ABSTRACT

Title of Dissertation: ITERATIVE METHODS FOR
THE STOCHASTIC DIFFUSION PROBLEM

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It is the purpose of this thesis to develop iterative methods for solving the linear systems that arise from application of the stochastic finite element method to steady-state stochastic diffusion problems. Although the theory herein is sufficiently general to be applicable to a variety of choices for the stochastic finite elements, attention is given to the method of polynomial chaos. For the second-order problem a multigrid algorithm is defined wherein the spatial discretization parameter is varied from grid to grid while the stochastic discretization parameter is held constant. It is demonstrated that the convergence rate of this method is independent of the discretization parameters. For the first-order problem, which produces a linear system that is symmetric and indefinite, the MINRES algorithm is applied with a preconditioner that incorporates a multigrid algorithm. This multigrid algorithm, as for the one applied to the second-order problem,

varies the spatial discretization from grid to grid while holding the stochastic discretization parameter constant. Again, it is demonstrated that the convergence rate of this method is independent of the discretization parameters.

ITERATIVE METHODS FOR
THE STOCHASTIC DIFFUSION PROBLEM

by

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DEDICATION

To Mum and Dad, for reasons which need not be stated.

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Chapter 1

Introduction

It goes without saying that partial differential equations (PDEs) are used ubiquitously in mathematics to model physical phenomena. In this thesis the PDE known as the (steady-state) diffusion equation will be considered. It is given by

$$-\nabla \cdot (c\nabla u) = f \quad \text{in } D$$

where c will be called the diffusion coefficient and f will be called the source function. This is an elliptic PDE and for the existence of solutions should be accompanied by boundary value conditions.

The diffusion equation can be used to model numerous physical processes. Perhaps most simply it can be viewed as giving the steady-state solution of the density of a fluid whose motion has been governed entirely by diffusion (i.e. the flow of fluid from higher density regions to lower density regions). In this context the diffusion equation can be seen to arise from considering conservation of mass with the temporal derivative being omitted as a steady-state solution is sought and the advection term being omitted by assumption. The source function, f, appears as there may be sources (or sinks) of fluid in the spatial domain, D.

Therefore, one sees that the diffusion equation will be incorporated into models of fluid flow through porous media. As discussed, for example, in [11], porous

media are inherently heterogeneous and, furthermore, generally defy precise description due to lack of information regarding the properties of the media. Such lack of information may arise from unavoidable ignorance of the system under investigation or the impracticality of understanding the structure of the media at the minute level over which heterogeneity occurs. One therefore sees that the diffusion coefficient can often not be described with certainty. The coarsest approach to resolving this problem would involve setting $c\colon D \to \mathbb{R}$ to be the average (or expected value) of the diffusion process. In this case the diffusion coefficient will be called a deterministic field (as its value at each $x \in D$ is completely determined, i.e. there is no randomness involved). The model, likewise, will be called the deterministic problem (assuming, of course, that f too is a deterministic field). A more sophisticated approach would be to try and incorporate the uncertainty into the problem by modelling the diffusion coefficient as a random field, i.e. $c\colon D \times \Omega \to \mathbb{R}$ where Ω is a sample space. In this case the resulting problem will be stochastic. In particular, the diffusion equation will now be a stochastic partial differential equation (SPDE) given by

$$-\nabla \cdot (c\nabla u) = f \quad \text{in } D \times \Omega.$$

The source function too may be a random field. In this thesis, however, the diffusion coefficient will be considered to be the cause of uncertainty. In the succeeding chapters the source function will be considered to be a random field as this is convenient for purposes of analysis and incorporates the fact that f may be deterministic as a special case.

In modelling the random nature of the diffusion coefficient it is generally considered to have known mean, variance, and covariance functions. Such a

process will possess a Karhunen-Loève expansion given by

$$c(x, \xi(\omega)) = c_0(x) + \sum_{r=1}^{\infty} \sqrt{\lambda_r} c_r(x) \xi_r(\omega)$$

where c_0 is the mean function of c, (ξ_r) is a sequence of uncorrelated random variables, and (λ_r, c_r) is a sequence of eigenvalue-eigenvector pairs associated with the covariance function. In particular, if the variance of c is taken to be a constant given by ν, then each ξ_r has mean zero and variance ν, and if the covariance function of c is given by $r(x, y)$, then each λ_r and c_r satisfy

$$\int_D \frac{r(x, y)}{\nu} c_r(y)\, dy = \lambda_r c_r(x) \qquad r = 1, 2, \ldots$$

The Karhunen-Loève expansion is discussed in the abstract setting of probability theory in [24] and in a context in accordance with the present discussion in [20]. (Note that many authors follow the convention of assigning the variance of ξ_r to be unity and including a factor of $\sigma = \sqrt{\nu}$ in front of the above summation to take into account the variance of c.) A common assumption, as made, for example, in [20], is that c is a Gaussian process and (ξ_r) is thus described by a sequence of continuous random variables with normal distributions, i.e. $\xi_r \sim \mathcal{N}(0, \nu)\ \forall r$. Consequently the random variables in (ξ_r) will possess the desirable property of being not only uncorrelated but also independent. Unfortunately though, this leads to an ill-posed problem as a diffusion coefficient thus defined will not be bounded between two positive constants for all possible realizations, a property that it must possess for the PDE to be well defined. It is generally assumed in such cases that providing the variance is sufficiently small, sensible solutions will be obtained (e.g., see [20] and [41]). Indeed, the discrete systems that arise from numerical methods are well defined and seemingly do produce sensible results for such sufficiently small variance. An alternative suggestion, as made, e.g.,

in [41], is to choose distributions of ξ_r that ensure c is bounded between two positive constants with probability one. One possible choice, for example, would be to let each ξ_r be uniformly distributed on some appropriately chosen finite interval. However, the independence of the random variables is consequently not guaranteed by virtue of the properties of the Karhunen-Loève expansion, and as this property is required for certain numerical methods may need to be assumed explicitly. The correct way to model c is still an area of active discussion. In this thesis the numerical experiments carried out will use random variables taken to be uniformly distributed and normally distributed alternatively, while the theory will assume that the diffusion coefficient is bounded between two positive constants almost everywhere.

The simplest method (at least when implemented in the most naive way) for obtaining a numerical approximation to the solution of the stochastic diffusion problem is the Monte Carlo method. This involves generating a prescribed number of realizations of the diffusion coefficient and solving the resulting set of deterministic PDEs. If the diffusion coefficient is a Gaussian process, and hence normally distributed at each point in the spatial domain, then this method is particularly easy to implement as realizations can be generated by matrix decomposition methods without taking into account the Karhunen-Loève expansion, as described in [12]. The disadvantage of the Monte Carlo method lies in the fact that it will be expensive to solve so many realizations of the problem. However, as well as being easy to implement, it is readily parallelizable and efficient methods for solving the deterministic problems are well studied. Moreover, and importantly, one gets a lot of information back from the method. That is, given that a finite number of realizations of the solution have been generated, not only

can one easily compute statistical moments but probability distributions at each point are readily available. This may lend the Monte Carlo method an advantage over the more sophisticated methods discussed below.

Another method that can be used to solve the stochastic diffusion problem is the perturbation method. This essentially involves expanding the random fields (the diffusion coefficient and the solution) in a Taylor series about their mean values. If the diffusion coefficient is expressed as a Karhunen-Loève expansion as given above, then to implement this method it is first required to curtail the expansion after, say, m terms. The remaining terms then constitute perturbations of the diffusion coefficient about its mean. The solution too, by virtue of the Doob-Dynkin Lemma (this notion will be elaborated on in Chapter 2) will be expressible as a function of the same random variables, i.e. $u = u(x, \xi_1, \ldots, \xi_m)$. The solution is then expanded about its mean in a Taylor series with each random variable constituting a perturbation. The Taylor series is then curtailed and plugged into the PDE. Equating terms of the same order then yields a series of problems that can be solved successively to obtain the functions in the expansion of the solution. This is discussed in more detail in [20]. The major drawback of this method is that it becomes overly complicated if the variance of the diffusion is large as more terms need to be included in the expansion of the solution.

To try and overcome the computational cost of the Monte Carlo method and the inherent weakness of such methods as the perturbation method, attention in recent years has turned to a class of methods collectively described as stochastic finite element methods. These methods first reformulate the SPDE as a variational problem, the solution of which lives in the tensor product of a space of deterministic fields (e.g., $H_0^1(D)$ for the diffusion problem given above with ho-

mogeneous Dirichlet boundary value conditions, as discussed in Chapter 2) and the space of random variables given by $L_2(\Omega)$ (i.e. the set of random variables that are square integrable and hence possess a variance). Given that a finite dimensional subspace, of dimension N, say, of the deterministic space has been chosen using the theory of deterministic finite elements, then it remains to choose a finite dimensional subspace, of dimension M, say, of $L_2(\Omega)$ which here will be denoted by T. The finite element approximation is then obtained by solving the variational problem posed on the tensor product of these finite-dimensional spaces which has dimension MN, which is to say, the finite element approximation can be obtained by solving a linear system of dimension MN.

One suggestion for T is the set spanned by the polynomial chaos restricted in order, to say n, and in dimension, to say m. This idea was put forward by Ghanem and Spanos (see [20]) in the case when the diffusion coefficient was a Gaussian process and hence the random variables in the Karhunen-Loève expansion are normally distributed. In this method the Karhunen-Loève expansion is curtailed after m terms and the basis of T is chosen to be the set of polynomials of degree n or less with arguments $\xi_1, \ldots, \xi_m$ that are orthogonal with respect to the underlying probability measure associated with the random vector $\xi = (\xi_1, \ldots, \xi_m)$, which in this case is an m-dimensional Gaussian distribution, i.e. given $T = \text{span}\{\chi_1, \ldots, \chi_M\}$, then

$$\int_\Omega \chi_k(\xi)\chi_l(\xi)\, dP = \int_{\mathbb{R}^m} \chi_k(y)\chi_l(y)\, \frac{e^{-y^2/(2\nu)}}{(2\pi\nu)^{m/2}}\, dy = \kappa_k \delta_{kl},$$

where $y = \mathbb{R}^m$ and ν is the variance of ξ_r, $r = 1, \ldots, m$. Consequently, the basis of T will be the set of m-variate Hermite polynomials of degree n or less. The

dimension of T will be given by

$$M = \frac{(m+n)!}{m!n!}.$$

As mentioned above, if the random variables in the Karhunen-Loève expansion are normally distributed, then the problem is not well-posed though for sufficiently small variance, well-defined discrete systems may result. One approach to resolve this problem has been to replace the normally distributed random variables with random variables whose distributions ensure that the diffusion coefficient remains bounded between two positive constants. This can be achieved, for example, by letting the random variables be uniformly distributed on some appropriately chosen interval. The polynomial chaos method outlined above was generalized by Xiu and Karniadakis (see [40]) to take into account various possibilities for the distributions of ξ_r. In this scheme, the case that the random variables are uniformly distributed will result in the basis of T being the set of m-variate Legendre polynomials of degree n or less.

The important thing to note here, is that in obtaining the weak formulation the differential equation was integrated over Ω (as will be shown in the following chapter) with the consequence that, by utilizing the fact that an integral over Ω with respect to the probability measure is computed as an integral over $\mathbb{R}^m$ with respect to a weight function equal to the density function of $\xi = (\xi_1, \ldots, \xi_m)$, the random fields, including the solution, are expressed as functions of $d + m$ real arguments where d is the dimension of the spatial domain. Therefore, in some sense, the d-dimensional stochastic problem has been transformed into a $(d+m)$-dimensional deterministic problem. As noted above, this ultimately results in a linear system of dimension MN, i.e. the finite element approximation to the solution is obtained by solving a matrix problem where the system matrix is in

$\mathbb{R}^{MN\times MN}$. It turns out, though it is far from transparent at this stage, that this matrix can be expressed as an $M\times M$ block matrix, each block being of size $N\times N$. The structure within each block will be determined by the spatial discretization (i.e. the nature of the mesh on D and the choice of spatial basis functions) while the block structure itself will depend on the stochastic discretization. As a consequence of the Karhunen-Loève representation of the diffusion coefficient and the three-term recurrence relation that orthogonal polynomials satisfy, the choice of the polynomial chaos as a basis for T results in a sparse block structure that is illustrated, e.g., in [28] and also in Figure 2.1 later in this thesis.

Other choices for the basis functions of T have been suggested. One prominent suggestion, made in [3], is to use doubly orthogonal polynomials. These, again chosen after the Karhunen-Loève expansion of the diffusion coefficient has been curtailed after m terms, satisfy

$$\int_\Omega \chi_k(\xi)\chi_l(\xi)\,dP = \delta_{kl}, \quad \int_\Omega \xi_r\chi_k(\xi)\chi_l(\xi)\,dP = \kappa_{rk}\delta_{kl}, \quad r = 1,\ldots,m.$$

The consequence of this choice is that the resulting system matrix will be block diagonal which makes the method easily parallelizable. The polynomials are chosen such that they are of degree n_r in their rth argument. If $n_r = n$, $r = 1,\ldots,m$, then the dimension of T is given by

$$M = (1+n)^m.$$

For a given choice of m and n the doubly orthogonal basis functions gives a larger value of M (much larger as n increases) than does the polynomial chaos. A comparison between the doubly orthogonal polynomials and polynomial chaos concerning the amount of work required to achieve a given accuracy has not yet, as far as the author is aware, been made.

It would be pertinent to point out at this point that though the above discussion of the stochastic finite element method assumed the random variables in the Karhunen-Loève expansion of the diffusion coefficient to be independent, this may not in fact be the case. Indeed, as mentioned above, independence needs to be explicitly assumed as the theory only guarantees that the random variables are uncorrelated. In this thesis independence will always be assumed, but mention will now briefly be made to how one might proceed if the assumption is not made. Each random variable in the Karhunen-Loève expansion, as discussed e.g., in [26], can be expanded using the polynomial chaos, i.e. each ξ_r can be written as an expansion of Hermite polynomials of increasing dimension and increasing degree and whose arguments are independent Gaussian random variables. The Karhunen-Loève expansion is then curtailed by only using Hermite polynomials with m arguments or less and choosing an upper limit for the degree of those polynomials. The basis of T is then chosen as for the polynomial chaos method, i.e. it is chosen to be the m-variate Hermite polynomials of degree n or less. The resultant system matrix will again be an $M \times M$ block matrix but now the sparsity of the block structure will depend on the degree of the polynomials kept in the polynomial chaos expansion of c. If only linear terms are included the expansion will look like a curtailed Karhunen-Loève expansion and the system matrix will exhibit the same block structure. However, as higher degree terms are retained in the expansion the block structure will become denser. Density plots of this block structure are given in [26] and [13].

Another method that should be pointed out in this context is the stochastic collocation method as advocated in [2]. This method too fits into the stochastic finite element framework, only here the SPDE is first expressed in a semi-discrete

weak formulation where only the deterministic part of the problem has been discretized and the differential equation has only been integrated over the spatial domain, D. The resulting equation is then collocated on the zeros of orthogonal polynomials and the discrete solution is obtained by interpolating the collocated solutions. This method results in a set of decoupled deterministic problems, the number of which corresponds to the number of orthogonal polynomials used in the collocation stage. The dimension of the stochastic space, i.e. T, that the solution lives in is the same as for the doubly orthogonal polynomials described above. The advantage of this method, it is argued, is that it allows a wider variety of characterizations of the random diffusion coefficient to be tackled, including the case when the random variables in its Karhunen-Loève expansion are not independent.

An important point that needs to be made regarding the stochastic finite element method regards what one obtains when the finite element approximation has been computed. This approximation is a random field on $D \times \Omega$ that is given by an expansion in the basis functions of T with the coefficients being deterministic functions residing in whatever space was used for the spatial discretization. But what does this tell one about the nature of the solution? As will become clear later, the statistical moments of the approximation, such as the expectation and the variance, can be trivially obtained from the expansion of the approximation. Therefore, if the stochastic finite element formulation of the problem can be solved efficiently it offers a fast way to obtain approximations to the statistical moments of the solution to the underlying SPDE. However, things are not so clear if one wants to compute probability distributions of the solution. For example, what is the probability that the solution is greater than some given

value at some given point in D? In order to obtain this information from the finite element approximation it would seem that the approximation needs to be sampled, i.e, a Monte Carlo method needs to be applied. This poses an interesting question, namely, given that a probability distribution of the solution to an SPDE is required, is it more efficient to use the Monte Carlo method where in each trial a deterministic PDE requires to be solved but once the trials are computed the desired distribution is readily available, or is it more efficient to use the stochastic finite element method which requires one much larger system to be solved and produces a solution that will still need to be sampled but with each trial now being trivial to carry out? This question, again to the author's knowledge, has not yet been answered.

The above provides a brief introduction to the stochastic finite element method; it is not the intention here to given an exhaustive overview. (For such an overview see [36] or [23].) Suffice to say that the method has gained much attention over the past decade or so as a possibly efficient method to obtain statistical information concerning the solution of PDEs whose defining characteristics (here the diffusion coefficient but possibly other components such as the boundary value conditions (e.g, see [15]), domain shape (e.g, see [9]), etc.) are uncertain. However, the linear system that results from the method's application does have the potential to be very large and is of a different character to its deterministic counterpart. As mentioned above, it is an $M \times M$ block matrix where each block resembles that arising from the analogous deterministic problem. One is therefore lead to ask how such a system may be solved efficiently. Little work has been done thus far that addresses this question. It is the purpose of this thesis to make some preliminary steps in this direction and to investigate how some of the iterative

methods that are known to perform well for the deterministic diffusion problem perform when suitably extended and applied to the stochastic problem.

Chapters 2 and 3 in this thesis deal with the diffusion equation as stated at the beginning of this chapter with homogeneous Dirichlet boundary value conditions. The spatial domain is taken to be two-dimensional, though the theory is readily extendible to three spatial dimensions. The diffusion coefficient is assumed to take the form of a curtailed Karhunen-Loève expansion with independent and identically distributed random variables. It is known that multigrid is an optimal (iterative) method for solving the linear system associated with the analogous deterministic problem, where by optimal it is meant that the contraction factor of the method, i.e. the upper bound associated with the convergence of the error, is independent of the spatial discretization. The consequence of this is that the number of iterations it requires for multigrid to reach a given tolerance will not depend on how fine the underlying mesh used to discretize the problem is. This idea is extended to the stochastic problem by defining a multigrid algorithm where the spatial discretization is varied from level to level while the stochastic discretization is held constant. It is shown theoretically and numerically that this algorithm is optimal with regard to the spatial discretization. Moreover, providing the stochastic basis functions are suitably scaled it will be optimal with regard to the stochastic parameters, m and n. This algorithm was also employed in [25], though the work presented here differs by also providing theoretical support to the numerical results. Also, a Fourier mode analysis was used to examine a similar multigrid algorithm in [35].

In Chapters 4 and 5 a first-order formulation of the diffusion problem is considered. This involves rewriting the second-order PDE as previously stated as a

system of two first-order PDEs. As there is now two unknowns to solve for this gives rise to mixed finite elements. The stochastic finite element method as given above can readily be extended to give a mixed stochastic finite element method. The resulting linear system has a system matrix with 2×2 blocks, each of which are themselves $M \times M$ blocks analogous to those discussed above. The diffusion coefficient in this case appears in the integrals in the $(1,1)$ block as c^{-1}. The assumption is made that c^{-1} can be expanded in a desirable Karhunen-Loève expansion. If this is not the case, then a polynomial chaos representation could be employed though this is not considered here. (Note however that if a polynomial chaos expression is used and only the linear terms are kept then this is equivalent to using a Karhunen-Loève expansion.) The system matrix in this case is symmetric and indefinite which suggests the use of the MINRES algorithm as given in [27]. To be efficient the system will need to be preconditioned. The preconditioner employed follows that implemented in [32] for the deterministic problem which in turn was motivated by the discussion in [1]. This deterministic preconditioner is block diagonal with the $(2,2)$ block being itself diagonal. The MINRES algorithm is required to carry out a solve with this preconditioner in each iteration. To carry out a solve with the $(1,1)$ block a multigrid algorithm that makes use of an additive Schwarz smoother defined with respect to a domain decomposition is implemented. In [1] it was shown that for the case of $c = 1$ this preconditioned MINRES algorithm will be optimal with respect to the spatial discretization. In this thesis this preconditioner is extended to take into account the stochastic nature of the problem. The multigrid algorithm, as with the second-order problem, is defined such that the spatial discretization is varied from level to level while the stochastic discretization is held constant. The anal-

ysis in [1] is extended to show that for a random diffusion coefficient (bounded between two positive constants) the iterative scheme thus defined is optimal with respect to the discretization parameters.

A word should be said concerning the software used to implement the algorithms discussed above. For the second-order case a suite of software was developed in MATLAB for the solution of the stochastic diffusion problem via the method of polynomial chaos. This was developed as an extension of MATLAB's PDE toolbox and consequently incorporated the methods therein for discretizing the deterministic part of the problem. The software ultimately could solve stochastic problems posed in two spatial dimensions on a domain constructed using MATLAB's PDE toolbox GUI with a variety of boundary value conditions. For the first-order case another suite of software was developed in MATLAB, again using the method of polynomial chaos for the solution of the stochastic diffusion problem. This software was an extension of the code developed by Bahriawati and Carstensen (see [4]) for the solution of the analogous deterministic problem. The author here wishes to express his gratitude to the aforementioned for making this code freely available. As with the code for the second-order problem, this code could solve stochastic diffusion problems in two spatial dimensions on a domain constructed using MATLAB's PDE toolbox GUI, only here the boundary value conditions were restricted to the homogeneous Dirichlet type.

Chapter 2

Second-Order Stochastic Diffusion Problem

In this chapter the stochastic steady-state diffusion problem is introduced along with its weak formulation. The problem is referred to as the second-order formulation so as to distinguish it from the first-order formulation discussed in chapter 5, though this terminology won't be used when the context is clear.

2.1 Boundary Value Problem

The stochastic diffusion equation with homogeneous Dirichlet boundary value conditions is given by

$$\begin{cases} -\nabla \cdot (c\nabla u) = f & \text{in } D \times \Omega, \\ u = 0 & \text{on } \partial D \times \Omega, \end{cases} \tag{2.1}$$

where D is the spatial domain, Ω is a sample space, $c\colon D \times \Omega \to \mathbb{R}$ is the diffusion coefficient, and $f\colon D \times \Omega \to \mathbb{R}$ is the source function. The sample space in turn belongs to a probability space $(\Omega, \mathcal{F}, P)$ where $\mathcal{F}$ is a σ-algebra and P is a probability measure.

The spatial domain, D, is assumed to be a two-dimensional simply connected bounded open set with piecewise smooth boundary. In particular, D is taken to

be the interior of a convex polygon.

The diffusion coefficient is assumed to be of the form of a curtailed Karhunen-Loève expansion, i.e.

$$c(x, \xi(\omega)) = c_0(x) + \sum_{r=1}^{m} \sqrt{\lambda_r} c_r(x) \xi_r(\omega) \tag{2.2}$$

where $\xi = (\xi_1, \ldots, \xi_r)$, ξ_r, $r = 1, \ldots, m$, are assumed to be continuous, independent, and identically distributed random variables, and (λ_r, c_r) are obtained by solving an eigenvalue problem in the form of a Fredholm integral equation, as discussed, e.g., in [20]. The σ-algebra $\mathcal{F}$ is defined to be the minimal σ-algebra generated by ξ, i.e. $\mathcal{F} = \sigma(\xi)$. As the source function f is a random field on $(\Omega, \mathcal{F})$ it follows from the Doob-Dynkin lemma (see [8]), that f can be expressed as a Borel function of ξ, i.e. $f = f(x, \xi)$. Similarly, if the existence of the solution u can be established, it too will be expressible as a Borel function of ξ, i.e. $u = u(x, \xi)$. Now, given $\xi_r(\Omega) = \Gamma \subset \mathbb{R}$, $r = 1, \ldots, m$, and denoting the density function of ξ_r by ρ_r, then for any Borel function $g = g(y)$ it follows that

$$\int_{\Omega} g(\xi(\omega))\, dP = \int_{\Gamma^m} g(y) \rho(y)\, dy \tag{2.3}$$

where $y \in \mathbb{R}^m$, $\rho(y) = \rho_1(y_1) \cdots \rho(y_m)$, and $dy = dy_1 \cdots dy_m$.

This problem is extensively discussed from a modeling perspective in [20] and from an analytic perspective in [3].

2.2 Lebesgue and Sobolev Spaces

Prior to stating the weak formulation of the boundary value problem given in §2.1 and pursuing the subsequent analysis, it is necessary to introduce the required *Lebesgue* and *Sobolev* spaces and their tensor products. Strictly speaking these

function spaces contain equivalence classes of functions but throughout this thesis their elements will simply be referred to as functions. Lebesgue and Sobolev spaces are introduced more formally in [19]. The tensor products of such spaces is discussed in [38] and [3].

In the following the notation

$$\int_D v = \int_D v(x)\,dx, \qquad \int_\Omega v = \int_\Omega v(\xi)\,dP = \int_{\Gamma^m} v(y)\rho(y)\,dy$$

will be used, where v is a Borel function of its argument on the relevant space, and in the second expression (2.3) has been used. (Recall that all measurable functions on $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is the minimal σ-algebra generated by ξ, can be expressed as Borel functions of ξ as a consequence of the Doob-Dynkin lemma.) Moreover,

$$\int_\Omega \int_D v = \int_\Omega \int_D v(x,\xi)\,dx\,dP = \int_{\Gamma^m} \int_D v(x,y)\rho(y)\,dx\,dy.$$

where v is a random field.

2.2.1 Spaces $L_2(D)$, $L_2(\Omega)$, and $L_2(D) \otimes L_2(\Omega)$

The Lebesgue spaces $L_2(D)$ and $L_2(\Omega)$ are defined by

$$L_2(D) = \left\{ v\colon D \to \mathbb{R} \mid \int_D |v|^2 < \infty \right\}, \tag{2.4}$$

$$L_2(\Omega) = \left\{ v\colon \Omega \to \mathbb{R} \mid \int_\Omega |v|^2 < \infty \right\}. \tag{2.5}$$

The inner products associated with these spaces are defined by

$$(v,w)_{L_2(D)} = \int_D vw, \qquad (v,w)_{L_2(\Omega)} = \int_\Omega vw. \tag{2.6}$$

These inner products induce norms given by

$$||v||_{L_2(D)} = \left(\int_D |v|^2 \right)^{\frac{1}{2}}, \qquad ||v||_{L_2(\Omega)} = \left(\int_\Omega |v|^2 \right)^{\frac{1}{2}}. \tag{2.7}$$

The spaces $L_2(D)$ and $L_2(\Omega)$ equipped with these inner products are Hilbert spaces, i.e. they are complete inner product spaces.

The space $L_2(D) \otimes L_2(\Omega)$ is given by

$$L_2(D) \otimes L_2(\Omega) = \{\, v \colon D \times \Omega \to \mathbb{R} \mid \int_D \int_\Omega |v|^2 < \infty \,\}. \tag{2.8}$$

with associated inner product

$$(v, w)_{L_2(D)\otimes L_2(\Omega)} = \int_D \int_\Omega vw \tag{2.9}$$

and norm

$$||v||_{L_2(D)\otimes L_2(\Omega)} = \left(\int_\Omega \int_D |v|^2 \right)^{\frac{1}{2}}. \tag{2.10}$$

Note that $v \in L_2(D) \otimes L_2(\Omega)$ can be considered as a function in $L_2(D)$ parameterized by $\omega \in \Omega$, or vice versa, as a function in $L_2(\Omega)$ parameterized by $x \in D$.

2.2.2 Spaces $L_\infty(D)$, $L_\infty(\Omega)$, and $L_\infty(D) \otimes L_\infty(\Omega)$

The spaces $L_\infty(D)$ and $L_\infty(\Omega)$ are defined to be the spaces of functions on D and Ω respectively whose absolute value has a finite essential supremum, i.e. their absolute value is bounded by a positive constant almost everywhere. Norms on these spaces are defined via this essential supremum, i.e.

$$||v||_{L_\infty(D)} = \operatorname*{ess.\,sup}_{x \in D} |v(x)|, \tag{2.11}$$

$$||v||_{L_\infty(\Omega)} = \operatorname*{ess.\,sup}_{\omega \in \Omega} |v(\omega)|. \tag{2.12}$$

With these norms the spaces $L_\infty(D)$ and $L_\infty(\Omega)$ constitute Banach spaces, i.e. they are complete normed spaces.

The space $L_\infty(D) \otimes L_\infty(\Omega)$ is the space of random fields on $D \times \Omega$ that are bounded almost everywhere in absolute value. The associated norm is

$$||v||_{L_\infty(D)\otimes L_\infty(\Omega)} = \operatorname*{ess.\,sup}_{(x,\omega)\,\in\, D\times\Omega} |v(x,\omega)|. \tag{2.13}$$

As in the case of $L_2(D) \otimes L_2(\Omega)$, given $v \in L_\infty(D) \otimes L_\infty(\Omega)$, it can be considered as a function in $L_\infty(D)$ parameterized by $\omega \in \Omega$ or as a function in $L_\infty(\Omega)$ parameterized by $x \in D$.

2.2.3 Spaces $H^k(D)$ and $H^k(D) \otimes L_2(D)$

Let $\gamma = (\gamma_1, \gamma_2) \in \mathbb{N}^2$ ($\mathbb{N} = \{0, 1, \dots\}$) be a multi-index with length $|\gamma| = \gamma_1 + \gamma_2$. Then the differential operator D^γ is given by

$$D^\gamma = \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} \partial x_2^{\gamma_2}}. \tag{2.14}$$

The derivatives here are to be interpreted in the weak sense. Then the *Sobolev* spaces $H^k(D)$ (where k is a positive integer) are defined by

$$H^k(D) = \{\, v\colon D \to \mathbb{R} \mid D^\gamma v \in L_2(D),\ |\gamma| \le k \,\}. \tag{2.15}$$

The space $H^1(D)$ will be required in order to define the weak formulation of the boundary value problem given in §2.1. The space $H^2(D)$ will be used when regularity bounds are invoked in §2.10. Inner products on $H^k(D)$ are defined by

$$(v, w)_{H^k(D)} = \sum_{|\gamma|\le k} (D^\gamma v, D^\gamma w)_{L_2(D)} \tag{2.16}$$

with norms

$$||v||_{H^k(D)} = \left(\sum_{|\gamma|\le k} ||D^\gamma v||^2_{L_2(D)} \right)^{\frac{1}{2}}. \tag{2.17}$$

The spaces $H^k(D)$ are Hilbert spaces. The space $H_0^1(D)$ is defined by

$$H_0^1(D) = \{\, v \in H^1(D) \mid v = 0 \text{ on } \partial D \,\} \tag{2.18}$$

with inner product and norm inherited from $H^1(D)$.

The space $H^k(D) \otimes L_2(\Omega)$ is given by

$$H^k(D) \otimes L_2(\Omega) = \{v\colon D \times \Omega \to \mathbb{R} \mid D^\gamma v \in L_2(D) \otimes L_2(\Omega),\ |\gamma| \leq k \,\}. \tag{2.19}$$

Note that the differential operator, by definition, only acts on the spatial components of the random fields. These spaces have the associated inner products

$$(v, w)_{H^k(D)\otimes L_2(\Omega)} = \sum_{|\gamma| \leq k} (D^\gamma v, D^\gamma w)_{L_2(D)\otimes L_2(\Omega)} \tag{2.20}$$

and norms

$$||v||_{H^k(D)\otimes L_2(\Omega)} = \left(\sum_{|\gamma| \leq k} ||D^\gamma v||^2_{L_2(D)\otimes L_2(\Omega)} \right)^{\frac{1}{2}}. \tag{2.21}$$

The definition of $H_0^1(D) \otimes L_2(\Omega)$ follows from the definition of $H^k(D) \otimes L_2(\Omega)$. As in the cases of $L_2(D) \otimes L_2(\Omega)$ and $L_\infty(D) \otimes L_\infty(\Omega)$, random fields in $H^k(D) \otimes L_2(\Omega)$ can be thought of as deterministic fields in $H^k(D)$ parameterized by $\omega \in \Omega$ or as random variables in $L_2(\Omega)$ parameterized by $x \in D$.

2.3 Weak Formulation

Let $c \in L_\infty(D) \otimes L_\infty(\Omega)$ and $f \in L_2(D) \otimes L_2(\Omega)$. The weak formulation of (2.1) is given by: find $u \in H_0^1(D) \otimes L_2(\Omega)$ such that

$$a(u, v) = l(v) \quad \forall v \in H_0^1(D) \otimes L_2(\Omega) \tag{2.22}$$

where

$$a(u,v) = \int_\Omega \int_D c\nabla u \cdot \nabla v, \tag{2.23}$$

$$l(v) = \int_\Omega \int_D fv. \tag{2.24}$$

The *Lax-Milgram lemma* can be used to show that there exists a unique solution to this problem providing that there exist positive constants α and β such that

$$\alpha \leq c(x, \xi(\omega)) \leq \beta \quad P\text{-a.e. } \forall x \in D, \tag{2.25}$$

where by P-a.e. it is meant that there exists a subset $F \in \mathcal{F}$ with $P(F) = 0$ such that the inequality holds on the complement of F.

2.4 Stochastic Finite Element Method

In order to obtain a finite element formulation of the stochastic diffusion problem finite-dimensional subspaces of $H_0^1(D)$ and $L_2(\Omega)$ are required. Let these be denoted by S and T respectively. For S the space of piecewise linear polynomials will be used whose dimension is governed by a mesh parameter denoted by h. This is discussed in more detail in §2.5. For T the space of polynomial chaos of order n will be used. This is discussed in more detail in §2.6. Then $S \otimes T \subset H_0^1(D) \otimes L_2(\Omega)$ and the finite element formulation is given by: find $u_{hn} \in S \otimes T$ such that

$$a(u_{hn}, v) = l(v) \quad \forall v \in S \otimes T. \tag{2.26}$$

This will possess a unique solution under the same conditions as are required for the weak formulation.

This is equivalent to a matrix problem, to be described in §2.7. It will there be seen that the system matrix has a block structure. Moreover, the choice of S

ensures that the blocks are sparse while the choice of T ensures that the block structure is sparse.

2.5 Deterministic Finite Elements

The spatial domain, D, is discretized using a triangulation $\mathcal{T} = \{\triangle_1, \ldots, \triangle_K\}$ where $\bigcup_{i=1}^{K} \triangle_i = \overline{D}$. The points where the vertices of the triangles meet are called the nodes of the mesh. It is assumed that any triangulation used belongs to a family of triangulations that is quasi-uniform and shape regular as defined, e.g., in [5].

The finite dimensional subspace of $H_0^1(D)$ is then defined to be the set $S = \text{span}\{\phi_1, \ldots, \phi_N\}$, where N is the number of nodes in the mesh and ϕ_i is a piecewise linear function that is equal to one at the i-th node and zero at all other nodes. This corresponds to the P_1 approximation as discussed, e.g., in [14].

2.6 Polynomial Chaos

To construct the finite-dimensional subspace of $L_2(\Omega)$ the *polynomial chaos method*, as originally given in [20] and generalized in [40], is used. This essentially involves using the set of m-variate functions (recalling that m is the number of terms in (2.2)) from the *Askey scheme of hypergeometric polynomials* that satisfy the orthogonality relationship,

$$\int_{\Omega} \chi_k \chi_l = \kappa_k \delta_{kl}. \tag{2.27}$$

The Askey scheme of hypergeometic polynomials is discussed in [33]. The finite-dimensional subset of $L_2(\Omega)$ is then defined to be the set of all such polynomials

that are of degree n or less, which will be denoted as $T = \text{span}\{\chi_1, \ldots, \chi_M\}$, where

$$M = \frac{(m+n)!}{m!n!}. \tag{2.28}$$

As the probability measure in this thesis will be that corresponding either to an m-dimensional normal distribution or an m-dimensional uniform distribution the corresponding polynomials will be the *Hermite polynomials* and the *Legendre polynomials* respectively. These are discussed below in §2.6.1 and §2.6.2. For polynomials corresponding to other distributions see [41].

2.6.1 Hermite Polynomials

The (1-variate) Hermite polynomials satisfy the orthogonality condition

$$\int_{-\infty}^{\infty} H_k(y) H_l(y) \frac{e^{-y/2}}{(2\pi)^{1/2}}\, dy = k!\delta_{kl}, \qquad k = 0, 1, \ldots \tag{2.29}$$

Note that some texts define the Hermite polynomials to be orthogonal with respect to the weight e^{-y^2} (*physical* Hermite polynomials) as opposed to the weight used above (*probabilistic* Hermite polynomials). The first six Hermite polynomials are given by

$$\begin{aligned} H_0(y) &= 1, & H_1(y) &= y, \\ H_2(y) &= y^2 - 1, & H_3(y) &= y^3 - 3y, \\ H_4(y) &= y^4 - 6y^2 + 3, & H_5(y) &= y^5 - 10y^3 + 15y. \end{aligned}$$

Subsequent Hermite polynomials can be computed using the three-term recurrence relation given by

$$H_{k+1}(y) = yH_k(y) - kH_{k-1}(y). \tag{2.30}$$

Three-term recurrence relations for 1-dimensional orthogonal polynomials are discussed in general in [10]. The result

$$\int_{-\infty}^{\infty} yH_k(y)H_l(y)\frac{e^{-y/2}}{(2\pi)^{1/2}}\,dy = \begin{cases} \max\{k!, l!\}, & |k-l|=1 \\ 0, & \text{otherwise} \end{cases} \tag{2.31}$$

can be established by using the three term-relation along with the orthogonality condition.

The generalized (1-variate) Hermite polynomials of variance $\nu = \sigma^2$ satisfy the orthogonality condition

$$\int_{-\infty}^{\infty} H_k^{(\nu)}(y)H_l^{(\nu)}(y)\frac{e^{-y/(2\nu)}}{(2\pi\nu)^{1/2}}\,dy = k!\delta_{kl}, \qquad k = 0, 1, \dots \tag{2.32}$$

and are given by $H_k^{(\nu)}(y) = H_k(y/\sigma)$. It follows that the three term recurrence relation for the generalized Hermite polynomials is given by

$$H_{k+1}^{(\nu)}(y) = \frac{y}{\sigma}H_k^{(\nu)}(y) - kH_{k-1}^{(\nu)}(y) \tag{2.33}$$

and that an analogous result to (2.31), viz.,

$$\int_{-\infty}^{\infty} yH_k^{(\nu)}(y)H_l^{(\nu)}(y)\frac{e^{-y/(2\nu)}}{(2\pi\nu)^{1/2}}\,dy = \begin{cases} \sigma\,\max\{k!, l!\}, & |k-l|=1 \\ 0, & \text{otherwise} \end{cases} \tag{2.34}$$

can be established.

Now let $y \in \mathbb{R}^m$ and let $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}$ (where $\mathbb{N} = \{0, 1, \dots\}$) be a multi-index. The generalized m-variate Hermite polynomials of variance ν are then given by $H_\alpha^{(\nu)}(y) = H_{\alpha_1}^{(\nu)}(y_1)\cdots H_{\alpha_m}^{(\nu)}(y_m)$ and satisfy the orthogonality condition

$$\int_{\mathbb{R}^m} H_\alpha^{(\nu)}(y)H_\beta^{(\nu)}(y)\frac{e^{-y^2/(2\nu)}}{(2\pi\nu)^{m/2}}\,dy = \prod_{i=1}^{m}\alpha_i!\delta_{\alpha\beta}. \tag{2.35}$$

Note that α_i gives the degree of $H_\alpha^{(\nu)}$ in y_i and that $|\alpha| = \sum_{i=1}^m \alpha_i$ gives the total degree of $H_\alpha^{(\nu)}$. The result

$$\int_{\mathbb{R}^m} y_k H_\alpha^{(\nu)}(y) H_\beta^{(\nu)}(y) \frac{e^{-y^2/(2\nu)}}{(2\pi\nu)^{m/2}}\, dy = \begin{cases} \sigma \max\{\alpha_k!, \beta_k!\} \prod_{i=1,\, i\neq k}^m \alpha_i!, & |\alpha_k - \beta_k| = 1,\ \alpha_i = \beta_i,\ i \neq k \\ 0, & \text{otherwise} \end{cases} \tag{2.36}$$

can be established by using the definition of $H_\alpha^{(\nu)}$ along with (2.32) and (2.34).

For the case when the random variables in (2.1) are normally distributed the orthogonality condition given by (2.27) corresponds to (2.35) so T will be defined by $T = \{\, H_\alpha^{(\nu)} \mid |\alpha| \leq m \,\}$. For example, if $m = 2$ and $n = 2$, $T = \{\chi_1, \ldots, \chi_6\}$ where

$$\begin{aligned} \chi_1 &= H_{(0,0)}^{(\nu)}, & \chi_2 &= H_{(1,0)}^{(\nu)}, & \chi_3 &= H_{(0,1)}^{(\nu)}, \\ \chi_4 &= H_{(2,0)}^{(\nu)}, & \chi_5 &= H_{(1,1)}^{(\nu)}, & \chi_6 &= H_{(0,2)}^{(\nu)}. \end{aligned}$$

The ordering of the basis functions is of course unimportant. However, in this thesis the ordering given in [20] will be followed.

2.6.2 Legendre Polynomials

The (1-variate) Legendre polynomials satisfy the orthogonality condition

$$\int_{-1}^{1} L_k(y) L_l(y) \frac{1}{2}\, dy = \frac{\delta_{kl}}{2k+1}, \qquad k = 0, 1, \ldots \tag{2.37}$$

The first six Legendre polynomials are given by

$$\begin{aligned} L_0(y) &= 1, & L_1(y) &= y, \\ L_2(y) &= \frac{1}{2}(3y^2 - 1), & L_3(y) &= \frac{1}{2}(5y^3 - 3y), \\ L_4(y) &= \frac{1}{8}(35y^4 - 30y^2 + 3), & L_5(y) &= \frac{1}{8}(63y^5 - 70y^3 + 15y). \end{aligned}$$

Subsequent Lengendre polynomials can be computed using the three term recurrence relation given by

$$L_{k+1}(y) = \frac{2k+1}{k+1} y L_k(y) - \frac{k}{k+1} L_{k-1}(y). \tag{2.38}$$

As mentioned in §2.6.1, three term recurrence relations of this nature are discussed in [10]. The result

$$\int_{-1}^{1} y L_k(y) L_l(y) \frac{1}{2}\, dy = \begin{cases} \frac{l}{(2l-1)(2l+1)}, & k+1 = l \\ \frac{k}{(2k-1)(2k+1)}, & k-1 = l \\ 0, & \text{otherwise} \end{cases} \tag{2.39}$$

can be established using (2.37) and (2.38).

Now let $y \in \mathbb{R}^m$ and let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}$ be a multi-index. The m-variate Legendre polynomials are then given by $L_\alpha(y) = L_{\alpha_1}(y_1) \ldots L_{\alpha_m}(y_m)$ and satisfy the orthogonality condition

$$\int_{(-1,1)^m} L_\alpha(y) L_\beta(y) \frac{1}{2^m}\, dy = \prod_{i=1}^{m} \frac{\delta_{\alpha\beta}}{2\alpha_i + 1}. \tag{2.40}$$

As for the case of the Hermite polynomials discussed in §2.6.1, α_i gives the degree of L_α in y_i and $|\alpha| = \sum_{i=1}^{m} \alpha_i$ gives the total degree of L_α. The result

$$\int_{(-1,1)^m} y_k L_\alpha(y) L_\beta(y) \frac{1}{2^m}\, dy = \begin{cases} \frac{\alpha_k}{(2\alpha_k-1)(2\alpha_k+1)} \prod_{i=1,\, i\neq k}^{m} \frac{1}{2\alpha_i+1}, & \alpha_k - \beta_k = 1,\ \alpha_i = \beta_i,\ i \neq k \\ \frac{\beta_k}{(2\beta_k-1)(2\beta_k+1)} \prod_{i=1,\, i\neq k}^{m} \frac{1}{2\alpha_i+1}, & \beta_k - \alpha_k = 1,\ \alpha_i = \beta_i,\ i \neq k \\ 0, & \text{otherwise} \end{cases} \tag{2.41}$$

can be established by using the definition of L_α along with (2.37) and (2.39).

For the case when the random variables in (2.1) are uniformly distributed the orthogonality condition given by (2.27) corresponds to (2.40) so T will be defined

by $T = \{ L_\alpha \mid |\alpha| \leq m \}$. E.g., if $m = 2$ and $n = 2$, $T = \{\chi_1, \ldots, \chi_6\}$ where

$$\chi_1 = L_{(0,0)}, \qquad \chi_2 = L_{(1,0)}, \qquad \chi_3 = L_{(0,1)},$$
$$\chi_4 = L_{(2,0)}, \qquad \chi_5 = L_{(1,1)}, \qquad \chi_6 = L_{(0,2)}.$$

The ordering of the basis functions follows that given in [20].

2.7 Matrix Formulation

To obtain the matrix formulation of the finite element problem given in §2.4 the expansion

$$u_{hn} = \sum_{j=1}^{N} \sum_{l=1}^{M} u_{jl} \phi_j \chi_l \tag{2.42}$$

is substituted into (2.26) and v is varied over the basis functions of $S \otimes T$. This gives the matrix problem: find $\mathbf{u} \in \mathbb{R}^{MN}$ such that

$$A\mathbf{u} = \mathbf{f} \tag{2.43}$$

where

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1M} \\ \vdots & & \vdots \\ A_{M1} & \cdots & A_{MM} \end{bmatrix}, \qquad [A_{kl}]_{ij} = \int_\Omega \int_D c \nabla \phi_i \cdot \nabla \phi_j \chi_k \chi_l, \tag{2.44}$$

and

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_M \end{bmatrix}, \qquad [\mathbf{f}_k]_i = \int_\Omega \int_D f \phi_i \chi_k. \tag{2.45}$$

The solution vector, $\mathbf{u}$, contains the coefficients in (2.42) stacked column-wise, i.e. $\mathbf{u} = [u_{11}, \ldots, u_{N1}, \ldots, u_{1M}, \ldots, u_{NM}]^T$.

Note that A thus defined is symmetric. To see that it is also positive definite, let $v \in S \otimes T$ with expansion

$$v = \sum_{i=1}^{N} \sum_{k=1}^{M} v_{ik} \phi_i \chi_k \tag{2.46}$$

and coefficient vector $\mathbf{v} \in \mathbb{R}^{MN}$ given by

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_M \end{bmatrix}, \qquad [\mathbf{v}_k]_i = v_{ik}. \tag{2.47}$$

Then

$$\begin{aligned} (\mathbf{v}, A\mathbf{v})_2 &= \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{M} \sum_{l=1}^{M} v_{ik} [A_{kl}]_{ij} v_{jl} \\ &= \int_\Omega \int_D c \nabla \Big(\sum_{i=1}^{N} \sum_{k=1}^{M} v_{ik} \phi_i \chi_k \Big) \cdot \nabla \Big(\sum_{j=1}^{N} \sum_{l=1}^{M} v_{jl} \phi_j \chi_l \Big) \\ &\geq \alpha \int_\Omega \int_D \nabla v \cdot \nabla v \geq 0 \end{aligned} \tag{2.48}$$

where α is given in (2.25). The above inequality is equal to zero if and only if $\nabla v = 0$ and as $v \in S \otimes T$ this implies that $v = 0$ and $\mathbf{v} = \mathbf{0}$.

Given that c has the expansion given in (2.2), A can be expanded as

$$A = G_0 \otimes A_0 + \sum_{r=1}^{m} \sqrt{\lambda_r} G_r \otimes A_r \tag{2.49}$$

where, defining $\xi_0 = 1$,

$$[G_r]_{kl} = \int_\Omega \xi_r \chi_k \chi_l, \tag{2.50}$$

$$[A_r]_{ij} = \int_D c_r \nabla \phi_i \cdot \nabla \phi_j. \tag{2.51}$$

From this it is seen that the sparsity structure of the matrices G_r, $r = 0, \ldots, m$, dictates the sparse block structure of the matrix A, whereas the sparsity structure

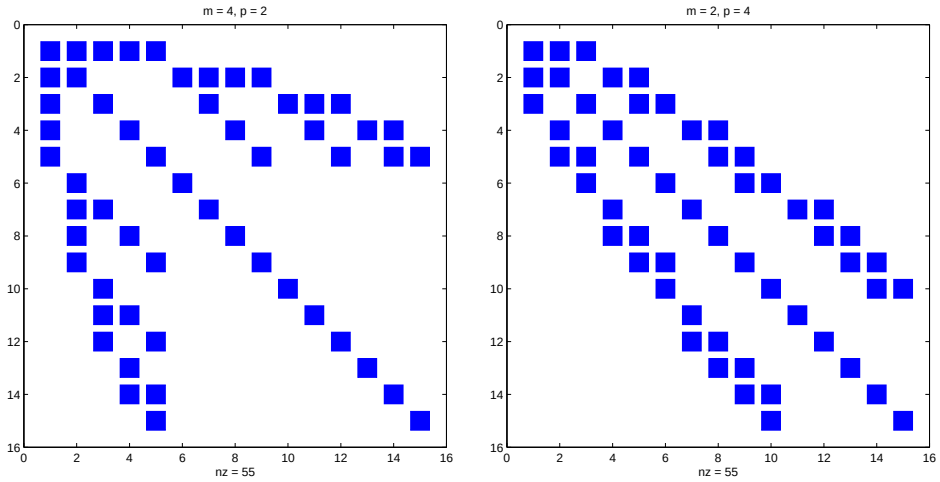

Figure 2.1: Block structure of the system matrix for the second-order diffusion problem arising from application of the polynomial chaos method.

of the matrices A_r, $r = 0, \ldots, m$, dictates the sparsity structure within the blocks of A. If ξ_r, $r = 1, \ldots, m$, are normally distributed then the appropriate choice for the basis functions of T are the m-variate Hermite polynomials described in §2.6.1. In this case, the entries of G_r, $r = 0, \ldots, m$, can be obtained using (2.35) and (2.36). If ξ_r, $r = 1, \ldots, m$, are uniformly distributed then the appropriate choice for the basis functions of T are the m-variate Legendre polynomials described in §2.6.2. In this case, the entries of G_r, $r = 0, \ldots, m$, can be obtained using (2.40) and (2.41). In either case the sparsity patterns of the matrices G_r, $r = 0, \ldots, m$, are the same and hence the block sparsity pattern of A will be the same. This block sparsity structure of A is shown in Figure 2.1 for the choices of $m = 2$, $p = 4$ and $m = 4$, $p = 2$. This block sparsity structure is also described in [28].

2.7.1 Post-processing the Matrix Solution

Once $\mathbf{u}$ has been computed it can be post-processed to obtain meaningful information, such as the mean and variance of u_{hn}. Definitions of mean and variance of a random field (noting that a random field is a stochastic process) can be found in [21].

The expected value of u_{hn} is given by

$$Eu_{hn} = \int_\Omega \sum_{j=1}^{N}\sum_{l=1}^{M} u_{jl}\phi_j\chi_l = \sum_{j=1}^{N}\sum_{l=1}^{M} u_{jl}\phi_j \int_\Omega \chi_l = \sum_{j=1}^{N} u_{j1}\phi_j \tag{2.52}$$

on account that $\chi_1 = 1$ and the orthogonality conditions (2.35) and (2.40). Therefore, if x_j is a node of $\mathcal{T}$ corresponding to the basis function ϕ_j, $Eu_{hn}(x_j, \xi) = u_{j1}$. Further, if $U \in \mathbb{R}^{N\times M}$ represents a matrix with entries u_{jl} then the first column of U will correspond to the finite element approximation of the mean of u_{hn}.

The variance of u_{hn} is given by

$$\begin{aligned} Vu_{hn} &= \int_\Omega \left(\sum_{j=1}^{N}\sum_{l=1}^{M} u_{jl}\phi_j\chi_l \right)^2 - (Eu_{hn})^2 \\ &= \sum_{j=1}^{N}\sum_{l=1}^{M} u_{jl}^2\phi_j^2 \int_\Omega \chi_l^2 - \left(\sum_{j=1}^{N} u_{j1}\phi_j \right)^2 \end{aligned} \tag{2.53}$$

and so the variance at the node x_j is given by

$$Vu_{hn}(x_j, \xi) = \sum_{l=2}^{M} u_{jl}^2 [G_0]_{ll} \tag{2.54}$$

on account of the definition of G_0 (whose entries are known analytically). Note that this is a linear combination of the squares of the entries in the 2nd to M-th columns of U.

2.8 Model Problem

To illustrate the stochastic finite element method a small model problem is now described. Let the spatial domain be given by $D = (-1, 1)^2$ and let the source function be given by $f = 1$. For the diffusion coefficient, c, consider a random field with mean function $c_0(x)$, constant variance ν, and covariance function $r(x, y)$. Such a process will have a Karhunen-Loève expansion of the form

$$c(x, \xi) = c_0(x) + \sum_{r=1}^{\infty} \sqrt{\lambda_r} c_r(x) \xi_r(\omega) \tag{2.55}$$

where (ξ_r) is a sequence of uncorrelated and identically distributed random variables with variance ν and mean zero, and (λ_r, c_r) is a sequence of eigenvalue-eigenfunction pairs that can be computed by solving

$$\int_D \frac{r(x, y)}{\nu} c_r(x)\, dx = \lambda_r c_r(y), \qquad r = 1, 2, \ldots \tag{2.56}$$

The sequence (λ_r) is ordered so as to be non-increasing. For $r(x, y)$ the exponential covariance function is considered which is given by

$$r(x, y) = \nu e^{-\frac{1}{b}|x_1 - y_1| - \frac{1}{b}|x_2 - y_2|} \tag{2.57}$$

where $x = (x_1, x_2)$, $y = (y_1, y_2) \in D$. The constant b is called the correlation length and will affect the decay of (λ_r), a larger value producing a faster decay. Here a value of $b = 10$ is considered. For this choice of covariance function analytic expressions exist for the solutions of (2.56) as given in [20]. For a general covariance function (2.56) will have to be solved numerically. An efficient algorithm for doing this involving fast multipole methods is given in [34].

For computational purposes a finite term expansion is required so (2.55) is curtailed to obtain

$$c(x, \xi) = c_0(x) + \sum_{r=1}^{m} \sqrt{\lambda_r} c_r(x) \xi_r(\omega). \tag{2.58}$$

From the modeling perspective the replacement of the infinite expansion with the finite expansion is justified providing (λ_r) decays rapidly, which matter is discussed in [20].

Now let $c_0(x) = 10$ and let ξ_r, $r = 1, \ldots, m$, be independent and uniformly distributed on $(-1, 1)$ whereby (2.25) is satisfied. The basis functions of T will hence be the m-variate Legendre polynomials as discussed in §2.6.2. To obtain a numerical approximation consider a triangulation of D consisting of a 32×32 grid of squares each of which is further divided into two triangles. Let $m = 4$ and $n = 4$. Figures 2.2 and 2.3 show plots of the numerical approximations to the expected value and variance of u. The maximum value of the expected value in this case is 0.0295 while the maximum value of the variance is 2.5879×10^{-06}.

2.9 Matrix and Right Hand Side Properties

In the following, E refers to the stochastic mass matrix and B refers to the deterministic mass matrix which are defined by

$$[E]_{kl} = \int_\Omega \chi_k \chi_l, \qquad [B]_{ij} = \int_D \phi_i \phi_j, \tag{2.59}$$

respectively. Also, given a function $v \in S \otimes T$ with expansion

$$v = \sum_{i=1}^{N} \sum_{k=1}^{M} v_{ik} \phi_i \chi_k, \tag{2.60}$$

its coefficient vector $\mathbf{v} \in \mathbb{R}^{MN}$ will be denoted by

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_M \end{bmatrix}, \qquad [\mathbf{v}_k]_i = v_{ik}. \tag{2.61}$$

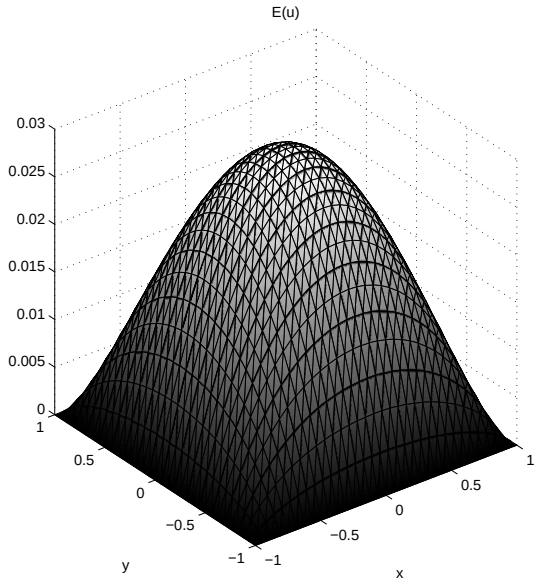

Figure 2.2: Expected value of the solution to the second-order model problem on a 32×32 grid with $m = 4$ and $n = 4$.

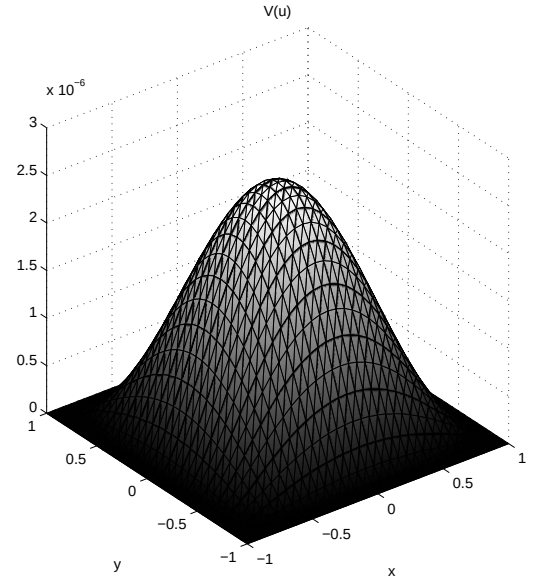

Figure 2.3: Variance of the solution to the second-order model problem on a 32×32 grid with $m = 4$ and $n = 4$.

Theorem 1 *Let $f \in S \otimes T$ with coefficient vector $\hat{\mathbf{f}}$. Then $\mathbf{f} = (E \otimes B)\hat{\mathbf{f}}$, where $\mathbf{f}$ is as in (2.45).*

Proof Substituting the expansion of f into (2.45) gives

$$\begin{aligned} f_{ik} &= \int_\Omega \int_D \left(\sum_{j=1}^{N} \sum_{l=1}^{M} \hat{f}_{jl} \phi_j \chi_l \right) \phi_i \chi_k \\ &= \sum_{j=1}^{N} \sum_{l=1}^{M} \hat{f}_{jl} \int_\Omega \chi_k \chi_l \int_D \phi_i \phi_j, \end{aligned} \tag{2.62}$$

whereby it is seen that $\mathbf{f} = (E \otimes M)\hat{\mathbf{f}}$. □

Theorem 2 *Let $f \in S \otimes T$ with coefficient vector $\hat{\mathbf{f}}$. Then $||f||^2_{L_2(D)\otimes L_2(\Omega)} = ((E \otimes B)\hat{\mathbf{f}}, \hat{\mathbf{f}})_2$.*

Proof Substituting the expansion of f into the definition of $||\cdot||_{L_2(D)\otimes L_2(\Omega)}$ gives

$$\begin{aligned} ||f||^2_{L_2(D)\otimes L_2(\Omega)} &= \int_\Omega \int_D f^2 \\ &= \int_\Omega \int_D \left(\sum_{i=1}^{N} \sum_{k=1}^{M} \hat{f}_{ik} \phi_i \chi_k \right) \left(\sum_{j=1}^{N} \sum_{l=1}^{M} \hat{f}_{jl} \phi_j \chi_l \right) \\ &= \sum_{i=1}^{N} \sum_{k=1}^{M} \left(\sum_{j=1}^{N} \sum_{l=1}^{M} \hat{f}_{jl} \int_\Omega \chi_k \chi_l \int_D \phi_i \phi_j \right) \hat{f}_{ik} \\ &= \sum_{i=1}^{N} \sum_{k=1}^{M} [(E \otimes B)\hat{\mathbf{f}}]_{ik} \hat{f}_{ik} = ((E \otimes B)\hat{\mathbf{f}}, \hat{\mathbf{f}})_2, \end{aligned} \tag{2.63}$$

as required. □

Theorem 3 *The inequality*

$$C_1 h^2 \leq \frac{((E \otimes B)\mathbf{v}, \mathbf{v})_2}{(\mathbf{v}, \mathbf{v})_2} \tag{2.64}$$

holds for all $\mathbf{v} \in \mathbb{R}^{MN}$, where C_1 is independent of the mesh parameter, h. Moreover, providing the basis functions of T are suitable scaled C_1 will also be independent of m and n.

Proof The right hand side of (2.64) is the Rayleigh quotient of $E \otimes B$ and this is bounded below by the lowest eigenvalue of $E \otimes B$. The eigenvalues of $E \otimes B$ are the products of the eigenvalues of E and the eigenvalues of B. Therefore, denoting the minimum eigenvalues of E and B as $\lambda_{\min}(E)$ and $\lambda_{\min}(B)$ respectively gives

$$\lambda_{\min}(E)\lambda_{\min}(B) \leq \frac{((E \otimes B)\mathbf{v}, \mathbf{v})_2}{(\mathbf{v}, \mathbf{v})_2}. \tag{2.65}$$

For a quasi-uniform and shape regular mesh the minimum eigenvalue of B is bounded below by Ch^2 where C is a constant, as shown, e.g., in [14]. The lowest eigenvalue of E will, in general, depend on the stochastic parameters m and p. If the basis of T is comprised of the Hermite polynomials, then $\lambda_{\min}(E) = 1$, and C_1 will be independent of m and n as well as h. If the basis of T is comprised of the Legendre polynomials, then this will not be the case. However, the basis functions can always be scaled or normalized in order to ensure that C_1 is independent of m and n. □

Theorem 4 *Let $f \in S \otimes T$. Then $h\sqrt{C_1}||f||_{L_2(D)\otimes L_2(\Omega)} \leq ||\mathbf{f}||_2$, where C_1 is as in Theorem 3.*

Proof Using Theorem 1 gives $||\mathbf{f}||_2^2 = ((E \otimes B)\hat{\mathbf{f}}, (E \otimes B)\hat{\mathbf{f}})_2$. Now setting $\mathbf{g} = (E \otimes B)^{\frac{1}{2}}\hat{\mathbf{f}}$ and using Theorems 2 and 3 gives

$$C_1 h^2 \leq \frac{((E \otimes B)\mathbf{g}, \mathbf{g})_2}{(\mathbf{g}, \mathbf{g})_2} = \frac{((E \otimes B)\hat{\mathbf{f}}, (E \otimes B)\hat{\mathbf{f}})_2}{((E \otimes B)\hat{\mathbf{f}}, \hat{\mathbf{f}})_2} = \frac{||\mathbf{f}||_2^2}{||f||^2_{L_2(D)\otimes L_2(\Omega)}} \tag{2.66}$$

as required. □

2.10 Semi-discrete Finite Element Formulation

In proving the approximation property used in the analysis for multigrid in §3.5 it will be useful to introduce the solution of a semi-discrete version of the finite

element formulation where only the stochastic space is discretized. This is given by: find $u_n \in H_0^1(D) \otimes T$ such that

$$a(u_n, v) = l(v) \quad \forall v \in H_0^1(D) \otimes T. \tag{2.67}$$

This has a unique solution under the same conditions as apply to the weak formulation in §2.2.

Theorem 5 *The solution to the semi-discrete problem, u_n, and the finite element approximation, u_{hn}, defined in §2.4, satisfy*

$$||u_n - u_{hn}||_a \leq \sqrt{\beta} C_2 h ||D^2 u_n||_{L_2(D)\otimes L_2(\Omega)}, \tag{2.68}$$

where β is defined in (2.25), $||D^2 v||_{L_2(D)\otimes L_2(\Omega)}$ is defined by

$$||D^2 v||_{L_2(D)\otimes L_2(\Omega)} = \int_\Omega |v|^2_{H^2(D)} \quad \forall v \in H^2(D) \otimes L_2(\Omega), \tag{2.69}$$

and C_2 is independent of h, m, and n.

Proof Galerkin orthogonality gives

$$||u_n - u_{hn}||_a^2 \leq ||u_n - v||_a^2 \quad \forall v \in S \otimes T. \tag{2.70}$$

Now let $\tilde{u}_n \in S \otimes T$ be the spatial interpolant of $u_n \in H_0^1(D) \otimes T$, i.e. if x_j, $j = 1, \ldots, N$, are the nodes of the spatial triangulation $\mathcal{T}$, then $\tilde{u}_n(x_j, \xi) = u_n(x_j, \xi)$ $\forall \omega \in \Omega$. Then

$$||u_n - u_{hn}||_a^2 \leq ||u_n - \tilde{u}_n||_a^2 \leq \beta \int_\Omega |u_n - \tilde{u}_n|^2_{H^1(D)}. \tag{2.71}$$

A standard interpolation result, as given e.g., in [22], gives

$$|u_n - \tilde{u}_n|_{H^1(D)} \leq C_2 h |u_n|_{H^2(D)} \quad \forall \omega \in \Omega \tag{2.72}$$

where C_2 is only dependent on the spatial domain. Therefore,

$$||u_n - u_{hn}||_a^2 \leq \beta C_2^2 h^2 \int_\Omega |u_n|_{H^2(D)}^2 = \beta C_2^2 h^2 ||D^2 u_n||_{L_2(D)\otimes L_2(\Omega)}^2, \tag{2.73}$$

which proves the theorem. □

In order to obtain a regularity bound for u_n an assumption is made. Let $f \in L_2(D) \otimes T$. Then it is assumed that for each realization of ξ the semi-discrete solution is in $H^2(D)$ and satisfies

$$||u_n||_{H^2(D)} \leq C_3 ||f||_{L_2(D)} \tag{2.74}$$

where C_3 is dependent on the spatial domain and β.

Theorem 6 *The H^2-regularity bound*

$$||D^2 u_n||_{L_2(D)\otimes L_2(\Omega)} \leq C_3 ||f||_{L_2(D)\otimes L_2(\Omega)} \tag{2.75}$$

holds, where C_3 is independent of h, m, and n.

Proof The assumption made prior to the statement of the theorem gives

$$|u_n|_{H^2(D)} \leq C_3 ||f||_{L_2(D)} \quad \forall \omega \in \Omega, \tag{2.76}$$

Squaring and integrating over Ω gives the desired result. □

Chapter 3

Solving the Second-Order Stochastic Diffusion Problem

Now multigrid is investigated as a means of solving the linear system arising from using the stochastic finite element method to discretize the second-order diffusion problem as described in Chapter 2. Multigrid is known to be an optimal method for solving the analogous deterministic problem which is to say, its convergence rate is independent of the mesh parameter h. Here the deterministic multigrid algorithm is extended to incorporate the stochastic nature of the problem and the results in Chapter 1 are used to demonstrate that this stochastic multigrid is also optimal with respect to the discretization parameters.

3.1 Stationary Iteration

Central to the idea of multigrid is the understanding that certain stationary iterations when applied to particular matrix problems tend to *smooth* the associated error. Given the problem $A\mathbf{u} = \mathbf{f}$, the matrix splitting $A = Q - Z$ inspires the

stationary iteration

$$\begin{aligned}\mathbf{u}^{(k+1)} &= Q^{-1}Z\mathbf{u}^{(k)} + Q^{-1}\mathbf{f} \\ &= Q^{-1}(Q-A)\mathbf{u}^{(k)} + Q^{-1}\mathbf{f} \\ &= (I - Q^{-1}A)\mathbf{u}^{(k)} + Q^{-1}\mathbf{f}. \end{aligned} \tag{3.1}$$

The matrix $I - Q^{-1}A$ is the iteration matrix of the method and in the context of multigrid is called the *smoother*. Various choices for the smoother exist. Two popular choices are the *damped Jacobi method* and *Gauss-Seidel method*, both of which are discussed, in the context of multigrid, in [7].

3.2 Two-grid Correction Scheme

Let $T \subset L_2(\Omega)$ and $S_{2h} \subset S_h \subset H_0^1(D)$ be as defined in §2.4, where the subscripts $2h$ and h are being used to distinguish between a *fine* spatial discretization, with mesh parameter $2h$, and a *coarse* mesh discretization, with mesh parameter h. Then defining $V_{2h} = S_{2h} \otimes T$ and $V_h = S_h \otimes T$ gives $V_{2h} \subset V_h \subset H_0^1(D) \otimes L_2(\Omega)$. Finite element formulations in V_h and V_{2h} give rise to matrix equations which will be represented as $A\mathbf{u} = \mathbf{f}$ and $\bar{A}\bar{\mathbf{u}} = \bar{\mathbf{f}}$ respectively.

Now let $I_{2h}^h \colon V_{2h} \to V_h$ denote a prolongation operator defined via natural inclusion i.e. for $v_{2h} \in V_{2h}$, $I_{2h}^h v_{2h} = v_{2h}$. To see how I_{2h}^h can be represented as a matrix, note that any basis function $\phi_j^{(2h)}$ of S_{2h} can be expanded in the basis functions of S_h, viz.,

$$\phi_j^{(2h)} = \sum_{i=1}^{N_h} p_{ij}\phi_i^{(h)}, \quad j = 1, \ldots, N_{2h}. \tag{3.2}$$

Defining a matrix P using the coefficients above, i.e. $[P]_{ij} = p_{ij}$, gives, for $v_{2h} \in$

V_{2h},

$$v_{2h} = \sum_{j=1}^{N_{2h}} \sum_{k=1}^{M} v_{jk}^{(2h)} \phi_j^{(2h)} \chi_k = \sum_{j=1}^{N_{2h}} \sum_{k=1}^{M} v_{jk}^{(2h)} \sum_{i=1}^{N_h} p_{ij} \phi_i^{(h)} \chi_k$$
$$= \sum_{i=1}^{N_h} \sum_{k=1}^{M} \left(\sum_{j=1}^{N_{2h}} p_{ij} v_{jk}^{(2h)} \right) \phi_i^{(h)} \chi_k = \sum_{i=1}^{N_h} \sum_{k=1}^{M} [P\mathbf{v}_k^{(2h)}]_i \phi_i^{(h)} \chi_k. \tag{3.3}$$

As $v_{2h} \in V_h$ it can be expanded as

$$v_{2h} = \sum_{i=1}^{N_h} \sum_{k=1}^{M} v_{ik}^{(h)} \phi_i^{(h)} \chi_k. \tag{3.4}$$

Comparing (3.3) and (3.4) it is seen that $[P\mathbf{v}_k^{(2h)}]_i = v_{ik}^{(h)}$ or that $P\mathbf{v}_k^{(2h)} = \mathbf{v}_k^{(h)}$. From this it follows that if $\mathbf{v}_{2h}$ is the coefficient vector of v_{2h} in V_{2h}, then $(I \otimes P)\mathbf{v}_{2h}$ is the coefficient vector of v_{2h} in V_h. (Here I is an $M \times M$ identity matrix.) Therefore $I \otimes P$ is the prolongation matrix associated with the prolongation operator I_{2h}^h, and shall be denoted as $\mathcal{P}$.

A restriction operator $I_h^{2h} \colon V_h \to V_{2h}$ is then defined such that the corresponding restriction matrix $\mathcal{R}$ satisfies $\mathcal{R} = \mathcal{P}^T$ (or equivalently $\mathcal{R} = I \otimes R$ where $R = P^T$). That is to say, that if I_h^{2h} maps $v_h \in V_{2h}$ to $v_{2h} \in V_{2h}$ and $\mathbf{v}_h$ and $\mathbf{v}_{2h}$ are the respective coefficient vectors of these functions, then $\mathbf{v}_{2h} = \mathcal{R}\mathbf{v}_h = \mathcal{P}^T\mathbf{v}_h$.

With the prolongation and restriction matrices related in this way the desirable relationships $\bar{\mathbf{f}} = \mathcal{R}\mathbf{f}$ and $\bar{A} = \mathcal{R}A\mathcal{P}$ are obtained. To see that the first of these relationships holds, note that

$$[\bar{\mathbf{f}}_k]_i = \int_\Omega \int_D f \phi_i^{(2h)} \chi_k = \int_\Omega \int_D f \left(\sum_{j=1}^{N_h} p_{ji} \phi_j^{(h)} \right) \chi_k$$
$$= \sum_{j=1}^{N_h} p_{ji} \int_\Omega \int_D f \phi_j^{(h)} \chi_k = [R\mathbf{f}_k]_i,$$

which is to say, $\bar{\mathbf{f}}_k = R\mathbf{f}_k$, from which it follows that $\bar{\mathbf{f}} = \mathcal{R}\mathbf{f}$. To see that the

second of the relationships holds, note that

$$\begin{aligned}
[A_{kl}]_{ij} &= \int_\Omega \int_D c\nabla\phi_i^{(2h)} \cdot \nabla\phi_j^{(2h)}\chi_k\chi_l \\
&= \int_\Omega \int_D c\left(\nabla \sum_{r=1}^{N_h} p_{ri}\phi_r^{(h)}\right) \cdot \left(\nabla \sum_{s=1}^{N_h} p_{sj}\phi_j^{(h)}\right)\chi_k\chi_l \\
&= \sum_{r=1}^{N_h} p_{ri} \sum_{s=1}^{N_h} p_{sj}[A_{kl}]_{rs} = \sum_{r=1}^{N_h} p_{ri}[A_{kl}P]_{rj} = [RA_{kl}P]_{ij},
\end{aligned}$$

which is to say, $\bar{A}_{kl} = RA_{kl}P$, from which it follows that $\bar{A} = \mathcal{R}A\mathcal{P}$.

Using the above definitions, the following algorithm gives a two-grid corrective scheme for solving the matrix problem $A\mathbf{u} = \mathbf{f}$.

choose initial guess $\mathbf{u}$

for $i = 0, 1, \ldots$

for $j = 1 : k$

$\mathbf{u} \leftarrow (I - Q^{-1}A)\mathbf{u} + Q^{-1}\mathbf{f}$

end

$\bar{\mathbf{r}} = \mathcal{R}(\mathbf{f} - A\mathbf{u})$

solve $\bar{A}\bar{\mathbf{e}} = \bar{\mathbf{r}}$

$\mathbf{u} \leftarrow \mathbf{u} + \mathcal{P}\bar{\mathbf{e}}$

end

The success of this algorithm necessarily depends on how well the smoother works and how well the functions are passed between the coarse and fine grids.

3.3 Convergence of Two-Grid Correction Scheme

That the two-grid convergence scheme, given in §3.2, converges can be shown to be true providing the *smoothing property* and the *approximation property* are satisfied, as is shown in the following theorem.

Theorem 7 *Providing the smoothing property,*

$$||A(I - Q^{-1}A)^k \mathbf{y}||_2 \leq \eta(k)||\mathbf{y}||_A \quad \forall \mathbf{y} \in \mathbb{R}^{MN_h}, \tag{3.5}$$

with $\eta(k) \to 0$ *as* $k \to \infty$*, and the approximation property,*

$$||(A^{-1} - \mathcal{P}\bar{A}^{-1}\mathcal{R})\mathbf{y}||_A \leq C_4||\mathbf{y}||_2 \quad \forall \mathbf{y} \in \mathbb{R}^{MN_h}, \tag{3.6}$$

are satisfied, then, providing k *is sufficiently large, the two-grid algorithm given in §3.2 converges.*

Proof It can be shown that the error associated with the two-grid algorithm obeys the recursive relationship

$$\mathbf{e}^{(i+1)} = (A^{-1} - \mathcal{P}\bar{A}^{-1}\mathcal{R})A(I - Q^{-1}A)^k \mathbf{e}^{(i)}. \tag{3.7}$$

Hence,

$$\begin{aligned} ||\mathbf{e}^{(i+1)}||_A &= ||(A^{-1} - \mathcal{P}\bar{A}^{-1}\mathcal{R})A(I - Q^{-1}A)^k \mathbf{e}^{(i)}||_A \\ &\leq C_4||A(I - Q^{-1}A)^k \mathbf{e}^{(i)}||_2 \\ &\leq C_4\eta(k)||\mathbf{e}^{(i)}||_A. \end{aligned} \tag{3.8}$$

Since $\eta(k) \to 0$ as $k \to \infty$ there exists some minimal number of smoothing steps such that $C_4\eta(k) < 1$. □

That the smoothing property and approximation property hold is discussed in §3.4 and §3.5 respectively. It will be seen that $\eta(k)$ and C_4 are independent of the

mesh parameter h and, moreover, that providing the stochastic basis functions are suitably scaled, they will also be independent of the stochastic parameters m and n.

3.4 Smoothing Property

The proof that the smoothing property holds is dependent on the choice of smoother. Here the the simplest choice of smoother, namely, Richardson's iterative method (as given e.g., in [39]), is considered. In this method $Q = \theta I$, $\theta \in \mathbb{R}$.

Theorem 8 *For the problem under consideration, if $Q = \theta I$ there is some $\theta \in \mathbb{R}$ such that the smoothing property given in Theorem 7 holds. Moreover, θ can be chosen such that $\eta(k)$ is independent of h and, providing the basis functions of T are suitably scaled, independent of m and n.*

Proof In [5] and [14] it is shown that, if $Q = \theta I$ and the eigenvalues of $I - Q^{-1}A$ are contained in the interval $[-\sigma, 1]$ with $0 \leq \sigma < 1$, then the smoothing property holds. In particular

$$||A(I - Q^{-1}A)^k \mathbf{y}||_2^2 \leq \max\left\{ \frac{\theta}{2ke}, \theta\sigma^{2k}(1+\sigma) \right\} ||\mathbf{y}||_A^2 \tag{3.9}$$

where $e = \exp 1 = 2.718...$ Now let λ_i, $i = 1, \ldots, MN_h$, be the eigenvalues of A. Then $1 - \theta^{-1}\lambda_i$, $i = 1, \ldots, MN_h$, will be the eigenvalues of $I - Q^{-1}A$. Providing $\theta > 0$ these eigenvalues will be less than one on account that A is positive definite. Therefore it is required to show that θ can be chosen such that

$$\theta \geq \frac{\lambda_{\max}(A)}{1+\sigma}. \tag{3.10}$$

To show that $\lambda_{\max}(A)$ is bounded above it is sufficient to show that the Rayleigh quotient of A is bounded above. To this end, let K be the stiffness matrix associated with the deterministic Poisson problem, i.e.

$$[K]_{ij} = \int_D \nabla\phi_i \cdot \nabla\phi_j. \tag{3.11}$$

Then, letting $u \in S \otimes T$ be the function with corresponding coefficient vector $\mathbf{u} \in R^{MN}$,

$$(A\mathbf{u}, \mathbf{u})_2 = \int_\Omega \int_D c\, \nabla u \cdot \nabla u \leq \beta \int_\Omega \int_D \nabla u \cdot \nabla u = \beta\, ((E \otimes K)\mathbf{u}, \mathbf{u})_2 \tag{3.12}$$

where β is as in (2.25). Consequently,

$$\frac{(A\mathbf{u}, \mathbf{u})_2}{(\mathbf{u}, \mathbf{u})_2} \leq \beta\, \frac{((E \otimes K)\mathbf{u}, \mathbf{u})_2}{(\mathbf{u}, \mathbf{u})_2} \leq \beta \lambda_{\max}(E) \lambda_{\max}(K). \tag{3.13}$$

It is well known that $\lambda_{\max}(K)$ can be bounded above by a constant independent of h. Providing the basis functions of T are suitably scaled, or orthonormalized, then $\lambda_{\max}(E)$ will be bounded above by a constant independent of m and n. Therefore, given a suitably chosen σ, θ can be chosen such that the smoothing property holds and $\eta(k)$ is independent of the discretization parameters. $\square$

3.5 Approximation Property

Here it is shown that the approximation property given in Theorem 7 is satisfied.

Theorem 9 *For the problem under consideration, the approximation property given in Theorem 7 holds. Moreover, the constant C_4 is independent of h and, providing the basis functions of T are suitably scaled, independent of m and n.*

Proof Given $\mathbf{y} \in \mathbb{R}^{MN_h}$ there exists some $f \in S_h \otimes T$ such that $\mathbf{y} = \mathbf{f}$. Let u_{hp} and $u_{2h,p}$ be the fine and coarse grid solutions respectively with coefficient vectors

$\mathbf{u} = A^{-1}\mathbf{f}$ and $\bar{\mathbf{u}} = \bar{A}^{-1}\bar{\mathbf{f}} = \bar{A}^{-1}\mathcal{R}\mathbf{f}$. Then

$$\begin{aligned}||(A^{-1} - \mathcal{P}\bar{A}^{-1}\mathcal{R})\mathbf{y}||_A^2 &= ||\mathbf{u} - \mathcal{P}\bar{\mathbf{u}}||_A^2 = (\mathbf{u} - \mathcal{P}\bar{\mathbf{u}}, \mathbf{u} - \mathcal{P}\bar{\mathbf{u}})_A \\ &= a(u_{hn} - I_{2h}^h u_{2h,n}, u_{hn} - I_{2h}^h u_{2h,n}) \\ &= a(u_{hn} - u_{2h,n}, u_{hn} - u_{2h,n}) = ||u_{hn} - u_{2h,n}||_a^2. \quad (3.14)\end{aligned}$$

Now introducing the solution to the semi-discrete problem, u_p, defined in §2.10, and applying Theorem 5 and Theorem 6 gives

$$\begin{aligned}||(A^{-1} - \mathcal{P}\bar{A}^{-1}\mathcal{R})\mathbf{y}||_A &\leq ||u_n - u_{hn}||_a + ||u_n - u_{2h,n}||_a \\ &\leq \sqrt{\beta}C_2 h||D^2 u||_{L_2(D)\otimes L_2(\Omega)} + 2\sqrt{\beta}C_2 h||D^2 u||_{L_2(D)\otimes L_2(\Omega)} \\ &\leq 3\sqrt{\beta}C_2 C_3 h||f||_{L_2(D)\otimes L_2(\Omega)}. \quad (3.15)\end{aligned}$$

Finally, applying Theorem 4 gives

$$||(A^{-1} - \mathcal{P}\bar{A}^{-1}\mathcal{R})\mathbf{y}||_A \leq \frac{3\sqrt{\beta}C_2 C_3}{\sqrt{C_1}}||\mathbf{f}||_2 = \frac{3\sqrt{\beta}C_2 C_3}{\sqrt{C_1}}||\mathbf{y}||_2 \quad (3.16)$$

which establishes the approximation property, with

$$C_4 = \frac{3\sqrt{\beta}C_2 C_3}{\sqrt{C_1}}. \quad (3.17)$$

Note that C_4 is independent of the mesh parameter h and, providing the stochastic basis functions are suitable scaled, also independent of the stochastic parameters m and n. □

3.6 Extension to Multigrid

The two-grid correction scheme given in §3.2 only contains pre-smoothing. In practice post-smoothing is often also applied, as in the numerical experiments given in §3.7.1 and §3.7.2. Post-smoothing has been neglected in the preceding

analytic argument in order to keep things a little simpler. It can be shown, though the details are omitted here, that the two-grid correction scheme with post-smoothing also converges with a convergence rate independent of h, and under suitable scaling of the stochastic basis functions, independent of m, and n

Recursively applying the two-grid correction scheme gives rise to a multigrid scheme. A number of variations are possible; see, for example, [7]. That multigrid converges with a convergence rate independent of the parameters h, m, and n can be established by an inductive argument, as shown, for example, in [14] and [5].

3.7 Numerical Experiments

To demostrate the convergence properties of multigrid some numerical experiments are now given. The problem considered will be that described in §2.8 with the random variables in the Karhunen-Loève expansion of c first being uniformly distributed and then being normally distributed.

3.7.1 Diffusion with Uniform Distributions

Let ξ_r, $r = 1, \ldots, m$, be uniformly distributed on $(-1, 1)$. Therefore, $\rho(y) = 1/2^m$. Let $\nu = 1/3$ and $c_0 = 10$. Applying the generalized polynomial chaos method, as described in §2.4, the basis functions of T will be the m-variate Legendre polynomials of degree n or less described in §2.6.2.

For the triangulation of D a uniform mesh consisting of an underlying grid of $d \times d$ squares, each of which is further subdivided into two equal triangles, is used. (Note that this gives $h = 2\sqrt{2}/d$.) A full V-cycle is used with a 2×2 coarsest mesh.

For the smoother the damped Jacobi method is employed with the damping parameter set to 2/3. Three pre-smoothing and three post-smoothing iterates are carried out. The iterations stop when the residual reaches a tolerance of 10^{-6}. Table 3.1 shows the number of iterations required for convergence for varying values of d, m, and n. The results clearly support the theoretical conclusion that the convergence rate of the multigrid algorithm is bounded independent of h. Also,though the stochastic basis functions here were not scaled, the method is apparently insensitive to m and n.

3.7.2 Diffusion with Normal Distributions

Now let ξ_r, $r = 1, \ldots, m$, be normally distributed with zero mean and variance ν. Then $\rho(y) = e^{-y^2/(2\nu)}/(2\pi\nu)^{m/2}$. Also let $c_0(x) = 1$. Applying the generalized polynomial chaos method, as described in §2.4, the basis functions of T will be the set of m-variate generalized Hermite polynomials of degree n or less described in §2.6.1.

Note that the diffusion coefficient as defined in §3.7 will now fail to satisfy condition (2.25) no matter what the choice of ν. However, there is reason to believe that the theory still applies. Only a heuristic argument is given here. Given a sufficiently small variance the probability of c being outside of two positive bounds becomes negligibly small. That is to say, that if the normal distributions were replaced by similar distributions that looked like the normal distributions with their tails cut off of so as to insure that c satisfies (2.25), then the difference would not be noticed computationally. To emphasize, this argument is heuristic and the reasoning has not been pursued analytically. Sufficiently small variance does result in positive definite systems that yield sensible results. This matter is

$d = 4$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	6	6	6	6
$n = 2$	6	6	6	6
$n = 3$	6	6	6	6
$n = 4$	6	6	6	6
$d = 8$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	7	7	7	7
$n = 2$	7	7	7	7
$n = 3$	7	7	7	7
$n = 4$	7	7	7	7
$d = 16$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	7	7	7	7
$n = 2$	7	7	7	7
$n = 3$	7	7	7	7
$n = 4$	7	7	7	7
$d = 32$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	7	7	7	7
$n = 2$	7	7	7	7
$n = 3$	7	7	7	7
$n = 4$	7	7	7	7

Table 3.1: Number of iterations required for multigrid to converge to the solution of the second-order problem with diffusion defined via uniform distributions.

analyzed in [30]. Here the value of $\nu = 0.01$ is taken.

The triangulation of D and the multigrid algorithm applied follows that described for the uniform case in §3.7.1. Table 3.2 shows the number of iterations required for convergence for varying values of d, m, and n. The results clearly support the theoretical conclusion that the convergence rate of the multigrid algorithm is bounded independent of h, m, and n. Note that the m-variate Hermite polynomials, as defined in §2.6.1, are naturally scaled such that C_1, and hence C_4, are independent of m, and n.

$d = 4$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	6	6	6	6
$n = 2$	7	7	7	7
$n = 3$	7	7	7	7
$n = 4$	7	7	7	7
$d = 8$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	8	8	8	8
$n = 2$	8	8	8	8
$n = 3$	9	9	9	9
$n = 4$	10	10	10	10
$d = 16$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	8	8	8	8
$n = 2$	8	8	8	8
$n = 3$	9	9	9	9
$n = 4$	9	10	10	10
$d = 32$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	7	7	8	8
$n = 2$	8	8	8	8
$n = 3$	8	8	9	9
$n = 4$	9	9	9	9

Table 3.2: Number of iterations required for multigrid to converge to the solution of the second-order problem with diffusion defined via normal distributions.

Chapter 4

First-Order Stochastic Diffusion Problem

In considering the diffusion problem sometimes the gradient of the solution is of more interest than the solution itself. Instead of seeking an approximation to the solution of the second-order problem and then post-processing this to obtain an approximation for its gradient, a preferred approach is to obtain an approximation to its gradient directly by converting the second-order equation into a system of first-order equations. The variable u in the previous chapters will here be replaced by p, and u will now represent a vector field given by $u = c\nabla p$. Also, as Chapters 4 and 5 are largely independent of Chapters 2 and 3 the numeration of the constants will begin afresh, i.e. C_1 in the current chapter will not be equal to C_1 in Chapter 2, etc.

4.1 Boundary Value Problem

The first-order formulation of the stochastic diffusion problem with homogeneous Dirichlet boundary value conditions is given by

$$\begin{cases} c^{-1}u - \nabla p = 0 & \text{in } D \times \Omega, \\ -\nabla \cdot u = f & \text{in } D \times \Omega, \\ p = 0 & \text{on } \partial D \times \Omega, \end{cases} \tag{4.1}$$

where, as in §2.1, D is the spatial domain, $c\colon D \times \Omega \to \mathbb{R}$ is the diffusion coefficient, and $f\colon D \times \Omega \to \mathbb{R}$ is the source function. The sample space in turn belongs to a probability space $(\Omega, \mathcal{F}, P)$ where $\mathcal{F}$ is a σ-algebra and P is a probability measure. The diffusion coefficient and source function are assumed to be random fields, that is they are measurable on $(\Omega, \mathcal{F})$ for each $x \in D$. The solutions, $u\colon D \times \Omega \to \mathbb{R}$ and $p\colon \overline{D} \times \Omega \to \mathbb{R}$, are random fields such that for each $\omega \in \Omega$ the above boundary value problem is satisfied in the classical sense. The functions u and p will be referred to as the *velocity* and *pressure* solutions respectively.

Also as in §2.1, attention is restricted to a spatial domain, D, that is a simply connected bounded open subset of $\mathbb{R}^2$ with piecewise smooth boundary. In particular, it is taken to be the interior of a convex polygon.

The further construction of the problem follows that for the second-order problem, with the only difference being that it is the reciprocal of the diffusion coefficient that is assumed to be expressible in the form of a curtailed Karhunen-Loève expansion of continuous, independent, and identically distributed random variables. If this is not possible, then another expansion must be used for c such as a polynomial chaos expansion as discussed in [26] and [13]. The diffusion

coefficient therefore takes the form

$$\frac{1}{c(x,\xi)} = c_0(x) + \sum_{r=1}^{m} \sqrt{\lambda_r} c_r(x) \xi_r(\omega) \tag{4.2}$$

where ξ_r, λ_r, c_r, $r = 1, \ldots, m$, are as in §2.1. Moreover, $\mathcal{F}$ is defined to be the minimal σ-algebra generated by $\xi = (\xi_1, \ldots, \xi_m)$, and therefore, by the Doob-Dynkin lemma, the random fields, f, u, and p will also be expressible as Borel functions of ξ for each value $x \in D$. Denoting the density function of ξ_r, $r = 1, \ldots, m$, as ρ_r, equation (2.3) defines the integral of all Borel functions over Ω.

4.2 Lebesgue and Sobolev Spaces

In this chapter and the next the Lebesgue and Sobolev spaces defined in §2.2 will be used. The spaces of random vector fields $H^k(D)^2 \otimes L_2(\Omega)$ and $H(\mathrm{div}; D) \otimes L_2(\Omega)$ will also be required. These are defined below.

4.2.1 Spaces $H^k(D)^2$ and $H^k(D)^2 \otimes L_2(\Omega)$

The Sobolev spaces $H^k(D)$, $k = 0, 1, 2, \ldots$, are given in §2.2. Note that $H^0(D) = L_2(D)$. The space $H^k(D)^2$ is the space of vector valued functions whose components are elements of $H^k(D)$, i.e. $v = (v_1, v_2) \in H^k(D)^2 \Leftrightarrow v_1 \in H^k(D)$ and $v_2 \in H^k(D)$. The inner product associated with $H^k(D)^2$ is defined by summing the inner products of the components, i.e.

$$(v, w)_{H^k(D)^2} = (v_1, w_1)_{H^k(D)} + (v_2, w_2)_{H^k(D)}, \tag{4.3}$$

with the induced norm following.

The space $H^k(D)^2 \otimes L_2(\Omega)$ is defined to be the set of random vector fields whose components are in $H^k(D) \otimes L_2(\Omega)$. Again, the inner product is defined by

summing the inner products of the component spaces, from which the associated norm is induced.

4.2.2 Spaces $H(\mathbf{div}; D)$ and $H(\mathbf{div}; D) \otimes L_2(\Omega)$

The space $H(\mathrm{div}; D)$ is defined to be the space of all vector fields whose components and divergence are in $L_2(D)$, i.e. $v \in H(\mathrm{div}; D) \Leftrightarrow v \in L_2(D)^2$ and $\nabla \cdot v \in L_2(D)$. The inner product of this space is given by

$$(v, w)_{H(\mathrm{div};D)} = (v, w)_{L_2(D)^2} + (\nabla \cdot v, \nabla \cdot w)_{L_2(D)} \tag{4.4}$$

from which the norm is induced.

The space $H(\mathrm{div}; D) \otimes L_2(\Omega)$ is the space of random vector fields whose components and divergence are in $L_2(D) \otimes L_2(\Omega)$. Its inner product is given by

$$(v, w)_{H(\mathrm{div};D)\otimes L_2(\Omega)} = (v, w)_{L_2(D)^2 \otimes L_2(D)} + (\nabla \cdot v, \nabla \cdot w)_{L_2(D)\otimes L_2(\Omega)} \tag{4.5}$$

from which the norm is induced.

4.3 Weak Formulation

Let $c^{-1} \in L_\infty(D) \otimes L_\infty(\Omega)$ and $f \in L_2(D) \otimes L_2(\Omega)$. The weak formulation of the first-order diffusion problem given in §4.1 is given by: find $u \in H(\mathrm{div}; D) \otimes L_2(\Omega)$ and $p \in L_2(D) \otimes L_2(\Omega)$ such that

$$a(u, v) + b(p, v) = 0 \qquad \forall v \in H(\mathrm{div}; D) \otimes L_2(\Omega), \tag{4.6}$$

$$b(q, u) = -l(q) \qquad \forall q \in L_2(D) \otimes L_2(\Omega), \tag{4.7}$$

where the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and the linear form $l(\cdot)$ are defined by

$$a(v,w) = \int_{\Omega}\int_{D} \frac{1}{c}\, v \cdot w, \tag{4.8}$$

$$b(q,v) = \int_{\Omega}\int_{D} q\nabla \cdot v, \tag{4.9}$$

$$l(q) = \int_{\Omega}\int_{D} fq. \tag{4.10}$$

This will possess a unique solution, as shown in [6], providing that the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous, $a(\cdot,\cdot)$ is coercive on the null-space of $b(\cdot,\cdot)$, and that there exists a constant $\zeta > 0$ such that

$$\sup_{\substack{v\in H(\mathrm{div};D)\otimes \\ L_2(\Omega)\setminus\{0\}}} \frac{b(q,v)}{||v||_{H(\mathrm{div};D)\otimes L_2(\Omega)}} \geq \zeta ||q||_{L_2(D)\otimes L_2(\Omega)} \quad \forall q \in L_2(D)\otimes L_2(\Omega) \tag{4.11}$$

The continuity and coercive conditions on $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are readily demonstrated providing condition (2.25) holds, that is,

$$\frac{1}{\beta} \leq \frac{1}{c(x,\xi(\omega))} \leq \frac{1}{\alpha} \quad P\text{-a.e. } \forall x \in D. \tag{4.12}$$

This will be assumed in the subsequent analysis. The existence of ζ is demonstrated in [16].

4.4 Mixed Stochastic Finite Element Method

In order to formulate a finite element method for obtaining an approximation to the weak velocity and pressure solutions, u and p, finite-dimensional subspaces of $H(\mathrm{div};D)$, $L_2(D)$, and $L_2(\Omega)$ are required, which will be denoted as R, Q, and T respectively. To obtain R and Q, a triangulation, $\mathcal{T} = \{\triangle_1, \ldots, \triangle_K\}$, is defined on the spatial domain, D, as in §2.5. Then $Q = \mathrm{span}\{\psi_1, \ldots, \psi_{N_Q}\}$ is defined to be the space of piecewise constant functions on $\mathcal{T}$, and $R = \mathrm{span}\{\varphi_1, \ldots, \varphi_{N_R}\}$

is defined to be the zeroth-order Raviart-Thomas space on $\mathcal{T}$, i.e.

$$R = \{\, v \in H(\mathrm{div}; D) \mid v|_{\triangle_k} = (a_k, b_k) + c_k(x_1, x_2),\ a_k, b_k, c_k \in \mathbb{R} \,\}. \tag{4.13}$$

See [5] for a general discussion of Raviart-Thomas elements. The space $T = \mathrm{span}\{\chi_1, \ldots, \chi_M\}$ is defined to be the m-dimensional polynomial chaos of order n as described in §2.6. Then $R \otimes T \subset H(\mathrm{div}; D) \otimes L_2(\Omega)$ and $Q \otimes T \subset L_2(D) \otimes L_2(\Omega)$. The mixed stochastic finite element formulation is given by: find $u_{hn} \in R \otimes T$ and $p_{hn} \in Q \otimes T$ such that

$$a(u_{hn}, v) + b(p_{hn}, v) = 0 \qquad \forall v \in R \otimes T, \tag{4.14}$$

$$b(q, u_{hn}) = -l(q) \qquad \forall q \in Q \otimes T. \tag{4.15}$$

This possesses a unique solution under the same conditions as apply to the weak formulation, only now defined with respect to the discrete spaces. The continuity and coercivity conditions are demonstrated in an analogous manner to the infinite-dimensional case. Therefore, it remains to show, in order to demonstrate existence and uniqueness, that there exists $\zeta_{hn} > 0$ such that

$$\sup_{v \in R \otimes T \setminus \{0\}} \frac{b(q, v)}{||v||_{H(\mathrm{div}; D) \otimes L_2(\Omega)}} \geq \zeta_{hn} ||q||_{L_2(D) \otimes L_2(\Omega)} \quad \forall q \in Q \otimes T. \tag{4.16}$$

Though not trivial this can be shown using Fortin's Lemma in an analogous fashion to the deterministic problem. Moreover, ζ_{hn} is equal to its deterministic counterpart and hence independent of h, m, and n.

4.5 Matrix Formulation

The finite element formulation given in §4.4 can be written as a matrix problem. This is obtained by using the expansions

$$u_{hn} = \sum_{j=1}^{N_R} \sum_{l=1}^{M} u_{jl} \varphi_j \chi_l \qquad\qquad p_{hn} = \sum_{j=1}^{N_S} \sum_{l=1}^{M} p_{jl} \psi_j \chi_l \tag{4.17}$$

and allowing v to vary over the basis functions of $R \otimes T$ and q to vary over the basis functions of $Q \otimes T$. This leads to a matrix problem of the form

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \end{bmatrix} \tag{4.18}$$

where $A \in \mathbb{R}^{MN_R \times MN_R}$ is given by

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1M} \\ \vdots & & \vdots \\ A_{M1} & \cdots & A_{MM} \end{bmatrix}, \qquad [A_{kl}]_{ij} = \int_\Omega \int_D \frac{1}{c}\, \varphi_i \cdot \varphi_j\, \chi_k \chi_l, \tag{4.19}$$

$B \in \mathbb{R}^{MN_Q \times MN_R}$ is given by

$$B = \begin{bmatrix} B_{11} & \cdots & B_{1M} \\ \vdots & & \vdots \\ B_{M1} & \cdots & B_{MM} \end{bmatrix}, \qquad [B_{kl}]_{ij} = \int_\Omega \int_D \psi_i \nabla \cdot \varphi_k\, \chi_k \chi_l, \tag{4.20}$$

and $\mathbf{f} \in \mathbb{R}^{MN_Q}$ is given by

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_M \end{bmatrix}, \qquad [\mathbf{f}_k]_i = -\int_\Omega \int_D f \psi_i \chi_k. \tag{4.21}$$

The solution vectors $\mathbf{u}$ and $\mathbf{p}$ will contain the coefficients in (4.17) stacked column-wise. Note that the system matrix in (4.18) is symmetric but it is not positive definite.

Given that the diffusion coefficient can be expanded as in (4.2), A can be expanded as

$$A = G_0 \otimes A_0 + \sum_{r=1}^{M} \sqrt{\lambda_r} G_r \otimes A_r \tag{4.22}$$

where, defining $\xi_0 = 1$,

$$[G_r]_{kl} = \int_\Omega \xi_r \chi_k \chi_l, \qquad [A_r]_{ij} = \int_D c_r \, \varphi_i \cdot \varphi_j. \tag{4.23}$$

Also, $B = G_0 \otimes B_0$ with

$$[B_0]_{ij} = \int_D \psi_i \nabla \cdot \varphi_j. \tag{4.24}$$

From this decomposition it can be seen that the system matrix will have a sparse block structure, with each block being itself sparse, for the same reasons outlined in §2.7. The block sparsity pattern of A for two choices of m and n is as shown in Figure 2.1. The matrix B will be block diagonal on account of the orthogonality of the basis functions of T.

Once the system has been solved to obtain $\mathbf{u}$ and $\mathbf{p}$, then these vectors can be post-processed to gain statistical information about the finite element approximations, u_{hn} and p_{hn}, in an analogous manner as that described in §2.7.1.

4.6 Model Problem

For a model problem the domain and source function are chosen as in §2.8, that is to say the spatial domain is taken to be $(-1, 1)^2$ and the source function is taken to be unity. The reciprocal of the diffusion coefficient is considered to be represented as a Karhunen-Loève expansion, i.e.

$$\frac{1}{c(x, \xi)} = c_0(x) + \sum_{r=1}^{\infty} \sqrt{\lambda} c_r(x) \xi_r(\omega) \tag{4.25}$$

where (λ_r, c_r) is a sequence of eigenvalue-eigenvector pairs computed using (2.56). The covariance function is taken to be the exponential covariance function given by (2.57). For computational purposes (4.25) is curtailed to give

$$\frac{1}{c(x,\xi)} = c_0(x) + \sum_{r=1}^{m} \sqrt{\lambda_r} c_r(x) \xi_r(\omega) \tag{4.26}$$

thereby taking the form as given by (4.2).

Now, as in §2.8, let $c_0(x) = 10$ and let ξ_r, $r = 1, \ldots, m$, be independent and uniformly distributed on $(-1, 1)$ resulting in the basis functions of T being m-variate Legendre polynomials. For numerical purposes the triangulation of D is taken to be a 32×32 grid, each square in the grid being divided into two triangles. Let $m = 4$ and $n = 4$. The expected value and variance of the pressure solution are shown in Figures 4.1 and 4.2 and achieve maximum values of 2.9441 and 0.0255 respectively. The expected value of the components of the velocity solution are shown in Figures 4.3 and 4.4, both of which achieve a maximum value of 0.6746. The variance of the components of the velocity solution are shown in Figures 4.5 and 4.6, both of which achieve a maximum value of 0.2374×10^{-4}.

4.7 Weighted $H(\mathbf{div}; D) \otimes L_2(\Omega)$ Bilinear Form

Let U be the Hilbert space consisting of the vector space $L_2(D)^2 \otimes L_2(\Omega)$ paired with the inner product

$$(v, w)_U = \int_\Omega \int_D \frac{1}{c}\, v \cdot w, \tag{4.27}$$

from which the norm $||\cdot||_U$ is induced. Note that this definition is permissible as a consequence of (4.12). Note also that $a(v, w) = (v, w)_U$.

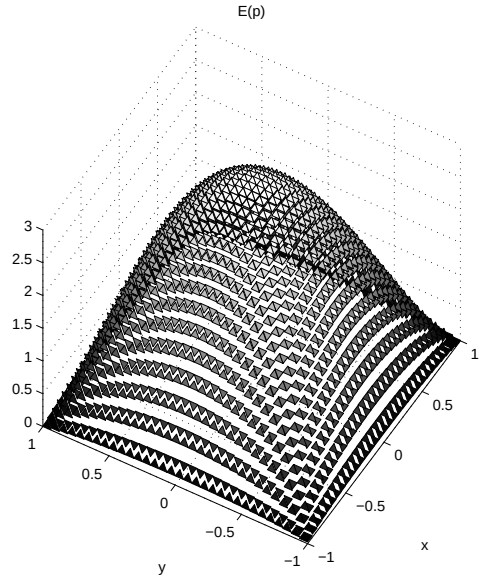

Figure 4.1: Expected value of the pressure solution to the first-order model problem on a 32×32 grid with $m = 4$ and $n = 4$.

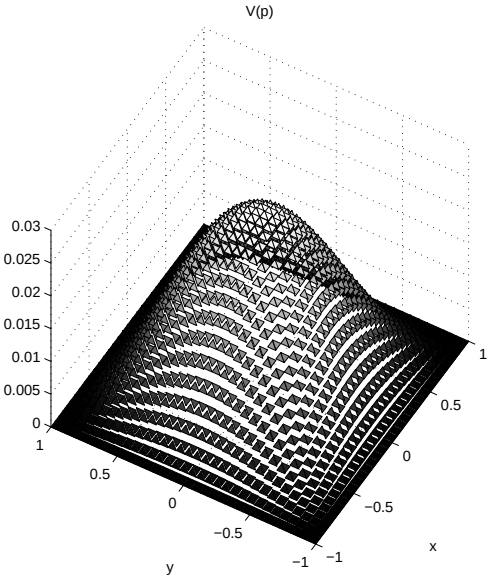

Figure 4.2: Variance of the pressure solution to the second-order model problem on a 32×32 grid with $m = 4$ and $n = 4$.

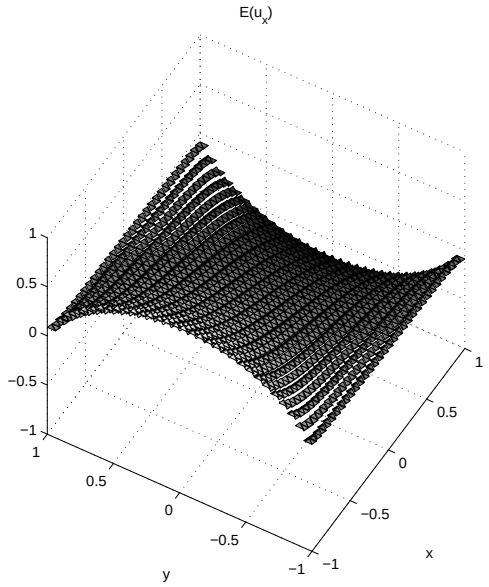

Figure 4.3: Expected value of the first component of the velocity solution to the first-order model problem on a 32×32 grid with $m = 4$ and $n = 4$.

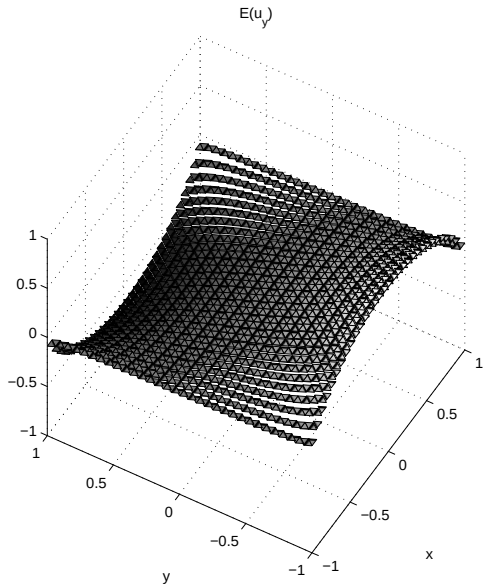

Figure 4.4: Expected value of the second component of the velocity solution to the first-order model problem on a 32×32 grid with $m = 4$ and $n = 4$.

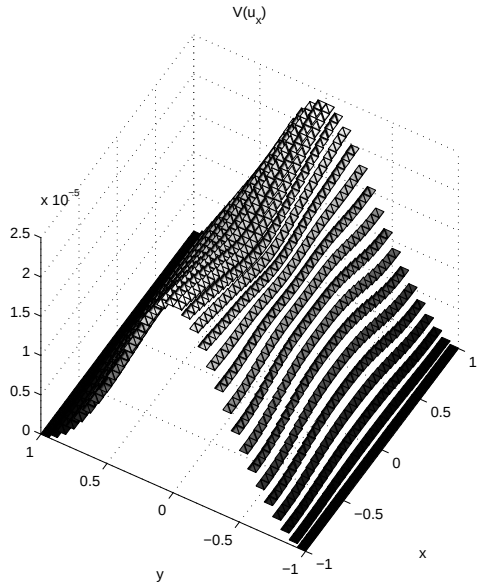

Figure 4.5: Variance of the first component of the velocity solution to the first-order model problem on a 32×32 grid with $m = 4$ and $n = 4$.

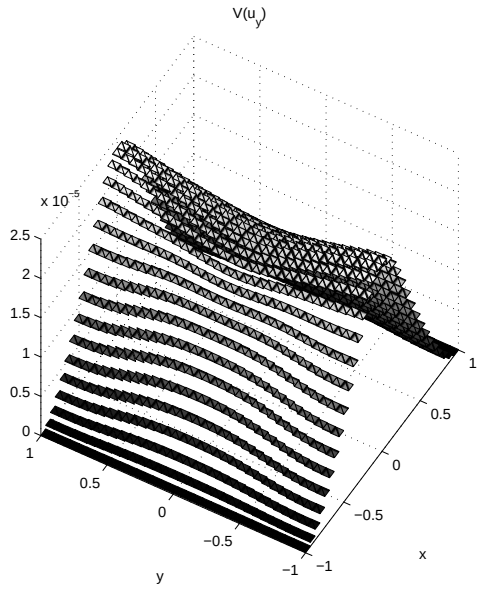

Figure 4.6: Variance of the second component of the velocity solution to the first-order model problem on a 32×32 grid with $m = 4$ and $n = 4$.

The bilinear form $\Lambda\colon H(\mathrm{div}; D) \otimes L_2(\Omega) \times H(\mathrm{div}; D) \otimes L_2(\Omega) \to \mathbb{R}$ is defined by

$$\Lambda(v, w) = (v, w)_U + (\nabla \cdot v, \nabla \cdot w)_{L_2(D) \otimes L_2(\Omega)}. \tag{4.28}$$

This induces a norm, to be denoted as $||\cdot||_\Lambda$.

4.8 Raviart-Thomas Interpolation Operator

The interpolation operator $\Pi\colon H^1(D)^2 \otimes L_2(\Omega) \to R \otimes L_2(\Omega)$ is defined such that, given $v \in H^1(D)^2 \otimes L_2(\Omega)$, then for each $\omega \in \Omega$, v is mapped from $H^1(D)^2$ to R in the conventional manner, as described, e.g., in [5]. To be more precise, let $R(\triangle_k)$ be the set of vector valued functions on $\triangle_k$ of the form $v = (a, b) + cx$, where a, b, $c \in \mathbb{R}$ and $x = (x_1, x_2) \in \triangle_k$, and then define the local interpolation operator $\Pi_k\colon H^1(\triangle_k)^2 \otimes L_2(\Omega) \to R(\triangle_k) \otimes L_2(\Omega)$ (note that $v \in R(\triangle_k) \otimes L_2(\Omega)$ will be of the form just mentioned with a, b, $c\colon \Omega \to \mathbb{R}$) such that

$$\int_{e_i} (v - \Pi_k v) \cdot n_i = 0 \quad \forall \omega \in \Omega, \quad i = 1, 2, 3 \tag{4.29}$$

where $e_i, i = 1, 2, 3$, are the edges of $\triangle_k$ and n_i, $i = 1, 2, 3$, are the respective unit normal vectors to these edges. Then given $v \in H^1(D)^2 \otimes L_2(\Omega)$, Π is defined such that

$$(\Pi v)|_{\triangle_k} = \Pi_k(v|_{\triangle_k}). \tag{4.30}$$

Given that for each $\omega \in \Omega$, Π maps v from $H^1(D)^2$ to R in the conventional manner described in deterministic analysis, the approximation result

$$||v - \Pi v||_{L_2(D)^2} \leq C_1 h ||v||_{H^1(D)^2} \quad \forall \omega \in \Omega \tag{4.31}$$

holds, as shown in [5]. Squaring and integrating over Ω gives

$$||v - \Pi v||_{L_2(D)^2 \otimes L_2(\Omega)} \leq C_1 h ||v||_{H^1(D)^2 \otimes L_2(\Omega)}. \tag{4.32}$$

4.9 Semi-discrete Mixed Finite Element Formulation

In the case of the second-order diffusion problem, a semi-discrete approximation was defined, in §2.10, which was instrumental in demonstrating the convergence of multigrid in Chapter 3. In Chapter 5 the MINRES method, with a preconditioner that incorporates a multigrid algorithm, will be used to solve the first-order diffusion problem. The subsequent analysis will also make use of a semi-discrete approximation. This is given by: find $u_n \in H(\mathrm{div}; D) \otimes T$ and $p_n \in L_2(D) \otimes T$ such that

$$a(u_n, v) + b(p_n, v) = 0 \qquad \forall v \in H(\mathrm{div}; D) \otimes T, \tag{4.33}$$

$$b(q, u_n) = -l(v) \qquad \forall q \in L_2(D) \otimes T. \tag{4.34}$$

That this possesses a unique solution can be demonstrated in the same manner as for the fully discrete problem discussed in §4.4.

Theorem 10 *The velocity solution to the semi-discrete problem, u_n, and the velocity solution to the finite element problem, u_{hn}, satisfy*

$$||u_n - u_{hn}||_U \leq C_2 h ||u_n||_{H^1(D)^2 \otimes L_2(\Omega)} \tag{4.35}$$

where C_2 is independent of h, m, and n.

Proof Following [18] it can be shown that there exists C_*, independent of m, n, and n, such that

$$||u_n - u_{hn}||_U \leq C_* ||u_n - \Pi u_n||_U. \tag{4.36}$$

Then using (4.12) and (4.32) gives

$$||u_n - u_{hn}||_U \leq \frac{C_1 C_* h}{\sqrt{\alpha}} ||u_n||_{H^1(D)^2 \otimes L_2(\Omega)} \tag{4.37}$$

which establishes the theorem. □

An assumption concerning the regularity of u_n, analogous to that made for the semi-discrete solution of the second-order diffusion problem in §2.10, will now be made. Let $f \in L_2(D) \otimes T$, then it is assumed that for each realization of ξ the semi-discrete solution is in $H^1(D)^2$ and satisfies

$$||u_n||_{H^1(D)^2} \leq C_\bigstar ||f||_{L_2(D)}. \tag{4.38}$$

Here $C_\bigstar$ will depend on the spatial domain and properties of the diffusion coefficient. From this it follows that

$$||u_n||_{H^1(D)^2 \otimes L_2(\Omega)} \leq C_\bigstar ||f||_{L_2(D) \otimes L_2(\Omega)}. \tag{4.39}$$

4.10 Helmholtz Decomposition

Vital to the analysis in Chapter 5 is the fact that a function in $R \otimes T$ admits a Helmholtz decomposition. The Helmholtz decomposition defined here is an extension of the definition given in [1] which was defined with respect to deterministic vector fields and made use of an un-weighted discrete gradient operator. Here the decomposition is defined with respect to random vector fields and makes

use of a weighted discrete gradient operator to reflect the fact that the underlying problem contains a diffusion coefficient that is not equal to unity.

Let $S = \text{span}\{\phi_1, \ldots, \phi_{N_S}\}$ be the set of piecewise linear functions defined on the triangulation, $\mathcal{T}$, of the spatial domain, D. Let the curl of a function be given by $\nabla \times w = (-\partial w/\partial x_2, \partial w/\partial x_1)$. It is known, e.g., see [1], that

$$\{\, v \in R \mid \nabla \cdot v = 0 \,\} = \{\, \nabla \times w \mid w \in S \,\}. \tag{4.40}$$

As divergence and curl are purely spatial operators it follows that

$$\{\, v \in R \otimes T \mid \nabla \cdot v = 0 \,\} = \{\, \nabla \times w \mid w \in S \otimes T \,\}. \tag{4.41}$$

The weighted discrete gradient operator $\text{grad}_h^c \colon Q \otimes T \to R \otimes T$ is defined such that for $q \in Q \otimes T$,

$$(\text{grad}_h^c\, q, v)_U = -(q, \nabla \cdot v)_{L_2(D)\otimes L_2(\Omega)} \quad \forall v \in R \otimes T. \tag{4.42}$$

The superscript c derives from the dependence of the operator on the diffusion coefficient through the definition of $(\cdot,\cdot)_U$.

Theorem 11 *(Helmholtz Decomposition) Given $v \in R \otimes T$ there exists $q \in Q \otimes T$ and $w \in S \otimes T$ such that*

$$v = \text{grad}_h^c\, q + \nabla \times w. \tag{4.43}$$

Proof In order to establish the given decomposition it is sufficient to show that $\text{grad}_h^c(Q \otimes T)$ and $\nabla \times (S \otimes T)$ are orthogonal complements on $R \otimes T$ with respect to $(\cdot,\cdot)_U$. To see this, let $w \in R \otimes T$. Then

$$\begin{aligned} \nabla \cdot (\nabla \times w) = 0 &\Leftrightarrow (\nabla \cdot (\nabla \times w), q)_{L_2(D)\otimes L_2(\Omega)} = 0 && \forall q \in Q \otimes T \\ &\Leftrightarrow (\nabla \times w, \text{grad}_h^c\, q)_U = 0 && \forall q \in Q \otimes T \\ &\Leftrightarrow \nabla \times w \in (\text{grad}_h^c\,(Q \otimes T))^{\perp}. \end{aligned} \tag{4.44}$$

Therefore, the decomposition is established. □

Note that for $q \in Q \otimes T$ and $w \in S \otimes T$, $\text{grad}_h^c\, q$ and $\nabla \times w$ are also orthogonal with respect to $(\cdot,\cdot)_\Lambda$.

4.11 Projection Operators

The projection operator $\Theta\colon L_2(D) \otimes L_2(\Omega) \to Q \otimes T$ is defined such that for $v \in L_2(D) \otimes L_2(\Omega)$,

$$(\Theta v, q)_{L_2(D)\otimes L_2(\Omega)} = (v, q)_{L_2(D)\otimes L_2(\Omega)} \quad \forall q \in Q \otimes T. \tag{4.45}$$

Here Θ is the $L_2(D) \otimes L_2(\Omega)$ projection operator onto $Q \otimes T$. It affects both the spatial and stochastic aspect of the random field it operates on.

In order to obtain a commutivity property analogous to that given in §3 of [1] it is required to define an operator that for each realization $\omega \in \Omega$ the resulting vector field in $L_2(D)$ is projected onto Q. This operator is denoted by $\Sigma\colon L_2(D) \otimes L_2(\Omega) \to Q \otimes L_2(\Omega)$ and is defined such that for $v \in L_2(D) \otimes L_2(\Omega)$,

$$(\Sigma v, q)_{L_2(D)} = (v, q)_{L_2(D)} \quad \forall q \in Q \quad \forall \omega \in \Omega. \tag{4.46}$$

It can be shown that given $v \in H^1(D)^2 \otimes L_2(\Omega)$, the commutivity property

$$\nabla \cdot \Pi v = \Sigma\, \nabla \cdot v \quad \forall \omega \in \Omega \tag{4.47}$$

holds, the proof of which follows that given in [5] for the deterministic problem. Now, given $v \in R \otimes T$ and $q \in Q \otimes T$,

$$\begin{aligned}(\Theta\, \nabla \cdot v, q)_{L_2(D)\otimes L_2(\Omega)} &= \int_\Omega (\nabla \cdot v, q)_{L_2(D)} \\ &= \int_\Omega (\Sigma\, \nabla \cdot v, q)_{L_2(D)} \\ &= \int_\Omega (\nabla \cdot \Pi v, q)_{L_2(D)} = (\nabla \cdot \Pi v, q)_{L_2(D)\otimes L_2(\Omega)}. \end{aligned} \tag{4.48}$$

This is analogous to the deterministic result given in [1].

The projection operator $\mathcal{P}\colon H(\mathrm{div}; D) \otimes L_2(\Omega) \to R \otimes T$ is defined such that for $v \in H(\mathrm{div}; D) \otimes L_2(\Omega)$,

$$\Lambda(\mathcal{P}w, v) = \Lambda(w, v) \quad \forall v \in R \otimes T. \tag{4.49}$$

4.12 Weighted $H(\mathbf{div}; D) \otimes L_2(\Omega)$ Operator

The weighted $H(\mathrm{div}; D) \otimes L_2(\Omega)$ operator $\mathcal{H}\colon R \otimes T \to R \otimes T$ is defined such that given $v \in R \otimes T$,

$$(\mathcal{H}v, w)_U = \Lambda(v, w) \quad \forall w \in R \otimes T. \tag{4.50}$$

Note that $\mathcal{H}$ is a positive-definite operator. Therefore, there exists an inverse operator $\mathcal{H}^{-1}\colon R \otimes T \to R \otimes T$ defined such that for $v \in R \otimes T$,

$$\Lambda(\mathcal{H}^{-1}v, w) = (v, w)_U \quad \forall w \in R \otimes T. \tag{4.51}$$

Note also that $\mathcal{H}$ maps $\mathrm{grad}_h^c\,(Q \otimes T)$ onto itself. To see this, let $v \in \mathrm{grad}_h^c\, q$ for some $q \in Q \otimes T$. Then

$$\begin{aligned}(\mathcal{H}v, w)_U &= (\mathrm{grad}_h^c\, q, w)_U + (\nabla \cdot \mathrm{grad}_h^c\, q, \nabla \cdot w)_{L_2(D)\otimes L_2(\Omega)} \\ &= (\mathrm{grad}_h^c\, q, w)_U - (\mathrm{grad}_h^c\, \nabla \cdot \mathrm{grad}_h^c\, q, w)_U \\ &= (\mathrm{grad}_h^c\,(q - \nabla \cdot \mathrm{grad}_h^c\, q), w)_U \quad \forall w \in R \otimes T. \end{aligned} \tag{4.52}$$

As $q - \nabla \cdot \mathrm{grad}_h^c\, q \in Q \otimes T$ it follows that given $q \in Q \otimes T$ there exists $q' \in Q \otimes T$ such that $\mathcal{H}\,\mathrm{grad}_h^c\, q = \mathrm{grad}_h^c\, q'$. As $\mathcal{H}$ maps $\mathrm{grad}_h^c\,(Q \otimes T)$ onto itself, then so does $\mathcal{H}^{-1}$.

Chapter 5

Solving the First-Order Stochastic Diffusion Problem

Let the coefficient matrix given in (4.18) be denoted by C, i.e.

$$C = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}. \tag{5.1}$$

This matrix is sparse and symmetric. Moreover, it is indefinite which can be shown by applying the Sylvester Law of Inertia to its congruence transform as described for the system matrix of the Stokes problem described in [14]. Hence a suitable method for obtaining an approximation to the solution of (4.18) is the MINRES algorithm which minimizes the Euclidean norm of the residual at each step. In order to ensure that the MINRES algorithm converges to within a given tolerance independent of the mesh parameter, h, it is required use a preconditioner such that the eigenvalues of the preconditioned system are independent of h, as described in [14]. In order to see how such a preconditioner can be chosen for the stochastic problem under consideration it will be useful first to see how such a preconditioner is chosen for the analogous deterministic problem.

5.1 Deterministic $H(\mathbf{div}; D)$ Preconditioner

Consider a deterministic diffusion problem with diffusion coefficient $c_0(x)\colon D \to \mathbb{R}$ and source function $f_0\colon D \to \mathbb{R}$. Applying a mixed finite element method gives the linear system

$$\begin{bmatrix} A_0 & B_0^T \\ B_0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{p}_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{f}_0 \end{bmatrix} \tag{5.2}$$

where A_0 and B_0 are as given in (4.23) and (4.24) and $\mathbf{f}_0$ is given by

$$[\mathbf{f}_0]_i = -\int_D f\psi_i. \tag{5.3}$$

The vectors $\mathbf{u}_0$ and $\mathbf{p}_0$ are the (deterministic) velocity and pressure solutions (which are not related to $\mathbf{u}$ and $\mathbf{p}$ in §4.5).

The so-called ideal $H(\mathrm{div}; D)$ preconditioner is given by

$$P_0 = \begin{bmatrix} A_0 + F_0 & 0 \\ 0 & N_0 \end{bmatrix} \tag{5.4}$$

where

$$[F_0]_{ij} = \int_D (\nabla \cdot \varphi_i)\,(\nabla \cdot \varphi_j), \qquad\qquad [N_0]_{ij} = \int_D \psi_i \psi_j. \tag{5.5}$$

This preconditioner was proposed in [1] for the case of $c_0(x) = 1$ and analyzed in [32] and [29] for arbitrary diffusion coefficients. The following theorem gives bounds for the eigenvalues of $P_0^{-1}C_0$, where C_0 is the system matrix given in 5.2.

Theorem 12 *The $N_R + N_Q$ eigenvalues of $P_0^{-1}C_0$ lie in $(-1, -\mu_0] \cup \{1\}$ where μ_0 is independent of h.*

Proof See [32].

The implementation of the preconditioned MINRES algorithm requires solving a system of the form $P_0\mathbf{z} = \mathbf{y}$ in each iteration. While the the action of the inverse of N_0 is trivial to compute (since it is a diagonal matrix), the action of the inverse of $A_0 + F_0$ is not trivial to compute. Therefore, a practical scheme involves approximating the action of the inverse of $A_0 + F_0$ in some computationally feasible manner. This can be achieved through the use of a geometric multigrid algorithm. To see this, let V_0 be some matrix such that

$$\theta_0 \leq \frac{(\mathbf{v}, (A_0 + F_0)\mathbf{v})_2}{(\mathbf{v}, V_0\mathbf{v})_2} \leq \overline{\theta}_0 \leq 1 \qquad \forall \mathbf{v} \neq \mathbf{0}. \tag{5.6}$$

Now consider a preconditioner given by

$$\widetilde{P}_0 = \begin{bmatrix} V_0 & 0 \\ 0 & N_0 \end{bmatrix}. \tag{5.7}$$

Theorem 13 *The $N_R + N_Q$ eigenvalues of $\widetilde{P}_0^{-1}C_0$ lie in*

$$\left(-1, \frac{1}{2}\left(\theta_0(1-\mu_0) - \sqrt{\theta_0^2(\mu_0 - 1)^2 + 4\mu_0\theta_0}\right)\right] \cup [\theta_0, 1], \tag{5.8}$$

where μ_0 is as in Theorem 12 and θ_0 is as in (5.6).

Proof See [32]. □

In [1] it was demonstrated that for $c_0(x) = 1$ choosing V_0^{-1} to be the matrix that represents the application of a certain multigrid V-cycle resulted in $\theta_0 = 1 - \delta_0$ and $\overline{\theta}_0 = 1$, where δ_0 is a constant independent of the mesh parameter h, and hence, by virtue of the above theorem, the number of iterations it requires for MINRES to converge with preconditioner $\widetilde{P}_0$ can be bounded independent of h.

5.2 Stochastic $H(\mathbf{div}; D) \otimes L_2(\Omega)$ Preconditioner

The preconditioner to be used with MINRES in order to solve (4.18) is chosen to be analogous to (5.4). Therefore, let the ideal $H(\text{div}; D) \otimes L_2(\Omega)$ preconditioner be given by

$$P = \begin{bmatrix} A + F & 0 \\ 0 & N \end{bmatrix} \tag{5.9}$$

where A is given by (4.22), and $F = G_0 \otimes F_0$ and $N = G_0 \otimes N_0$ where G_0 is given by (4.23) and F_0 and N_0 are given by (5.5). The following theorem gives eigenvalue bounds on the preconditioned system $P^{-1}C$.

Theorem 14 *The $M \times (N_R \times N_Q)$ eigenvalues of $P^{-1}C$ lie in $(-1, -\mu] \cup \{1\}$ where μ is independent of h, m, and n.*

Proof The eigenvalues of $P^{-1}C$ satisfy

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{q} \end{bmatrix} = \nu \begin{bmatrix} A + F & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{q} \end{bmatrix}. \tag{5.10}$$

Now note that since $\nabla \cdot R = Q$ it follows that $F_0 = B_0 N_0^{-1} B_0$ and

$$\begin{aligned} B^T N^{-1} B &= (G_0 \otimes B_0)^T (G_0 \otimes N_0^{-1})(G_0 \otimes B_0) \\ &= G_0 \otimes (B_0^T N_0^{-1} B_0) = G_0 \otimes F_0 = F. \end{aligned} \tag{5.11}$$

Then, following the reasoning given for the deterministic case in [31], it can be shown that there are MN_R eigenvalues equal to one with the remaining eigenvalues satisfying

$$B(A + F)^{-1} B^T \mathbf{q} = -\nu N \mathbf{q}. \tag{5.12}$$

A bound for these eigenvalues can be obtained by considering (4.16), which using the equivalence of $||\cdot||_\Lambda$ and $||\cdot||_{H(\mathrm{div};D)\otimes L_2(\Omega)}$ can be written as

$$\zeta_{hn}||q||_{L_2(D)\otimes L_2(\Omega)} \leq \frac{1}{\sqrt{\min\{1,\alpha\}}} \sup_{v\in R\otimes T\setminus\{0\}} \frac{b(q,\nabla\cdot v)}{||v||_\Lambda} \tag{5.13}$$

In vector notation this becomes

$$\zeta_{hn}(\mathbf{q},\mathbf{q})_2^{1/2} \leq \frac{1}{\sqrt{\min\{1,\alpha\}}} \sup_{\mathbf{z}\in\mathbb{R}^{MN_R}\setminus\{\mathbf{0}\}} \frac{(\mathbf{q},B(A+F)^{-1/2}\mathbf{z})_2}{(\mathbf{z},\mathbf{z})_2^{1/2}} \tag{5.14}$$

where $\mathbf{z}=(A+F)^{1/2}\mathbf{v}$. The supremum can be shown, using the Cauchy-Schwarz inequality, to be attained when $\mathbf{z}=(A+F)^{-1/2}B^T\mathbf{q}$. Therefore,

$$\zeta_{hn}^2\min\{1,\alpha\} \leq \frac{(\mathbf{q},B(A+F)^{-1}B^T\mathbf{q})_2}{(\mathbf{q},N\mathbf{q})_2} \qquad \forall\mathbf{q}\in\mathbb{R}^{MN_Q}. \tag{5.15}$$

This gives a lower bound for the eigenvalues in (5.12) and consequently μ can be chosen to be $\zeta_{hn}^2\min\{1,\alpha\}$. $\square$

To make the preconditioning scheme practical a computationally optimal method is required to approximate the inverse of $A+F$ in each MINRES iteration. Assume there exists some matrix V such that

$$\theta \leq \frac{(\mathbf{v},(A+F)\mathbf{v})_2}{(\mathbf{v},V\mathbf{v})_2} \leq \overline{\theta} \leq 1 \qquad \forall\mathbf{v}\in\mathbb{R}^{MN_R}\setminus\{0\}. \tag{5.16}$$

Now consider a preconditioner given by

$$\widetilde{P} = \begin{bmatrix} V & 0 \\ 0 & N \end{bmatrix}. \tag{5.17}$$

Then the analogue of Theorem 13 holds, as stated in the following theorem.

Theorem 15 *The $M\times(N_R\times N_Q)$ eigenvalues of $\widetilde{P}^{-1}C$ lie in*

$$\left(-1,\frac{1}{2}\left(\theta\,(1-\mu)-\sqrt{\theta^2\,(\mu-1)^2+4\mu\theta}\right)\right]\cup\left[\,\theta,1\,\right], \tag{5.18}$$

where μ is as in Theorem 14 and θ is given in (5.16).

Proof The proof follows that for the deterministic case given in [29]. □

In the following sections, the analysis presented in [1] for the deterministic problem will be extended to construct a multigrid V-cycle operator $\mathcal{V}^{-1}\colon R\otimes T \rightarrow R \otimes T$ whose matrix representation satisfies (5.16), with constants $\overline{\theta} = 1$ and $\theta = 1 - \delta$ where δ is independent of h, m, and n.

5.3 Two-grid Function Bounds

In this section some bounds are obtained for the difference between functions defined on a fine mesh and coarse mesh, with mesh parameters h and H respectively. The bounds are analogous to those given in Lemma 3.1 in [1]. Throughout the remainder of this section the spaces and operators defined in Chapter 4 will be subscripted with an h or H to indicate the underlying triangulation they are defined with respect to.

First note that given an arbitrary function $q_h \in Q_h \otimes T$ and defining $v_h = \text{grad}_h^c\, q_h \in R_h \otimes T$, there exists unique $q_H \in Q_H \otimes T$ and $v_H \in R_H \otimes T$ such that $v_H = \text{grad}_H^c\, q_H$ and $\nabla \cdot v_H = \Theta_H \nabla \cdot v_h$. This is established by defining $f = -\nabla \cdot \text{grad}_h^c\, q_h$, whereby (q_h, v_h) and (q_H, v_H) can be shown to be the finite element approximations to the diffusion problem given in §4.4.

Lemma 1 *Given $q_h \in Q_h \otimes T$, there exists $\sigma \in R_h \otimes T$ such that*

$$\nabla \cdot \sigma = q_h - \Theta_H q_h, \qquad ||\sigma||_{H^1(D)^2 \otimes L_2(\Omega)} \leq C_3 ||q_h - \Theta_H q_h||_{L_2(D)\otimes L_2(\Omega)} \tag{5.19}$$

where C_3 is independent of h, m, and n.

Proof Consider the semi-discrete variational problem: find $w \in H_0^1(D) \otimes T$ such

that

$$\int_\Omega \int_D \nabla w \cdot \nabla v = \int_\Omega \int_D (q_h - \Theta_H q_h) v \quad \forall v \in H_0^1(D) \otimes T. \tag{5.20}$$

This can be seen as the weak formulation of the second-order diffusion problem with source function $f = q_h - \Theta_H q_h$. Regularity considerations give the bound

$$||w||_{H^2(D)} \leq C_D ||q_h - \Theta_H q_h||_{L_2(D)} \quad \forall \omega \in \Omega. \tag{5.21}$$

From this it follows that

$$||w||_{H^2(D)\otimes L_2(\Omega)} \leq C_D ||q_h - \Theta_H q_h||_{L_2(D)\otimes L_2(\Omega)}. \tag{5.22}$$

Since $q_h - \Theta q_h \in Q_h \otimes T$, it can be shown that $-\nabla^2 w = q_h - \Theta_H q_h$ a.e., where the derivatives are to be understood in the weak sense. Now set $\sigma = -\nabla w \in R_h \otimes T$. Then

$$\begin{aligned} ||\sigma||_{H^1(D)^2\otimes L_2(\Omega)} &= ||\nabla w||_{H^1(D)^2\otimes L_2(\Omega)} \\ &\leq \sqrt{2}\, ||w||_{H^2(D)\otimes L_2(\Omega)} \\ &\leq \sqrt{2}\, C_D ||q_h - \Theta_H q_h||_{L_2(D)\otimes L_2(\Omega)} \end{aligned} \tag{5.23}$$

which, equating C_3 with $\sqrt{2}\, C_D$, establishes the lemma. □

Theorem 16 *Let $q_h \in Q_h \otimes T$, then*

$$||q_h - \Theta_H q_h||_{L_2(D)\otimes L_2(\Omega)} \leq C_4 H ||\mathrm{grad}_h^c\, q_h||_U \tag{5.24}$$

where C_4 is independent of h, m, and n.

Proof Define σ as in Lemma 1. Then

$$\begin{aligned} ||q_h - \Theta_H q_h||^2_{L_2(D)\otimes L_2(\Omega)} &= (\nabla \cdot \sigma, q_h - \Theta_H q_h)_{L_2(D)\otimes L_2(\Omega)} \\ &= (\nabla \cdot \sigma, (\Theta_h - \Theta_H) q_h)_{L_2(D)\otimes L_2(\Omega)} \\ &= ((\Theta_h - \Theta_H)\nabla \cdot \sigma, q_h)_{L_2(D)\otimes L_2(\Omega)} \\ &= (\nabla \cdot (\Pi_h - \Pi_H)\sigma, q_h)_{L_2(D)\otimes L_2(\Omega)}, \end{aligned} \tag{5.25}$$

where, in the last line, (4.48) was used. Applying the definition of the discrete gradient operator and the Cauchy-Schwarz inequality gives

$$\begin{aligned}||q_h - \Theta_H q_h||^2_{L_2(D)\otimes L_2(\Omega)} &= ((\Pi_H - \Pi_h)\sigma, \text{grad}_h^c\, q_h)_U \\ &\leq ||(\Pi_H - \Pi_h)\sigma||_U ||\text{grad}_h^c\, q_h||_U \\ &\leq (||\sigma - \Pi_H\sigma||_U + ||\sigma - \Pi_h\sigma||_U)||\text{grad}_h^c\, q_h||_U. \quad (5.26)\end{aligned}$$

Now applying the equivalence between $||\cdot||_U$ and $||\cdot||_{L_2(D)^2\otimes L_2(\Omega)}$, the approximation bound given by (4.32), noting that $h < H$, and using Lemma 1, gives

$$\begin{aligned}||q_h - \Theta_H q_h||^2_{L_2(D)\otimes L_2(\Omega)} &\leq \frac{1}{\sqrt{\alpha}}\,(||\sigma - \Pi_H\sigma||_{L_2(D)^2\otimes L_2(\Omega)} + \\ &\qquad ||\sigma - \Pi_h\sigma||_{L_2(D)^2\otimes L_2(\Omega)})||\text{grad}_h^c\, q_h||_U \\ &\leq \frac{C_1}{\sqrt{\alpha}}\,(H+h)\,||\sigma||_{H^1(D)^2\otimes L_2(\Omega)}||\text{grad}_h^c\, q_h||_U \\ &\leq \frac{2HC_1C_3}{\sqrt{\alpha}}\,||q_h - \Theta_H q_h||_{L_2(D)\otimes L_2(\Omega)}||\text{grad}_h^c\, q_h||_U. \quad (5.27)\end{aligned}$$

This establishes the theorem with $C_4 = 2C_1C_3/\sqrt{\alpha}$. □

Theorem 17 *Let $q_h \in Q_h \otimes T$ and $v_h = \text{grad}_h^c\, q_h \in R_h \otimes T$. Define $q_H \in Q_H \otimes T$ and $v_H \in R_H \otimes T$ such that $v_H = \text{grad}_H^c\, q_H$ and $\nabla \cdot v_H = \Theta_H \nabla \cdot v_h$. Then*

$$||v_h - v_H||_U \leq C_5 H ||\nabla \cdot v_h||_{L_2(D)\otimes L_2(\Omega)} \quad (5.28)$$

where C_5 is independent of h, m, and n.

Proof As noted above v_h and v_H can be considered to be finite element approximations to the velocity solution of a first-order diffusion problem with source function $f = -\nabla \cdot \text{grad}_h^c\, q_h$. Let $v_* \in H(\text{div}; D) \otimes L_2(\Omega)$ be the solution to the semi-discrete problem as defined in §4.9 with this source function. Then using Theorem 10 and noting $H > h$ gives

$$||v_h - v_H||_U \leq ||v_* - v_h||_U + ||v_* - v_H||_U \leq 2C_2 H ||v_*||_{H^1(D)^2\otimes L_2(\Omega)}. \quad (5.29)$$

Now using (4.39) gives

$$||v_h - v_H||_U \leq 2C_2C_\bigstar H||\nabla \cdot \text{grad}_h^c\, q_h||_{L_2(D)\otimes L_2(\Omega)}$$
$$= 2C_2C_\bigstar H||\nabla \cdot v_h||_{L_2(D)\otimes L_2(\Omega)}, \tag{5.30}$$

which establishes the theorem with $C_5 = 2C_2C_\bigstar$. □

Theorem 18 *Let $q_h \in Q_h \otimes T$ and $v_h \in R_h \otimes T$. Define $q_H \in Q_H \otimes T$ and $v_H \in R_H \otimes T$ such that $v_H = \text{grad}_H^c\, q_H$ and $\nabla \cdot v_H = \Theta_H \nabla \cdot v_h$. Then*

$$||v_h - v_H||_\Lambda \leq C_6 H||\mathcal{H}_h v_h||_U \tag{5.31}$$

where C_6 is independent of h, m, and n.

Proof First, note that $\nabla \cdot v_h \in Q_h \otimes T$. Applying Theorem 16 gives

$$||\nabla \cdot v_h - \nabla \cdot v_H||_{L_2(D)\otimes L_2(\Omega)} = ||\nabla \cdot v_h - \Theta_H \nabla \cdot v_h||_{L_2(D)\otimes L_2(\Omega)}$$
$$\leq C_4 H||\text{grad}_h^c\, \nabla \cdot v_h||_U. \tag{5.32}$$

Then, using this along with Theorem 17 gives

$$||v_h - v_H||_\Lambda^2 = ||v_h - v_H||_U^2 + ||\nabla \cdot v_h - \nabla \cdot v_H||_{L_2(D)\otimes L_2(\Omega)}^2$$
$$\leq C_5^2 H^2||\nabla \cdot v_h||_{L_2(D)\otimes L_2(\Omega)}^2 + C_4^2 H^2||\text{grad}_h^c\, \nabla \cdot v_h||_U^2. \tag{5.33}$$

It can be shown that

$$||\mathcal{H}_h v_h||_U^2 = ||v_h||_\Lambda^2 + ||\nabla \cdot v_h||_{L_2(D)\otimes L_2(\Omega)}^2 + ||\text{grad}_h^c\, \nabla \cdot v_h||_U^2. \tag{5.34}$$

Therefore,

$$||v_h - v_H||_\Lambda \leq C_6 H||\mathcal{H}_h v_h||_U \tag{5.35}$$

where $C_6 = \max\{C_4, C_5\}$. □

5.4 Λ-Projection Bounds

In this section two bounds involving a function in $R_h \otimes T$ and its projection with respect to $\Lambda(\cdot,\cdot)$ onto $R_H \otimes T$ are obtained. Note that these two bounds comprise the stochastic analogue of Lemma 3.2 in [1].

Theorem 19 *Let $v \in R_h \otimes T$ and define $w = v - \mathcal{P}_H v \in R_h \otimes T$, with Helmholtz decomposition $w = \mathrm{grad}_h^c\, q + \nabla \times z$, where $q \in Q_h \otimes T$ and $z = S_h \otimes T$. Then*

$$||\mathrm{grad}_h^c\, q||_U \leq C_6 H ||w||_\Lambda \tag{5.36}$$

where C_6 is as in Theorem 18.

Proof First note that

$$\Lambda(\mathrm{grad}_h^c\, q, \mathcal{H}_h^{-1}\mathrm{grad}_h^c\, q) = (\mathrm{grad}_h^c\, q, \mathrm{grad}_h^c\, q)_U = ||\mathrm{grad}_h^c\, q||_U^2. \tag{5.37}$$

Let $\tau_h = \mathcal{H}_h^{-1}\mathrm{grad}_h^c\, q$. Then, using (4.41),

$$\Lambda(\nabla \times z, \tau_h) = (\nabla \times z, \tau_h)_U.$$

As $\mathcal{H}_h^{-1}$ maps $\mathrm{grad}_h^c(Q_h \otimes T)$ onto itself, as noted in §4.12, there exists $q_h' \in Q_h \otimes T$ such that $\tau_h = \mathrm{grad}_h^c\, q_h'$. So, employing the definition of grad_h^c and using (4.41) again,

$$\begin{aligned} \Lambda(\nabla \times z, \tau_h) &= (\nabla \times z, \mathrm{grad}_h^c\, q')_U \\ &= -(\nabla \cdot \nabla \times z, q')_{L_2(D)\otimes L_2(\Omega)} = 0. \end{aligned} \tag{5.38}$$

Now define τ_H in relation to τ_h as in Theorem 18. Then

$$\begin{aligned} \Lambda(\mathrm{grad}_h^c\, q, \tau_h) &= \Lambda(v - \mathcal{P}_H v, \tau_h) = \Lambda(v - \mathcal{P}_H v, \tau_h - \tau_H) \\ &\leq ||v - \mathcal{P}_H v||_\Lambda ||\tau_h - \tau_H||_\Lambda \leq C_6 H ||\mathrm{grad}_h^c\, q||_U ||w||_\Lambda \end{aligned} \tag{5.39}$$

which establishes the theorem. □

Theorem 20 *Let $v \in R_h \otimes T$ and define $w = v - \mathcal{P}_H v \in R_h \otimes T$ with Helmholtz decomposition $w = \mathrm{grad}_h^c\, q + \nabla \times z$, where $q \in Q_h \otimes T$ and $z \in S_h \otimes T$. Then*

$$||z||_{L_2(D)\otimes L_2(\Omega)} \leq C_7 H ||w||_\Lambda \tag{5.40}$$

where C_7 is independent of h, m, and n.

Proof A standard result from deterministic analysis (see [1]) yields

$$||z||_{L_2(D)} \leq C_\star H ||\nabla \times z||_{L_2(D)^2} \quad \forall \omega \in \Omega \tag{5.41}$$

for some constant $C_\star$. Therefore,

$$||z||_{L_2(D)\otimes L_2(\Omega)} \leq C_\star H ||\nabla \times z||_{L_2(D)^2 \otimes L_2(\Omega)} \leq C_\star \sqrt{\beta} H ||\nabla \times z||_U. \tag{5.42}$$

Now, using the properties of the Helmholtz decomposition noted in §4.10,

$$\begin{aligned} ||\nabla \times z||_U^2 &= (w - \mathrm{grad}_h^c\, q, \nabla \times z)_U = (w, \nabla \times z)_U \\ &= (w, \nabla \times z)_U + (\nabla \cdot w, \nabla \cdot \nabla \times z)_{L_2(D)\otimes L_2(\Omega)} \\ &= \Lambda(w, \nabla \times z) \leq ||w||_\Lambda ||\nabla \times z||_U. \end{aligned} \tag{5.43}$$

Therefore $||\nabla \times z||_U \leq ||w||_\Lambda$ which establishes the theorem with $C_7 = C_\star \sqrt{\beta}$. □

5.5 Additive Schwarz Method

The multigrid algorithm to be described in §5.6 makes use of the additive Schwarz method which is defined with respect to a covering of the spatial domain D. In this section the additive Schwarz method is described along with results that will be used to establish the convergence of multigrid.

Let $\mathcal{D} = \{D_1, \ldots, D_L\}$ be an overlapping covering of D. Let γ be an integer such that no point in $\overline{D}$ occurs in more than γ elements of $\mathcal{D}$. Then $\sum_k \int_{D_k} \leq \gamma \int_D$ and γ is called the overlap parameter. Now define

$$R^k \otimes T = \{\, v \in R \otimes T \mid \mathrm{supp}(v) \subset D_k \otimes \Omega \,\}. \tag{5.44}$$

Then given $v \in R \otimes T$ there exists a decomposition $v = \sum_k v_k$, $v_k \in R^k \otimes T$, (though the decomposition is not necessarily unique).

Let $\{\, \theta^k \colon \overline{D} \to \mathbb{R}\}_{k=1}^L$ be a partition of unity subordinate to the covering of $\mathcal{D}$, i.e. $\sum_{k=1}^L \theta^k = 1$ and $\mathrm{supp}\{\theta^k\} \subset D_k$.

Theorem 21 *Let $v \in R \otimes T$ and $v_k = \Pi\theta^k v \in R^k \otimes T$. Then, by construction, $v = \sum_k v_k$ and*

$$\sum_{k=1}^{L} \Lambda(v_k, v_k) \leq C_8((1 + h^{-2})||v||_U^2 + ||\nabla \cdot v||^2_{L_2(D)\otimes L_2(\Omega)}) \tag{5.45}$$

where C_8 is independent of h, m, and n.

Proof Following [1], there exists a constant $C_\dagger$ such that

$$||v_k||_{L_2(D)^2} \leq C_\dagger ||v||^2_{L_2(D_k)^2} \quad \forall \omega \in \Omega. \tag{5.46}$$

Therefore, introducing the notation $||\cdot||_{U^k}$ to denote a norm defined in the same manner as $||\cdot||_U$ but restricted to D_k gives

$$||v_k||_U^2 \leq \frac{1}{\alpha} ||v_k||^2_{L_2(D)^2\otimes L_2(\Omega)} \leq \frac{C_\dagger}{\alpha} ||v||^2_{L_2(D_k)^2\otimes L_2(\Omega)} \leq \frac{\beta C_\dagger}{\alpha} ||v||^2_{U^k}. \tag{5.47}$$

Also, following [1], there exists a constant $C_\ddagger$ such that

$$||\nabla \cdot v_k||_{L_2(D)} \leq C_\ddagger(h^{-1}||v||_{L_2(D_k)^2} + ||\nabla \cdot v||_{L_2(D_k)}) \quad \forall \omega \in \Omega. \tag{5.48}$$

Therefore,

$$||\nabla \cdot v_k||^2_{L_2(D)\otimes L_2(\Omega)} \leq 2C_\ddagger^2(h^{-2}||v||^2_{L_2(D_k)^2\otimes L_2(\Omega)} + ||\nabla \cdot v||^2_{L_2(D_k)\otimes L_2(\Omega)})$$
$$\leq 2C_\ddagger^2(\beta h^{-2}||v||^2_{U^k} + ||\nabla \cdot v||^2_{L_2(D_k)\otimes L_2(\Omega)}). \tag{5.49}$$

Then using these two results gives

$$\sum_{k=1}^{L} \Lambda(v_k, v_k) = \sum_{k=1}^{L}(||v_k||^2_U + ||\nabla \cdot v_k||^2_{L_2(D)\otimes L_2(\Omega)})$$
$$\leq \sum_{k=1}^{L}\left(\frac{\beta C_\dagger^2}{\alpha}||v||_{U^k} + 2C_\ddagger^2(\beta h^{-2}||v||^2_{U^k} + ||\nabla \cdot v||^2_{L_2(D_k)\otimes L_2(\Omega)})\right)$$
$$\leq \gamma \max\left\{\frac{\beta C_\dagger^2}{\alpha}, 2C_\ddagger^2 \max\{\beta, 1\}\right\}((1+h^{-2})||v||^2_U$$
$$+ ||\nabla \cdot v||^2_{L_2(D)\otimes L_2(\Omega)}) \tag{5.50}$$

where γ is the overlap parameter mentioned above. □

The additive Schwarz operator is defined by

$$\mathcal{S} = \eta \sum_{k=1}^{L} \mathcal{P}^k \mathcal{H}^{-1}, \tag{5.51}$$

where η is some constant, chosen such that Theorem 22 given below holds, $\mathcal{P}^k$: $R^k \otimes T \to R \otimes T$ is a projection operator defined by

$$\Lambda(\mathcal{P}^k v, w) = \Lambda(v, w) \qquad \forall w \in R^k \otimes T, \tag{5.52}$$

and $\mathcal{H}^{-1}$ is as in §4.12. The additive Schwarz operator thus defined is symmetric on U. To see this, let v, $w \in U$, then

$$(\mathcal{S}v, w)_U = \eta \sum_{k=1}^{L}(\mathcal{P}^k\mathcal{H}^{-1}v, w)_U = \eta \sum_{k=1}^{L} \Lambda(\mathcal{P}^k\mathcal{H}^{-1}v, \mathcal{H}^{-1}w)$$
$$= \eta \sum_{k=1}^{L} \Lambda(\mathcal{H}^{-1}v, \mathcal{P}^k\mathcal{H}^{-1}w) = \Lambda(\mathcal{H}^{-1}v, \mathcal{S}w) = (v, \mathcal{S}w)_U. \tag{5.53}$$

Now assume that there exists a subspace of U, W say, such that for each $v \in W$ there exists a decomposition $v = \sum_{k=1}^{L} v_k$, $v_k \in R^k \otimes T$ such that

$$\sum_{k=1}^{L} \Lambda(v_k, v_k) \leq C_\diamond \Lambda(v, v), \tag{5.54}$$

where $C_\diamond$ is a constant. Then $\mathcal{S}$ is positive definite on W and hence invertible on W. To see this, let $\mathcal{P}_\mathcal{S} = \sum_{k=1}^{L} \mathcal{P}^k$ and let $v \in W$. Then

$$\begin{aligned}
\Lambda(v, v) &= \sum_{k=1}^{L} \Lambda(v, v_k) = \sum_{k=1}^{L} \Lambda(\mathcal{P}^k v, v_k) \leq \sum_{k=1}^{L} ||\mathcal{P}^k v||_\Lambda ||v_k||_\Lambda \\
&\leq \left(\sum_{k=1}^{L} \Lambda(\mathcal{P}^k v, \mathcal{P}^k v) \right)^{1/2} \left(\sum_{k=1}^{L} \Lambda(v_k, v_k) \right)^{1/2} \\
&\leq \Big(\Lambda(\mathcal{P}_\mathcal{S} v, v) \Big)^{1/2} \Big(C_\diamond \Lambda(v, v) \Big)^{1/2}.
\end{aligned} \tag{5.55}$$

Therefore,

$$\Lambda(\mathcal{P}_S v, v) \geq \frac{1}{C_\diamond} \Lambda(v, v). \tag{5.56}$$

Returning to the additive Schwarz operator and letting $w = \mathcal{H}^{-1} v \in W$ gives

$$\begin{aligned}
(\mathcal{S}v, v)_U &= \eta \sum_{k=1}^{L} (\mathcal{P}^k \mathcal{H}^{-1} v, v)_U = \eta \sum_{k=1}^{L} \Lambda(\mathcal{P}^k \mathcal{H}^{-1} v, \mathcal{H}^{-1} v) \\
&= \eta \sum_{k=1}^{L} \Lambda(\mathcal{P}^k w, w) = \eta \, \Lambda(\mathcal{P}_\mathcal{S} w, w) \geq \frac{1}{C_\diamond} \, \eta \, \Lambda(w, w).
\end{aligned} \tag{5.57}$$

As $v = 0 \Leftrightarrow w = 0$ this shows that $\mathcal{S}$ is positive definite on W and hence invertible on W. This argument follows that given in [37], where the additive Schwarz operator is considered in a general context.

5.6 Multigrid

In the following, a family of triangulations, $\mathcal{T}_j$, $j = 1, \ldots, J$, is considered with mesh parameters h_j, where $h_i > h_j$ for $i < j$, that is to say $\mathcal{T}_J$ is the finest mesh

and $\mathcal{T}_1$ is the coarsest mesh. These triangulations give rise to a nested sequence of finite-dimensional spaces denoted $R_1 \otimes T \subset \cdots \subset R_J \otimes T \subset V$ as defined in §4.4. For the remainder of this section the various operators and spaces defined in Chapter 4 will be subscripted to denote the underlying triangulation they are defined with respect to. Further, let $\mathcal{D}_j = \{D_1, \ldots, D_{L_j}\}$ be a covering of D with respect to $\mathcal{T}_j$, as discussed in §5.5, and let $\mathcal{S}_j$, $j = 2, \ldots, J$, be the additive Schwarz operator defined with respect to D_j. Let $\mathcal{S}_1 = \mathcal{H}_1^{-1}$.

Now consider the equation $\mathcal{H}_J v = z$ where $\mathcal{H}_J$ is defined in §4.12 and $z \in R_J \otimes T$ is known. To obtain an approximation to $v \in R_J \otimes T$, a family of multigrid operators $\mathcal{V}_j^{-1} \colon R_j \otimes T \to R_j \otimes T$ is defined such that $\mathcal{V}_1^{-1} = \mathcal{H}^{-1}$ and $\mathcal{V}_j^{-1}$, $j = 2, \ldots, J$, is given by the algorithm:

$v = 0$

for $i = 1, \ldots, k$

$\quad v \leftarrow v + \mathcal{S}_j(\vec{z} - \mathcal{H}_j v)$

end

$v \leftarrow v + \mathcal{V}_{j-1}^{-1} \Theta_{j-1}(z - \mathcal{H}_j v)$

for $i = 1, \ldots, k$

$\quad v \leftarrow v + \mathcal{S}_j(z - \mathcal{H}_j v)$

end

This corresponds to a multigrid V-cycle with one multigrid iteration at each level. At each level there are k pre-smoothing steps and k post-smoothing steps.

The matrix representation of the operator $\mathcal{H}_J$ is $A_J^{-1}(A_J + F_J)$ where A_J and F_J are as in §5.2. Therefore, the vector equation $\mathcal{H}_J v = z$ can be expressed as the linear system $(A_J + F_J)\mathbf{v} = A_J \mathbf{z}$, where $\mathbf{v}$ and $\mathbf{z}$ are the coefficient vectors

of v and z. This relationship along with the convergence properties of the above algorithm will be used to demonstrate that the spectral bounds in (5.16) will be independent of the parameters h, m, and n if the matrix V_J^{-1} is chosen to be the matrix representation of the multigrid operator $\mathcal{V}_J^{-1}$.

Theorem 22 *The inequality*

$$\Lambda((I - \mathcal{S}_j\mathcal{H}_j)v, v) \geq 0 \quad \forall v \in R_j \otimes T \tag{5.58}$$

holds for $j = 1, \ldots, J$.

Proof For $j = 1$, $\mathcal{S}_j = \mathcal{H}_j^{-1}$ and the result holds as $\Lambda(\cdot, \cdot)$ is an inner product on $R_1 \otimes T$. Let $2 < j \leq J$. Then

$$\begin{aligned}
\Lambda((I - \mathcal{S}_j\mathcal{H}_j)v, v) &= \Lambda(v, v) - \eta \sum_{k=1}^{L_j} \Lambda(\mathcal{P}_j^k v, v) \\
&= \Lambda(v, v) - \eta \sum_{k=1}^{L_j} \Lambda(\mathcal{P}_j^k v, \mathcal{P}_j^k v) \\
&= \Lambda(v, v) - \eta \sum_{k=1}^{L_j} ||\mathcal{P}_j^k v||_\Lambda^2.
\end{aligned} \tag{5.59}$$

Let $||\cdot||_{\Lambda_k}$ be the norm induced on $D_k \times \Omega$ by $||\cdot||_\Lambda$. Then

$$||\mathcal{P}_j^k v||_\Lambda^2 = \Lambda(\mathcal{P}_j^k v, \mathcal{P}_j^k v) = \Lambda(\mathcal{P}_j^k v, v) \leq ||\mathcal{P}_j^k v||_{\Lambda_k} ||v||_{\Lambda_k} = ||\mathcal{P}_j^k v||_\Lambda ||v||_{\Lambda_k}. \tag{5.60}$$

Therefore,

$$\Lambda((I - \mathcal{S}_j\mathcal{H}_j v, v) \leq (1 - \eta\gamma)\Lambda(v, v) \tag{5.61}$$

where γ is the overlap parameter discussed in §5.5. So the theorem holds providing $\eta \leq \gamma^{-1}$. □

Lemma 2 *Let $v \in R_j \otimes T$ and let $w \in (I - \mathcal{P}_{j-1})v \in R_j \otimes T$. Then, for $j = 2, \ldots, J$, there exists a decomposition $w = \sum_{k=1}^{L_j} w_k$, $w_k \in R_j^k \otimes T$ such that*

$$\sum_{k=1}^{L_j} \Lambda(w_k, w_k) \leq C_9 \Lambda(w, w) \tag{5.62}$$

where C_9 is independent of h, m, and n.

Proof Let w have the Helmholtz decomposition $w = \mathrm{grad}_{h_j}^c\, q + \nabla \times s$, where $q \in Q_j \otimes T$ and $s \in R_j \otimes T$. Define $w' = \mathrm{grad}_{h_j}^c\, q$ and $w'' = \nabla \times s$. Then given the decompositions $w' = \sum_{k=1}^{L_j} w'_k$ and $w'' = \sum_{k=1}^{L_j} w''_k$ and taking into account that the Helmholtz decomposition is orthogonal with respect to $\Lambda(\cdot, \cdot)$ it is sufficient to show that

$$\sum_{k=1}^{L_j} \Lambda(w'_k, w'_k) \leq C'_9 \Lambda(w, w), \qquad \sum_{k=1}^{L_j} \Lambda(w''_k, w''_k) \leq C''_9 \Lambda(w, w). \tag{5.63}$$

for some constants C'_9 and C''_9. This will result in $C_9 = 2\max\{C'_9, C''_9\}$. To show the first of these, let $w'_k = \mathit{\Pi}_j \theta_j^k w'$ where θ_j^k is as in §5.5. Invoking Theorems 19 and 21 gives

$$\begin{aligned}
\sum_{k=1}^{L_j} \Lambda(w'_k, w'_k) &\leq C_8((1 + h_j^{-2})||w'||_U^2 + ||\nabla \cdot w'||_{L_2(D)\otimes L_2(\Omega)}^2) \\
&= C_8(||w'||_\Lambda^2 + h_j^{-2}||w'||_U^2) \\
&\leq C_8(||w'||_\Lambda^2 + C_6^2 h_j^{-2} h_{j-1}^2 ||w||_\Lambda^2).
\end{aligned} \tag{5.64}$$

Assume that for $j = 2, \ldots, J$ there exists $\varrho > h_{j-1}/h_j$, then, as $||w'||_\Lambda \leq ||w||_\Lambda$,

$$\sum_{k=1}^{L_j} \Lambda(w'_k, w'_k) \leq 2C_8 \max\{1, C_6^2 \varrho^2\}\, \Lambda(w, w). \tag{5.65}$$

Next, following [1], there exists a decomposition $w'' = \sum w''_k$ such that

$$\sum_{k=1}^{L_j} (w''_k, w''_k)_{L_2(D)^2} \leq C_{10}(||w''||_{L_2(D)^2}^2 + h_j^{-2}||s||_{L_2(D)}^2) \quad \forall \omega \in \Omega, \tag{5.66}$$

from which it follows (using the fact that $||\nabla \times s||_{L_2(D)\otimes L_2(\Omega)} = 0$) that

$$\sum_{k=1}^{L_j} \Lambda(w_k'', w_k'') \leq \alpha^{-1} C_{10} (\beta ||w''||_\Lambda^2 + h_j^{-2} ||s||_{L_2(D)\otimes L_2(\Omega)}^2). \tag{5.67}$$

Applying Theorem 20 and noting that $||w''||_\Lambda \leq ||w||_\Lambda$ gives

$$\sum_{k=1}^{L_j} \Lambda(w_k'', w_k'') \leq 2\alpha^{-1} C_{10} \max\{\beta, C_7^2 \varrho^2\} \, \Lambda(w, w). \tag{5.68}$$

Therefore the theorem is established. □

Lemma 3 *Let $v \in R_j \otimes T$ and $w = (I - \mathcal{P}_{j-1})v \in R_j \otimes T$. Then for $j = 2, \ldots, N$,*

$$(\mathcal{S}_j^{-1} w, w)_U = \frac{1}{\eta} \inf \sum_{k=1}^{L_j} \Lambda(w_k, w_k) \tag{5.69}$$

where the infimum is taken over all decompositions of the form $w = \sum_{k=1}^{L_j} w_k$, $w_k \in R_j^k \otimes T$.

Proof Let $W = \{\, w \in R_j \otimes T \mid w = (I - \mathcal{P}_{j-1})v,\ v \in R_j \otimes T \,\}$. Then, following Lemma 2, (5.54) is satisfied and $\mathcal{S}$ is invertible on W. The rest of the proof follows in an analogous fashion to the proof for the deterministic case given in Appendix B of [1]. □

Theorem 23 *The inequality*

$$(\mathcal{S}_j^{-1}(I - \mathcal{P}_{j-1})v, (I - \mathcal{P}_{j-1})v)_U \leq C_{11} \Lambda((I - \mathcal{P}_{j-1})v, (I - \mathcal{P}_{j-1})v) \tag{5.70}$$

holds for all $v \in R_j \otimes T$, $j = 2, \ldots, J$, where C_{11} is independent of h, m, and n.

Proof Let $w = (I - \mathcal{P}_{j-1})v$. Then for each $v \in R_j \otimes T$ it is required to show that

$$(\mathcal{S}^{-1} w, w)_U \leq C_{11} \Lambda(w, w). \tag{5.71}$$

From Lemma 2 there exists a decomposition $w = \sum_{k=1}^{L_j} w_k$, $w_k \in R_j^k \otimes T$ such that

$$\sum_{k=1}^{L_j} \Lambda(w_k, w_k) \leq C_9 \Lambda(w, w). \tag{5.72}$$

Now using Lemma 3 gives (5.71) with $C_{11} = \eta^{-1} C_9$. $\square$

Theorem 24 *The eigenvalues of $\mathcal{V}_J^{-1}\mathcal{H}_J$ are contained in the interval $[1-\delta, 1]$ where*

$$\delta = \frac{C_{11}}{C_{11} + 2k} \tag{5.73}$$

where C_{11} is as in Theorem 23 and k is as in the multigrid algorithm given earlier in the section.

Proof Given Theorem 22 and Theorem 23, the proof follows that for Corollary 5.2 in [1]. $\square$

Theorem 24 can be restated as

$$1 - \delta \leq \frac{(\mathcal{H}_J v, v)_U}{(\mathcal{V}_J v, v)_U} \leq 1 \qquad \forall v \in R_J \otimes T. \tag{5.74}$$

The multigrid algorithm that defines $\mathcal{V}_J^{-1}$ finds an approximation, v_1 say, to the solution of $\mathcal{H}_J v = z$. This equation can be expressed as $(A_J + F_J)\mathbf{v} = A_J \mathbf{z}$ where A_J and F_J are as in §5.2 (only there they weren't subscripted) and $\mathbf{v}$ and $\mathbf{z}$ are the coefficient vectors of v and z. Now let V_J be the matrix such that $V_J^{-1} A_J \mathbf{z} = \mathbf{v}_1$ where $\mathbf{v}_1$ is the coefficient vector of v_1. That is to say, V_J^{-1} is the matrix representation of the multigrid operator $\mathcal{V}_J^{-1}$. Then

$$\frac{(\mathcal{H}_J v, v)_U}{(\mathcal{V}_J v, v)_U} = \frac{\Lambda(v, v)}{(\mathcal{V}_J v, v)_U} = \frac{((A_J + F_J)\mathbf{v}, \mathbf{v})_2}{(A_J^{-1} V_J \mathbf{v}, \mathbf{v})_{A_J}} = \frac{((A_J + F_J)\mathbf{v}, \mathbf{v})_2}{(V_J \mathbf{v}, \mathbf{v})_2}. \tag{5.75}$$

This establishes (5.16) with $\theta = 1 - \delta$ and $\overline{\theta} = 1$.

5.7 Numerical Experiments

For the numerical experiments the model problem as discussed in §4.6 is considered with the random variables appearing in the Karhunen-Loève expansion of the reciprocal of the diffusion coefficient being firstly, uniformly distributed, and secondly, normally distributed.

5.7.1 Diffusion with Uniform Distributions

Consider ξ_r, $r = 1, \ldots, m$, to be uniformly distributed on $(-1, 1)$ which implies $\rho(y) = 1/2^m$. Let $\nu = 1/3$ and let $c_0 = 10$. Let the triangulation of D be as described in §3.7.1.

Now MINRES is applied with a stopping tolerance of 10^{-6}. The preconditioner described in §5.2 is used with the multigrid algorithm given in §5.6. For each solve involving the preconditioner, one of which is required in each MINRES iteration, one multigrid V-cycle is used. Within multigrid at each level, one pre-smooth and one post-smooth using the additive Schwarz smoother, as described in §5.5, is carried out, with the parameter η set to be 0.3. Table 5.1 shows the number of iterations it requires for MINRES to converge to the required tolerance. The results support the conclusion that the convergence rate of the method is independent of the parameters h, m, and n.

5.7.2 Diffusion with Normal Distributions

Now consider ξ_r, $r = 1, \ldots, m$, to be normally distributed which implies $\rho(y) = e^{-y^2/(2\nu)}/(2\pi\nu)^{m/2}$. Let $\nu = 0.01$ and let $c_0 = 1$. With the random variables in the expansion of the diffusion coefficient defined in this way, the condition given

by (4.12) is no longer satisfied. However, as noted in §3.7.2, diffusion defined in this manner has been studied in the literature, e.g., [20], and providing ν is sufficiently small the problem is thought to yield sensible results.

The MINRES algorithm is implemented as in §5.7.1. Table 5.2 shows the number of iterates required for the method to converge to within the desired tolerance. The results support the theoretical conclusion that the convergence rate is independent of the parameters h, m, and n.

5.8 Efficiency of the Additive Schwarz Smoother

The multigrid algorithm, described in §5.6, provides an optimal way of solving the system $(A + F)\mathbf{v} = \mathbf{z}$ where A and F are as in §5.2. By optimal it is meant that the convergence rate of the method is independent of the discretization parameters. However, the method as described is not as efficient as one would like. To see why this is so it is necessary to look at how the additive Schwarz smoother is implemented. The additive Schwarz operator is given by

$$\mathcal{S} = \eta \sum_{k=1}^{L} \mathcal{P}^k \mathcal{H}^{-1} \tag{5.76}$$

where L is the number of patches in the domain decomposition of D, i.e. the number of elements in $\mathcal{D}$. The implementation of this smoother follows that described in [31] where L is equal to the number of nodes in the mesh and each patch consists of all the triangles attached to a given node. Consequently, the matrix representation of the additive Schwarz operator is

$$S = \eta \sum_{k=1}^{L} R_k^T (A^{(k)} + F^{(k)})^{-1} R_k \tag{5.77}$$

where R_k is a restriction matrix corresponding to the k-th patch. The matrix $A^{(k)} + F^{(k)}$ is a submatrix of $A + F$ where the rows and columns not associated

$d = 4$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	25	26	26	26
$n = 2$	25	27	27	27
$n = 3$	27	27	27	27
$n = 4$	27	27	27	27
$d = 8$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	33	34	34	34
$n = 2$	33	34	34	34
$n = 3$	33	34	34	34
$n = 4$	36	36	36	36
$d = 16$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	36	38	36	36
$n = 2$	37	39	39	37
$n = 3$	37	39	39	37
$n = 4$	39	41	42	37
$d = 32$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	38	37	37	37
$n = 2$	37	39	39	39
$n = 3$	38	39	39	39
$n = 4$	39	42	42	42

Table 5.1: Number of iterations required for MINRES to converge to the solution of the first-order problem with diffusion defined via uniform distributions.

$d = 4$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	16	17	17	17
$n = 2$	16	17	17	17
$n = 3$	16	18	19	19
$n = 4$	18	18	19	19
$d = 8$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	20	20	20	20
$n = 2$	20	20	20	20
$n = 3$	20	20	20	20
$n = 4$	22	22	20	20
$d = 16$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	21	21	21	21
$n = 2$	23	23	23	23
$n = 3$	23	23	23	23
$n = 4$	23	23	23	23
$d = 32$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	21	21	21	21
$n = 2$	21	23	23	23
$n = 3$	23	23	23	23
$n = 4$	23	23	23	23

Table 5.2: Number of iterations required for MINRES to the converge to the solution of the first-order problem for diffusion defined via normal distributions.

with the k-th patch have been eliminated. This submatrix can be written as

$$A^{(k)} + F^{(k)} = G_0 \otimes A_0^{(k)} + \sum_{r=1}^{m} \sqrt{\lambda_r} G_r \otimes A_r^{(k)} + G_0 \otimes F_0^{(k)}. \tag{5.78}$$

For the choice of the zeroth-order Raviart-Thomas space for the finite-dimensional subspace of $H(\mathrm{div}; D)$, the dimension of the matrices $A_k^{(k)}$, $k = 0, \ldots, m$, and $F_0^{(k)}$ will be equal to the number of edges associated with the k-th node, say $N_R^{(k)}$, which for the uniform meshes considered in §5.7 will be no more than six. Therefore, the dimension of $A^{(k)} + F^{(k)}$ will be $MN_R^{(k)}$ where M is the dimension of T. As each implementation of the smoother will require L (where L varies from grid to grid) solves with matrices of size $MN_R^{(k)} \times MN_R^{(k)}$, $k = 1, \ldots, L$, this is going to be expensive for non-trivial values of M.

To try and overcome this problem one can replace $A^{(k)} + F^{(k)}$ with

$$\widehat{A}^{(k)} + F^{(k)} = G_0 \otimes A_0^{(k)} + \sum_{r=1}^{m} \sqrt{\lambda_r} G_0 \otimes A_r^{(k)} + G_0 \otimes F_0^{(k)}. \tag{5.79}$$

This matrix is block diagonal and hence the action of its inverse will be cheaper to compute than that for (5.78). It also offers the possibility of parallelizing the smoother algorithm. Tables 5.3 and 5.4 show the number of iterates it requires for MINRES to converge when the numerical experiments described in §5.7.1 and §5.7.2 are carried out with this alternative suggestion for the additive Schwarz smoother. As can be seen, the convergence rate of this method also appears to be independent of the discretization parameters. However, an analysis has not been carried out for this alternative choice of smoother and further experiments may indicate that the method is not optimal for a smaller ratio of the mean to the variance.

$d = 4$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	28	30	30	30
$n = 2$	28	30	30	30
$n = 3$	29	30	31	31
$n = 4$	30	30	31	31
$d = 8$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	35	37	37	37
$n = 2$	35	38	39	39
$n = 3$	35	38	39	39
$n = 4$	39	39	39	39
$d = 16$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	39	40	40	40
$n = 2$	39	40	42	42
$n = 3$	39	42	42	42
$n = 4$	42	42	43	43
$d = 32$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	39	40	40	40
$n = 2$	40	42	42	42
$n = 3$	40	42	42	42
$n = 4$	42	43	43	43

Table 5.3: Number of iterations required for MINRES, with alternative additive Schwarz smoother, to converge to the solution of the first-order problem with diffusion defined via uniform distributions.

$d = 4$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	23	27	27	27
$n = 2$	24	29	30	30
$n = 3$	26	30	32	32
$n = 4$	29	32	33	33
$d = 8$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	30	31	33	31
$n = 2$	32	33	33	33
$n = 3$	33	35	35	36
$n = 4$	35	36	36	36
$d = 16$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	31	31	31	33
$n = 2$	33	33	33	33
$n = 3$	33	34	35	35
$n = 4$	35	36	36	36
$d = 32$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	31	31	31	31
$n = 2$	33	33	33	33
$n = 3$	33	33	33	33
$n = 4$	33	33	33	33

Table 5.4: Number of iterations required for MINRES, with alternative additive Schwarz smoother, to converge to the solution of the first-order problem with diffusion defined via normal distributions.

Chapter 6

Concluding Remarks

As mentioned in the introduction, it was the purpose of this thesis to take some precursory steps in developing iterative methods for solving the large linear block systems that arise from the application of the stochastic finite element method to stochastic diffusion problems. In particular, methods incorporating geometric multigrid algorithms that are known to be optimal for the related deterministic problems were extended and shown to retain their optimal properties. There is, of course, far more to be said and done on the matter. For example, for the positive definite systems associated with the second-order problem, one might consider how preconditioned conjugate gradient performs for certain choices of the preconditioner. For the symmetric indefinite systems associated with the first-order problem, one might investigate different choices of preconditioner, as indeed was done in [17], and see how these methods compare.

There are other computational comparisons that need to be made. Such as, how do the methods discussed in this thesis compare to methods that decouple the block system into a number of deterministic problems that can be solved in parallel, though at a cost of using a much higher dimensional space? And how do stochastic finite elements fare against Monte Carlo methods when a probability

distribution is sought? There is also the issue of how things are affected when the random variables are not independent. One way around this is to use a polynomial chaos expansion of the diffusion coefficient which upon application of the polynomial chaos method produces a denser block system, the density depending on the order of terms kept in the expansion. The methods in this thesis are applicable to this scenario but clearly their efficiency will be degraded by the denser coefficient matrix. How do these methods then fare against the collocation method which can handle non-independent random variables and non-linear expansions of the diffusion coefficient, and produces a decoupled system, though again at the cost of using a higher dimensional space?

In short, there are many computational questions that need to be asked and answered regarding the solution of linear systems arising from stochastic partial differential equations. It is hoped that this modest thesis provides a little preliminary insight into these issues.

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