ABSTRACT

Title of thesis: AN EXPOSITION ON STOCHASTIC INTEGRALS AND THEIR APPLICATION TO LINEARIZATION COEFFICIENTS

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Stochastic integration is introduced as a tool to address the problem of finding linearization coefficients. Stochastic, off-diagonal integration against a random spectral measure is defined and its properties discussed, followed by a proof that two formulations of Ito's Lemma are equivalent. Diagonals in $\mathbb{R}^n$ are defined, and their relationship to partitions of $\{1, \cdots, n\}$ is discussed. The intuitive notion of a stochastic integral along a diagonal is formalized and calculated. The relationship between partitions and diagonals is then exploited to apply Möbius inversion to stochastic integrals over different diagonals. Diagonals along which stochastic integrals may be nonzero with positive probability are shown to correspond uniquely to diagrams. This correspondence is used to prove the Diagram Formula. Ito's Lemma and the Diagram Formula are then combined to calculate the linearization coefficients for Hermite Polynomials. Finally, future work is suggested that may allow other families of linearization coefficients to be calculated.

AN EXPOSITION OF STOCHASTIC INTEGRALS AND THEIR APPLICATION TO LINEARIZATION COEFFICIENTS

by

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Preface

If you can dream - and not make dreams your master;

If you can think - and not make thoughts your aim;

If you can meet with triumph and disaster

And treat those two impostors just the same...

...Yours is the Earth and everything that's in it,

And - which is more - you'll be a Man my son!

-Rudyard Kipling, 'If'

Acknowledgments

My gratitude goes out to many people for helping me through every stage of this thesis. At the same time, I am very indebted to several people I would like to thank here, people without whom I would not have completed this work.

My wisest choice in the pursuit of my degree was my selection of advisor. Professor Slud spent many, many hours talking with me and reading my work (the quality of which, I can admit now, varied wildly). Through personal, professional, and academic setbacks, I could rely on Dr. Slud to engage me (and become engaged) with Brownian motion, stochastic integration, and linearization coefficients. I am quite certain that my study with him has made me a better mathematician– more precise in my understanding and articulation of particular theories of mathematics, and more aware of their significance. But I appreciate Professor Slud's encouragement and patience with me even more: my own patience, determination, and self-confidence would surely have flagged long ago had I chosen a less-capable mentor.

My family has been an invaluable resource for me during my study, as they are in the rest of my life. My mother and my two sisters have all completed very difficult degrees in the last decade. Their hard work and determination, in the face of great difficulties, has been a great encouragement to me and given me a framework in which I can step back and evaluate my progress from a more objective, and calm, position. My dad has instilled in me a great respect for higher education since I was young, and my graduate school experience (while much different from his) allows

me to recognize another similarity between us, of which I am very proud. Finally, my older brother, who was the arbiter of what was cool for me from an early age, demonstrated a respect for my study that helped my graduate study retain some of its appeal even when it seemed it would never end. My sister-in-law, my brother-in-law, and my four bright and funny nephews have made the entire graduate process more bearable.

Though many of my friends have very little interest in probability theory, a notable exception is Tanya McKay, a special friend who offered me the twin gifts, very rare in my experience, of close friendship and mathematical discourse. Several other friends have joined me in my studies, and I owe them thanks for their encouragement, as well: Dave Blagg and Amanda Carr, who started my graduate career with me; Scott Croshier, who began his thesis work shortly after I did; and Ken MacDonald, who finished his thesis before I did mine and was kind enough not to bring it up very often.

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I would also like to acknowledge my thesis reviewers, Professors Barg, Koralov, and Slud. Professor Barg provided me the problem that motivated my thesis and exercised patience as I drew out the writing of the thesis and diverged from the original problem. Professor Koralov's Probability Theory course was one of the best courses of my graduate career, and the theory I learned in that class was invaluable during the writing of my thesis.

Finally, though, the thesis was written by John Bynum Kuykendall. I would like to thank everyone and everything that helped bring me to this place today, and I am extremely blessed that my life would contain milestones such as this.

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Chapter 1

Introduction

1.1 Motivation: Linearization Coefficients

Given a complete system of orthogonal polynomials $\{P_i(x) : i \in \mathcal{I}\}$, the question arises how to calculate the *linearization coefficients* of the product of any two polynomials; in other words, for $P_n(x) \cdot P_m(x) = \sum_{i \in \mathcal{I}} c_i P_i(x)$, calculate c_i for all $i \in \mathcal{I}$.

For several families of polynomials, the study of *stochastic integrals*, in which integrals are defined against random-valued measures, produces formulae for linearization coefficients. In this paper, we articulate the combinatoric machinery implicit in [7] and [1] in order to better understand stochastic integration and, in the future, enhance the mathematical framework surrounding it so it may be applied to the problem of linearization coefficients in other families of polynomials. This machinery provides formulas for the linearization coefficients of Hermite polynomials, and could possible provide formulas for the polynomials arising from other Gaussian measures (e.g., measures with spectral atoms).

1.2 Stochastic Integrals

In order to extend a continuous measure M on $\mathbb{R}$ to a measure on all of $\mathbb{R}^n$, we usually use the product measure $M^n \equiv M \times \cdots \times M$. In the case of a scalar, deterministic measure M, M^n vanishes on all linear subspaces of $\mathbb{R}^n$. This is not necessarily true in the case of random-valued measure (even when the variances of increments decrease to zero with the size of the increments). For example: in the case that the measure is the random spectral measure of a stationary Gaussian process (the primary case used in this paper), each subspace $A_{i,j} = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_i = -x_j\}$ for $i \neq j$ has a positive (deterministic) measure. The notion of "off-diagonal" integration is developed to remove these linear subspaces from the area of integration. This method, which is related to the classic probabilistic method of inclusion-exclusion, leads to some very interesting combinatorial properties. In the case of random spectral measure, it provides a well-known method of calculating the linearization coefficients of the Hermite polynomials.

1.3 Outline of Thesis

In Chapter 2, Introduction to Stochastic Integration, we define a random measure and build the univariate and multivariate stochastic integral using functions that disappear along diagonals. Major gives a form of Itô's Lemma [5, p. 30] that demonstrates the connection between off-diagonal stochastic integrals with Gaussian measure and the Hermite polynomials (Chapter 2, p. 15). We then confirm that Major's formulation of Itô's Lemma is a special case of the more widely-known

formulation demonstrated in McKean [6, pp. 32-33]. This verification, through direct calculation, is one of the novel elements of this work. The link between the two is elsewhere explored in [3].

In Chapter 3, The Combinatorics of Stochastic Integrals, we formally define the notion of diagonal. The connection between partitions of $\{1, \cdots, n\}$ and diagonals in $\mathbb{R}^n$ is explored. We define what integration along a diagonal means, and we prove that in the case of a Gaussian random measure, stochastic integrals vanish along diagonals in which two variables are equal. Along diagonals in which two variables are equal in magnitude but are opposite in sign, integration reduces to deterministic integration in a single variable against the original spectral measure. We then apply the combinatorial theory of Möbius inversion to integration along diagonals using a partial ordering derived from partitions. This illustrates how an off-diagonal integral can be represented as a linear combination of stochastic integrals along other diagonals.

In Chapter 4, The Diagram Formula, we examine the relationship between combinatorial objects called "diagrams" and stochastic integrals. The product of two off-diagonal integrals is shown to be a stochastic integral evaluated along diagonals determined by the associated diagram, which results in the Diagram Formula. We then combine Itô's Lemma with the Diagram Formula to calculate the linearization coefficients of Hermite Polynomials.

In Chapter 5, Conclusion, we provide a retrospective overview of the work and propose some further mathematical research suggested by this combinatorial approach to stochastic integrals.

Chapter 2

An Introduction to Stochastic Integrals

2.1 Definition of Random Spectral Measure

The first step in constructing stochastic integrals is to define a specific type of random measure against which we will integrate. In this work, all Gaussian fields $\{X_n : n \in \mathbb{Z}^+\}$ are presumed real and satisfy the following properties:

1) discrete;

2) stationary;

3) with non-atomic spectral measures.

Definition 2.1.1 (Spectral Measure) *Any probability measure G on $[-\pi, \pi)$ such that*

$$E(X_0 X_n) = \int e^{inx} dG(x)$$

is called the spectral measure *associated with X_n.*

Remark 2.1.1 *G can be assumed to have all of $\mathbb{R}$ in its domain, but with positive support only on $[-\pi, \pi)$.*

By Bochner's theorem (Appendix, Theorem A.0.1), there exists a unique spectral measure for every discrete Gaussian stationary field. Since $EX_0X_n = EX_nX_0$, it follows that $\int e^{inx} dG(x) = \int e^{-inx} dG(x)$, and therefore G is symmetric about the origin: i.e., $G(A) = G(-A)$ for all Borel sets A, where $-A \equiv \{-x : x \in A\}$.

Remark 2.1.2 *Unless otherwise noted, all notions of convergence and closedness are in the usual L^2_G sense.*

Notation 2.1.1 *Let $fin(\mathcal{L}^{\mathbb{C}}_n)$ be the linear span of $\{X_{i_1} \cdots X_{i_n} : \ i_1, \cdots, i_n \in \mathbb{Z}\}$ over $\mathbb{C}$ and let $\mathcal{L}^{\mathbb{C}}_n$ be the Hilbert space closure of $fin(\mathcal{L}^{\mathbb{C}}_n)$. The inner product of $A, B \in \mathcal{L}^{\mathbb{C}}_n$ is defined to be $E(A\overline{B})$.*

Note that for any $A, B \in \mathcal{L}^{\mathbb{C}}_n$, although the inner product $\langle A, B\rangle_{\mathcal{L}^{\mathbb{C}}_n} = E(A\overline{B})$ may be complex, the norm $\|A\|_{\mathcal{L}^{\mathbb{C}}_n} = \sqrt{E(A\overline{A})}$ is real.

Let $fin(L^2_G)$ be all finite linear combinations of the form

$\{\sum_{j=-m}^{m} c_j e^{ijx} : m \in \mathbb{Z}^+, c_j \in \mathbb{C}\}$. The following mapping, $I : fin(L^2_G) \to fin(\mathcal{L}^{\mathbb{C}}_1)$, leads to our first random measure. Let $c_j \in \mathbb{C}\ \forall j, m \in \mathbb{Z}^+$.

Definition 2.1.2

$$I(\sum_{j=-m}^{m} c_j e^{ijx}) = \sum_{j=-m}^{m} c_j X_j$$

Lemma 2.1.1 *I is an isometry from $fin(L^2_G)$ to $fin(\mathcal{L}^{\mathbb{C}}_1)$.*

Proof: Prove that I is norm-preserving and therefore an isometry (see Appendix, Theorem A.0.2).

$$\begin{aligned} ||I(\sum_{j=-m}^{m} c_j e^{ijx})||^2_{\mathcal{L}^{\mathbb{C}}_1} &= ||\sum_{j=-m}^{m} c_j X_j||^2_{\mathcal{L}^{\mathbb{C}}_1} \\ &= \sum_{j,k=-m}^{m} c_j \overline{c_k} E(X_j X_k) \end{aligned}$$

$$
\begin{aligned}
&= \int (\sum_{j=-m}^{m} c_j e^{ixj})(\sum_{k=-m}^{m} \overline{c_k} e^{-ixk}) dG(x) \\
&= ||\sum_{j=-m}^{m} c_j e^{ijx}||^2_{L^2_G}
\end{aligned}
$$

□

So I is norm-preserving and can be extended to a function from all of L^2_G to all of $\mathcal{L}^{\mathbb{C}}_1$. We will refer to this extended function as I for the rest of this work.

Definition 2.1.3 (Random Spectral Measure) *Define the* random spectral measure $Z_G(A)$ *associated with* G*, for any Borel set* A*, by:*

$$Z_G(A) \equiv I(\chi_A),$$

where χ_A *is the indicator function of the set* A*.*

2.2 Properties of Random Spectral Measure

The subscript G will be suppressed and the random spectral measure denoted by Z when there is no ambiguity.

Lemma 2.2.1 *Let* A, B *be Borel sets in* $\mathbb{R}$*.*

1) $EZ(A) = 0$*.*

2) $Z(A) = \overline{Z(-A)}$*.*

3) $E(Z(A)\overline{Z(B)}) = G(A \bigcap B)$*.*

4) $Z(A) = U(A) + iV(A)$ *for some real-valued, Gaussian random variables* $U(A)$*,* $V(A)$*.*

5) $U(-A) = U(A), V(-A) = -V(A)$

6) $U(A)$ *and* $V(A)$ *are independent.*

7) $U(A),\ V(A) \sim \mathcal{N}(0, \frac{G(A)}{2})\ \forall A$ *such that* $A \bigcap (-A) = \emptyset$.

Proof: In the following proofs, we follow the approach of Major [5, p. 17-21] Due to the denseness of the trigonometric polynomials in the space of complex L^2_G functions, there exist $c_k \in \mathbb{C}, k \in \mathbb{Z}$ such that

$$\chi_A(x) = \lim_{n\to\infty} \sum_{k=-n}^{n} c_k e^{ikx}.$$

1) $EZ(A) = \lim_{n\to\infty} \sum_{k=-n}^{n} c_k EX_k = 0.$

2) $\sum_{k=-n}^{n} \overline{c_k} e^{-ikx} = \overline{\sum_{k=-n}^{n} c_k e^{ikx}} \to \overline{\chi_A} = \chi_A$ Therefore,

$$\frac{1}{2}(\sum_{k=-n}^{n} c_k e^{ikx} + \sum_{k=-n}^{n} \overline{c_k} e^{-ikx}) \to \chi_A$$

Let $d_k = \frac{1}{2}(c_k + \overline{c_{-k}})$. Then $\overline{d_k} = d_{-k}$ and $\lim_{n\to\infty} \sum_{k=-n}^{n} d_k e^{ikx} = \chi_A$.

$$\begin{aligned}
Z(-A) &= I(\chi_{-A}) \\
&= I(\lim_{n\to\infty} \sum_{j=-n}^{n} d_j e^{-ijx}) \\
&= \lim_{n\to\infty} \sum_{j=-n}^{n} d_j X_{-j} \\
&= \lim_{n\to\infty} \sum_{j=-n}^{n} \overline{d_j} X_j \\
&= \overline{\lim_{n\to\infty} \sum_{j=-n}^{n} d_j X_j} \\
&= \overline{I(\lim_{n\to\infty} \sum_{j=-n}^{n} d_j e^{ijx})} \\
&= \overline{Z(A)}
\end{aligned}$$

3) By the isometry property of I,

$$\begin{aligned} E(Z(A)\overline{Z(B)}) &= \langle I(\chi_A), I(\chi_B)\rangle_{\mathcal{L}_1^{\mathbb{C}}} \\ &= \langle \chi_A, \chi_B\rangle_{L^2_G} \\ &= \int \chi_A \overline{\chi_B} dG(x) \\ &= G(A \bigcap B) \end{aligned}$$

4) Since $Z(A)$ is the limit of sums of jointly complex-valued Gaussian variables, it is itself a complex-valued Gaussian variable.

5) By 2), $U(A) + iV(A) = U(-A) - iV(-A)$.

6) In order to show $U(A), V(A)$ independent, we show their covariance is zero. Observe:

$$\begin{aligned} U(A) &= \frac{Z(A) + Z(-A)}{2} \\ V(A) &= \frac{Z(A) - Z(-A)}{2i} \end{aligned}$$

So the covariance is:

$$\begin{aligned} E(U(A)V(A)) &= E(\frac{Z(A) + Z(-A)}{2} \cdot \frac{Z(A) - Z(-A)}{2}) \\ &= \frac{1}{4i} E(Z(A)^2 - Z(-A)^2) \\ &= \frac{1}{4i} E(Z(A)\overline{Z(-A)} - Z(-A)\overline{Z(A)}) \\ &= \frac{1}{4i}(G(A \bigcap (-A)) - G((-A) \bigcap A)) \\ &= 0 \end{aligned}$$

Therefore, $U(A), V(A)$ are independent of each other.

7) By 4), $U(A), V(A)$ are both Gaussian random variables. By 1), $EU(A) =$

$EV(A) = 0$. Now let A be a Borel set such that $A \bigcap (-A) = \emptyset$.

$$\begin{aligned} E(Z(A)^2) &= \langle Z(A), \overline{Z(A)} \rangle^2_{\mathcal{L}^{\mathbb{C}}_1} \\ &= \langle \chi_A, \chi_{-A} \rangle^2_{L^2_G} \\ &= \int_{-A \bigcap A} dG(x) \\ &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} E((U(A) + iV(A))^2) &= E(U(A)^2) - E(V(A)^2) + 2iE(V(A))E(U(A)) \\ &= 0 \end{aligned}$$

so $EU(A)^2 = EV(A)^2$. By 3), $EU(A)^2 + EV(A)^2 = E|Z(A)|^2 = G(A)$, so $EU(A)^2 = EV(A)^2 = \frac{G(A)}{2}$. □

2.3 Definitions of Classes of Integrands, Simple Integrands, and Intervals

In order that the off-diagonal stochastic integrals be real-valued, the integrands in this work are limited to the following class of functions, H_n^G:

Definition 2.3.1 ($\boldsymbol{H_n^G}$) *$f \in H_n^G$ if and only if:*

1) $f : \mathbb{R}^n \mapsto \mathbb{C}$;

2) $f \in L^2_{G^n}$;

3) $f(x_1, ..., x_n) = \overline{f(-x_1, ..., -x_n)}$; *and*

4) $f(x_1, \cdots, x_n) = f(x_{\pi(1)}, \cdots, x_{\pi(n)})$ *for all permutations* $\pi \in S_n$.

Properties 3 and 4 state that all functions in H_n^G are *Hermitian symmetric* and *permutation symmetric*.

Definition 2.3.2 *The inner product of* $f, g \in H_n^G$ *is*

$$\begin{aligned} \langle f, g \rangle_{H_n^G} &\equiv n! \langle f, g \rangle_{L^2_{G^n}} \\ &= n! \int f(x_1, \cdots, x_n) \overline{g(x_1, \cdots, x_n)} dG(x_1) \cdots dG(x_n) \end{aligned}$$

We define a set of intervals used throughout this work as a step towards defining the off-diagonal stochastic integral.

Definition 2.3.3 *For any given* $l \in \mathbb{Z}^+$, $k \in \{1, 2, \cdots, l \cdot 2^l - 1\}$, *let* $J_k^l = (\frac{k-1}{2^l}, \frac{k}{2^l})$ *be an interval on the real line, and let* $J_{-k}^l = -J_k^l$ *be its reflection over the origin. Let* $J_{l \cdot 2^l}^l = (l - \frac{1}{2^l}, +\infty)$ *and* $J_{-l \cdot 2^l}^l = (-\infty, -l + \frac{1}{2^l})$. *Let* $\mathcal{J}_l = \{J_k^l : -l \cdot 2^l \leq k \leq l \cdot 2^k\}$ *be the collection of all of these integrals. We denote the index set of the finite-length intervals by* $\mathcal{K}^l = \{-l \cdot 2^l + 1, \cdots, l \cdot 2^l - 1\}$.

Then $\{\mathcal{J}_1, \mathcal{J}_2, \cdots\}$ is a nested set of partitions. At each stage of refinement, a finite number of endpoints are lost, since the partition intervals are open: since G is non-atomic, this is not a problem. Though we use this particular set of partitions in our proofs, the proofs hold for any nested set of partitions whose mesh (over any compact set) goes to zero. Define the following class of Hermitian symmetric, permutation symmetric simple functions on $\mathbb{R}$:

Definition 2.3.4 *For any given* $l \in \mathbb{Z}^+$, *let* $simp^l(\mathbb{R})$ *denote the following class of*

simple, finite, piecewise-constant functions from $\mathbb{R}$ *to* $\mathbb{C}$*:*

$$simp^l(\mathbb{R}) \;=\; \{g \in H_1^G : g(x) = \sum_{j \in \mathcal{K}^l} c_j \chi_{J_j^l}(x)\}$$

where $c_j \in \mathbb{C}\ \forall j$*.*

2.4 Definition of Univariate Stochastic Integral

We define the univariate stochastic integral with integrands in $simp^l(\mathbb{R})$ as a mapping from the class of simple functions $simp^l(\mathbb{R})$ above to random variables in $\mathcal{L}_1^{\mathbb{C}}$:

Definition 2.4.1 *Define the mapping* $\int dZ(x) : \bigcup_l simp^l(\mathbb{R}) \mapsto \mathcal{L}_1^{\mathbb{C}}$*: for* $g \in simp^l(\mathbb{R})$*,*

$$\int g(x)dZ(x) = \sum_{i \in \mathcal{K}^l} g(m_i^l)(Z(J_i^l))$$

An equivalent definition arises more directly from the isometry I:

$$\begin{aligned} \int g(x)dZ(x) &= \sum_{k \in \mathcal{K}^l} g(m_k^l)(Z(J_k^l)) \\ &= I(\sum_{k \in \mathcal{K}^l} g(m_k^l)\chi_{J_k^l}) \\ &= I(g(x)) \end{aligned}$$

Remark 2.4.1 *The above integral is evaluated on all of* $\mathcal{J}_l$*. To limit the area of integration to only those intervals in* $\mathcal{J}_l$ *that fall within the range* (a, b)*, for some* $a, b \in \mathbb{R}$*, the definite integral notation* $\int_a^b g(x)dZ(x)$ *can be adopted. We will call these* definite *stochastic integrals. Note that, because* (a, b) *may not be symmetric across the origin, the integral may be complex-valued.*

Indefinite stochastic integrals are real-valued, as are definite stochastic integrals evaluated on any $(-t, t)$:

$$\begin{aligned}
&\int g(x)dZ(x) \\
&= \sum_{j\in\mathcal{K}^l} g(m_j^l)(Z(J_j^l)) \\
&= \sum_{j\in\mathcal{K}^l} (Re(g(m_j^l)) + i\cdot Im(g(m_j)^l))(U(J_j^l) + i\cdot V(J_j^l)) \\
&= \sum_{j\in\mathcal{K}^l, j>0} (Re(g(m_j^l))U(J_j^l) - Im(g(m_j^l))V(J_j^l) + i(Im(g(m_j^l))U(J_j^l) + Re(g(m_j^l))V(J_j^l))) \\
&+ \sum_{j\in\mathcal{K}^l, j<0} (Re(g(m_j^l))U(J_j^l) - Im(g(m_j^l))V(J_j^l) + i(Im(g(m_j^l))U(J_j^l) + Re(g(m_j^l))V(J_j^l))) \\
&= 2\cdot \sum_{j\in\mathcal{K}^l, j>0} (Re(g(m_j^l))U(J_j^l) - Im(g(m_j^l))V(J_j^l))
\end{aligned}$$

Since G is non-atomic, this class of simple functions on open intervals is dense in H_1^G. Since I is an isometry, the domain of the univariate stochastic integral can be extended from $simp^l(\mathbb{R})$ to all of H_1^G.

2.5 Definition of Multivariate Stochastic Integral

For the multivariate integral case, we again begin by restricting our integrands, this time to a set of simple-function integrands defined on $\mathbb{R}^n$. Let $\mathcal{J}_l = \{J_k^l : k \in \mathcal{K}^l\}$ be as above.

Definition 2.5.1 *Define $simp^l(\mathbb{R}^n)$ as the set of functions $g : \mathbb{R}^n \to \mathbb{C}$ such that:*

1) $g(x_1, \cdots, x_n) = \sum_{i_1\in\mathcal{K}^l} \cdots \sum_{i_n\in\mathcal{K}^l} c_{i_1,\cdots,i_n} \chi_{J_{i_1}^l}(x_1)\cdots\chi_{J_{i_n}^l}(x_n)$

2) $g \in H_n^G$ *(i.e., g is Hermitian symmetric and permutation symmetric)*

3) $c_{i_1,\cdots,i_n} = 0$ *if* $|i_j| = |i_l|$ *for any* $j \neq l$

Remark 2.5.1 *These functions always take the value* 0 *along subspaces in which* $|x_j| = |x_l|$ *for some* $j \neq l$.

Define the mapping I_n from $\bigcup_l simp^l(\mathbb{R}^n)$ to $\mathcal{L}_n^{\mathbb{C}}$:

Definition 2.5.2

$$\begin{aligned} & I_n(\sum_{i_1 \in \mathcal{K}^l} \cdots \sum_{i_n \in \mathcal{K}^l} c_{i_1,\cdots,i_n} \chi_{J_{i_1}^l}(x_1) \cdots \chi_{J_{i_n}^l}(x_n)) \\ = & \sum_{i_1 \in \mathcal{K}^l} \cdots \sum_{i_n \in \mathcal{K}^l} c_{i_1,\cdots,i_n} I(\chi_{J_{i_1}^l}(x_1)) \cdots I(\chi_{J_{i_n}^l}(x_n)) \\ = & \sum_{i_1 \in \mathcal{K}^l} \cdots \sum_{i_n \in \mathcal{K}^l} c_{i_1,\cdots,i_n} Z(J_{i_1}^l) \cdots Z(J_{i_n}^l) \end{aligned}$$

We define $I_0 \equiv 1$.

The property $Z(-J_k) = \overline{Z(J_k)}$ is called Hermitian symmetry. Hermitian symmetry of the multivariate random measure $Z(J_{i_1}^l) \cdot Z(J_{i_2}^l) \cdot \cdots \cdot Z(J_{i_n}^l)$ is immediate from the univariate case; that is, $Z(-J_{i_1}^l) \cdot Z(-J_{i_2}^l) \cdot \cdots \cdot Z(-J_{i_n}^l) = \overline{Z(J_{i_1}^l) \cdot Z(J_{i_2}^l) \cdot \cdots \cdot Z(J_{i_n}^l)}$.

Lemma 2.5.1 $I_n(g(x_1, \cdots, x_n))$ *is a real-valued random variable for all* $g \in simp^l(\mathbb{R}^n)$.

Proof: It is clear that $I_n(g(x_1, \cdots, x_n)) = \overline{I_n(g(x_1, \cdots, x_n))}$ and is therefore real-valued. □

We can extend I_n so its domain is all of H_n^G:

Lemma 2.5.2 $I_n : simp^l(\mathbb{R}^n) \mapsto \mathcal{L}_n^{\mathbb{C}}$ *is an isometry.*

Proof: Let $g, h \in simp^l(\mathbb{R}^n)$.

$$\langle I_n(g(x_1, \cdots, x_n)), I_n(h(x_1, \cdots, x_n))\rangle_{\mathcal{L}_n^{\mathbb{C}}} \tag{2.1}$$

$$= \sum_{j_1 \in \mathcal{K}^l} \cdots \sum_{k_n \in \mathcal{K}^l} g(m_{j_1}^l, \cdots, m_{j_n}^l)\overline{h(m_{k_1}^l, \cdots, m_{k_n}^l)}$$
$$E(Z(J_{j_1}^l) \cdots Z(J_{j_n}^l)\overline{Z(J_{k_1}^l)} \cdots \overline{Z(J_{k_n}^l)}) \tag{2.2}$$

$$= n! \sum_{j_1 \in \mathcal{K}^l} \cdots \sum_{k_n \in \mathcal{K}^l} g(m_{j_1}^l, \cdots, m_{j_n}^l)\overline{h(m_{k_1}^l, \cdots, m_{k_n}^l)}G(J_{j_1}^l) \cdots G(J_{j_n}^l) \tag{2.3}$$

In order to get from line 2.2 to line 2.3, each $Z(J_{j_p}^l)$ must pair with a $\overline{Z(J_{k_q}^l)}$ in order to create $G(J_{j_p}^l)$. There are $n!$ ways of matching up the n distinct intervals. Then line 2.3 is equal to

$$n! \int g(x_1, \cdots, x_n)\overline{h(x_1, \cdots, x_n)}dG(x_1) \cdots dG(x_n) \tag{2.4}$$

$$= \langle f, g\rangle_{H_n^G} \tag{2.5}$$

Since the norm is preserved under I_n for all finite combinations of elements of $simp^l(\mathbb{R}^n)$, the mapping is an isometry (Appendix, Theorem A.0.2). □

If the permutation and Hermitian symmetric restrictions were not placed on $simp^l(\mathbb{R}^n)$, it would be dense in L_G^2. Therefore, $simp^l(\mathbb{R}^n)$ is dense in H_n^G, and we can extend the domain of I_n to all of H_n^G: the range is increased to all of $\mathcal{L}_n^{\mathbb{C}}$. From now on, I_n will refer to the function with this extended domain and range. In future chapters, we change our notation to reflect the lack of support along the diagonals:

Notation 2.5.1 *For any $n \in \mathbb{Z}, g \in H_n^G$,*

$$\int\!\!\!\!\!\!\diagup g(x_1, \cdots, x_n)dZ(x_1) \cdots dZ(x_n) = I_n(g(x_1, \cdots, x_n))$$

2.6 Itô's Lemma

The class of functions that can be represented by stochastic integrals is closed under composition with smooth (i.e., twice differentiable) functions. This is shown via Itô's Lemma, which illuminates the close relationship between stochastic integrals and the Hermite polynomials.

Definition 2.6.1 (Hermite Polynomials) *The Hermite polynomials are the series of orthogonal polynomials with respect to the measure $e^{-x^2/2}$, defined by the formulae:*

$$\begin{aligned} H_0(x) &= 1 \\ H_n(x) &= (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} \ for \ n \geq 1. \end{aligned}$$

Now we state a major formula of stochastic integrals: Itô's Lemma. Note that this is a specialized version of the formula, developed by Major, strictly dealing with random spectral measures Z. Then we examine the relationship between Major's formulation and the more traditional one [6, p. 32]

Lemma 2.6.1 (Itô's Lemma, Major) *Let Z be the random spectral measure of a measure G. Let $p_1, \cdots, p_m$ be an orthonormal set of functions in H^1_G. Let $j_1, \cdots, j_m \in \mathbb{Z}$ be given, and let $n = \sum_{i=1}^m j_i$. Define the functions g_a, for $a \in \{1, \cdots, n\}$, such that $g_a = p_k$ for $\sum_{i=1}^{k-1} j_i < a \leq \sum_{i=1}^k j_i$. Then:*

$$H_{j_1}(\int p_1(x)dZ(x)) \cdots H_{j_m}(\int p_m(x)dZ(x)) = \not\int g_1(x_1) \cdots g_N(x_n) dZ(x_1) \cdots dZ(x_n)$$

where $H_i(x)$ is the i'th Hermite polynomial.

Since the Hermite polynomials form a basis for all L^2_G functions, this formula implies that the class of off-diagonal stochastic integrals is closed under composition with polynomials.

2.7 Reconciling Major and McKean on Itô's Lemma

Major's simplified version of Itô's Lemma turns out to be sufficient for our purposes, so we concern ourselves here with a proof that Major's version is indeed a specialization of McKean's. First, we state a slightly modified version of McKean's formulation.

Lemma 2.7.1 (Ito's Lemma, McKean) *Let $f(t, x_1, ..., x_n)$ with domain $[0, \infty) \times \mathbb{R}^n$ have continous second partial derivatives. Let $\{W_i, 1 \leq i \leq n\}$ be a set of Wiener processes such that, for all i, j, W_i is either independent of or identical to W_j. Choose n stochastic integrals:*

$$g_i(t) = a_i + \int_0^t b_i(s)dW_i(s) + \int_0^t c_i(s)ds$$

where $a_i, b_i, c_i \in H_1^G \; \forall i$ Then the composition $F(t) = f(t, g_1(t), ..., g_n(t))$ is a stochastic integral of the following form:

$$\begin{aligned} & dF(t) = \\ & \frac{\partial f}{\partial t}(t, g_1(t), ..., g_n(t))dt + \sum_{i=t}^{n} \frac{\partial f}{\partial x_i}(t, g_1(t), ..., g_n(t))dg_i(t) \\ + \; & \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(t, g_1(t), ..., g_n(t))d\langle g_i, g_j \rangle(t) \end{aligned}$$

where $d\langle g_i, g_j \rangle(t) = b_i(t)b_j(t)\Delta_{ij}(t)dt$ and $\Delta_{ij}(t) = 1$ if W_i and W_j are identical and is 0 otherwise. [6, pp. 32-33]

In the fully-generalized version of Itô's Lemma, McKean allows a_i, b_i, and c_i to exist in a much larger class of functions. Why is Major's formulation (Lemma 2.6.1) good enough for our purposes? The Hermite polynomials in a single standard-normal variable X_0 are dense in the $L^2(P)$ functions measurable with respect to X_0, and a similar statement can be made for multivariable tensor products of Hermite polynomials in orthonormal variables in the linear space spanned by $\{X_t\}$. Since these Hermite polynomials are found to be in the range of stochastic integrals of all orders, the linear span of such integrals spans the linear space that $\{X_{t_1} \cdots X_{t_n} : n \in \mathbb{Z}^+, t \in \mathbb{R}^+\}$ spans. In order to reconcile these two formulations (at least for products of Hermite polynomials in a single first-order stochastic integral), we take f to be a Hermite polynomial. With a proper Wiener integral $\int_0^t g(x)dW(x)$ as an argument, both of the above lemmas produce identical values for $df(\int g(x)dW(x))$. We elaborate these arguments in the following two subsections.

2.7.1 Stochastic Integrals Reduce to Integration Against a Wiener Process

McKean is integrating against several Wiener processes, while Major is integrating against a complex-valued random spectral measure. Since Z is composed of a Gaussian real and a Gaussian complex part, however, we can reduce integration against dZ to integration of a complex function against Wiener processes. As before, we can decompose Z into two independent, real-valued Gaussian independent-increments processes: U and V, where $Z(J_j^l) = U(J_i^j) + iV(J_i^j)$. Let $g(x) \in H_1^G$ be

defined as $g(x) = g_1(x) + ig_2(x)$ for real-valued functions $g_1, g_2 \in H_1^G$. Then:

$$\int g(x)dZ(x) \quad = \quad \int (g_1(x) + ig_2(x))dU(x) + \int (-g_2(x) + ig_1(x))dV(x)$$

So the stochastic integral reduces to two integrals, each against a real-valued Gaussian random-variable. Now we show that any of our first-order stochastic integrals against dZ can be reduced to an integral against a Wiener process. Let $h \in simp^l(\mathbb{R})$ and U be as above. Notice that $U(\cdot) = W_1 \circ \frac{1}{2}G(\cdot)$ for a Wiener process W_1, in the sense of equality in distribution of the entire stochastic process. Similarly, $V(\cdot) = W_2 \circ \frac{1}{2}G(\cdot)$ for a Wiener process W_2 that is independent of W_1. So Major's integration is equivalent to integration against Wiener processes, as in McKean.

2.7.2 Proving Major is a Special Case of McKean

Now we take f in McKean's formulation of the lemma to be a Hermite polynomial, and we prove that the resulting stochastic differential equation is the same one satisfied by the multiple Wiener-Ito integral that Major claims it should be. We first want to examine a non-anticipating f that depends on only one Wiener process, so we would like to set $f(W(t)) = H_n(\int_0^t q(s)dW(s))$ for some function q in H_G^1. However, we must normalize the Wiener integral, as Major requires all his component functions be orthonormal. So we instead calculate $dH_n(\frac{\int_0^t q(s)dW(s)}{\sqrt{\int_0^t q^2(s)ds}})$ using McKean's

version of the lemma. For brevity's sake, let $p(x) = \int_0^x q(s)dW(s)/\sqrt{\int_0^t q^2(s)ds}$:

$$\begin{aligned} dH_n(p(t)) &= 0 \cdot dt + H_n'(p(t)) \cdot d(p(t)) + \frac{1}{2}H_n''(p(t)) \cdot d\langle p(t), p(t)\rangle \qquad (2.6) \\ &= H_n'(p(t)) \cdot \Big[\frac{q(t)}{\sqrt{\int_0^t q^2(s)ds}}dW - \frac{q^2(t)}{2}\frac{\int_0^t q(s)dW(s)}{(\int_0^t q^2(s)ds)^{3/2}}\Big] \\ &\quad + \frac{1}{2}H_n''(p(t)) \cdot \frac{q^2(t)}{\int_0^t q^2(s)ds}dt \qquad (2.7) \end{aligned}$$

Line 2.6 is simply a substitution into McKean's formula. In line 2.7, we expand $d(p(t))$ and $d\langle p(t), p(t)\rangle$. Then we separate the term containing dW from the terms containing dt in line 2.8:

$$\begin{aligned} &= H_n'(p(t))\frac{q(t)}{\sqrt{\int_0^t q^2(s)ds}}dW \\ &\quad + \frac{1}{2}\frac{q^2(t)}{\int_0^t q^2(s)ds}dt\big[H_n''(p(t)) - p(t) \cdot H_n'(p(t))\big] \qquad (2.8) \\ &= H_n'(p(t))\frac{q(t)}{\sqrt{\int_0^t q^2(s)ds}}dW - \frac{n}{2}\frac{q^2(t)dt}{\int_0^t q^2(s)ds}H_n(p(t)) \qquad (2.9) \end{aligned}$$

In passing from line 2.8 to 2.9, we have applied the Hermite identity $H_n''(x) - xH_n'(x) = -nH_n(x)$ (Lemma A.0.1). If $n = 0$, then McKean's formula reduces to the trivial identity $0 = 0$, as does Major's (since we defined $I_0 = 0$ [in Definition 2.5.2]). So now let us inductively assume that McKean's and Major's formulas match for all $n < N$. In other words,

$$\begin{aligned} & H_n'(p(t))\frac{q(t)}{\sqrt{\int_0^t q^2(s)ds}}dW - \frac{n}{2}\frac{q^2(t)dt}{\int_0^t q^2(s)ds}H_n(p(t)) \\ = \; & dH_n(p(t)) \\ = \; & d\Big(\int_0^t \cdots \int_0^t \frac{q(s_1)}{\sqrt{\int_0^t q(s)ds}} \cdots \frac{q(s_n)}{\sqrt{\int_0^t q(s)ds}}dW(s_1)\cdots dW(s_n)\Big) \end{aligned}$$

for all $n < N$. Our aim now is to show the formulas match for $n = N$.

$$
\begin{aligned}
& dH_N(p(t)) \\
by\ Major \quad &= d\Big(\int_0^t \cdots \int_0^t \frac{q(s_1)}{\sqrt{\int_0^t q^2(s)ds}} \cdots \frac{q(s_N)}{\sqrt{\int_0^t q^2(s)ds}} dW(s_1) \cdots dW(s_N) \Big) \quad (2.10) \\
&= \frac{-N}{2} \Big(\int_0^t q^2(s)ds \Big)^{-N/2} \frac{q^2(t)dt}{\int_0^t q^2(s)ds} \\
&\quad \cdot N \cdot \int_0^t q(s_1) \cdot \Big[\int_0^{s_1} \cdots \int_0^{s_1} q(s_2) \cdots q(s_N) dW(s_2) \cdots dW(s_N) \Big] dW(s_1) \\
&\quad + \Big(\int_0^t q^2(s)ds \Big)^{-N/2} q(t) \cdot N \\
&\quad \cdot \int_0^t \cdots \int_0^t q(s_2) \cdots q(s_N) dW(s_2) \cdots dW(s_N) dW(t) \qquad (2.11)
\end{aligned}
$$

In expression 2.11, we apply the chain rule. First, we collect the denominators together as the term $(\int_0^t q^2(s)ds)^{-N/2}$ and differentiate it. On the next line, we turn the multiple Wiener-Ito integral into an iterated integral. Since all the integrands are identical, we can replace its domain of integration with one where s_1 is always greater than or equal to $s_2, \cdots, s_N$ and then multiply that integral by N, since any of the N variables could be largest. The last two lines are the other half of the chain rule differentiation, as we differentiate the iterated integral.

The iterated integral technique above creates a nonanticipating, random integrand, which falls outside the scope of Major's theory. A more general definition of the stochastic integral, as in [6], allows such integrands, and it is a necessary bridge between the two formulations of Itô's Lemma.

In line 2.12, we move the term $(\int_0^t q^2(s)ds)^{-N/2}$ back inside the first integral to properly scale the integrands and apply the inductive hypothesis to convert the stochastic integral of degree $N-1$ into a Hermite polynomial. We perform virtually

the same operation on the second integral, except we only move $(\int_0^t q^2(s)ds)^{(-N+1)/2}$ inside the integral.

$$\begin{aligned} = & \frac{-N}{2}\frac{q^2(t)dt}{\int_0^t q^2(s)ds}\cdot N\cdot\int_0^t \frac{q(s_1)}{\sqrt{\int_0^t q^2(s)ds}}H_{N-1}(p(s_1))dW(s_1) \\ & +\frac{q(t)}{\sqrt{\int_0^t q^2(s)ds}}NH_{N-1}(p(t))dW(t) \end{aligned} \tag{2.12}$$

In line 2.13, we apply the identity $H_n'(x) = nH_{n-1}(x)$.

$$\begin{aligned} = & \frac{-N}{2}\frac{q^2(t)dt}{\int_0^t q^2(s)ds}\int_0^t H_N'(p(s_1))\frac{q(s_1)}{\sqrt{\int_0^t q^2(s)ds}}dW(s_1) \\ & +\frac{q(t)}{\sqrt{\int_0^t q^2(s)ds}}H_N'(p(t))dW(t) \end{aligned} \tag{2.13}$$

Line 2.14 is a straightforward substitution, while in line 2.15 we integrate the Hermite polynomial.

$$\begin{aligned} & = \frac{-N}{2}\frac{q^2(t)dt}{\int_0^t q^2(s)ds}\int_0^t H_N'(p(s_1))dp(s_1) \\ & \quad +\frac{q(t)}{\sqrt{\int_0^t q^2(s)ds}}H_N'(p(t))dW(t) \end{aligned} \tag{2.14}$$

$$\begin{aligned} & = \frac{-N}{2}\frac{q^2(t)dt}{\int_0^t q^2(s)ds}H_N(p(t)) \\ & \quad +\frac{q(t)}{\sqrt{\int_0^t q^2(s)ds}}H_N'(p(t))dW(t) \end{aligned} \tag{2.15}$$

$$by\ McKean \quad = \quad dH_N(p(t)) \tag{2.16}$$

So we have shown that substituting a stochastic integral into a single Hermite polynomial yields the same result, whether we use McKean's or Major's version of the lemma.

Chapter 3

The Combinatorics of Stochastic Integrals

In the last chapter, we defined the off-diagonal stochastic integral $\int\!\!\!\!\!\!\diagup g(x_1, \cdots, x_n) dZ(x_1) \cdots dZ(x_n)$ by taking limits of sums over the complement of the areas in $\mathbb{R}^n$ where two or more coordinates were equal in magnitude. In this chapter, we extend the domain of integration of the stochastic integral to include all of $\mathbb{R}^n$.

3.1 Diagonals

Definition 3.1.1 *Let $D_{i_i, i_2, \cdots, i_n} \subset \mathbb{R}^n$, where $i_j \in \{\pm 1, \cdots, \pm n\}$, be defined as follows:*

$$D_{i_1, \cdots, i_n} \equiv \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_j = x_k \Leftrightarrow i_j = i_k, x_j = -x_k \Leftrightarrow i_j = -i_k\}$$

Each such $D_{i_1, \cdots, i_n}$ is called a diagonal *of the space $\mathbb{R}^n$. In order that each diagonal is represented uniquely, the indices $\{i_1, \cdots, i_n\}$ are assumed to satisfy the following properties:*

$$i_1 = 1; \tag{3.1}$$

$$|i_j| \leq \max_{k<j} |i_k| + 1, \ \forall j > 1; \tag{3.2}$$

$$i_j > 0 \ if \ |i_j| > |i_k| \ \forall k < j. \tag{3.3}$$

In other words, the indexing will start at 1; no index equals k until all positive integers less than k have been used; and $k > 0$ must occur in the indices before $-k$.

By definition, all the diagonals indexed as in properties 3.1-3.3 are disjoint: for a given $D_{i_1,\cdots,i_n}$, and $j, k \in \{1, \cdots, n\}$, there are three possibilities:

1) $x_j = x_k$ for all $(x_1, \cdots, x_n) \in D_{i_1,\cdots,i_n}$;

2) $x_j = -x_k$ for all $(x_1, \cdots, x_n) \in D_{i_1,\cdots,i_n}$; or

3) $|x_j| \neq |x_k|$ for all $(x_1, \cdots, x_n) \in D_{i_1,\cdots,i_n}$.

Example 3.1.1 *Three examples of diagonals:*

- *The "off-diagonal" subset of* $\mathbb{R}^n$*, or* $\{(x_1, \cdots, x_n) : |x_i| \neq |x_j| \ \forall \ i \neq j\}$*, is denoted* $D_{1,2,\cdots,n}$.

- *The subset of* $\mathbb{R}^n$ *in which* $x_1 = x_2$ *and no other coordinates are equal in magnitude is* $D_{1,1,2,3,4,\cdots,n-1}$.

- *The subset of* $\mathbb{R}^n$ *in which* $x_1 = -x_3$ *and no other coordinates are equal in magnitude is* $D_{1,2,-1,3,4,\cdots,n-1}$.

3.2 Ordering Partitions

Partitions of $[n] = \{1, 2, \cdots, n\}$ can represent the diagonals in $\mathbb{R}^n$, and such partitions can be partially ordered. Let $n \in \mathbb{Z}^+$.

Definition 3.2.1 *A* partition *of* $[n] = \{1, \cdots, n\}$ *is a set of disjoint sets, the union of which is all of* $[n]$. $\Pi(n)$ *is the set of all partitions of* $[n]$. *For any partition* ν*, we refer to each element of* ν *as a* partition atom. *Let* $r(\nu)$ *denote the number of*

partition atoms in ν. We refer to this number as r when there is no ambiguity of partition. Define the mapping $\nu_{map} : \{1, \cdots, n\} \to \{1, \cdots, r(\nu)\}$ such that $\nu_{map}(q)$ is the index of the partition atom that contains q, where the partition atoms are ordered by least element. The subscript is suppressed, so that $\nu \in \Pi(n)$ refers to both a partition of $[n]$ and a mapping from $[n]$ to $[r(\nu)]$.

We choose the notation $r(\sigma)$ to make the reader think of the "rank of σ," which we will make sense of later.

Example 3.2.1 *Let $\sigma = \{\{1, 3\},\ \{2\}\} \in \Pi(3)$. Then $r(\sigma) = 2,\ \sigma(1) = 1,\ \sigma(2) = 2,\ \sigma(3) = 1$.*

The elements of $\Pi(n)$ can be partially ordered as follows: if $\nu, \pi \in \Pi(n)$, then $\nu \leq \pi$ if and only if every partition atom in ν is contained in a partition atom of π. The notation $\overline{0}$ will denote the partition in which each element of $[n]$ is a singleton set, and the notation $\overline{1}$ will refer to the partition that groups all the elements together in one set. Intuitively, a coarser partition is greater than a finer one (if the two partitions are ordered relative to each other).

Example 3.2.2 *We partially order the elements of $\Pi(3)$:*

$$\overline{0} = \{\{1\},\ \{2\},\ \{3\}\} \leq \begin{array}{l} \{\{1,\ 2\},\ \{3\}\} \\ \{\{1,\ 3\},\ \{2\}\} \\ \{\{1\},\ \{2,\ 3\}\} \end{array} \leq \{\{1,\ 2,\ 3\}\} = \overline{1}$$

Definition 3.2.2 *For any $\sigma, \theta \in \Pi(n)$, we define:*

1) $\pi = \sigma \wedge \theta \in \Pi(n)$ *is the mutual refinement of two partitions: i.e.,* $\pi(i) = \pi(j)$ *if and only if* $\sigma(i) = \sigma(j)$ *and* $\theta(i) = \theta(j)$*;*

2) $\sigma^k \subset \{1, \cdots, n\}$ *is the* k*th partition atom of* σ*, where the partition atoms are ordered by least element;* $|\sigma^k| \in \mathbb{N}$ *is the number of elements in* σ^k*;*

3) $s_k(\sigma)$ *is the number of partition atoms in* σ *that have exactly* k *elements. (We will primarily use* $s_1(\sigma)$ *and* $s_2(\sigma)$*, the number of singletons and doubletons in* σ*.)*

Notice that $\sigma \wedge \theta \leq \sigma$ and $\sigma \wedge \theta \leq \theta$.

Example 3.2.3 *Let* $\sigma = \{\{1, 3\},\ \{2\},\ \{4\}\} \in \Pi(4)$*. Then:*

- $r(\sigma) = 3$*;*
- $\sigma^1 = \{1,\ 3\},\ \sigma^2 = \{2\},\ \sigma^3 = \{4\}$;
- $|\sigma^1| = 2, |\sigma^2| = 1, |\sigma^3| = 1$;
- $\sigma(1) = 1, \sigma(2) = 2, \sigma(3) = 1, \sigma(4) = 3$.

Definition 3.2.3 *We say that the diagonal* $D_{i_1,\cdots,i_n}$ satisfies *the partition* $\sigma \in \Pi(n)$ *if* σ *partitions the elements of* $\vec{i}$ *into classes by their magnitude; i.e.,* $\sigma(j) = \sigma(k)$ *if and only if* $|i_j| = |i_k|$*.*

Example 3.2.4 *The diagonals* $D_{1,2,1,3,2}$*,* $D_{1,2,-1,3,2}$*,* $D_{1,2,1,3,-2}$*, and* $D_{1,2,-1,3,-2}$ *all satisfy the partition* $\{\{1,\ 3\},\ \{2,\ 5\},\ \{4\}\} \in \Pi(5)$*.*

Definition 3.2.4 *Let $\sigma \in \Pi(n)$. Define $\mathcal{I}(\sigma) \subset \{(i_1, \cdots, i_n) \in \mathbb{Z}^n : 1 \leq |i_k| \leq n\}$ such that*

$$\mathcal{I}(\sigma) \equiv \{\vec{i} \in \mathbb{Z}^n : D_{\vec{i}} \; satisfies \; \sigma\}$$

Lemma 3.2.1 *Each diagonal $D_{i_1,\cdots,i_n}$ satisfies a unique $\sigma \in \Pi(n)$.*

Proof: Given $D_{i_1,\cdots,i_n}$, it is trivial to construct a $\sigma \in \Pi(n)$ that it satisfies: let $\sigma^k = \{i_j : |i_j| = k\}$. Then $D_{i_1,\cdots,i_n}$ satisfies σ.

Now assume $D_{i_1,\cdots,i_n}$ satisfies $\sigma, \theta \in \Pi(n)$. Let $j, k \in \{1, \cdots, n\}$. Then:

$$\begin{aligned} \sigma(j) = \sigma(k) &\Leftrightarrow |i_j| = |i_j| \\ &\Leftrightarrow \theta(j) = \theta(k) \end{aligned}$$

So $\theta = \sigma$. □

Let $\sigma \in \Pi(n)$. We count the number of diagonals that satisfy σ.

Lemma 3.2.2 *Define $p(\sigma) \equiv |\mathcal{I}(\sigma)|$, the number of diagonals that satisfy a given partition σ. Then for any given $\sigma \in \Pi(n)$,*

$$p(\sigma) = |\mathcal{I}(\sigma)| = 2^{\sum_{k=1}^{r(\sigma)}(|\sigma^k| - 1)}$$

Proof: Let $D_{i_1,\cdots,i_n}$ satisfy σ. Let $j_k \in [n]$ be the least element of σ^k. According to the rules by which our diagonals are written, $i_{j_k} = k$. For a given k,

$$|\{m \in \sigma^k : m \neq j_k\}| \;\; = \;\; |\sigma^k| - 1$$

Summing over all k:

$$|\bigcup_{k\in\{1,\cdots,r(\sigma)\}}\{m\in\sigma^k : m\neq j_k\}\}| \;=\; \sum_{k=1}^{r(\sigma)}(|\sigma^k|-1).$$

For all $m\in\sigma^k, m\neq j_k$, we notice that $i_m = k$ or $i_m = -k$. Since we have two choices for each element of $\bigcup_{k\in\{1,\cdots,r(\sigma)\}}\{m\in\sigma^k : m\neq j_k\}$, we end up with a total of $2^{\sum_{k=1}^{r(\sigma)}(|\sigma^k|-1)}$ diagonals that satisfy σ. □

3.3 Restricting Integrands Along Diagonals

We want to consider stochastic integrals integrated along diagonals of $\mathbb{R}^n$. The integrands cannot be treated as $L^2_{G^n}$ functions in an $n-$fold integral, since in that case each of these subsets has measure zero (given that G is non-atomic). As an L^2_G function, the integrand can take on any value along these diagonals! So the integrands are now further restricted to ensure that they are well-defined on all of $\mathbb{R}^n$.

For a given $\sigma\in\Pi(n)$ and $\vec{i}\in\mathcal{I}(\sigma)$, there is a natural set-isomorphism between $D_{\vec{i}}$ and $\mathbb{R}^{|\sigma|}$.

Definition 3.3.1 *Let i_{j_k} be the first occurrence of k in the sequence $i_1,\cdots,i_n$. Then define the mapping $h_{\vec{i}} : D_{i_1,\cdots,i_n}\to\mathbb{R}^{r(\sigma)}$ by:*

$$h_{\vec{i}}(x_1,\cdots,x_n) \;=\; (x_{j_1},\cdots,x_{j_r}),\ \textit{where } r=r(\sigma)$$

Note that this is a bijection from the diagonal to the space $\{(x_1, \cdots, x_r) \in \mathbb{R}^{|\sigma|} : x_i \neq x_j \ \forall i \neq j\}$.

Example 3.3.1 *Let* $\sigma = \{\{1,3\},\{2,5\},\{4\}\}$. *Then* $D_{1,2,-1,3,2}$ *satisfies* σ. *Since* $r(\sigma) = 3$, *the mapping* $h_{\vec{i}} : D_{1,2,-1,3,2} \to \mathbb{R}^3$ *is defined in the following way. For every* $(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, -x_1, x_4, x_2) \in D_{1,2,-1,3,2}$,

$$h_{\vec{i}}(x_1, x_2, x_3, x_4, x_5) \quad = \quad (x_1, x_2, x_4)$$

Similarly, consider the inverse mapping $h_{\vec{i}}^{-1} : \mathbb{R}^3 \to D_{1,2,-1,3,2}$:

$$h_{\vec{i}}^{-1}(x_1, x_2, x_3) \quad = \quad (x_1, x_2, -x_1, x_3, x_2)$$

Remark 3.3.1 *For any* $\sigma \in \Pi(n)$, *define the mapping* $g_\sigma : \mathbb{R}^n \to \mathbb{R}^r$ *(where* $r = r(\sigma)$*):*

$$g_\sigma(x_1, \cdots, x_n) \quad = \quad (x_{j_1}, \cdots, x_{j_r})$$

such that $g_\sigma \restriction D_{\vec{i}} = h_{\vec{i}}$ *for any* $\vec{i} \in \mathcal{I}(\sigma)$.

We employ $h_{\vec{i}}$ to piece together appropriate functions along each diagonal.

Definition 3.3.2 *Let* $\tilde{L}^2_{G^n}$ *be a class of functions from* $\mathbb{R}^n \to \mathbb{C}$ *such that* $f \in \tilde{L}^2_{G^n}$ *if and only if, for every* $\sigma \in \Pi(n), \vec{i} \in \mathcal{I}(\sigma)$:

$$f \circ h_{\vec{i}}^{-1} \in H^r_G.$$

In other words, for any $f \in \tilde{L}^2_{G^n}$ and $\sigma \in \Pi(n)$, there exists some $g \in H_G^{r(\sigma)}$ such that

$$f(x_1, \cdots, x_n) = g(x_{j_1}, \cdots, x_{j_r})$$

for every $(x_1, \cdots, x_n) \in D_{i(\vec{\sigma})}$ (assuming that $i(\vec{\sigma})$ exists).

That is, each function in $\tilde{L}^2_{G^n}$, evaluated on any diagonal $D_{i(\vec{\sigma})}$, corresponds to a function in $H_G^{r(\sigma)}$ via $h_{i(\vec{\sigma})}$.

Definition 3.3.3 *The norm in $\tilde{L}^2_{G^n}$ is defined, for any $f \in \tilde{L}^2_{G^n}$, as:*

$$|f|_{\tilde{L}^2_{G^n}} = \sum_{\sigma \in \Pi(n)} \sum_{\vec{i} \in \mathcal{I}(\sigma)} |f \circ h_{\vec{i}}^{-1}|_{H_G^r}$$

3.4 Integration Along Diagonals

Now consider integration in $\mathbb{R}^n$ along diagonals other than the "off-diagonal" diagonal, $D_{1,2,\cdots,n}$. Note that the off-diagonal multiple stochastic integral, developed in the previous chapter (which is referred to as a Multiple Wiener-Itô Integral in Major), is identical to the idea in this chapter of "integration along $D_{1,2,\cdots,n}$." We will refer to it as "off-diagonal integration" and "integration along $D_{1,2,\cdots,n}$" interchangeably. Beginning with $\mathbb{R}^2$, we want to integrate along the diagonal $D_{1,-1}$, which corresponds to the diagonal $x_2 = -x_1$. Let $f(x_1, x_2) \in \tilde{L}^2_{G^2}$. How do we define:

$$\int_{D_{1,-1}} f(x_1, x_2) dZ(x_1) dZ(x_2)$$

Such an integral could intuitively be written as

$$\int f(x, -x) dZ(x) dZ(-x)$$

but we have yet to formally define this notation.

We again begin with simple functions to define the multiple integral. However, these functions need to be dense in $\tilde{L}^2_{G^n}$, not simply in $L^2_{G^n}$. This requires separate simple functions on every diagonal. These functions are pieced together to build a class of functions dense in $\tilde{L}^2_{G^n}$.

Definition 3.4.1 *Define* $Simp^l(\mathbb{R}^n) \subset \tilde{L}^2_{G^n}$ *such that* $f \in Simp^l(\mathbb{R}^n)$ *if and only if* $f \in \tilde{L}^2_{G^n}$ *and:*

$$f(x_1, \cdots, x_n) \;=\; \sum_{\sigma \in \Pi(n)} f_\sigma(x_{j_{\sigma,1}}, \cdots, x_{j_{\sigma,r(\sigma)}}) \cdot \sum_{\vec{i} \in \mathcal{I}(\sigma)} \chi_{D_{\vec{i}}}$$

where $f_\sigma \in simp^l(\mathbb{R}^{r(\sigma)})$*, and* $j_{\sigma,k}$ *is the index of the first element of* σ^k*.*

The indicator functions reflect the fact that each $(x_1, \cdots, x_n)$ is in only one diagonal.

Recall that $\mathcal{K}^p$ is a set of disjoint intervals at the pth level of refinement (see Definition 2.3.3, page 10).

Lemma 3.4.1 $\bigcup_{l \in \mathbb{Z}^+} Simp^l(\mathbb{R}^n)$ *is dense in* $\tilde{L}^2_{G^n}$*.*

Proof: Let $f \in Simp^l(\mathbb{R}^n)$. Let $f_{D_{\vec{i}}} = f \lceil D_{\vec{i}}$. Then for every $(x_1, \cdots, x_n) \in D_{\vec{i}}$, functions of the form:

$$f_{D_{\vec{i}}} \circ h^{-1}_{\vec{i}}(x_1, \cdots, x_r) \;=\; \sum_{k_1, \cdots, k_r \in \mathcal{K}^l}^{N} c_{\vec{i}, k_1, \cdots, k_r} \cdot \chi_{(J^l_{k_1} \times \cdots \times J^l_{k_r})}$$

are dense in $L^2_{G^r}$. Therefore, $\bigcup_{l \in \mathbb{Z}^+} Simp^l(\mathbb{R}^n)$ is dense in $\tilde{L}^2_{G^n}$. □

Definition 3.4.2 *Let* $f \in Simp^l(\mathbb{R}^2)$*.*

$$\int \int f(x, -x) dZ(x) dZ(-x) \;=\; \sum_{i \in \mathcal{K}^l} f(m^l_i, -m^l_i) \lim_{p \to \infty} \sum_{j \in \mathcal{K}^p} |Z(J^p_j \bigcap J^l_i)|^2$$

In fact, this integral yields a deterministic value almost surely! We will use the notation $|dZ(x)|^2$ instead of $dZ(x)dZ(-x)$, since Z is Hermitian symmetric.

Lemma 3.4.2 *The limit in definition* 5 *exists almost surely and is equal to:*

$$\int f(x,-x)dG(x)$$

Proof: Recall the previous notation $Z(J_i^l) = U(J_i^l) + iV(J_i^l)$, where $U(J_i^l), V(J_i^l)$ are independent, real-valued random variables with identical distributions: $\mathcal{N}(0, \frac{1}{2}G(J_i^l))$. Let $|J_i^l|$ denote length of the interval J_i^l. The method of proof is to calculate the expectation and variance of the above integral. The expectation of the integral is:

$$\begin{aligned}
E(\int f(x,-x)|dZ(x)|^2) &= \sum_{i\in\mathcal{K}^l} f(m_i^l,-m_i^l)E(\lim_{p\to\infty}\sum_{j\in\mathcal{K}^p}|Z(J_j^p\bigcap J_i^l)|^2) \\
&= \sum_{i\in\mathcal{K}^l} f(m_i^l,-m_i^l)E(\lim_{p\to\infty}\sum_{j\in\mathcal{K}^p}|U(J_j^p\bigcap J_i^l)+iV(J_j^p\bigcap J_i^l)|^2) \\
&= \sum_{i\in\mathcal{K}^l} f(m_i^l,-m_i^l)\lim_{p\to\infty}\sum_{j\in\mathcal{K}^p}E(U(J_j^p\bigcap J_i^l)^2)+E(V(J_j^p\bigcap J_i^l)^2) \\
&= \sum_{i\in\mathcal{K}^l} f(m_i^l,-m_i^l)\lim_{p\to\infty}\sum_{j\in\mathcal{K}^p}\frac{1}{2}G(J_j^p\bigcap J_i^l)+\frac{1}{2}G(J_j^p\bigcap J_i^l) \\
&= \sum_{i\in\mathcal{K}^l} f(m_i^l,-m_i^l)[\frac{1}{2}G(J_i^l)+\frac{1}{2}G(J_i^l)] \\
&= \int f(x,-x)dG(x)
\end{aligned}$$

The variance of the integral goes to zero, since the variables $U(J_i^l)$ and $V(J_i^l)$ are

all mutually independent:

$$\begin{aligned}
& Var(\int f(x,-x)|dZ(x)|^2) \\
= & \sum_{i\in\mathcal{K}^l} |f(m_i^l,-m_i^l)|\ Var[\lim_{p\to\infty}\sum_{j\in\mathcal{K}^p}(U(J_j^p\bigcap J_i^l)^2+V(J_j^p\bigcap J_i^l)^2)] \\
= & \sum_{i\in\mathcal{K}^l} |f(m_i^l,-m_i^l)|[\frac{3}{2}\lim_{p\to\infty}\sum_{j\in\mathcal{K}^p} G(J_j^p\bigcap J_i^l)^2] \\
\leq & \sum_{i\in\mathcal{K}^l} |f(m_i^l,-m_i^l)|[\frac{3}{2}\lim_{p\to\infty}(\max_{k\in\mathcal{K}^p} G(J_k^p))\sum_{j\in\mathcal{K}^p} G(J_j^p\bigcap J_i^l)] \\
= & \sum_{i\in\mathcal{K}^l} |f(m_i^l,-m_i^l)|[\frac{3}{2}\lim_{p\to\infty}(\max_{k\in\mathcal{K}^p} G(J_k^p))G(J_i^l)] \\
\leq & \lim_{p\to\infty}(\max_{k\in\mathcal{K}^p} G(J_k^p))\frac{3}{2}\int |f(x,-x)|dG(x) \\
\to & \ 0
\end{aligned}$$

□

Remark 3.4.1 *: The same limit exists for any partition whose mesh over any compact interval goes to zero.*

Now consider integration (in $\mathbb{R}^2$) along the diagonal $D_{1,1}$ (i.e., the diagonal $x_1 = x_2$). It is defined in a similar manner:

Definition 3.4.3 *Let $f \in Simp^l(\mathbb{R}^2)$.*

$$\int f(x,x)dZ(x)dZ(x) \ = \ \sum_{i\in\mathcal{K}^l} f(m_i,m_i)\lim_{p\to\infty}\sum_{j\in\mathcal{K}^p}(Z(J_j^p\bigcap J_i^l))^2$$

This integral, too, is almost surely deterministic: in this case, it is zero! We will use the notation $(dZ(x))^2$ instead of $dZ(x)dZ(x)$.

Lemma 3.4.3 *The limit in definition* 6 *exists almost surely and is equal to zero.*

Proof: The proof is almost identical to the proof above, and is omitted here. The difference lies in the expectation to which the random variables converge: in this case,

$$E[U(J_i^l)^2 - V(J_i^l)^2] = 0$$

so the expectation of the integral is zero. The variance still vanishes in the limit. □

Notice that we can define integration against any number of $dZ(x)$ and $dZ(-x)$ similarly.

Notation 3.4.1 *For any $\sigma \in \Pi(n), \vec{i} \in \mathcal{I}(\sigma)$, let:*

$$\begin{aligned} i^{+k} &= |\{a \in [n] : i_a = k\}| \\ i^{-k} &= |\{a \in [n] : i_a = -k\}| \end{aligned}$$

Definition 3.4.4 *Let $f \in Simp^l(\mathbb{R}^n)$. Let $\sigma \in \Pi(n)$, and $\vec{i} \in \mathcal{I}(\sigma)$. Recall that m_j^l is the midpoint of interval J_j^l. Then:*

$$\int_{D_{\vec{i}}} f(x_1, \cdots, x_n) dZ(x_1) \cdots dZ(x_n) =$$
$$\lim_{p \to \infty} \sum_{j_1, \cdots, j_r \in \mathcal{K}^l} f \circ h_{\vec{i}}^{-1}(m_{j_1}^l, \cdots, m_{j_r}^l) \sum_{k \in \mathcal{K}^p} \Pi_{q=1}^r (Z(J_{j_q}^l \bigcap J_k^p)^{\vec{i}^{+q}})(Z(-J_{j_q}^l \bigcap -J_k^p)^{\vec{i}^{-q}})$$

Lemma 3.4.4 *The limit in the above definition exists almost surely and is equal to 0 if $i^{+k} \geq 2$ or $i^{-k} \geq 2$ for any k, and to*

$$\int\!\!\!\!\!\!\diagup f(x_{\sigma(1)} \cdot sgn(i_1), \cdots, x_{\sigma(n)} \cdot sgn(i_n)) \ [\Pi_{l:|\sigma^l|=2} dG(x_l \cdot sgn(i_l))]$$
$$\cdot [\Pi_{m:|\sigma^m|=1} \cdot \ dZ(x_m \cdot sgn(i_m))]$$

otherwise.

Proof: If the diagonal $D_{\vec{i}}$ contains $i_k = i_l$ for $j \neq l$ (i.e., the case in which $i^{+k} \geq 2$ or $i^{-k} \geq 2$), that dimension can be integrated along first and the entire sum goes to zero, as in Lemma 3.4.3. If $i_k \neq i_l \ \forall j \neq l$, consider all pairs of $i_k, i_l, k \neq l$, such that $i_k = -i_l$. Integrating along dimension i_k thus involves integrating against $|dZ(x_{i_k})|^2$. By Lemma 3.4.2, integration in this dimension reduces to integration against $dG(x_{i_k} \cdot sgn(i_k))$. The rest of the variables are never equal to each other, so it is a standard off-diagonal integral at this point. □

A number of diagonals may satisfy a given $\sigma \in \Pi(n)$ (in fact, we calculated how many diagonals will satisfy each partition and gave it a name, $p(\sigma)$). However, for any $\sigma \in \Pi(n)$, there is at most one diagonal $D_{\vec{(i)}}$ such that $\vec{i} \in \mathcal{I}(\sigma)$ and $\int_{D_{\vec{i}}} dZ(x_1) \cdots dZ(x_n)$ is nonzero with positive probability.

Corollary 3.4.1 *Let $\sigma \in \Pi(n)$. Then there is at most one $\vec{i} \in \mathcal{I}(\sigma)$ such that*

$$\int_{D_{\vec{i}}} dZ(x_1) \cdots dZ(x_n) \quad \neq \quad 0$$

with non-zero probability.

Proof: If σ has a partition atom containing at least three distinct elements, j, k, l, then $|i_j| = |i_k| = |i_l|$ for any $\vec{i}$ that satisfies σ. Therefore, $i_j = i_k, i_j = i_l$, or $i_k = i_l$, and integration along $D_{\vec{i}}$ yields zero by Lemma 3.4.3. So assume the partition atoms of σ have at most two elements each. If j is the least element in a partition atom, then i_j must equal the number of partition atoms in σ that have a least element less than j. If j, k are in a partition atom together and $k > j$, i_k must equal $-i_j$ (otherwise, $i_k = i_j$ and integrating along the associated diagonal yields zero). This

is the only $\vec{i}$ that corresponds to a diagonal along which integration yields a nonzero value with positive probability. $\square$

Note that the same applies for any integrand $f \in \tilde{L}^2_{G^n}$.

Notation 3.4.2 *Given $\sigma \in \Pi(n)$, denote the vector of indices $\vec{i}$ described in the Corollary above, if such a vector exists, as $i(\vec{\sigma})$.*

Notation 3.4.3 *Let $n \in \mathbb{Z}^+, \sigma \in \Pi(n), f \in \tilde{L}^2_{G^n}$. Then we employ the following notation:*

$$Z_n^\sigma(f) \;=\; \int_{D_{i(\vec{\sigma})}} f(x_1, ..., x_n) dZ(x_1) ... dZ(x_n)$$

Similarly, we write

$$Z_n^{\geq \sigma}(f) \;=\; \sum_{\{\theta \in \Pi(n) : \theta \geq \sigma\}} \int_{D_{i(\vec{\theta})}} f(x_1, ..., x_n) dZ(x_1) ... dZ(x_n)$$

Remark 3.4.2 *Though we define the random variable $Z_n^{\geq \sigma}(f)$ as a sum of integrals along certain diagonals, we could have instead begun by defining $Z_n^{\geq \sigma}(f)$ in an alternative but equivalent manner: as an "integral" (i.e., limit of sums) evaluated over the subspace of $\mathbb{R}^n$ in which $\sigma(i) = \sigma(j)$ implies $|x_i| = |x_j|$.*

Definition 3.4.5 (Alternative Definition for $\boldsymbol{Z_n^{\geq \sigma}}$) *Let f be a simple function in $\tilde{L}$ which is constant on each interval of $\mathcal{J}^l$ for given $l \in \mathbb{Z}^+$. Let $\sigma \in \Pi(n)$ such that no partition atom in σ has more than two elements. Then*

$$Z^{\geq \sigma} = \sum_{i_1 \in \mathcal{K}^l} \cdots \sum_{i_{r(\sigma)} \in \mathcal{K}^l} f(m_{i_{\sigma(1)}} \cdot \epsilon_{i_{\sigma(1)}}, \cdots, m_{i_{\sigma(n)}} \cdot \epsilon_{i_{\sigma(n)}}) Z(J_{i_{\sigma(1)}} \cdot \epsilon_{i_{\sigma(1)}}) \cdots Z(J_{i_{\sigma(n)}} \cdot \epsilon_{i_{\sigma(n)}}) \tag{3.4}$$

where $\epsilon_{\sigma(k)}$ is $+1$ if k is the least element in its partition atom of σ and -1 if k is the greatest element in its partition atom of σ.

3.5 Möbius Inversion

Now we pause and review Möbius inversion.

Note: In this section, we rely on [2, p. 83] for our description of Möbius inversion; however, we switch his ordering: $\leq$ in his explanation becomes $\geq$ in our notation.

Definition 3.5.1 *A partially-ordered set X is* locally finite *if and only if for every $a, b \in X,\ a \leq b$, the collection of all elements $\{c \in X : a \leq c \leq b\}$ is finite.*

Definition 3.5.2 *Let X be a locally finite, partially-ordered set. Then the* Möbius function μ *is defined inductively for $x, y \in X, y \leq x$:*

$$\begin{aligned} \mu(x,x) &= 1 \\ \mu(x,y) &= -\sum_{y<t\leq x} \mu(x,t) \end{aligned}$$

Theorem 3.5.1 (Möbius inversion) *Let X be a locally finite ordered set, $x \in X$, and $f(x), g(x)$ be functions defined on X such that:*

$$f(x) = \sum_{x\leq u\leq 1} g(u).$$

Then

$$g(x) = \sum_{x\leq u\leq 1} \mu(u,x) f(u).$$

[2, p. 83]

Let $f \in \tilde{L}^2_{G^n}$. For this function, there is a subset $\Omega' \subset \Omega$ with full measure such that $\omega \in \Omega'$ ensures that $Z^{\pi}_n(f,\omega)$ and $Z^{\geq\pi}_n(f,\omega)$ are real-valued and well-defined. Fix

some $\omega \in \Omega'$. Then, $Z_n^{\pi}(f,\omega)$ and $Z_n^{\geq\pi}(f,\omega)$ are treated as functions on $\pi \in \Pi(n)$, and the situation is as follows:

1) There is a partial ordering on $\Pi(n)$.

2) There are two well-defined functions on $\Pi(n)$, $Z_n^{\pi}(f,\omega)$ and $Z_n^{\geq\pi}(f,\omega)$.

3) There is a relationship between the two functions; namely, $Z_n^{\geq\pi}(f,\omega) = \sum_{\nu\geq\pi} Z_n^{\nu}(f,\omega)$.

4) $\Pi(n)$ is locally finite, since $|\Pi(n)| < \infty$.

All the requirements to perform Möbius inversion are satisfied!

Corollary 3.5.1 $Z_n^{\pi} = \sum_{\nu\geq\pi} \mu(\nu,\pi) Z_n^{\geq\nu}$

This result is not immediately intuitive , but it is useful in other contexts. By the above corollary:

$$\begin{aligned} \int\!\!\!\!\!\!\diagup f(x_1,\cdots,x_n)dZ(x_1)\cdots dZ(x_n) &= Z_n^{\overline{0}}(f) \\ &= \sum_{\nu\in\Pi(n)} \mu(\nu,\overline{0}) Z_n^{\geq\nu} \end{aligned}$$

Below are several examples of Möbius inversion. They appear to reduce to trivial statements of equality, because we defined $Z_n^{\geq\sigma}$ as a sum of Z_n^{π} over all $\pi \geq \sigma$, but they help illustrate the inversion technique.

Example 3.5.1 *We begin with a simple example. Let* $\pi = \{\{1\},\{2\}\} \in \Pi(2)$. *Calculate* $\int_{D_{i(\vec{\pi})}} dZ(x_1)dZ(x_2)$. *By the above corollary,*

$$\int_{D_{i(\vec{\pi})}} f(x_1,x_2)dZ(x_1)dZ(x_2) = \sum_{\nu\geq\pi} \mu(\nu,\pi) \sum_{\theta\geq\nu} \int_{D_{i(\vec{\theta})}} f(x_1,x_2)dZ(x_1)dZ(x_2)$$

There are only two elements in $\Pi(2)$*:* π *and* $\vec{1} = \{\{1,2\}\}$. *Since* $\mu(\vec{1},\pi) = -1$, *the*

above equals:

$$\begin{aligned}
&= \mu(\pi,\pi)[\int_{D_{i(\vec{\pi})}} f(x_1,x_2)dZ(x_1)dZ(x_2) + \int_{D_{i(\vec{1})}} f(x_1,x_2)dZ(x_1)dZ(x_2)] \\
&+ \mu(\vec{1},\pi)\int_{D_{i(\vec{1})}} f(x_1,x_2)dZ(x_1)dZ(x_2) \\
&= \int_{D_{i(\vec{\pi})}} f(x_1,x_2)dZ(x_1)dZ(x_2) + \int_{D_{i(\vec{1})}} f(x_1,x_2)dZ(x_1)dZ(x_2) \\
&- \int_{D_{i(\vec{1})}} f(x_1,x_2)dZ(x_1)dZ(x_2) \\
&= \int_{D_{i(\vec{\pi})}} f(x_1,x_2)dZ(x_1)dZ(x_2)
\end{aligned}$$

as desired.

Example 3.5.2 *Now let* $\pi = \{\{1,2\},\{3\},\{4\},\{5\}\} \in \Pi(5)$. *Since there are four distinct partition atoms in* π, *there are* $\binom{4}{2} = 6$ *ways of merging two partition atoms to create a "greater" partition in* $\Pi(5)$*:*

$$\begin{aligned}
\pi_1 &= \{\{1,2,3\},\{4\},\{5\}\} \\
\pi_2 &= \{\{1,2,4\},\{3\},\{5\}\} \\
\pi_3 &= \{\{1,2,5\},\{3\},\{4\}\} \\
\pi_4 &= \{\{1,2\},\{3,4\},\{5\}\} \\
\pi_5 &= \{\{1,2\},\{3,5\},\{4\}\} \\
\pi_6 &= \{\{1,2\},\{4,5\},\{3\}\}
\end{aligned}$$

By merging two partition atoms, each of these partitions can yield three "greater"

partitions:

$$\begin{aligned}
\phi_1 &= \{\{1,2,3,4\},\{5\}\} \\
\phi_2 &= \{\{1,2,3,5\},\{4\}\} \\
\phi_3 &= \{\{1,2,3\},\{4,5\}\} \\
\phi_4 &= \{\{1,2,4,5\},\{3\}\} \\
\phi_5 &= \{\{1,2,4\},\{3,5\}\} \\
\phi_6 &= \{\{1,2,5\},\{3,4\}\} \\
\phi_7 &= \{\{1,2\},\{3,4,5\}\}
\end{aligned}$$

Notice that each ϕ is greater than three πs, and each π is less than three ϕs. The only element greater than any ϕ is $\vec{1}$. Calculate the stochastic integral along the diagonal associated with π.

$$\begin{aligned}
&\int_{D_{i(\vec{\pi})}} f(\vec{x})dZ(x_1)dZ(x_2)dZ(x_3)dZ(x_4)dZ(x_5) \\
= &\sum_{\nu\geq\pi}\mu(\nu,\pi)\sum_{\theta\geq\nu}\int_{D_{i(\vec{\theta})}} f(\vec{x})dZ(x_1)dZ(x_2)dZ(x_3)dZ(x_4)dZ(x_5) \\
= &\ \mu(\pi,\pi)\sum_{\theta\geq\pi}\int_{D_{i(\vec{\theta})}} f(\vec{x})dZ(x_1)dZ(x_2)dZ(x_3)dZ(x_4)dZ(x_5) \\
+ &\sum_{k=1}^{6}\mu(\pi_k,\pi)\sum_{\theta\geq\pi_k}\int_{D_{i(\vec{\theta})}} f(\vec{x})dZ(x_1)dZ(x_2)dZ(x_3)dZ(x_4)dZ(x_5) \\
+ &\sum_{k=1}^{7}\mu(\phi_k,\pi)\sum_{\theta\geq\phi_k}\int_{D_{i(\vec{\theta})}} f(\vec{x})dZ(x_1)dZ(x_2)dZ(x_3)dZ(x_4)dZ(x_5) \\
+ &\ \mu(\vec{1},\pi)\sum_{\theta\geq\vec{1}}\int_{D_{i(\vec{\theta})}} f(\vec{x})dZ(x_1)dZ(x_2)dZ(x_3)dZ(x_4)dZ(x_5)
\end{aligned}$$

The stochastic integral vanishes along the diagonals corresponding to all ϕs (as well as along the diagonal corresponding to $\vec{1}$), since each has a partition atom with at

least three elements in it. So the above simplifies to:

$$
\begin{aligned}
&= \mu(\pi,\pi)\sum_{\theta\geq\pi}\int_{D_{i(\vec{\theta})}} f(\vec{x})dZ(x_1)dZ(x_2)dZ(x_3)dZ(x_4)dZ(x_5) \\
&+ \sum_{k=1}^{6}\mu(\pi_k,\pi)\sum_{\theta\geq\pi_k}\int_{D_{i(\vec{\theta})}} f(\vec{x})dZ(x_1)dZ(x_2)dZ(x_3)dZ(x_4)dZ(x_5) \\
&= 1\cdot[\int_{D_{i(\vec{\pi})}} f(\vec{x})dZ(x_1)dZ(x_2)dZ(x_3)dZ(x_4)dZ(x_5) \\
&+ \sum_{k=1}^{6}\int_{D_{i(\vec{\pi_k})}} f(\vec{x})dZ(x_1)dZ(x_2)dZ(x_3)dZ(x_4)dZ(x_5)] \\
&+ -1\cdot\sum_{k=1}^{6}\int_{D_{i(\vec{\pi_k})}} f(\vec{x})dZ(x_1)dZ(x_2)dZ(x_3)dZ(x_4)dZ(x_5) \\
&= \int_{D_{i(\vec{\pi})}} f(\vec{x})dZ(x_1)dZ(x_2)dZ(x_3)dZ(x_4)dZ(x_5)
\end{aligned}
$$

as desired.

Chapter 4

The Diagram Formula

4.1 Introduction and Exploration of Diagonals in Area of Integration

In the previous chapter, stochastic integrals were defined over all of $\mathbb{R}^n$ by dividing $\mathbb{R}^n$ into disjoint diagonals, denoted by $\{D_{i_1,\cdots,i_n}\}$. (Recall that the off-diagonal was written $D_{1,2,\cdots,n}$.) Each partition $\nu \in \Pi(n)$ corresponded to at most one $\vec{i} \in \mathbb{R}^n$, denoted by $\vec{i}(\nu)$. Given a fixed $f \in \tilde{L}^2_{G^n}$ and an appropriate $\omega \in \Omega_f \subset \Omega$, we were then able to use Möbius inversion to express the off-diagonal integral as a linear combination of stochastic integrals along other diagonals (see Corollary 3.5.1). Products of off-diagonal integrals can also be expressed as the sum of stochastic integrals along certain diagonals. For example, consider a product of two off-diagonal stochastic integrals:

$$\diagup\!\!\!\!\int h_1(x_1, x_2)dZ(x_1)dZ(x_2) \cdot \diagup\!\!\!\!\int h_2(x_3, x_4)dZ(x_3)dZ(x_4)$$

It may be tempting to merge the two integrals into one off-diagonal integral:

$$\diagup\!\!\!\!\int h_1(x_1, x_2) \cdot h_2(x_3, x_4)dZ(x_1)dZ(x_2)dZ(x_3)dZ(x_4)$$

But this is a mistake. What is the difference between the two expressions above? The first does not necessarily vanish along the diagonals:

$$D_{1,2,-1,3},\ D_{1,2,3,-1},\ D_{1,2,-2,3},\ D_{1,2,3,-2},\ D_{1,2,-1,-2},\ D_{1,2,-2,-1}$$

while the second does. In order to express the above product as a sum of off-diagonal integrals, the integral must be evaluated along every allowed diagonal.

Consider the properties of these diagonals. Remember that any integral vanishes along the diagonal $x_i = x_j$ for any $i \neq j$ (Lemma 3.4.3). So we will only consider diagonals such that, for each $i \neq j$, $x_i = -x_j$ or $x_i \neq x_j$. Next, recognize that the off-diagonal integrals vanish along their own sub-diagonals, forcing the product of the integrals to also vanish along those sub-diagonals. In other words, $\int\!\!\!\!\!\diagup h_1(x_1, x_2)dZ(x_1)dZ(x_2)$ vanishes along the diagonal $x_1 = -x_2$, so the product of the two integrals vanishes, as well. The product similarly vanishes along $x_3 = -x_4$.

Thus, the following partitions remain:

$$\{\{1\},\{2\},\{3\},\{4\}\},$$

$$\{\{1,3\},\{2\},\{4\}\},\{\{1,4\},\{2\},\{3\}\},\{\{1\},\{2,3\},\{4\}\},\{\{1\},\{2,4\},\{3\}\},$$

$$\{\{1,3\},\{2,4\}\},\{\{1,4\},\{2,3\}\}$$

Let $\mathcal{B}$ be the set of these partitions. These correspond to the diagonals along which the integral should be evaluated:

$$\begin{aligned} & \int\!\!\!\!\!\diagup h_1(x_1, x_2)dZ(x_1)dZ(x_2) \cdot \int\!\!\!\!\!\diagup h_2(x_3, x_4)dZ(x_3)dZ(x_4) \\ = & \sum_{\sigma \in \mathcal{B}} \int_{D_{\vec{i}(\sigma)}} h_1(x_1, x_2)h_2(x_3, x_4)dZ(x_1)dZ(x_2)dZ(x_3)dZ(x_4) \end{aligned}$$

4.2 Diagrams

The product above can be represented in terms of *diagrams*.

Definition 4.2.1 (Diagram) *A* diagram of order $(n_1, ..., n_m)$ *is an undirected graph*

of $n = n_1 + ... + n_m$ *vertices, each indexed by a pair of integers* (l, j), $l \in [1, m], j \in [1, n_l]$, *such that:*

1) No more than one edge is connected to each vertex, and

2) Two vertices can only be connected if they do not share the same l *coordinate; i.e., if* (a_1, b_1) *is connected to* (a_2, b_2), *then* $a_1 \neq a_2$.

Let $\Gamma(n_1, \cdots, n_m)$ *be the set of all diagrams of order* $(n_1, \cdots, n_m)$, *denoted* Γ *when there is no ambiguity about the order.*

Any product of k off-diagonal integrals $[\int\!\!\!\!\diagup\, dZ(x_1) \cdots dZ(x_{n_1})] \cdots$ $\cdot[\int\!\!\!\!\diagup\, dZ(x_{n-n_k+1}) \cdots dZ(x_n)]$ can be expressed as a sum of stochastic integrals over diagonals in the space $\mathbb{R}^n$.

Lemma 4.2.1 *There is a one-to-one correspondence between diagrams in* $\Gamma(n_1, \cdots, n_k)$ *and diagonals in* $\mathbb{R}^n$ *on which* $[\int\!\!\!\!\diagup\, dZ(x_1) \cdots dZ(x_{n_1})] \cdot [\int\!\!\!\!\diagup\, dZ(x_{n_1+1}) \cdots dZ(x_{n_1+n_2})] \cdot \cdots \cdot [\int\!\!\!\!\diagup\, dZ(x_{n-n_k+1}) \cdots dZ(x_n)]$ *is nonzero with positive probability.*

Proof: Let $n = \sum_{i=1}^{k} n_i$. Each $\gamma \in \Gamma(n_1, \cdots, n_k)$ corresponds to a $\pi \in \Pi(n)$ in the following way. Each node in γ is numbered with a unique integer from 1 to n, beginning at the leftmost column of nodes and numbering down each column. According to this numbering scheme, any two nodes a, b that share the same l coordinate correspond to two variables x_a, x_b that are in the same off-diagonal integral in the above product.

Define $\pi \in \Pi(n)$ to contain only singleton and doubleton sets: $j, k \in [n]$ $(j \neq k)$ are in the same partition atom of π if and only if nodes j and k are connected by an edge in γ. Since π contains only singletons and doubletons, $\vec{i}(\pi)$ is well defined, and

$\int_{D_{\vec{i}(\pi)}} dZ(x_1)\cdots dZ(x_n)$ is nonzero with positive probability. Similarly, any partition $\pi \in \Pi(n)$ such that $\int_{D_{\vec{i}(\pi)}} dZ(x_1)\cdots dZ(x_n)$ is nonzero with positive probability can be mapped to a diagram. □

In the product of off-diagonal integrals

$\iint h_1(x_1,x_2)dZ(x_1)dZ(x_2)\cdot\iint h_2(x_3,x_4)dZ(x_3)dZ(x_4)$, the first two variables are never equal in the domain of integration– neither are the last two variables. So we construct diagrams of degree $(2,2)$. Each one should correspond to a diagonal in the domain of integration.

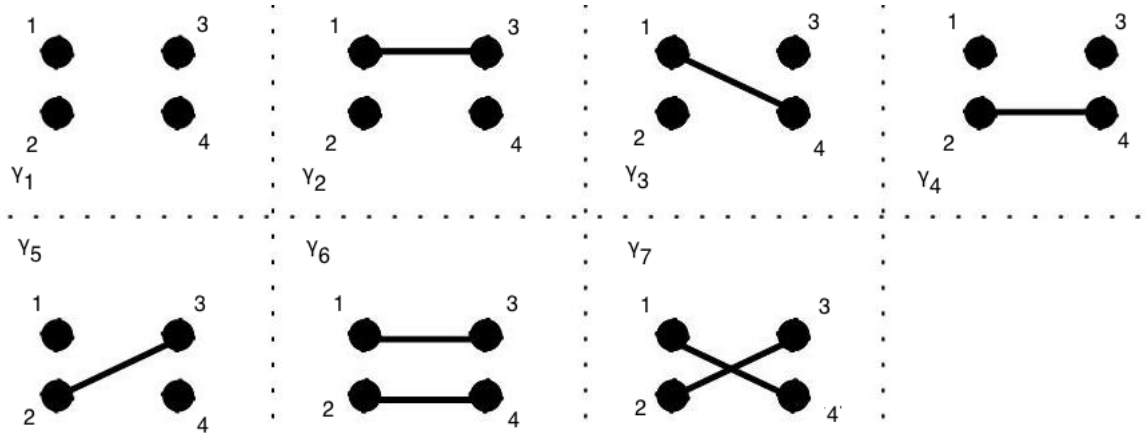

Figure 4.1: All possible diagrams of degree (2,2)

This provides an ideal notation for the above equation (Equation 4.1):

$$\begin{aligned}&\iint h_1(x_1,x_2)dZ(x_1)dZ(x_2)\cdot\iint h_2(x_3,x_4)dZ(x_3)dZ(x_4)\\ =\ &\sum_{\gamma\in\Gamma(2,2)}\int_{D_{\vec{i}(\sigma_\gamma)}} h_1(x_1,x_2)\cdot h_2(x_3,x_4)dZ(x_1)dZ(x_2)dZ(x_3)dZ(x_4)\end{aligned}$$

where $\sigma_\gamma \in \Pi(4)$ is the partition corresponding to diagram γ.

Now consider the product of **three** stochastic integrals:

$$\int h_1(x_1)dZ(x_1)\cdot\int h_2(x_2)dZ(x_2)\cdot\int h_3(x_3)dZ(x_3)$$

Consider integration along the appropriate diagonals. Let $\mathcal{C} = \{\{1,\ 2,\ 3\},\ \{1,\ -1,\ 2\},\ \{1,\ 2,\ -1\},\ \{1,\ 2,\ -2\}\}$.

$$\int h_1(x_1)dZ(x_1) \cdot \int h_2(x_2)dZ(x_2) \cdot \int h_3(x_3)dZ(x_3) \tag{4.1}$$

$$= \sum_{\vec{i} \in \mathcal{C}} \int_{D_{\vec{i}}} h_1(x_1)h_2(x_2)h_3(x_3)dZ(x_1)dZ(x_2)dZ(x_3) \tag{4.2}$$

The diagrams associated with these diagonals are:

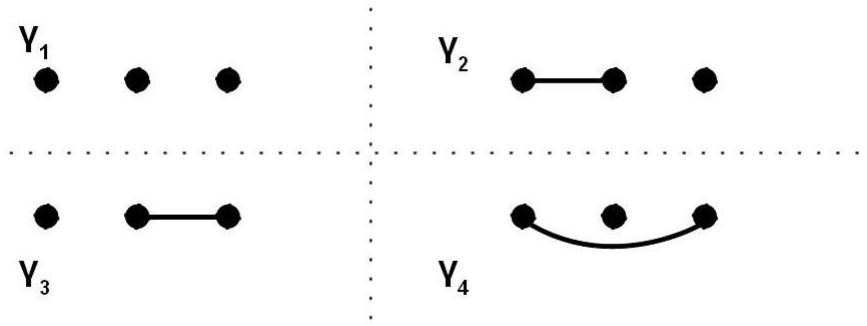

Figure 4.2: All possible diagrams of degree (1,1,1). Notice that no diagram above corresponds to the diagonal $\{\{1, 2, 3\}\}$. This is appropriate, since integrating along that diagonal yields zero almost surely.

These diagrams again correspond to the appropriate diagonals in equation 4.1:

$$\int h_1(x_1)dZ(x_1) \cdot \int h_2(x_2)dZ(x_2) \cdot \int h_3(x_3)dZ(x_3)$$

$$= \sum_{i=1}^{4} \int_{D_{\vec{i}(\sigma\gamma_i)}} h_1(x_1)h_2(x_2)h_3(x_3)dZ(x_1)dZ(x_2)dZ(x_3)$$

4.3 Diagram Formula

Recall that for any $\pi \in \Pi(n)$, $s_2(\pi)$ is the number of doubletons in π (see Notation 3.2.2).

Definition 4.3.1 *For any diagram* $\gamma \in \Gamma(n_1, \cdots, n_k)$, *let* $n = \sum_{i=1}^{k} n_i$ *and let* $\pi_\gamma \in \Pi(n)$ *be the partition associated with* γ. *Then the random variable* $h_\gamma \in \mathcal{L}^{\mathbb{C}}_{s_2(\pi)}$ *is defined to be:*

$$h_\gamma = \int_{D_{\vec{i}(\pi_\gamma)}} h(x_1, \cdots, x_n) dZ(x_1) \cdots dZ(x_n)$$

The correspondence between diagrams and diagonals along with stochastic integrals are nonzero with positive probability is formalized in the Diagram Formula. Though lemma 4.2.1 supplies most of the proof for the Diagram Formula, see [5, p. 42-50] for a formal proof.

Theorem 4.3.1 (Diagram Formula) *For any* $h_1 \in H_G^{n_1}, \cdots, h_m \in H_G^{n_m}$, *the following relations hold:*

1) $h_\gamma \in H^G_{N-2|\gamma|}$, *and* $\|h_\gamma\| \leq \Pi_{j=1}^m \|h_j\|$ *for all* $\gamma \in \Gamma$, *and*

2) $\Pi_{i=1}^m \diagup\!\!\!\int h_i(x_1, \cdots, x_{n_i}) dZ_G(x_1) \cdots dZ_G(x_{n_i}) = \sum_{\gamma \in \Gamma} h_\gamma$.

Using the Diagram Theorem, we now prove:

Lemma 4.3.1 *For any* $n, m \in \mathbb{Z}^+$, $n \neq m$, *and any* $f \in \tilde{L}^2_{G^n}$, $g \in \tilde{L}^2_{G^m}$, $\diagup\!\!\!\int f(x_1, \cdots, x_n) dZ(x_1) \cdots dZ(x_n)$ *is orthogonal to* $\diagup\!\!\!\int g(x_1, \cdots, x_m) dZ(x_1) \cdots dZ(x_m)$ *as elements of the original probability space of the stationary time series* $\{X_t\}$.

Proof: Let $n < m$, without loss of generality.

$$\begin{aligned} & < \diagup\!\!\!\int f(x_1, \cdots, x_n) dZ(x_1) \cdots dZ(x_n), \diagup\!\!\!\int g(x_1, \cdots, x_m) dZ(x_1) \cdots dZ(x_m) > \\ = \; & E(\diagup\!\!\!\int f(x_1, \cdots, x_n) dZ(x_1) \cdots dZ(x_n) \cdot \diagup\!\!\!\int \overline{g(x_1, \cdots, x_m) dZ(x_1)} \cdots \overline{dZ(x_m)}) \\ = \; & E(\sum_{\gamma \in \Gamma(n,m)} (f \cdot g)_\gamma) \end{aligned}$$

Where $(f \cdot g)_\gamma$ is defined in definition 4.3.1. For any $\gamma \in \Gamma(n,m)$, $|\gamma| \leq n$. Therefore, at least one node in γ is not connected to any others. This leaves a term $dZ(x_k)$ in $(f \cdot g)_\gamma$ such that $|x_k| \neq |x_l|$ for any $k \neq l, 1 \leq k \leq n+m$. The expectation of $dZ(x_k)$ (or, in our limiting arguments, $Z(J_k^l)$) is zero. So each $(f \cdot g)_\gamma$ is zero, making the entire sum zero, as well. □

4.4 Linearization Coefficients

Consider the product of Hermite polynomials with univariate stochastic integral arguments:

$$\Pi_{i=1}^k H_{n_i}(\int dZ(x)) \tag{4.3}$$

By Itô's Lemma, line 4.3 equals

$$\Pi_{i=1}^k \big(\bigtimes dZ(x_{n_1+\cdots+n_{i-1}+1}) \cdots dZ(x_{n_1+\cdots+n_i}) \big) \tag{4.4}$$

We can apply the Diagram Formula to line 4.4:

$$\sum_{\gamma \in \Gamma(n_1,\cdots,n_k)} \big[\int \cdots \int dG(x_1) \cdots dG(x_{|\gamma|}) \big] \cdot \big[\bigtimes dZ(y_1) \cdots dZ(y_{N-2|\gamma|}) \big] \tag{4.5}$$

$$= \sum_{\gamma \in \Gamma(n_1,\cdots,n_k)} \big[\bigtimes dZ(y_1) \cdots dZ(y_{N-2|\gamma|}) \big] \tag{4.6}$$

$$= \sum_{\gamma \in \Gamma(n_1,\cdots,n_k)} H_{N-2|\gamma|}(\int dZ(x)) \tag{4.7}$$

We integrate out the measure G in line 4.6 and then apply Itô's Lemma once more in line 4.7. To simplify the notation, define $\mathcal{P}(m; n_1, ..., n_k) = |\{\gamma \in \Gamma(n_1, ..., n_k) : \#\ edges\ in\ \gamma = m\}|$. So the above equation becomes:

$$\Pi_{i=1}^k H_{n_i}(\int dZ(x)) \quad = \quad \sum_{i=1}^{\lfloor N/2 \rfloor} \mathcal{P}(i; n_1, ..., n_k) H_{N-2i}(\int dZ(x))$$

We have therefore calculated the **linearization coefficients** of the Hermite polynomials, limited to the case in which the argument is the stochastic integral $\int dZ(x)$. However, since the range of the stochastic integral is all $\mathbb{R}$, the equality holds for all $x \in \mathbb{R}$. Thus, we have calculated the linearization coefficients for arbitrary arguments.

Chapter 5

Conclusion

While we have focused on random spectral measure of Gaussian stationary processes in this paper, we have also developed a more general notation and framework along the way. Our construction of the univariate and multivariate stochastic integrals is not restricted to random spectral measure, although Major's version of Itô's Lemma is. The notion of integration along any diagonal is implicit in [5], [6], [7], and [1], but we introduced here the rigorously defined notation for stochastic integration on all of $\mathbb{R}^n$ that is missing in those works. We believe this to be the most significant contribution of this largely expository work, and it is defined in enough generality that it can be applied in the cases of other spectral measures in a Gaussian-process setting (for example, non-atomic measures), and perhaps in the case of non-Gaussian settings, as in [1].

5.1 Further Generalizations

The behavior of any given measure along a diagonal is specific to that measure: in our case of random spectral measure, the dual facts that

a) integrating along any linear subspace of $\mathbb{R}^n$ such that $x_i = x_j$ (for some $i \neq j$) yields zero; and

b) integrating along a linear subspace of $\mathbb{R}^n$ in which $x_i = -x_j$ (for some $i \neq j$)

reduces to integration against the Gaussian measure G (against *either* x_i *or* x_j); meant we could restrict our attention to diagonals in which any coordinate x_j was either equal to the negative of some other coordinate, or not equal in magnitude to any other coordinate. This allowed us to represent each relevant diagonal with a partition of $[n]$ in which partition atoms contained at most two elements. We were also able to represent each diagonal with a diagram: nodes i and j were connected in the diagram if and only if $x_i = -x_j$ along the diagonal. If we examine a different measure Z, each partition might correspond to more than one diagonal along which the measure is non-zero with positive probability. For example, if the measures of the two diagonals defined by 1) $x_i = x_j$, and 2) $x_i = -x_j$ are both non-zero with positive probability, then any partition $\sigma \in \Pi(n)$ in which $\sigma(i) = \sigma(j)$ corresponds to two diagonals we are interested in. We might come up with a different notion of diagram for such a measure, however. For example, if the measure Y disappears along any diagonal such that $|x_i| = |x_j| = |x_k|$, but not necessarily along $x_i = x_j$ or $x_i = -x_j$, we could create a "diagram-prime" object in which there are two types of edges: an A-edge would connect nodes i and j if and only if $x_i = x_j$ along the diagonal, while a B-edge would connect them if and only if $x_i = -x_j$ along the diagonal. Since no three coordinates could be equal in magnitude, each node would still be connected to at most one other node. A corresponding "Diagram-Prime" Formula could be developed to calculate the product of several off-diagonals with this measure. Our application of Möbius Inversion depended on a unique diagonal in $\mathbb{R}^n$ associated with each partition in $\Pi(n)$: without this, the partial ordering of $\Pi(n)$ cannot be applied to the relevant diagonals in $\mathbb{R}^n$. In our hypothetical measure

Y above, we might still be able to perform Möbius inversion by imposing a partial ordering on "diagram-primes", and therefore implicitly on the diagonals with which we are concerned.

5.2 Final Words

This work was initially motivated by a problem posed by Dr. Alexander Barg (University of Maryland, College Park). The problem centered around finding linearization coefficients for a family of polynomials relevant to the field of Forward Error Correction/Coding Theory. The work veered off this path as we sorted through the mathematical literature and found that several gaps in the theory had been glossed over. We have no doubt that the researchers who developed the field of stochastic integration and explored its application to linearization coefficients understood the topics with mathematical rigor. However, as we worked to understand their text and express it in perhaps a more accessible form, we were able to develop some of their work a bit more fully. This is most apparent in Chapter 3, in which we define, formally, what stochastic integration along a diagonal means.

Appendix A

Reference Theorems

Definition A.0.1 (Pre-Hilbert Space) *A* pre-Hilbert space *is (in this work) a complex linear vector space X together with an* inner product *defined on $X \times X$. Corresponding to each pair of vectors $x, y \in X$ is the inner product, a complex scalar denoted $\langle x, y \rangle$. The inner product satisfies the following axioms:*

1) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

2) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

3) $\langle cx, y \rangle = c\langle x, y \rangle \ \forall c \in \mathbb{C}$

4) $\langle x, x \rangle \geq 0$ *and* $\langle x, x \rangle = 0$ *if and only if* $x = 0$.

The topology and norm of the space are defined through the inner product. [4, pp. 46-47]

Definition A.0.2 (Hilbert Space) *A* Hilbert space *is a pre-Hilbert space which is complete with respect to the norm defined by its inner product. [4, p. 49]*

Lemma A.0.1 (Hermite Identity) $H_n''(x) - xH_n'(x) = -nH_n(x)$

Theorem A.0.1 (Bochner's Theorem) *Every positive definite function $Q : \mathbb{R} \to \mathbb{C}$ is the Fourier transform of a positive finite Borel measure on $\mathbb{R}$. [5, p. 14]*

Definition A.0.3 (Isometry) *A linear isometry is a linear, distance-preserving map between metric spaces.*

Theorem A.0.2 (Isometry Theorem) *A metric space X is linearly isometric to a metric space Y if there is a linear bijection $f : X \to Y$ that preserves the norm of every element in X or, equivalently, preserves the inner product on all elements in some $X' \subset X$ such that the closed linear span of X' is X. [8]*

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