

ABSTRACT

Title of dissertation: DEPENDENCE STRUCTURE IN
 LEVY PROCESSES AND ITS
 APPLICATION IN FINANCE

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In this paper, we introduce DSPMD, discretely sampled process with pre-specified marginals and pre-specified dependence, and SRLMD, series representation for Levy process with pre-specified marginals and pre-specified dependence.

In the DSPMD for Levy processes, some regular copula can be extracted from the discrete samples of a joint process so as to correlate discrete samples on the pre-specified marginal processes. We prove that if the pre-specified marginals and pre-specified joint processes are some Levy processes, the DSPMD converges to some Levy process. Compared with Levy copula, proposed by Tankov, DSPMD offers easy access to statistical properties of the dependence structure through the copula on the random variable level, which is difficult in Levy copula. It also comes with a simulation algorithm that overcomes the first component bias effect of the series representation algorithm proposed by Tankov. As an application and example of DSPMD for Levy process, we examined the statistical explanatory power of VG copula implied by the multidimensional VG processes. Several baskets of equi-

ties and indices are considered. Some basket options are priced using risk neutral marginals and statistical dependence.

SRLMD is based on Rosinski's series representation and Sklar's Theorem for Levy copula. Starting with a series representation of a multi-dimensional Levy process, we transform each term in the series component-wise to new jumps satisfying pre-specified jump measure. The resulting series is the SRLMD, which is an exact Levy process, not an approximation. We give an example of α -stable Levy copula which has the advantage over what Tankov proposed in the follow aspects: First, it is naturally high dimensional. Second, the structure is so general that it allows from complete dependence to complete independence and can have any regular copula behavior built in. Thirdly, and most importantly, in simulation, the truncation error can be well controlled and simulation efficiency does not deteriorate in nearly independence case. For compound Poisson processes as pre-specified marginals, zero truncation error can be attained.

DEPENDENCE STRUCTURE IN LEVY PROCESSES
AND ITS APPLICATION IN FINANCE

by

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Dedication

To My Parents,

Chen, Min

and Chen, Lijing

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I am sure this is far from being complete for people that I own debt to; any inadvertent missing in the list is my fault.

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List of Abbreviations

| | |
|--------|---|
| DSPMD | Discretely sampled process with pre-specified marginals and pre-specified dependence |
| SRLMD | Series representation of Levy processes with pre-specified marginals and pre-specified dependence |
| T.I.P. | Tail Integral of Probability Measure |
| T.I.L. | Tail Integral of Levy Measure |
| CDO | Collateralized Debt Obligation |
| CDS | Credit Default Swap |
| VG | Variance Gamma |

Chapter 1

Introduction

1.1 Background

The modeling of dependence among financial assets is essential in many financial engineering problems, such as the dependence modeling in CDO pricing in the credit derivative market, basket option pricing in the equity market and risk management of assets with dependence, to name but a few.

In recent years, copula has been successfully introduced to the math finance world. Among them, Gaussian, Student-t, Clayton copula, etc, are widely used in pricing structured products in the credit market and the equity market. We refer readers to a comprehensive analysis in this direction by Burtschell, Gregory, and Laurent [16] and an excellent paper by Laurent and Gregory [17].

The strongest argument for using copula approach is that one can separate the dependence structure from the marginal distribution completely. In the financial modeling, it is a big advantage to have the property of separation. With this separation, the choice of dependence modeling is independent from the choice of modeling of the marginals. This adds great flexibility to the modeling of financial products that depend on the joint law. In the framework, the change in dependence does not disturb the marginal behavior. In a lot of cases, it means efficiency in calibration procedures. Examples are basket options pricing and CDO pricing. Also, it

offers the access to the statistical property of the dependence alone, eliminating the effect of marginal distribution. The goodness-of-fit test of the dependence, not the full joint distribution with marginal information, can be carried out in the copula framework.

However, copulas deal with random variables not stochastic processes. Static modeling of single name cannot meet the need of more complex products. Consequently, the dependence modeling of processes, in particular, Levy processes, is desired. Examples in this direction include the followings: Madan and Schoutens [22] uses one-sided Levy processes to model CDS, Credit Default Swap. Moosbrucker [26] used some correlated VG processes to model CDO, collateralized debt obligation. Both of these work are based on a structural model by Merton 1974 [25]. Joshi and Stacey [18] used Gamma processes to model CDS and CDO in a stochastic intensity model. Xia [32] proposed to use a linear combination of VG processes to model multi-asset problems in equity. All these work took a dynamic modeling approach but none of them are in a copula type structure.

The difficulty of modeling Levy processes using regular copula is that it is unclear which copula function constructs a Levy process. Infinite divisibility of a probability distribution is not invariant under a copula structure in general. Tankov [30] generalized the idea of copula for random variables to Levy copula for Levy processes. Levy copula is defined on the level of Levy measure. It connects marginal Levy measures to build the joint Levy measure. The benefit of using Levy copula is that the resulting processes are guaranteed to be Levy processes. A set of theorems that are parallel to regular copulas have been developed by Tankov and many other

authors. For details, we list some useful references [6] [2] [31] and [20].

As a newly introduced concept, there are some issues regarding Levy copula. This dissertation is trying to address the following two issues. Firstly, in this Levy Copula setting, statistical inference is difficult in general. Because Levy copula is defined on the infinitesimal level while copula function is a probability distribution. Finding its implied copula is equivalent to solving a multi-dimensional PDE in general. The connection between Levy copula and the implied regular copula remains uninvestigated.

Another issue about Levy copula is its implementation. The algorithm for simulating Levy processes with Levy copula is given by Tankov, which uses Rosinsky's series representation theory [27]. To my best knowledge, there is no other algorithm to simulate Levy copula based multi-dimensional Levy processes. We confirm in theory as well as in practice that this algorithm has a first component bias effect, which leads to significant loss of jump mass when dependence level is low. In addition to the bias effect, because the algorithm is based on a conditional probability argument, high dimensional extension requires recursively applying conditional law which is expensive to carry out numerically.

For the first issue, we introduces, in Chapter 2, DSPMD, discretely sampled process with pre-specified marginals and pre-specified dependence. A DSPMD is a discrete time process, whose increments come from some pre-specified marginal processes and are correlated through some copula embodied by the discrete time sample of some pre-specified joint process. In short, in a DSPMD, the pre-specified marginals are coupled using the joint law of the pre-specified joint process. Here we

are going to prove that if the pre-specified marginal and pre-specified joint processes are some Levy processes, a DSPMD converges to a Levy process, under certain technical conditions. And the Levy copula of the limiting process can be written in terms of the tail integral of the Levy measure of the pre-specified joint process and pre-specified marginal processes. In that respect, DSPMD can be viewed as the discrete version of the Levy copula.

The advantage of DSPMD is that it uses a copula structure on the random variable level so that one can have access to its statistical property. Also, it comes with a simple simulation algorithm that avoids the deficiency of the series representation method by Tankov.

In Chapter 3, we discuss the choice of the pre-specified joint process. We focused on the subordination of Brownian motion, for example VG, with an application in equity. In the class of copula implied by subordination of Brownian motion, closed form of copula function is often available, which makes possible efficient statistical inference. Within this construction, we introduce the concept of Stochastic Stressing of the Gaussian copula, which provides a conventional perspective on this new class. And at last, VG copula, a particular example of this class is presented and statistical test was performed on a basket of equity names. Pair-wise Chi-squared test shows that it is a very competitive copula against many other popular copulas for modeling dependence of equity names.

In Chapter 4, we introduce SRLMD, series representation for Levy processes with pre-specified marginals and pre-specified dependence in order to address the second issue of Levy copulas, the simulation algorithm. SRLMD is also based on

Rosinski's series representation, but it avoids Tankov's conditional probability argument. In the example of α -Stable Levy Copula, we show that it has the advantage over Tankov's Levy copula function in the following aspects: First, it is naturally high dimensional. Second, the structure is so general that it allows from complete dependence to complete independence and can have any regular copula behavior built in. Thirdly, and most importantly, in any case, the truncation error can be well controlled and simulation efficiency does not deteriorate in nearly independence case. For compound Poisson processes as pre-specified marginals, zero truncation error can be attained.

1.2 Overview of Multi-dimensional Levy processes

The main subject of this dissertation is multi-dimensional Levy processes. In this section, we are going to review the existing ways to construct a multi-dimensional Levy process. Since multi-dimensional Brownian motion has been well studied and understood, we will focus our discussion on pure jump Levy processes throughout this section and the rest of the dissertation.

In general, there are three well known methods to construct a multi-dimensional Levy process: subordination of multi-dimensional Brownian motion, linear transformation of independent Levy processes and multi-dimensional Levy measure.

Subordination of multi-dimensional Brownian motions constructs multi-dimensional Levy processes. Most of its one-dimensional version are well studied and applied in all kinds of problems in finance. Examples are Variance Gamma processes [23], NIG

processes [1], etc. However, problems associated with such construction in multi-dimensional version is that the heavy tail behavior are very similar in all marginals. For example, in multi-dimensional VG processes, kurtosis are almost identical in all marginal processes, which makes it difficult to model multi-name asset problems. A simple explanation for this effect is that all marginals share the same subordinator which is the source of all heavy tail behavior.

Linear transformation of independent Levy processes produces Levy processes with dependence. This method is very popular with, but not limit to, building correlated compound Poisson processes. The main idea is to construct the marginal processes as some idiosyncratic process plus some common process. The dependence comes from the common process while the idiosyncratic process makes it possible to match some pre-specified marginals. The dependence can be carefully designed to meet various needs of dependence behavior such as tail dependence and skewness in dependence. Various books and papers used this method to model multi-name problems such as CDO and basket option pricing such as [32], [19], [26], [18], [26]. However, it is not a copula type approach. One cannot separate the dependence part from the marginals. Whenever the dependence is changed, i.e, the common process is changed, the entire marginal process is also changed. In almost all applications in Finance, marginal processes are pre-specified. In such a model, one has to adjust for the idiosyncratic process to match the pre-specified marginals for any changes in the dependence. This procedure is inefficient. The worst case is that the common process dominates the idiosyncratic process such that one cannot match the pre-specified marginals.

At last, one can construct a multi-dimensional Levy measure directly to obtain a multi-dimensional Levy process. One example of such Levy process is α -Stable process. The Levy measure of α -Stable process in R^d has a spherical or euclidean decomposition. It can be viewed as a one-dimensional α -Stable process multiplied by a random vector from a probability measure in R^d . For any $B \subset R^d$, the Levy measure of α -stable ν can be written as

$$\nu(B) = \int_{R^d} \lambda(d\xi) \int_0^\infty 1_B(r\xi) \frac{dr}{r^{1+\alpha}},$$

where λ is the probability measure in R^d and $\alpha \in (0, 2)$. The concept of Levy copula is the new development in this direction. Tankov generalized copula for the probability measure to Levy measure, so that one can build joint Levy measure with arbitrary marginal Levy measure. This dissertation is trying to extend and improve this idea in various ways. First, we propose DSPMD, which can be understood as an extension of Levy copula in a discrete time random variable level. In this way, it makes an connection with regular copula so that one can perform statistical inference. It also comes with a simulation algorithm, which does not suffer from the first component bias in what Tankov proposed. We also propose SRLMD and its example α -Stable Levy copula which overcomes the weakness in Tankov's simulation algorithm.

1.3 Review of Levy Processes, Copula and Levy Copula

In order to make this dissertation self-contained, this section is devoted to the review of the basic concepts about Levy processes, copula and Levy copula.

1.3.1 Levy Processes and Infinitely divisible distribution

All definition and theorems in this section can be found in the book by Cont and Tankov [6]. For proofs and more rigorous treatment of the basic knowledge of Levy processes and infinitely divisible distribution, we refer the readers to the book by Sato [28].

Definition A stochastic process $\{X_t; t \geq 0\}$ on a probability space (Ω, F, P) is a Levy process if the following properties are satisfied:

1. X_t has independent increments: $\forall n \geq 1$ and $0 \leq t_0 \leq t_1 \leq \dots \leq t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent.
2. X_t has stationary increment: the law of $X_{t+h} - X_t$ does not depend on t .
3. $X_0 = 0$ almost surely.
4. X_t is stochastically continuous: $\forall t \geq 0$ and $\forall \epsilon \geq 0$,

$$\lim_{h \rightarrow 0} P(|X_{t+h} - X_t| > \epsilon) = 0.$$

5. X_t is right-continuous with left limits almost surely

Levy process is closely related to infinitely divisible distribution.

Definition A random variable X taking values in R^d is infinitely divisible, if for all $n \in N$, there exists i.i.d. random variable $Y_1^{(n)}, \dots, Y_n^{(n)}$ such that

$$X \stackrel{d}{=} Y_1^{(n)} + \dots + Y_n^{(n)}.$$

Another way to state the definition is that F is an infinitely divisible distribution if the n -th convolution root is still a probability distribution for any n .

Given an infinitely divisible distribution F , it is easy to see that for any $n \geq 1$ by chopping it into n i.i.d. components, we can construct a random walk model on a time grid with step size $1/n$ such that the law of the position at $t = 1$ is given by F . In the limit, this procedure can be used to construct a continuous time Levy process $(X_t)_{t \geq 0}$ such that the law of X_1 is given by F .

Proposition 1.3.1 *Let (X_t) be a Levy process. For every t , X_t has an infinitely divisible distribution. Conversely, if F is an infinitely divisible distribution then there exists a Levy process (X_t) such that the distribution of X_1 is given by F .*

Summation of independent random variables corresponds to the convolution of their probability distribution function. On the Fourier side, convolution becomes multiplication, which is an ideal tool for studying Levy processes and infinitely divisible distributions. The characteristic function of a probability distribution is simply the Fourier transform of its density function. Or precisely,

$$\phi(u) = E[e^{iuX}].$$

Given the relation between an infinitely divisible distribution and its implied Levy process, we can define the characteristic function of X_t as

$$\psi_t(u) = E[e^{iuX_t}].$$

As a direct result of continuity and multiplicative property, we can assert that $\psi_t(u)$ is an exponential function.

Proposition 1.3.2 *Let X_t be a Levy process on R^d . There exists a continuous function $\phi : R^d \rightarrow R$ called the characteristic exponent of X , such that:*

$$E[e^{iuX_t}] = e^{t\psi(u)}, u \in R^d.$$

So, clearly, the law of (X_t) is determined by the law of X_1 . One can define the Levy process from any given infinitely divisible distribution through its characteristic function. We will use this property extensively in the later sections. One thing to notice is that not all infinitely divisible distribution is closed in its parametric family under convolution. For example, the summation of two Student's t distribution is not a t distribution. Nonetheless, Student's t distribution is infinitely divisible and its Levy process is well defined in terms of its characteristic function. See P46 in [28].

The celebrated Levy-Khinchin representation theorem reveals the structure of its characteristic function and its local structure of the path.

Theorem 1.3.3 *Let (X_t) be a Levy process on R^d with characteristic triplet (A, ν, γ) then*

$$E[e^{iuX_t}] = e^{t\psi(u)},$$

with

$$\psi(u) = -\frac{1}{2}uAu + i\gamma u + \int_{R^d} (e^{iux} - 1 - iux1_{|x|\leq 1})\nu(dx),$$

where A is a symmetric positive $n \times n$ matrix, $\gamma \in R^d$ and ν is a positive Radon measure on $R^d \setminus \{0\}$ verifying

$$\int_{|x|\leq 1} |x|^2\nu(dx) < \infty, \int_{|x|\geq 1} \nu(dx) < \infty.$$

ν is called the Levy measure of the Levy process..

According to Levy-Khinchin, a Levy process can be decomposed into three parts, deterministic drift, continuous Brownian motion and pure jump part. The continuous part, or multi-dimensional Brownian motion is well studied and understood. For this dissertation, we are interested in multi-dimensional structure in the pure jump part. Without loss of generality, we are going to focus on some of the pure jump Levy processes.

There are many different ways to build a Levy process. One can specify the Levy measure, which dictates the jump structure directly. Or one can specify the distributional property of some small time interval, which is infinitely divisible. Another well known way is subordination of Brownian motion. A subordinator is a Levy process with non-decreasing paths. Subordination to Brownian motion means that the time variable is replaced by the subordinator, or the Brownian motion is evaluated at random time change by a subordinator. The Variance Gamma processes by Madan and Seneta [23], Normal Inverse Gaussian process by Barndorff-Nielsen [1], CGMY process by Carr, Geman, Madan, Yor [8], among many others, belong to this category. Subordination also provides a natural way to extend one dimensional Levy process to higher dimensions. It is simply subordination of multi-dimensional Brownian motion.

For example, Variance Gamma process is defined as a Brownian motion subordinated by a unit rate Gamma process. Let $b(t; \theta, \sigma) = \theta t + \sigma W(t)$ be the Brownian

motion with drift θ and volatility σ . Let $\gamma(t; 1, \nu)$ be the gamma process.

$$X_t = b(\gamma(t; 1, \nu), \theta, \sigma) = \theta\gamma(t) + W(\gamma(t)),$$

where gamma process is a subordinator. It is the process of independent gamma increments over non-overlapping intervals of time. The density over $(t, t+h)$ is given by $f_h(g) = \frac{1}{\nu} g^{h/\nu-1} \frac{\exp(-g/\nu)}{\Gamma(h/\nu)}$. For more details on VG processes, please see [23] and [24]. For more theory and examples of subordinated Levy process, please see [6] and [28].

1.3.2 Copula and Levy copula

The concept of copula was introduced by Sklar [29]. Copula functions uniquely specify the structure of dependence of multivariate distributions. It separates the marginal distributions from its core dependence part. In finance, it provides a tool that enables us to model the dependence independently from the marginals.

Definition Definition: A copula is a function $C: [0, 1]^n \rightarrow [0, 1]$ such that

- $C(u) = 0$ whenever $u \in [0, 1]^n$ at least one component equal to 0.
- $C(u) = u_i$ whenever $u \in [0, 1]^n$ has all the components equal to 1 except the i -th one, which is equal to u_i .
- $C(u)$ is n -increasing. (Any n -dimensional distribution function is n -increasing)

In other words, a copula function is a multivariate distribution function with uniform marginals. The following theorem reveals the the relation between the multivariate distribution and a copula function.

Theorem 1.3.4 *Sklar's Theorem: Let X and Y be random variables with joint distribution function H and marginal distribution functions F and G , respectively. Then there exists a copula C such that*

$$H(x, y) = C(F(x), G(y))$$

for all x, y in R . Conversely, if C is a copula and F and G are distribution functions, then the function

$$H(x, y) = C(F(x), G(y))$$

is a joint distribution function with margins F and G .

Sklar's theorem shows that for each of the multivariate distribution, one can extract its copula function by transforming the marginals into uniform distribution by applying its marginal CDF. Then one can construct a new multivariate distribution using the copula function with any other marginal CDF. Examples of copula are Gaussian copula, Student's t copula, Clayton copula, etc.

Among those copulas, factorized copulas are very popular. For example, one factor Gaussian copula is the industry standard in modeling default times for credit names, which was introduced by Li [12]. Let $Z_i, Z, i = 1, \dots, N$ be the i.i.d standard normal distribution.

$$X_i = \rho Z + \sqrt{1 - \rho^2} Z_i.$$

In this way, X_i 's are correlated normal random variable through the common factor Z . Here, conditional on the common factor Z , all X_i 's are independent. This special structure allows tractability in computing joint distribution or other expressions depending on the joint law.

A very important concept in dependence is called tail dependence. Let (X, Y) be a random pair with joint cumulative distribution function F and marginals G for X and H for Y . The upper tail dependence is given by

$$\lambda_U = \lim_{t \rightarrow 1^-} P(G(X) > t | H(Y) > t),$$

and the lower tail dependence is given by

$$\lambda_L = \lim_{t \rightarrow 0^+} P(G(X) \leq t | H(Y) \leq t).$$

If C is the copula for (X, Y) , which is unique when G and H are continuous, we have

$$\lambda_L = \lim_{t \rightarrow 0^+} \frac{C(t, t)}{t},$$

and

$$\lambda_U = \lim_{t \rightarrow 1^-} \frac{1 - 2t + C(t, t)}{1 - t}.$$

For example, Gaussian copula has zero tail dependence when $\rho < 1$. Clayton copula has a lower tail dependence $\lambda_L = 2^{-1/\theta}$ for $\theta > 0$ and no upper tail dependence. For more examples and tail dependence, we refer the readers to [16].

The concept of Levy copula was introduced by Tankov in [6] and discussed in Chapter 5 from his book [30] and many other literatures such as [15] [2] [10]. The idea is that one can construct a Levy copula function that glues together marginal Levy measures to build joint Levy measure. As an extension of regular copula, one can separate the marginal Levy processes from its dependence part. Also, it is a natural way to build multi-dimensional Levy processes since Levy copula guarantees that the resulting process is a Levy process. When the dimensionality

is low, Levy copula is suitable for PDE approach since it defines the infinitesimal generator directly. One can also use Monte Carlo simulation for high dimensional problem. In the end, in theory, one can retrieve the implied distributional property of a Levy copula by doing the inverse transform of characteristic function, but, in practice, multi-dimensional FFT procedures are numerically expensive.

The following definitions and theorems come from the book by Cont and Tankov [6]. For more rigorous treatment of Levy copula, we refer the readers to [30] and [20]. In order to define the Levy copula, we will first introduce the concept of tail integral. It is the counterpart of the CDF of a probability measure.

Definition Let X be a R^d -valued Levy process with Levy measure μ . The tail integral of the Levy measure, or T.I.L of X is the function $U : (R^d/\{0\}) \rightarrow R$ defined by

$$U(x_1, \dots, x_d) = \prod_{i=1}^d \text{sgn}(x_i) \mu\left(\prod_{j=1}^d (I(x_j))\right),$$

where for every $x \in R$, $I(x) = [x, \infty)$ if $x > 0$; $I(x) = (-\infty, x)$ if $x < 0$:

Unlike a probability measure, the Levy measure can have a total mass of infinity and is undefined at 0. By defining the tail integral, one can avoid 0.

Definition A function $F : R^d \rightarrow R$ is called Levy copula if

1. $F(u_1, \dots, u_d) \neq \infty$ for $(u_1, \dots, u_d) \neq (\infty, \dots, \infty)$.
2. $F(u_1, \dots, u_d) = 0$ if $u_i = 0$ for at least one $i \in \{1, \dots, d\}$.
3. F is d-increasing.

4. $F^i(u) = u$ for any $i \in 1, \dots, d, u \in R$.

Theorem 1.3.5 *Sklar's Theorem for Levy Copula*

1. Let U be the n -dimensional tail integral and U_i be its i th marginal tail integral.

There exists a Levy Copula F such that

$$U(x_1, \dots, x_n) = F(U(x_1), \dots, U(x_n)).$$

If U_i s are continuous, then F is unique.

2. If F is a Levy Copula and U_i s are one-dimensional tail integral, then

$$U(x_1, \dots, x_n) = F(U_1(x_1), \dots, U_n(x_n))$$

define a n -dimensional tail integral.

We can see that the definition and theory for Levy copula is very similar in spirit to regular copula. Here is some examples of Levy copula

Independence Levy Copula

$$F_{\perp} = \sum_{i=1}^d x_i \prod_{j \neq i} 1_{\infty}(x_j).$$

Independent Levy process means no common jumps at one jump event, a.s. Each process jumps individually.

Completely dependent or comonotonic Levy Copula

$$F_{\parallel}(x_1, \dots, x_d) = \min(|x_1|, \dots, |x_d|).$$

Comonotonic Levy process means all the jumps happen at the same time and in the same direction, a.s. Jump sizes are the subset of all strictly ordered set, which implies that it can be determined by any component.

α -Stable Levy Copula: X is α -Stable if and only if its components X^1, \dots, X^d are α -stable and if it has a Levy Copula F that is a homogeneous function of order 1. i.e.

$$F(ru_1, \dots, ru_d) = rF(u_1, \dots, u_d).$$

Tankov proposed Clayton Levy copula. The function form is very similar to the regular Clayton Copula.

$$F(u_1, \dots, u_n) = \left(\sum_{i=1}^n |u_i|^{-\theta} \right)^{-1/\theta},$$

where θ is interpreted the same way as in the regular Clayton Copula. When $\theta = 0$, it becomes the independent Levy copula. When $\theta = 1$, it becomes an comonotonic Levy copula.

Chapter 2

DSPMD For Levy Processes

2.1 Motivations and Ideas

A well known way to build multi-dimensional Levy processes is through subordination. For example, one dimensional VG is a Brownian motion under a gamma process time change. Then, the natural multi-dimensional extension for VG is a multi-dimensional Brownian motion under a common gamma process time change. However, this construction has its limitation since the marginal processes have almost identical kurtosis, which makes it unrealistic for practical purposes.

There is a demand to use some type of copula structure to model processes. As introduced in the Chapter 1, the concept of Levy copula was introduced to solve this problem. We recognize that Levy copula by Tankov is a very general way to build multi-dimensional Levy processes. It keeps the copula property which separates the dependence part from the marginals. And in the same time, the resulting process is automatically Levy process, which avoided the question of infinitely divisibility one will have to face when use regular copula.

Tankov proposed the following way to construct a Levy copula in [6] and [30]. One can abstractly construct the Levy copula function that satisfies the definition of Levy copula, examples such as Clayton Levy copula. Alternatively, one can construct the Levy copula by a transformation to change the domain of the regular

copula. For more details, we refer the reader to [6] and Chapter 5 from [30].

There are some drawbacks for this approach. First, the construction shows no connection between the Levy copula and the implied copula structure which is required for statistical inference purpose. Second, for simulation, the existing algorithm given in [6] and [30] has first component bias effect, which makes this algorithm practically useless. See Chapter 5 or [10] for more details on this issue.

To fix this problem, we propose DSPMD. DSPMD represents discretely sampled process with pre-specified marginals and pre-specified dependence. A two-dimensional DSPMD on $[0, T]$ can be constructed in the following way: One starts with two pre-specified marginal processes and discretely sample the increments on sub-intervals by the generalized inverse function of its tail integral of probability measure using correlated uniform random variables. The correlated uniform random variables are embodied by some pre-specified discretely sampled joint processes on the same sub-intervals in the form of its tail integral copula. So in a DSPMD, the pre-specified marginals are coupled using the joint law of the pre-specified joint process. The advantage of DSPMD is that it uses a copula structure on the random variable level so that one can have access to its statistical property. Here we are going to prove that if the pre-specified marginal and pre-specified joint processes are some Levy process, a DSPMD converges to a Levy process, under certain technical conditions. In that respect, DSPMD is the discrete version of the Levy copula.

2.2 Preliminaries

In order to be self-contained in this chapter, we recall the definition of T.I.L., tail integral of Levy measure.

Definition Let X be a R^d -valued Levy process with Levy measure ν . The tail integral of X is the function $U : (R \setminus \{0\})^d \rightarrow R$ defined by

$$U(x_1, \dots, x_d) = \prod_{i=1}^d \text{sgn}(x_i) \mu\left(\prod_{j=1}^d (I(x_j))\right)$$

where for every $x \in R$, $I(x) = [x, \infty)$ if $x > 0$; $I(x) = (-\infty, x]$ if $x < 0$. T.I.L. is short for tail integral of the Levy measure

Since T.I.L. is only defined on $(R \setminus \{0\})^d$, it does not determine the Levy measure uniquely (unless we know that the latter does not charge the coordinate axes). However, Levy measure is completely determined by its T.I.L. and all its marginal T.I.L.. See [20].

Definition Let X be a R^d -valued Lvy process and let $I \subset \{1, \dots, d\}$ non-empty. The I -marginal tail integral U^I of X is the tail integral of the process $X^I = (X^i)_{i \in I}$.

Tankov proved the following lemma in [20]

Lemma 2.2.1 *Let X be a R^d -valued Levy process. Its marginal tail integrals $\{U^I : I \subset \{1, \dots, d\} \text{ non - empty}\}$ are uniquely determined by its Levy measure ν . Conversely, its Levy measure is uniquely determined by the set of its marginal tail integrals.*

Follow the definition of tail integral of Levy measure, we introduce the tail integral of probability measure, or T.I.P..

Definition Let X be a R^d random variable with probability measure P . The tail integral of X is the function $U : (R)^d \rightarrow [-1, 1]$ defined by

$$U(x_1, \dots, x_d) = \prod_{i=1}^d \text{sgn}(x_i) P\left(\prod_{j=1}^d (I(x_j))\right)$$

where for every $x \in R$, $I(x) = [x, \infty)$ if $x \geq 0$; $I(x) = (-\infty, x]$ if $x < 0$. T.I.P. is short for the tail integral of the probability measure.

The tail integral of probability measure, or T.I.P., uniquely determine the probability measure including all the mass on the axis and the origin. When X is one dimension, we have $U(0) - U(0-) = 1$. Further, if X does not have mass on the origin, we have $\lim_{x \rightarrow 0+} U(x) = U(0)$, i.e. $U(0+) - U(0-) = 1$.

Definition Generalized inverse function of $F : R \rightarrow R$ is defined as

$$F^{-1}(u) = \inf\{x : u \geq F(x)\}$$

Since one dimensional T.I.P. and T.I.L. are monotonically decreasing on R^- and R^+ respectively. We define the generalized inverse of the tail integral as

$$F^{-1}(u) = \begin{cases} \inf\{x > 0 : u \geq F(x)\} & \text{if } u \geq 0 \\ \inf\{x < 0 : u \geq F(x)\} & \text{if } u \leq 0 \end{cases}$$

Lemma 2.2.2 Let X and Y be random variables on $R^+ \cup \{0\}$ with the T.I.P. $U(x), V(x)$, which are monotonic. We define a mapping $F : R^+ \cup \{0\} \rightarrow R^+ \cup \{0\}$

$$F(x) = V^{-1}(U(x))$$

the following are true:

i) if X and Y has no atom at 0, then F is continuous at 0 and $F(X) = Y$ in distribution.

ii) if $P(X = 0) \leq P(Y = 0)$, or in terms of the T.I.P., $U(0) - U(0+) \leq V(0) - V(0+)$, F is continuous at 0, and $F(X) = Y$ in distribution.

Proof: *i)*

Since X and Y don't have atom at 0, then $U(0+) = U(0) = 1$, and $V(0) = V(0+) =$

1. It is easy to check that

$$\lim_{x \rightarrow 0+} F(x) = V^{-1}(1) = 0$$

so $F(x)$ is continuous at 0.

For $x > 0$

$$P(F(X) > x) = P(V^{-1}(U(X)) > x) \tag{2.1}$$

$$= P(U(X) < V(x)) \tag{2.2}$$

$$= V(x) \tag{2.3}$$

so $F(X)$ have the same tail integral as Y , and $F(X)$ does not have atom at 0. We conclude that $F(X) = Y$ in distribution.

ii)

A note: the result from *i)* is included in *ii)*

Because $P(X = 0) \leq P(Y = 0)$, $U(0+) \geq V(0+)$,

$$\lim_{x \rightarrow 0+} F(x) = V^{-1}(U(0+))$$

By the definition of generalized inverse, we have

$$V^{-1}(U(0+)) = \inf\{x : U(0+) \geq V(x)\} = 0$$

The proof of $F(x) = Y$ in distribution is similar as in *i*).

Q.E.D.

We quote the following important result from Sato [28] P45 Corollary 8.9. It shows the connection between the probability measure of a infinitely divisible distribution with its Levy measure.

Proposition 2.2.3 *For any bounded continuous function f that vanishes at a neighborhood of 0, if ν is the Levy measure of an infinitely divisible distribution μ , then we have*

$$t_n^{-1} \int_{\mathbb{R}^d} f(x) \mu^{t_n}(dx) \rightarrow \int_{\mathbb{R}^d} f(x) \nu(dx), t_n \rightarrow 0$$

μ^{t_n} denotes the t_n th fold convolution of μ .

The proof of the following proposition is given by [20] Theorem 5.1, step 3 to step 5, which is basically an application of the above proposition.

Proposition 2.2.4 *Let X_t be a pure jump Levy process with T.I.P. at time t $U^t(x)$ and T.I.L. $u(x)$, then*

$$\frac{U^t(x)}{t} \rightarrow u(x)$$

as $t \rightarrow 0$ point-wise on the continuous points of $u(x)$. In addition, if $U^t(x)$ and $u(x)$ are absolutely continuous with respect to Lebesgue measure everywhere except at zero, then

$$\frac{U_x^t(x)}{t} \rightarrow u_x(x)$$

where $U_x^t(x)$ is the p.d.f. and $u_x(x)$ is the Levy density.

Corollary 2.2.5 *Let $\{X_t\}$ and $\{Y_t\}$ be two compound Poisson processes on $R^+ \cup \{0\}$ with the T.I.P. at time t $U^t(x), V^t(x)$, And with T.I.L. $u(x), v(x)$. We define a mapping $F^t : R \rightarrow R$*

$$F^t(x) = V^{t-1}(U^t(x))$$

then, as $t \rightarrow 0$, we have

$$F^t(x) \rightarrow v^{-1}(u(x))$$

point-wise on the continuous points of $v^{-1}(u(x))$.

Proof:

By Proposition 2.2.4, as $t \rightarrow 0$, we have

$$\frac{U^t(z)}{t} \rightarrow u(z)$$

This limit is of point-wise convergence on the continuous points of $u(x)$. Same for $\frac{V^t(z)}{t} \rightarrow v(z)$ point-wise. Under the definition of generalized inverse function, it is easy to check that $V^{t-1}(xt) \rightarrow v^{-1}(x)$ point-wise as $t \rightarrow 0$. So, we have

$$F^t(x) = V^{t-1}(U^t(x)) \tag{2.4}$$

$$= V^{t-1}\left(t \times \left(\frac{U^t(x)}{t}\right)\right) \tag{2.5}$$

$$\rightarrow v^{-1}(u(x)) \text{ as } t \rightarrow 0 \tag{2.6}$$

Q.E.D.

The following result holds for compound Poisson process on the whole real line, with a slightly different definition of $F^t(x)$

Lemma 2.2.6 *Let X_t and Y_t be the compound Poisson Random variables on R with the T.I.P. $U^t(x), V^t(x)$. It also has the T.I.L. $u(x)$ and $v(x)$. We define a mapping $F : R \rightarrow R$*

$$F^t(x) = V^{t-1}(U^t(x) - U^t(0) + V^t(0) + e^{-u(0-)t} - e^{-v(0-)t})$$

Then the following is true:

i) if $u(0+) \geq v(0+)$ and $u(0-) \geq v(0-)$, when t is small enough, $F^t(x)$ is continuous at 0 in x , and $F(X) = Y$ in distribution.

ii) $F^t(x) \rightarrow v^{-1}(u(x))$, as $t \rightarrow 0$

Proof:

i)

To show $F^t(x)$ is continuous at 0 in x , we need to prove that $\lim_{x \rightarrow 0-} F^t(x) = \lim_{x \rightarrow 0+} F^t(x) = 0$. From the definition of generalized inverse function, we know that $V^{t-1}(V^t(0+) + c_1) = 0$, $V^{t-1}(V^t(0-) - c_2) = 0$, where c_1, c_2 are any non-negative number.

$$\lim_{x \rightarrow 0-} F^t(x) = V^{t-1}(U^t(0-) - U^t(0) + V^t(0) + e^{-u(0-)t} - e^{-v(0-)t}) \quad (2.7)$$

$$= V^{t-1}(U^t(0-) - (1 + U^t(0-)) + (1 + V^t(0-))) \quad (2.8)$$

$$+ e^{-u(0-)t} - e^{-v(0-)t} \quad (2.9)$$

$$= V^{t-1}(V^t(0-) + e^{-u(0-)t} - e^{-v(0-)t}) \quad (2.10)$$

Since $e^{-u(0-)t} - e^{-v(0-)t} \leq 0$, by the definition of generalized inverse function, we have $\lim_{x \rightarrow 0-} = V^{t-1}(V^t(0-) + e^{-u(0-)t} - e^{-v(0-)t}) = 0$

$$\lim_{x \rightarrow 0^+} F^t(x) = V^{t-1}(U^t(0+) - U^t(0) + V^t(0) + e^{-u(0-)t} - e^{-v(0-)t}) \quad (2.11)$$

$$= V^{t-1}(U^t(0+) - U^t(0+) - P(X_t = 0) + V^t(0+) + P(Y_t = 0)) \quad (2.12)$$

$$+ e^{-u(0-)t} - e^{-v(0-)t}) \quad (2.13)$$

$$= V^{t-1}(V^t(0+) - e^{-(u(0+)+u(0-))t} + e^{-(v(0+)+v(0-))t} + e^{-u(0-)t} - e^{-v(0-)t}) \quad (2.14)$$

Next, we are going to show that $-e^{-(u(0+)+u(0-))t} + e^{-(v(0+)+v(0-))t} + e^{-u(0-)t} - e^{-v(0-)t} \geq 0$ when t is small enough. Let $z(t) = -e^{-(u(0+)+u(0-))t} + e^{-(v(0+)+v(0-))t} + e^{-u(0-)t} - e^{-v(0-)t}$ and $z(t) \in C^\infty$. $z'(t) = u(0+) - v(0+) \geq 0$, so there exists a small neighborhood of 0, such that $z'(t) \geq 0$. $z(0) = 0$ implies that there exist a small neighborhood of 0, such that $z(t) \geq 0$. So $\lim_{x \rightarrow 0^+} F^t(x) = 0$.

At last, it is easy to verify that $F^t(0) = 0$. We conclude that $F^t(x)$ is continuous at 0 in x when t is small enough.

ii)

Firstly, we use Proposition 2.2.4, and get

$$\frac{U^t(z)}{t} \rightarrow u(z)$$

$$\frac{V^t(z)}{t} \rightarrow v(z)$$

point-wise. By the definition of generalized inverse function, it is easy to check that

$V^{t-1}(xt) \rightarrow v^{-1}(x)$ point-wise as $t \rightarrow 0$ for $x \in R \setminus \{0\}$. So, we have

$$F^t(x) = V^{t-1}(U^t(x) - U^t(0) + V^t(0) + e^{-u(0-)t} - e^{-v(0-)t}) \quad (2.15)$$

$$= V^{t-1}\left(t \times \left(\frac{U^t(x)}{t} - \frac{U^t(0) - V^t(0)}{t} + \frac{e^{-u(0-)t} - e^{-v(0-)t}}{t}\right)\right) \quad (2.16)$$

We have $U^t(0) = U^t(0+) + e^{-(u(0+)+u(0-))t}$ and $V^t(0) = V^t(0+) + e^{-(v(0+)+v(0-))t}$

Consequently, we have

$$\frac{U^t(0) - V^t(0)}{t} = t^{-1}(U^t(0+) - V^t(0+) + e^{-(u(0+)+u(0-))t} - e^{-(v(0+)+v(0-))t}) \quad (2.17)$$

$$+ e^{-u(0-)t} - e^{-v(0-)t}) \quad (2.18)$$

$$\rightarrow u(0+) - v(0+) - (u(0+) + u(0-)) + (v(0+) + v(0-)) \quad (2.19)$$

$$- u(0-) + v(0-) \text{ as } t \rightarrow 0 \quad (2.20)$$

$$= 0 \quad (2.21)$$

Now we have for any $x \in R \setminus \{0\}$, as $t \rightarrow 0$

$$F^t(x) \rightarrow v^{-1}(u(x))$$

Q.E.D

Next proposition is from Sato [28] P123 Proposition 19.5

Proposition 2.2.7 *Let $(\Theta, \mathcal{B}, \rho)$ be a measure space with $\rho(\Theta) < \infty$ and $\{N(B), B \in \mathcal{B}\}$ be a Poisson random measure with intensity measure ρ . Let ϕ be a measurable function from Θ to R^d and define*

$$Y(\omega) = \int_{\Theta} \phi(\theta) N(d\theta, \omega).$$

then, Y is a random variable on R^d with compound Poisson distribution satisfying

$$E[e^{i\langle z, y \rangle}] = \exp\left[\int_{R^d} (e^{i\langle z, x \rangle} - 1)(\rho\phi^{-1})(dx)\right]$$

for $z \in R^d$.

2.3 Main Results

In this section, we are going to prove that if the pre-specified marginal processes are Levy processes and the copula is embodied by some multi-dimensional pure jump Levy processes, then DSPMD converges to some Levy process. We prove this result in several scenarios. We first prove for the case of compound poisson processes with only positive jumps. Then we generalize the case to compound poisson processes supported on the whole real line. For most of our practical use, this result can be used in an asymptotical sense since all Levy processes can be viewed as some compound Poisson process approximation after truncating infinitely frequent small jumps. In the end, we prove the result for general subordinator type of Levy processes. For the most general case of Levy process, the proof is still under investigation.

Theorem 2.3.1 *A DSPMD converges to a compound poisson process almost surely, as $N \rightarrow \infty$, if the pre-specified marginals and pre-specified joint processes are compound poisson processes with pure positive jumps and if the jump intensity on the pre-specified processes are no greater than the ones on the marginals of the joint process, respectively. The marginals of DSPMD are the exact discretely sampled processes of the pre-specified marginal processes, and the T.I.L. of the limiting process is the T.I.L. of the pre-specified marginal processes coupled by the Levy copula of the pre-specified joint process.*

Proof:

For $j = 1, 2$, let G_j^t , and g_j be the T.I.P. and the T.I.L. of the pre-specified marginal compound Poisson processes with positive jumps. Let (X_t, Y_t) be the pre-specified joint process, which is a compound Poisson processes with positive jumps, with its T.I.P. $F^t(x, y)$ and marginal T.I.P. F_j^t at time t , and its T.I.L. $f(x, y)$ and marginal T.I.L. $f_j(x)$. $g_j(0+) \leq f_j(0+)$ for $j = 1, 2$.

We make a partition on $[0, T]$ into N equal length sub-intervals and we can get a discretely sampled process from (X_t, Y_t) as:

$$(X_t^N, Y_t^N) = \sum_{i=1}^{N(t)} (X_i^{T/N}, Y_i^{T/N})$$

where $N(t) = \inf\{i \in Z : iT/N \geq t\}$ We want to transform the increment $(X_i^{T/N}, Y_i^{T/N})$ into correlated uniforms by its T.I.P. and transform it back to the increment of the DSPMD by the generalized inverse function of the $G_j^{T/N}$ for $j = 1, 2$. This can be done by introducing a transformation $P_j^{T/N}(x) = G_j^{T/N-1}(F_j^{T/N}(x))$ for $j = 1, 2$. So the DSPMD can be written as

$$(U_t^N, V_t^N) = \sum_{i=1}^{N(t)} (U_i^{T/N}, V_i^{T/N})$$

, where $U_i^{T/N} = P_1^{T/N}(X_i^{T/N})$, $V_i^{T/N} = P_2^{T/N}(Y_i^{T/N})$. That is, $P^{T/N}$ transforms the increment of (X_t^N, Y_t^N) into the increment of (U_t^N, V_t^N) component-wise. By Lemma 2.2.2, for $i = 1, \dots, N$, $P_1^{T/N}(X_i^{T/N})$ has distribution function $G_1^{T/N}$. The same goes for $P_2^{T/N}(Y_i^{T/N})$ with $G_2^{T/N}$. So we have proved that, at discrete time, it is an exact discrete time process of the pre-specified marginal process for any discretization step size.

Now, we are going to prove that in the limit, it is again a compound poisson process.

The sample path of compound Poisson process $\{X_t\}$ on $[0, T]$ has only finite number of jumps and two jumps never happen at the same time, almost surely. For any $\omega \in \Omega$ fixed, we look at the sample path of the discrete process $X_t^N(\omega) = \sum_{i=1}^{N(t)} X_i^{T/N}(\omega)$. When N is large enough, only fixed number of $X_i^{T/N}(\omega)$ are non-zero. So there exists a sub-sequence $\{i_k\}$ of $\{i\}$, such that $X_{i_k}^{T/N}$ contains only one jump from $\{X_t\}$, and $X_i^{T/N} = 0$, for $i \notin \{i_k\}$. So we denote the non-zero terms as $X_{i_k}^{T/N} = J_k^X(\omega)$, $k = 1 \dots K(\omega)$. Let $K(t) = \sup\{k : i_k \leq N(t)\}$, which counts the total number of jumps up to time t , which depends on ω . We do the same thing for $\{Y_t\}$, and denote the non-zero terms as $J_l^Y(\omega)$, $l = 1, \dots, L(\omega)$, and $L(t)$ counts the total number of jumps up to t .

We also define $p_j(x) = g_j^{-1}(f_j(x))$ for $j = 1, 2$. By Lemma 2.2.5, for $j = 1, 2$, $P_j^{T/N}(x)$ converge point-wise to $p_j(x)$ as $N \rightarrow \infty$. By Lemma 2.2.2, with the condition $g_j(0+) \leq f_j(0+)$, we have $P_j^{T/N}(0) = 0$ and continuous at 0. And $p_j(x) \rightarrow 0$ as $x \rightarrow 0$. We can assign $p_j(0) = 0$. Therefore, in the summation, 0 are transformed to 0 under the mapping $P_j^{T/N}$ and $p_j(x)$. Now for any fixed ω and any fixed t , let $N \rightarrow \infty$, we have

$$U_t^N(\omega) \rightarrow U^t(\omega) = \sum_{k=1}^{K(t)} p_1(J_k^X(\omega))$$

and

$$V_t^N(\omega) \rightarrow V^t(\omega) = \sum_{l=1}^{L(t)} p_2(J_l^Y(\omega))$$

Now, (U_t, V_t) are again compound Poisson process. By Proposition 2.2.7, we assert that (U_t, V_t) has the Levy measure $f(f_1^{-1}(g_1), f_2^{-1}(g_2))$ where the pre-specified

marginal Levy measures are coupled by the pre-specified Levy copula.

Q.E.D.

Remark: The DSPMD is an approximation of (U_t, V_t) which is a compound Poisson process with the marginal process from G_1^t and G_2^t and dependence structure from $F^t(x, y)$. In case of pure positive jumps, tail integral of a probability measure is exactly the same as survival function of the probability measure. And for each increment, we basically used the survival copula of $F^t(F_1^{t-1}(x), F_2^{t-1}(y))$. Basically, this theorem makes sure that if one use the survival copula implied by the compound Poisson distribution to build a discretely sampled process, then in the limit, it converges to a compound Poisson. We can work with the random walk to get its local copula information and do estimation on it as long as the time step is small. Then, as an approximation, we can say something about the copula at a longer time horizon.

We can drop the restriction of the positive jumps and extend this theorem to the general compound Poisson process on R^d with a slightly different definition of $P^t(x)$ with some technical conditions.

Theorem 2.3.2 *A DSPMD converges to a compound poisson process almost surely, as $N \rightarrow \infty$, if the pre-specified marginals and pre-specified joint processes are compound Poisson processes and if the intensity of the positive and negative jumps on the pre-specified marginal processes are no greater than the ones on the marginals of the joint process, respectively and component-wise. The marginals of DSPMD are*

the exact discretely sampled processes of the pre-specified marginal processes, and the T.I.L. of the limiting process is the T.I.L. of the pre-specified marginal processes coupled by the Levy copula of the pre-specified joint process.

Proof:

For $j = 1, 2$, let G_j^t , and g_j be the T.I.P. and the T.I.L. of the pre-specified marginal compound Poisson processes. Let (X_t, Y_t) be the pre-specified joint process, which is also a compound Poisson process, with its T.I.P. $F^t(x, y)$ and marginal T.I.P. F_j^t at time t , and its T.I.L. $f(x, y)$ and marginal T.I.L. $f_j(x)$. $g_j(0+) \leq f_j(0+)$ and $g_j(0-) \leq f_j(0-)$ for $j = 1, 2$. We make a partition on $[0, T]$ into N equal length sub-intervals and we can get a discretely sampled process from (X_t, Y_t) as:

$$(X_t^N, Y_t^N) = \sum_{i=1}^{N(t)} (X_i^{T/N}, Y_i^{T/N})$$

where $N(t) = \inf\{i \in Z : iT/N \geq t\}$ We want to transform the increment $(X_i^{T/N}, Y_i^{T/N})$ into correlated uniforms by its T.I.P. and transform it back to the increment of the DSPMD by the generalized inverse function of the $G_j^{T/N}$ for $j = 1, 2$. This can be done by introducing a transformation $P_j^{T/N}(x)$ for $j = 1, 2$. The exact form of $P_j^{T/N}(x)$ will be given shortly. So the DSPMD can be written as

$$(U_t^N, V_t^N) = \sum_{i=1}^{N(t)} (U_i^{T/N}, V_i^{T/N})$$

, where $U_i^{T/N} = P_1^{T/N}(X_i^{T/N})$, $V_i^{T/N} = P_2^{T/N}(Y_i^{T/N})$. That is, $P^{T/N}$ transforms the increment of (X_t^N, Y_t^N) into the increment of (U_t^N, V_t^N) component-wise.

The difficulty here is the choice of the mapping $P^{T/N}$. The T.I.P. maps the increment of the joints into uniforms on a subset of $[-1, 1]^2$. In order to transform

back to the pre-specified marginal for any finite N , one has to match up the domain of the uniforms and the domain of $G_j^{T/N-1}$. Since there is probability mass at 0, the match-up is not unique. We choose the following form:

$$P_j^{T/N}(x) = G_j^{T/N-1}(F_j^{T/N}(x) - F_j^{T/N}(0) + G_j^{T/N}(0) + e^{-f_j(0-)t} - e^{-g_j(0-)t})$$

such that $P_j^{T/N}(0) = 0$ and continuous at 0 and $P_1^{T/N}(X_i^{T/N})$ and $P_2^{T/N}(Y_i^{T/N})$ have the same distribution as the increment of the pre-specified marginal $G_1^{T/N}$ and $G_2^{T/N}$. These statements have already been proven in *i)* of Lemma 2.2.6. So, at discrete time, it is an exact discrete time process of the pre-specified marginal process for any discretization step size.

In addition, by Lemma 2.2.6 *ii)*, we have

$$P_j^{T/N}(x) \rightarrow p_j(x) = g_j^{-1}(f_j(x))$$

point-wise, as $N \rightarrow \infty$, for $j = 1, 2$

From here, the proof follows exactly the same as in the first theorem.

Q.E.D.

Remark: The DSPMD is an approximation of (U_t, V_t) which is a compound Poisson process with the marginal process from G_1^t and G_2^t and dependence structure from $F^t(x, y)$. For the construction shown in the proof, the marginals are the exact discretely sampled processes. The dependence at the discrete time level is, however, an approximation in general and exact for some special cases. Notice that the transformation on the random variable level is not always strictly co-monotonic, hence the dependence structure from the joint processes is not exactly preserved after the transformation into the pre-specified marginals. We say it an approximation

because the limiting process is precisely given by the Levy copula from the joint processes, which characterizes the dependence structure of the joint process, a direct result by Sklar's Theorem for Levy copula.

DSPMD can be constructed with a different $P^{T/N}$, which is given by

$$P_j^{T/N}(x) = g_j^{-1}(f_j(x))$$

In this case, the transformation is co-monotonic, hence the dependence structure from the joint process is precisely preserved after the transformation. So at discrete time, the dependence structure for DSPMD is exact. The marginals at the discrete time level is, however, an approximation. By Proposition 2.2.4, we have

$$\frac{U^t(x)}{t} \rightarrow u(x)$$

, and we have the following approximation

$$U^t(x) \approx u(x) \times t$$

when t is small enough. It can be proved that when t goes to 0, DSPMD converges to the same limiting process as the shown in the theorem above. Furthermore, this construction can be applied to the case of infinite activity. The idea of proof is the same as the one shown below.

In summary, when constructing a DSPMD for the Levy processes with both positive and negative jumps, one can have the choice of exact marginals and approximated dependence or approximated marginals and exact dependence. In some special case, using the construction given by the theorem above, one can have exact marginals and exact dependence. In both cases, the limiting process is given by the

marginal Levy measure of the pre-specified marginals and the Levy copula from the pre-specified joint process.

Now we are going to extend this theorem in a different way. We will prove that instead of compound Poisson process, for a general Levy process with pure positive jumps, which allows for infinite activity, the result still holds. Without loss of generality, we limit ourselves to the case of the infinitely activity.

Theorem 2.3.3 *A DSPMD converges to a Levy process in law, as $N \rightarrow \infty$, if the pre-specified marginals and pre-specified joint processes are Levy processes with pure positive jumps and if the ratio of Levy densities on the marginals of the joint process versus the pre-specified marginals are bounded as jump size goes to 0, respectively. The marginals of DSPMD are the exact discretely sampled processes of the pre-specified marginal processes, and the T.I.L. of the limiting process is the T.I.L. of the pre-specified marginal processes coupled by the Levy copula of the pre-specified joint process.*

Proof:

For $j = 1, 2$, let G_j^t , and g_j be the T.I.P. and the T.I.L. of the pre-specified marginal Levy processes with positive jumps. Let (X_t, Y_t) be the pre-specified joint process, which is a Levy process with positive jumps, with $|\frac{f_j(0+)}{g_j(0+)}| < \infty$ for $j = 1, 2$. We make a partition on $[0, T]$ into N equal length sub-intervals and we can get a discretely sampled process from (X_t, Y_t) as:

$$(X_t^N, Y_t^N) = \sum_{i=1}^{N(t)} (X_i^{T/N}, Y_i^{T/N})$$

where $N(t) = \inf\{i \in Z : iT/N \geq t\}$. We want to transform the increment $(X_i^{T/N}, Y_i^{T/N})$ into correlated uniforms by its T.I.P. and transform it back to the increment of the DSPMD by the generalized inverse function of the $G_j^{T/N}$ for $j = 1, 2$. This can be done by introducing a transformation $P_j^{T/N}(x) = G_j^{T/N-1}(F_j^{T/N}(x))$ for $j = 1, 2$. So the DSPMD can be written as

$$(U_t^N, V_t^N) = \sum_{i=1}^{N(t)} (U_i^{T/N}, V_i^{T/N})$$

, where $U_i^{T/N} = P_1^{T/N}(X_i^{T/N}), V_i^{T/N} = P_2^{T/N}(Y_i^{T/N})$. That is, $P^{T/N}$ transforms the increment of (X_t^N, Y_t^N) into the increment of (U_t^N, V_t^N) component-wise.

For any small $\epsilon > 0$, the sample path of pure positive jump processes on $[0, T]$ has only finite number of jumps that are greater than ϵ and jumps never happen at the same time, almost surely. For any $\omega \in \Omega$ fixed, when N is large enough, only fixed number of $X_i^{T/N}(\omega)$ contain at most one single jump that is greater than ϵ . So there exists a subsequence $i_k \subset 1, \dots, N, k = 1, \dots, K_\epsilon$, depending on ω , such that

$$X_{i_k}^{T/N}(\omega) = J_k^X(\omega) + \eta_{i_k}^{T/N}(\epsilon)(\omega)$$

where $|J_k^X| > \epsilon$ and $\eta_{i_k}^{T/N}(\epsilon)$ is the sum of the jumps in the i_k th sub-interval that are less than ϵ . K_ϵ is the number of jumps of X_t on $[0, T]$ that are greater than ϵ . Notice that K_ϵ depends only on ϵ , not on N , if N is large enough but finite. We denote $K_\epsilon(t) = \sup\{k : i_k \leq N(t)\}$, which counts the total number of large jumps before time t , which depends on ω .

For any $t \in [0, T]$, we can rewrite the sum as

$$X_t^N = \sum_{k=1}^{K_\epsilon(t)} X_{i_k}^{T/N} + \sum_{i=1, i \notin \{i_k\}}^{N(t)} X_i^{T/N} \quad (2.22)$$

$$= \sum_{k=1}^{K_\epsilon(t)} (J_k^X + \eta_{i_k}^{T/N}(\epsilon)) + \sum_{i=1, i \notin \{i_k\}}^{N(t)} \eta_i^{T/N}(\epsilon) \quad (2.23)$$

$$(2.24)$$

Now, we apply the transformation $P^{T/N}$ on each individual sub interval

$$U_t^N = \sum_{i=1}^{N(t)} P_j^{T/N}(X_i^{t/N}) \quad (2.25)$$

$$= \sum_{k=1}^{K_\epsilon(t)} P_j^{T/N}(J_k^X + \eta_{i_k}) + \sum_{i=1, i \notin \{i_k\}}^{N(t)} P_j^{T/N}(\eta_i^{t/N}(\epsilon)) \quad (2.26)$$

From the smoothness property (Sato [28] P189) for G_j^t and F_j^t , we know that $P_j^t(x)$ is smooth, too. For $k = 1, \dots, K_\epsilon, i_k \leq N(t)$, we have

$$P_1^{T/N}(J_k^X + \eta_{i_k}) = P_1^{T/N}(J_k^X) + \frac{dP_1^{T/N}(x)}{dx}(\xi_{i_k})\eta_{i_k}(\epsilon)$$

where $\xi_k \in (J_k^X, J_k^X + \eta_{i_k}(\epsilon))$. And for $i = 1, \dots, N(t), i \neq i_k$

By Lemma 2.2.2, we have $P_j^{T/N}(0) = 0$ and continuous at 0. Together with the smoothness property of $P_j^t(x)$, we have

$$P_1^{T/N}(\eta_i^{T/N}(\epsilon)) \quad (2.27)$$

$$= P_1^{T/N}(\eta_i^{T/N}(\epsilon)) \quad (2.28)$$

$$= 0 + \frac{dP_1^{T/N}(x)}{dx}(\xi_i)\eta_i^{T/N}(\epsilon) \quad (2.29)$$

where $\xi_i \in (0, \eta_i^{T/N}(\epsilon))$. The summation can be rewritten as:

$$U_t^N = \sum_{k=1}^{K_\epsilon(t)} (P_1^{T/N}(J_k^X) + \frac{dP_1^{T/N}(x)}{dx}(\xi_{i_k})\eta_{i_k}(\epsilon)) + \sum_{i=1, i \notin \{i_k\}}^{N(t)} \frac{dP_1^{T/N}(x)}{dx}(\xi_i)\eta_i^{T/N}(\epsilon)$$

After re-arranging the terms:

$$|U_t^N - \sum_{k=1}^{K_\epsilon(t)} P_1^{T/N}(J_k^X)| \quad (2.30)$$

$$= \left| \sum_{k=1}^{K_\epsilon(t)} \frac{dP_1^{T/N}(x)}{dx}(\xi_{i_k}) \eta_{i_k}(\epsilon) + \sum_{i=1, i \notin \{i_k\}}^{N(t)} \frac{dP_1^{T/N}(x)}{dx}(\xi_i) \eta_i^{T/N}(\epsilon) \right| \quad (2.31)$$

$$\leq \left| \max_{i=1, \dots, N(t)} \left\{ \frac{dP_1^{T/N}(x)}{dx}(\xi_i) \right\} \sum_{i=1}^{N(t)} (\eta_i^{T/N}(\epsilon)) \right| \quad (2.32)$$

Since (X_t) contains only positive jumps without diffusion, it is of finite variation, $\sum_{i=1}^N(t) (\eta_i^{T/N}(\epsilon))$ contains all the small jumps of X_t that are less than ϵ on $[0, t]$. We can also rewrite it as

$$\sum_{i=1}^{N(t)} (\eta_i^{T/N}(\epsilon))(\omega) = \int_{|x| < \epsilon, s \in [0, t]} x J_X(dx \times ds)(\omega)$$

where J_X is a Poisson random measure on $[0, T] \times R$ with intensity $f_1(dx)dt$.

We define $p_j(x) = g_j^{-1}(f_j(x))$ for $j = 1, 2$. So by Lemma 2.2.5, for $j = 1, 2$, $P_j^{T/N}(x)$ converge point-wise to $p_j(x)$ as $N \rightarrow \infty$. By Lemma 2.2.2, $p_j(x) \rightarrow 0$ as $x \rightarrow 0$ for the case of infinite activity. So we can assign $p_i(0) = 0$ to be a continuous function at the neighborhood of 0

By 2.2.4, we have $\frac{dF_j^{t/N}(x)}{dx} \rightarrow \frac{df_j(x)}{dx}$ point-wise as $N \rightarrow \infty$. From the composition of F_j^t, G_j^t and f_j, g_j , we assert that

$$\frac{dP_j^t}{dx} \rightarrow \frac{dp_j}{dx}$$

, as $N \rightarrow \infty$. In addition, from the smoothness assumption of P_j^t and $p_j(x)$ and $|\frac{f_j(0+)}{g_j(0+)}| < \infty$, $\frac{dP_j^t(x)}{dx}$ and $\frac{dp_j(x)}{dx}$ is bounded on $(0, 1]$

Let $S_X(\omega) = \max_{i=1, \dots, N}(J_i^X(\omega))$, Firstly,

$$\left| \sup_{i=1, \dots, N} \left\{ \frac{dP_j^{T/N}(x)}{dx}(\xi_i) \right\} \right| < \left| \sup_{x \in (0, S_X]} \left\{ \frac{dP_j^{T/N}(x)}{dx} \right\} \right| = M_N^j$$

We have

$$M_N^j \rightarrow M^j = \sup_{x \in (0, S_X]} \left\{ \frac{dp_j(x)}{dx} \right\} = M^i \leq \infty$$

as $N \rightarrow \infty$.

We follow the same procedure for (Y_t) and we have the following

$$|U_t^N - \sum_{k=1}^{K_\epsilon(t)} (P_1^{T/N}(J_k^X))| \leq |M_N^1| \int_{|x| < \epsilon, s \in [0, t]} x J_X(dx \times ds)$$

$$|V_t^N - \sum_{l=1}^{L_\epsilon(t)} (P_2^{T/N}(J_l^Y))| \leq |M_N^2| \int_{|y| < \epsilon, s \in [0, t]} y J_Y(dy \times ds)$$

Now, let $N \rightarrow \infty$, for any fixed ω , we have

$$\left(\sum_{k=1}^{K_\epsilon(t)} P_1^{T/N}(J_k^X), \sum_{l=1}^{L_\epsilon(t)} P_2^{T/N}(J_l^Y) \right) \rightarrow (U_t(\epsilon), V_t(\epsilon)), a.s.$$

where $(U_t(\epsilon), V_t(\epsilon))$ is a compound Poisson process with characteristic function

$$\psi_\epsilon(u) = E[e^{i \langle u, (U_\epsilon^t, V_\epsilon^t) \rangle}] = \exp\left(t \int_{|x_i| > \epsilon, j=1,2} (e^{i \langle u, x \rangle} - 1) v(dx)\right)$$

where $v(dx) = f(f_1^{-1}(g_1(dx_1)), f_2^{-1}(g_2(dx_2)))$ by Proposition 2.2.7.

We have

$$|U_t - U_t(\epsilon)| \leq |M^1| \int_{|x| < \epsilon, s \in [0, t]} x J_X(dx \times ds)$$

$$|V_t - U_t(\epsilon)| \leq |M^2| \int_{|y| < \epsilon, s \in [0, t]} y J_Y(dy \times ds)$$

Now, let $\epsilon \rightarrow 0$,

$$\psi(u)_\epsilon \rightarrow \psi(u) = \exp\left(\int_{\mathbb{R}^2} (e^{i \langle u, x \rangle} - 1) v(dx)\right)$$

where $v(dx) = f(f_1^{-1}(g_1(dx_1)), f_2^{-1}(g_2(dx_2)))$ and also $|\int_{|x| < \epsilon, s \in [0, t]} x J_X(dx \times ds)| \rightarrow 0$

almost surely. So, we can see that $U_\epsilon^t \rightarrow U^t, V_\epsilon^t \rightarrow V^t$ in distribution by the

convergence of the characteristic functions, and therefore, $(U_t^N, V_t^N) \rightarrow (U_t, V_t)$ in distribution.

Q.E.D.

Remark: The condition for convergence in finite variation case is only a sufficient condition. One can actually drop the boundedness of the ratio of Levy densities on the marginals of the joint process versus the pre-specified marginals at the right limit of 0, and just keep the smoothness assumption.

2.4 Simulation Algorithm For DSPMD

In general, there are two ways to simulate Levy processes. One way is to simulate the i.i.d increment as random variables on small time intervals. The other way is to simulate the jumps directly. In one dimensional case, both methods are well-studied. For multi-dimensional case, a few special multi-dimensional Levy processes can be simulated in both ways. However, with Levy copula, Tankov proposed series representation method to simulate correlated jumps, which failed to control the error from the truncation of the series. Here, DSPMD is a way to simulate a multi-dimensional Levy process by correlating the increments by a regular copula on some time grid. The approximation is stable and on the marginals, it is always an exact simulation regardless of the time step size.

We will take a multi-dimensional Levy process and call it the joint process. It is required that the increment of the joint processes can be simulated. To extract and apply its copula on each small time interval, one can transform the increment

of the marginals of the joint process into uniforms and transform back to the pre-specified marginal Levy processes. We call it an approximation because the joint distribution of the increments are not infinitely divisible. However, as stated in the theorem, the DSPMD converges to a Levy processes as the time step size goes to zero. Notice that the marginal processes are always Levy processes regardless of the time step size.

By Theorem 2.3.2 and the discussion of the simplified version, we have the following simulation algorithm for Levy processes constructed with a regular copula.

Simulation of DSPMD for Levy processes

Let G_j^t be the T.I.P. of the pre-specified marginal Levy process. Let F^t be the T.I.P. of the pre-specified joint process. Let $Z(t_i)$ be the DSPMD for $i = 1, \dots, n$, with $h_i = t_{i+1} - t_i$

1. Simulate n independent random vector $(X_i^{h_i}, Y_i^{h_i})$ with $F^{h_i}(X, Y)$ where $h_i = t_i - t_{i-1}$.
2. $\Delta Z(t_i) = (G_X^{h_i^{-1}}(F_X^{h_i}(X_i^{h_i})), G_Y^{h_i^{-1}}(F_Y^{h_i}(Y_i^{h_i})))$
3. The discretized trajectory is given by $Z(t_i) = \sum_{k=1}^i \Delta Z_k$

Chapter 3

VG Copula and Stochastic Stressing of Gaussian Copula

In finance, physical dependence structure is of great importance. For example, one important issue in risk management is to find out the physical dependence structure and calculate the VaR from the physical measure. This procedure requires estimation, goodness of fit test, etc. Another example is that when pricing basket options, one may want to use the physical dependence measure and risk neutral marginal to price basket options. The same procedure is required. In short, when one wants to answer such questions as which Levy copula is right for the market, we offer the access to some type of copula on the random variable level so that one can conduct statistical inference procedure. This question was unanswered or avoided in previous literatures regarding Levy copula.

One of the purpose to use a regular copula to couple Levy processes is that one can easily perform the statistical inference on the joint structure. As shown in the previous chapter, if we choose the copula coming from some joint pure jump Levy processes, the limiting DSPMD is again a Levy process. One of the well known way to construct a joint Levy process is subordination to Brownian motion. In this class, we can name a few popular models: VG processes, CGMY processes, NIG processes, etc.

3.1 Statistical Property of VG Copula

In the general N-dimensional VG process, it has the following density function at time t :

$$f_t(X) = \int_0^\infty \phi(X; \theta g, \Sigma^2 g) \frac{g^{t/v-1} \exp(-g/v)}{v^{t/v} \Gamma(t/v)} dg$$

It is expressed as a N-D normal density function conditional on the realization of the gamma time change g . This is a semi-analytical form of the density function. Like its one dimensional version, it has a closed form in terms of modified Bessel function of the second kind. Let $A = \theta' \Sigma^{-1} \theta$, $B = X' \Sigma^{-1} X$, $C = \theta' \Sigma^{-1} X$

$$f(X) = \frac{2}{(2\pi)^{N/2} |\Sigma|^{1/2}} \Gamma(t/v) v^{t/v} e^C \left(\frac{2 + Av}{Bv} \right)^{N/4 - t/(2v)} Bes(N/2 - t/v, \sqrt{B(A + 2/v)})$$

The copula function can be obtained by the following procedure: Let $F_t(x, y)$ be the C.D.F of two dimensional V.G. processes at time t . And $C(u_x, u_y)$ be the VG copula function.

$$C(u_x, u_y) = F(F_X^{-1}(u_x), F_Y^{-1}(u_y))$$

For the purpose of statistical test, we would like to have the density function of the copula.

$$\frac{\partial^2 C}{\partial u_x \partial u_y} = \frac{\partial^2 F}{\partial x \partial y} \frac{\partial F_X^{-1}}{\partial u_x} \frac{\partial F_Y^{-1}}{\partial u_y}$$

$$\frac{\partial^2 C}{\partial u_x \partial u_y}(u_x, u_y) = \frac{\partial^2 F}{\partial x \partial y} \Big|_{u_x=F_X(x), u_y=F_Y(y)} \frac{1}{\frac{\partial F_X}{\partial x}} \Big|_{u_x=F_X(x)} \frac{1}{\frac{\partial F_Y}{\partial y}} \Big|_{u_y=F_Y(y)}$$

In the above formula, we need the two dimensional density, the one dimensional density and the inverse of one dimensional C.D.F of VG process at time t . Notice that this formula can be extended very easily to high dimensional case. It is simply the product of the joint density of VG and its marginal density.

Two dimensional density can also be obtained, in theory, by doing a two dimensional fourier transform from the characteristic function. In practice, we found it's very computationally intensive. Instead, we will use the bessel function form to calculate the points on the grid. The range and spacing of the grid depend on the data set and some accuracy criteria.

From the structure of VG copula, we can see that it is similar to Gaussian copula in the sense that it is based on a multivariate Gaussian. It also has tail dependence which comes from the common Gamma process. Higher volatility in the Gamma process implies higher tail dependence. Parameter θ dictates the skewness of the tail dependence. With a positive θ , one can have a higher tail dependence in the upper co-movement than the down co-movement. Various plot of the density function of VG copula shows these properties. Please see figures 3.1, 3.2, 3.3, 3.4, for details.

3.2 Stochastic Stressing of Gaussian Copula

Gaussian Copula is the copula implied by multi-variate Gaussian distribution. The correlation matrix uniquely determines the copula function. As a simplified version and for ease of use, the factor form of Gaussian Copula is popular in the industrial world and among a lot of academic work. It was firstly introduced to the financial world by David Li in 2000 [12] for modeling default times of a pool of credit names in CDO pricing.

To construct one factor Gaussian copula, let $Z_i, Z, i = 1, ..n$ be the i.i.d stan-

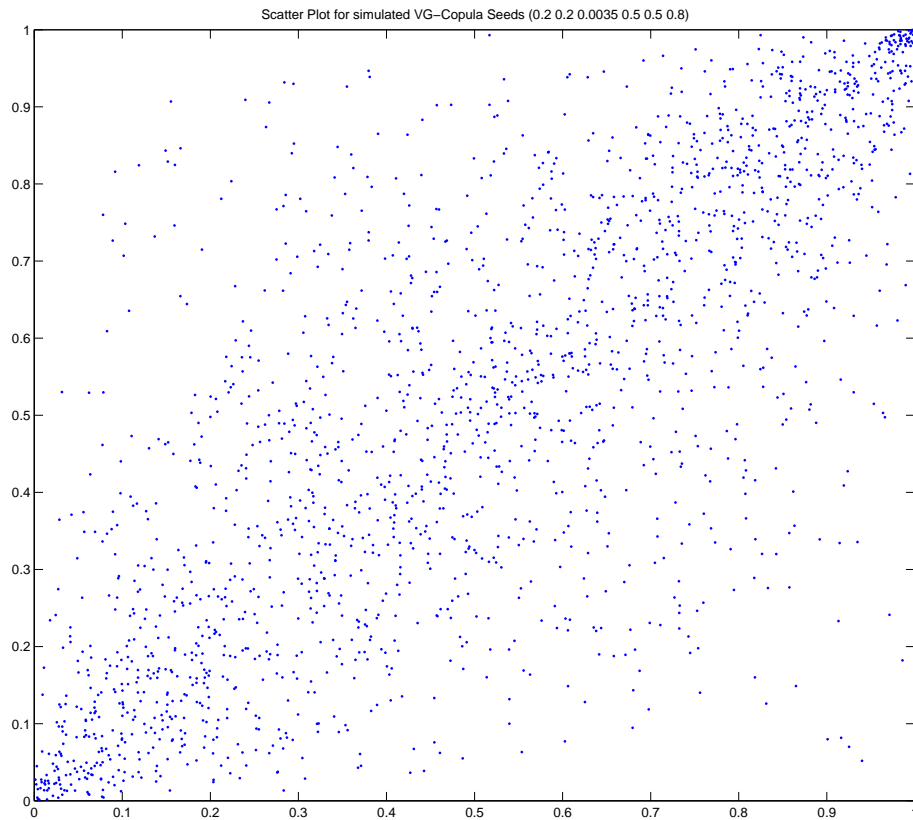


Figure 3.1: This figure is the scatter plot for simulated pairs of uniform random variable from VG copula with parameters $\theta_1 = \theta_2 = 0.2, \nu = 0.0035, \sigma_1 = \sigma_2 = 0.5, \rho = 0.8$. We can see that it is very similar to Gaussian copula simulated seeds except that there are more points gathering around the upper and lower corners.

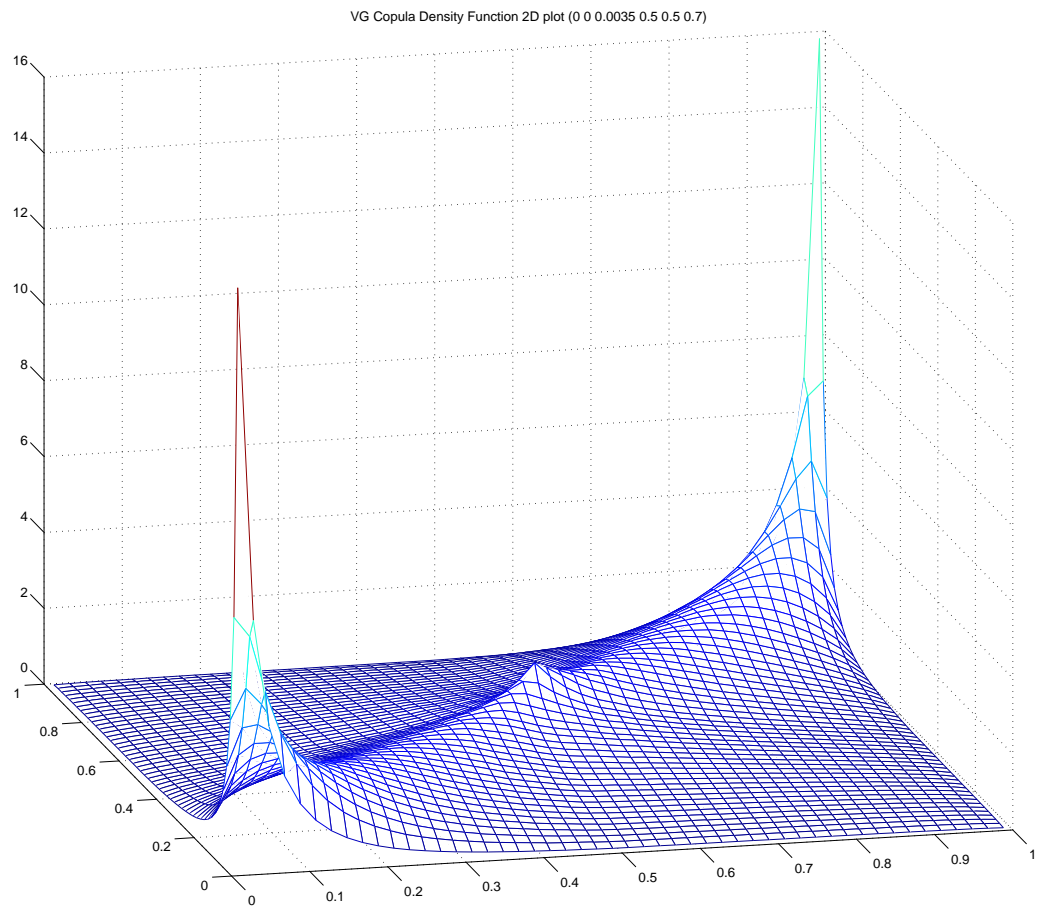


Figure 3.2: A 2D plot of VG Copula Density Function. Parameter value $\theta_1 = \theta_2 = 0, \nu = 0.0035, \sigma_1 = \sigma_2 = 0.5, \rho = 0.7$.

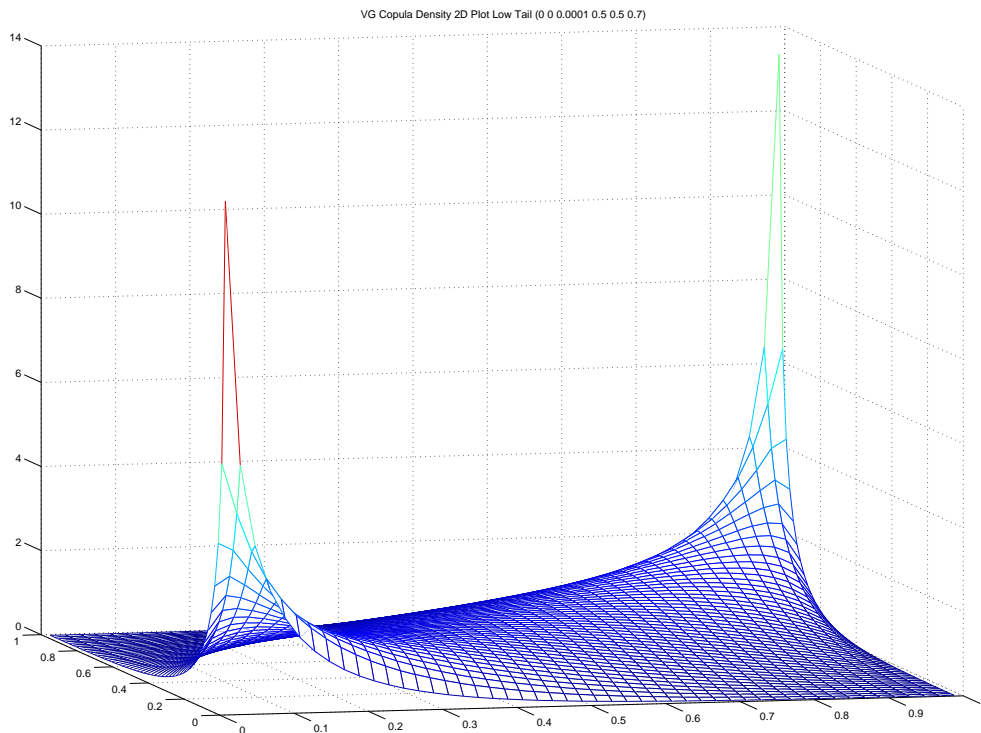


Figure 3.3: A 2D plot of VG Copula Density Function with Low Tail Dependence. The shape is very similar to that of Gaussian copula. Parameter value $\theta_1 = \theta_2 = 0, \nu = 0.0001, \sigma_1 = \sigma_2 = 0.5, \rho = 0.7$.

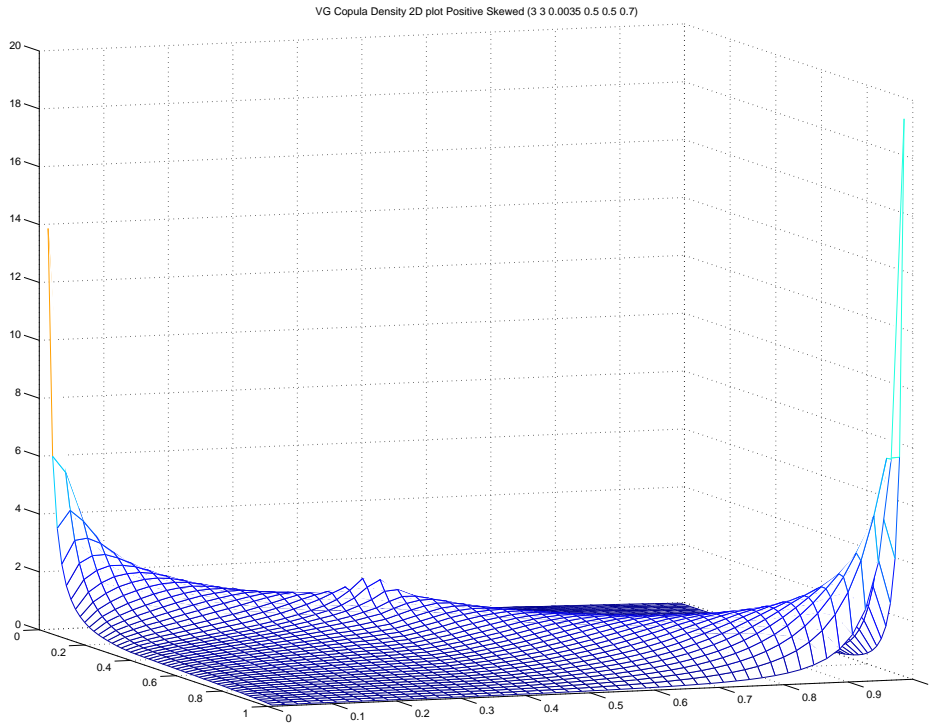


Figure 3.4: A 2D plot of VG Copula Density Function with Positively Skewed Tail Dependence. The upper corner has more probability mass than the lower corner when both θ are positive. Parameter value $\theta_1 = \theta_2 = 3, \nu = 0.0035, \sigma_1 = \sigma_2 = 0.5, \rho = 0.7$.

standard normal distribution.

$$X_i = \rho_i Z + \sqrt{1 - \rho_i^2} Z_i$$

Then, the C.D.F of $(\Phi(X_1), \dots, \Phi(X_n))$ is the one factor Gaussian Copula, where $\Phi(x)$ is the standard CDF of Gaussian distribution. In this construction, X_i 's are correlated normal random variables through the common factor Z . Here, conditional on the common factor Z , all X_i 's are independent. Conditional independence allows tractability in computing joint distribution or other expressions depending on the joint law. One factor Gaussian Copula can be viewed as a special parameterization of the correlation matrix in the general Gaussian Copula.

In the VG process, at any finite time, it is a VG distribution. As discussed previously, VG distribution can be viewed as a Gaussian distribution condition on a Gamma random variable as its volatility. So the Gamma random variable can be viewed as a stochastic stressing factor on the Gaussian Copula structure. We can construct a one factor VG copula as follows: let $Z_i, Z, i = 1, \dots, N$ be the i.i.d standard normal distribution and g be the Gamma random variable with mean 1 and variance ν .

$$X_i = \rho_i Z \sqrt{g} + \sqrt{1 - \rho_i^2} Z_i \sqrt{g}$$

Then, the C.D.F of $(F(X_1), \dots, F(X_n))$ is the one factor VG Copula, where $F(x)$ is the CDF of VG distribution with $(0, 1, \nu)$. Conditional on the common factor Z and g , all X_i 's are independent. We can have the same tractability as in Gaussian copula. To uncondition, one needs to integral out Z as well as g . We have two sources of dependence, Z offers the linear correlation, similar to Gaussian copula,

while g creates tail dependence. The one factor VG copula is uniquely determined by the correlation parameters ρ_i and ν . It is also a special parameterization of general VG copula.

We generalize the above method as **Stochastic Stressing of Gaussian copula**, when g is some heavy tailed positive random variable. Furthermore, when g is infinitely divisible distribution, the stochastic stressing of Gaussian copula has a corresponding Levy process in the form of subordination to Brownian motion, which can serve as the joint process of DSPMD.

To get back to the correlated uniforms, we need the marginal CDF for X_i . For tractability, most Levy process of this type has an explicit form in characteristic function and Fourier transform for one dimensional case is easy and fast. As for VG, closed form is also available.

3.3 Empirical Study of VG Copula For Multi-asset Return

The single asset return, i.e. the one dimensional problem, in equity has been well studied. The analysis of univariate time series data on financial asset shows that log returns are skewed and have heavy tails when compared with Normal distribution. Levy processes were introduced to model the single names due to its capability to model skewness and kurtosis, and for the most part, it still behaves similar to Brownian Motion. Among them, the Variance Gamma process by Seneta and Madan [23] successfully explains the physical returns on single name from historical data and also nicely captures the risk neutral measure from the option surface.

As for the case of multi-asset, evidence of tail dependence is present in the historical data. However, the dependence structure implied by multivariate Gaussian has zero tail dependence when ρ is not zero. From there, various types of copula structure were investigated, for example, Student's t copula, Clayton copula, etc. However, none of them can work with Levy process directly. In the mean time, multi-variate VG has a limitation on its marginals. As we discussed in the introduction, marginals of multidimensional VG process have similar kurtosis, which is not suitable to model multi-asset products.

Now, we have presented a new way to use multi-dimensional VG. We extract its copula structure on small time intervals and we proved that it is one of those regular copulas that construct infinitely divisible distribution. So it can work with Levy process. With all those tools, we are ready to answer this question: Is VG copula a good model for multi-asset returns?

We first model the marginals with VG process according to Madan Carr and Chang in [24]. The stock price dynamics is given by

$$S(t) = S(0)\exp(mt + X(t; \sigma_s, \nu_s, \theta_s) + \omega_s t),$$

where the subscript s on the VG parameters indicates that these are the statistical parameters, $\omega_s = 1/\nu_s \ln(1 - \theta_s \nu_s - \sigma_s^2 \nu_s / 2)$ and m is the mean rate of return on the stock under the statistical probability measure. Next, we are going to estimate the parameters in the VG component by MLE. The density function can be found by the closed form or FFT of the characteristic function. We found it fast by doing FFT. We pre-process the data by take the log return and de-mean it. In [24], it

indicates that the estimation of the mean of the daily return is not stable. So the estimated parameters do not include m . The estimated parameters for the marginals are reported in Table 3.1.

The second step is to transform all the daily log demeaned returns into uniform random variables. We use the estimated parameters to get the CDF and apply it on the log return. $u_i^j = F(r_i^j)$, where r_i is the i th daily log return for name j . (u_i^1, \dots, u_i^N) are uniformly distributed on the hyper-cube. This step is to remove the information from the marginals. The random vector (u_i^1, \dots, u_i^N) contains the dependence information only.

Now we can perform MLE to estimate VG copula using the transformed data on the hyper-cube. The density function of VG copula is given by the density function of the joint VG distribution and the inverse of its marginal CDF. There are four parameters in VG copula to be estimated. That is θ, σ, ν and ρ , the correlation matrix. In simulation experiments, where we perform MLE on simulated data points from VG copula to back out the parameters, we find that the estimation of θ, σ is not stable. It could be the reason that the likelihood function is pretty flat on these dimensions at some reasonable range. We do find that ν and ρ , the correlation matrix, can be estimated very easily and results are accurate and stable. So, in our estimation exercise, we fix the value of θ to be 0 and value of σ to be 0.5. The estimated parameters for the VG copula are reported in Table 3.2.

We use a Chi-squared test to test goodness-of-fit of VG copula against some of the most popular copulas available. In higher dimensions, one need a large number of data points to make the test valid and in most cases, it is not possible. We

carry out the test on a pair-wise fashion. We understand that chi-squared test is an approximate test and the accuracy and stability is subject to the size of the partition and the number of data points. We divide the unit square into 10x10, 15x15 and 20x20. We toss away the small squares where the expected number of observation is less 5. Like in the MLE, we did a simulation validation test to verify the effectiveness of chi-squared test for VG copula. Then, we do the test on the actual data. The results of the Chi-squared test of VG copula against a basket of popular copulas are reported in Table 3.3, Table 3.4, Table 3.5, Table 3.6, Table 3.7, Table 3.8. We also show, in the Figure 3.5, Figure 3.6, the plots of the estimated VG copula against the histogram of the transformed data on the unit square.

Chi-squared test on VG copula for pairs of equity returns show that it is a very competitive copula compared against Gaussian, Clayton, Frank, Gumble copula. Firstly, in all cases, Gaussian copula are strongly rejected in the Chi-squared test of 95% level. It is well known that Gaussian copula doesn't have tail dependence when $\rho < 1$ and the evidence of tail dependence is present in financial data. Failure to model the tail dependence leads to the rejection of the model. Clayton and other Archimedean copula were also rejected. Clayton copula has a strong tail dependence only in one corner and its ability to model linear correlation, which is of Gaussian type, is poor. From the histogram of pairs of equity names, a major part of dependence can be seen around the mean and is similar to that of Gaussian. Clayton copula has been studied in the context of multi-asset returns. For example, Madan [13] has studied rotated Clayton with Gaussian in the center. In credit derivative modeling, Clayton copula and other Archimedean copulas enjoyed some

success compared to Gaussian, please see [17] [16] for a comprehensive comparative analysis of various of Copulas in CDO pricing.

The only comparable copula to VG copula is Student's t copula. This is within our expectation. Student's t distribution is from the same family as VG distribution, which is generalized hyperbolic distribution[3]. Basically, Student's t as a distribution can be regarded as a normal random variable divided by a Gamma random variable, versus the case of VG, where it is a normal random variable multiplied by a Gamma random variable. From a Levy process perspective, Student's t is a Brownian motion subordinated by the inverse Gamma process. VG is a Brownian motion subordinated by Gamma process. Like VG copula, Student's t copula can be regarded as a stochastically stressed Gaussian copula, where the stressing is the reciprocal of Gamma random variable. Similarly, it has tail dependence in both upper and lower corner. The problem with Student's t in application is that the finite time distribution of its Levy process is not closed in the parametric family. It means that if one assumes the daily joint return is Student's t copula, then at other time horizon, the dependence structure is not of Student's t copula type. Nonetheless, one can get its distribution or copula from its characteristic function, but not trivial in computation. In VG, in all time horizon, the finite time distribution is closed in one parametric family, thus in practice it is much easier to use VG than Student's t. Here, we will also expect that other types of copula implied by the family of generalized hyperbolic distribution have similar performance on multi-asset return modeling, e.g. Normal Inverse Gaussian, etc.

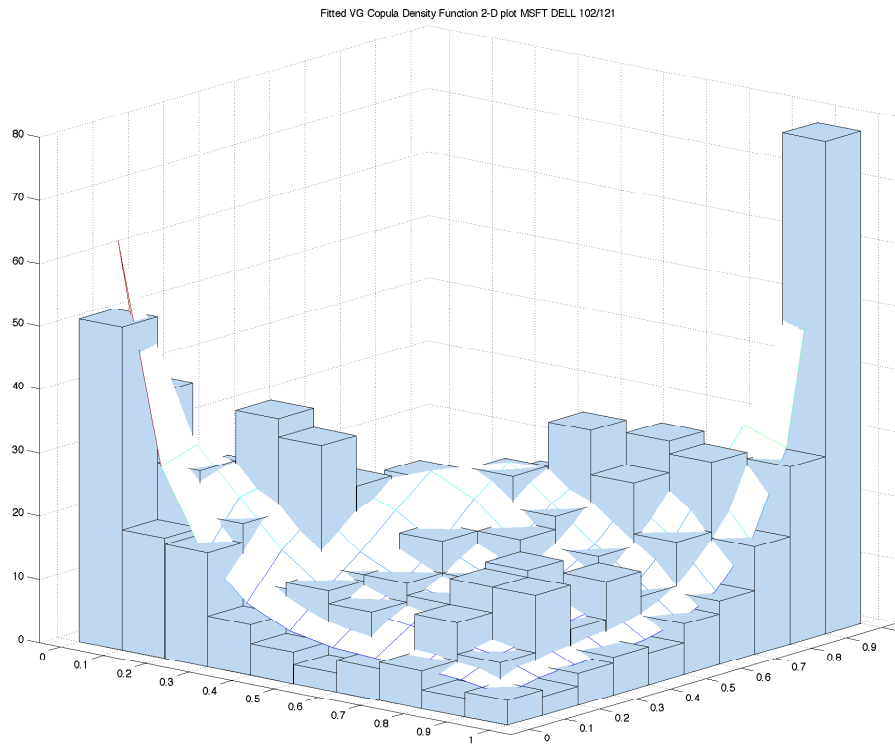


Figure 3.5: A 2D plot of VG Copula Density Function estimated using MLE on pairs of daily return from DELL and MSFT. The original data on which the MLE was performed is the pairs of daily return, shown here in the histogram. Estimated Parameters: ν and ρ . $\chi^2=102$. 95% critical value:121. Data Set: Data Set: 01/02/2001-12/29/2006.

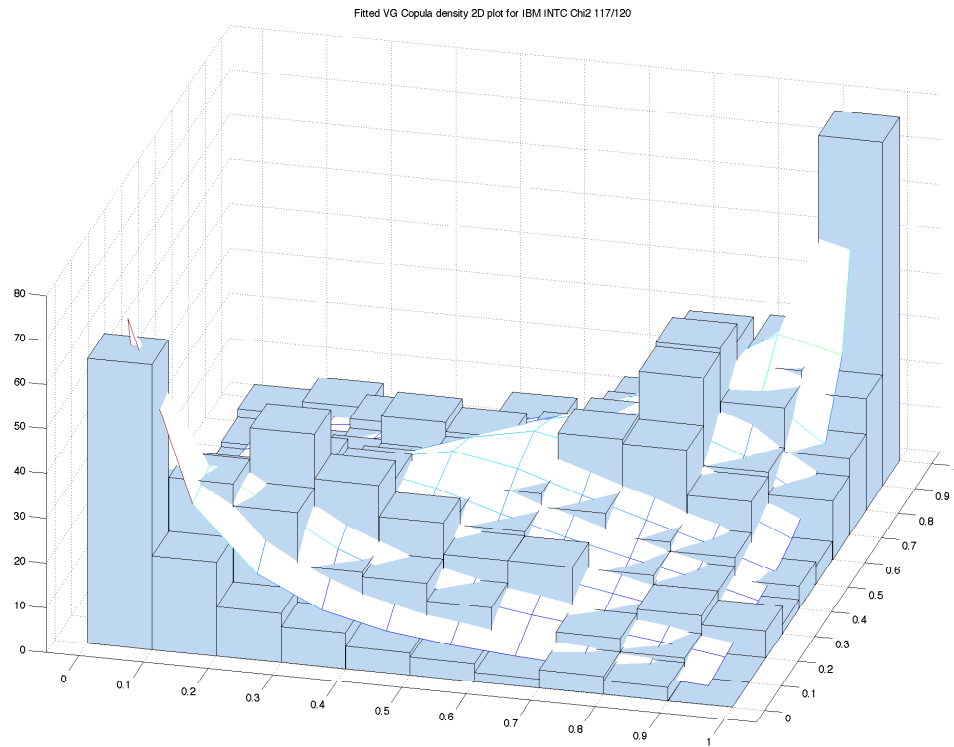


Figure 3.6: A 2D plot of VG Copula Density Function estimated using MLE on pairs of daily return from IBM and INTC. The original data on which the MLE was performed is the pairs of daily return, shown here in the histogram. Estimated Parameters: ν and ρ . $\chi^2=117$. 95% critical value:120. Data Set: Data Set: 01/02/2001-12/29/2006.

Table 3.1: MLE on the Marginal Distribution

| Ticker | θ | ν | σ |
|--------|----------|--------|----------|
| DELL | -0.8439 | 0.0052 | 0.3628 |
| IBM | -0.1606 | 0.0017 | 0.2637 |
| INTC | -0.9817 | 0.0028 | 0.4191 |
| MSFT | -0.9673 | 0.0035 | 0.2914 |
| GS | -0.2989 | 0.0024 | 0.2645 |
| JPM | -0.9757 | 0.0152 | 0.1416 |
| LEH | -0.9404 | 0.0028 | 0.3300 |
| MER | -1.0032 | 0.0029 | 0.3142 |
| NASDQ | -1.0102 | 0.0036 | 0.2943 |
| SPX | -0.0361 | 0.0031 | 0.1774 |

MLE on the marginal distribution for daily return. Data Set:
01/02/2001-12/29/2006

3.4 Pricing Basket Options Using VG Copula

The payoff of european basket options on n names is defined as $\max(S_1 + S_2 + \dots + S_n, K)$ at maturity T , which depends on the joint law of the underlying equity names. Dependence is a central part of such products. It has been approached in several ways. Xia [32] used linear combination of independent VG processes to model multi-asset and price basket options. They also used physical measure which is obtained by perform an ICA on the historical data. This methodology overcame the limitation of multi-dimensional VG processes for modeling multi-asset. However, one can not separate the dependence part from the marginal part easily. To order to keep the marginal process risk neutral, a measure change procedure was performed on the marginal processes. Also, it is unclear whether the dependence structure

Table 3.2: MLE for VG Copula on Pairs

| Pair | ρ | ν | Pair | ρ | ν |
|--------------|--------|--------|------------|--------|--------|
| DELL VS IBM | 0.5323 | 0.0026 | GS VS JPM | 0.6690 | 0.0024 |
| DELL VS INTC | 0.6570 | 0.0021 | GS VS LEH | 0.8255 | 0.0022 |
| DELL VS MSFT | 0.5659 | 0.0031 | GS VS MER | 0.8096 | 0.0023 |
| IBM VS INTC | 0.5945 | 0.0032 | JPM VS LEH | 0.6682 | 0.0027 |
| IBM VS MSFT | 0.6198 | 0.0036 | JPM VS MER | 0.7051 | 0.0026 |
| MSFT VS INTC | 0.6363 | 0.0027 | LEH VS MER | 0.7915 | 0.0019 |

Estimation for pair-wise daily return with VG Copula. $\theta = 0, \sigma = 0.5$
 Data Set: 01/02/2001-12/29/2006

implied by linear combination of VG processes is the right structure for multi-asset returns in the physical measure.

Here, we are going to use the general form of VG copula to price basket options. Previous sections have shown that VG copula is a good copula for describing the physical dependence. The basket options are priced by risk neutral marginals as VG processes and physical dependence modeled by VG copula. From there, we are going to focus on the impact of tail dependence and skewness from the dependence and leave the marginals unchanged.

For the marginal dynamics, we model the stock price processes by

$$S(t) = S(0)\exp(rt + X(t; \sigma, \nu, \theta) + \omega t),$$

where r is the risk-free interest rate. This process is regarded as the risk neutral process and the parameters are calibrated to the option surface. We follow [7] to use FFT procedure to obtain a set of out-of-the-money calls and puts and use some optimization procedure to minimize the L^2 norm of the difference between model

price and market price. Here, we only calibrate to option prices of one maturity. The maturity is chosen to be the closest date after the maturity of the basket option. VG distribution fits fairly well on most of the single names. We report the calibrated parameters on the marginals in Table 3.9

For the dependence, we follow the same procedure as mentioned in the previous section to get the physical dependence information from the MLE using VG copula at daily horizon. We report the estimated parameters on a basket of DELL, IBM, INTC and MSFt in Table 3.10. Then, we can scale the time horizon of the VG copula to the maturity of the basket option. From there, we use simulation to price basket options. We report the prices from simulation in the table 3.11 for out-of-the-money Call options and table 3.12 for out-of-the-money put options.

Table 3.3: Chi-squared Test on Copulas

| | DELL VS IBM | | IBM VS INTC | |
|-------------|-------------|----------------|-------------|----------------|
| Copula | χ^2 | Critical Value | χ^2 | Critical Value |
| VG | 136.8 | 123.2 | 119.0 | 123.2 |
| Student's t | 136.9 | 120.9 | 128.0 | 120.9 |
| Gaussian | 196.1 | 119.8 | 188.7 | 119.8 |
| Clayton | 306.8 | 122.1 | 296.1 | 122.1 |
| Frank | 222.1 | 122.1 | 214.4 | 122.1 |
| Gumble | 186.7 | 122.1 | 167.1 | 122.1 |

Comparison of Chi Squared Test on different types of copula function. All estimation are done by MLE on daily return. Data Set: 01/02/2001-12/29/2006

Table 3.4: Chi-squared Test on Copulas

| | DELL VS INTC | | IBM VS MSFT | |
|-------------|--------------|----------------|-------------|----------------|
| Copula | χ^2 | Critical Value | χ^2 | Critical Value |
| VG | 115.8 | 123.2 | 112.2 | 123.2 |
| Student's t | 122.6 | 120.9 | 117.7 | 120.9 |
| Gaussian | 147.8 | 115.3 | 206.3 | 119.8 |
| Clayton | 280.2 | 122.1 | 312.4 | 122.1 |
| Frank | 201.6 | 119.8 | 229.5 | 122.1 |
| Gumble | 175.0 | 119.8 | 185.4 | 122.1 |

Comparison of Chi Squared Test on different types of copula function. All estimation are done by MLE on daily return. Data Set: 01/02/2001-12/29/2006

Table 3.5: Chi-squared Test on Copulas

| | DELL VS MSFT | | INTC VS MSFT | |
|-------------|--------------|----------------|--------------|----------------|
| Copula | χ^2 | Critical Value | χ^2 | Critical Value |
| VG | 103.0 | 123.2 | 124.2 | 123.2 |
| Student's t | 116.7 | 120.9 | 126.9 | 120.9 |
| Gaussian | 176.9 | 119.8 | 183.0 | 119.8 |
| Clayton | 281.5 | 122.1 | 294.5 | 122.1 |
| Frank | 219.1 | 122.1 | 228.8 | 119.8 |
| Gumble | 169.5 | 122.1 | 177.9 | 119.8 |

Comparison of Chi Squared Test on different types of copula function. All estimation are done by MLE on daily return. Data Set: 01/02/2001-12/29/2006

Table 3.6: Chi-squared Test on Copulas

| | GS VS LEH | | | GS VS JPM |
|-------------|-----------|----------------|----------|----------------|
| Copula | χ^2 | Critical Value | χ^2 | Critical Value |
| VG | 95.2 | 123.2 | 135.3 | 123.2 |
| Student's t | 95.6 | 122.1 | 139.3 | 120.9 |
| Gaussian | 335.2 | 123.2 | 150.6 | 115.3 |
| Clayton | 368.4 | 123.2 | 352.3 | 122.1 |
| Frank | 240.0 | 123.2 | 232.7 | 119.8 |
| Gumble | 133.7 | 123.2 | 160.5 | 119.8 |

Comparison of Chi Squared Test on different types of copula function. All estimation are done by MLE on daily return. Data Set: 01/02/2001-12/29/2006

Table 3.7: Chi-squared Test on Copulas

| | GS VS MER | | LEH VS JPM | |
|-------------|-----------|----------------|------------|----------------|
| Copula | χ^2 | Critical Value | χ^2 | Critical Value |
| VG | 112.6 | 123.2 | 112.3 | 123.2 |
| Student's t | 112.1 | 122.1 | 122.3 | 122.1 |
| Gaussian | 348.1 | 123.2 | 244.3 | 123.2 |
| Clayton | 396.5 | 123.2 | 337.3 | 123.2 |
| Frank | 243.8 | 123.2 | 225.20 | 123.2 |
| Gumble | 152.3 | 123.2 | 152.9 | 123.2 |

Comparison of Chi Squared Test on different types of copula function. All estimation are done by MLE on daily return. Data Set: 01/02/2001-12/29/2006

Table 3.8: Chi-squared Test on Copulas

| | LEH VS MER | | LEH VS MER | |
|-------------|------------|----------------|------------|----------------|
| Copula | χ^2 | Critical Value | χ^2 | Critical Value |
| VG | 122.0 | 123.2 | 151.1 | 123.2 |
| Student's t | 126.2 | 122.1 | 147.8 | 122.1 |
| Gaussian | 240.9 | 123.2 | 333.6 | 123.2 |
| Clayton | 365.0 | 123.2 | 369.1 | 123.2 |
| Frank | 260.5 | 123.2 | 253.4 | 123.2 |
| Gumble | 155.5 | 123.2 | 210.0 | 123.2 |

Comparison of Chi Squared Test on different types of copula function. All estimation are done by MLE on daily return. Data Set: 01/02/2001-12/29/2006

Table 3.9: Calibrated Parameters For Marginal Processes

| | DELL | IBM | INTC | MSFT |
|----------|---------|---------|---------|---------|
| θ | -0.5626 | -0.4907 | -0.5766 | -0.3387 |
| σ | 0.2752 | 0.0758 | 0.1134 | 0.2783 |
| μ | 0.1071 | 0.1175 | 0.0971 | 0.1487 |

Data date: 12/26/2006 Calls and puts whose strikes are within 40% out of the money. Time-to-Maturity 207 days.

Table 3.10: Estimated Parameters For VG copula On the Basket

| | | | | |
|--------|----------|------|------|------|
| ν | 0.001148 | | | |
| ρ | DELL | IBM | INTC | MSFT |
| DELL | 1.00 | 0.63 | 0.73 | 0.66 |
| IBM | 0.63 | 1.00 | 0.68 | 0.70 |
| INTC | 0.73 | 0.68 | 1.00 | 0.71 |
| MSFT | 0.66 | 0.70 | 0.71 | 1.00 |

Data Set Data Set: 01/02/2001-12/29/2006. ν comes from the common Gamma part and ρ is the correlation matrix from the Gaussian part. We fix $\theta = 0$ $\sigma = 0.5$ in the MLE procedure.

Table 3.11: Basket Call Option Prices By Simulation

| Moneyiness | ATM | 10%OTM | 20%OTM | 30%OTM | 40%OTM | 50%OTM |
|------------------------------|---------|--------|--------|--------|--------|--------|
| $\nu = 0.0011 \theta = 0$ | 12.5430 | 4.8783 | 1.2798 | 0.2070 | 0.0202 | 0.0011 |
| $\nu = 0.0085 \theta = 0$ | 12.5185 | 4.8335 | 1.2622 | 0.2085 | 0.0242 | 0.0021 |
| $\nu = 0.0085 \theta = -0.1$ | 10.3111 | 3.7034 | 0.8834 | 0.1224 | 0.0114 | 0.0007 |
| $\nu = 0.0085 \theta = -0.2$ | 8.3309 | 2.7917 | 0.6005 | 0.0783 | 0.0057 | 0.0002 |

Table 3.12: Basket Put Option Prices By Simulation

| Moneyiness | 50%OTM | 40%OTM | 30%OTM | 20%OTM | 10%OTM | ATM |
|------------------------------|--------|--------|--------|--------|--------|--------|
| $\nu = 0.0011 \theta = 0$ | 0.0011 | 0.0182 | 0.1410 | 0.7278 | 2.6982 | 7.3389 |
| $\nu = 0.0085 \theta = 0$ | 0.0024 | 0.0183 | 0.1467 | 0.7453 | 2.7233 | 7.3389 |
| $\nu = 0.0085 \theta = -0.1$ | 0.0042 | 0.0359 | 0.2438 | 1.1214 | 3.6975 | 5.1316 |
| $\nu = 0.0085 \theta = -0.2$ | 0.0100 | 0.0707 | 0.3966 | 1.5936 | 4.8657 | 3.1513 |

Table 3.11 and 3.12 shows an array of out-of-the-money calls and puts with different strikes at one maturity. The first line of each table is the prices with physical dependence estimated from the historical data. Then we kept correlation matrix unchanged and vary the value of ν and θ to see the price change and sensitivity. Notice that we only change the information from the dependence not the marginals. This copula based pricing technique enables us to separate the dependence modeling from the marginal modeling. This kind of feature is desired in the practical use. In many cases, single names are very liquid assets and can be priced or hedged fairly easily. But the basket options are usually over-the-counter. No matter it is for hedging, pricing or risk management, it is fairly easy to handle the change coming from the marginals but difficult to deal with the ones from the dependence. With VG copula, it is helpful to understand various behaviors of the price dynamics coming from the dependence, such as linear correlation, tail dependence and skewness.

From the specification of the model, we understand that ν controls the level of tail dependence and θ controls the skewness. From the simulated prices, we can see that the prices of options are not sensitive to the change of ν unless it is deep out of the money, in which case both calls and puts are very sensitive to the change of ν . This again confirmed that ν changes the tail behavior of the sum of the stock price. All prices are sensitive to the skewness in the dependence. Negative value of θ implies downside co-movement are more likely to happen than the upside co-movement. The simulation experiments confirmed this effect. Notice that this doesn't change the skewness from the marginals.

Chapter 4

Simulation By Series Representation

Tankov proposed a simulation algorithm for Levy processes constructed by Levy copula based on the series representation of Levy processes. As the only simulation algorithm proposed by Tankov, it has a first component bias effect which suffers loss of jumps, especially in a nearly independent case. In this chapter, we are going to first review series representation of Levy process and its one dimensional simulation algorithm. Then, we are going to review Tankov's method and explain the source of the problem. At last, we propose SRLMD, series representation for Levy process with pre-specified marginals and pre-specified dependence. Examples of VG copula and α -stable Levy copula will be discussed in detail.

4.1 Simulation of Levy Processes By Series Representation

Any pure jump Levy process can be written as a sum of independent identically distributed jumps, according to Levy-Ito decomposition. Series representation of Levy process writes the Levy process as a sum of jumps ordered from large to small. When Levy process is compound Poisson process, the series is finite. When it is not finite variation, one needs to introduce a centering term to compensate the small jumps. For simplicity, we are going to deal with finite variation type, such as subordinators, and some finite variation α -Stable processes.

First, we will give the series representation of a subordinator, which serves as a basic model for other type of Levy processes.

Theorem 4.1.1 *Let $(Z_t)_{t>0}$ be a subordinator whose Levy measure $\nu(dx)$ has tail integral $U(x) = \int_x^\infty \nu(dx)$. Let Γ_i be a sequence of jumping times of a standard Poisson process and U_i be an independent sequence of independent random variable, uniformly distributed on $[0, 1]$, Z is representable in law, on the time interval $[0, 1]$, as*

$$Z_s = \sum_{i=1}^{\infty} U^{(-1)}(\Gamma_i) 1_{U_i \leq s},$$

The jumps, which are represented as $U^{(-1)}(\Gamma_i)$, are ordered from large to small. In simulation, when there are infinitely many jumps, one needs to truncate the infinite series. The order of the jumps in this series makes sure that the truncated tails only contains jumps smaller than the truncation level. With this property, one can carry out the same procedure as compound Poisson approximation of Levy process, where small jumps are replaced by Brownian motion. What one should not be confused with is that the actual timing of the jump is irrelevant to the order in the series. The timing of the jumps are uniformly distribution on the time interval $[0, 1]$.

In the actual implementation, the series must be truncated. The right thing to do is not to keep a fixed number of terms for each simulation path, but to fix some τ and keep a random number $N(\tau)$ of terms where $N(\tau) = \inf\{i : \Gamma_i \geq \tau\}$. One can see that $N(\tau)$ follows a Poisson distribution. It counts the number of jumps on $[0, 1]$ whose size is greater than $U^{(-1)}(\tau)$. In one dimensional case, there is no fundamental

difference from simulating the compound Poisson process that approximates the Levy process. Obviously, compound Poisson process can be simulated exactly in series representation.

To better understand series representation, one can think of the example of compound Poisson. Here Γ_i is the arrival time of a Poisson process. It is a basic result that the arrival time of a unit rate Poisson process on $[0, \tau]$ is the order statistic of uniform random variables on $[0, \tau]$. Here $U^{(-1)}(\Gamma_i)$ represents the jump size. Recall the inverse CDF approach to simulate a random variable. We call Γ_i s the Gamma seeds which are just uniform random variables except that they are ordered. In this way, $U^{(-1)}(\Gamma_i)$ can be viewed as the order statistics of the jumps sizes, since T.I.L of subordinators are monotonically decreasing.

To simulate a subordinator on $[0, T]$, one can follow the algorithm given below:

Set $i = 0$

While $\Gamma_i < \tau$

1. $i = i + 1$
2. $\Gamma_i^1 = \Gamma_{i-1}^1 + E_i$, where E_i is $exp(1/T)$
3. Generate V_i from Uniform $[0, T]$

The simulated path is given by

$$Z_t^k = \sum_{i=1}^{\infty} U_i^{(-1)}(\Gamma_i^k) 1_{V_i < t}, k = 1, 2$$

where U_i is the T.I.P of the Levy process.

Rosinski [27] showed a general series representation theorem for multi-dimensional Levy process. The series representation for Brownian motion subordinated by a subordinator can be obtained by applying Rosinski's result. More details can be found in [30].

Theorem 4.1.2 *Series representation for subordinated Brownian motion. Let (S_t) be a subordinator without drift and with continuous Levy measure v and let $X_t = W(S_t)$, where W is a d -dimensional Brownian motion, be a subordinated Levy process on \mathbb{R}^d . The characteristic triplet of (X_t) is $(0, 0, v_X)$ with $v_X(A) = \int_0^\infty \mu^{s\Sigma}(A)v(ds)$ where μ is the probability measure of a Gaussian random variable with mean 0 and variance $s\Sigma$, where Σ is the correlation matrix. The tail integral of v is given by $U(x) = \int_x^\infty v(dt)$. Let V_i to be a series of independent standard normal random vectors. Let Γ_i be a sequence of jumping times of a standard Poisson process and U_i be an independent sequence of independent random variable, uniformly distributed on $[0, 1]$. we have the series representation for X_t*

$$X_t = \sum_{i=1}^{\infty} \sqrt{U^{-1}(\Gamma_i)} V_i 1_{U_i \leq t}$$

For one dimensional case with Brownian motions with drift μ and volatility σ , we have

$$X_t = \sum_{i=1}^{\infty} (\sqrt{U^{-1}(\Gamma_i)} \sigma V_i + U^{-1}(\Gamma_i) \mu) 1_{U_i \leq t}$$

We conclude that the jumps from X_t can be represented as $(\sqrt{U^{-1}(\Gamma_i)} \sigma V_i + U^{-1}(\Gamma_i) \mu)$. On the marginals, jumps are not ordered from the large to small, but $\sqrt{U^{-1}(\Gamma_i)}$ are still ordered from large to small. In the case of VG, $U^{-1}(x)$ is the inverse of tail integral of the Gamma Levy measure, which exponentially decays. In

simulation, there is some nonzero probability of loss of jumps that are larger than the cutoff value. With the exponential decay of jumps from the subordinator, one can easily reduce the probability of loss of jumps by simulating just a few more terms. In other words, the probability of loss beyond the truncation level can be controlled. Future work in this direction involves the study of the distributional property or quantitative property of the error.

4.2 Series Representation For Levy Copula And First Component Bias

In the series representation, in order to correlate the jump event and jump sizes, one should start with constructing correlated Γ_i from different components directly. Tankov showed the following result to generate correlated Γ_i sequence. For more details, see [6] Theorem 5.5 and [20].

Theorem 4.2.1 *Let (X_t^1, \dots, X_t^N) be a N -dimensional Levy process with positive jumps, having marginal T.I.L. U_j for $j = 1, \dots, N$ and Levy copula F , Let ΔX_t^i be the sizes of jumps of the i th components at time t . Define*

$$F_{x_1}(x_1, \dots, x_2) = \frac{\partial}{\partial x_1} F(x_1, x_2, \dots, x_N)$$

Then, if U_1 has a non-zero density at x , $F_{U_1(x)}$ is the distribution function of random vector $(U_i(\Delta X_t^i))_i$ conditionally on $\Delta X_t^1 = x$

$$F_{U_1(x)}(x_1, \dots, x_2) = P(U_2(\Delta X_t^2) \leq x_2, \dots, U_N(\Delta X_t^N) \leq x_N | \Delta X_t^1 = x)$$

So given the Γ_i^1 , the conditional distribution of $\Gamma_i^2, \dots, \Gamma_i^N$ is $F_{x_1}(x_1, x_2, \dots, x_N)$. So from there, Tankov proposed the following procedure: Γ_i^1 are generated as unit rate Poisson jump times as usual. Then Γ_i^j for $j = 2, \dots, N$ are generated from the conditional joint distribution function given the Γ_i^1 . In a two dimensional case, Γ_i^2 can be simulated using inverse CDF method directly.

The above idea is summarized in the follow theorem by Tankov.

Theorem 4.2.2 *Let ν be the Levy measure on R^d for Levy processes Z_t , with marginal tail integrals $U_k, k = 1, \dots, d$ and Levy copula $F(x_1, \dots, x_d)$. Let $\{V_i\}$ be a sequence of independent random variables, uniformly distributed on $[0, 1]$. Introduce d random sequence $\{\Gamma_i^1\}, \dots, \{\Gamma_i^d\}$, independent from $\{V_i\}$.*

- $\{\Gamma_i^1\}$ is the jump times of a Poisson process at unit rate.
- Conditionally on Γ_i^1 , the random vector $(\Gamma_i^2, \dots, \Gamma_i^d)$ is independent from Γ_i^k with $j \neq k$ and all k and is distributed with $\frac{dF}{dx_1}(x_1, \dots, x_d)$

Then, in law, on time interval $[0, 1]$

$$(Z_t^k) = \sum_{i=1}^{\infty} U_k^{(-1)}(\Gamma_i^k) 1_{V_i \leq t}, k = 1, \dots, d$$

This method is very general. It works for all types of Levy copula functions. However, it does have some issues in the actual implementation. Let's take a two dimensional case as an example. In all simulation algorithms, the series must be truncated as described in the first section. The first component in this algorithm is just like any other one-dimensional series representation. Γ_i^1 are strictly increasing and the jump sizes are strictly decreasing. When the tail of the series is truncated,

only small jumps are tossed away and the error are well controlled. Especially, if it is compound Poisson, the simulation is exact. For the second component, however, Γ_i^2 are not strictly increasing at all, hence the jumps sizes in the series are NOT ordered from large to small. What's worse is that if it is nearly independent, for example θ in Clayton Levy copula is close to 0, small values of Γ_i^1 corresponds to large values of Γ_i^2 . If translated in terms of jump sizes, large jumps on first component correspond to small jumps on the second components. In such a case, we know that the large jumps are all kept in the tail of the series rather than the first $N(\tau)$ terms. Truncation of the series will lead to big loss of jump mass. This is true even for simulating compound Poisson processes. We call it first component bias, because only the first component is simulated well while other components suffer from uncontrolled loss of jump mass.

In addition to the first component bias, there are some other issues. For high dimensional case, the random vector $(\Gamma_i^2, \dots, \Gamma_i^d)$ is hard to sample. The general way would be recursively using the conditional probability argument, which is computationally intensive. One would hope to find some factor structure in the conditional joint distribution function. Efforts have been made in this direction for some special case such as Clayton Levy copula. In [10], a transformation is found to broil down the conditional distribution function from Clayton levy copula to a regular Clayton copula, in which case an one factor structure was discovered. However, this won't change the first component bias effect

In the next section, we propose our method and introduce two examples.

4.3 SRLMD

In order to avoid first component bias effect and the high-dimensional difficulty, we introduce SRLMD, series representation of Levy processes with pre-specified marginals and pre-specified dependence. Instead of working with the explicit form of Levy copula function. We start out with the series representation of some multi-dimensional Levy process, where the pre-specified dependence is defined. Examples of such series representation include subordinated Brownian motion and α -stable Levy process, which will be covered in detail in the later sections. Then we perform a transformation on the jump sizes by applying its marginal T.I.L to get correlated Gamma Seeds in the intensity space. At last, we use the correlated Gamma Seeds to generate the correlated jumps by the inverse T.I.P of the pre-specified marginals. The validity of such method is explained by the Sklar's Theorem for Levy copula and Rosinski's Theorem. We can see that this scheme is in spirit the same as DSPMD except that this is done on the infinitesimal level and the resulting process is an exact Levy process not an approximation. We summarize the above idea in the following theorem.

Theorem 4.3.1 *The SRLMD for W_t on $R^2 \times [0, 1]$ with pre-specified marginal with finite variation pure jump T.I.L G_X and G_Y and pre-specified dependence implied by Z_t is given by*

$$W_t^X = \sum_{i=1}^{\infty} G_X^{-1}(H_X(J_i^X))1_{U_i \leq t}$$

$$W_t^Y = \sum_{i=1}^{\infty} G_Y^{-1}(H_Y(J_i^Y))1_{U_i \leq t}$$

where Z_t is a two-dimensional finite variation pure jump Levy processes with joint T.I.L $H(x, y)$ and marginal T.I.L $H_X(x), H_Y(y)$, and its jumps on $[0, 1]$ occurred at U_i can be represented as (J_i^X) and (J_i^Y) .

Idea of the proof:

By definition, the T.I.L of W_t can be written as $F(x, y) = H(H_X^{-1}(G_X(x)), H_Y^{-1}(G_Y(y)))$.

This is a direct result from Sklar's Theorem. From there, Rosinski's result can be applied to obtain the series representation of W_t , and convergence is guaranteed by Rosinski's Theorem

Q.E.D

In the infinite variation case, the series representation introduces a centering term. We refer to [27] for more details.

With the above theorem, the simulation algorithm for subordinated Brownian motion Levy copula is readily available.

Simulation of jumps from W_t on $[0, 1]$, a finite variation pure jump Levy process with joint T.I.L $F(x, y) = H(H_X^{-1}(G_X(x)), H_Y^{-1}(G_Y(y)))$, where $H(x, y)$ is the joint T.I.L of a Brownian motion subordinated by a subordinator with T.I.L U . G_X and G_Y are the marginal T.I.L of W_t

While $\Gamma_i < \tau$

1. $i = i + 1$
2. $\Gamma_i^1 = \Gamma_{i-1}^1 + E_i$, where E_i is $exp(1/T)$
3. Generate (V_i^X, V_i^Y) from multivariate Normal distribution with zero mean and

covariance matrix Σ ,

4. Generate $U_i \sim Uniform[0, T]$.

$$W_t^X = \sum_{i=1}^{\infty} G_X^{-1}(H_X(\sqrt{U_i^{(-1)}}(\Gamma_i)V_i^X))1_{U_i \leq t}$$

$$W_t^Y = \sum_{i=1}^{\infty} G_X^{-1}(H_Y(\sqrt{U_i^{(-1)}}(\Gamma_i)V_i^Y))1_{U_i \leq t}$$

This algorithm can be easily extended to higher dimensional case. When the subordinator is a Gamma process, we get the series representation for VG Levy copula.

In this case, we have already avoided the problem of high dimensionality difficulty in Tankov's algorithm. Also, we don't have first component bias because this algorithm treat every component equally. However, this is not the best way to show this property, since subordinated Brownian motion Levy copula do not have a complete independence case. Even if the the Brownian motion part is independent, the resulting Levy processes is still correlated through the common subordinator. Next, we are going to construct a special case of α -stable Levy copula such that it include complete dependence and complete independence.

4.4 α -stable Levy Copula and SRLMD

In this section, we are going to introduce a special case of α -Stable Levy process which can be served as the joint process embodying the pre-specified dependence for SRLMD. It has the advantage over Tankov's Levy copula function in the following aspects: First, it is naturally high dimensional. Second, the structure is so general

that it allows from complete dependence to complete independence and can have any regular copula behavior built in. Thirdly, and most importantly, in any case, the truncation error can be controlled and simulation efficiency does not deteriorate in nearly independence case. For compound Poisson processes as pre-specified marginals, zero truncation error can be attained.

4.4.1 Construction of α -Stable Levy Process And Its Levy Copula

Recall that the Levy measure of α -stable process has the following decomposition form. For any $B \subset R^d$, the Levy measure of α -stable ν can be written as

$$\nu(B) = \int_{R^d} \lambda(d\xi) \int_0^\infty 1_B(r\xi) \frac{dr}{r^{1+\alpha}}$$

where λ is the probability measure in R^d and $\alpha \in (0, 2)$

For $\alpha \in (1, 2)$, α -stable is not finite variation if the probability measure in R^d is asymmetric around the origin. For simplicity, we limit ourselves to the case of finite variation and let $\alpha \in (0, 1)$. We have the following result by LePage, see [4]:

Theorem 4.4.1 *Let X_i be a sequence of independent and identically distributed random variables on R^d with the distribution λ . Let Γ_i be a sequence of arrival times of a Poisson process of unit rate. V_i be an independence sequence of independent random variables, uniformly distributed on the interval $[0, 1]$. Then the α -stable process admits the following series representation, with $\alpha \in (0, 1)$:*

$$\sum_{i=1}^{\infty} \Gamma_i^{-1/\alpha} X_i 1_{V_i \leq t}$$

We firstly interpret the structure of the series representation of α -stable. The i th term in the series is the jump size, $\Gamma_i^{-1/\alpha}$, from an one dimensional α -stable Levy process, which we call the Central Process, multiplied by a random vector $X_i \in R^d$ from probability measure λ . Given a jump event i , i.e. the i th term in the series, the components of X_i projects the jump size on the central process to the jumps of the marginal processes by scaling. One can suppress the jump on a particular marginal j by setting the value of X_i^j close to 0. Likewise, one can enforce a jump on a marginal j by setting the value of X_i^j away from 0. Also, the correlation on the jumps can be induced by not only the central process but also the regular copula implied by random vector X_i .

We specify the distribution of $\{X_i\}$ in the following way. j follows a discrete uniform distribution of $[1, 2, \dots, d]$. Let X_i be iid random vector distributed as B_j random vector, where each component is a Z distribution random variable correlated by a regular copula Y , except for the j th component being 1. Z is some distribution on $[-1, 1]$ for general case and $[0, 1]$ for positive jumps only. In such a construction, each realization of X_i , a randomly selected component j is 1. All other components follows some joint distribution with marginal distribution Z and copula Y . For simplicity, we only discuss the case of pure positive jump case. The general case can be followed similarly.

So the marginal p.d.f of X_i 's component is given by

$$f(x) = \frac{1}{d}\delta_1(x) + \frac{d-1}{d}f_Z(x)$$

where $f_Z(x)$ is the p.d.f of Z . Let ν be the Levy measure of the central process and

J be the jumps from the central process. The marginal T.I.P of the α -stable process is given by

$$U(x) = \nu(XJ > x) = \nu(J > x/X) = \int_{x/X}^{\infty} \nu(dx)$$

So, the Levy measure conditioning on X scaling given by

$$\nu(dx|X) = 1/X\nu(x/X)dx$$

If X follows distribution $P(x)$, then the marginal Levy measure is given by

$$\nu_X(dx) = \int \frac{1}{X}\nu(d\frac{x}{X})dP(X)$$

For $\nu(dx) = \frac{A}{x^{\alpha+1}}dx, \alpha \in (0, 1)$, we have

$$\nu_X = \frac{A}{x^{\alpha+1}} \int X^\alpha dP(x)$$

which is still a α -stable Levy measure with a different constant. In our case, X has a p.d.f of

$$f(x) = \frac{1}{d}\delta_1(x) + \frac{d-1}{d}f_Z(x)$$

so the Levy density of the marginal of X is given by

$$\nu_X(x) = \frac{A}{x^{\alpha+1}}\left(\frac{1}{d} + \frac{d-1}{d}E_Z[X^\alpha]\right)$$

The T.I.L is given by

$$U_X(t) = \frac{A}{\alpha x^\alpha}\left(\frac{1}{d} + \frac{d-1}{d}E_Z[X^\alpha]\right)$$

In order to construct SRLMR, we are ready to recover the correlated Gamma seeds from the correlated jumps from α -stable process.

$$\Gamma_i^* = U_X\left[\left(\frac{A}{\alpha\Gamma_i}\right)^{1/\alpha}X_i\right] = \Gamma_i X_i^{-\alpha}\left(\frac{1}{d} + \frac{d-1}{d}E_Z[X^\alpha]\right)$$

4.4.2 Error Bound For Truncated Series Representation

The most important advantage of using this method is that we can work out an explicit error bound for the truncated series representation which Tankov failed to do in his method. We overcome this problem by doing two things right. First, for each jump event, we randomly sample a component from the random vector X and assign 1. Second, we choose our distribution Z to be bounded on finite interval. The reason for this is due to the fact that if the scaling factor X is bounded by 1, jump size can be only scaled down, not scaled up. After truncation on the central process, jumps from the truncated tail are no larger than the specified level. In other words, jumps of sizes over a certain level are guaranteed to be in the first $N(\tau)$ terms, although the jump sizes in general are not ordered. For a certain truncation level on the jump size of the processes being simulated, we find the corresponding truncation level for the central process. If compound Poisson process are correlated using this Levy copula, the series representation can actually be exact. We consider the series representation for α - process and number of terms $N(\tau) = \max\{i : \Gamma_i < \tau\}$. For $i > N(\tau)$, i.e. $\Gamma_i > \tau$, we have

$$U^{-1}(\Gamma_i) < U^{-1}(\tau)$$

Random vector $X \sim B_j$ is bounded from above by 1 uniformly. So, component-wise, we have

$$U^{-1}(\Gamma_i)X < U^{-1}(\tau)X < U^{-1}(\tau)$$

For the Gamma seeds vector Γ_i^* , we have component-wise

$$\Gamma_i^* = U_X(U^{-1}(\Gamma_i)X) > U_X(U^{-1}(\tau)) = \tau\left(\frac{1}{d} + \frac{d-1}{d}E_Z[X^\alpha]\right)$$

Let λ be the intensity of the compound Poisson process as the pre-specified marginal process. If $\tau(\frac{1}{d} + \frac{d-1}{d}E_Z[X^\alpha]) > \lambda$, Γ_i^* will be mapped to no jumps. So for $\tau > \lambda/(\frac{1}{d} + \frac{d-1}{d}E_Z[X^\alpha]) > \lambda$, the simulation is exact for compound Poisson process. For infinite activity case, the truncation on the series will be exact to the level of compound Poisson process approximation.

4.4.3 Dependence and Independence in α -stable Levy copula and Efficiency

The stable index α , the mean and variance of distribution Z and the copula parameters control the dependence level of the Levy copula.

The nearly independence case is treated as a limiting scenario of complete independence. For each jump event i , the jump size of the randomly sampled component i is not altered while other components are suppressed to be small jumps. In order to realize that, the mean of distribution Z should be set to be low. The variance is set to control the dispersion. The dependence inside random vector X_i should be low as well. If the marginals are compound poisson process, jumps on the α - stable process are mapped to no jumps if the jumps size is below some threshold.

The nearly dependence case is treated as a limiting scenario of complete dependence. In such a case, at one jump event, jumps across the components are almost identical. The mean of distribution of Z should be close to 1 and the dependence inside random vector X_i should be high. In this way, the randomly picked component has almost the same scaling factor as all other components.

Of course, for exactly independence and complete dependence, we can just assign 0 and 1 for all other components, instead of sampling from Z distribution. However, such cases are easy without series representations, too.

Put together the understanding from the error bound, we make a remark on the reason why we choose to always first pick a random component of X_i to be 1. Generally speaking, for each jump event, we want to make sure there is at least one component jumping if it jumps at all. For nearly independence case, the mean is set to be very close to 0, implying $E[X^\alpha]$ being close to 0. The truncation level for λ is given by

$$\tau = \lambda / \left(\frac{1}{d} + \frac{d-1}{d} E[X^\alpha] \right)$$

which is bounded by $d \times \lambda$. Without the special random picking procedure, the error bound is

$$\tau = \lambda / E[X^\alpha]$$

which goes to infinity in the nearly independence case. So by picking the random component of X_i to be 1, one can greatly improve the simulation efficiency.

4.4.4 Examples of Series representation For α -Stable Levy Copula

Here we illustrate a concrete example of α -stable Levy copula and its series representation. We choose Z to be Kumaraswamy distribution on $[0, 1]$. It is as versatile as Beta distribution and it has an analytical inverse CDF function which makes the use of the copula easy¹. The p.d.f of Kumaraswamy distribution is given

¹This is exactly the reason why we didn't choose Beta distribution with which people are more familiar. Beta distribution does not have an analytical form of inverse CDF

by

$$f(x; a, b) = abx^{a-1}(1-x)^{b-1}$$

where $a > 0, b > 0$, and it admits a simple form of CDF

$$F(x; a, b) = 1 - (1-x)^b$$

This made easy the inverse transform method to simulate Kumaraswamy random variables. In order to correlate these Kumaraswamy random variables, we will first generate d -dimensional uniform random vectors using some copula function. The moments of Kumaraswamy distribution are given by

$$m_n = \frac{b\Gamma(1+n/a)\Gamma(b)}{\Gamma(1+b+n/a)} = bBeta(1+n/a, b)$$

So the marginal T.I.L of the α -stable Levy process is given by

$$U_X(x) = \frac{A}{\alpha x^\alpha} \left(\frac{1}{d} + \frac{d-1}{d} m_\alpha \right)$$

And the implied correlated Gamma seeds are given by

$$\Gamma_i^* = \Gamma_i X_i^{-\alpha} \left(\frac{1}{d} + \frac{d-1}{d} m_\alpha \right)$$

Now, we are ready to give the simulation algorithm.

Main routine:

While $\Gamma_i < \tau$

1. $i = i + 1$
2. $\Gamma_i^1 = \Gamma_{i-1}^1 + E_i$, where E_i is $exp(1/T)$
3. Generate $j \sim$ discrete uniform $[1, 2, \dots, d]$

4. Generated d -dimensional random vector $X_i(\theta) \sim K_j(\theta)$, where the j -th component is 1, and all other components are identically distributed as Kumaraswamy(a,b) and they are correlated by *Copula*(θ) with parameter θ . See sub-routine for details. a and b are given by the system of equation $m_1(a, b) = \mu$, $m_2 - m_1^2 = \sigma^2$
5. The i th Gamma seeds vector is given by

$$\Gamma_i^* = \Gamma_i X_i^{-\alpha} \left(\frac{1}{d} + \frac{d-1}{d} m_\alpha \right)$$

6. Generate $V_i \sim Uniform[0, T]$, the jump time for the i th term in the series/
 i th jump event.

END

The truncation for compound Poisson process with intensity λ as marginal processes is given by $\tau = \lambda / \left(\frac{1}{d} + \frac{d-1}{d} m_\alpha \right)$

Sub-routine for generate correlated Kumaraswamy random vectors using one factor Marshal-Olkin Copula

1. Generate $V \sim exp(\alpha)$, and $\tilde{V} \sim exp(1 - \alpha)$, $i = 1, \dots, d$
2. Generate the correlation uniforms by $U_i^* = exp(-min(V, \tilde{V}))$
3. $X_i = (1 - (1 - U_i^*)^{1/b}) \times 1/a$, for $i = 1, \dots, d$ gives the correlated Kumaraswamy random variables with Marshall-Olkin copula

END

To simulate correlated Kumaraswamy random vectors using regular copula is not our focus in this paper as it is a very standard application of Sklar's theorem and regular copula. The above algorithm is for illustrative purpose. A brief description can be found in [16]

In conclusion, this algorithm has a wide range of application. It is suitable for modeling CDO pricing. CDO, collateralized debt obligation, is a tranching pool of credit names as underlying. It is a high-dimensional problem in nature and requires dynamic modeling on the marginals, as well as the dependence. The dependence implied from the traded tranches shows complicated skewness and heavy tail behavior. α -stable Levy copula, flexibility and versatility show great potential to model this kind of products.

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