ABSTRACT

Title of dissertation: NONNEGATIVELY CURVED FIXED-POINT HOMOGENEOUS MANIFOLDS IN LOW DIMENSIONS

Fernando Galaz Garcia,
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Dissertation directed by: Karsten Grove
Department of Mathematics

Let G be a compact Lie group acting isometrically on a compact Riemannian manifold M with nonempty fixed point set M^{G}. We say that M is *fixed-point homogeneous* if G acts transitively on a normal sphere to some component of M^{G}. Fixed-point homogeneous manifolds with positive sectional curvature have been completely classified. We classify fixed-point homogeneous Riemannian manifolds in dimensions 3 and 4 and determine which nonnegatively curved simply-connected 4-manifolds admit a smooth fixed-point homogeneous circle action with a given orbit space structure.

NONNEGATIVELY CURVED FIXED-POINT HOMOGENEOUS MANIFOLDS IN LOW DIMENSIONS

by

Fernando Galaz Garcia

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Advisory Committee:
Professor Karsten Grove, Chair/Advisor
Professor James Schafer
Professor William M. Goldman
Professor Horst E. Winkelnkemper
Professor Theodore Jacobson

To my parents

Edith Garcia Bernal and Fernando Galaz Fontes.

To my grandfather

Sergio Garcia Treviño.

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Chapter 1

Introduction

The interaction between curvature and topology plays a major role in Riemannian geometry and may take different guises. The topology of a smooth manifold usually imposes restrictions on the existence of Riemannian metrics with given curvature conditions (e.g., the Gauss-Bonnet theorem). Alternatively, restrictions on the curvature of a Riemannian manifold may have topological consequences (e.g., Synge's theorem). Finiteness theorems are a different instance of the influence of curvature on topology. In these theorems one considers a class of Riemannian manifolds satisfying given bounds on curvature and other metric invariants, such as diameter or volume. A bound on the number of homotopy, homeomorphism, or diffeomorphism types in this class is then a consequence of the choice of bounds on the metric invariants.

In the framework just described, it is natural to study Riemannian manifolds whose curvature is bounded above or below. This is the case, for example, of Riemannian manifolds with *nonnegative curvature*.

One of the fundamental invariants in Riemannian geometry is *sectional curvature*, which will be henceforth referred to also as *curvature*. The study of Riemannian manifolds with nonnegative sectional curvature is an area of active research in which metric aspects of differential geometry, such as comparison arguments, play a central role (cf. [43, 44]). Despite the existence of general structure results (e.g., Cheeger-Gromoll [6]) and of obstructions to nonnegative curvature (e.g., Gromov's Betti number theorem [12]), examples of nonnegatively curved manifolds and techniques for their construction are scarce. Thus, finding new examples in this class remains a central problem in the field. In this context, considering manifolds with a "large" isometry group provides a systematic approach to the study of both positively and nonnegatively curved manifolds (see, e.g., [13]), revealing the structure of these manifolds and providing insight into methods for constructing new examples (cf. [19]). What we mean by "large" is open for interpretation, as the following examples illustrate. We let M be a compact Riemannian manifold and G its isometry group, which is a compact Lie group. Observe that G acts on M by isometries; we will assume that this action is effective.

Example 1.1. Let the *symmetry rank* of M, denoted by $\mathrm{symrank}(M)$, be the rank of the isometry group of M. Here "large" is interpreted as $\mathrm{symrank}(M)$ being big. Grove and Searle [15] determined the maximal symmetry rank of compact positively curved manifolds:

Theorem 1.2 (Grove, Searle)**.** *If a torus T acts isometrically on M and $\sec M > 0$ then $\dim T \leq [(\dim M + 1)/2]$, and equality holds if and only if M is a sphere, a lens space, or a complex projective space.*

It is also possible to obtain information when the symmetry rank is not maximal, as the following theorem due to Willking [41] for $n \geq 10$, and Fang and Rong [8] for $n = 8, 9$, shows (see also [36]).

Theorem 1.3. *Let M^n be a simply-connected n-dimensional manifold of positive sectional curvature, $n \geq 8$, and let $d \geq n/4 + 1$. Suppose that there is an effective isometric action of a torus T^d on M^n. Then M is homotopically equivalent to $\mathbb{CP}^{n/2}$ or homeomorphic to $\mathbb{HP}^{n/4}$ or $\mathbb{S}^n$.*

Example 1.4. Suppose that G acts on M with nonempty fixed-point set M^G. We define the *fixed-point cohomogeneity* of M as $\dim M/G - \dim M^G - 1 \geq 0$. Here "large" is interpreted as having low fixed-point cohomogeneity, in particular, we say that the action is *fixed-point homogeneous* if the fixed-point cohomogeneity is 0, i.e., if M^G has codimension 1 in the orbit space M/G. Fixed point homogeneous connected positively curved manifolds were classified by Grove and Searle [16]. In the simply-connected case one has the following theorem:

Theorem 1.5 (Grove, Searle)**.** *Any simply-connected fixed-point homogeneous manifold with positive curvature is diffeomorphic to a compact rank one symmetric space.*

This result has been proven a strong tool in other classification work on positively curved manifolds with symmetries, e.g, [42] and the classification of simply-connected positively curved *cohomogeneity* 1 manifolds [18, 40] (i.e., positively curved manifolds with an isometric Lie group action whose orbit space is 1-dimensional).

The presence of an isometric Lie group action provides a link between Riemannian geometry, transformation groups and Alexandrov geometry, making the study of nonnegatively and positively curved manifolds with an isometric Lie group action a rich area which has recently seen some exciting developments (e.g., [17, 7, 33]).

In this work we investigate fixed-point homogeneous Riemannian manifolds with nonnegative curvature. In addition to the intrinsic interest these manifolds have as an extension of the class of positively curved fixed-point homogeneous manifolds studied in [16], the classification of these manifolds is likely to provide a useful tool in further research, as has been the case for positive curvature.

The presence of a fixed-point homogeneous action on a nonnegatively curved manifold M yields information on the structure of M. More precisely, if F is a fixed-point set component with maximal dimension, M can be written as the union of $D(F)$, a tubular neighborhood of F, and $D(B)$, a neighborhood of a subspace $B \subset M$ determined by the geometry of the action (cf. Section 2 of Chapter 2). Thus understanding the pieces $D(F)$ and $D(B)$ is a first step in understanding the structure of nonnegatively curved manifolds with a fixed-point homogeneous action. When $\dim M \leq 5$, F will be a nonnegatively curved closed manifold of dimension at most 3. These manifolds have been classified and one can then proceed to understand the decomposition of M in terms of F and B. The next two steps are to identify M and to construct nonnegatively curved metrics on M realizing the possible fixed-point homogeneous actions on M. In dimensions greater than 6, however, this approach is not practical, since in these cases F will have dimension $n \geq 4$ and in these dimensions the classification of nonnegatively curved manifolds has not been completed. Observe, for example, that the Riemannian product $N \times \mathbb{S}^2$ of a nonnegatively curved manifold N and the round 2-sphere $\mathbb{S}^2$ has a fixed-point homogeneous circle actions, given by letting S^1 act by rotations on $\mathbb{S}^2$ and trivially on N. The fixed-point set of this action consists of two copies of N. Thus any nonnegatively curved manifold may arise as a fixed-point set component with maximal dimension of a fixed-point homogeneous circle action.

We have focused our attention on dimensions 3 and 4, in which one is able to obtain detailed information on the manifolds and the actions by combining geometry and the classification results of Orlik and Raymond [28, 35] in dimension 3, and of Fintushel [9], in dimension 4. The classification of fixed-point homogeneous 2-manifolds is a consequence of the classification of fixed-point homogeneous manifolds with cohomogeneity one, which we recall in Chapter 2. The only fixed-point homogeneous 2-manifolds, regardless of curvature assumptions, are $\mathbb{S}^2$ and $\mathbb{RP}^2$. In dimensions 3 and 4 our main results are the following.

Theorem A. *Let M^3 be a 3-dimensional nonnegatively curved fixed-point homogeneous Riemannian G-manifold. Then G can be assumed to be $\mathsf{SO}(3)$ or S^1 and* $\operatorname{codim} M^{\mathsf{G}} = 3$ *or* 2*, respectively.*

(1) If $\mathsf{G} = \mathsf{SO}(3)$, then M^3 is equivariantly diffeomorphic to $\mathbb{S}^3$ or $\mathbb{RP}^3$.

(2) If $\mathsf{G} = \mathsf{S}^1$, then M^3 is equivariantly diffeomorphic to $\mathbb{S}^3$, a lens space L^3, $\mathbb{S}^2\times\mathbb{S}^1$, $\mathbb{RP}^2\times\mathbb{S}^1$, $\mathbb{RP}^3\#\mathbb{RP}^3$ or the non-trivial bundle $\mathbb{S}^2\tilde{\times}\mathbb{S}^1$.

Theorem B. *Let M^4 be a 4-dimensional nonnegatively curved fixed-point homogeneous G-manifold. Then G can be assumed to be $\mathsf{SO}(4)$, $\mathsf{SU}(2)$, $\mathsf{SO}(3)$ or S^1.*

(1) If $\mathsf{G} = \mathsf{SO}(4)$*, then* M^4 *is equivariantly diffeomorphic to* $\mathbb{S}^4$ *or* $\mathbb{RP}^4$*.*

(2) If $\mathsf{G} = \mathsf{SU}(2)$*, then* M^4 *is equivariantly diffeomorphic to* $\mathbb{S}^4$*,* $\mathbb{RP}^4$*,* $\mathbb{HP}^1$ *or* $\mathbb{CP}^2$*.*

(3) If $\mathsf{G} = \mathsf{SO}(3)$*, then* M^4 *is diffeomorphic to a quotient of* $\mathbb{S}^4$ *or* $\mathbb{S}^3 \times \mathbb{S}^1$*.*

(4) If $\mathsf{G} = \mathsf{S}^1$*, then* M^4 *is diffeomorphic to a quotient of* $\mathbb{S}^4$*,* $\mathbb{CP}^2$*,* $\mathbb{S}^2 \times \mathbb{S}^2$*,* $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$*,* $\mathbb{S}^3 \times \mathbb{R}$ *or* $\mathbb{S}^2 \times \mathbb{R}^2$*.*

Theorems A and B are proved in Chapters 3 and 4, respectively. We have provided examples of isometric actions realizing some of the possible orbit space configurations that occur in the proofs. Chapter 2 contains preliminary definitions and results that will be used in subsequent chapters. We remark that all of the manifolds in Theorems A and B are known to carry metrics of nonnegative curvature. However, not every 3-manifold with nonnegative curvature appears in our list, e.g. the Poincaré homology sphere.

In Chapter 5 we further study fixed-point homogeneous circle actions on non-negatively curved simply-connected 4-manifolds. To put our results in context, let us recall first that, as a consequence of the work of Kleiner [22] and Searle and Yang [37], in combination with Fintushel's classification of circle actions on simply-connected 4-manifolds [9] and Perelman's proof of the Poincaré conjecture, a simply-connected nonnegatively curved 4-manifold with an isometric circle action is diffeomorphic to $\mathbb{S}^4$, $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$. Let $\chi(M)$ be Euler characteristic of a manifold M. By a well-known theorem of Kobayashi, if S^1 acts effectively on M, $\chi(M) = \chi(\mathrm{Fix}(M, \mathsf{S}^1))$. Thus, for a simply- connected nonnegatively curved 4-manifold M with an isometric S^1-action, we have $2 \leq \chi(M) \leq 4$ and the fixed-point set components are 2-spheres and isolated fixed-points. Therefore, the only possible fixed- point sets coming from a fixed-point homogeneous circle action on $\mathbb{S}^4$, $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$ are

$$\mathrm{Fix}(M, \mathsf{S}^1) = \begin{cases} \mathbb{S}^2 & \text{if } M \text{ is } \mathbb{S}^4. \\ \mathbb{S}^2 \cup \{p\} & \text{if } M \text{ is } \mathbb{CP}^2. \\ \mathbb{S}^2 \cup \mathbb{S}^2 & \text{if } M \text{ is } \mathbb{S}^2 \times \mathbb{S}^2 \text{ or } \mathbb{CP}^2 \pm \mathbb{CP}^2. \\ \mathbb{S}^2 \cup \{p', p''\} & \text{if } M \text{ is } \mathbb{S}^2 \times \mathbb{S}^2 \text{ or } \mathbb{CP}^2 \pm \mathbb{CP}^2. \end{cases}$$

Both $\mathbb{S}^4$ and $\mathbb{CP}^2$ have metrics of positive curvature with an isometric fixed-point homogeneous circle action, i.e., the fixed-point set of the action is the one in the list above. On the other hand, when M is $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$, it is not known if M has a nonnegatively curved Riemannian metric with a fixed point homogeneous circle action realizing each one of the corresponding fixed-point sets listed above. Motivated by this question, in Chapter 4 we study smooth fixed-point homogeneous circle actions on $\mathbb{S}^4$, $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$. We have summarized our results in the following theorem.

Theorem C. *Let M be a simply-connected smooth 4-manifold with a smooth S^1-action.*

(1) If $\mathrm{Fix}(M, \mathsf{S}^1) = \mathbb{S}^2$, then M is equivariantly diffeomorphic to $\mathbb{S}^4$ with a linear

action.

(2) *If* $\mathrm{Fix}(M, \mathsf{S}^1) = \mathbb{S}^2 \cup \{p\}$, *then* M *is equivariantly diffeomorphic to* $\pm\mathbb{CP}^2$ *with a linear action.*

(3) *If* $\mathrm{Fix}(M, \mathsf{S}^1) = \mathbb{S}^2 \cup \mathbb{S}^2$, *then* M *is equivariantly diffeomorphic to* $\mathbb{CP}^2 \# - \mathbb{CP}^2$ *or* $\mathbb{S}^2 \times \mathbb{S}^2$ *with a linear action.*

(4) *If* $\mathrm{Fix}(M, \mathsf{S}^1) = \mathbb{S}^2 \cup \{p', p''\}$ *and there are no orbits with finite isotropy, then* M *is equivariantly diffeomorphic to* $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$ *with only one linear action.*

(5) *If* $\mathrm{Fix}(M, \mathsf{S}^1) = \mathbb{S}^2 \cup \{p', p''\}$ *and there is only a weighted arc, then* M *is equivariantly diffeomorphic to one of the following:*

(a) $\mathbb{CP}^2 \# \mathbb{CP}^2$ *with only one linear action with finite isotropy* $\mathbb{Z}_2$.

(b) $\mathbb{CP}^2 \# - \mathbb{CP}^2$ *with only one linear action with finite isotropy* $\mathbb{Z}_k$, k *odd.*

(c) $\mathbb{S}^2 \times \mathbb{S}^2$ *with only one linear action with finite isotropy* $\mathbb{Z}_k$, k *even.*

Theorem C is an application of Fintushel's classification of circle actions on simply-connected 4-manifolfds [9]. It follows from Fintushel's work that a closed simply-connected smooth 4-manifold with a smooth S^1-action is diffeomorphic to a connected sum of copies of $\mathbb{S}^4$, $\pm\mathbb{CP}^2$ and $\mathbb{S}^2 \times \mathbb{S}^2$. Moreover, the action is determined up to equivariant diffeomorphism by a set of orbit space data (cf. Section 2 of Chapter 5). In our case, the orbit space comes from a fixed-point homogeneous circle action on a nonnegatively curved simply-connected 4-manifold and has a rather simple structure, which is described in detail in Section 2 of Chapter 4. Parts (1) and (2) of Theorem C are simple corollaries of Fintushel's work. To prove parts (3) and (4) we compute the possible orbit space data and determine the intersection form of M following a recipe given by Fintushel. We get our results by showing that the intersection form obtained from each possible orbit space configuration is equivalent to the intersection form of $\mathbb{S}^4$, $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$.

Chapter 2

Preliminaries

2.1 Introduction

In this chapter we introduce some notation and several basic tools that we will use throughout. We will always assume our manifolds are connected, unless noted otherwise.

2.2 Fixed-point homogeneous manifolds

Let G be a compact Lie group acting by isometries on a compact Riemannian manifold M. We will consider the action of G as a left action. Given $x \in M$, we denote its *isotropy subgroup* by

$$\mathsf{G}_x = \{\, g \in \mathsf{G} : gx = x \,\}$$

and the *orbit* of x under the action of G by

$$\mathsf{G}x = \{\, gx : g \in \mathsf{G} \,\} \simeq \mathsf{G}/\mathsf{G}_x.$$

We will often denote the orbit space M/G by M^*. Unless mentioned otherwise, we will assume that G acts *effectively* on M, i.e., that the *ineffective kernel* $\mathsf{K} = \cap_{x\in M}\mathsf{G}_x$ of the action is trivial. We say that the action of G is *free* if all the isotropy groups are trivial. Note that the isotropy group $\mathsf{G}_{gx} = g\mathsf{G}_x g^{-1}$ is conjugate to G_x . We say that two orbits $\mathsf{G}x$ and $\mathsf{G}y$ are of the same *type* if G_x and G_y are conjugate subgroups in G.

We will denote the *fixed-point set* of an element $g \in \mathsf{G}$ by

$$M^g = \{x \in M : gx = x\}.$$

The fixed-point set of a subgroup $\mathsf{H} \leq \mathsf{G}$ is $M^{\mathsf{H}} = \cap_{g \in \mathsf{H}} M^g$; we will occasionally denote it also by $\mathrm{Fix}(M, \mathsf{H})$. It is well known that each M^{H} is a finite disjoint union of closed totally geodesic submanifolds of M (cf. [24]). Given M^{H}, we define its *dimension* by

$$\dim M^{\mathsf{H}} = \max\{\dim C_i : C_i \text{ is a connected component of } M^{\mathsf{H}}\}.$$

We now state the *Slice theorem*, which is one of the basic results in the theory of transformation groups.

Slice Theorem 2.1. *For any $x \in M$, a sufficiently small tubular neighborhood $D(\mathsf{G}x)$ of $\mathsf{G}x$ is equivariantly diffeomorphic to $\mathsf{G} \times_{\mathsf{G}_x} D_x^{\perp}$.*

Here $D_x^{\perp}$ is a ball at the origin of the normal space $T_x^{\perp}$ to the orbit $\mathsf{G}x$ at x and $G \times_{\mathsf{G}_x} D_x^{\perp}$ is the bundle with fiber $D_x^{\perp}$ associated to the principal bundle $\mathsf{G} \to \mathsf{G}/\mathsf{G}_x$.

Suppose now that G acts on M with non-empty fixed-point set M^{G}. We say that the action is *fixed-point homogeneous* if M^{G} has codimension 1 in M^*; equivalently, if G acts transitively on the normal sphere to some component of M^{G}. We say that M is *fixed-point homogeneous* if it supports a fixed-point homogeneous action for some compact Lie group G.

The fact that G must act transitively on the normal sphere to some component of M^{G} determines what Lie groups G can act fixed-point homogeneously. The groups

G that can act transitively on a k-dimensional sphere $\mathbb{S}^k$ with isotropy H have been classified (cf. [2, 3, 25, 34]). By possibly replacing G by a subgroup, it suffices to consider the pairs (G, H) in the following list. We have labeled each pair (G, H) by $(a_{k+1}), \ldots, (f)$.

$$(\mathsf{G}, \mathsf{H}) = \begin{cases} (a_{k+1}) & (\mathsf{SO}(k+1), \mathsf{SO}(k)), & k \geq 1; \\ (b_{m+1}) & (\mathsf{SU}(m+1), \mathsf{SU}(m)), & k = 2m+1 \geq 3; \\ (c_{m+1}) & (\mathsf{Sp}(m+1), \mathsf{Sp}(m)), & k = 4m+3 \geq 7; \\ (d) & (\mathsf{G}_2, \mathsf{SU}(3)), & k = 6; \\ (e) & (\mathsf{Spin}(7), \mathsf{G}_2)), & k = 7; \\ (f) & (\mathsf{Spin}(9), \mathsf{Spin}(7)), & k = 15. \end{cases} \tag{2.2.1}$$

A closed 2-manifold with a fixed-point homogeneous action must have cohomogeneity one and must be $\mathbb{S}^2$ or $\mathbb{RP}^2$ (cf. Corollary 2.15). Closed 3-manifolds with a fixed-point homogeneous S^1-action have been classified by Raymond [35] (cf. Theorem 3.1 in Chapter 3). This is a particular instance of the general Orlik-Raymond-Seifert classification of 3-manifolds with a smooth S^1-action [28, 35, 38] (cf. [27]). Fixed-point homogeneous manifolds have also been studied in a Riemannian geometric context. In particular, fixed-point homogeneous Riemannian manifolds with positive sectional curvature have been completely classified by Grove and Searle [16]:

Classification Theorem 2.2 (Grove, Searle)**.** *Let M be a closed, fixed-point homogeneous Riemannian manifold. Then M supports an effective and isometric* G-

action, where G *is one of the groups* $\mathsf{SO}(n)$, $\mathsf{SU}(n)$, $\mathsf{Sp}(n)$, G_2, $\mathsf{Spin}(7)$, *or* $\mathsf{Spin}(9)$ *and* $\operatorname{codim} M^{\mathsf{G}} = n$, $2n$, $4n$,7, *or* 16, *respectively. If, moreover,* $\sec(M) > 0$, *then* M *is* G*-equivariantly diffeomorphic to one of the following:*

(a_n) $\mathbb{S}^m$, $\mathbb{RP}^M$ $(m \geq n)$, *or in addition, when* $n = 2$, $\mathbb{S}^m/\mathbb{Z}_q$ $(q \geq 3)$ *or* $\mathbb{CP}^m$*;*

(b_n) $\mathbb{S}^m$, $\mathbb{S}^m/\mathbb{Z}_q$ $(m \geq 2n)$ *or* $\mathbb{CP}^m$ $(m \geq n)$, *or in addition, when* $n = 2$, $\mathbb{S}^m\Gamma$ $((\Gamma \subset \mathsf{SU}(2)),\ (m \geq 5))$, $\mathbb{CP}^m/\mathbb{Z}_2$ *(m odd) or* $\mathbb{HP}^m$*;*

(c_n) $\mathbb{S}^m$, $\mathbb{S}^m/\Gamma$ $(\Gamma \subset \mathsf{Sp},\ m \geq 4n)$, $\mathbb{CP}^m$ $(m \geq 2n)$, $\mathbb{CP}^m$ $(m \geq 2n)$, $\mathbb{CP}^m/\mathbb{Z}_2$ $(m > 2n$ *odd) or* $\mathbb{HP}^m$ $(m \geq n)$*;*

(d) $\mathbb{S}^m$, *or* $\mathbb{RP}^m$ $(m \geq 7)$*;*

(e) $\mathbb{S}^m$ *or* $\mathbb{RP}$ $(m \geq 8)$*; or*

(f) $\mathbb{S}^m$, $\mathbb{RP}^m$ $(m \geq 16)$ *or* $\mathbb{C}a\mathbb{P}^2$,

where G *in case* (a_n) *is* $\mathsf{SO}(n)$, *etc., as in (2.2.1).*

Fixed-point homogeneous manifolds are a particular instance of manifolds with a "large" group of isometries. There exist, however, manifolds that do not admit smooth actions of compact Lie groups, as the following theorem of Atiyah and Hirzebruch shows (cf. [1]).

Theorem 2.3 (Atiyah, Hirzebruch)**.** *If a circle group acts differentiably on a compact spin manifold M, then the $\hat{A}$-genus of M vanishes.*

This theorem implies, for example, that the $K3$ surface does not support any smooth S^1-action, since it is spin and $\hat{A}(K3) = 2$.

2.3 Geometry of the orbit space

In this section we will outline the geometric structure of the orbit space M^* of an isometric Lie group action on a nonnegatively curved compact Riemannian manifold M. Such an orbit space is, in general, an Alexandrov space with nonnegative curvature. We start by recalling some basic notions from Alexandrov geometry in the context of an isometric group action (cf. [13]). We will then review some fundamental results linking the geometry of the orbit space M^* with the structure of M.

Recall that a finite dimensional length space (X, dist) is an *Alexandrov space* if it has curvature bounded from below $\mathrm{curv} \geq k$ (cf. [4]). When M is a complete, connected Riemannian manifold and G is a compact Lie group acting (effectively) on M by isometries, the orbit space M^* is equipped with the orbital distance metric induced from M, i.e., the distance between p^* and q^* in M^* is the distance between the orbits $\mathsf{G}p$ and $\mathsf{G}q$ as subsets of M. If, in addition, M has sectional curvature bounded below $\sec M \geq k$, then the orbit space M^* is an Alexandrov space with curv$M^* \geq k$.

The *space of directions* S_xX of a general Alexandrov space X at a point x is, by denition, the completion of the space of geodesic directions at x. The euclidean cone $CS_x = T_xX$ is called the *tangent space* to X at x. In the case of an orbit space $M^* = M/\mathsf{G}$, the space of directions $S_{p^*}M^*$ at a point $p^* \in M^*$ consists of geodesic directions and is isometric to

$$\mathbb{S}_p^{\perp}/\mathsf{G}_p,$$

where $\mathbb{S}_p^{\perp}$ is the normal sphere to the orbit $\mathsf{G}p$ at p.

The possible isotropy groups along a minimal geodesic joining two orbits $\mathsf{G}p$ and $\mathsf{G}q$ in M and, equivalently, along a minimal geodesic joining p^* and q^* in the orbit space M^*, are restricted by Kleiner's Isotropy Lemma [22]. We will use this result to obtain restrictions on the isotropy groups of the interior points of a minimal geodesic joining two singular points via the geometry of the space of directions.

Isotropy Lemma 2.4 (Kleiner)**.** *Let $c : [0, d] \to M$ be a minimal geodesic between the orbits $\mathsf{G}c(0)$ and $\mathsf{G}c(d)$. Then, for any $t \in (0, d)$, $\mathsf{G}_{c(t)} = \mathsf{G}_c$ is a subgroup of $\mathsf{G}_{c(0)}$ and of $\mathsf{G}_{c(1)}$.*

The following analog of the Cheeger-Gromoll Soul Theorem [6] in the case of orbit spaces will be a fundamental tool in our study of the structure of fixed-point homogeneous Riemannian manifolds with nonnegative curvature. A more general result for Alexandrov spaces with curvature bounded below is due to Perelman [30].

Soul Theorem 2.5. *If* curv$M^* \geq 0$ *and $\partial M^* \neq \emptyset$, then there exists a totally convex compact subset $S \subset M^*$ with $\partial S = \emptyset$, which is a strong deformation retract of M^*. If*

$\text{curv} M^* > 0$, *then* $S = [s]$ *is a point, and* ∂M^* *is homeomorphic to* $S_{[s]}M^* \simeq S_s^{\perp}/\mathsf{G}_s$.

Recall that the orbit space M^* of a compact nonnegatively curved Riemannian manifold M is a nonnegatively curved Alexandrov space. Moreover, if M is fixed-point homogeneous, ∂M^* contains a component F of M^{G} with maximal dimension. We now carry out the soul construction on M^* and let $C \subset M^*$ be the set at maximal distance from $F \subset \partial M^*$. Let $B = \pi^{-1}(C) \subset M$ be the preimage of C under the projection map $\pi : M \to M^*$. It follows from the Soul Theorem 2.5 that M can be exhibited as the union $M = D(F) \cup_E D(B)$ of neighborhoods $D(F)$ and $D(B)$ along their common boundary E. Hence, in the presence of an isometric fixed-point homogeneous G-action, the structure of M is fundamentally linked to F and B and a thorough understanding of the latter yields information on the structure of M. This will be our guiding principle. The following theorem illustrates this philosophy:

Double Soul Theorem 2.6. *Let* M *be a nonnegatively curved fixed-point homogeneous Riemannian* G*-manifold. If* $\text{Fix}(M, \mathsf{G})$ *contains at least two components* X, Y *with maximal dimension, one of which is compact, then* M *is diffeomorphic to an* $\mathbb{S}^k$*-bundle over* X*, where* $\mathbb{S}^k = \mathsf{G}/\mathsf{H}$ *with* G *as structure group.*

The proof of this theorem follows immediately from the proof of Theorem 2 in [37].

The following lemma yields information on the distribution of the isotropy groups in the orbit space M^*. We refer the reader to [15] for a proof.

Lemma 2.7. *Let* $\mathsf{G} \times M \to M$ *be an isometric fixed-point homogeneous action on a compact nonnegatively curved manifold* M*. Let* C *be the set at maximal distance* *from ∂M^*. Then all the points in $M^* - \{C \cup M^{\mathsf{G}}\}$ correspond to principal orbits.*

Nonnegatively curved Alexandrov spaces of dimension 2 appear as orbit spaces of fixed-point homogeneous actions as well as sets at maximal distance from a boundary component of an orbit space. It is well known that a 2-dimensional Alexandrov space X is a topological 2-manifold, possibly with boundary (cf. [4], Corollary 10.10.3). In addition, when X has nonnegative curvature, we have the following result (cf. [39]).

Theorem 2.8. *Let X be a 2-dimensional Alexandrov space of nonnegative curvature. Then, the following hold: X is homeomorphic to either $\mathbb{R}^2$, $[0,+\infty] \times \mathbb{R}$, $\mathbb{S}^2$, $\mathbb{RP}^2$, $\mathbb{D}^2$, or isometric to $[0,l] \times \mathbb{R}$, $[0,l] \times \mathbb{S}^1(r)$, $[0,+\infty] \times \mathbb{S}^1(r)$, $\mathbb{R} \times \mathbb{S}^1(r)$, $\mathbb{R} \times \mathbb{S}^1(r)/\mathbb{Z}_2$, $[0,l] \times \mathbb{S}^1/\mathbb{Z}_2$, a flat torus, or a flat Klein bottle for some $l, r > 0$.*

Corollary 2.9. *A compact 2-dimensional Alexandrov space with nonnegative curvature and non-empty boundary is homeomorphic to a closed disc $\mathbb{D}^2$ or isometric to a flat Möbius band $\mathbb{M}^2$ or a flat cylinder $\mathbb{S}^1 \times \mathbb{I}$.*

2.4 Other tools

In this section we have collected some other results that we will use throughout. The first one is the following consequence of the Cheeger-Gromoll Splitting Theorem (cf. [5, 6]).

Splitting Theorem 2.10 (Cheeger, Gromoll)**.** *Let M be a compact manifold of nonnegative Ricci curvature. Then $\pi_1(M)$ contains a finite normal subgroup Ψ such* *that $\pi_1(M)/\Psi$ is a finite group extended by $\mathbb{Z}_1 \oplus \cdots \oplus \mathbb{Z}_k$ and $\tilde{M}$, the universal covering of M, splits isometrically as $\overline{M} \times \mathbb{R}^k$, where $\overline{M}$ is compact.*

In the rest of this section we will assume that M is a compact nonnegatively curved Riemannian manifold with a fixed-point homogeneous isometric S^1-action. We will let $F \subset \partial M^*$ be a component of the fixed-point set and C be the set at maximal distance from F in M^*. We will study the structure of the orbit space in the case when $\dim F = \dim C$.

Lemma 2.11. *The only possible isotropy groups in C are $\mathsf{1}$, $\mathbb{Z}_2$ and S^1.*

Proof. Let $p^* \in C$ be a point with finite isotropy $\mathsf{S}^1_p = \mathbb{Z}_k$, $k \geq 3$. Let $T_p^\perp$ be the normal space to the orbit $\mathsf{S}^1 p$ at p and let $F_p = (T_p^\perp)^{\mathsf{S}^1_p}$. We let $F_p^\perp$ be the orthogonal complement of F_p in $T_p^\perp$. The tangent space T_{p^*} to M^* at p^* can be written as $T_{p^*} \simeq F_p \times (F_p^\perp)/\mathsf{S}^1_p$ and F_p is isomorphic to the tangent space at p^* of the orbit stratum containing p^*. Observe that the cone $(F_p^\perp)/\mathsf{S}^1_p$ contains all directions perpendicular to this orbit stratum in M^*. Now, let γ be a minimal geodesic in M^* joining p^* with $F \subset \partial M^*$. Observe that γ is perpendicular to C, which has codimension 1 in M^*. Since the orbit stratum containing p^* must be contained in C, the direction of γ must be contained in $\mathbb{S}(F_p^\perp)/\mathsf{S}^1_p$, the quotient of the unit sphere $\mathbb{S}(F_p^\perp)$ of $F_p^\perp$ by the isotropy group S^1_p. On the other hand, $\mathbb{S}(F_p^\perp)/\mathsf{S}^1_p = \mathbb{S}(F_p^\perp)/\mathbb{Z}_k$ has diameter $\pi/2$ so γ cannot be orthogonal to C, which has codimension 1 in M^*. □

We will now consider two cases: $C \subset \partial M^*$ and $\partial M^* = F$.

Lemma 2.12. *If $C \subset \partial M^*$, then either C is a fixed-point set component or all the points in C have isotropy $\mathbb{Z}_2$. Moreover, C and F are isometric and M^* is isometric to a product $F \times \mathbb{I}$.*

Proof. A point p^* in M^* is a boundary point if its space of directions S_{p^*} has boundary. Consider the tangent space decomposition $T_{p^*} \simeq F_p \times (F_p^\perp)/\mathsf{S}^1_p$. For p^* to be a boundary point, S^1_p must act transitively on the unit sphere $\mathbb{S}(F_p^\perp)$ of $F_p^\perp$ so S^1_p is either S^1 or $\mathbb{Z}_2$. Recall that F_p is the tangent space of the orbit stratum of p^* so it follows from the tangent space decomposition that the orbit stratum with S^1_p isotropy is a subset of C of the same dimension. Hence all the points in C must also have isotropy S^1_p. The second assertion in the theorem follows from the proof of Theorem 2 in [37] (cf. Theorem 2.6 above). □

Lemma 2.13. *Suppose $\partial M^* = F$.*

(1) If $\partial C = \emptyset$, then all the points in C have principal isotropy, F is a double-cover of C and the covering map is a local isometry.

(2) If $\partial C \neq \emptyset$, then all the points in $\operatorname{int} C$ *are principal.*

Proof. We first prove (1). Let $p^* \in C$ and suppose that p has isotropy group S^1_p. Observe that p^* is an interior point of M^*. The only possible isotropy groups in C are S^1, $\mathbb{Z}_2$ and 1. Suppose first that $\mathsf{S}^1_p = \mathbb{Z}_2$ and consider the tangent space decomposition $T_{p^*} \simeq F_p \times (F_p^\perp)/\mathbb{Z}_2$. Observe first that $\mathbb{Z}_2$ acts freely on $F_p^\perp$. If $\dim F_p^\perp \geq 2$, then $\operatorname{diam} \mathbb{S}(F_p^\perp)/\mathbb{Z}_2 = \pi/2$. Let γ be a minimal geodesic joining p^*

with $F \subset \partial M^*$. Observe that γ is perpendicular to C, so its direction must be contained in $\mathbb{S}(F_p^\perp)/\mathbb{Z}_2$. Moreover, since C has codimension 1 in M^*, the direction of γ is at a distance $\pi/2$ from a codimension 1 subset of $\mathbb{S}(F_p^\perp)/\mathbb{Z}_2$ and it follows that $\mathbb{S}(F_p^\perp)/\mathbb{Z}_2$ is a spherical cone, which implies that p^* is a boundary point, which is a contradiction. If $\dim F_p^\perp = 1$, then $\mathbb{Z}_2$ acts transitively on $F_p^\perp$ so p^* is a boundary point, which is a contradiction. Finally, if $\dim F_p^\perp = 0$, then the $\mathbb{Z}_2$ orbit stratum has dimension $\dim M - 1$. This implies that $\mathrm{Fix}(M, \mathbb{Z}_2) = M$ which contradicts our assumption that the action is effective. If p^* has isotropy S^1, then we have $\operatorname{diam} \mathbb{S}(F_p^\perp)/\mathsf{S}^1 = \pi/2$ so p^* must be a boundary point, which is a contradiction. Hence the only possible isotropy group in C must be $\mathsf{1}$. The other assertions follow from the observation that M^* is a manifold with boundary $\partial M^* = F$. Then the Soul Theorem implies that M^* is a line bundle over C and, since $\partial M^* = F$ is connected, it must double-cover C.

To prove part (2), let p^* be a regular point in C. Let γ be a minimal geodesic from p^* to F and v a tangent vector to C at p^*. Parallel translation of v along γ is an isometry, since $\operatorname{curv} \geq 0$. In this way we construct a local isometry $\varphi : (C - E^*) \to F$, where E^* is the set of exceptional orbits. Moreover, this map is an isometry except on E^*. Hence $\operatorname{cl}(C - E^*)$ is isometric to a subset of F and, in particular, since F is a manifold, there cannot be any singular points in $\operatorname{int} C$.

$\square$

2.5 Manifolds of cohomogeneity one

The classification of fixed-point homogeneous manifolds of cohomogeneity one follows from the work of Grove and Searle in [16]. Before stating this result, we recall some basic facts about fixed-point homogeneous manifolds of cohomogeneity one.

Let M be an n-manifold with a fixed-point homogeneous action of cohomogeneity one, so that M^* is either a circle or an interval. If M^* is a circle, then all the orbits are principal and M is a fiber bundle over M^*. Since a fixed-point homogeneous action has fixed points, only the second case may arise in our context. Let $M^* \cong [-1, +1]$. The interior points of the interval correspond to principal orbtis $E = \mathsf{G}/\mathsf{H}$ and the endpoints of the interval correspond to exceptional orbits $B_\pm = \mathsf{G}/\mathsf{K}_\pm$, where $\mathsf{K}_\pm$ is the isotropy group of ± 1. It follows that M can be written as the union of tubular neighborhoods $D(B_\pm) \to B_\pm$ with common boundary $E \simeq \partial D(B_+) \simeq \partial D(B_-)$. In particular, E can be written in two different ways as a bundle $\pi_\pm : E = \mathsf{G}/\mathsf{H} \to \mathsf{G}/\mathsf{K}_\pm = B_\pm$ with sphere fibers $\mathsf{K}_\pm/\mathsf{H} = \mathbb{S}^{l_\pm}$. Observe that a cohomogeneity one G-action determines a group diagram

$$\begin{array}{ccccc} & & \mathsf{G} & & \\ & {\scriptstyle j_-}\nearrow & & \nwarrow{\scriptstyle j_+} & \\ \mathsf{K}_- & & & & \mathsf{K}_+ \\ & {\scriptstyle i_-}\nwarrow & & \nearrow{\scriptstyle i_+} & \\ & & \mathsf{H} & & \end{array} \tag{2.5.1}$$

where $i_\pm$ and $j_\pm$ are the inclusion maps.

Conversely, any diagram as the one above determines a cohomogeneity one

G-manifold, which is exhibited as

$$M = (\mathsf{G} \times_{\mathsf{K}_-} D^{(1+l_-)}) \cup_{\mathsf{G}/\mathsf{H}} (\mathsf{G} \times_{\mathsf{K}_+} D^{(1+l_+)}) \tag{2.5.2}$$

Different possibilities for M can arise from different glueing maps

$$\partial D(B_-) \simeq \partial D(B_+).$$

These glueing maps must be G-equivariant and are determined by an element in the normalizer $N(\mathsf{H})$.

The analysis carried out in [16] also applies when M admits a fixed-point homogeneous cohomogeneity one action, independently of any curvature assumptions. In particular, the following result is an immediate consequence of the method of proof of the Classification Theorem 2.2.

Corollary 2.14. *Let M be a closed, connected Riemannian manifold with a fixed-point homgeneous G-action of cohomogeneity one.*

(a_n) If $\mathsf{G} = \mathsf{SO}(n)$, then M is G-equivariantly diffeomorphic to $\mathbb{S}^n$ or $\mathbb{RP}^n$.

(b_n) If $\mathsf{G} = \mathsf{SU}(n)$, then M is G-equivariantly diffeomorphic to $\mathbb{S}^{2n}$, $\mathbb{RP}^{2n}$ or $\mathbb{CP}^n$.

(c_n) If $\mathsf{G} = \mathsf{Sp}(n)$, then M is G-equivariantly diffeomorphic to $\mathbb{S}^{4n}$, $\mathbb{S}^{4n}/\Gamma$ ($\Gamma \subset \mathsf{Sp}(1)$), $\mathbb{CP}^{2n}$ or $\mathbb{HP}^n$.

(d) If $\mathsf{G} = \mathsf{G}_2$*, then* M *is* G*-equivariantly diffeomorphic to* $\mathbb{S}^7$ *or* $\mathbb{RP}^7$*.*

(e) If $\mathsf{G} = \mathsf{Spin}(7)$*, then* M *is* G*-equivariantly diffeomorphic to* $\mathbb{S}^8$ *or* $\mathbb{RP}^8$*.*

(f) If $\mathsf{G} = \mathsf{Spin}(9)$*, then* M *is* G*-equivariantly diffeomorphic to* $\mathbb{S}^{16}$*,* $\mathbb{RP}^{16}$ *or* $\mathbf{Ca}\mathbb{P}^2$*.*

Observe that a 2-dimensional fixed-point homogeneous manifold must have cohomogeneity one. The classification of these manifolds is then a particular case of Corollary 2.14:

Corollary 2.15. *Let* M^2 *be a 2-dimensional fixed-point homogeneous* G*-manifold. Then* $\mathsf{G} = \mathsf{S}^1$ *and* M^2 *is equivariantly diffeomorphic to* $\mathbb{S}^2$ *or* $\mathbb{RP}^2$*.*

Chapter 3

Nonnegatively curved fixed-point homogeneous 3-manifolds

3.1 Introduction

In this chapter we classify up to equivariant diffeomorphism fixed-point homogeneous Riemannian 3-manifolds with nonnegative curvature. The orbit space of such an action is one- or two-dimensional. In the last case, we have a circle action and we will make use of the Orlik-Rayomond-Seifert classification of cirlcle actions on 3-manifolds. We will outline this classification and the results we will use in the next section, and prove our main result in Section 3.

3.2 Circle actions on 3-manifolds

In this section we outline the Orlik-Raymond-Seifert classification of smooth circle actions on 3-manifolds (cf. [28, 35, 38]). We refer the reader to [28, 35] for a detailed exposition of this classification and related results.

A smooth S^1-action on a closed 3-manifold M is completely determined by a *weighted* orbit space (cf. [27, 28])

$$M^* = \{b; (\varepsilon, g, \bar{h}, t), (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)\}$$

which we now describe. The orbit space M^* is a surface of genus g with $0 \leq \bar{h} + t$ boundary components. Of these boundary components, $\bar{h}$ correspond to fixed-point set components while t correspond to special exceptional orbits. The symbol ε takes on the value o, when M^* is orientable, and $\bar{n}$ when M^* is non-orientable. There are n exceptional orbits and each one is assigned a pair of integers (α_i, β_i) called *Seifert invariants*. These are pairs of relatively prime integers with the property that if $\varepsilon = o$, then $0 < \beta_i < \alpha_i$ and if $\varepsilon = \bar{n}$, then $0 < \beta_i < \alpha_i/2$. We will decribe the Seifert invariantes in more detail in the next paragraph. If $\varepsilon = o$ and $\bar{h} + t = 0$, we let b be an arbitrary integer. If $\bar{h} + t \neq 0$, let $b = 0$. If $\varepsilon = \overline{n}$, $\bar{h} + t = 0$ and no $\alpha_i = 2$, let b take on the values 0 or 1, while $b = 0$ otherwise.

We will now describe the Seifert invariants (α_i, β_i) (cf. [9, 27]). Following the notation in the transformation groups literature, given a set $A \subset M$, we will let A^* denote the projection of A under the orbit map $\pi : M \to M^*$, so $A^* = \pi(A)$. Let E be the union of the exceptional orbits and suppose $E^* = \{ x_1^*, \ldots, x_n^* \}$. For each $x_i^* \in E^*$, let V_i^* be a closed 2-disk neighborhood such that $V_i^* \cap V_j^* = \emptyset$ if $i \neq j$. For $x_i \in \pi^{-1}(x_i^*)$ there is a closed 2-disk slice S_i at x_i such that $S_i^* = V_i^*$. We orient S_i so that its intersection number with the oriented orbit $\pi^{-1}(x_i^*)$ is $+1$ in the solid torus V_i. This induces an orientation on m_i, the boundary of the slice S_i. Observe that m_i is null-homotopic in V_i. Now let h_i be an oriented principal orbit on ∂V_i. Since the action is principal on ∂V_i, it admits a cross-section q_i. If the isotropy group at x_i is $\mathbb{Z}_{\alpha_i}$, the cross-section q_i of the action on ∂V_i is determined up to homology by the homology relation $m_i \sim \alpha_i q_i + \beta_i h_i$, where α_i and β_i are relatively prime and $0 < \beta_i < \alpha_i$. The *Seifert invariants* (α_i, β_i) determine V_i up to orientation-preserving equivariant diffeomorphism. If we reverse the orientation of V_i, the Seifert invariants become $(\alpha_i, \alpha_i - \beta_i)$. The action of the isotropy group $\mathbb{Z}_{\alpha_i}$

on the slice S_i is orientation-preserving equivariantly diffeomorphic to the action of $\mathbb{Z}_{\alpha_i}$ on the 2-disk $\mathbb{D}^2$ given by

$$\frac{2\pi}{\alpha_i}(r,\theta) \mapsto (r, \theta + \frac{2\pi\nu_i}{\alpha_i}).$$

The pair $[\alpha_i, \nu_i]$ are called the *orbit invariants* of $\pi^{-1}(x_i^*)$ and satisfy

$$\beta_i \nu_i \equiv 1 \pmod{\alpha_i}.$$

Observe that a fixed-point homogeneous S^1-action on a closed 3-manifold corresponds to having $\bar{h} > 0$. The classification of these manifolds is due to Raymond [35].

Theorem 3.1 (Raymond)**.** *Let*

$$M = \{b; (\varepsilon, g, \bar{h}, t), (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$$

and assume that $\bar{h} > 0$, i.e., that S^1 acts on M with fixed points. Then M is diffeomorphic to

(1) $\mathbb{S}^3 \# (\mathbb{S}^2 \times \mathbb{S}^1)_1 \# \cdots \# (\mathbb{S}^2 \times \mathbb{S}^1)_{2g+\bar{h}-1} \# (\mathbb{RP}^2 \times \mathbb{S}^1)_1 \# \cdots \# (\mathbb{RP}^2 \times \#^1)_t$ $\# L(\alpha_1, \beta_1) \# \cdots \# L(\alpha_n, \beta_n)$ *if* $(\varepsilon, g, \bar{h}, t) = (o, g, \bar{h}, t)$, $t \geq 0$;

(2) $(\mathbb{S}^2 \times \mathbb{S}^1)_1 \# \cdots \# (\mathbb{S}^2 \times \mathbb{S}^1)_{g+\bar{h}-1} \# (\mathbb{RP}^2 \times \mathbb{S}^1)_1 \# \cdots \# (\mathbb{RP}^2 \times \#^1)_t \; \# L(\alpha_1, \beta_1)$ $\# \cdots \# L(\alpha_n, \beta_n)$ *if* $(\varepsilon, g, \bar{h}, t) = (\bar{n}, g, \bar{h}, t)$, $t > 0$;

(3) $(\mathbb{S}^2 \tilde{\times} \mathbb{S}^1) \# (\mathbb{S}^2 \times \mathbb{S}^1)_1 \# \cdots \# (\mathbb{S}^2 \times \mathbb{S}^1)_{g+\bar{h}-1} \quad \# L(\alpha_1, \beta_1) \#$ $\cdots \# L(\alpha_n, \beta_n)$ *if* $(\varepsilon, g, \bar{h}, t) = (\bar{n}, g, \bar{h}, 0)$.

3.3 Main result

Theorem 3.2. *Let M^3 be a 3-dimensional nonnegatively curved fixed-point homogeneous Riemannian G-manifold. Then G can be assumed to be $\mathsf{SO}(3)$ or S^1 and* $\operatorname{codim} M^{\mathsf{G}} = 3$ *or* 2*, respectively.*

(1) If $\mathsf{G} = \mathsf{SO}(3)$, then M^3 is equivariantly diffeomorphic to $\mathbb{S}^3$ or $\mathbb{RP}^3$.

(2) If $\mathsf{G} = \mathsf{S}^1$, then M^3 is equivariantly diffeomorphic to $\mathbb{S}^3$, a lens space L^3, $\mathbb{S}^2 \times \mathbb{S}^1$, $\mathbb{RP}^2 \times \mathbb{S}^1$, $\mathbb{RP}^3 \# \mathbb{RP}^3$ or the non-trivial bundle $\mathbb{S}^2 \tilde{\times} \mathbb{S}^1$.

Remark. Except for $\mathbb{S}^3$ and L^3, all the other manifolds in part (3) are quotients of $\mathbb{S}^2 \times \mathbb{R}$.

Proof. The first assertion follows from the comments made in chapter 2. Indeed, the cohomogeneity of the action determines the group G acting on M. Recall that, since M is fixed-point homogeneous, the fixed-point set M^{G} has codimension 1 in the orbit space M^*. We now list the possible groups G acting on M with principal isotropy H, using list (2.2.1) in Chapter 2. We denote by $\mathbb{S}^k$ the normal sphere to a component of the fixed-point set with maximal dimension.

$$(\mathsf{G}, \mathsf{H}) = \begin{cases} (\mathsf{SO}(3), \mathsf{SO}(2)) & \text{if } \operatorname{cohom} \mathsf{G} = 1 \ (\mathbb{S}^k = \mathbb{S}^2); \\ (\mathsf{S}^1, 1) & \text{if } \operatorname{cohom} \mathsf{G} = 2 \ (\mathbb{S}^k = \mathbb{S}^1). \end{cases}$$

Now we prove the rest of the theorem.

Case 1. $(\mathsf{G},\mathsf{H}) = (\mathsf{SO}(3),\mathsf{SO}(2))$. In this case M is a cohomogeneity 1 manifold and the conclusion follows from Corollary 2.14 in Chapter 2. There are two singular orbits, one of which is an isolated fixed point.

Case 2. $(\mathsf{G},\mathsf{H}) = (\mathsf{S}^1, 1)$. This corresponds to the cohomogeneity 2 case. The orbit space is a nonnegatively curved 2-dimensional Alexandrov space with non-empty boundary. By Corollary 2.9, the only possible such spaces are the disc $\mathbb{D}^2$, the flat Möbius band $\mathbb{M}^2$ and the flat cylinder $\mathbb{S}^1 \times \mathbb{I}$. We will determine all the possible structure invariants of the circle action and will use Raymond's classification of fixed-point homogeneous circle actions on 3-manifolds (cf. Section 3.2) to identify M up to equivariant diffeomorphism.

Let $F^1 \cong \mathbb{S}^1$ be a component of the fixed-point set with maximal dimension. Let C^k be the set at maximal distance from F^1 in M^*. By construction, $k = \dim C^k \leq \dim F^1 = 1$. By Theorem 2.7, all points in $M^* - \{\, C \cup M^{\mathsf{G}} \,\}$ correspond to principal orbits.

Case 2.1. Suppose $\dim C = 0$. Then C^0 is the soul and, by the Soul Theorem 2.5, it must be a point. Thus we have $M^* \simeq \mathbb{D}^2$. This orbit space configuration has been analyzed in [16] and it follows that M^3 is equivariantly diffeomorphic to $\mathbb{S}^3$ or to a lens space.

Case 2.2. Suppose $\dim C = 1$. We have two possibilities: $C^1 \simeq \mathbb{S}^1$ or $C^1 \simeq [-1,+1]$.

Case 2.2.1. Suppose $C \simeq [-1, +1]$. After another step of the soul construction, we obtain the soul, which must be a point. Then $M^* \simeq \mathbb{D}^2$.

We now analyze the orbits corresponding to the points in C^1. By the Isotropy Lemma 2.4 all the points in the interior of the interval have the same isotropy. These points cannot all be fixed, since that would imply that $(-1, +1) \cong \mathbb{S}^1$, which is a contradiction.

Let K_-, K_+ and K_0 denote, respectively, the isotropy group of points in the subsets $\{-1\}$, $\{+1\}$ and $(-1, +1)$ of $C^1 \simeq [-1, +1]$. We will refer to this triple as an *isotropy triple* and will denote it by

$$\mathsf{K}_- \cdots \mathsf{K}_0 \cdots \mathsf{K}_+.$$

It follows from the Isotropy Lemma 2.4 that $\mathsf{K}_0 \leq \mathsf{K}_\pm \leq \mathsf{S}^1$. The largest isotropy group in this triple is either $\mathsf{1}$, $\mathbb{Z}_q$ (for some $q > 1$), or S^1.

Case 2.2.1.1. Suppose the largest isotropy group is $\mathsf{1}$. This case reduces to the case when C is a point with trivial isotropy (cf. case 2.1) and it follows from Theorem 3.1 that M^3 is diffeomorphic to $\mathbb{S}^3$. Moreover, it follows from Theorem 1 in [35] that M^3 must be equivariantly diffeomorphic to $\mathbb{S}^3$.

Case 2.2.1.2. Suppose the largest isotropy group is $\mathbb{Z}_k$, for some $k > 1$, so that we have the isotropy triple

$$\mathbb{Z}_{q_-} \cdots \mathbb{Z}_l \cdots \mathbb{Z}_{q_+}.$$

Since the space of directions at a point in $(-1,+1)$ has diameter π, it follows that $\mathbb{Z}_l = \mathsf{1}$. Now we determine $\mathbb{Z}_{q_\pm}$. Let γ be a minimal geodesic from $\partial M^* = \mathbb{S}^1$ to $+1 \in [-1,+1] \simeq C^1$. Recall that C^1 is totally convex and observe that γ is orthogonal to C^1. Then the space of directions at $+1$ must have diameter at least $\pi/2$, so we must have $\mathbb{Z}_{p_+} = \mathbb{Z}_2$ or $\mathsf{1}$. The same argument replacing $+1$ with -1 shows that $\mathbb{Z}_{p_-} = \mathbb{Z}_2$ or $\mathsf{1}$. Since we have assumed that at least one isotropy group is non-trivial, we have the following isotropy triples:

$$\mathsf{1} \cdots \mathsf{1} \cdots \mathbb{Z}_2,$$

$$\mathbb{Z}_2 \cdots \mathsf{1} \cdots \mathbb{Z}_2.$$

In the first case, observe that the distance function to $+1$, the point in C^1 with isotropy $\mathbb{Z}_2$, has no critical points, so we have a gradient-like vector field whose flow-lines yield a deformation retraction of M^* onto the point with isotropy $\mathbb{Z}_2$, as in case Case 2.1, in which the field corresponds to the gradient-like vector field of the distance function from F to C^0. Hence this case reduces to the case in which C^k is a point with isotropy $\mathbb{Z}_2$ and it follows that M is diffeomorphic to $\mathbb{RP}^3$. Observe that it follows from [35] that, up to equivariant diffeomorphism, there is only one action on $\mathbb{RP}^3$ with orbit space a 2-disk whose boundary is the fixed-point set and a point with $\mathbb{Z}_2$-isotropy in the interior.

Now we analyze the case corresponding to the isotropy triple $\mathbb{Z}_2 \cdots \mathsf{1} \cdots \mathbb{Z}_2$. We will first see that, in this case, M is $\mathbb{RP}^3 \# \mathbb{RP}^3$ and then we will discuss the disk-bundle decomposition of M. Observe that the orbit space M^* is a 2-disk; its boundary circle is the fixed-point set, and in the interior of the 2-disk there are two points with $\mathbb{Z}_2$-isotropy. According to Theorem 3.1, it follows from this orbit space structure that M^3 is diffeomorphic to $\mathbb{RP}^3\#\mathbb{RP}^3$. We may also read this off the orbit space structure in the following way. Recall that M^* is a 2-disk $\mathbb{D}^2$ whose boundary consists of fixed-points. Divide M^* by a curve γ joining different points in the boundary circle so that the two points with $\mathbb{Z}_2$-isotropy lie in different halves of M^*. Now observe that γ lifts to $\mathbb{S}^2$ in M^3 and each half of M^* corresponds to $\mathrm{cl}(\mathbb{RP}^3 - \mathbb{B}^3)$. Thus M consists of two copies of $\mathrm{cl}(\mathbb{RP}^3 - \mathbb{B}^3)$ identified on the boundary sphere. This corresponds to $\mathbb{RP}^3\#\mathbb{RP}^3$.

Observe now that $\pi^{-1}(C^1) \cong \mathbb{RP}^2\#\mathbb{RP}^2 \cong \mathbb{K}^2 \subset M^3$. We can write M^3 as the union of tubular neighborhoods $D(\mathbb{S}^1)$ and $D(\mathbb{K}^2)$ identified by their common boundary E^2. The tubular neighborhood $D(\mathbb{S}^1)$ is a 2-disc bundle over $\mathbb{S}^1$, so E^2 is an $\mathbb{S}^1$-bundle over $\mathbb{S}^1$. Since M is orientable we must have that E^2 is $\mathbb{T}^2$. On the other hand, E^2 is also an $\mathbb{S}^0$-bundle over $\mathbb{K}^2$. We must have then

$$\partial(\mathbb{S}^1 \times \mathbb{D}^2) = \mathbb{S}^1 \times \mathbb{S}^1 = \partial(D(\mathbb{RP}^2\#\mathbb{RP}^2)).$$

The tubular neighborhood $D(\mathbb{K}^2)$ of $\mathbb{K}^2 \cong \mathbb{RP}^2\#\mathbb{RP}^2$ is a 1-line bundle and has boundary a torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. We may construct such a bundle in the following way. Note first that $\mathbb{T}^2$ is an orientable double-cover of $\mathbb{K}^2$ and we have a $\mathbb{Z}_2$ action on $\mathbb{T}^2$ with quotient $\mathbb{T}^2/\mathbb{Z}_2 = \mathbb{K}^2$. On the other hand we have a $\mathbb{Z}_2$ action on $\mathbb{D}^1$ via a flip. Hence we can construct the associated fiber bundle

$$\begin{array}{c} \mathbb{T}^2 \times_{\mathbb{Z}_2} \mathbb{D}^1 \\ \big\downarrow \\ \mathbb{K}^2 \end{array} \tag{3.3.1}$$

and we have that $\partial(\mathbb{T}^2 \times_{\mathbb{Z}_2} \mathbb{D}^1) = \mathbb{T}^2 \times_{\mathbb{Z}_2} \mathbb{S}^0 \simeq \mathbb{T}^2$.

Now we show that there is only one possible isometric circle action on $\mathbb{RP}^3\#\mathbb{RP}^3$ with nonnegative curvature realizing this orbit space structure. According to [35], Theorem 4, there are $4^2 = 16$ inequivalent actions on $\mathbb{RP}^3\#\mathbb{RP}^3$. We now show that only one of these can occur on a nonnegatively curved $\mathbb{RP}^3\#\mathbb{RP}^3$. Recall that $\mathbb{RP}^3\#\mathbb{RP}^3$ with nonnnegative sectional curvature has $\mathbb{S}^2 \times \mathbb{S}^1$ as a double cover (cf. [20]). This in turn has as universal covering space $\mathbb{S}^2 \times \mathbb{R}$ with nonnegative curvature. By the Splitting Theorem 2.10, $\mathbb{S}^2 \times \mathbb{S}^1$ must have a product metric with nonnegative curvature. There is only one S^1 action on $\mathbb{S}^2 \times \mathbb{S}^1$ according to [35] Theorem 1 (iii). So there is only one S^1-action on $\mathbb{RP}^3\#\mathbb{RP}^3$ by isometries, induced by the action on $\mathbb{S}^2 \times \mathbb{S}^1$. Now we describe the action.

We observe first that $\mathbb{RP}^3\#\mathbb{RP}^3$ can be written as the quotient of $\mathbb{Z}_2$-action on $\mathbb{S}^2 \times \mathbb{S}^1$ given by $-1(x, y) \mapsto (-x, \bar{y})$, i.e., the antipodal map on $\mathbb{S}^2$ and reflection on $\mathbb{S}^1$, corresponding to complex conjugation when we consider $\mathbb{S}^1 \subset \mathbb{C}$. The S^1-action on $\mathbb{S}^2 \times \mathbb{S}^1$ is given by $\lambda(x, y) \mapsto (\lambda x, y)$, where S^1 acts by rotations on $\mathbb{S}^2$. Since the antipodal map commutes with rotations, the S^1-action on $\mathbb{S}^2 \times \mathbb{S}^1$ commutes with the $\mathbb{Z}_2$-action, inducing an S^1-action on $\mathbb{RP}^3\#\mathbb{RP}^3$ giving the desired orbit space. Observe also that this induces a $\mathbb{Z}_2$-action on the orbit space of $\mathbb{S}^2 \times \mathbb{S}^1$, which is a cylinder whose boundary circles are fixed-point components. The quotient of this $\mathbb{Z}_2$-action yields the orbit space of the S^1-action on $\mathbb{RP}^3\#\mathbb{RP}^3$, i.e., we have a commutative diagram

$$\begin{array}{ccc} \mathbb{S}^2\times\mathbb{S}^1 & \xrightarrow{\kappa} & \mathbb{RP}^3\#\mathbb{RP}^3 \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ \mathbb{S}^1\times\mathbb{I} & \xrightarrow{\kappa} & \mathbb{D}^2 \end{array}$$

where π is the orbit projection map corresponding to the S^1-action and κ is the quotient map of the $\mathbb{Z}_2$-action.

Case 2.2.1.3. Suppose the largest isotropy group is S^1. A fixed-point set component is a circle and must be contained in $C^1 \simeq [-1,+1]$. This is a contradiction so this case does not occur.

Case 2.2.2. Suppose $C^1 \simeq \mathbb{S}^1$. Then C^1 is the soul of M^*, is totally convex and, by the Isotropy Lemma 2.4, all the points in C^1 must have the same isotropy.

Case 2.2.2.1. Suppose C^1 has trivial isotropy. Then F^1 double-covers C^1. The orbit space corresponds to a Möbius band $\mathbb{M}^2$ whose boundary circle is F^1. Now we use the classification of circle actions on closed 3-manifolds to identify M^3. We have $(\varepsilon, g, \bar{h}, t) = (\bar{n}, 1, 1, 0)$ and M is diffeomorphic to $\mathbb{S}^2\tilde{\times}\mathbb{S}^1$, the non-trivial $\mathbb{S}^2$-bundle over $\mathbb{S}^1$. It follows from Theorem 1(iii) in [35] that there is only one circle action with fixed points on this manifold. We can realize this orbit space structure on $\mathbb{S}^2\times\mathbb{S}^1$ with nonnegative curvature by letting S^1 act fiberwise by cohomogenety one. We obtain this action by first considering $\mathbb{S}^2\times[0,1]$ with S^1 acting by rotations on the first factor and then identifying $\mathbb{S}^2\times\{0\}$ with $\mathbb{S}^2\times\{1\}$ via the antipodal map, which is an equivariant isometry.

Case 2.2.2.2. Suppose C^1 has finite isotropy $\mathbb{Z}_q$. By Lemma 2.13, we must have $\mathbb{Z}_2$ isotropy and C^1 must be a boundary component. In this case the set of special exceptional orbits, is C^1. We have $(\varepsilon, g, \bar{h}, t) = (o, 0, 1, 1)$, so M^3 is equivariantly diffeomorphic to $\mathbb{RP}^2 \times \mathbb{S}^1$, according to Theorem 1 in [35]. By Theorem 1(iii) in [35], $\mathbb{RP}^2 \times \mathbb{S}^1$ supports only one circle action with fixed points, up to equivariant diffeomorphism. We can realize this orbit space structure on $\mathbb{RP}^2 \times \mathbb{S}^1$ with nonnegative curvature by letting S^1 act via the standard cohomogeneity 1 action on the $\mathbb{RP}^2$-factor and trivially on the $\mathbb{S}^1$-factor. Observe that there is only one $\mathbb{RP}^2$-bundle over $\mathbb{S}^1$.

Case 2.2.2.3. Suppose C^1 has isotropy S^1. In this case the orbit space is a cylinder whose boundary components correspond to components of the fixed-point set. There are not any exceptional orbits. We have that $(\varepsilon, g, \bar{h}, t) = (o, 0, 2, 0)$ and it follows from Theorem 1 in [35] that M^3 is equivariantly diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$. Moreover, by Theorem 1(iii) in [35], $\mathbb{S}^2 \times \mathbb{S}^1$ supports only one circle action with fixed points, up to equivariant diffeomorphism. We can realize this orbit structure by taking $\mathbb{S}^2 \times \mathbb{S}^1$ with the standard nonnegatively curved product metric and letting S^1 act on the $\mathbb{S}^2$ factor via the standard cohomogeneity 1 action and trivially on the $\mathbb{S}^1$-factor.

□

Chapter 4

Nonnegatively curved fixed-point homogeneous 4-manifolds

4.1 Main result

Theorem 4.1. *Let M^4 be a 4-dimensional nonnegatively curved fixed-point homogeneous G-manifold. Then G can be assumed to be $\mathsf{SO}(4)$, $\mathsf{SU}(2)$, $\mathsf{SO}(3)$ or S^1.*

(1) If $\mathsf{G} = \mathsf{SO}(4)$, then M^4 is equivariantly diffeomorphic to $\mathbb{S}^4$ or $\mathbb{RP}^4$.

(2) If $\mathsf{G} = \mathsf{SU}(2)$, then M^4 is equivariantly diffeomorphic to $\mathbb{S}^4$, $\mathbb{RP}^4$, $\mathbb{HP}^1$ or $\mathbb{CP}^2$.

(3) If $\mathsf{G} = \mathsf{SO}(3)$, then M^4 is diffeomorphic to a quotient of $\mathbb{S}^4$ or $\mathbb{S}^3 \times \mathbb{S}^1$.

(4) If $\mathsf{G} = \mathsf{S}^1$, then M^4 is diffeomorphic to a quotient of $\mathbb{S}^4$, $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$, $\mathbb{S}^3 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}^2$.

Proof. The first assertion follows from the comments made in chapter 2. Recall that since M is fixed-point homogeneous, the fixed-point set M^{G} has codimension one in the orbit space. Hence the cohomogeneity of the action determines G, which must act transitively on the normal sphere $\mathbb{S}^k$ to a component of M^{G} with maximal dimension.

We now list the possible compact Lie groups G acting fixed-point homogeneously on M^4 with principal isotropy group H.

$$(\mathsf{G},\mathsf{H}) = \begin{cases} (\mathsf{SO}(4),\mathsf{SO}(3)) \text{ or } (\mathsf{SU}(2),\mathsf{SU}(1)) & \text{if } \operatorname{cohom}\mathsf{G} = 1 \ (\mathbb{S}^k = \mathbb{S}^3); \\ (\mathsf{SO}(3),\mathsf{SO}(2)) & \text{if } \operatorname{cohom}\mathsf{G} = 2 \ (\mathbb{S}^k = \mathbb{S}^2); \\ (\mathsf{S}^1, 1) & \text{if } \operatorname{cohom}\mathsf{G} = 3 \ (\mathbb{S}^k = \mathbb{S}^1). \end{cases}$$

We will now prove (1)–(4) in the statement of the Theorem.

Cases 1 and 2. $(\mathsf{G},\mathsf{H}) = (\mathsf{SO}(4),\mathsf{SO}(3))$ or $(\mathsf{SU}(2),\mathsf{SU}(1))$. These are cohomogeneity one cases and the conclusions follow from Corollary 2.14.

Case 3. $(\mathsf{G},\mathsf{H}) = (\mathsf{SO}(3),\mathsf{SO}(2))$. The orbit space M^* is a 2-dimensional Alexandrov space with non-empty boundary and nonnegative curvature. It follows from Corollary 2.9 that M^* is homeomorphic to a closed disk $\mathbb{D}^2$ or isometric to a flat cylinder $\mathbb{S}^1 \times \mathbb{I}$, where $\mathbb{I}$ is a closed interval, or to a flat Möbius band $\mathbb{M}^2$. Observe that the fixed-point set components are 1-dimensional closed submanifolds of M, i.e., circles. Let $F^1 \subset M^{\mathsf{G}}$ be a component of ∂M^* and let C be the set of points at maximal distance from F^1 in the orbit space M^*. We have $\dim C \leq \dim F^1 = 1$. We will now analyze all the possible orbit space structures.

Case 3.1. Suppose $\dim C = 0$. Then C^0 is the soul and, by the Soul Theorem 2.5, it must be a point. This case corresponds to case (a_3) in Theorem 2.2 and it follows from the proof of this theorem (cf. [16]) that M^4 is equivariantly diffeomorphic to $\mathbb{S}^4$ or $\mathbb{RP}^4$. Observe that the principal isotropy group is $\mathsf{SO}(2)$, so the only possible non-principal isotropy group for C^0 is $\mathsf{O}(2)$, in which case M^4 is equivariantly diffeomorphic to $\mathbb{RP}^4$. When the isotropy of C^0 is $\mathsf{SO}(2)$, M^4 is equivariantly diffeomorphic to $\mathbb{S}^4$.

Case 3.2. Suppose $\dim C = 1$. We have $C^1 \simeq [-1, +1]$ or $C^1 \simeq \mathbb{S}^1$.

Case 3.2.1. Assume $C \simeq [-1, +1]$. After another step of the soul construction, we get the soul of M^*, which must be a point and it follows that M^* is homeomorphic to $\mathbb{D}^2$. By Theorem 2.7, all the points in $M^* - \{ M^{\mathsf{G}} \cup C \}$ correspond to principal orbits.

We will now analyze the orbits corresponding to the points in the singular set $C^1 \simeq [-1, +1]$. Note first that the points in $(-1, +1)$ cannot all be fixed, since this would imply that $(-1, +1) \cong \mathbb{S}^1$, which is a contradiction.

Let K_-, K_+ and K_0 be the isotropy groups corresponding to $\{-1\}$, $\{+1\}$ and $(-1, +1)$, respectively. As we have done before, we will denote this triple by $\mathsf{K}_- \cdots \mathsf{K}_0 \cdots \mathsf{K}_+$.

It follows from the Isotropy Lemma 2.4 that $\mathsf{K}_0 \leq \mathsf{K}_\pm$. The principal isotropy group is $\mathsf{SO}(2) = \mathsf{S}^1$, so the possibilities for $\mathsf{K}_\pm$ are $\mathsf{SO}(3)$, $\mathsf{O}(2)$ and S^1. By reprsentation theory, the fixed-point set components must be circles, so there are no isolated points with $\mathsf{SO}(3)$ isotropy. Recall that an orbit stratum must be a manifold without boundary, so the points in C cannot all have the same non-principal isotropy group. Thus we have the following possibilities for the isotropy triple $\mathsf{K}_- \cdots \mathsf{K}_0 \cdots \mathsf{K}_+$:

$$\mathsf{SO}(2)\cdots\mathsf{SO}(2)\cdots\mathsf{SO}(2), \tag{4.1.1}$$

$$\mathsf{O}(2)\cdots\mathsf{SO}(2)\cdots\mathsf{SO}(2), \tag{4.1.2}$$

$$\mathsf{O}(2)\cdots\mathsf{SO}(2)\cdots\mathsf{O}(2). \tag{4.1.3}$$

In case 4.1.1 all orbits are principal, except for those in $M^{\mathsf{G}} \cong \mathbb{S}^1$. This case is ruled out because by Lemma 2.13 there must be non-pricipal isotropy in ∂C^1.

Case 4.1.2 reduces to case 3.1, in which C is a point with isotropy $\mathsf{O}(2)$, following the same argument as in Case 2.2.1.2 in Chapter 3. It follows that M^4 is diffeomorphic to $\mathbb{RP}^4$.

We consider now 4.1.3. Let us see that M^4 can be exhibited as the connected sum of two copies of $\mathbb{RP}^4$. Recall first that M^* is a 2-disk. Divide $M^* \simeq \mathbb{D}^2$ in half, by a curve joining points in the boundary circle, so that each point with $\mathsf{O}(2)$-isotropy lies in a different half of M^*. This curve lifts to a 3-sphere and we see that M^4 is $\mathbb{RP}^4 \# \mathbb{RP}^4$.

The lift of $C^1 \simeq [-1,+1]$ under the projection map $\pi : M \to M^*$ is $\pi^{-1}([-1,+1]) \cong \mathbb{RP}^3 \# \mathbb{RP}^3$. The boundary E^3 of a tubular neighborhood of this lift in M^4 is an $\mathbb{S}^0$-bundle over $\mathbb{RP}^3 \# \mathbb{RP}^3$, so E^3 double-covers $\mathbb{RP}^3 \# \mathbb{RP}^3$. On the other hand, E^3 is also the boundary of a tubular neighborhood of the fixed-point set $\mathbb{S}^1$. Hence E^3 is also an $\mathbb{S}^2$-bundle over $\mathbb{S}^1$. We have the following diagram

$$\begin{array}{ccccc} \mathbb{S}^2 & \longrightarrow & E^3 & \longleftarrow & \mathbb{S}^0 \\ & \swarrow & & \searrow & \\ \mathbb{S}^1 & & & & \mathbb{RP}^3\#\mathbb{RP}^3 \end{array} \quad . \tag{4.1.4}$$

We will see now that this orbit space can be realized by an $\mathsf{SO}(3)$-action on $\mathbb{RP}^4\#\mathbb{RP}^4$, induced from an isometric $\mathsf{SO}(3)$-action on $\mathbb{S}^3 \times \mathbb{S}^1$.

We will describe a general construction for $\mathsf{SO}(n-1)$-actions on $\mathbb{RP}^n\#\mathbb{RP}^n$. The $\mathsf{SO}(3)$-action we want on $\mathbb{RP}^4\#\mathbb{RP}^4$ will then be a particular case of this construction. Observe first that $\mathbb{RP}^n\#\mathbb{RP}^n$ is the quotient of $\mathbb{S}^{n-1}\times\mathbb{S}^1$ by the $\mathbb{Z}_2$-action given by $-1(x,z) \mapsto (Ax,\bar{z})$ where $A : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is the antipodal map and $z \mapsto \bar{z}$ is complex conjugation when we consider $\mathbb{S}^1 \subset \mathbb{C}$. Now, consider the $\mathsf{SO}(n-1)$ action on $\mathbb{S}^{n-1} \times \mathbb{S}^1$ given by letting $\mathsf{SO}(n-1)$ act with cohomogeneity one on $\mathbb{S}^{n-1}$ and trivially on $\mathbb{S}^1$. Since rotations commute with the antipodal map, this action induces an $\mathsf{SO}(n-1)$-action on the quotient $\mathbb{RP}^n\#\mathbb{RP}^n$. Observe that the orbit space of $\mathbb{S}^{n-1} \times \mathbb{S}^1$ is a cylinder whose boundary circles are fixed-point components. This orbit space double-covers the orbit space of $\mathbb{RP}^n\#\mathbb{RP}^n$. We have a commutative diagram

$$\begin{array}{ccc} \mathbb{S}^{n-1}\times\mathbb{S}^1 & \xrightarrow{\kappa} & \mathbb{RP}^n\#\mathbb{RP}^n \\ \pi\big\downarrow & & \pi\big\downarrow \\ \mathbb{S}^1\times I & \xrightarrow{\kappa} & \mathbb{D}^2 \end{array} \quad ,$$

where π is the orbit projection map of the $\mathsf{SO}(n-1)$-action and κ is the quotient map under the $\mathbb{Z}_2$ covering action.

Case 3.2.2. Suppose $C^1 \simeq \mathbb{S}^1$. In this case M^* is isometric to a flat cylinder $\mathbb{S}^1 \times \mathbb{I}$

or to a flat Möbius band whose boundary is the fixed-point set F^1.

Suppose first that M^* is a cylinder $\mathbb{S}^1 \times \mathbb{I}$. One of the boundary components corresponds to the fixed-point set component F^1. The other boundary component, corresponding to C^1, is either another component of the fixed-point set or it has isotropy $\mathsf{O}(2)$. When the boundary is a fixed-point set component, the manifold is an $\mathbb{S}^3$-bundle over $\mathbb{S}^1$. Observe that we can realize this orbit space structure on $\mathbb{S}^3 \times \mathbb{S}^1$ with nonnegative curvature by letting $\mathsf{SO}(3)$ act by cohomogeneity one on the $\mathbb{S}^3$-factor and trivially on the $\mathbb{S}^1$-factor. When the boundary has $\mathsf{O}(2)$-isotropy, then the orbit type of points in the boundary with isotropy $\mathsf{O}(2)$ is $\mathbb{RP}^2$ and the lift of a geodesic joining two points in the boundary of $\mathbb{S}^1 \times \mathbb{I}$ is $\mathbb{RP}^3$. Hence M^4 is an $\mathbb{RP}^3$-bundle over the fixed-point set component $\mathbb{S}^1$. It follows from the long exact sequence of this bundle that $\pi_1(M^4) = \mathbb{Z}_2 \times \mathbb{Z}$ and, by the Splitting Theorem, M^4 is covered by $\mathbb{S}^3 \times \mathbb{R}$. In fact, we can realize this orbit space structure on $\mathbb{RP}^3 \times \mathbb{S}^1$ with nonnegative curvature by letting $\mathsf{SO}(3)$ act by cohomogeneity one on the $\mathbb{RP}^3$-factor and trivially on the $\mathbb{S}^1$-factor.

Suppose now that M^* is a Möbius band. Observe that the set at maximal distance $C^1 = \mathbb{S}^1$ cannot have any isotropy, since its space of directions must have diameter π. In this case, M^4 is an $\mathbb{S}^3$-bundle over $C = \mathbb{S}^1$. We can realize this orbit space structure on the non-trivial bundle $\mathbb{S}^3 \tilde{\times} \mathbb{S}^1$ with nonnegative curvature by letting $\mathsf{SO}(3)$ act by cohomogeneity one on the $\mathbb{S}^3$-fibers.

Case 4. $(\mathsf{G}, \mathsf{H}) = (\mathsf{S}^1, 1)$. We will determine the possible orbit spaces of a fixed-pont homogeneous circle action on a nonnegatively curved 4-manifold M^4. We will also give examples realizing these orbit spaces via isometric actions on 4-manifolds with nonnegative curvature. In the next chapter we will further discuss circle actions realizing these orbit spaces when M^4 is simply connected.

Let $F^2 \subset \partial M^*$ be a component of the fixed-point set with maximal dimension and let C be the set at maximal distance from F^2 in the orbit space M^*. We have $0 \leq \dim C \leq \dim F^2 = 2$.

Case 4.1. Suppose $\dim C = 0$. Then C^0 is the soul of M^*. This orbit space structure corresponds to case (a_2) in Theorem 2.2. It follows from the proof of this theorem (cf. [16]) that M^4 is equivariantly diffeomorphic to $\mathbb{CP}^2$ when C^0 is a fixed point, to $\mathbb{S}^4$ when C^0 has trivial isotropy, or to $\mathbb{RP}^4$ when C^0 has $\mathbb{Z}_2$-isotropy. Observe that C^0 cannot have isotropy group $\mathbb{Z}_q$, with $q \geq 3$, since the set of points with finite isotropy group $\mathbb{Z}_q$, $q \geq 3$, must have even codimension in M^4.

Case 4.2. Suppose $\dim C = 1$. Then $C^1 \simeq \mathbb{S}^1$ or $C^1 \simeq [-1, +1]$.

Case 4.2.1. Suppose $C^1 \simeq \mathbb{S}^1$. Then C^1 is the soul of M^* and the largest isotropy group in $C^1 \cong \mathbb{S}^1$ is either S^1, $\mathbb{Z}_q$ $(q \geq 2)$, or 1. Observe that by the Isotropy Lemma 2.4 all the points in C^1 must have the same isotropy group.

Case 4.2.1.1. Suppose the largest isotropy group is S^1. Then $C^1 \cong \mathbb{S}^1$ is a component of the fixed-point set. This is a contradiction, since the components of the fixed-point set of an S^1- action must have even codimension in M^4. Hence this case is ruled out.

Case 4.2.1.2. Suppose the largest isotropy group is $\mathbb{Z}_q$, for some $q \geq 2$. Then all the points in $C^1 \simeq \mathbb{S}^1$ have isotropy $\mathbb{Z}_q$. Observe that there are no critical points for the distance function to F in $M^* - \{F \cup C^1\}$ and we have a gradient-like vector field from F to the soul circle C^1 which is radial near F and near C^1 (cf. [15]). Given a point p^* in C^1, the set of flow-lines from p^* to F is a 2-disk whose lift is a lens space $L(q, q')$. Hence M^4 is a lens space-bundle over $\mathbb{S}^1$ and it follows from the long exact homotopy sequence of a bundle that $\pi_1(M^4) \cong \mathbb{Z}_q \times \mathbb{Z}$. Hence, by the Splitting Theorem 2.10, M^4 is covered by $\mathbb{S}^3 \times \mathbb{R}$.

The fixed-point set F^2 is diffeomorphic to the boundary of a tubular neighborhood of C^1, so it is an $\mathbb{S}^1$-bundle over $C^1 \simeq \mathbb{S}^1$ and hence either a torus $\mathbb{T}^2$ or a Klein bottle $\mathbb{K}^2$.

When $F^2 = \mathbb{T}^2$ we can realize this orbit space configuration on $L(q, q') \times \mathbb{S}^1$. Observe first that the fixed-point homogeneous S^1-action on $\mathbb{S}^3$ commutes with the $\mathbb{Z}_q$ action whose quotient is the lens space $L(q, q')$. Hence the covering map $\kappa : \mathbb{S}^3 \to L(q, q')$ induces a fixed-point homogeneous S^1-action on $L(q, q')$ whose orbit space is a 2-disk whose boundary circle is the fixed-point set of the action and whose set at maximal distance is a point with finite isotropy $\mathbb{Z}_q$. Consider now the S^1-action on $L(q, q') \times \mathbb{S}^1$ given by letting S^1 act fixed-point homogeneously on $L(q, q')$ and trivially on $\mathbb{S}^1$. The orbit space is a solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ with $F^2 = \mathbb{T}^2$ and $C^1 \simeq \mathbb{S}^1$ with $\mathbb{Z}_q$ isotropy.

When $F^2 = \mathbb{K}^2$, we can realize this orbit space configuration on $L(q,q')\tilde{\times}\mathbb{S}^1 \cong (L(q,q') \times [0,1)]/(x,0) \sim (Ax,1)$ where A is the map induced on $L(q,q')$ by the antipodal map on $\mathbb{S}^3$ via the covering map $\kappa : \mathbb{S}^3 \to L(q,q')$. Since $A : L(q,q') \to L(q,q')$ commutes with the fixed-point homogeneous S^1-action on $L(q,q')$, we have a fixed-point homogeneous action on $L(q,q')\tilde{\times}\mathbb{S}^1$ by letting S^1 act-fixed point homogeneously on the $L(q,q')$-fibers. The orbit space is a non-trivial $\mathbb{D}^2$-bundle $\mathbb{D}^2\tilde{\times}\mathbb{S}^1$ whose boundary $F^2 = \mathbb{K}^2$ is the fixed-point set and C^1 is a circle with $\mathbb{Z}_q$ isotropy.

Case 4.2.1.3. Suppose the largest isotropy group is the principal isotropy group 1, so that all points in $C^1 \simeq \mathbb{S}^1$ have trivial isotropy group. As in Case 4.2.1.2 , we see that M^4 is an $\mathbb{S}^3$-bundle over $\mathbb{S}^1$. It follows from the long exact homotopy sequence of a bundle that $\pi_1(M^4) \cong \mathbb{Z}$. By the Splitting Theorem 2.10, M^4 must be covered by $\mathbb{S}^3 \times \mathbb{R}$ equipped with a product metric of nonnegative curvature.

We know that the lift $\pi^{-1}(C^1)$ of $C^1 = \mathbb{S}^1$ in M^4 is an $\mathbb{S}^1$-bundle over $\mathbb{S}^1$. Thus the total space must be $\mathbb{T}^2$ or $\mathbb{K}^2$ and a tubular neighborhood of this lift in M^4 is a $\mathbb{D}^2$-bundle over $\pi^{-1}(C^1)$.

We now determine F^2, the fixed-point set component with maximal dimension. Consider a tubular neighborhood $D(C^1)$ in the orbit space M^*. Its boundary is an $\mathbb{S}^1$-bundle over $\mathbb{S}^1$, so it must be $\mathbb{T}^2$ or $\mathbb{K}^2$. Moreover, $\partial D(C^1)$ is diffeomorphic to $\partial M^* = F^2$, the fixed-point set component with maximal dimension. Assume first that M^4 is orientable. Then we must have $F^2 = \mathbb{T}^2$, since a fixed-point set component of a smooth S^1-action on an orientable manifold is an orientable submanifold.

A tubular neighborhood of the 2-dimensional fixed-point set $F^2 = \mathbb{T}^2$ in M^4

is a $\mathbb{D}^2$-bundle over $\mathbb{T}^2$ and its boundary is an $\mathbb{S}^1$-bundle over $\mathbb{T}^2$.

Thus we have the following two possible diagrams:

$$\begin{array}{ccccc} \mathbb{S}^1 & \longrightarrow & E^3 & \longleftarrow & \mathbb{S}^1 \\ & \swarrow & & \searrow & \\ \mathbb{T}^2 & & & & \mathbb{T}^2 \end{array}, \tag{4.1.5}$$

$$\begin{array}{ccccc} \mathbb{S}^1 & \longrightarrow & E^3 & \longleftarrow & \mathbb{S}^1 \\ & \swarrow & & \searrow & \\ \mathbb{T}^2 & & & & \mathbb{K}^2 \end{array}. \tag{4.1.6}$$

From the long exact homotopy sequence of the fiber bundle $\mathbb{S}^1 \longrightarrow E^3 \longrightarrow \mathbb{T}^2$ and the fact that $\pi_i(\mathbb{T}^2) = 0$ for $i \geq 2$ we obtain the short exact sequence

$$0 \longrightarrow \pi_1(\mathbb{S}^1) \longrightarrow \pi_1(E^3) \longrightarrow \pi_1(\mathbb{T}^2) \to 0,$$

which we rewrite as

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi(E^3) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \to 0.$$

Recall that free modules are projective. Hence, since $\mathbb{Z} \oplus \mathbb{Z}$ is a free $\mathbb{Z}$-module, the short exact sequence splits and we have that

$$\pi_1(E^3) = \mathbb{Z}^3.$$

On the other hand, from the long exact homotopy sequence of the fiber bundle $\mathbb{S}^1 \longrightarrow E^3 \longrightarrow \mathbb{K}^2$ in diagram 4.1.6 we obtain the short exact sequence

$$0 \longrightarrow \pi_1(\mathbb{S}^1) \longrightarrow \pi_1(E^3) \longrightarrow \pi_1(\mathbb{K}^2) \to 0,$$

which we rewrite as

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^3 \longrightarrow \pi_1(\mathbb{K}^2) \to 0.$$

By exactness and the first isomorphism theorem we cannot have such a sequence, since $\pi_1(\mathbb{K}^2)$ is not abelian. This rules out diagram 4.1.6.

The orbit space structure in diagram 4.1.5 can be realized by an isometric S^1-action on $\mathbb{S}^3 \times \mathbb{S}^1$ equipped with the product metric, taking the fixed-point homogeneous S^1 action on the $\mathbb{S}^3$ factor and letting S^1 act trivially on the $\mathbb{S}^1$ factor. The orbit space is a solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ whose boundary $\mathbb{T}^2$ is the fixed-point set of the action, and the set at maximal distance is a circle with trivial isotropy. Observe that, by the splitting theorem, any isometric action on $\mathbb{S}^3 \times \mathbb{S}^1$ must split, acting by isometries on each factor. There is only one isometric fixed-point homogeneous action on $\mathbb{S}^3$ up to equivariant diffeomorphism (cf. Chapter 3) so there is only one isometric action on $\mathbb{S}^3 \times \mathbb{S}^1$ with nonnegative curvature yielding the desired orbit space.

Assume now that M^4 is non-orientable. Then we also have $\mathbb{K}^2$ as a possible 2-dimensional fixed-point set component F^2. Proceeding as above, we see that we must have a diagram

$$\begin{array}{ccccc} \mathbb{S}^1 & \longrightarrow & E^3 & \longleftarrow & \mathbb{S}^1 \\ & \swarrow & & \searrow & \\ \mathbb{K}^2 & & & & \mathbb{K}^2 \end{array}.$$

This orbit space structure can be realized on the non-trivial bundle $\mathbb{S}^3 \tilde{\times} \mathbb{S}^1$, by taking the fixed-point homogeneous action on each fiber. The orbit space will be the non-trivial $\mathbb{D}^2$-bundle over $\mathbb{S}^1$, whose boundary $\mathbb{K}^2$ corresponds to the fixed-point set F^2, with set at maximal distance $C^1 \simeq \mathbb{S}^1$ with trivial isotropy. Let us denote this pair by $[F, C]^*$ and its lift by $[F, B]$, so that in this case we have $[F, C]^* = [\mathbb{K}^2, \mathbb{S}^1]$ and $[F, B] = [\mathbb{K}^2, \mathbb{K}^2]$. Observe that this action on $\mathbb{S}^3 \tilde{\times} \mathbb{S}^1$ is induced by the action of S^1 on $\mathbb{S}^3 \times \mathbb{S}^1$ via the double-covering map $\kappa : \mathbb{S}^3 \times \mathbb{S}^1 \to \mathbb{S}^3 \tilde{\times} \mathbb{S}^1$ and we have a commutative diagram

$$\begin{array}{ccc} \mathbb{S}^3 \times \mathbb{S}^1 & \xrightarrow{\kappa} & \mathbb{S}^3 \tilde{\times} \mathbb{S}^1 \\ {\scriptstyle \pi}\downarrow & & {\scriptstyle \pi}\downarrow \\ \mathbb{D}^2 \times \mathbb{S}^1 & \xrightarrow{\kappa} & \mathbb{D}^2 \tilde{\times} \mathbb{S}^1 \end{array},$$

where π is the orbit projection map of the S^1-action and κ is the quotient map under the $\mathbb{Z}_2$ covering action. In particular, we have

$$\begin{array}{ccc} [\mathbb{T}^2, \mathbb{T}^2] & \xrightarrow{\kappa} & [\mathbb{K}^2, \mathbb{K}^2] \\ {\scriptstyle \pi}\downarrow & & {\scriptstyle \pi}\downarrow \\ [\mathbb{T}^2, \mathbb{S}^1]^* & \xrightarrow{\kappa} & [\mathbb{K}^2, \mathbb{S}^1]^* \end{array},$$

where the left-hand side corresponds to the S^1-action on $\mathbb{S}^3 \times \mathbb{S}^1$ and the right-hand side corresponds to the S^1-action on $\mathbb{S}^3 \tilde{\times} \mathbb{S}^1$.

Case 4.2.2. Suppose $C^1 \simeq [-1, +1]$. We first analyze the orbits corresponding to the points in C^1. Let K_-, K_+ and K_0 be the isotropy group corresponding to $\{-1\}$, $\{+1\}$ and $(-1, +1)$ respectively, and denote this triple by $\mathsf{K}_- \cdots \mathsf{K}_0 \cdots \mathsf{K}_+$. The largest isotropy group in this triple is either $\mathsf{1}$, $\mathbb{Z}_q$ $(q \geq 2)$, or S^1 and it follows from the Isotropy Lemma 2.4 that $\mathsf{K}_0 \leq \mathsf{K}_\pm$. Observe that we cannot have $\mathsf{K}_- = \mathsf{K}_0 = \mathsf{K}_+$ unless every isotropy group is principal.

Case 4.2.2.1. Suppose the largest isotropy group is $\mathsf{1}$. This case reduces to the case when C is a point with trivial isotropy (cf. Case 4.1) and it follows that M^4 is diffeomorphic to $\mathbb{S}^4$.

Case 4.2.2.2. Suppose the largest isotropy group is $\mathbb{Z}_q$, for some $q \geq 2$, so that we have the isotropy triple

$$\mathbb{Z}_{q_-} \cdots \mathbb{Z}_l \cdots \mathbb{Z}_{q_+}.$$

We now show that we must have $q_\pm \leq 2$. Let p be a point in an orbit with finite isotropy $\mathbb{Z}_k$, $k \geq 3$. Then the unit normal sphere to the orbit at p is $\mathbb{S}^2$ and $\mathbb{Z}_k$ acts on it fixing one direction. Hence the space of directions has diameter π and an endpoint of $C^1 \simeq [-1,+1]$ cannot have isotropy $\mathbb{Z}_k$ with $k \geq 3$. Hence we have $q_\pm \leq 2$. Observe that we cannot have all points in $C^1 \simeq [-1,+1]$ have $\mathbb{Z}_2$-isotropy, since an orbit stratum must be a manifold without boundary. Hence we only have the isotropy triples

$$\mathsf{1} \cdots \mathsf{1} \cdots \mathbb{Z}_2 \quad \text{and} \quad \mathbb{Z}_2 \cdots \mathsf{1} \cdots \mathbb{Z}_2.$$

The same argument as in case 2.2.1.2 in Chapter 3 implies that the case of $\mathsf{1} \cdots \mathsf{1} \cdots \mathbb{Z}_2$ reduces to the case when the set at maximal distance C is a point with $\mathbb{Z}_2$-isotropy (cf. Case 4.1). The manifold in this case is diffeomorphic to $\mathbb{RP}^4$.

In the second case corresponding to the isotropy triple $\mathbb{Z}_2 \cdots \mathsf{1} \cdots \mathbb{Z}_2$, the lift of C^1 under the orbit map $\pi : M \to M^*$ is $\pi^{-1}([-1,+1]) \simeq \mathbb{RP}^2\#\mathbb{RP}^2$. Observe now that the space of directions at $\pm 1 \in C^1$ is $\mathbb{RP}^2$. Hence the boundary of a tubular neighborhood of ± 1 in M^* is $\mathbb{RP}^2$ and the boundary of a tubular neighborhood of C^1 in M^* of C^1 is $\mathbb{RP}^2\#\mathbb{RP}^2$. Hence $F^2 \cong \mathbb{RP}^2\#\mathbb{RP}^2 \cong \mathbb{K}^2$ and it follows that M^4 is non-orientable. Observe that M^4 can be written as the union of tubular neighborhoods of $\mathbb{RP}^2\#\mathbb{RP}^2$ and $\mathbb{RP}^2\#\mathbb{RP}^2$ along their common boundary E^3. We consider now the orientable double cover $\tilde{M}$ of M. The fixed-point set $\tilde{F}^2$ of the lifted isometric circle action must double-cover $F^2 \cong \mathbb{K}^2$ and be orientable, so $\tilde{F}^2 \cong \mathbb{T}^2$ and the lift of the set at maximal distance is a circle $\mathbb{S}^1$ with no isotropy. This orbit space configuration has been analyzed already (cf. Case 4.2.1.3) and it follows that M is covered by $\mathbb{S}^3 \times \mathbb{R}$.

We will now describe an isometric S^1-action on $\mathbb{RP}^4\#\mathbb{RP}^4$ with this orbit space structure. Observe first that $\mathbb{S}^3\times\mathbb{S}^1$ is a double-cover of $\mathbb{RP}^4\#\mathbb{RP}^4$. We get $\mathbb{RP}^4\#\mathbb{RP}^4$ as a quotient of $\mathbb{S}^3 \times \mathbb{S}^1 \subset \mathbb{C}^2 \times \mathbb{C}$ by the action of $\mathbb{Z}_2$ given by

$$-1((z_1, z_2), z_3) \mapsto ((-z_1, -z_2), \bar{z}_3),$$

i.e., $\mathbb{Z}_2$ acts by the antipodal map on $\mathbb{S}^3 \subset \mathbb{C}^2$ and by conjugation on $\mathbb{S}^1 \subset \mathbb{C}$. On $\mathbb{S}^3 \subset \mathbb{C}^2$ we have the standard fixed-point cohomogeneity S^1-one action given by

$$\lambda(z_1, z_2) \mapsto (\lambda z_1, z_2), \quad \lambda \in \mathsf{S}^1,\ (z_1, z_2) \in \mathbb{S}^3$$

which has fixed-point set a circle. We extend this action to a fixed-point homogeneous action on $\mathbb{S}^3\times\mathbb{S}^1$ by letting S^1 act fixed-point homogeneously on the $\mathbb{S}^3$-factor and trivially on the $\mathbb{S}^1$-factor. Since the S^1=action on $\mathbb{S}^3 \times \mathbb{S}^1$ commutes with the $\mathbb{Z}_2$-action, we have an induced S^1-action on $\mathbb{RP}^4\#\mathbb{RP}^4$. Moreover, the orbit space $(\mathbb{S}^3 \times \mathbb{S}^1)^* \simeq \mathbb{D}^2 \times \mathbb{S}^1$ double-covers the orbit space $(\mathbb{RP}^4 \times \mathbb{S}^4)^* \simeq \mathbb{D}^2$ and we have a commutative diagram

$$\begin{array}{ccc} \mathbb{S}^3 \times \mathbb{S}^1 & \xrightarrow{\kappa} & \mathbb{RP}^4\#\mathbb{RP}^4 \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\ \mathbb{D}^2 \times \mathbb{S}^1 & \xrightarrow{\kappa} & \mathbb{D}^2 \end{array},$$

where π is the orbit projection map of the S^1-action and κ is the quotient map under the $\mathbb{Z}_2$ covering action.

Observe that the $\mathbb{S}^1$-action on $\mathbb{S}^3 \times \mathbb{S}^1$ has $F^2 \cong \mathbb{T}^2$ and set at maximal distance $C^1 \simeq \mathbb{S}^1$. Let us denote this pair by $[F, C]^*$ and denote its lift by $[F, B]$, so that in this case we have $[F, C]^* = [\mathbb{T}^2, \mathbb{S}^1]$ and $[F, B] = [\mathbb{T}^2, \mathbb{T}^2]$. Hence we have

$$\begin{array}{ccc} [\mathbb{T}^2, \mathbb{T}^2] & \xrightarrow{\kappa} & [\mathbb{K}^2, \mathbb{K}^2] \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ [\mathbb{T}^2, \mathbb{S}^1]^* & \xrightarrow{\kappa} & [\mathbb{K}^2, \mathbb{I}]^* \end{array},$$

where the left-hand side corresponds to the S^1-action on $\mathbb{S}^3 \times \mathbb{S}^1$ and the right-hand side corresponds to the S^1-action on $\mathbb{RP}^4 \# \mathbb{RP}^4$.

Case 4.2.2.3. Suppose the largest isotropy group is S^1. We have the following possible isotropy configurations:

$$\mathsf{S}^1 \cdots \mathsf{1} \cdots \mathsf{1}; \tag{4.1.7}$$

$$\mathsf{S}^1 \cdots \mathsf{1} \cdots \mathsf{S}^1. \tag{4.1.8}$$

$$\mathsf{S}^1 \cdots \mathsf{1} \cdots \mathbb{Z}_2, \text{ for some } q \geq 2; \tag{4.1.9}$$

$$\mathsf{S}^1 \cdots \mathbb{Z}_l \cdots \mathsf{S}^1, \text{ for some } l \geq 2; \tag{4.1.10}$$

Case 4.1.7 reduces to case 4.1, where C is a point with S^1-isotropy. In this case M^4 is diffeomorphic to $\mathbb{CP}^2$.

In case 4.1.8, the boundary of a neighborhood of $C^1 \simeq [-1, +1]$ in M^* is $\mathbb{S}^2$. Hence the 2-dimensional fixed-point set component F^2 is diffeomorphic to $\mathbb{S}^2$. Moreover, the lift $\pi^{-1}(C^1)$ is also $\mathbb{S}^2$, so we can write M^4 as the union of two 2-disk bundles over $\mathbb{S}^2$. It follows from Van Kampen's theorem that M^4 is simply connected. By Kobayashi's theorem, $\chi(M^4) = \chi(\text{Fix}(M^4, \mathsf{S}^1))$=4. It follows from Theorem 5.2 in Chapter 5 that M^4 is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$. We will see in Chapter 5 that $\mathbb{CP}^2 \# \mathbb{CP}^2$ and $\mathbb{CP}^2 \# - \mathbb{CP}^2$ are the only simply-connected 4-manifolds that support smooth circle actions with this orbit space structure.

Now we analyze case 4.1.9. Observe that the boundary of a neighborhood of C^1 is $\mathbb{RP}^2$ and the lift of C^1 in M^4 is also $\mathbb{RP}^2$. Hence $F^2 \cong \mathbb{RP}^2$, so M is non-orientable and can be written as the union of two 2-disk bundles over $\mathbb{RP}^2$ glued along their common boundary E^3. Let $\tilde{M}$ be the orientable double-cover of M with the lifted isometric circle action. Then the fixed-point set $\text{Fix}(\tilde{M}, \mathsf{S}^1)$ of the lifted action double-covers the fixed-point set of the S^1-action on M and we must have that $\text{Fix}(\tilde{M}, \mathsf{S}^1)$ consists of a 2-sphere and two isolated fixed-points. Hence M^4 must be double-covered by $\mathbb{CP}^2 \# \mathbb{CP}^2$ or $\mathbb{CP}^2 \# - \mathbb{CP}^2$ (cf. Case 4.1.8).

In case 4.1.10, the boundary of a neighborhood of the interval C^1 is $\mathbb{S}^2$. Hence the fixed-point set F^2 is diffeomorphic to $\mathbb{S}^2$. Moreover, the lift of C^1 is a manifold, since it is a component of the fixed-point set of $\mathbb{Z}_l$, and corresponds to $\mathbb{S}^2$. As in case 4.1.8, M^4 is diffeomorphic to either $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$. Smooth actions with this orbit space structure can be realized on $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$ (cf. Chapter 5).

Case 4.3. Suppose $\dim C = 2$. Observe that C^2 is a 2-dimensional nonnegatively curved Alexandrov space. We consider two cases: $C \subset \partial M^*$ and $\partial M^* = F$.

Case 4.3.1. Suppose C^2 is a boundary component of M^*. Then, by Lemma 2.12, C^2 is a fixed-point component or all the points in C^2 have isotropy $\mathbb{Z}_2$. In both cases C^2 is a closed smooth 2-manifold with nonnegative curvature, F^2 and C^2 are isometric and M^* is isometric to $F^2 \times \mathbb{I}$. Since F^2 is a closed nonnegatively curved 2-manifold, it must be diffeomorphic to $\mathbb{S}^2$, $\mathbb{RP}^2$, $\mathbb{T}^2$ or $\mathbb{K}^2$.

Suppose that C^2 is a component of the fixed-point set. Then M^4 is an $\mathbb{S}^2$-bundle over $C^2 = F^2$, by the Double Soul Theorem 2.6. It follows from the long exact homotopy sequence of a fiber bundle that $\pi_1(M^4) \cong \pi_1(F^2)$. Hence M^4 is simply connected if and only if F^2 is $\mathbb{S}^2$.

When $F^2 = \mathbb{S}^2$, M^4 is an $\mathbb{S}^2$-bundle over $\mathbb{S}^2$ and it follows that M^4 is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2 \cong \mathbb{CP}^2 \# -\mathbb{CP}^2$. In fact, both manifolds support isometric S^1-actions with fixed-point set $\mathbb{S}^2 \cup \mathbb{S}^2$. On $\mathbb{S}^2 \times \mathbb{S}^2$ let S^1 act by cohomogeneity one on the first $\mathbb{S}^2$ factor and trivially on the second $\mathbb{S}^2$ factor. To obtain an isometric S^1-action on $\mathbb{CP}^2 \# -\mathbb{CP}^2$ with nonnegative curvature and fixed-point set $\mathbb{S}^2 \cup \mathbb{S}^2$ start by letting S^1 act fixed-point homogeneously on $\mathbb{CP}^2$. This action has fixed-point set $\mathbb{S}^2 \cup \{p\}$. We remove an invariant neighborhood of the isolated fixed point and do the same construction on $-\mathbb{CP}^2$ equipped with a fixed-point homogeneous $\mathbb{S}^1$-action. We now take an equivariant connected sum to obtain $\mathbb{CP}^2 \# -\mathbb{CP}^2$ with nonnegative curvature and a fixed-point homogeneous isometric S^1-action with fixed-point set $\mathbb{S}^2 \cup \mathbb{S}^2$.

When F^2 is not $\mathbb{S}^2$, M^4 is not simply-connected. Let $\tilde{M}^4$ be the universal covering space of M^4. Then we have

$$\tilde{M}^4 = \begin{cases} \mathbb{CP}^2 \# -\mathbb{CP}^2 \text{ or } \mathbb{S}^2 \times \mathbb{S}^2 & \text{if } F^2 = \mathbb{RP}^2; \\ \mathbb{S}^2 \times \mathbb{R}^2 & \text{if } F^2 = \mathbb{T}^2 \text{ or } \mathbb{K}^2. \end{cases}$$

We can construct examples realizing the orbit space structure $M^* = F^2 \times I$ with the two boundary components corresponding to fixed-point set components by letting S^1 act on the product $F^2 \times \mathbb{S}^2$ by cohomogeneity one on $\mathbb{S}^2$ and trivially on F^2.

Suppose now that all the points in C^2 have isotropy $\mathbb{Z}_2$. Observe that a geodesic from F^2 to C^2 lifts to $\mathbb{RP}^2$, so M^4 is an $\mathbb{RP}^2$-bundle over F^2. Suppose that F^2 is $\mathbb{S}^2$. It follows from the long exact homotopy sequence of a bundle that $\pi_1(M^4)$ is 0 or $\mathbb{Z}_2$ and hence M^4 is covered by $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$ or $\mathbb{S}^2 \times \mathbb{S}^2$. When F^2 is $\mathbb{T}^2$, it follows from the long exact homotopy sequence of a bundle that $\pi_1(M^4) \cong \mathbb{Z}_2 \times \mathbb{Z}^2$ and, by the Splitting Theorem, M^4 is covered by $\mathbb{S}^2 \times \mathbb{R}^2$. When F^2 is $\mathbb{RP}^2$ or $\mathbb{K}^2$, we see that M^2 is covered by $\mathbb{S}^2 \times \mathbb{R}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$ by considering the orientable double-cover of M^4.

We can construct examples of actions on nonnegatively curved 4-manifolds with this orbit space structure by considering the product $F^2 \times \mathbb{RP}^2$ with S^1 acting by cohomogeneity one on $\mathbb{RP}^2$ and trivially on F^2.

Case 4.3.2. Suppose now that $\partial M^* = F^2$, so C^2 is not a boundary component of M^*. We consider two cases, depending on whether or not C^2 has boundary.

Case 4.3.2.1. Suppose $\partial C^2 = \emptyset$. Then by Lemma 2.13 all the points in C^2 have principal isotropy and F^2 double-covers C^2. Moreover, we have that M^4 is an $\mathbb{S}^2$-bundle over C^2. The only possibilities for F^2 are $\mathbb{S}^2$, $\mathbb{T}^2$ or $\mathbb{K}^2$.

When $F^2 \cong C^2 \cong \mathbb{T}^2$, we construct an example realizing this orbit space structure by considering $(\mathbb{S}^2 \tilde{\times} \mathbb{S}^1) \times \mathbb{S}^1$ with S^1 acting fixed-point homogeneously on $\mathbb{S}^2 \tilde{\times} \mathbb{S}^1$ and trivially on $\mathbb{S}^1$. The orbit space is the product of a Möbius band and $\mathbb{S}^1$. This has boundary $\mathbb{T}^2$, which corresponds to the fixed-point set, and set at maximal distance $\mathbb{T}^2$.

Case 4.3.2.2. Suppose $\partial C \neq \emptyset$. Observe that C is a 2-dimensional Alexandrov space with nonnegative curvature, hence it must be homeomorphic to $\mathbb{D}^2$ or isometric to a flat Möbius band $\mathbb{M}^2$ or a flat cylinder $\mathbb{S}^1 \times \mathbb{I}$. By Lemma 2.13 there is no isotropy in the interior of C^2. Observe that there cannot be an isolated fixed-point in ∂C^2 since in this case the space of directions $\overline{p}$ is $\mathbb{S}^2(\frac{1}{2})$. We have a direction that is normal to C^2, corresponding to a geodesic from F^2 to $\overline{p}$. On the other hand, this geodesic is normal to C^2, so its direction must make an angle of $\pi/2$ with a 1-dimensional subset of $S_{\overline{p}}X = \mathbb{S}^2(\frac{1}{2})$, which is a contradiction. Thus, the only non-trivial isotropy is $\mathbb{Z}_2$.

Assume $C^2 = \mathbb{D}^2$. Suppose first that every point in the boundary circle has isotropy $\mathbb{Z}_2$. It follows from the Orlik-Raymond classification of 3-manifolds with a smooth S^1-action that the lift of C^2 is $\mathbb{S}^2 \tilde{\times} \mathbb{S}^1$, the non-trivial $\mathbb{S}^2$-bundle over $\mathbb{S}^1$. Then we can write M^4 as the union of disk bundles over $\mathbb{S}^2$ and $\mathbb{S}^2 \tilde{\times} \mathbb{S}^1$ glued along their common boundary E^3. We have the following diagram:

$$\begin{array}{ccccc} \mathbb{S}^1 & \longrightarrow & E^3 & \longleftarrow & \mathbb{S}^0 \\ & \swarrow & & \searrow & \\ \mathbb{S}^2 & & & & \mathbb{S}^2\tilde{\times}\mathbb{S}^1 \end{array} \quad . \tag{4.1.11}$$

Assuming M^4 is orientable, we must have $E^3 \simeq \mathbb{S}^2 \times \mathbb{S}^1$ and it follows from the Van-Kampen theorem that $\pi_1(M^4) \cong \mathbb{Z}_2$, so M^4 is double-covered by $\mathbb{S}^4$, $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$. Now, observe that $\chi(M^4) = \chi(\mathrm{Fix}(M^4, \mathsf{S}^1)) = 2$, so M^4 can only be covered by $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$, all of which have Euler characteristic 4. The fixed-point set of the lifted circle action on the universal cover $\tilde{M}^4$ double-covers $\mathbb{S}^2$. Hence $\mathrm{Fix}(\tilde{M}^4, \mathsf{S}^1)$ contains two fixed-point 2-spheres. By the Double Soul Theorem, $\tilde{M}$ is an $\mathbb{S}^2$-bundle over $\mathbb{S}^2$. Hence $\tilde{M}^4$ is diffeomorphic to either $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{CP}^2 \# - \mathbb{CP}^2$. If M^4 is non-orientable, by passing to the orientable double-cover we see that M^4 is covered by $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{CP}^2 \# - \mathbb{CP}^2$.

Suppose now that there are isolated points in ∂C^2 with finite isotropy $\mathbb{Z}_2$. By compactness there are finitely many of these points in the boundary circle. In fact, there can be at most four points $\bar{p}_1, \ldots, \bar{p}_4$ with isotropy $\mathbb{Z}_2$ on the boundary ∂C^2. We will now show that there can be at most two isolated points with $\mathbb{Z}_2$-isotropy in ∂C^2. Let $\bar{q}$ be a point in the interior of C^2 and let $\gamma_1, \ldots, \gamma_4$ be minimal geodesics joining $\bar{q}$ with $\bar{p}_1, \ldots, \bar{p}_k$, respectively, for some $k \geq 1$. Since C^2 is totally geodesic, these geodesics are contained in C^2. Now, observe that C^2 deformation retracts onto $U = \cup_{i=1}^{k} \gamma_1$. Hence a tubular neighborhood $D(C^2)$ is homotopy equivalent to a tubular neighborhood $D(U)$. The boundary of $D(U)$ is the connected sum of k

projective spaces and $\partial D(C^2) \cong F^2$ is homotopy equivalent to $\partial D(U)$. Hence F^2, which is a closed 2-manifold with nonnegative curvature, is homotopy equivalent to a connected sum of k projective spaces. Hence $\pi_1(F^2) \cong \pi_1(\#_{i=1}^k \mathbb{RP}^2)$. Hence we must have $k = 1$ or 2. When we have only one isolated point with $\mathbb{Z}_2$-isotropy, this case reduces to the case in which C is a point with $\mathbb{Z}_2$ isotropy and hence M^4 is diffeomorphic to $\mathbb{RP}^4$. When there are two points with $\mathbb{Z}_2$-isotropy, this case reduces to the case when C is an interval with enpoints with $\mathbb{Z}_2$-isotropy. In this case the manifold is diffeomorphic to $\mathbb{RP}^4 \# \mathbb{RP}^4$.

Suppose $C^2 = \mathbb{S}^1 \times \mathbb{I}$. The possible isotropy groups are $\mathbb{Z}_2$ or the trivial group $\mathsf{1}$.

Suppose the largest isotropy group is $\mathsf{1}$. Then M^* is a manifold with totally geodesic boundary and soul $\mathbb{S}^1$. This case reduces to the case in which $C = \mathbb{S}^1$ with trivial isotropy and the manifold is then diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^1$ or $\mathbb{S}^3 \tilde{\times} \mathbb{S}^1$.

Suppose now the largest isotropy group is $\mathbb{Z}_2$. Since $\mathbb{S}^1 \times \mathbb{I}$ has the product metric, the boundary components are closed geodesics. It follows from the Isotropy Lemma that, if a point in a boundary circle of $\mathbb{S}^1 \times \mathbb{I}$ has isotropy $\mathbb{Z}_2$, then every point in this circle has isotropy $\mathbb{Z}_2$. Assume first that there are two boundary components with $\mathbb{Z}_2$-isotropy. Observe that $F^2 \cong \mathbb{T}^2$ and the lift of $C^2 \cong \mathbb{S}^1 \times \mathbb{I}$ is $\mathbb{RP}^2 \# \mathbb{RP}^2 \times \mathbb{S}^1 \cong \mathbb{K}^2 \times \mathbb{S}^1$. Then M^4 is the union of tubular neighborhoods $D(\mathbb{T}^2)$

and $D(\mathbb{K}^2 \times \mathbb{S}^1)$ glued along their common boundary E^3. We have the diagram

$$\begin{array}{ccccc} \mathbb{S}^1 & \longrightarrow & E^3 & \longleftarrow & \mathbb{S}^0 \\ & \swarrow & & \searrow & \\ \mathbb{T}^2 & & & & \mathbb{K}^2 \times \mathbb{S}^1 \end{array} \quad . \tag{4.1.12}$$

It follows from Van-Kampen's Theorem that $\pi_1(M^4) \cong \pi_1(\mathbb{RP}^3 \# \mathbb{RP}^3) \times \mathbb{Z}$. It follows from the Spitting Theorem that M^4 is covered by $\mathbb{S}^2 \times \mathbb{R}^2$. This orbit space structure can be realized on $\mathbb{RP}^3 \# \mathbb{RP}^3 \times \mathbb{S}^1$ with S^1 acting fixed-point homogeneously on the first factor and trivially on the second factor.

Suppose we only have one boundary component with finite isotropy. As in case 2.2.1.2 in Chapter 3, the distance function to the boundary component of $C^2 = \mathbb{S}^1 \times \mathbb{I}$ with $\mathbb{Z}_2$ isotropy has no critical points and this case reduces to the case in which C is a circle with $\mathbb{Z}_2$-isotropy. It follows that M^4 is diffeomorphic to an $\mathbb{RP}^3$-bundle over $\mathbb{S}^1$.

Suppose $C^2 = \mathbb{M}^2$. Suppose the largest isotropy group is $\mathsf{1}$. Then the soul is $\mathbb{S}^1$ and this case reduces to the case in which $C = \mathbb{S}^1$ with trivial isotropy. It follows that M^4 is diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^1$ or $\mathbb{S}^3 \tilde{\times} \mathbb{S}^1$.

Suppose now that the largest isotropy group is $\mathbb{Z}_2$. We have isotropy $\mathbb{Z}_2$ on all the points in the boundary of C^2 and the lift of C^2 is $\mathbb{K}^2 \tilde{\times} \mathbb{S}^1$, a non-trivial $\mathbb{K}^2$-bundle over $\mathbb{S}^1$. Then M^4 is the union of tubular neighborhoods $D(\mathbb{K}^2)$ and $D(\mathbb{K}^2 \tilde{\times} \mathbb{S}^1)$

glued along their common boundary E^3. We have the diagram

$$\begin{array}{ccccc} \mathbb{S}^1 & \longrightarrow & E^3 & \longleftarrow & \mathbb{S}^0 \\ & \swarrow & & \searrow & \\ \mathbb{K}^2 & & & & \mathbb{K}^2 \tilde{\times} \mathbb{S}^1 \end{array} \quad . \tag{4.1.13}$$

Now, since $F = \mathbb{K}^2$, M^4 must be non-orientable. Passing to the orientable double-cover $\tilde{M}^4$, we must have $\tilde{F} = \mathbb{T}^2$, and it follows from the previous case that M^4 is covered by $\mathbb{S}^2 \times \mathbb{T}^2$.

□

Chapter 5

Fixed-point homogeneous circle actions on nonnegatively curved simply connected 4-manifolds

5.1 Introduction

Effective, locally smooth circle actions on 4-manifolds were classified up to equivariant homeomorphism by Fintushel in [9, 10]. This classification holds in the smooth category, as a result of carrying out the constructions therein in this setting [11]. In particular, as an immediate consequence of Fintushel's results, work of Pao [29], and the validity of the Poincaré conjecture due to Perelman [31, 32, 23, 26] one has the following theorem (cf. Theorem 13.2 in [10]).

Theorem 5.1. *Let M be a closed simply connected smooth 4-manifold with a smooth S^1-action. Then M is diffeomorphic to a connected sum of copies of $\mathbb{S}^4$, $\pm\mathbb{CP}^2$ and $\mathbb{S}^2 \times \mathbb{S}^2$. Moreover, the action is determined up to equivariant diffeomorphism by so-called legally weighted orbit space data.*

Suppose now that M is a simply connected Riemannian 4-manifold with an isometric S^1-action. If M has positive curvature, it follows from the work of Kleiner and Hsiang [21] that the Euler characterstic of M, denoted by $\chi(M)$, is 2 or 3. More generally, if M has nonnegative curvature, it follows from the work of Kleiner [22] or of Searle and Yang [37] that $2 \leq \chi(M) \leq 4$. Combining these facts with

Theorem 5.1 yields the following result.

Theorem 5.2. *Let M be a compact, simply connected Riemannian 4-manifold with an isometric S^1-action.*

(1) If M has positive curvature, then M is diffeomorphic to $\mathbb{S}^4$ or $\mathbb{CP}^2$.

(2) If M has nonnegative curvature, then M is diffeomorphic to $\mathbb{S}^4$, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{CP}^2$ or $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$.

In section 5.3 we apply Fintushel's work [9] to obtain further information on the orbit space of a smooth fixed-point homogeneous S^1-action on a nonnegatively curved simply connected Riemannian manifold M. We will use the orbit space data to identify M using the recipe given in [9] for computing its intersection form. We have collected in Section 5.2 the definitions and results from [9] that we use in section 5.3 to obtain our results.

The classification of positively curved fixed-point homogeneous manifolds due to Grove and Searle [16], which does not require the Poincaré conjecture, implies that a compact, simply connected Riemannian 4-manifold with positive curvature and an isometric fixed-point homogeneous S^1-action must be equivariantly diffeomorphic to $\mathbb{S}^4$ or $\mathbb{CP}^2$ with a linear action. More generally, a conjecture of Grove states that this should be the case for any isometric S^1-action on a positively curved simply connected Riemannian manifold (cf. [14]). It is an interesting question whether or not the conjecture also holds for nonnegatively curved manifolds. In this more general case, we will say that an S^1-action is *linear* if it extends to a T^2-action.

Question 5.3. *Is an isometric* S^1*-action on a simply connected nonnegatively curved 4-manifold equivariantly diffeomorphic to a linear action on* $\mathbb{S}^4$, $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$ *or* $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$*?*

We will see at the end of section 5.2 that the answer to this question is *yes*, provided the S^1-action is fixed-point homogeneous. This will be a simple consequence of [9] and our work in Chapter 4.

5.2 Fintushel's construction

Let M be a simply connected 4-manifold with a smooth S^1-action with orbit space M^*. In this section we review the definitions and results from [9] that we will use in the next section.

5.2.1 The weighted orbit space

Let us recall first some basic facts and terminology from [9] pertaining to the orbit space M^*. We will denote the fixed-point set by F, the set of exceptional orbits by E and the set of principal orbits by P. Given a subset $X \subset M$, we will denote its projection under the orbit map $\pi : M \to M^*$ by X^*. Given a subset $X^* \subset M^*$, we will let $X = \pi^{-1}(X^*)$ be its preimage under π. The orbit space M^* is a simply connected 3-manifold with $\partial M^* \subset F^*$, the set $F^* - \partial M^*$ of isolated fixed points is finite and F^* is nonempty. The components of ∂M^* are 2-spheres and the closure of E^* is a collection of polyhedral arcs and simple closed curves in M^*. The components of E^* are open arcs on which orbit types are constant, and these arcs have closures with distinct endpoints in $F^* - \partial M^*$. We will reserve the term *regular neighborhood* of $X^* \subset E^* \cup F^*$ for those regular neighborhoods N^* of X^* that satisfy $N^* \cap (E^* \cup F^*) = X^*$.

We remark that, if we do not require that M be simply connected, we may have loops $Q^* \subset E^*$. Consider, for example, the S^1-action on $\mathbb{RP}^3 \times \mathbb{S}^1$ given by the fixed-point homogeneous action of S^1 on $\mathbb{RP}^3$, induced by the fixed-point homoeneous S^1-action on $\mathbb{S}^3$ via the covering map, and the trivial action on the $\mathbb{S}^1$-factor. In this case M^* is a solid torus with $Q^* = E^*$ a loop with $\mathbb{Z}_2$ isotropy.

The orbit space M^* is assigned a set of data, called *weights*, which we now describe.

(a) Let F_i^* be a boundary component of M^*, choose a regular neighborhood $F_i^* \times [0,1]$ and orient $F_i^* \times 1$ by the normal out of $F_i^* \times [0,1]$. The restriction of the orbit map gives a principal S^1-bundle over $F_i^* \times 1$ and F_i^* is assigned the Euler number of this bundle. This is independent of the choice of the collar. We will call F_i^* a *weighted sphere*.

(b) If x^* is an isolated fixed point, i.e., if $x^* \in F^* - (\partial M^* \cup \operatorname{cl} E^*)$, let B^* be a polyhedral 3-disk neighborhood of x^* with $B^* - x^* \cup P^*$. We obtain a principal S^1-bundle over ∂B^* with total space $\mathbb{S}^3$ by restricting the orbit map. Orient ∂B^* by the normal out of B^* and assign to x^* the Euler number, ± 1, of the bundle.

(c) Let L^* be a simple closed curve in $E^* \cup F^*$. To each component J^* of E^* in L^* we assign Seifert invariants (cf. Chapter 3, Section 3.2) in the following way. Fix an orientation on L^*. This induces an orientation each component J^* of E^* in L^*.

Let y^* be an endpoint of clJ^* and let B^* be a polyhedral 3-disk neighborhood of y^* such that $B^* \cap (E^* \cup F^*) = B^* \cap L^*$ is an arc and $B^* \cap F^* = y^*$. If ∂B^* is oriented by the normal with direction J^* then ∂B is an oriented 3-sphere. Assign to J^* the Seifert invariants (α, β) of the orbit in ∂B with image in J^*. The covering homotopy theorem of Palais implies that this definition is independent of the choices made.

The weights assigned to L^* consist of the orientation and the Seifert invariants. We abbreviate this system of weights by $\{(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$, where the order of the (α_i, β_i) is determined up to a cyclic permutation, and we call L^* a *weighted circle*. If the orientation of L^* is reversed, each (α_i, β_i) becomes $(\alpha_i, \alpha_i - \beta_i)$ and we regard the resulting weighted circle as equivalent to the first.

(d) Let A^* be an arc which is a component of $E^* \cup F^*$. Orient A^* and assign Seifert invariants as in (c). Let y^* be the initial point or final point of A^* and B^* a small 3-disk neighborhood of y^*. Proceeding as in (c), ∂B has the S^1-action $\{b; (o, 0, 0, 0); (\alpha, \beta)\}$ (cf. Chapter 3, Section 3.2). Assign this integer b to y^*. We call A^* a *weighted arc* and write the weight system as $[b'; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n); b'']$. Reversing the orientation on A^* changes the weight system to $[-1 - b''; (\alpha_n, \alpha_n - \beta_n), \ldots, (\alpha_1, \alpha_1 - \beta_1); -1 - b'']$ which we regard as equivalent to the original weight system of A^*. We also recall the following Lemma (cf. Lemma 3.5 in [9]).

Lemma 5.4. (a) *If (α_i, β_i) and $(\alpha_{i+1,\beta_{i+1}})$ are the Seifert invariants assigned to adjacent arcs in some weighted arc or circle, then*

$$\begin{vmatrix} \alpha_i & \beta_i \\ \alpha i + 1 & \beta_{i+1} \end{vmatrix} = \pm 1.$$

(b) If $[b'; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n); b'']$ is a weighted arc then $b'\alpha_1 + \beta + 1 = \pm 1$ and

$b''\alpha_n + \beta_n = \pm 1$. *(So for $i = 1$ or n, $\beta_i = 1$ or $\alpha_i - 1$, and b' and b'' can only take on the values* 0 *or* -1*.)*

The oriented orbit space M^* together with the above collection of weights is called a *weighted orbit space*. More generally, recall that a *legally weighted simply connected* 3*-manifold* is an oriented simply connected compact 3-manifold X^* along with the following data:

(A) an integer a_i assigned to each boundary component of X^*,

(B) a finite collection of points in $\operatorname{int} X^*$ with each assigned an integer $b_i = \pm 1$, and

(C) a collection of weighted arcs and circles in $\operatorname{int} X^*$ as above and satisfying the criteria of Lemma 5.4. To each weighted arc $A_i^* = [b'; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n); b'']$ the integer $c_i = b'' - b'$ is assigned.

At least one of the above collections must be nonempty and we require $\Sigma a_i + \Sigma b_i + \Sigma c_i = 0$. It is shown in [9] that the weighted orbit space of an S^1-action on a simply connected 4-manifold is legally weighted.

It follows from Theorem [9] (7.1) and the validity of the Poincaré conjecture that, if M^* contains no weighted circles, then any S^1-action on a simply connected 4-manifold M extends to an action of $\mathsf{T}^2 = \mathsf{S}^1 \times \mathsf{S}^1$. As part of the proof of Theorem 4.1, we determined all the possible orbit spaces of an isometric fixed-point homogeneous S^1-action on a nonnegatively curved Riemanian 4-manifold M. When

M is simply connected, the orbit space contains no weighted circles and hence the S^1-action must extend to a T^2-action, answering affirmatively Question 5.3 in the case of a fixed-point homogeneous S^1-action. We summarize this in the following corollary.

Corollary 5.5. *A fixed-point homogeneous isometric S^1-action on a simply connected nonnegatively curved 4-manifold must be equivariantly diffeomorphic to a linear action on $\mathbb{S}^4$, $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$.*

5.2.2 Equivariant plumbing

The equivariant plumbing of 2-disk bundles over 2-spheres is used in [9] to construct 4-manifolds with S^1-actions out of orbit space data. We will review this construction in this subsection. The basic building blocks will be 2-disk bundles over $\mathbb{S}^2$ equipped with a given S^1-action. First we show how to construct a 2-disk bundle over $\mathbb{S}^2$ with Euler number ω equipped with certain S^1-action and then we see how these disk bundles can be equivariantly plumbed together to obtain a given orbit space configuration (cf. [9] 4., 5.).

Write $\mathbb{S}^2 = B_1 \cup B_2$ as the union of its upper and lower hemispheres and consider polar coordinates on $B_i \times D_i^2$, $i = 1, 2$. Given relatively prime integers u_i and v_i, define an S^1-action on $B_i \times D_i$ by $\phi(r, \gamma, s, \delta) \mapsto (r, \gamma + u_i\phi, s, \delta + v_i\phi)$. If $u_2 = -u_1$ and $v_2 = -\omega u_1 + v_1$ we obtain $Y_\omega = B_1 \times D_1 \cup_G B_2 \times D_2$ via the equivariant pasting $G : \partial B_1 \times D_1 \to \partial B_2 \times D_2$ given by $(1, \gamma, s, \delta) \mapsto (1, -\gamma, s, -\omega\gamma + \delta)$. The 4-manifold with boundary Y_ω is the D^2-bundle over $\mathbb{S}^2$ with Euler number ω, i.e., ω

is the self-intersection number of the zero section of Y_ω.

Given Y_{ω_1} and Y_{ω_2} with $u_{2,1} = v_{1,2}$ and $v_{2,1} = u_{1,2}$ (or $u_{2,1} = -v_{1,2}$ and $v_{2,1} = -u_{1,2}$) we may equivariantly plumb Y_{ω_1} and Y_{ω_2} with sign $+1$ (sign -1) by identifying $B_{2,1} \times D_{2,1}$ with $B_{1,2} \times D_{1,2}$ by means of the equivariant diffeomorphism $(r, \gamma, s, \delta) \mapsto (s, \delta, r, \gamma)$ $((r, \gamma, s, \delta) \mapsto (s, -\delta, r, -\gamma))$. The resulting manifold, which we denote by $Y_{\omega_1} \Box Y_{\omega_2}$, has an induced S^1-action.

We may carry out these constructions also with T^2-actions on Y_ω using integers u_i, v_i, w_i and t_i with

$$\begin{vmatrix} u_i & w_i \\ v_i & t_i \end{vmatrix} = \pm 1.$$

The T^2-action on $B_i \times D_i$ is given by $(\phi, \theta)(r, \gamma, s, \delta) \mapsto (r, \gamma + u_i\phi + w_i\theta, s, \delta + v_i\phi + t_i\theta)$. The glueing map G defined in the preceding paragraph will be equivariant provided $w_2 = -w_1$ and $t_2 = -\omega w_1 + t_1$. We may construct $Y_{\omega_1} \Box Y_{\omega_2}$ with sign $+1$ and T^2-equivariantly if $w_{2,1} = t_{1,2}$.

Some examples. We will now describe some of the disk bundles catalogued in [9] that we will use in our constructions. As described above, actions of S^1 and T^2 on Y_ω are determined by a matrix

$$\begin{pmatrix} u_1 & u_2 & w_1 & w_2 \\ v_1 & v_2 & t_1 & t_2 \end{pmatrix}$$

whose entries satisfy certain conditions. We will use the following disk bundles and actions (cf. [9]). We will assume that $\varepsilon = \pm 1$, n is an arbitrary integer, and pairs (α, β) consist of relatively prime integers $0 < \beta < \alpha$.

(c) If $b'\alpha+\beta=\pm 1$, $b''\alpha+\beta=\pm 1$, $\varepsilon'=\begin{vmatrix}1 & |b'| \\ \alpha & \beta\end{vmatrix}$, $\varepsilon''=\begin{vmatrix}\alpha & \beta \\ 1 & |b''|\end{vmatrix}$ and $\omega=\varepsilon'\varepsilon''\begin{vmatrix}1 & |b'| \\ 1 & |b''|\end{vmatrix}$, then

$$\begin{pmatrix}\varepsilon\alpha & -\varepsilon\alpha & \varepsilon(\beta+n\alpha) & -\varepsilon(\beta+n\alpha) \\ \varepsilon\varepsilon' & -\varepsilon\varepsilon'' & -\varepsilon\varepsilon'(|b'|+n) & -\varepsilon\varepsilon''(|b''|+n)\end{pmatrix}$$

defines actions on Y_ω with $Y_\omega^*\cong D^3$ and a weighted arc $\bullet\longrightarrow\bullet$ with weights $[b';(\alpha,\beta);b'']$.

(d)Let $\varepsilon',\varepsilon''=\pm 1$ and $\omega=-\varepsilon'-\varepsilon''$. Then

$$\begin{pmatrix}\varepsilon & -\varepsilon & \varepsilon n & -\varepsilon n \\ -\varepsilon\varepsilon' & \varepsilon\varepsilon'' & -\varepsilon\varepsilon'(n+\varepsilon') & \varepsilon\varepsilon''(n-\varepsilon'')\end{pmatrix}$$

describes actions on Y_ω with $Y_\omega^*\cong D^3$ with two isolated fixed-points with weights ε' and ε''.

(g) Suppose $b'\alpha'+\beta'=\pm 1$, $\varepsilon'=\begin{vmatrix}\alpha' & \beta' \\ 1 & |b'|\end{vmatrix}$ and $\omega=\varepsilon'\alpha'$. Then

$$\begin{pmatrix}\varepsilon & -\varepsilon & \varepsilon(|b'|+n) & -\varepsilon(|b'|+n) \\ -\varepsilon\varepsilon'\alpha' & 0 & \varepsilon\varepsilon'(\beta'+n\alpha') & -\varepsilon\end{pmatrix}$$

defines actions on Y_ω and Y_ω^* with a fixed D^2 and half a weighted arc $\longrightarrow\bullet$ with weights (α',β') and b'.

(h) Let $\varepsilon' = \pm 1$ and $\omega = -\varepsilon'$. Then

$$\begin{pmatrix} \varepsilon & -\varepsilon & \varepsilon & \varepsilon n \\ -\varepsilon\varepsilon' & 0 & -\varepsilon\varepsilon'(n+\varepsilon') & -\varepsilon \end{pmatrix}$$

describes actions on Y_ω with $Y_\omega^* \cong D^3$ with an isolated fixed point with weight ε' and a fixed D^2.

(i) Let $\delta = \pm 1$. Then

$$\begin{pmatrix} \varepsilon & -\varepsilon & n & -n \\ 0 & 0 & \delta & \delta \end{pmatrix}$$

describes actions on Y_0 with $Y_0^* \cong D^3$ with two fixed 2-discs.

(j) For ω arbitrary and $\delta = \pm 1$ actions on Y_ω are defined by

$$\begin{pmatrix} 0 & 0 & \delta & -\delta \\ \varepsilon & \varepsilon & n & -\omega\delta + n \end{pmatrix}$$

and $Y_\omega^* \cong \mathbb{S}^2 \times I$ with $E^* \cup F^* = F^* = \mathbb{S}^2 \times 0$ with weight ω.

5.2.3 Computation of the intersection form

In [9] there is a catalog of different disk-bundles with S^1- and T^2-actions realizing different basic orbit space configurations. If M^* contains no weighted circles, these disk bundles may be plumbed together to construct a 4-manifold R whose orbit space R^* is a particular subset of M^*. We will outline the construction of R and then recall the recipe given in [9] for computing the intersection form of M out of the intersection form of R (cf. [9], 5.,8.).

Let $S_1^*, \ldots, S_t^*$ be the collection of weighted sets in M^* other than the weighted circles, with the weighted boundary components of M^*, if any, listed at the end. For each $i = 1, \ldots, t-1$ let γ_i^* be an arc in M^* joining S_i^* to S_{i+1}^* such that the interior of the arc lies in the regular orbit stratum P^* and such that if S_i^* is a weighted arc, γ_i^* begins at the endpoint of S_i^*, and if S_{i+1}^* is a weighted arc, then γ_i^* ends at the initial point of S_{i+1}^*. Let R^* be a regular neighborhood of $\bigcup S_i^* \cup \bigcup \gamma_i^*$. By equivariantly plumbing disk bundles Y_{ω_i} listed in [9] (with each plumbing of sign $+1$) one can construct a 4-manifold R with S^1-action and weighted orbit space isomorphic to R^*. Moreover, this action extends to a T^2-action (cf. Lemma 4.7 in [9]). In the next section we will explicitly list the bundles we will use in our constructions, along with the actions on them.

Let M be a simply connected 4-manifold with a smooth S^1-action such that M^* contains no weighted circles. We now recall how to recover the intersection form Q_M of M out of the set R^*. Let R be the 4-manifold with S^1-action and weighted orbit space isomorphic to R^*. Then R is the result of an equivariant linear plumbing

$$\underset{\omega_1}{\bullet} — \underset{\omega_2}{\bullet} — \cdots — \underset{\omega_t}{\bullet}$$

If ∂M^* has m components and $(F^* - \partial M^*) \cap R^*$ contains l points then $t = 2m + l - 1$. The intersection matrix B_0 of the plumbing R is the $t \times t$ matrix given by

$$[B_0]_{ij} = \begin{cases} \omega_i, & i = j, \\ 1, & i = j \pm 1, \\ 0, & \text{otherwise,} \end{cases}$$

since each plumbing has sign $+1$.

Given a square matrix B, we will denote by B^- the matrix obtained after removing the last row and column from B. It is shown in [9] that the intersection form Q_M of M is B_0^-.

5.3 Main results

In this section we will determine the possible legally weighted orbit spaces of a simply connected nonnegatively curved Riemannian 4-manifold M with an isometric fixed-point homogeneous S^1-action. We will also identify M out of the orbit space data following the constructions described in Section 5.2. By Theorem 5.2 (2), M is diffeomorphic to $\mathbb{S}^4$, $\mathbb{CP}^2$, $\mathbb{S}^2\times\mathbb{S}^2$ or $\mathbb{CP}^2\#\pm\mathbb{CP}^2$. It is well known that $\chi(M)=\chi(\mathrm{Fix}(M,\mathsf{S}^1))$ (cf. [24]) and, since the action is fixed-point homogeneous, $\mathrm{Fix}(M,\mathsf{S}^1)$ must contain a 2-sphere. Hence we have the following possible fixed-point sets:

$$\mathrm{Fix}(M,\mathsf{S}^1)=\begin{cases}\mathbb{S}^2 & \text{if } M \text{ is } \mathbb{S}^4.\\ \mathbb{S}^2\cup\{p\} & \text{if } M \text{ is } \mathbb{CP}^2.\\ \mathbb{S}^2\cup\mathbb{S}^2 & \text{if } M \text{ is } \mathbb{S}^2\times\mathbb{S}^2 \text{ or } \mathbb{CP}^2\pm\mathbb{CP}^2.\\ \mathbb{S}^2\cup\{p',p''\} & \text{if } M \text{ is } \mathbb{S}^2\times\mathbb{S}^2 \text{ or } \mathbb{CP}^2\pm\mathbb{CP}^2.\end{cases} \tag{5.3.1}$$

By our analysis in Chapter 4, the orbit space of an isometric fixed-point homogeneous circle action on a simply connected nonnegatively curved manifold M does not contain any weighted circles. Hence we restrict our analysis to these orbit spaces. Observe that there cannot be any exceptional orbits unless $\mathrm{Fix}(M,\mathsf{S}^1)$

contains two isolated fixed points. Hence, when $\mathrm{Fix}(M, \mathsf{S}^1)$ contains at most one isolated fixed point, corresponding to $\mathrm{Fix}(M, \mathsf{S}^1) = \mathbb{S}^2$ or $\mathbb{S}^2 \cup \{p\}$, we may dispense with the geometric assumptions, since the orbit space structure itself prevents the existence of any weighted circles. It follows then that any fixed-point homogeneous circle action on $\mathbb{S}^4$ or $\mathbb{CP}^4$ is equivariantly diffeomorphic to a linear action. However, when F contains two isolated fixed points we will explicitly assume that the orbit space contains no weighted circles.

We summarize our results in the following theorem.

Theorem 5.6. *Let M be a simply connected smooth 4-manifold with a smooth S^1-action.*

(1) If $\mathrm{Fix}(M, \mathsf{S}^1) = \mathbb{S}^2$, then M is equivariantly diffeomorphic to $\mathbb{S}^4$ with a linear action.

(2) If $\mathrm{Fix}(M, \mathsf{S}^1) = \mathbb{S}^2 \cup \{p\}$, then M is equivariantly diffeomorphic to $\pm\mathbb{CP}^2$ with a linear action.

(3) If $\mathrm{Fix}(M, \mathsf{S}^1) = \mathbb{S}^2 \cup \mathbb{S}^2$, then M is equivariantly diffeomorphic to $\mathbb{CP}^2 \# - \mathbb{CP}^2$ or $\mathbb{S}^2 \times \mathbb{S}^2$ with a linear action.

(4) If $\mathrm{Fix}(M, \mathsf{S}^1) = \mathbb{S}^2 \cup \{p', p''\}$ and there are no orbits with finite isotropy, then M is equivariantly diffeomorphic to $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$ with only one linear action.

(5) If $\mathrm{Fix}(M, \mathsf{S}^1) = \mathbb{S}^2 \cup \{p', p''\}$ *and there is only a weighted arc, then* M *is equivariantly diffeomorphic to one of the following:*

(a) $\mathbb{CP}^2 \# \mathbb{CP}^2$ *with only one linear action with finite isotropy* $\mathbb{Z}_2$*.*

(b) $\mathbb{CP}^2 \# -\mathbb{CP}^2$ *with only one linear action with finite isotropy* $\mathbb{Z}_k$*,* k *odd.*

(c) $\mathbb{S}^2 \times \mathbb{S}^2$ *with only one linear action with finite isotropy* $\mathbb{Z}_k$*,* k *even.*

Remark. We remark that it is not known whether the smooth actions in (4) and (5) can be realized as isometric circle actions on $\mathbb{CP}^2 \pm \mathbb{CP}^2$ and $\mathbb{S}^2 \times \mathbb{S}^2$ with metrics of nonnegative curvature.

Proof of Theorem 5.6 We will prove three propositions, corresponding to (3)–(5) in Theorem 5.6. Parts (1) and (2) follow from the comments preceding the statement of the theorem. We will proceed as follows. Given a fixed-point set F we will construct R as in Section 5.2 using the pieces we have described therein. We will then identify M by computing its intersection form Q_M following the recipe in Section 5.2.

Case 1. $\mathrm{Fix}(M, \mathsf{S}^1) = \mathbb{S}^2 \cup \mathbb{S}^2$.

Proposition 5.7. *Let M be a simply connected smooth 4-manifold with a smooth S^1-action. If* $\mathrm{Fix}(M, \mathsf{S}^1) = \mathbb{S}^2 \cup \mathbb{S}^2$, *then M is equivariantly diffeomorphic to $\mathbb{CP}^2 \# {-\mathbb{CP}^2}$ or $\mathbb{S}^2 \times \mathbb{S}^2$ with a linear action.*

Proof. We construct R using bundles Y_{ω_1}, Y_{ω_2} and Y_{ω_3} with actions (j), (i) and (j), respectively. Observe that $\omega_2 = 0$, so the plumbing $Y_{\omega_1} \Box Y_{\omega_2} \Box Y_{\omega_3}$ has intersection form

$$B_0 = \begin{bmatrix} \omega_1 & 1 & 0 \\ 1 & \omega_2 & 1 \\ 0 & 1 & \omega_3 \end{bmatrix} = \begin{bmatrix} \omega_1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & \omega_3 \end{bmatrix}.$$

The intersection form of M is then B_0^-, i.e.,

$$Q_M = \begin{bmatrix} \omega_1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Now we show that Q_M is equivalent to the intersection form of $\mathbb{CP}^2 \# - \mathbb{CP}^1$, if ω_1 is odd, and to the intersection form of $\mathbb{S}^2 \times \mathbb{S}^2$, if ω_1 is even.

Recall that the operation of adding an integral constant k times row i to row j and then that constant times column i to column j preserves the congruence class over $\mathbb{Z}$ of an integral matrix. We call this an *elementary operation* and will keep track of it by denoting it by $(i, j; k)$. We have

$$\begin{bmatrix} \omega_1 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow{(2,1;\pm 1)} \begin{bmatrix} \omega_1 \pm 2 & 1 \\ 1 & 0 \end{bmatrix}$$

Thus, after repeated application of the elementary operation $(2, 1; \pm 1)$ to $\begin{bmatrix} \omega_1 & 1 \\ 1 & 0 \end{bmatrix}$

we have

$$Q_M \cong \begin{bmatrix} \omega_1 \pmod 2 & 1 \\ 1 & 0 \end{bmatrix}$$

When ω_1 is even, we have

$$Q_M \cong \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which is the intersection form of $\mathbb{S}^2 \times \mathbb{S}^2$.

When ω_1 is odd, we have

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow{(1,2;-1)} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which is the intersection form of $\mathbb{CP}^2 \# - \mathbb{CP}^2$.

□

Remark. Proposition 5.7 and its proof show that the fact that $\mathbb{CP}^2 \# \mathbb{CP}^2$ does not admit any smooth circle action with fixed-point set the union of two 2-spheres is a purely topological phenomenon. Under the additional condition of nonnegative curvature, this follows from the Double Soul Theorem, which implies that M^4 is an $\mathbb{S}^2$-bundle over $\mathbb{S}^2$ and hence M^4 must be $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{CP}^2 \# - \mathbb{CP}^2 \cong \mathbb{S}^2 \tilde{\times} \mathbb{S}^2$.

Case 2. $\text{Fix}(M, \mathsf{S}^1) = \mathbb{S}^2 \cup \{p', p''\}$. We split this case into two subcases, depending on whether or not there are any orbits with finite isotropy.

No finite isotropy. Suppose first there are no orbits with finite isotropy.

Proposition 5.8. *Let M^4 be a simply connected smooth 4-manifold with a smooth S^1-action without finite isotropy. If $\mathrm{Fix}(M,\mathsf{S}^1)=\mathbb{S}^2\cup\{p',p''\}$, then M is equivariantly diffeomorphic to $\mathbb{CP}^2\#\pm\mathbb{CP}^2$ with a linear action.*

Proof. We will compute the intersection form Q_M of M. To compute the intersection form of M we first construct R using the bundles Y_{ω_1} with action (d), Y_{ω_2} with action (h) and Y_{ω_3} with action (j). The intersection form of the plumbing $Y_{\omega_1}\Box Y_{\omega_2}\Box Y_{\omega_3}$ is

$$B_0=\begin{bmatrix}\omega_1 & 1 & 0\\ 1 & \omega_2 & 1\\ 0 & 1 & \omega_3\end{bmatrix}.$$

Then the intersection form of M is given by B_0^-, i.e.

$$Q_M=\begin{bmatrix}\omega_1 & 1\\ 1 & \omega_2\end{bmatrix}.$$

We now determine ω_1 and ω_2. Let $\varepsilon_1',\ \varepsilon_1''=\pm1$. Then $\omega_1=-\varepsilon_1'-\varepsilon_1''$, coming from action (d). On the other hand, for Y_{ω_2} we have $\omega_2=-\varepsilon_2'$, where $\varepsilon_2'=\pm1$. In order to plumb these two bundles together, we need $\varepsilon_1''=\varepsilon_2'$. Hence $\omega_2=-\varepsilon_2'=-\varepsilon_1''$. We now compute the possible intersection forms Q_M in terms of the weights ε_1', ε_1''.

When $\varepsilon_1'=\varepsilon_1''=1$, we have $\omega_1=-\varepsilon_1'-\varepsilon_1''=-2$, $\omega_2=-\varepsilon_1''=-1$. Hence

$$Q_M=\begin{bmatrix}-2 & 1\\ 1 & -1\end{bmatrix}\xrightarrow{(2,1;1)}\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix},$$

which is the intersection form of $-\mathbb{CP}^2\#-\mathbb{CP}^2$.

When $\varepsilon_1'=1$ and $\varepsilon_1''=-1$, we have $\omega_1=-\varepsilon_1'-\varepsilon_1''=0$, $\omega_2=-\varepsilon_1''=1$. Hence

$$Q_M=\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}\xrightarrow{(2,1;-1)}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix},$$

which is the intersection form of $-\mathbb{CP}^2\#\mathbb{CP}^2$.

When $\varepsilon_1' = -1$ and $\varepsilon_1'' = 1$, we have $\omega_1 = -\varepsilon_1' - \varepsilon_1'' = 0$, $\omega_2 = -\varepsilon_1'' = -1$. Hence

$$Q_M = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow{(2,1;-1)} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which is the intersection form of $\mathbb{CP}^2\# - \mathbb{CP}^2$.

When $\varepsilon_1' = -1$ and $\varepsilon_1'' = -1$, we have $\omega_1 = -\varepsilon_1' - \varepsilon_1'' = 2$, $\omega_2 = -\varepsilon_1'' = 1$. Hence

$$Q_M = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{(2,1;-1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which is the intersection form of $\mathbb{CP}^2\#\mathbb{CP}^2$.

We have accounted for all the combinations of ε_1' and ε_1'', which proves the theorem. □

Finite isotropy. Suppose there are points with finite isotropy.

Proposition 5.9. *Let M^4 be a simply connected smooth 4-manifold with a smooth S^1-action with $\mathrm{Fix}(M, \mathsf{S}^1) = \mathbb{S}^2 \cup \{p', p''\}$ and a weighted arc with finite isotropy $\mathbb{Z}_k$. Then M is equivariantly diffeomorphic to one of the following:*

(1) $\mathbb{CP}^2\#\mathbb{CP}^2$ with a linear action with finite isotropy $\mathbb{Z}_2$.

(2) $\mathbb{CP}^2\# - \mathbb{CP}^2$ with a linear action with finite isotropy $\mathbb{Z}_k$, k odd.

(3) $\mathbb{S}^2 \times \mathbb{S}^2$ *with a linear action with finite isotropy* $\mathbb{Z}_k$*, k even.*

Proof. Let $[b'; (\alpha_1, \beta_1); b'']$ be the weighted arc. In this case $\beta_1 = 1$ or $\alpha_1 - 1$ and b' and b'' can only take on the values 0 or -1 (cf. Lemma 3.5 in [9]). We will use actions (c), (g) and (j). Recall that, to each weighted arc $[b'; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n); b'']$, the integer $c = b'' - b'$ is assigned (cf. [9](5.2)(c)). For the orbit space to be legally weighted, we must have $a + c = 0$, where a is the weight of the boundary 2-sphere, so $a = -c$. The following table lists the possible combinations of weights.

b'	b''	$c = b'' - b'$	a
0	0	0	0
0	-1	-1	1
-1	0	1	-1
-1	-1	0	0

1. The first piece we need is a bundle Y_{ω_1} with action (c) as in Section 5.2. We have

$$\pm 1 = \varepsilon_1' = \begin{vmatrix} 1 & |b'| \\ \alpha & \beta \end{vmatrix} = \beta - \alpha |b_1'| = \begin{cases} \beta, & \text{if } b_1' = 0; \\ \beta - \alpha, & \text{if } b_1' = -1. \end{cases}$$

$$\pm 1 = \varepsilon_1'' = \begin{vmatrix} \alpha & \beta \\ 1 & |\beta''| \end{vmatrix} = \alpha |b_1''| - \beta = \begin{cases} -\beta, & \text{if } |b_1''| = 0; \\ \alpha - \beta, & \text{if } b_1'' = -1. \end{cases}$$

We also have

$$\omega_1 = \varepsilon_1'\varepsilon_1'' \begin{vmatrix} 1 & |b_1'| \\ 1 & |b_1''| \end{vmatrix} = \varepsilon_1'\varepsilon_1''(|b_1''| = |b_1'|)$$

We have the following possible combinations:

b_1'	b_1''	ε_1'	ε_1''	ω_1
0	0	β	$-\beta$	0
0	-1	β	$\alpha-\beta$	$\beta(\alpha-\beta)$
-1	0	$-(\alpha-\beta)$	$-\beta$	$-\beta(\alpha-\beta)$
-1	-1	$-(\alpha-\beta)$	$\alpha-\beta$	0

Case: $(b_1', b_1'') = (0,0)$. We have $\beta = \varepsilon_1' = \pm 1$. Recall that $\beta = 1$ or $\alpha - 1$. Hence $1 = \beta = \varepsilon_1'$ and $\varepsilon_1'' = -1$.

Case: $(b_1', b_1'') = (0,-1)$. We have $\varepsilon_1' = \pm 1 = \beta > 0$ so $\varepsilon_1' = \beta = 1$. Hence

$$\pm 1 = \varepsilon_1'' = \alpha - \beta = \alpha - 1.$$

We have $\alpha \geq 2$ so $\alpha - 1 \geq 1 > 0$. Hence $\varepsilon_1'' = +1$. Hence $\alpha - 1 = 1$ so $\alpha = 2$.

Case: $(b_1', b_1'') = (-1,0)$. Recall that β takes on the values 1 or $\alpha - 1$. We have

$$\pm 1 = \varepsilon_1' = -(\alpha - \beta) = \begin{cases} -(\alpha-1), & \text{if } \beta = 1; \\ -1, & \text{if } \beta = \alpha - 1. \end{cases}$$

$$\pm 1 = \varepsilon_1'' = -\beta = \begin{cases} -1, & \text{if } \beta = 1; \\ -(\alpha - 1), & \text{if } \beta = \alpha - 1. \end{cases}$$

It follows from these equations that $\varepsilon_1' = \varepsilon_1'' = -1$ and $\alpha = 2$, $\beta = 1$.

Case: $(b', b'') = (-1, -1)$. We have

$$\pm 1 \quad = \quad \varepsilon_1' = -(\alpha - \beta) = -\varepsilon_1''.$$

Recall that $\beta = 1$ or $\alpha - 1$. In both cases the equation above implies that $\varepsilon_1' = -1$ and $\varepsilon_1'' = +1$. Observe that any $\alpha \geq 2$ is possible.

We update the table of weights in the previous page and obtain the following list of weights.

b_1'	b_1''	ε_1'	ε_1''	ω_1	α	β
0	0	1	-1	0	$k \geq 2$	$k-1$
0	-1	1	1	1	2	1
-1	0	-1	-1	-1	2	1
-1	-1	-1	1	0	$k \geq 2$	$k-1$

2. Now we deal with piece 2, coming from bundle Y_{ω_2} with action (g). We have weightsb_2', α_2' and β_2'. In order to plumb Y_{ω_1} and Y_{ω_2} we need $\alpha_1 = \alpha_2'$, $\beta_1 = \beta_2'$ and $b_2' = b_1''$. The subscript i denotes the bundle Y_{ω_i} to which each weight belongs.

We also have

$$\varepsilon_2' = \begin{vmatrix} \alpha_2' & \beta_2' \\ 1 & |b_2'| \end{vmatrix} = \begin{vmatrix} \alpha_1 & \beta_1 \\ 1 & |b_1''| \end{vmatrix} = \varepsilon_1''.$$

Since $\omega_2 = \varepsilon_2'\alpha_2'$, we have

$$\omega_2 = \varepsilon_1''\alpha_1.$$

Hence we have the following combinations:

b_1'	b_1''	ε_1'	ε_1''	ω_1	α	β	$\omega_2 = \varepsilon_1''\alpha$
0	0	1	-1	0	$k \geq 2$	$k-1$	$-k$
0	-1	1	1	1	2	1	2
-1	0	-1	-1	-1	2	1	-2
-1	-1	-1	1	0	$k \geq 2$	$k-1$	k

The last piece we need is a bundle Y_{ω_3} with action (j). The intersection form of the plumbing $Y_{\omega_1} \square Y_{\omega_2} \square Y_{\omega_3}$ is

$$B_0 = \begin{bmatrix} \omega_1 & 1 & 0 \\ 1 & \omega_2 & 1 \\ 0 & 1 & \omega_3 \end{bmatrix}.$$

Hence the intersection form Q_M of M is B_0^-, i.e.,

$$Q_M = \begin{bmatrix} \omega_1 & 1 \\ 1 & \omega_2 \end{bmatrix}.$$

When $b_1' = 0$ and $b_1'' = -1$, we have

$$Q_M = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \xrightarrow{(1,2;-1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which is the intersection form of $\mathbb{CP}^2 \# \mathbb{CP}^2$.

When $b_1' = -1$ and $b_1'' = 0$, we have

$$Q_M = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \xrightarrow{(1,2;1)} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

which is the intersection form of $-\mathbb{CP}^2 \# -\mathbb{CP}^2$. Observe that in these two cases (which are the same up to orientation) we can only have isotropy $\mathbb{Z}_2$.

When $b_1' = b_1'' = 0$, we have

$$Q_M = \begin{bmatrix} 0 & 1 \\ 1 & -k \end{bmatrix}$$

for $k \geq 2$. Observe now that

$$Q_M = \begin{bmatrix} 0 & 1 \\ 1 & -k \end{bmatrix} \xrightarrow{(1,2;1)} \begin{bmatrix} 0 & 1 \\ 1 & -k+2 \end{bmatrix}.$$

After repeated applications of the elementary operation $(1,2;1)$ we have

$$Q_M \cong \begin{bmatrix} 0 & 1 \\ 1 & -k \bmod 2 \end{bmatrix}.$$

When k is even, we have

$$Q_M \cong \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which is the intersection form of $\mathbb{S}^2 \times \mathbb{S}^2$. When k is odd, we have

$$Q_M \cong \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{(2,1;-1)} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

which is the intersection form of $-\mathbb{CP}^2 \# \mathbb{CP}^2$.

When $b_1' = b_1'' = -1$, we have

$$Q_M = \begin{bmatrix} 0 & 1 \\ 1 & k \end{bmatrix}$$

for $k \geq 2$. Observe now that

$$Q_M = \begin{bmatrix} 0 & 1 \\ 1 & k \end{bmatrix} \xrightarrow{(1,2;-1)} \begin{bmatrix} 0 & 1 \\ 1 & k-2 \end{bmatrix}.$$

Again, after repeated applications of the elementary operation $(1, 2; 1)$ we have

$$Q_M \cong \begin{bmatrix} 0 & 1 \\ 1 & k \bmod 2 \end{bmatrix}.$$

Hence, when k is even, we have

$$Q_M \cong \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which is the intersection form of $\mathbb{S}^2 \times \mathbb{S}^2$. When k is odd, we have

$$Q_M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{(2,1;-1)} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

which is the intersection form of $-\mathbb{CP}^2 \# \mathbb{CP}^2$.

□

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