \, dy \, .$$

Explain this equation. What does each part of the equation/integral mean?

ITEM Physics6 (presented to all student pairs, excluding Bill/Becky, and Ethan)

This represents a metal bar with varying mass along its length
(lighter = less dense, less mass / darker = more dense, more mass)

How could you figure out the center of mass for the bar along its length?

Summary of Symbolic Forms Activated during the Interviews

I now display a summary of the symbolic forms the students activated during each of the interview items. I list the items of the mathematics-day and physics-day interviews, as well as the students' names, and in each cell I put the symbolic forms that I have evidence those students activated during that item. I want to note here that it is absolutely possible that the students drew on more than the symbolic forms I have listed for them. However, I am limited to what they verbally stated or wrote down and can only report the symbolic forms that I have evidence for.

For each student pair, I list the symbolic forms, or the other two cognitive resources, that are supported by the data for each interview item. If the students drew heavily on a particular symbolic form, I display that form in bold lettering. At times it was clear that only one of two students was drawing on a particular symbolic form, and that the other was not. If this is the case, I make that distinction in the table by listing the symbolic form next the student's name in the cell. Also, occasionally there is scant evidence that a symbolic form may have been activated in a student's thinking, but the evidence is not necessarily robust. I indicate this through the use of parenthesis to show that that form may have been active, though I cannot substantiate it with confidence. If a student did not provide any evidence of symbolic form activation during a particular interview item, I denote that with a dash. The summary of the symbolic forms that the students drew upon in the mathematics-day interview is listed in Table 4.8.2 and the physics-day interview in Table 4.8.3.

Students	Adam/ Alice	Becky/ Bill	Chris/ Clay	David/ Devon
Math 1	**Area**, Adding Up Pieces	Area, Area between [] and [], Bill: **Adding Up Pieces**, Becky: Function Mapping	**Area**, Adding Up Pieces, Symmetric Graph, Area between [] and []	**Area**, Area between [] and [], Devon: Adding Up Pieces, David: Function Mapping
Math 2	**Area**, **Function Mapping**, Facing the Other Way, Generic Answer, Function Mapping with no Limits	**Area between [] and [],** Bill: Area, Becky: Function Mapping	**Area**, Area between [] and [], **Adding Up Pieces**	**Function Mapping**, Area Between [] and []
Math 3	**Area**, **Function Mapping**	Bill: **Area**, Bill: (Function Mapping with no Limits), Bill: Melds with Integrand, Becky: **Function Mapping**, Becky: Generic Answer	Function Mapping, Generic Answer, Area, Adding Up Pieces	**Function Mapping**, (Generic Answer), **Function Mapping with no Limits**, Adding Up Pieces, Dependent Differential, **Area**
Math 4	**Function Mapping,** Area	Function Mapping, Bill: **Area**, Bill: Melds with Integrand	Function Mapping, **Adding Up Pieces**	Function Mapping with no Limits
Math 5	Scales the Result, Area, Adam: Shorthand for Limits, Adam: Region in Space	Scales the Result, Bill: Shorthand for Limits, Bill: Adding Up Pieces	Shorthand for Limits, **Area**, **Melds with Integrand**, Adding Up Pieces, Scales the Result	**Function Mapping**, Shorthand for Limits, **Melds with Integrand**, Scales the Result
Math 6	N/A	N/A	N/A	**Area between [] and [], Facing the Other Way**

Table 4.8.2: Breakdown of symbolic forms (or other resources) by students, by item

Students	Adam/Alice	Becky/Bill	Chris/Clay	Devon (David was not present)	Ethan
Physics 1	Adam: Adding Up Pieces, Alice: --	Bill: **Add Up Then Multiply**, Becky: --	**Add Up Then Multiply**, Melds with Integrand,	Region in Space, **Dependent Differential**, **Adding Up Pieces**	**Add Up Then Multiply**, Region in Space
Physics 2	**Function Mapping**	Bill: **Area**, Adding Up Pieces, Becky: (Function Mapping)	Area, **Adding Up Pieces**, (Function Mapping)	**Adding Up Pieces**, (Function Mapping)	**Add Up Then Multiply**, Area
Physics 3	**Adding Up Pieces**, Shorthand for Limits	Shorthand for Limits, Adding Up Pieces, Bill: Add Up Then Multiply	Shorthand for Limits, **Adding Up Pieces**, Area	**Adding Up Pieces**, Region in Space	**Add Up Then Multiply**, Area
Physics 4	**Function Mapping**	Becky: **Function Mapping**, Bill: Adding Up Pieces, Bill: Area	**Adding Up Pieces**, Generic Answer, (Area)	**Adding Up Pieces**, (Function Mapping), (Generic Answer)	N/A
Physics 5	N/A	Becky: Function Mapping, Bill: Area, Bill: Adding Up Pieces	**Adding Up Pieces**, Scales the Result	**Adding Up Pieces**	N/A
Physics 6	Area, Adam: Adding Up Pieces,	N/A	Adding Up Pieces	**Adding Up Pieces**	N/A

Table 4.8.3: Breakdown of symbolic forms (or other resources) by students, by item

CHAPTER 5: DISCUSSION AND CONCLUSIONS

This chapter discusses the results as presented in Chapter four. The symbolic forms are examined (and other cognitive resources I documented) that were drawn on by the students throughout the interviews, focusing first on the symbolic forms students drew upon during the mathematics-day interview and then focusing on the symbolic forms of the physics-day interview. This leads to a discussion on the intersection and disjunction between the symbolic forms activated in these two different contexts. The relationships between the nature of the interview items and the symbolic forms that the students drew on during the interviews are examined. Additionally, the apparent difficulties in moving from univariate (or one-dimensional) integrals to multivariate (or multi-dimensional) integrals are explored. Finally, this chapter looks at the relationships between the presentation of integrals in mathematics and physics textbooks and the symbolic forms that the students activated during the interviews. Based on this recommendations are made for future research that can be done to further the understanding of how students apply mathematics to physics and engineering and how curriculum design can be informed through this research.

5.1 Discussion of Resource Activation during the Interviews

Resource Activation in the Mathematics-Framed Interview

Table 4.8.2 in Chapter four details the symbolic forms and other cognitive resources that the students gave evidence of drawing on during the mathematics-framed interview. First, I want to remark about the variety of symbolic forms activated. Of the four “major symbolic forms,” three of them were activated by the students from every pair. This provides one interesting result from this interview: The problematic *add up* *then multiply* symbolic form is absent from the students work during the mathematics-day interview. That is, there was not a single pair of students that showed any evidence of thinking of the integral in terms of adding up the “quantity” in the integrand and then multiplying that resultant sum by the “quantity” indicated by the differential.

As discussed in Chapter two, the framing employed by the students will affect the choice of resource activation. Descriptions of traditional mathematics courses state that they often contain equations and expressions that do not necessarily carry any physical meaning (Torigoe & Gladding, 2007; Dray & Manogue, 2004b). Therefore, in some sense there *is* no “quantity” of the integrand (by “quantity” I am referring to the object that the variable refers to, such as a force, a pressure, a velocity, and so forth). There is also no apparent “quantity” indicated by the differential. Rather, the symbols used for the integrand and differential most often represent numerical values. The integrand is comprised of some function that contains a dependent variable. The differential is usually an indication that the computation of the anti-derivative should be done with consideration to that particular variable. Thus, it could be that as students are given expressions in a mathematics-framed setting, they are not looking for the variables to represent any particular quantity. Thus we might expect a lack of the *add up then multiply* form during the mathematics-framed interview.

Next, as seen in Table 4.8.2, the *area* symbolic form figured heavily in the students’ thinking during the mathematics-framed interview. Each interview pair activated this particular form multiple times during the interview. The activation of this form would not be surprising for the first interview item, since it specifically asked the students to determine a way to figure out the area between the two wires. It is natural that if the students created an integral, the integral would be connected to the concept of area. However, this symbolic form was activated multiple times throughout the interview, and was widely present across all of the pairs of students. This is true even for interview items that simply asked the students to compute an integral. There is one exception to this, namely Becky, who did not often draw on the *area* symbolic form during either interview except in a few places where she agreed with the work that Bill had already produced. I discuss the case of Becky later in this chapter. Additionally, the *area between* [] *and* [] symbolic form, which is a special case of the *area* form, was activated regularly for three of the four interview pairs (there was not sufficient evidence for it during the interview with Adam and Alice). Thus it is possible to conclude that the *area* symbolic form figured heavily in these students' thinking during the mathematics-day interview and that it possibly represents one of the more common types of symbolic forms for the integral that students might employ in a mathematics setting.

After the *area* symbolic form was the *function mapping* symbolic form in terms of how often it was activated during the mathematics-day interview. Each student pair drew on this form multiple times throughout the interview. This suggests that in a mathematics-framed setting, students are more inclined to understand the integral as a pairing of the integrand function with some other function, which can be found through a relationship mediated by the differential. This result is also not necessarily surprising given the traditional nature of calculus instruction, which is often seen as procedure-focused (Meel, 1998; Park & Travers, 1996). Yet one counterexample to this claim exists in Chris and Clay, who did not rely as much on the *function mapping* symbolic form, but instead drew heavily on the *adding up pieces* symbolic form. In fact, this symbolic form was activated in every mathematics-day interview item during their interview. Often it played a central role in their discussion of the integrals. This occurred even in items where the task simply involved a computation of an integral. As soon as they began talking about what the integral meant, they would return to the idea that it was a summation of many small pieces that eventually added up to the overall total. The key idea in most of their conceptual thinking was the use of *representative rectangles* to describe how the integral worked and why certain properties were true. This symbolic form also played a prominent role in their physics-day interview, which I discuss shortly.

Much of the students' work seemed based on the activation of these three major symbolic forms. The other symbolic forms and resources were evidenced in several places, but did not have a central role in the way that the major forms did. However, the fact that each interview pair drew on a range of different resources shows that the students did have what we could call a "large pool" of cognitive resources at their disposal for thinking about the integral in a mathematics-framed context. They had resources for dealing with the meaning of a multiplier on the front of the integral, the difference between definite and indefinite integrals, and the relationships between visual representations and the symbols of the integral. In fact, several of the students held multiple resources for understanding these, and were able to draw on more than one during the same episode in order to discuss the meaning of the integral. For example, Bill, Chris, Clay, Devon, and David each discussed the meaning of the -2 in the integral "$-2\int_D f(x)\,dx$" both from the perspective of the -2 interacting with *f(x)* to create a new function as well as from the perspective of the -2 scaling the result of the integral.

Additionally, Chris and Clay viewed a multiplier in front of an integral they had created as corresponding to the area of a "symmetric graph."

These findings seem to corroborate with the perception that students do, in fact, learn from their mathematics classes (Dray & Manogue, 2004a), in contrast to the idea that they cannot apply mathematics to physics and engineering *because* they do not know the mathematics. Thus the results of this study do not support a deficit perspective of students (NCSM, 2008) and support an underlying notion of cognitive resources, which is that students do have productive cognitive resources available to them. The difficulties students may experience in applying mathematics to physics and engineering may not necessarily be due to a lack of mathematical knowledge. Instead, it may be rooted in which resources the students "choose" to activate (and why) in a physics and engineering setting. It is possible that students are not activating resources they hold that would be productive and are instead activating resources that are not the best fit for the task. I talk about this more after I have discussed the symbolic forms the students activated during the physics-day interview.

Resource Activation in the Physics-Framed Interview

There is a marked shift in the symbolic forms that appear to have been activated in the physics-framed interview setting. Unlike in the mathematics-day interview, the students relied somewhat less on the *area* symbolic form and significantly less on the *function mapping* symbolic form during the physics-day interview. Instead, in most cases they drew heavily on the *adding up pieces* symbolic form (or the similar *add up then multiply* form). There are only a few exceptions to this, and I discuss the specific cases of these students later. Furthermore, there is a drop in the overall range of symbolic forms that the students activated during this interview.

In the cases of Clay, Chris, Devon, Adam, and Bill most of their thinking seemed largely based in the *adding up pieces* symbolic form. Ethan, on the other hand, seemed to stably draw on the *add up then multiply* symbolic form. Bill, Chris, and Clay also showed evidence of sometimes activating an *add up then multiply* form. This finding is worth noting. Bill, Chris, and Clay all drew largely on the *adding up pieces* form, but *also* at times drew on the *add up then multiply* form. It appears possible that these two forms can exist simultaneously in a student's cognition. This finding is supported by the fact that multiple students, three out of the nine to be specific, seem to have both of these forms in their cognition and drew on both of them during the physics-day interview. The number of instances that some students employed the problematic *add up then multiply* symbolic form was certainly less than the number of instances they used *adding up pieces*, but it nonetheless appears that the two symbolic forms can co-exist in their cognition. Thus, we cannot observe a student who seems to be activating an *adding up pieces* form and conclude that they must not hold the *add up then multiply* form. It would take more research to understand how these forms are compiled in students' cognition and how students choose which of these forms to activate.

The *area* symbolic form played a diminished role during the physics-day interview. Devon did not draw on the *area* form at all, though Bill, Adam, Chris, Clay, and Ethan all did to some extent. The amount that they drew on the *area* form was reduced compared to the mathematics-day interview, and was certainly significantly less than the amount they drew on the *adding up pieces* form. Becky did not often draw on the *area* form in either the mathematics-day or physics-day interview, except occasionally in agreement with Bill's work. Similarly, the *function mapping* symbolic form was activated significantly less during this interview. It did not play a role at all in Bill's or Ethan's work, and Chris, Clay, and Devon relied on it only occasionally. It was drawn on somewhat more by Adam and Alice, though less than during the mathematics-day interview. Becky was the only student to rely heavily on the *function mapping* form during the physics-day interview.

Finally, the students seem to have drawn on a smaller subset of the symbolic forms than they did during the mathematics-day interview. Generally, there was no activation of the graph-related resources, including the *symmetric graph* and *facing the other way* resources. The students' work often made extensive use of graphs, so it is not possible to say that the absence of these resources is due solely to the lack of graphs used in the students' work. Instead, it appears that the physics-framed items were less amenable to producing a standard two-dimensional graph in the plane. Similarly, there was an absence of resources around the indefinite integral. In the interview item with acceleration and velocity, I had intentionally not placed limits on the integrals to generate conversation. In most cases the students simply attached limits to the integrals, such as t_0 and t_1 or v_o and v_f, and continued working. When I asked them about the limits, the conversation reverted to an explanation of what limits mean in general. Only in two cases did the students discuss the meaning of not having limits on the integral in the context of this interview, and both times at my prompting. On the other hand, the students did activate the *region in space* symbolic form more often. Devon drew on this resource twice during the interview and Ethan drew on it once.

5.2 Intersection and Disjunction between Resource Activation

These results provide us with four findings regarding the four major symbolic forms, which I recap here for convenience. (1) The *area* symbolic form was prevalent during the mathematics-day interview for all student pairs, but was less prevalent during the physics-day interview for all student pairs. (2) The *adding up pieces* played a small role for most students in the mathematics-day interview (with the exception of Chris and Clay, who drew on it extensively), but played a significant role in most of the students' thinking during the physics-day interview (with the exceptions of Becky and Ethan). (3) The *function mapping* form was activated frequently for all student pairs in the mathematics-day interview, but was activated only occasionally in the physics-day interview, especially with Ethan, Bill, Chris, and Clay, who provided only scant evidence of its activation. (4) The *add up then multiply* symbolic form was not evidenced at all during the mathematics-day interview, but was drawn on heavily by Ethan, and occasionally by Bill, Chris and Clay, during the physics-day interview. However, due to the design of this study, these results cannot necessarily be generalized to the entire population.

Likewise, we can see three basic findings about the activation of the other symbolic forms and cognitive resources during the two interview contexts. (1) Students drew on a large range of symbolic forms while working with the integral in the mathematics-day interview, but drew on a smaller subset of these forms during the physics-day interview. (2) Though graphs were often used during the physics-day interview, the students provided less evidence of "graph-specific" symbolic forms pertaining to the integral than they did during the mathematics-day interview.

(3) Symbolic forms related to the indefinite integral were more common during the mathematics-day interview than they were during the physics-day interview.

These findings already begin to provide a picture of the differences in symbolic form activation in mathematics-framed versus physics-framed settings. In mathematics-framed settings it appears that students might rely more on resources that pertain to either the graphical nature of a function, or to the computational procedure of finding an anti-derivative. The functions typical to a mathematics classroom, such as $f(x) = 2x^2 - 3x$ or $f(x) = e^{x+2}$, are easily drawn as graphs in a two-dimensional plane. When working with functions like these during the mathematics-day interview, the students often acted first to draw the graph of the function in the plane. Many of the students then used this graph to mark off a fixed region "underneath the graph" and discussed the integral as calculating the area of that fixed region.

Similarly, much attention in a mathematics course on integrals is spent going over calculational procedures for finding the anti-derivative of the integrand. Usually the integral is taught *after* the derivative and is demonstrated to be the "inverse" of the derivative. The meaning of the integral can be taken to be "undoing a derivative," which corresponds to the *function mapping* symbolic form. The students showed ample evidence of understanding the integral symbol template as representing a function that originated as the derivative of another function. The integral then "looks" for this original function. That is, the integral reverses the process of differentiation.

By contrast, it appears that the *adding up pieces* and the related *add up then multiply* symbolic forms dominate in the physics-framed setting. In the physics-day interview, the students most often drew on notions related to partitioning the domain into tiny pieces, finding the quantity over each piece and adding them up. In fact, the only real difference between these two symbolic forms is *what* exactly is being added up. Thus, when the students were confronted with tasks related to those they would see in a physics classroom, the *area* and *function mapping* forms became somewhat dormant while the adding-related symbolic forms became dominant.

Characteristics of the Tasks and Symbolic Form Activation

If we look at the nature of the tasks given in the mathematics-day and physics-day interviews, we can see some relationships between the tasks and the type of symbolic forms the students activated in working with each item. First, consider the integrals that were explicitly presented to the students in the mathematics-day interview. They consisted of the integrals $\int_1^2 \frac{2}{x^3} - x^2\,dx$, $\int \sin(x)$, $\int_2^0 e^x\,dx$, $\int dx$, $\int \sqrt{t}\,dx$, and $-2\int_D f(x)\,dx$. These integrals were meant to resemble integrals normally seen in a mathematics course. They all make exclusive use of the dependent variable *x*. The expressions $\frac{2}{x^3}$, x^2, $\sin(x)$, e^x, and 1 are all one-dimensional functions that can be relatively easily graphed in the *x*,*y* plane. Many students demonstrated an initial inclination to graph the functions that were given as the integrand. Often the students did not even attend much to accurately representing the graph, except for sin(*x*) and e^x, which graphs appeared to be memorized. They often drew a "squiggly" graph and said something along the lines of "let's just say this is the graph of the function." It was enough to recognize the functions as being graph-able in the *x*,*y* plane. From there, many of the students drew vertical lines up from the horizontal axis, which made the left and right side boundaries of a fixed region. This was usually followed by "shading in" the fixed region. Thus, the relative ease of graphing these types of functions seems to have provided a frame for the students to work with. We can see the relationship between the types of expressions used in the integrals, their ability to be easily graphed in the plane, and the activation of an *area* symbolic form.

Once the concept of the integral has been established in a mathematics course, considerable attention is given to the computation of various types of integrals. During subsequent calculus courses, entire chapters are devoted to studying the anti-derivatives of increasingly complicated functions. In the mathematics-day interview, the integrals all had functions that could be considered “routine” expressions, in the sense that they are all common functions to work with in a calculus class. Students learn early on the derivatives and anti-derivatives of sin(x), x^n, x^{-n}, and e^x. When the students were working with these integrals in the interviews, they quickly, and often without explanation, determined their anti-derivatives. This led to discussions about the integral as reversing the derivative process. Thus, the types of functions used here may be linked with an understanding of the integral as “finding an anti-derivative” that matches with the integrand. We can see the relationship between the types of expressions used in the integrals, the ease of finding their anti-derivative, and the activation of the *function mapping* symbolic form.

On the other hand, consider the items used during the physics-day interview. The integrals that the students were explicitly given to work with were the following: $\int_0^{600} R\, dt$, $F = \int_S P\, dA$, $\int a\, dt = \int dv$, and $\Delta U = -\int_{y_i}^{y_f} F_y\, dy$. Though sharing similar symbol templates to those used in the mathematics-day interview, there are significant differences in the integrals used during the physics-day interview. Each integral makes use of variables that are connected to physical quantities and measurements, such as revolutions per minutes, time, pressure, area, acceleration, velocity, energy, force, and distance. These meanings associated with the variables in the physics-day interview are absent in the mathematics-day interview.

Another significant feature of these integrals is that *none* of the integrands are represented as "functions of the differential." By this I mean that the integrand is not an explicit function written in terms of a dependent variable, which coincides with the variable of the differential. The integrand R is not written as a function of the differential variable t. Similarly P is not written as a function of A nor is a written as a function of t. The function F_y does provide an indication that it is a function with y as a dependent variable, but the formulation of the function is not provided. The function-relationship is tacit. Also, in some cases it may not be possible to write the integrand as a function of the differential variable. For example, it does not make sense to try to formulate a function for P in terms of the variable A. That is, it would not make sense to write the integral $\int_S P\,dA$ as something like $\int_S (A^2 + 2A)\,dA$. This formulation does not properly take into account the meaning of the variable A, which is meant to denote the area of the surface. Instead, an integral such as this one is usually calculated by breaking dA into component pieces, such as dx and dy. Then the pressure could be written as an explicit function in terms of x and y. But the integral as it is written does not necessarily express the relationship between P and A. This is in stark contrast to the mathematics-day integrals, where the integrand is explicitly written as a function of *x*, where *x* is also the variable of the differential, as in the example $\int_1^2 \frac{2}{x^3} - x^2 \, dx$.

The Function Mapping Symbolic Form in Both Contexts

Consider a student attempting to activate the *function mapping* symbolic form with the integral $\int_S P \, dA$. This form suggests that the integrand, *P*, originated from another function and that the derivative of this other function with respect to *A* resulted in *P*. However, if we had such an original function, what does it mean to take its derivative with respect to *A*? A derivative process with respect to *A* might not be meaningful for students. Also, would we view *P* then as a function of *A*? What exactly is *P*(*A*)? What is its explicit formulation? Again, a formulation of *P* as a function of *A* may not be meaningful to students since *A* is representing the size of an area, and not necessarily a specific location in the domain. Finally, how is it that *mapping P*(*A*) with its *original function* explains why this integral calculates the total force? Even if one could find an anti-derivative of *P*(*A*) it may not provide any meaningful reason for why the anti-derivative should result in the overall force. The anti-derivative lacks the explaining power of some of the other symbolic forms. It is easy to see how the *function mapping* symbolic form quickly becomes less productive in this particular situation. We have a good example of attempting this in the case of Becky, who seemed to predominantly draw on the *function mapping* form during both interview sessions and showed little evidence of activating the other major symbolic forms.

Becky appeared to draw on the *function mapping* form in every item in which she participated during both the mathematics-day and physics-day interviews. She had completed three of the four calculus courses, and was currently enrolled in the fourth, the multivariate course. She was able to quickly "solve" the integrals she and Bill were given and was able to explain the rules of integrals to show how she arrived at her answers. She gave clear explanations about what she saw as the difference between definite and indefinite integrals. She provided some evidence of holding the *area* symbolic form, though it did not appear to drive her thinking as much. The evidence for this is two-fold. First, she did not draw graphs herself and did not talk about the "area model" of the integral until after Bill had provided a graph and started talking about the integral in that way. When I specifically asked her to look at the integral "$\int dx$" using a graph, she essentially passed the task over to Bill, citing his ability to work with graphs as better than hers.

> Interviewer: So Becky, you were talking about "I'll write that as 1, dx." So if you were to represent that graphically, how could you represent what that integral is saying?
>
> Becky: A graph…Good call… [to Bill] Do you want to try a graph? You're better at graphs than I am.

Second, she expressed the perception that she was not well-versed with graphs and did not feel as comfortable working with them. When she and Bill were trying to explain the need for a constant "+ *c*" to be added onto the anti-derivative, Bill attempted to use a graph to provide a reason. She expressed that she did not use graphs often.

> Becky: [to Bill] You like graphs, huh? That's great. I never, ever draw graphs.

These were not isolated cases and are representative of her approach during the two interviews.

During the physics-day interview, Becky continued to draw heavily on the *function mapping* symbolic form at the relative exclusion of the other major forms. This seemed to correspond to a general discomfort for her during this interview. There is evidence of this throughout the entire interview. While working on the problem of determining the mass of a box, Becky stated, “Honestly, I have no idea what to do if it’s varying.” As Bill made progress on the item by drawing on the *adding up pieces* symbolic form, Becky would ask him to explain what he was doing, adding comments such as “I have no clue” and “Honestly, I have no idea.” Then after Bill provided his thoughts on the next integral, “$\int_0^{600} R\,dt$,” she again stated, “Sure, works for me. I have no idea.” Since she had remained quiet during most of the item, I asked her to comment on what Bill had done.

> Interviewer: What would you say to that, Becky?
> Becky: Honestly, I don’t do any problems like this. I don’t like problems that are word problems, personally. But I wouldn’t really know how to interpret it much more than what we’ve explained.

While they worked on the integral of pressure over area, “$\int_S P\,dA$,” I asked them to explain the relationship between *P* and *dA*. After Bill provided his thoughts about it, Becky interjected,

> Becky: Nobody ever does it in terms of that problem, never do problems in terms of real life like that. [Bill laughs] To be honest, they don’t. You do it on paper, but you don’t actually get to see something.

While Bill made use of the *adding up pieces* symbolic form often throughout the interview, Becky seemed to struggle with these items. While it is possible that a range of issues contributed to her discomfort with the items, it did seem to correspond to the way she thought about the problems. Her usual approach, which was based mostly in the *function mapping* symbolic form, did not help her deal with these problems as well. Thus, while she held productive resources pertaining to the integral, such as the *function* *mapping* and *area* forms, their activation for the physics-day interview items did not seem as helpful. Becky becomes an example of a "successful" mathematics student who demonstrated that she had sound mathematical knowledge and a pool of productive cognitive resources, but then struggled to apply this knowledge to a physics setting.

The Adding Up Pieces Symbolic Form in Both Contexts

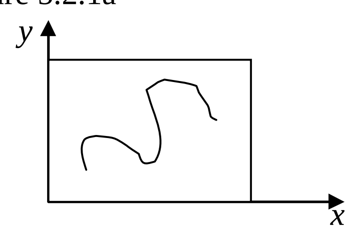
By contrast, consider the *adding up pieces* symbolic form in the physics context. This form takes the integral as dividing a region or object into pieces, finding a small amount of a quantity within each piece of the partition, and adding up these small amounts in order to capture the overall total. If we reanalyze the integral "$\int_S P\, dA$" from this perspective, we can see this symbolic form's utility in a physics-framed setting. If we start with a particular region, *S*, this symbolic form will take the region and partition it into many small pieces (potentially "infinitely" many pieces). The differential *dA* corresponds to these small pieces. Within the tiny area of each piece we have a particular pressure at that point. The pressure over that tiny piece yields a small amount of force over that same tiny piece. Thus each tiny piece is thought to contain a small amount of force. The forces throughout all of these pieces are then added up in order to produce the overall force exerted on that region. This symbolic form allows the student to do three things: (1) to use the differential as a means of looking at the region, or object, (2) to retain the physical meaning of the variables in use in the integral, and (3) to provide a reason for *why* the integral calculates the total force on the region. Thus activating this form is helpful in interpreting these mathematical symbols in a physics-framed problem in a productive way. I use the case of Chris and Clay to show the utility of this symbolic form in understanding integrals in these physics-framed items.

Chris and Clay regularly used the *adding up pieces* symbolic form during both the mathematics-day and physics-day interviews. Thus, they showed comfort and flexibility in drawing on this symbolic form. When they were given the task of explaining the integral "$\int_S P\,dA$" and why it calculated the force, Chris and Clay began by stating that they would want to express *P* as a function of *A*, showing evidence of drawing on the *function mapping* form. They seemed stuck for about one minute as they continued to think about the integral from this perspective. The turning point for them came as they begin to think about the relationship between the pressure and the area.

> Chris: Pressure times area…
> Clay: Right, because F equals P times A. Like, force is equal to pressure times area.

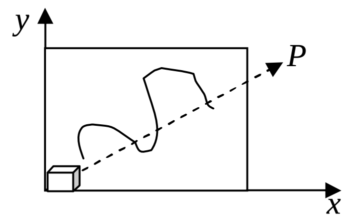
Once they settled on the idea of thinking about the relationship between *P* and *A*, Chris began working on a graph to show the relationship. He started with a two-dimensional graph representing the surface area, *S* (see Figure 5.2.1a).

Figure 5.2.1a



> Chris: [Find a] particular pressure at one point. So we'd want to calculate… so it would need to be a three-dimensional graph, this axis would be the P value at a particular point [draws in third axis. See Figure 5.2.1b]. So that means our area would have to be with respect to x and y. So in order to calculate force, we'd have infinitesimal cubes, again, which we could calculate. And we'd have pressure being dP.

Figure 5.2.1b



Chris used the idea of relating pressure to area to talk about finding the pressures at particular points over the surface. This gave him the idea to draw in a “pressure axis” which then gave rise to small “infinitesimal cubes” over the surface. Each cube represented a little bit of pressure, or “*dP*,” over the small area, *dA*. As they continued down this path, they gained momentum in their explanations. Clay pushed this idea further by extending it to the integral.

> Clay: And by making it an integral, you have all the small bits of area times the pressure at that location. And then you have a total force because of that.

At this point they had provided evidence of drawing on the *adding up pieces* symbolic form, which continued to guide their thinking for the remainder of the item. Clay used the “adding” concept in the integral to state that “you have a total force because of that.” Thus, this symbolic form seems to have aided in making sense of the integral equation. Chris then relied on this symbolic form to explain the overall idea of the integral.

> Chris: We’re adding up, here we have force equals pressure times area. So since we have this being pressure [references “*P* axis,” see Figure 5.2.1b]… so we have pressure times dx times dy. And since dx dy is an area, then we have a pressure times an area. So we’re actually finding, so the volume of this [points to box] essentially is force.
>
> …
>
> Chris: So, um, since we’re finding the integral, so these infinitesimally small cubes which consist of pressure and then dx and dy which would be dA. So we’re finding the, we’re integrating over the infinitesimally small volumes, which each one composes of a force. So we’re integrating a force and adding up all the infinitesimally small pieces of force to find the total force.

Chris and Clay transitioned from drawing on the *function mapping* form to the *adding up pieces* form as they worked on this item. Once they activated the *adding up pieces* form they began to make progress on explaining what the integral meant as well as how it calculated the total force. It gave them a framework for understanding why the integral equation made sense. Hence, as stated earlier, we see evidence that drawing on this form is productive in a physics-framed setting and that it is helpful in order to analyze the region or object, retain the physical meaning of the variables in use in the integral, and provide a reason for *why* the integral calculates the total force on the region.

The Add Up Then Multiply Symbolic Form in Both Contexts

Related to the topic of the *adding up pieces* form is the *add up then multiply* form, which only appeared during the physics-day interview. It seems that there is something about the form that facilitates its activation during physics-framed contexts. Like the *adding up pieces* form, it relies on dividing the region or object of interest into pieces, then looking at a quantity within each piece. The quantities within each piece in this case are the quantity of the integrand. In the case of the integral "$\int_S P\, dA$," the quantity within each piece is thought to be pressure. Thus, tiny pressures are added up over each piece, determining the overall pressure over the whole surface. This resultant pressure is then multiplied to the area, which is the variable of the differential, to determine the force. Why then would this form only be activated in a physics-framed setting?

This form is dependent on looking at the *quantities* involved in an integral. As previously discussed, integrals found in mathematics-framed settings typically do not carry associated meaning. Thus, there is less opportunity to look for a quantity when the function involved does not seem related to any quantities. For example, in the integral

"$\int_1^2 \frac{2}{x^3} - x^2\, dx$," the function in the integrand does not suggest any particular quantity. This contrasts with the integral "$\int_S P\, dA$," where the integrand represents pressure, which is a physical quantity. Thus, there appears to be less motivation to draw on this symbolic form, which specifically deals with quantities, in a mathematics-framed setting than in a physics-framed setting.

5.3 The Difficulties Transitioning from One Dimension to Multiple Dimensions

One noteworthy difference between the productivity of the symbolic forms that students activated during the mathematics-day interview and the physics-day interview centered on the "dimensionality" of the problem. By this is meant the dimension of the function and differential used in the integral. For example, the integral $\int_1^2 e^x dx$ contains the function e^x which is only dependent on one variable, x. Hence, this is a one-dimensional function, or a univariate function. The differential, dx, has as its variable, x, which again, is a one-dimensional variable. The limits are 1 and 2, which correspond to single values of the variable x.

Many of the integrals used in physics settings are quite different in nature. The integral $\int_R \rho\, dV$ can be used for finding the mass of a box, where ρ refers to the density of the box, V is its volume, and R is the spatial region making up the box. The integrand, ρ, consists of a function that indicates the density at a given point in the box. However, the box is three-dimensional in nature, meaning that ρ might depend on three spatial variables, x, y, and z. That is, it could be thought of as the function $\rho(x,y,z)$, which depends on more than simply x, unlike the previous integral. Furthermore, the variables are not symbolically represented in the differential *dV*. Here the variable *V* is also multi-dimensional in that it also can be thought to be made up of *x*, *y*, and *z*, yielding the differential units *dx*, *dy*, and *dz*. Also, the "limits" on the integral, *R*, refer to an actual spatial region instead of two simple numerical values that can be placed in for the variable *x*. That makes this integral much more complex than the previous integral. However, the symbolic template is nearly identical to the more simple integral $\int_1^2 e^x dx$. Integrals making use of multivariate functions and differentials are more common in a physics context, making them somewhat different in nature than those typically seen in a mathematics course.

The Area Symbolic Form in Both Contexts

One instance where the dimensionality of the integral played a significant role was in the productivity of the *area* symbolic form. As previously discussed, the *area* form was activated often throughout the mathematics-day interview by all of the students. When the students were given integrals such as $\int_2^0 e^x\, dx$, $\int dx$, or $\int_1^2 \frac{2}{x^3} - x^2\, dx$, they commonly initiated the discussion by drawing the graph of the integrand. As they continued to talk about the meaning of the integral, they would mark out vertical lines at the numbers dictated by the limits of integration (and often invented limits for the integral "$\int dx$"). This produced a bounded region in the plane, which they had a strong tendency to shade in, as an indication that the integral was represented by the area of the region. This symbolic form provided a useful framework for the students in explaining the meaning of the integral, how the parts of the integral fit together, and what the integral was calculating.

On the other hand, there were a few instances where this symbolic form became less productive during the physics-day interview. Essentially, if the integrand could be represented as a one-dimensional function whose graph could be created in the plane, then the form remained productive for expressing the meaning of the integral. However, when the students drew on this form during items with "multivariate" integrals, this form appeared to have lost some of its utility. The two integrals of this nature in the interview were the integral of pressure "$\int_S P\,dA$" and the integral of density "$\int_R \rho\,dV$." Bill drew heavily on the *area* form in both interviews and provides an example of the productivity of the form in the two contexts.

During the mathematics-day interview, Bill regularly activated the *area* symbolic form as a means of making sense of the integral. After he and Becky had calculated (numerically) the integral "$\int_1^2 \frac{2}{x^3} - x^2 \; dx$," I began asking them about different parts of the integral symbol template.

> Interviewer: When you see the 1 and the 2, what does that mean?
> Bill: It means to me that if we're looking at this on a graph, this is 1 and this is 2. Let's just say it looks something like that [draws in squiggly graph], it would be the area in between 1 and 2 [draws in vertical lines and shades in region].

The *area* symbolic form offered a useful framework for discussing the meaning of the integral. The integrand was represented by a squiggly graph, the 1 and 2 were represented by vertical lines marked out on the *x*-axis, and the integral itself was the area of this fixed region. In the next item, Bill was discussing the meaning of the integral "$\int_2^0 e^x dx$" and was trying to justify why the limits would end up yielding a negative area.

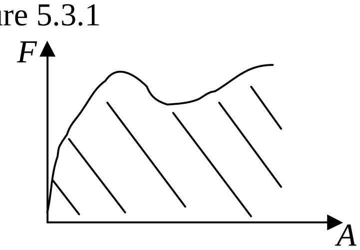
> Bill: Let's see, for this… if I drew e to the x… [draws e^x graph]. When I look at this, when I try to justify why it's negative area and not positive area, cause you normally... Like with the identity [a reference to a rule], it would be from 0 to 2, but it would be negative. So if it was redrawn, like that [draws in $-e^x$ graph].

Again, Bill drew on the *area* form to analyze the meaning of the integral, and attempted to use it to explain why the integral would produce a "negative area." By drawing a graph in the negative region of the plane (i.e. "$-e^x$"), it made more sense to Bill that it would be a negative area, and hence a negative value for the integral. It allowed him to discuss the conceptual meaning of the symbols. These episodes are not isolated cases, but are representative of Bill's work throughout the mathematics-day interview. Thus the *area* form provided a useful cognitive resource to activate in thinking about these integrals.

Next, consider some of Bill's work during the physics-day interview. As he and Becky were trying to describe the integral "$\int_S P\,dA$" and why it calculated the total force, Bill was again drawing on the *area* symbolic form. (Note that in the following excerpt he talks about a graph of *F*, though Becky later corrects him that it should be *P* and he changes *F* to *P*.)

> Bill: If we had, if we plotted this. Area versus, change in area versus F, total F. [draws graph with *A*-axis, *F*-axis, and a squiggly graph. See Figure 5.3.1]. It would be something like that.

Figure 5.3.1



Bill: …So if we took the integral of that, it would be, this would be, it would be all this [shades in the region underneath the *F*-graph]. And that would be the total force it would exert. … I feel like I skipped a step.

In attempting to make sense of this integral, Bill again drew on the *area* symbolic form. He created a graph, and though at first he named the graph *F*, Becky later corrected him that it should be *P*. He shaded in the region and claimed that "all this [area] would be the total force it would exert." However, Bill appeared to feel that this explanation was lacking and followed this explanation with, "I feel like I skipped a step." He seemed to experience a degree of uncertainty that he did *not* exhibit during the mathematics-day interview. During that interview, he confidently drew on the *area* form to explain the integrals. Here the *area* form seems to be missing something for him. A few moments later, I asked Bill and Becky to explain the meaning of the differential, *dA*. Bill referred back to his drawing of the graph while he talked.

Interviewer: So conceptually, what does dA mean?
Becky: Change in area.
Bill: Yeah, the change in area [points to horizontal axis]. Like if you compared, because area isn't just automatically changed over time. It's hard to say… But I think if you took the, if you measured the pressure and there's that much area [puts chalk on a point on the horizontal axis], and you took the pressure of some other lesser area [puts chalk on another point on the horizontal axis], and you subtracted, that would be what dA is, the change in area.

Continuing to draw on the *area* symbolic form, Bill attempted to use the graph he had constructed to explain the meaning of *dA*. The use of his graph led him to an explanation that might be more closely related to univariate integrals. For example, in a mathematics course, an integral such as $\int_1^2 x^2\,dx$ might be explained by graphing the function x^2 and finding the area between $x = 1$ and $x = 2$. The *x*-values underneath this portion of the graph range from 1 to 2. By using this one-dimensional graph to understand the integral, Bill is drawing the conclusion that *A* is ranging between two *A*-values, like in this integral. Thus he believes that the "value" of *A* must be changing in this scenario. Different points along the horizontal axis represent different sizes of *A*. Thus the meaning of *dA* is to look at one area and then another smaller area and then to subtract to get the difference between the two areas.

By activating the *area* form in this context, Bill ended up feeling less confident in his explanation as well as drawing an incorrect conclusion about the meaning of *dA*. His one-dimensional graph did not offer a useful context for describing the integral. We can see that while Bill productively used this symbolic form in several instances, it did not allow him to satisfactorily explain the meaning of this multivariate integral. Instead, it was not until Bill began drawing on the *adding up pieces* form that he made progress in fully explaining the meaning of the integral and why it calculated the total force. I now describe this transition.

The Productivity of the Adding Up Pieces Form in Multivariate Cases

After Bill had drawn on the *area* form to explain the integral, he appeared unsatisfied. At this point both Bill and Becky became silent and did not continue to work on the item. They showed no signs of making any more progress, so I inserted a prompt. I asked them to think about the table I was sitting at as the surface area, *S*, and I described a non-uniform pressure that was exerted on the table from one end to the other. I then asked them whether this integral would be useful for describing this situation. Bill jumped in immediately with an answer.

Bill: I would say yes, because this integral is describing a situation where there is non-uniform pressure and a defined total area.

…

Bill: I believe that, uh, I'm just trying to relate this to rectangles. If we just took the area of this piece of the rectangle here, this part of the table, and found the total force exerted on that. You would get some kind of estimate. Because if you added all those… it wouldn't be exact, it would be an estimate. But as you make that area smaller and smaller and smaller, and then it would get better and better, until it gets close to 0 and it would be an integral.

Here Bill had switched his thinking and showed evidence of drawing on the *adding up pieces* form. He was now talking about dividing the surface area into "rectangles" where each rectangle would have some amount of force exerted on it. These could be added up to get an estimate. Then as the area of these rectangles became "smaller and smaller" it would "become an integral." After some discussion, Bill summarized his thinking and showed strong evidence of drawing on the *adding up pieces* form, which enabled him to explain why the integral calculated the total force.

Bill: [Draws a rectangle on the board to represent the table] Let's just say this is *dA* [references a small strip at one end of the rectangle]. This whole thing is *dA*, this whole area [again references small strip]. And you have pressure pushing on that, on all that area. So you can multiply P times dA and you get the total force pushed, exerted on that part of the table.

…

Bill: Yeah, if you make that area smaller and smaller and smaller and then add up those infinite, those really small areas on the whole table, you get the total force.

By activating the *adding up pieces* symbolic form, Bill explained the meaning of *dA*, the relationship between *P* and *dA*, and how the integral calculated the total force exerted on the surface. Thus we can see the productivity of this form when dealing with multivariate integrals. The *area* form did not provide Bill with a context for explaining these relationships in the integral, because it focused him on an apparently less useful one-dimensional graph. The *area* form, then, is productive in some circumstances, but not as productive in others. Meanwhile, the *adding up pieces* form became significantly productive in dealing with the multivariate integral.

The Symbol "$\int_{[]}$" in Multivariate Contexts

Another instance where the dimensionality of the integral played a role in the productivity of students' cognitive resources was in the interpretation of the symbol template "$\int_{[]}$." In univariate cases, the students held resources that were productive for making sense of this symbol in a way that matches commonly accepted notions about this symbol. However, in multivariate cases, the resources that some students activated did not yield an interpretation that matches commonly accepted notions about the symbol.

First, let me be clear that the students do appear to hold productive resources for handling this symbol in both the univariate and multivariate cases. As described in chapter four, the students may hold a cognitive resource that recasts this symbol as "shorthand notation" for the more standard-looking limits of integration "$\int_{[]}^{[]}$." In univariate cases, this is appropriate since the dimension of the variable of the differential is one. For example, in the integral $\int_D (2x+1)\,dx$, the differential has the variable *x*, which is often considered to be a one-dimensional variable. Thus, *D* could be interpreted as being a particular portion of the domain of the function 2*x* + 1, meaning that it could take on values such as *D*:(1,2) or *D*:(-2,5) and so on. When Adam and Alice were working with the integral "$-2\int_D f(x)\,dx$," Adam explained to Alice the meaning of the *D* on the integral.

> Adam: Well I see, well it says, with the domain. I picture, there's no value, there's no boundaries, there's no values for the boundaries. You just put D to

represent what boundaries there are going to be. So, in an assignment you could put, like, the domain would be, like, 1 to 2.

…

Adam: Let's say, like 1, 2, or something like that [writes *D*:(1,2)]. Like I would imagine in a book, it would just, like, list a couple of different domains [writes *D*:(2,3)].

Adam demonstrated the possession of a cognitive resource that considered the template "$\int_{[]}$ " as shorthand for the other template "$\int_{[]}^{[]}$," since *D* stood in place of numbers such as (1,2) or (2,3).

Adam also provided an example of a symbolic form that interprets the symbol template "$\int_{[]}$ " in the multivariate case. After the previous explanation that he gave to Alice, he continued by expanding the meaning of the symbol for a multivariate integral.

Adam: I guess just as far as the domain goes, sometimes there's a double integral [writes $\int\int_{R}$]. There would be an R right here. Which would be the region.

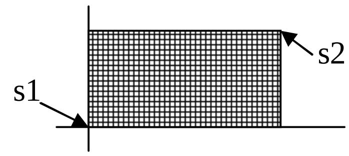
Adam then drew a figure in the plane which he labeled *R*. Thus he showed that he also possessed a symbolic form that takes this symbol to mean a region in the plane.

All of the students gave evidence of holding the "shorthand for limits" cognitive resource. However, in the multivariate case some of the students attempted to activate this resource which provided interpretations of the symbol that do not fit with the commonly accepted convention. While Chris and Clay were working on explaining the meaning of the integral equation "$F = \int_{S} P\,dA$" they drew on this resource to interpret the symbol *S*. They replaced this symbol template with the symbol template "$F = \int_{s1}^{s2} P\,dA$." They offered the following explanation for doing this (see Figure 5.3.2).

Clay: s1 is the minimum, where you're starting from, I guess. And s2 would be the last one you're integrating, but that would be inclusive of all the ones in between.

Chris: s1 would be the first square here [points to lower left-hand corner of the surface], and then s2 would be this square here because it's where both x and y reach the maximum [points to upper right-hand corner of the surface].

Figure 5.3.2



In the interview with Bill and Becky we have a similar interpretation of *S* as being shorthand notation for a range of values. While they were working with the integral, they changed the template to "$F = \int_0^S P\, dA$." I asked them to explain further why they could change the *S* to be 0 to *S*.

Becky: I think in reality, it's trying to say that it's from the total, like, I mean we could say S is, let's say 5. You're trying to find from 0 to 5 or you could do from 1 to 6. So it doesn't really matter where you do it from, at least that's how I interpret it, as long as you get the total amount in it.

…

Bill: I pretty much agree with that, by not saying the bounds, I think it just implies that it's just a range from whatever to whatever. Like if it's 0 to S, or 1 to S plus 1.

Bill and Becky interpret the symbol "$\int_{[]}$" as being shorthand for the limits of integration "$\int_{[]}^{[]}$." In their explanation they claim that *S* could take on a particular value, such as 5, and then the limits of integration would need to be a distance of 5 apart. Thus the limits could be anything such as "0 to 5," or "1 to 6," or "1 to S+1." The cognitive resource "shorthand for limits" appears to be active in their thinking at this point.

For Chris, Clay, Bill, and Becky, the symbol template "$\int_{[]}$ " was interpreted as being a shorthand notation for the limits of integration "$\int_{[]}^{[]}$." This interpretation of the symbol template "$\int_{[]}$ " does not fit as well with commonly accepted notions for certain types of problem, particular when the integral is multivariate. Placing the symbol *S* on the integral sign in this way generally means that the entire region *S* should be partitioned and the pressure should be found over each piece. It does not necessarily signify, however, that there is one particular "starting place" for the addition, nor an "ending place" for the addition. This contrasts with the univariate case, where there is a definite "starting" and "ending" point for the addition because of the one-dimensional nature of the variable of the differential. The flexibility of the symbol is somewhat lost when replaced with the other limits notation.

These four students all successfully drew on the *adding up pieces* to make meaning out of the integral; however, their interpretation of the symbol *S* did not reflect the typical meaning given to that symbol. While this may not have affected their understanding of the rest of the integral symbol template, this resource may create the potential for confusion when the correct interpretation of the symbol becomes important. For example, in order to compute this integral, it would most likely have to be broken down into a multiple integration where the region *S* is separated into an *x* component and a *y* component. The integral would then need to be calculated over each component individually. If the symbol *S* has already been converted into a range *s*1 to *s*2, or into numbers such as 1 to 6, this could potentially be problematic for the student. Suppose the pressure equation is given in terms of *x* and *y*, such as $P(x,y) = x + y$. If we apply this cognitive resource to the symbol *S* as these students did, then the integral might become something along the lines of $F = \int_1^6 (x+y)\,dA$. This formulation may require a much more rigid calculation method for this integral, causing less flexible solution strategies. Thus, this cognitive resource is more productive in some situations, especially those dealing with univariate integrals, but can become less productive if applied to situations dealing with multivariate integrals.

The Cause of the Difficulties in Multivariate Cases

This chapter describes some of the difficulties that were apparent when the integrals were multivariate. In general, the *area* symbolic form and the *function mapping* symbolic form became less productive for the students in explaining the meaning of multivariate integrals. By contrast, the *adding up pieces* form became significantly productive in explaining the integral and how the integral fit into a larger equation. However, caution should be employed here to point out that there is not always sufficient data to explain the "cause and effect" of the difficulties in these multivariate cases. There appear to be *at least* two components to dealing with multivariate integrals that could be at play, and possibly more. First, there is the overall utility of the *adding up pieces* form and the potential ineffectiveness of the *area* and *function mapping* forms in multivariate cases. It seems that the students' possession and activation of the *adding up pieces* form is important for understanding and explaining these cases. This provides one possible cause for students' difficulties with multivariate cases: the students do not hold the *adding up pieces* form.

Second, there is additional conceptual baggage in multivariate integrals that does not necessarily exist in univariate cases. For example, there are multiple variables to mentally juggle. Often multivariate integrals are calculated by holding one variable constant (something that does not explicitly happen in univariate cases) while integrating over another variable. As another example, it could be possible that univariate functions are easier to graph on a coordinate system than multivariate functions. Conceptualizing this spatial region might require additional cognitive elements. Thus it is possible that another explanation for the difficulties that students have is that they simply do not have a sufficient framework for understanding integrals in a multivariate context. It is possible that a student could hold the *adding up pieces* symbolic form and productively draw on it, but that the student does not have an adequate understanding of multivariate cases, which inhibits the activation of the *adding up pieces* in a productive way.

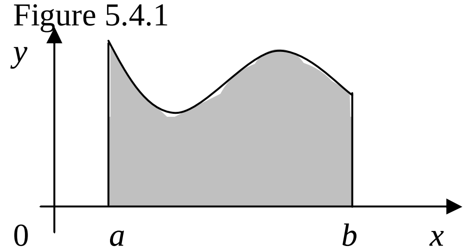
This means that there are multiple possible explanations for the "cause and effect" of why students may have difficulties with multivariate integrals. They might struggle because they do not have a well-compiled *adding up pieces* symbolic form to draw on, or they might simply lack a framework for understanding multivariate integrals. It is possible that there are other alternatives for this difficulty as well. Some students, such as Bill, do provide enough evidence to propose a hypothesis regarding their difficulties with multivariate integrals. It appears that he struggled while trying to draw on the *area* symbolic form, but that once he drew on the *adding up pieces* form he was able to successfully explain the meaning of the integral and its relation to the integral equation it was in. Thus, Bill held the necessary conceptual framework of a multivariate integral and it was the particular forms that he relied on that caused his difficulty. For him, holding and activating the *adding up pieces* form was sufficient for overcoming this difficulty. But for other students, this may not be the case.

5.4 Current Curriculum and Symbolic Forms

I have discussed how the interview items correlated with the kinds of symbolic forms the students activated. There is likewise evidence that typical calculus and physics textbooks bear a resemblance to the symbolic forms the students drew on during the interviews. If this is the case, then one can argue that current curriculum encourages the type of symbolic form activation seen in the students' work during the mathematics-day and physics-day interviews. Additionally, some of the intersections and disjunctions in symbolic form activation between the two contexts may exist in typical classroom settings.

Consider typical sections from standard calculus books (Briggs & Cochran, 2009; Stewart, 2008; Salas, Hille, & Etgen, 1999). There are relationships between the material presented and the more common symbolic forms that the students drew on during the mathematics-day interview. In Stewart's *Single Variable Calculus: Early Transcendentals*, the chapter on integration opens with a discussion of the area underneath an arbitrary graph. The first section states, "We begin by attempting to solve the *area problem:* Find the area of the region *S* that lies under the curve $y = f(x)$ from *a* to *b*" (p. 355). This paragraph is followed by a picture of a graph with the area shaded in between *a* and *b* on the *x*-axis (see Figure 5.4.1).

Figure 5.4.1



This opening discussion resembles the students' *area* symbolic form. It draws a "squiggly" graph above the *x*-axis, marks off *a* and *b,* and represents these boundaries by vertical lines. The result is a fixed region in the plane that is shaded in to show that the integral represents the area of the fixed, bounded region.

The next section of the book then describes the Riemann sum process. The *x*-axis between *a* and *b* is partitioned off with equal segments and rectangles are drawn up to the height of the curve over each partition. The areas of the rectangles are then systematically added up. Many drawings are given that depict the region being divided into thinner and thinner rectangles. The book then states, "From Figures 8 and 9 it appears that, as n increases, both L_n and R_n become better and better approximations to the area of S" (Stewart, 2008, p. 358). It then provides a definition of the "area" of the region S as

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x].$$

We can see the *adding up pieces* in the process it describes through Riemann sums. However, this explanation is closely tied to the area of the region. Thus it continues to reinforce the thinking typical in the *area* symbolic form.

The calculus textbooks reviewed in this section all spend the opening section developing the concept of the integral through the area "underneath" various graphs. They also all spend the second section on definite integrals by exploring the notion of Riemann sums. The third section in each book is a discussion about the fundamental theorem of calculus followed by multiple sections developing various techniques of integration, such as the substitution rule. This uniformity suggests a common approach to teaching integrals in a mathematics course. The integral is conceptually grounded in the area of a fixed region, is explored in more detail using a Riemann sum process, and then culminates with the notion that an integral is an anti-derivative. The majority of the attention over the next several sections and chapters is given to clever ways to determine the anti-derivative of various functions. Thus one of the major ways that students are taught about the integral is that it is the process of finding an anti-derivative. It could be said that the reification of the anti-derivative process could be responsible for the *function mapping* symbolic form. The relationship between the presentation of the calculus textbooks and the *area* symbolic form and the *function mapping* symbolic form can be seen. The *meaning* of the integral becomes intertwined with either the area of a fixed region or the "game" of finding an anti-derivate.

Similarly there are relationships between the presentation of integrals in physics textbooks (Serway & Jewett, 2008; Tipler & Mosca, 2008), engineering textbooks (Hibbeler, 2006; Hibbeler, 2004), and the symbolic forms students activated during the interviews. In Tipler & Mosca's *Physics for Scientists and Engineers*, the center of mass is introduced by calculating the center of mass between two objects. The text then prompts the reader to extend this concept to multiple objects. "We can generalize from two particles in one dimension to a system of many particles in three dimensions. For *N* particles in three dimensions,

$$Mx_{cm} = m_1x_1 + m_2x_2 + m_3x_3 + ... + m_Nx_N$$ " (p. 150).

The text then describes the meaning of the center of mass relative to this equation. Following is a paragraph that states that the center of mass for a continuous object will lie somewhere along the line or plane that the object is on. "To find the position center of mass of an object, we replace the sum in Equation 5-18 with an integral:

$M\vec{r}_{cm} = \int \vec{r}\, dm$" (p. 150).

The center of mass in this section of the book is given as an integral *after* a discussion of the center of mass as a summation. In fact, the book states that the summation can be "replaced" by an integral. This connection between the integral and a summation resembles the *adding up pieces* symbolic form that the students often drew on in the physics-day interview. The summation between the masses and lengths is extended to an "infinite number of objects," giving the summation as an integral.

We can see the difference between the presentation of the integral in mathematics textbooks (Briggs & Cochran, 2009; Stewart, 2008; Salas, Hille, & Etgen, 1999) and physics and engineering textbooks (Serway & Jewett, 2008; Tipler & Mosca, 2008; Hibbeler, 2006; Hibbeler, 2004). These differences mirror the differences in the symbolic forms that the students activated during the mathematics-day interview and the physics-day interview. This shows the likelihood of the students' epistemologies and framings about the two contexts to be influencing their choice of cognitive resources. Hence, there is validity to the idea that applying mathematics to physics and engineering implies *more* than simply learning the mathematical content in a mathematics course. Different meaning-making needs to happen within each context. Students appear to draw on resources that match the type of context they find themselves in. However, if the student has not activated the more useful symbolic forms in a physics context, this can produce difficulty in making sense of the integrals presented.

In light of the concern for the perceived disconnect between mathematics learning and the application of that knowledge to physics and engineering contexts, some educators have been developing curriculum to bridge the two disciplines. The "Vector

Calculus Bridge Project" at Oregon State University (Dray & Manogue, 2006) seeks to switch the emphasis of traditionally taught calculus concepts from the limit to the differential, from slope to rate of change, and from area to total amount. They argue that these changes in emphasis will have the effect of making generalizations of the mathematical concepts to physical situations easier. The "Studio Calculus/Physics" course designed at the University of New Hampshire (Meredith & Black, 2001) aims to help students "see the use of the calculus immediately" (p. 5) by mixing introductory calculus and introductory physics together into one class. The students would take a double time-intensive course that is essentially a calculus class and a physics class in one. The material would be woven together so that as the students learn about the calculus concepts they would immediately apply them to the physics contexts. The "Workshop Physics" presented by More and Hill (2002) at the University of Portland attempts to coordinate the timing of the concepts taught in a calculus course and a physics course in order to "take advantage of the opportunities each subject has to motivate and reinforce the other" (p. 2). For example, in this curriculum the concept of the differential is closely tied to error calculations and Riemann sums are related to the electric field of a line charge. The instructors of the two courses time their lessons to draw on each other's material.

These curricular projects are attempting to remove some of the potential difficulty in applying mathematics to physics and engineering by blending the two topics more closely. However, the impact and success of these curricula may be affected by the extent to which they facilitate appropriate cognitive resource activation. As seen in the results of this study, the symbolic forms chosen by the students during their work affected the meaning they gave to the integral. Some forms were apparently more productive in making sense of integrals in physics contexts than others. Curriculum development should take into account the nature of the symbolic forms students hold in their cognition regarding integrals and when and how those forms are compiled or activated. If the curricular projects do not consider the knowledge that students have available to them and how that knowledge could be activated in a productive way, then the curriculum may not be as successful at “bridging the gap” between mathematics learning and physics learning. An analysis of the curriculum projects might prove beneficial in terms of their support of the creation and activation of symbolic forms that will help the students make sense of integrals in physics contexts.

5.5 Implications

This study has implications for both instruction and curriculum. The results presented here suggest practices that could be done in both mathematics and physics classrooms to help students gain a robust understanding of the integral and to support the application of knowledge about the integral to physics and engineering. Additionally, it suggests possible ways to present the concept of the integral in curricular materials that could support the application of this knowledge. This study also has implications regarding perceptions of students’ mathematical preparation as they enter introductory physics courses.

Implications for Instruction

In the results, the general importance of the *adding up pieces* symbolic form is visible. It was a productive way for the students to think about the integral in both the mathematics and the physics contexts. Furthermore, it was by far the most productive form that the students drew on during the physics-day interview. By contrast, the *area* and *function mapping* forms typically became less productive during the physics-day interview. If the intentional framing of the interviews as “mathematics” or “physics” was at all successful, then it is possible that when students encounter integrals in physics settings, it is generally much more effective to understand the integral as the addition of many small pieces. What does this say about the ways calculus instructors might want to approach the topic of integration?

There are several ways to conceptualize the integral, and if one of these ways is overemphasized to the relative exclusion of the others, then the students will have less opportunity to gain a well-rounded understanding of the various interpretations of the integral. Specifically, if integrals are introduced through the area perspective and then the majority of the time is devoted to the anti-derivative process, then students may not have sufficient time to digest the meaning of the integral as an “addition.” Riemann sums are certainly a central part to instruction on the integral, but it is possible that not enough attention is given to this critical way of viewing the integral. If instructors proceed too quickly past this important concept, then the students may not fully compile an *adding up pieces* symbolic form, or they may not fully understand how to employ it in the context of a problem. This may cause difficulties later when the student needs this symbolic form to work with the integrals presented in their physics and engineering courses. Thus, the results of this study imply that calculus instruction should devote sufficient time to multiple interpretations of the integral, and that the interpretation of the integral as an addition should be given enough time for the students to have the opportunity to compile the *adding up pieces* form. This will help students be better prepared to understand the integrals presented in their physics or engineering courses.

There are also implications for instruction on the other end, in the physics classroom. In a calculus course, there may be students who need to understand the concepts of calculus for a variety of reasons. There may be students proceeding on to further mathematics courses, or others who need it for the biological sciences, or computer science, or other fields of study. Thus, in the mathematics classroom, it is important that the integral is taught in such a way that several interpretations of the integral are encouraged. The goal may be to provide the opportunity for as many symbolic forms of the integral to be compiled as possible so that the students have a well-rounded understanding that can be applied to future coursework in a variety of possible fields. However, in this study there appeared to be strong evidence that the *adding up pieces* form is especially effective in understanding the type of integrals that are commonly found in physics courses. How might physics instruction help students take advantage of this way of thinking?

In a physics classroom, the importance of this particular type of thinking should be highlighted. If students have come from a mathematics classroom where they have spent time learning about the integral through a variety of interpretations, the physics and engineering instructors may wish to emphasize the particular ways of thinking about the integral that are especially useful in the physics and engineering context. If it is generally less productive to attempt to interpret integrals through the *area* or *function mapping* symbolic forms, then instructors should explain that it might be more helpful to think about the integral as an addition over several pieces. Since the *adding up pieces* form is apparently very useful, physics instructors could take some time to recap this conception of the integral and explicitly state that it may be one of the more helpful ways to conceptualize the integrals discussed in that course. Furthermore, this approach could be important for students who have not compiled this symbolic form in their mathematics course (whether it is due to the lack of time spent on this conceptualization or because they simply have not yet cognitively compiled it) so that they might have a chance to reconceive the integral in this way.

Implications for Curriculum

In a manner similar to the implications for instruction, this study has implications for curricular approaches to the integral. Since many courses may follow a pattern similar to that outlined in the calculus textbooks discussed in this chapter, it may be worthwhile to reconsider how the integral is laid out in standard curriculum. The integral is often motivated by a discussion of the area of a shape created by a graph in a coordinate system, which may lead to the compilation of the *area* symbolic form. Also, since much attention is dedicated to the computation of the integral as an anti-derivative, the students might reify this process into the *function mapping* symbolic form. However, the textbooks described in this chapter only dedicated one section to the integral as a Riemann sum, and these sections still emphasized the Riemann sum as a convenient way to find the area. Thus, a standard calculus course's curriculum does not appear to devote much attention to the integral as a process of adding up a quantity over several pieces. It is also possible, then, that students do not spend much time working with problems in class or in homework where the integral is depicted as an addition.

This may mean that the common curriculum is not designed to allow students enough time to compile the *adding up pieces* form, nor to practice working with integrals in this context. Instead, it may be necessary to revise common curriculum to allow more emphasis to be given to the integral as an addition. This conception of the integral does not need to eclipse the other conceptions of the integral nor replace them. But it may be important to allow the idea of the integral as an addition to receive as much attention as the other views of the integral. Furthermore, the idea of the integral as an addition should go beyond the area conception of the integral. While it is true that a Riemann sum is a useful way to calculate the irregular areas underneath the graphs of functions, it could be important to talk about additions that do not necessarily reflect an area. Discussions of other contexts that do not deal with areas might help students see the utility of thinking about the integral as an addition.

The so-called Harvard Calculus textbook (Hughes-Hallett, et al, 2005) provides an example of alterations that could promote the idea of the integral as an addition over several pieces. In stark contrast to the other, more standard textbooks, this book opens the discussion of integrals with a problem about how far a car has travelled given that its velocity is known. The problem is resolved by looking at approximations for the distance the car has travelled over two-second intervals, one-second intervals, half-second intervals, and so on. Thus, the idea that the small pieces of the distance travelled over each time interval are found and then added up is used to solve the problem. This resembles the *adding up pieces* symbolic form and would help the students toward compiling that form.

However, even this alternate approach does not necessarily give enough weight to the idea of the integral as an addition. The opening section concludes with the idea that the area underneath the velocity curve is the distance travelled and then the area model claims the majority of the rest of the conceptual discussion. Like the other textbooks, the bulk of the sections are devoted to techniques for finding anti-derivatives. While the techniques of integration and the area model are important components to learning the integral, the interpretation of the integral as an addition still does not necessarily receive enough attention. Yet, the approach to the integral in this textbook is certainly more in line with promoting the integral as an addition over several pieces by using an example to motivate the integral which depends on adding up pieces of distance.

Here is where the work of Sealey and Oehrtman (Engelke & Sealey, 2009; Sealey, 2006; Sealey & Oehrtman, 2007; Sealey & Oehrtman, 2005) and the work of Dray and Manogue (2003, 2004a, 2004b, 2006) become important for the discussion of curriculum. The work of Sealey and Oehrtman pushes into the boundaries of *how* students compile their understanding of the Riemann sum conception of the integral. This work could be used as a foundation for creating introductions to the integral as an addition. By extending the amount of conceptual development this idea receives, the students will have more opportunities for constructing the *adding up pieces* form. And if that presentation is based in what is known about how students come to understand the integral as an addition, these opportunities will be able to better support the compilation of this symbolic form and to better enable students to draw on it in subsequent work.

Sealey presents specific tasks that allow students to connect position, velocity, and acceleration to the Riemann sum concept. The integral is portrayed as an addition of lots of small distances. Dray and Manogue also present ways to introduce the integral through the general idea of “total amount.” This means that the integral would be presented in contexts where a solution to the problem requires an addition over a particular quantity in a problem. These tasks from Sealey and Dray and Manogue, the tasks found in the Harvard Calculus textbook, or other tasks like them can enrich students’ understanding of the integral. It may help them build a more well-rounded understanding by better supporting the compilation of the *adding up pieces* symbolic form. It may also give them practice drawing on this form in physical contexts. This would help prepare physics and engineering students for their future coursework.

Implications for Perceptions about Student Knowledge

This study also has implications for the way students entering an introductory physics course might be perceived. If students have difficulties applying mathematical knowledge to physics and engineering, it could be easy to ascribe this difficulty to a lack of mathematical knowledge. That is, the students cannot apply their knowledge because they simply *do not have* that knowledge. The results of this study described several students who demonstrated not only that they held productive knowledge about the integral, but that they held a large pool of productive knowledge. These students explained multiple ways of interpreting the entire integral symbol template, described conceptual meaning for individual symbols that relate to the integral, and provided productive reasoning about integrals in a variety of settings. Thus, for these nine students, it is not possible to say that they lack the necessary mathematical knowledge. If these students encountered difficulties, which several of them did during the interviews, it could not be blamed on inadequate mathematical preparation or insufficient understanding about the integral.

This study does not attempt to generalize these students' knowledge to the overall population. Other types of studies would be required to attempt to make assertions about what students know in general. Therefore, it is not possible to say that difficulties cannot be ascribed to inadequate mathematical knowledge. However, this study does provide an example of a group of introductory physics students who knew a significant amount about the integral. Furthermore, within this group there were students who did sometimes struggle to apply their understanding of the integral to problems that made use of physics contexts. For these students these difficulties did not always arise because of a lack of knowledge. Therefore, it is possible that the perception that difficulties are rooted in inadequate mathematical preparation may not describe the reality of the situation. Again, this study cannot make claims about the general state of student mathematical knowledge, but it shows that students who have productive understanding about the integral may still have difficulties. Thus, there is more at play than the amount of knowledge students have.

5.6 Future Research

This dissertation study sheds light on some of the cognitive resources that students hold regarding integrals and the contexts in which they might be activated. This study cannot possibly describe *all* of the possible cognitive resources, or even symbolic forms, that students have about the integral. Furthermore, there are many other calculus concepts that use specialized notations where students will compile symbolic forms. The derivative $\frac{d}{dx}f(x)$ can be arranged according to specific symbol templates. The symbols could be arranged as $\frac{d}{d[\,]}[\,]$, $\frac{d[\,]}{d[\,]}$, $\frac{d^{[\,]}[\,]}{d[\,]^{[\,]}}$, []'([]), or []'. Each of these symbol templates holds the potential for students to create symbolic forms that blend these symbols with a conceptual schema. The limit $\lim_{x \to c} f(x)$ could be construed as $\lim_{[\,] \to [\,]}[\,]$, $\lim_{[\,] \to \infty}[\,]$, or $\lim_{[\,] \to [\,]+}[\,]$. Again, these symbols may carry meaning for the students which could be compiled into symbolic forms.

With the myriad symbol structures from mathematics that are employed in physics and engineering contexts, there is much research left to be done in documenting and describing the cognitive resources, and specifically the symbolic forms, that students hold in their cognition regarding these symbols. Further studies should be conducted in these areas in order to better understand the knowledge that students have and activate in both mathematics and physics contexts. This understanding is important for building curriculum that will support the creation and activation of resources that are productive in applying mathematics to physics and engineering.

In addition to simply knowing the cognitive units that students possess, there is much work left to be done in understanding *why* students activate certain cognitive resources in a given situation. In effect, there is research that needs to be done in understanding how students *frame* the contexts they are in and how that framing affects the choice of resources that are activated. While research is being conducted in these areas (Gupta, Redish, & Hammer, 2008; Hammer, Elby, Scherr, & Redish, 2005; Tuminaro 2004; Hammer & Elby, 2003; MacLachlan & Reid, 1994), there is much more to do to fully understand the factors that influence student framing and why certain resources are activated over others. Student beliefs play a key role in the framing of a context, meaning that the explicit documentation of students' epistemological resources around calculus concepts in mathematics and physics contexts also needs attention. More studies may be done to understand the beliefs that students hold about the mathematics context and the physics context and how that influences their activation of the symbolic forms of the integral as they are applying it to physics and engineering.

Finally, as there are already significant efforts underway to create curriculum that attempts to bridge the gap between mathematics learning and physics learning, the knowledge about student cognitive resources, epistemological resources, and framing should be incorporated into existing curricular projects. The accepted notions about resources, beliefs, and framing should be tested through curricular devices that are meant to facilitate the creation of cognitive resources and their activation in productive ways in physics and engineering contexts. Studies that compare the effects of these curricula on "student success" would also be needed to verify the impact that these attempts to bridge the gap are having.

5.7 Summary

There is evidence that students hold productive knowledge about the mathematical concept of the integral. The students in this study demonstrated that they held a range of symbolic forms that were productive in interpreting the symbols of the integral and in making meaning out of the integral. Within a given interview item, many students were able to draw on multiple symbolic forms to understand the integral and to explain its conceptual meaning. Furthermore, the students in this study demonstrated that much of this knowledge provided useful tools for making sense of integrals in a physics context. Thus the results allow the conclusion that students (1) have productive knowledge about the integral and (2) have knowledge that is productive in understanding the integral in a physics context.

However, there is also evidence that there are some differences between the symbolic forms the students activated in the mathematics-framed setting and those activated in the physics-framed setting. Some symbolic forms, such as the *area* and the *function mapping* forms, were more common during the mathematics-day interview and were lesson common during the physics-day interview. By contrast, the *adding up pieces* symbolic form was drawn on much more frequently during the physics-day interview. Furthermore, the *adding up pieces* form appeared more productive in understanding the integral in the physics-framed items than the other forms. Thus it appears that particular ways of making sense of the integral are more useful in applying the mathematics to physics and engineering contexts. However, careful attention must be given to the creation and activation of the *adding up pieces* form as the elements of the form may be compiled differently, resulting in the problematic *add up then multiply* form. This may happen even if the student already possesses the *adding up pieces* form.

There is also a potential difficulty in moving from univariate integrals to multivariate integrals. The multivariate integrals are more common in a physics setting than in a mathematics setting and some of the symbolic forms that were productive in understanding a univariate integral did not appear as useful in making sense of a multivariate integral. On the other hand, the *adding up pieces* form was, again, more productive in handling these cases than the other forms. This supports the general utility of the *adding up pieces* form for understanding integrals in a physics or engineering context.

There is still much research to be done in fully understanding the cognitive resources available to students who are applying the concepts of calculus to physics and engineering. This study opens up a vein of research that could lead to insights into the symbolic forms that students hold about a range of calculus concepts. Additional studies could provide more understanding of the framing that students perform when working with mathematics in a physics or engineering context and the role of their beliefs in this framing and in the subsequent cognitive resource activation. Curriculum research can benefit from the results of these studies as educators take into account the knowledge and beliefs students have and the framing that supports the activation of productive resources.

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