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***Abstract*—In this paper, we investigate the optimal design of wireless networks. We consider wireless networks that have fixed and movable nodes, and we assume that all nodes feature adjustable transmission power. Hence, we aim at maximizing network centric objectives, by optimizing over admissible choices of the positions of the movable nodes as well as the transmission power at all the nodes. We adopt exponential path loss, which is a realistic assumption in urban and sub sea environments, and we propose ways of using this assumption to obtain a tractable optimization problem. Our formulation allows for the optimization of typical network centric objectives, such as power and throughput. It also allows signal-to-interference based constraints, such as rate-regions and outage probabilities, under the high signal to interference regime. We show that our optimization paradigm is convex and that it can be solved up to an arbitrary degree of accuracy via geometric programming techniques. By using a primal-dual decomposition, we also provide a case-study that illustrates how certain instances of our optimization paradigm can be solved via distributed iterative algorithms. We show that such a solution method also leads to a convenient layering in the primal step, whereby the power allocation and the node placement become two independent sub-problems.**

***Index Terms*—Wireless, network, power allocation, optimal placement, convex, geometric program, primal-dual decomposition**

I. Introduction

Optimal node placement in wireless networks has received significant attention in the networking, robotics and computer science research communities. Examples of optimal placement paradigms are the maximization of the coverage of a sensor network [1], or the design of a wireless network so as to minimize the number of relays [10], [3], under a combination of power, longevity and rate-region constraints. The resulting optimization paradigm depends on the model of the wireless medium, the cost function and the constraints. Some of the existing paradigms are inherently combinatorial, while others rely on suboptimal strategies. In this paper, we observe that, by adopting an exponential path loss model, we can integrate optimal node placement in existing convex programming techniques, which so far have been used for optimal power allocation in wireless networks. Hence, we obtain an optimization paradigm, for wireless network design, that is jointly parameterized by the power allocation of all nodes and by the placement of nodes that can be moved. Notice that exponential path loss is characteristic of high absorption media, such as radio frequency communication in urban [16] and in sub sea environments [12].

Our formulation is general enough to model the effects of interference and to include constraints on the rate region and on the outage probability at pre-defined routing paths, in the high signal to interference regime. In addition, we show that our paradigm is convex and that it can be solved with arbitrary accuracy using geometric programming techniques, which are highly desirable due to their guaranteed polynomial time properties [18], [23]. We present a case study, where we exemplify how certain instances of our paradigm can be solved via a primal-dual iterative scheme. An attractive characteristic of such a scheme is that, in the primal step, the power allocation and the positions of the nodes can be optimized independently, which can be viewed as layering. In addition, our case-study illustrates how the primal-dual algorithm might be implemented in a distributed way.

This paper is organized as follows: first in section II, we mention some preliminary definitions and assumptions regarding our communication model, then in section III, we give a precise description of our design problem. After proving the convexity of our placement problem in section IV, we proceed in section V, by some examples that comply with our problem formulation. In section VI, we give an approximate solution to the placement problem by utilizing Geometric Programming. Then in section VII, we focus on a particular placement optimization problem and we propose a layering approach together with an efficient primal-dual algorithm that leads to a decentralized solution to that problem. Section VIII, is dedicated to some simulations to picture the performance of the proposed algorithms and finally in section IX, we give some possible extensions to our optimization framework.

II. Preliminary definitions and assumptions

Before we give a description of the model of the wireless network adopted in this paper, we introduce the following basic notation:

- Design parameters that are integers are represented using large caps Greek letters, such as Ω, while scalar or finite vectors of real numbers are represented using small caps Greek letters, such as ϕ. Set valued parameters are represented in blackboard font, such as $\mathbb{S}$.
- Optimization variables are indicated using boldface fonts, such as $\mathbf{P}$ and $\mathbf{x}$.
- The letters i and j are reserved for use as subscripts for integer indexing, with respect to the nodes of the wireless network. The letters k and l are also set aside for integer indexing.
- Functions are represented in calligraphic font, such as $\mathcal{U}$.

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A. Basic description of the nodes in the network

Consider a wireless network consisting of a collection of nodes placed in a Cartesian plane. A non-empty sub-collection of these nodes is fixed, i.e., their positions in the Cartesian plane are pre-selected, while the locations of the remaining nodes are optimization parameters. Denote by Δ the number of fixed nodes and by Ω the number of remaining (movable) nodes in the network. The nodes are uniquely identified by an integer index in the set $\{1, \ldots, \Delta + \Omega\}$. We adopt the convention of allocating the first Δ indexes for the fixed nodes and the last Ω indexes for the movable nodes. We indicate the positions of the fixed nodes using ordered pairs (χ_1, γ_1) through $(\chi_\Delta, \gamma_\Delta)$, while the locations of the remaining nodes are specified by $(\mathbf{x}_{\Delta+1}, \mathbf{y}_{\Delta+1})$ through $(\mathbf{x}_{\Delta+\Omega}, \mathbf{y}_{\Delta+\Omega})$.

B. Wireless medium sharing assumptions

Each node of the wireless network has a communication module comprising a receiver and a transmitter. In addition, we assume that each node has a distinct reception channel assigned to it. As such, any given node will *tune* into and receive information transmitted through its ascribed channel. In addition,we assume that inter-channel interference is negligible. However, in our formulation, we allow multiplexing at each channel, so as to allow more than one source node to send information to any given destination node. In practice, multiple sources can send their messages through the same channel via multiplexing techniques, such as CDMA (asynchronous code division multiplexing) [11] among many other possibilities [19]. In order to quantify the impact of channel multiplexing, we adopt a formulation that is suitable for performance metrics and constraints that are based on the signal to interference ratio (see Section III).

C. Power allocation and propagation loss model

For each node index i, in the set $\{1, \ldots, \Delta+\Omega\}$, we adopt the following model for quantifying the total power $\mathbf{P}_i^{total}$ used by node i, in $dBmW$ power units:

$$\mathbf{P}_i^{total} = 10 \log_{10} \left[\sum_{k \in \{1,\ldots,\Delta+\Omega\}-\{i\}} \phi 10^{0.1 \mathbf{P}_{i\to k} + \alpha \mathbf{e}_{i,k}} \right] \quad (1)$$

where $\mathbf{e}_{i,k}$ is the Euclidean distance that separates nodes i and k, given by:

$$\mathbf{e}_{i,k} = \begin{cases} \sqrt{(\mathbf{x}_i - \mathbf{x}_k)^2 + (\mathbf{y}_i - \mathbf{y}_k)^2} & \text{if } i,k \geq \Delta + 1 \\ \sqrt{(\mathbf{x}_i - \chi_k)^2 + (\mathbf{y}_i - \gamma_k)^2} & \text{if } i \geq \Delta + 1, k \leq \Delta \\ \sqrt{(\mathbf{x}_k - \chi_i)^2 + (\mathbf{y}_k - \gamma_i)^2} & \text{if } k \geq \Delta + 1, i \leq \Delta \\ \sqrt{(\chi_i - \chi_k)^2 + (\gamma_i - \gamma_k)^2} & \text{if } i \leq \Delta, k \leq \Delta \end{cases} \quad (2)$$

In addition, the constants ϕ and α in (1) are positive real parameters that depend on the characteristics of the wireless medium and $\mathbf{P}_{i\to k}$ represents the received power in $dBmW$, as measured at the destination node k, of the signal transmitted by node i. We express power in $dBmW$ not only because it is a standard and convenient option for wireless communication [19], but also because commercial radio frequency amplifiers often feature controllable amplification gains that have uniform (linear) resolution in the $dBmW$ scale. Similarly, commercially available radio frequency power meters usually provide readings with a resolution that is uniform in the $dBmW$ scale. Implicit in (1) is the simplifying assumption that the transmitted signal between any two nodes, say i and k, is attenuated in dB according to an affine law of the distance, given by $-\log(\phi) - \alpha \mathbf{e}_{i,k}$. This assumption will be discussed in more detail in the following Section.

1) Validity of our propagation loss model for urban and sub sea environments: It has been shown, both analytically and empirically, that in a (logarithmic) dB scale (consistent with $dBmW$), the path loss attenuation of radio frequency waves in urban [16] and in sub sea environments [12] is well approximated by an affine function of the Euclidean distance that separates the source from the receiver. More specifically, the authors of [12] suggest that attenuation, in $dBmW$, is essentially an affine law for distances of four meters or above. For distances below two meters our model may become conservative, which is not a significant drawback because most applications will not require placement of wireless nodes at such short distances. Commercial underwater radio frequency modems operate over distances as large as fifty meters [14], which indicates that our propagation loss model is accurate for underwater communications in the four to fifty meters range. Similarly, in urban environments, the authors of [16] have shown that an affine law is very accurate for distances of fifty meters or above. In the setting of [16], transmitter and receiver can communicate over distances of at least three hundred and fifty meters, which indicates that an affine law is an accurate model for propagation loss in urban environments, for distances that range from fifty to three hundred and fifty meters.

D. Comment on radio frequency communication underwater

In contrast to what was believed until recently, underwater radio modems are viable with loop antennas of one meter radius or less and modem housings of thirty centimeters (see [13] for an example). Underwater radio can be used for communication over ranges up to (typically) fifty meters, where it is far superior to acoustic based communication both in terms of delay and immunity to turbulence and noise. These features make underwater radio communications very suitable for mobile applications or (and) when tight clock synchronization is required for extended periods of time, such as in monitoring operations[1].

III. Problem formulation

In this section, we formulate the central optimization paradigm of this paper. We start by specifying the following class of functions, which we use to express the cost function as well as the constraints of our optimization paradigm. In Section V, we provide network design examples and we show how they can be cast using the framework put forward in

[1]Note that clock synchronization underwater is critical for packet stamping and that it cannot be performed using global positioning systems. The work by [15] explains the difficulties of clock synchronization over networks, including sensitivity to communication delay

this Section. Such examples are intended to illustrate the wide applicability of our formulation.

Definition 3.1: Let Δ and Ω be positive integers representing the number of fixed and movable nodes of a wireless network, respectively. Given a non-negative integer Ξ, we define $\mathbb{F}_{\Delta,\Omega,\Xi}$ as the set of all functions $\mathcal{F}$ with structure $\mathcal{F} : \mathbb{R}^{\Xi} \times (\mathbb{R})^{2(\Omega+\Delta)} \times (\mathbb{R}^2)^{\Omega} \mapsto \mathbb{R}_+$ and that can be written in the following form:

$$\mathcal{F}\left(\{\mathbf{z}_l\}_{l=1}^{\Xi}, \{\mathbf{P}_{i\to j}\}_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega}, \{\mathbf{x}_i, \mathbf{y}_i\}_{i=\Delta+1}^{\Delta+\Omega}\right) = \\ = \sum_{k=1}^{\Gamma} \varsigma_k 10^{\sum_{l=1}^{\Xi} \xi_{l,k}\mathbf{z}_l + \sum_{i,j=1}^{\Delta+\Omega}(\beta_{i,j,k}\mathbf{P}_{i\to j} + \tau_{i,j,k}\mathbf{e}_{i,j})} \quad (3)$$

where Γ is a positive integer, ς_k and $\tau_{i,j,k}$ are non-negative real constants, while $\beta_{i,j,k}$ and $\xi_{l,k}$ are real constants. (Note: Hereafter, we will use $\mathbf{z}_l$ to denote auxiliary variables which are introduced (as needed) to impose design constraints)

Notice that Definition 3.1 might be viewed as an extension of the class of posynomial functions [18], so as to include the Euclidean distance between any pair of nodes. Indeed, if we select $\tau_{i,j,k} = 0$ in (3) then the resulting function is a posynomial in $\tilde{\mathbf{P}}_{i\to j}$ and $\tilde{\mathbf{z}}_l$, where $\tilde{\mathbf{P}}_{i\to j} \overset{def}{=} 10^{\mathbf{P}_{i\to j}}$ and $\tilde{\mathbf{z}}_l \overset{def}{=} 10^{\mathbf{z}_l}$.

Below, we describe how we will be using functions in class $\mathbb{F}_{\Delta,\Omega,\Xi}$ to impose constraints on transmission power and node placement.

Definition 3.2: (**Joint constraints on power and node placement**) Let Δ and Ω be positive integers representing the number of fixed and movable nodes in a wireless network. Given a non negative constant Ξ, a positive integer Φ and functions $\mathcal{F}_1$ through $\mathcal{F}_\Phi$ in the set $\mathbb{F}_{\Delta,\Omega,\Xi}$, we consider constraints expressed by the following inequalities:

$$\mathcal{F}_k\left(\{\mathbf{z}_l\}_{l=1}^{\Xi}, \{\mathbf{P}_{i\to j}\}_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega}, \{\mathbf{x}_i, \mathbf{y}_i\}_{i=\Delta+1}^{\Delta+\Omega}\right) \le 1, \\ k \in \{1, \ldots, \Phi\} \quad (4)$$

where $\{\mathbf{P}_{i\to j}\}_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega}$, $\{\mathbf{x}_i, \mathbf{y}_i\}_{i=\Delta+1}^{\Delta+\Omega}$ and $\{\mathbf{z}_l\}_{l=1}^{\Xi}$ are optimization variables. Here, $\mathbf{z}_1$ through $\mathbf{z}_\Xi$ are auxiliary optimization variables taking real values.

An immediate and central example of application of Definition 3.2 is imposing constraints on the total power at every node, which could be expressed as $\mathbf{P}_i^{total} \le \Psi$, where $\mathbf{P}_i^{total}$ is given by (1) and Ψ quantifies the total power available at each node. Similarly, we can adopt cost functions such as $10^{\sum_{i=1}^{\Delta+\Omega} \lambda_i \mathbf{P}_i^{total}}$, where λ_1 through $\lambda_{\Delta+\Omega}$ are nonnegative weights.

Further examples of constraints, with network-centric significance, can be expressed using the following definition of signal to interference plus noise ratio:

Definition 3.3: (**Signal to interference plus noise ratio**) Let Δ and Ω quantify the number of fixed and movable nodes of a wireless network, respectively. Let i and j be distinct integers in the set $\{1, \ldots, \Delta + \Omega\}$ representing nodes of a wireless network. The signal to interference plus noise ratio for the logical link from node i to node j is defined as:

$$\mathbf{S}_{i\to j} \overset{def}{=} \frac{\eta_{i,i} 10^{0.1\mathbf{P}_{i\to j}}}{\sum_{k\in\{1,\ldots,\Delta+\Omega\}-\{i,j\}} \eta_{k,j} 10^{0.1\mathbf{P}_{k\to j}} + \sigma_N^2} \quad (5)$$

In (5) $\eta_{k,i}$ are positive real coding gains that quantify the fact that, for instance, multiplexing codes are not perfectly orthogonal and $\mathbf{P}_{k\to j}$ represents the received power, as measured at the destination node j, of the signal transmitted by node k and σ_N^2 is the variance of the noise at the receiver.

Clearly, constraints of the type $\mathbf{S}_{i,j} \ge \lambda_{i,j}$ can be cast as in Definition 3.2. In Section V we provide more examples of constraints and cost functions that can be expressed as in Definition 3.2.

Using the following class of constraints, we can impose constraints on the placement of the movable nodes.

Definition 3.4: (**Polyhedral convex set placement constraints**) Let Δ and Ω be positive integers representing the number of fixed and movable nodes in a wireless network, respectively. Given a subset $\mathbb{S}$ of $\{\Delta + 1, \ldots, \Delta + \Omega\}$, an integer Γ, real constants $\zeta_{i,k}$ and $\vartheta_{i,k}$ with (i,k) in the set $\mathbb{S} \times \{1, \ldots, \Gamma\}$, we consider the following class of constraints:

$$\zeta_{i,k}\mathbf{x}_i + \vartheta_{i,k}\mathbf{y}_i \le 1, \ (i,k) \in \mathbb{S} \times \{1, \ldots, \Gamma\} \quad (6)$$

The following is the description of the main paradigm addressed in this paper.

Problem 3.1: (**Jointly optimal placement and power allocation**) Let Δ and Ω be given positive integers quantifying the number of fixed and movable nodes in a wireless network, respectively. In addition, consider that the following design parameters are given:

- **Specification of constraints as in Definition 3.2:** Let a non negative integer Ξ, positive integers Φ and Γ, and functions $\mathcal{F}_1$ through $\mathcal{F}_\Phi$ in the set $\mathbb{F}_{\Delta,\Omega,\Xi}$ be given.
- **Placement constraints as in Definition 3.4:** Let a subset of $\{1, \ldots, \Delta + \Omega\}$ denoted by $\mathbb{S}$ and real constants $\zeta_{i,k}$ and $\vartheta_{i,k}$, with (i,k) in the set $\mathbb{S} \times \{1, \ldots, \Gamma\}$, be given.

Given a cost function $\mathcal{U} \in \mathbb{F}_{\Delta,\Omega,\Xi}$, we want to find the solution of the following optimization paradigm:

$$\mathbf{Q}^* = \arg\min_{\mathbf{Q}} \mathcal{U}(\mathbf{Q}) \quad (7)$$

subject to the following constraints:

$$\mathcal{F}_k(\mathbf{Q}) \le 1, k \in \{1, \ldots, \Phi\} \quad (8)$$

$$\zeta_{i,k}\mathbf{x}_i + \vartheta_{i,k}\mathbf{y}_i \le 1, \ (i,k) \in \mathbb{S} \times \{1, \ldots, \Gamma\} \quad (9)$$

where $\mathbf{Q}$ represents the entire collection of optimization variables given by $\{\mathbf{P}_{i\to j}\}_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega}$, $\{\mathbf{x}_i, \mathbf{y}_i\}_{i=\Delta+1}^{\Delta+\Omega}$ and $\{\mathbf{z}_l\}_{l=1}^{\Xi}$. In (7) $\mathbf{Q}^*$ is used to indicate the collection of optimization variables at an optimum.

IV. Basic optimality properties of Problem 3.1

In Section IV-A, we show that Problem 3.1 is convex. This is very useful attribute because it guarantees that an optimum can be found via standard constrained optimization techniques.

We also show, in Section VI, that Problem 3.1 can be arbitrarily well approximated by a geometric program. This is quite desirable, since geometric programs can be solved via polynomial time algorithms, which are available in existing software packages. In addition, in a like manner to linear programs, geometric programming solvers provide a certificate of infeasibility, in case the problem is not feasible.

Since constrained optimization algorithms may feature slow converge, we propose the use of geometric programs for obtaining a first approximate solution, which can be used as an initial condition in any method that is adopted for solving the exact problem.

A. Proof that Problem 3.1 is convex.

In order to prove that Problem 3.1 is convex, it suffices to prove that the class of functions specified in Definition 3.1 is convex. The main argument is given in the following Proposition:

Proposition 4.1: Given positive integers Δ, Ω and Ξ, along with nonnegative real constants $\{\tau_{i,j}\}_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega}$ and real constants $\{\beta_{i,j}\}_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega}$ and $\{\xi_l\}_{l=1}^{\Xi}$, consider a function $\mathcal{G}:\mathbb{R}^{\Xi}\times(\mathbb{R})^{2(\Omega+\Delta)}\times(\mathbb{R}^2)^{\Omega}\mapsto\mathbb{R}_+$ given by:

$$\mathcal{G}\left(\{\mathbf{z}_l\}_{l=1}^{\Xi},\{\mathbf{P}_{i\to j}\}_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega},\{\mathbf{x}_i,\mathbf{y}_i\}_{i=\Delta+1}^{\Delta+\Omega}\right)= 10^{\sum_{l=1}^{\Xi}\xi_l\mathbf{z}_l+\sum_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega}\beta_{i,j}\mathbf{P}_{i\to j}+\tau_{i,j}\mathbf{e}_{i,j}} \quad (10)$$

where $\mathbf{e}_{i,j}$ is the Euclidean distance specified in (2). The function $\mathcal{G}$ is convex.

Proof: We start by defining the following functions:

$$\mathcal{G}_2(\mathbf{Q})\overset{def}{=}\sum_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega}\tau_{i,j}\mathbf{e}_{i,j} \quad (11)$$

$$\mathcal{G}_1(\mathbf{Q})\overset{def}{=}\sum_{l=1}^{\Xi}\xi_l\mathbf{z}_l+\sum_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega}\beta_{i,j}\mathbf{P}_{i\to j}+\mathcal{G}_2(\mathbf{Q}) \quad (12)$$

where $\mathbf{Q}$ is a shorthand notation for representing the entire collection of optimization variables, given by $\{\mathbf{P}_{i\to j}\}_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega}$, $\{\mathbf{x}_i,\mathbf{y}_i\}_{i=\Delta+1}^{\Delta+\Omega}$ and $\{\mathbf{z}_l\}_{l=1}^{\Xi}$. Now notice that $\tau_{i,j}$ are non-negative and that the Euclidean distance is itself a convex function of its parameters. Hence, $\mathcal{G}_2$ is convex function of $\{(\mathbf{x}_i,\mathbf{y}_i)\}_{i=\Delta+1}^{\Delta+\Omega}$ because it is a sum of convex functions. This also implies that $\mathcal{G}_1$ is convex and since exponentiation is an increasing convex function [2], we can conclude that $\mathcal{G}(\mathbf{Q})=10^{\mathcal{G}_1(\mathbf{Q})}$ is convex. ■

We can now state the following Theorem:

Theorem 4.2: Problem 3.1 is convex

Proof: Recall that the coefficients ς_k, in Definition 3.1, are non-negative. Therefore, it follows from Proposition 4.1 that functions in the set $\mathbb{F}_{\Delta,\Omega,\Xi}$ are convex because they are the sum of convex functions. This implies that the cost $\mathcal{U}$ of Problem 3.1 and all of its constraints are convex, which includes the ones specified in (8) and (8), as well as Definitions 3.2 and 3.4. These facts lead to the conclusion that Problem 3.1 is convex. □

V. Examples of design problems that comply with the problem formulation of Section III

In this Section, we give design examples that we can cast in the framework of Section III. By way of these examples, we expect to illustrate the pertinence of our framework for the design of wireless networks, with respect to jointly optimal power allocation and node placement.

[2]Recall that the composition of a convex function with an increasing convex function is convex.

A. Optimal relay placement, power allocation and routing, under the high signal to interference ratio assumption

In this Subsection, we delineate a design example which involves the maximization of the bit-rate between a pre-specified collection of fixed nodes. Throughout, we will describe how our design example can be cast in the formulation of Section III. An interesting attribute of such an exercise is that it also shows, for the present example, how we can integrate optimization of routing, in addition to power allocation and placement of the movable nodes. For simplicity of notation, we describe our example for two fixed and two movable nodes, but our approach can be used for any number of nodes at the expense of a potentially large number of auxiliary variables. In addition, this example admits that information is routed through at most two hops, but the number of hops can be increased at the expense of using more auxiliary variables.

Example 5.1: Consider a wireless network with two fixed and two movable nodes. The movable nodes, indexed by 3 and 4, are intended to act as relays so as to maximize the rate of communication between the fixed nodes, which are identified by indices 1 and 2. In order to precisely state our design example, we assume that the following design parameters are given:

- We are given the parameters ϕ and α that are needed in the total power formula (1).
- We pre-specify a positive real constant Ψ representing the maximal power available at each node.
- We are given the positions of the fixed nodes (χ_1,γ_1) and (χ_2,γ_2).

In addition, we want to maximize the following cost function:

$$\mathcal{U}^{example\ 5.1}(\mathbf{R}_{1\to2}^{total},\mathbf{R}_{2\to1}^{total})=\min\{\mathbf{R}_{1\to2}^{total},\mathbf{R}_{2\to1}^{total}\} \quad (13)$$

where $\mathbf{R}_{1\to2}^{total}$ and $\mathbf{R}_{2\to1}^{total}$ represent the total aggregated rates that stream from node 1 to node 2 and from node 2 to node 1, respectively. These aggregated rates can be computed as follows:

$$\mathbf{R}_{1\to2}^{total}=\mathbf{R}_{1\to2}+\underbrace{\min\{\mathbf{R}_{1\to3},\mathbf{R}_{3\to2}\}}_{(A)-\text{route via node 3}}+\underbrace{\min\{\mathbf{R}_{1\to4},\mathbf{R}_{4\to2}\}}_{(B)-\text{route via node 4}} \quad (14)$$

$$\mathbf{R}_{2\to1}^{total}=\mathbf{R}_{2\to1}+\underbrace{\min\{\mathbf{R}_{2\to3},\mathbf{R}_{3\to1}\}}_{(C)-\text{route via node 3}}+\underbrace{\min\{\mathbf{R}_{2\to4},\mathbf{R}_{4\to1}\}}_{(D)-\text{route via node 4}} \quad (15)$$

Here we use $\mathbf{R}_{i\to j}$ to represent the average rate through the direct communication link (point-to-point, i.e., no relaying) from node i to node j, in bits per time unit. An interpretation for the flux constraints (14)-(15) is that the information that is transferred between nodes 1 and 2 can flow through three different routes. It can flow directly (point-to-point) between nodes 1 and 2 and it can be routed through nodes 3 and 4.

Hereon, we use the following formula for relating the point-to-point rate $\mathbf{R}_{i\to j}$ with the signal to interference plus noise ratio [18, page 68], which is valid in the high signal to interference plus noise regime:

$$\mathbf{R}_{i\to j}=\frac{1}{\Upsilon}\log_{10}(\kappa\mathbf{S}_{i\to j}) \quad (16)$$

where Υ and κ are positive real constants. We can now precisely state the optimization paradigm for our example, which consists of finding $\{\mathbf{P}_{i,j}\}_{i=1,j=1}^{i=4,j=4}$ and $\{\mathbf{x}_i, \mathbf{y}_i\}_{i=3}^{4}$ so as to maximize $\mathcal{U}^{example\ 5.1}$, subject to the following constraints:

$$\mathbf{P}_i^{total} \leq \Psi, \ i \in \{1, \ldots, 4\} \tag{17}$$

Now we show that (13)-(17), which constitute the specification of Example 5.1, can be put in the framework of Section III. We start by noticing that the following optimization paradigm is equivalent to Example 5.1:

$$\min 10^{-\mathbf{z}_1} \tag{18}$$

subject to the constraints specified in inequalities (19)-(25):

$$\mathbf{z_2} \geq \mathbf{z}_1 \text{ and } \mathbf{z_3} \geq \mathbf{z}_1 \tag{19}$$

$$\underbrace{\kappa \mathbf{S}_{1\to 2}^{\frac{1}{\Upsilon}}}_{10^{\mathbf{R}_{1\to 2}}} \geq 10^{\mathbf{z}_2 - \mathbf{z}_4 - \mathbf{z}_5} \text{ and } \underbrace{\kappa \mathbf{S}_{2\to 1}^{\frac{1}{\Upsilon}}}_{10^{\mathbf{R}_{2\to 1}}} \geq 10^{\mathbf{z}_3 - \mathbf{z}_6 - \mathbf{z}_7} \tag{20}$$

$$\underbrace{\kappa \mathbf{S}_{1\to 3}^{\frac{1}{\Upsilon}}}_{10^{\mathbf{R}_{1\to 3}}} \geq 10^{\mathbf{z_4}} \text{ and } \underbrace{\kappa \mathbf{S}_{3\to 2}^{\frac{1}{\Upsilon}}}_{10^{\mathbf{R}_{3\to 2}}} \geq 10^{\mathbf{z_4}} \tag{21}$$

$$\underbrace{\kappa \mathbf{S}_{1\to 4}^{\frac{1}{\Upsilon}}}_{10^{\mathbf{R}_{1\to 4}}} \geq 10^{\mathbf{z_5}} \text{ and } \underbrace{\kappa \mathbf{S}_{4\to 2}^{\frac{1}{\Upsilon}}}_{10^{\mathbf{R}_{4\to 2}}} \geq 10^{\mathbf{z_5}} \tag{22}$$

$$\underbrace{\kappa \mathbf{S}_{2\to 3}^{\frac{1}{\Upsilon}}}_{10^{\mathbf{R}_{2\to 3}}} \geq 10^{\mathbf{z_6}} \text{ and } \underbrace{\kappa \mathbf{S}_{3\to 1}^{\frac{1}{\Upsilon}}}_{10^{\mathbf{R}_{3\to 1}}} \geq 10^{\mathbf{z_6}} \tag{23}$$

$$\underbrace{\kappa \mathbf{S}_{2\to 4}^{\frac{1}{\Upsilon}}}_{10^{\mathbf{R}_{2\to 4}}} \geq 10^{\mathbf{z_7}} \text{ and } \underbrace{\kappa \mathbf{S}_{4\to 1}^{\frac{1}{\Upsilon}}}_{10^{\mathbf{R}_{4\to 1}}} \geq 10^{\mathbf{z_7}} \tag{24}$$

$$10^{\mathbf{P}_i^{total} - \Psi} \leq 1, \ i \in \{1, \ldots, 4\} \tag{25}$$

Notice that, in the inequalities (20)-(24), the rates $\mathbf{R}_{i\to j}$ appear in the exponent and that the variables $\mathbf{S}_{i\to j}$ are obtained by direct substitution in (16). We have decided to write the contraints in exponential form, because that allowed us to express our optimization problem in a format that is compatible with the formulation of Section III. In order to check that (18)-(25) is equivalent to Example 5.1, notice that the auxiliary variables $\mathbf{z}_1$ through $\mathbf{z}_7$ are introduced to construct the cost function (13). In particular, (19) implements the minimum in (13), while the auxiliary variables $\mathbf{z}_2$ and $\mathbf{z}_3$ represent $\mathbf{R}_{1\to 2}^{total}$ and $\mathbf{R}_{2\to 1}^{total}$, respectively. Similarly, the auxiliary variables $\mathbf{z}_4$ and $\mathbf{z}_5$ implement the two terms, denoted by (A) and (B), at the right hand side of (14), while $\mathbf{z}_6$ and $\mathbf{z}_7$ implement the two terms (C) and (D) at the right hand side of (15). The power constraint (17) is also re-written in the form (25).

In order to show that (18)-(25) comply with the formulation of Section III, it suffices to notice the following facts:

- By a direct substitution of (5), we can write inequalities (20)-(24) as in Definition 3.2.
- Similarly, by using (1), (25) can be re-written so as to comply with Definition 3.2.

B. Further examples of optimization constraints that comply with the formulation of Section III

The following Example illustrates how linear inequalities on the rates among distinct nodes can be expressed as in Definition 3.2.

Example 5.2: (**Linear inequalities on rates**) Let Λ be a given integer and λ_1 through λ_Λ be given positive real constants. In addition, consider real parameters given by $\varrho_{i,j,k}$, with $(i,j) \in \{1, \ldots, \Delta + \Omega\}^2$ and $k \in \{1, \ldots, \Lambda\}$. We consider the following collection of constraints

$$\sum_{(i,j) \in \{1, \ldots, \Delta+\Omega\}^2} \varrho_{i,j,k} \mathbf{R}_{i\to j} \geq \lambda_k, \ k \in \{1, \ldots, \Lambda\} \tag{26}$$

where $\mathbf{R}_{i\to j}$ represents the rate of transmission (in bits per channel use) from node i to node j. Using (16), we can re-write (26) as:

$$\prod_{(i,j) \in \{1, \ldots, \Delta+\Omega\}^2} (\kappa \mathbf{S}_{i\to j})^{\frac{\varrho_{i,j,k}}{\Upsilon}} \geq 2^{\lambda_k}, \ k \in \{1, \ldots, \Lambda\} \tag{27}$$

which clearly complies with Definition 3.2.

Inequalities of the form (26) can be used to specify any convex polyhedral rate region among any sub-collection of source and destination nodes. In addition, necessary and sufficient conditions for multi-terminal omniscience, in the presence of an overlay node [21], can be cast as in (26). This class of inequalities can also be used to specify the rate of certain distributed secret key generation mechanisms [20].

Yet another example is the specification of constraints on the outage probabilities over a path, which can be cast as in Definition 3.2, under the assumption of no single dominant interferer (see [18, Page 68] for more details).

VI. Approximate Solutions to Problem 3.1 via Geometric Programming

Geometric programming is an optimization paradigm that has been widely studied for more than thirty years. Due to the increase in computational power verified in recent years, efficient solvers for geometric programs are now easily accessible. Hence, the recent significant interest in casting the optimal design of communication systems as geometric programs [18]. In a way that is similar to linear programs, solvers for geometric programs can efficiently handle thousands of variables and constraints, which is very relevant for the method proposed in this Section, where a large number of auxiliary variables might be needed. Even when the original problem cannot be cast as a geometric program, in many cases a satisfactory solution can be found via an approximate geometric program [18], [22]. In this section, we follow such an approach, by proposing an approximate solution to Problem 3.1 via geometric programming. In order to accomplish this goal, we follow a strategy where it suffices to approximate the Euclidean distance function by the maximum of some linear functions. No other approximations are needed. A comprehensive account of the uses of geometric programming in various other fields can be found in [22], where a rich portfolio of examples is also provided.

Below we give a brief description of the standard geometric programming paradigm. We start with the definition of the class of posynomial functions.

Definition 6.1: Given a finite collection of non-negative real variables denoted by $\mathbf{W} \stackrel{def}{=} \{\mathbf{w}_i\}_{i=1}^{\Xi}$. The class of posynomials over $\mathbf{W}$ is formed by functions $\mathcal{P}$ with the following structure:

$$\mathcal{P}(\mathbf{W}) = \sum_{k=1}^{\Gamma} \varsigma_k \prod_{l=1}^{\Xi} \mathbf{w}_l^{\xi_{l,k}} \tag{28}$$

where ς_k are positive real and $\xi_{l,k}$ are real (possibly negative) constants.

The following is the definition of the general form of a geometric program in standard form.

Definition 6.2: Let a finite collection of non-negative real variables denoted by $\mathbf{W} \stackrel{def}{=} \{\mathbf{w}_i\}_{i=1}^{\Xi}$, and posynomials over $\mathbf{W}$ denoted by $\mathcal{U}$ and $\mathcal{P}_1$ through $\mathcal{P}_\Phi$ be given. The following optimization paradigm is a geometric program:

$$\mathbf{W}^* = \arg\min_{\mathbf{W}} \mathcal{U}(\mathbf{W}) \tag{29}$$

subject to the following inequality constraints:

$$\mathcal{P}_i(\mathbf{W}) \leq 1, \ i \in \{1, \ldots, \Phi\} \tag{30}$$

A. Specification of a geometric program that approximates Problem 3.1

In what follows, we specify a geometric program that approximates Problem 3.1, to an arbitrary degree of accuracy. However, we should note that higher accuracy is attained at the expense of a larger number of optimization variables. Nonetheless, as we explain in Section IV, obtaining an approximate solution is important because it can be used as an initial condition on any constrained optimization algorithm that is applied to the exact problem. Another reason for adopting this procedure is that solvers provide a certificate if a geometric program is infeasible.

Notice that Problem 3.1 is not a geometric program because the constraints involve the Euclidean distance. The main idea, in what follows, is to approximate the Euclidean distance in Problem 3.1 with an appropriate distance function so as to obtain a geometric program. No other approximations are required. The following is the class of distance functions that we will use in our approximate geometric program.

Definition 6.3: (**Convex polygonal distances**) Let $\mathcal{D}$ be a distance function in the Cartesian plane that satisfies the properties of a norm. We qualify $\mathcal{D}$ as a convex polygonal distance if and only if the unit ball, according to $\mathcal{D}$, is a convex polygon[17, Chapter 19]. We define the class of convex polygonal distances as $\mathbb{D}$.

The following remark states the main reason why the class $\mathbb{D}$ is a suitable choice for approximating the Euclidean distance.

Remark 6.1: Given any two positive real constants $\wp_1$ and $\wp_2$ satisfying $\wp_1 < 1$ and $\wp_2 > 1$, we can always find a distance function $\mathcal{D}$ in the set $\mathbb{D}$ such that the following holds:

$$\wp_1 \mathcal{D}(u_1, u_2) < \|u_1 - u_2\|_2 < \wp_2 \mathcal{D}(u_1, u_2), \ u_1, u_2 \in \mathbb{R}^2 \tag{31}$$

where $\|u_1 - u_2\|_2$ is the Euclidean distance between u_1 and u_2.

In order to prove Remark 6.1, one only needs to realize that the unit ball, associated with the Euclidean distance, can be approximated arbitrarily well by a convex polygon. Hence, the proof follows by selecting the convex polygonal distance that corresponds to the approximating convex polygon. The following remark provides a systematic method for obtaining such a distance function, which will also be useful later in this subsection.

Remark 6.2: If $\mathcal{D}$ is a convex polygonal distance then there an integer m and a finite collection of m vectors $(\nu_{1,1}, \nu_{2,1})$ through $(\nu_{1,m}, \nu_{2,m})$ in the Cartesian plane, for which the following holds [17, pp. 173]:

$$\mathcal{D}((x_i, y_i), (x_j, y_j)) = \min \mathbf{h} \tag{32}$$

subject to:

$$\nu_{1,k}(x_j - x_i) + \nu_{2,k}(y_j - y_i) \leq \mathbf{h}, k \in \{1, \ldots, m\} \tag{33}$$

for any vectors (x_i, y_i) and (x_j, y_j) in the Cartesian plane.

Example 6.1: Now we consider an example of application of Remark 6.2. Consider that we wish to specify a distance whose unit ball is the unit square, centered at the origin. The corresponding distance can be obtained from Remark 6.2 by selecting the four vectors $(\nu_{1,1}, \nu_{2,1}) = (1, 0)$, $(\nu_{1,2}, \nu_{2,2}) = (-1, 0)$, $(\nu_{1,3}, \nu_{2,3}) = (0, 1)$ and $(\nu_{1,4}, \nu_{2,4}) = (0, -1)$. Likewise, if the unit ball is an hexagon then the distance would be specified by $(\nu_{1,i}, \nu_{2,i}) = (\cos(i\frac{\pi}{3}), \sin(i\frac{\pi}{3}))$, with $i \in \{1, \ldots, 6\}$.

The class of functions specified in Definition 3.1 plays a central role in the statement of Problem 3.1. By replacing the Euclidean distance with an approximating convex polygonal distance we obtain the class of functions defined below. From Remark 6.1, we conclude that functions in such a class may be used as an approximation to the functions given in Definition 3.1.

Definition 6.4: Let Δ and Ω be positive integers representing the number of fixed and movable nodes of a wireless network, respectively. Given a convex polyhedral distance $\mathcal{D}$ and a non-negative integer Ξ representing the number of auxiliary optimization variables, we define $\mathbb{H}_{\Delta,\Omega,\Xi,\mathcal{D}}$ as the set of all functions $\mathcal{H}$ that can be written in the following form:

$$\mathcal{H}(\mathbf{Q}) = \sum_{k=1}^{\Gamma} \varsigma_k 10^{\sum_{l=1}^{\Xi} \xi_{l,k} \mathbf{z}_l + \sum_{i,j=1}^{\Delta+\Omega} (\beta_{i,j,k} \mathbf{P}_{i \to j} + \tau_{i,j,k} \mathbf{d}_{i,j})} \tag{34}$$

where $\mathbf{d}_{i,j} = \mathcal{D}((\mathbf{x}_i, \mathbf{y}_i), (\mathbf{x}_j, \mathbf{y}_j))$, Γ is a positive integer, ς_k, $\xi_{l,k}$ and $\tau_{i,j,k}$ are non-negative real constants, while $\beta_{i,j,k}$ are real constants. Moreover, $\mathbf{z}_1$ through $\mathbf{z}_\Xi$ are non-negative real auxiliary optimization variables. In addition, $\mathbf{Q}$ is a shorthand notation for representing the entire collection of optimization variables given by $\{\mathbf{P}_{i \to j}\}_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega}$, $\{\mathbf{x}_i, \mathbf{y}_i\}_{i=\Delta+1}^{\Delta+\Omega}$ and $\{\mathbf{z}_l\}_{l=1}^{\Xi}$.

The following Problem 6.1 is a modified version of Problem 3.1, where the class of functions $\mathbb{F}$ is replaced with $\mathbb{H}$. Since functions in $\mathbb{F}$ can be approximated by functions in $\mathbb{H}$, we can view Problem 6.1 as an approximate version of Problem 3.1.

In addition, as we show in Theorem 6.1, Problem 6.1 can be cast as a geometric program.

Problem 6.1: (**Jointly optimal placement and power allocation with polyhedral distances**) Consider that Δ and Ω are positive integers quantifying the number of fixed and movable nodes in a wireless network, respectively. Let a polyhedral convex distance $\mathcal{D}$, a positive integer Φ, functions $\mathcal{H}_1$ through $\mathcal{H}_\Phi$ in the set $\mathbb{H}_{\Delta,\Omega,\Xi,\mathcal{D}}$, a subset of $\{1,\dots,\Delta+\Omega\}$ denoted by $\mathbb{S}$ and real constants $\zeta_{i,k}$ and $\vartheta_{i,k}$, with (i,k) in the set $\mathbb{S}\times\{1,\dots,\Gamma\}$, be given design parameters. Given a cost $\mathcal{U}$ in the set $\mathbb{H}_{\Delta,\Omega,\Xi,\mathcal{D}}$, we want to find the solution to the following optimization paradigm:

$$\mathbf{Q}^* = \arg\min_{\mathbf{Q}} \mathcal{U}(\mathbf{Q}) \tag{35}$$

$$\mathcal{H}_k(\mathbf{Q}) \leq 1,\ k \in \{1,\dots,\Phi\} \tag{36}$$

subject also to the placement constraints (6). Here $\mathbf{Q}^*$ is used to indicate the entire collection of optimization variables$\{\mathbf{P}_{i\to j}\}_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega}$, $\{\mathbf{x}_i,\mathbf{y}_i\}_{i=\Delta+1}^{\Delta+\Omega}$ and $\{\mathbf{z}_l\}_{l=1}^{\Xi}$ at an optimum.

Theorem 6.1: Problem 6.1 can be cast as a standard geometric program.

Proof: Let $\tilde{\mathbf{P}}_{i\to j} \overset{def}{=} 10^{\mathbf{P}_{i\to j}}$, $\tilde{\mathbf{x}}_i \overset{def}{=} 10^{\mathbf{x}_i}$, $\tilde{\mathbf{y}}_j \overset{def}{=} 10^{\mathbf{x}_j}$, $\tilde{\mathbf{z}}_l \overset{def}{=} 10^{\mathbf{z}_l}$ and $\tilde{\mathbf{d}}_{i,j} \overset{def}{=} 10^{\mathbf{d}_{i,j}}$, where $\mathbf{d}_{i,j} = \mathcal{D}\left((\mathbf{x}_i,\mathbf{y}_i),(\mathbf{x}_j,\mathbf{y}_j)\right)$. The proof follows as a conclusion based on the following two facts: **(Fact 1)** By inspecting (34), we conclude that any function in class $\mathbb{H}_{\Delta,\Omega,\Xi,\mathcal{D}}$ can be re-written as a posynomial in terms of the supplemental variables $\tilde{\mathbf{P}}_{i\to j}$,$\tilde{\mathbf{x}}_i$, $\tilde{\mathbf{y}}_j$, $\tilde{\mathbf{z}}_l$ and $\tilde{\mathbf{d}}_{i,j}$. In fact, the cost function in (35) and the left hand side of the inequality constraints in (6) and (36) can be re-cast as posynomial functions. As a result, we infer that Problem 6.1 can be written as a Geometric program with respect to the supplemental variables. However, according to the statement of Problem 6.1, $\tilde{\mathbf{d}}_{i,j}$ is not one of the desired optimization variables, i.e., we want to obtain an answer in terms of the positions and not the distances. In addition, if we optimize with respect to $\tilde{\mathbf{d}}_{i,j}$ directly, and without further constraints, then the optimum may be such that the resulting distances are not consistent with the positions. In order to address this problem, we make use of the following fact: **(Fact 2)** From Remark 6.2, we conclude that there exists a positive integer m and vectors $(\nu_{1,1},\nu_{2,1})$ through $(\nu_{1,m},\nu_{2,m})$ such that $\tilde{\mathbf{d}}_{i,j}$ can be expressed as a function of $\tilde{\mathbf{x}}_i$ and $\tilde{\mathbf{y}}_j$ via the following geometric program:

$$\tilde{\mathbf{d}}_{i,j} = \arg\min \tilde{\mathbf{h}}_{i,j} \tag{37}$$

subject to:

$$(\tilde{\mathbf{x}}_j\tilde{\mathbf{x}}_i^{-1})^{\nu_{1,l}}(\tilde{\mathbf{y}}_j\tilde{\mathbf{y}}_i^{-1})^{\nu_{2,l}}\tilde{\mathbf{h}}_{i,j}^{-1} \leq 1,\ l \in \{1,\dots,m\} \tag{38}$$

Now notice that in the definition of $\mathbb{H}_{\Delta,\Omega,\Xi,\mathcal{D}}$ (see (34)), the coefficients of the distances $\mathbf{d}_{i,j}$, by definition, satisfy $\tau_{i,j,k} \geq 0$. This means that if any given $\mathbf{d}_{i,j}$ is present in the cost $\mathcal{U}$ or in an active constraint, say $\mathcal{H}_k \leq 1$, then $\tilde{\mathbf{d}}_{i,j}$ is implicitly minimized. This observation, together with Fact 1 and Fact 2 show that Problem 6.1 can be written as a geometric program by re-expressing the left hand side of (6), (36) and $\mathcal{U}$ as posynomials and by adding the following constraints:

$$(\tilde{\mathbf{x}}_j\tilde{\mathbf{x}}_i^{-1})^{\nu_{1,l}}(\tilde{\mathbf{y}}_j\tilde{\mathbf{y}}_i^{-1})^{\nu_{2,l}}\tilde{\mathbf{d}}_{i,j}^{-1} \leq 1,\ l \in \{1,\dots,m\} \tag{39}$$

■

VII. Minimizing the total transmition power with point-to-point rate constraints: layering and distributed implementation

In Section III, we defined Problem 3.1 which constitutes the main paradigm in this paper. In addition, in Section IV-A, we proved that such a problem can be cast as a convex problem and in Section VI we provided a method for obtaining approximate solutions via geometric programming.

Hereon, we illustrate how a particular instance of Problem 3.1 can be efficiently solved and decomposed into simpler subproblems, via iterative algorithms based on the primal-dual principle. The decomposition of optimization problems via primal-dual methods has a long history [5], [6]. Using such a decomposition, many large-scale optimization problems can be solved via iterative algorithms that can be implemented in a distributed fashion, such as in [4]. In other cases, primal-dual methods allow the decomposition of the original problem into independent and simpler subproblems that otherwise would have to be solved jointly. This rationale has been widely adopted to obtain viable design methods and distributed implementation of optimal network paradigms, such as, for instance, in [9], [8]. The paper [7] provides a recent overview, including a rich collection of references on the subject of decomposition via primal-dual principles.

In this Section, we seek to minimize the total transmition power by optimizing (jointly) with respect to the positions of the movable nodes and the transmition power at every node. We show that the aforementioned problem admits a primal-dual decomposition, which we use to obtain an iterative solution comprising two main steps denoted as primal and dual. The primal step consists of an optimization problem that can be decomposed (layered) into two smaller subproblems, and the dual step is a simple price[3] update rule. Notice that designing wireless networks that minimize the total transmition power is meaningful when nodes are battery operated. Indeed, if all nodes use the same type of battery unit then the average number of nodes that need to be serviced[4], per unit of time, is proportional to the total transmition power. As such, a design that minimizes transmition power migth also minimize the cost (per time unit) of maintaining the network.

Our iterative solution and its associated decomposition has the following advantages: (1) The primal step can be split into two simpler and independent optimization tasks, namely, the placement and the power allocation subproblems get decoupled. This decomposition can be interpreted as layering, i.e., the primal step dissociates into a placement layer and a power allocation layer. The dimension (in number of variables) of each one of such subproblems is smaller than the original

[3] We denote the dual variables as prices.

[4] Having a node serviced in this context might be replacing or recharging batteries.

problem and they can be solved in parallel. The coupling among agents is implemented via the exchange of information, such as dual variables (shadow prices) and other variables. (2) If constraints are imposed within a neighborhood of each node then it suffices to implement price and variable exchanges within those same neighborhoods. Such an implementation, where prices and a few variables are exchanged within neighborhoods, is what we qualify as distributed solution.

The following is the specification of the total power minimization problem that we will address hereafter [5]:

Problem 7.1: (**total power minimization**) Let Δ and Ω be positive integers representing the number of fixed and movable nodes of a wireless network, respectively. Given non-negative real constants φ_1 through $\varphi_{\Delta+\Omega}$, define the cost function $\mathcal{U}$ to be a weighted sum of the transmission power spent by each node:

$$\mathcal{U}(\mathbf{Q}) = \sum_{i=1}^{\Delta+\Omega} \varphi_i 10^{0.1\mathbf{P}_i^{total}} \tag{40}$$

where $\mathbf{P}_i^{total}$ is given by (1) and $\mathbf{Q}$ is a shorthand notation for representing the entire collection of optimization variables given by $\{\mathbf{P}_{i\to j}\}_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega}$, $\{\mathbf{x}_i, \mathbf{y}_i\}_{i=\Delta+1}^{\Delta+\Omega}$. Consider also that a collection $\{\mathbb{O}(i)\}_{i=1}^{\Delta+\Omega}$ of subsets of $\{1,\ldots,\Delta+\Omega\}$ is given, representing the neighborhoods or destination nodes in the outgoing links from each node i. Let $\{\varrho_{i\to j}\}_{i=1,j\in\mathbb{O}(i)}^{\Delta+\Omega}$ and $\{\Psi_i\}_{i=1}^{\Delta+\Omega}$ be two sets of pre-selected real constants (design parameters), representing the minimum required rates in the links and the maximum power available at the transmitters, respectively. We want to find the solution of the following optimization paradigm:

$$\mathbf{Q}^* = \arg\min_{\mathbf{Q}} \mathcal{U}(\mathbf{Q}) \tag{41}$$

subject to the following constraints:

$$\mathcal{R}(\mathbf{S}_{i\to j}) \geq \varrho_{i\to j},\ (i,j) \in \bigcup_{i=1}^{\Delta+\Omega} \{i\} \times \mathbb{O}(i) \tag{42}$$

$$\mathbf{P}_i^{total} \leq \Psi_i,\ i \in \{1,\ldots,\Delta+\Omega\} \tag{43}$$

where $\mathbf{S}_{i\to j}$ is the signal to interference ratio of the transmission from link i to j and $\mathcal{R}$ is any positive and increasing function that satisfies $\lim_{\gamma\to\infty} \mathcal{R}(\gamma) = \infty$. We use $\mathcal{R}$ to quantify the rate of data transmission from node i to node j.

Remark 7.1: (**Problem 7.1 is convex.**) Since $\mathcal{R}$ is an increasing function that is invertible in the positive reals, the constraints in (42) can be replaced by the following constraints on the signal to interference plus noise ratio:

$$\mathbf{S}_{i\to j} \geq \mathcal{R}^{-1}(\varrho_{i\to j}), i \in \{1,\ldots,\Delta+\Omega\}, j \in \mathbb{O}(i) \tag{44}$$

where $\mathcal{R}^{-1}$ represents the inverse function of $\mathcal{R}$. Notice that not only the approximation in (16), i.e., $\mathcal{R}(\mathbf{S}_{i\to j}) = \frac{1}{\Upsilon}\log_{10}(\kappa\mathbf{S}_{i\to j})$, but also the exact formula $\mathcal{R}(\mathbf{S}_{i\to j}) = \frac{1}{\Upsilon}\log_{10}(1+\kappa\mathbf{S}_{i\to j})$ are valid choices here. In addition, constraints (44) comply with Definition 3.2. Hence, by replacing (42) with (44), we conclude that Problem 7.1 is a particular case of Problem 3.1 and, as such, from Theorem 4.2 we conclude that Problem 7.1 is also convex.

[5]Notice that this paradigm is a particular case of Problem 3.1. The generalization of this primal-dual solution method to other objective functions and more general constraints can be found in [30]

In what follows, we describe how we obtain a distributed algorithm that converges to the optimal solution of Problem 7.1. In particular, in Section VII-A we show how Problem 7.1 can be decomposed using a primal-dual approach. We also provide a layering of the primal step, where the optimal placement and the optimal power allocation subproblems become decoupled. In Section VII-C, we introduce efficient distributed solutions for each of these subproblems.

A. Primal-dual decomposition and layering of the primal subproblem

Utilizing a dual decomposition approach, we break down Problem 7.1 into smaller subproblems which can be solved efficiently in a distributed fashion. One of these subproblems, which we denote as *node placement subproblem*, only involves the location of the mobile nodes, i.e., $(\mathbf{x}_{\Delta+1}, \mathbf{y}_{\Delta+1})$ through $(\mathbf{x}_{\Delta+\Omega}, \mathbf{y}_{\Delta+\Omega})$. The second subproblem, which we refer to as *power allocation subproblem*, is an optimization with respect to the received powers $\mathbf{P}_{i\to k}$. The correct coupling between these two subproblems, via message exchange, leads to an algorithm that converges to the optimal solution of Problem 7.1.

The main idea for breaking down our overall optimization problem is applying Lagrange relaxation to the power constraints in (43). These power constraints involve both sets of variables $\{\mathbf{x}_{\Delta+i}, \mathbf{y}_{\Delta+i}\}_{i=1}^{\Omega}$ and $\{\mathbf{P}_{i\to j}\}_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega}$ in a coupled fashion. In order to start the decoupling process, we need to replace the power constraints (43) for each transmitter i in the set $\{1, 2, ..., \Delta+\Omega\}$, with the following inequalities:

$$\phi 10^{0.1\mathbf{P}_{i\to j}+\alpha\mathbf{e}_{i,j}} \leq 10^{0.1\mathbf{v}_{i\to j}},\ j \in \mathbb{O}(i) \tag{45}$$

$$\sum_{j\in\mathbb{O}(i)} 10^{0.1\mathbf{v}_{i\to j}} \leq 10^{0.1\Psi_i} \tag{46}$$

Where $\mathbf{v}_{i\to j}$ represents an auxiliary variable that can be interpreted as the maximum transmission power (in $dBmW$) that transmitter i can allocate for the link from the transmitter i to receiver j. We can also re-write the utility function (40) in terms of these supplemental variables. As such, Problem 7.1 can be re-formulated as:

Problem 7.2: Let all parameters needed in the definition of Problem 7.1 be given. Adopting the same parameters, consider the following augmented optimization paradigm:

$$(\mathbf{Q}^*, \mathbf{V}^*) = \arg \overset{\flat}{\min_{(\mathbf{Q},\mathbf{V})}} \tilde{\mathcal{U}}(\mathbf{V}) \tag{47}$$

where $\flat$ indicates that the minimum should be taken subject to constraints (44), (45) and (46)[6]. In addition, the cost $\tilde{\mathcal{U}}$ and

[6]Notice that $\mathbf{Q}$ and $\mathbf{V}$ get coupled through constraints (44), (45) and (46). In addition, at an optimum constraints (45) and (46) will be satisfied with equality, implying that $\mathbf{Q}^*$ obtained from (47) is also an optimum for Problem 7.1

the collection of optimization variables $\mathbf{V}$, are given by :

$$\tilde{\mathcal{U}}\left(\mathbf{V}\right) \stackrel{def}{=} \sum_{i=1}^{\Delta+\Omega} \sum_{j \in \mathbb{O}(i)} \varphi_i 10^{0.1 \mathbf{v}_{i \to j}} \tag{48}$$

$$\mathbf{V} \stackrel{def}{=} \{\mathbf{v}_{i \to j}\}_{i=1, j \in \mathbb{O}(i)}^{i=\Delta+\Omega} \tag{49}$$

Proposition 7.1: Let all parameters needed in the definition of Problem 7.1 be given. The following min-max optimization paradigm is equivalent to Problems 7.1 and 7.2:

$$(\mathbf{Q}^\star, \mathbf{V}^\star, \mathbf{H}^\star) = \arg \left[\max_{\mathbf{H}} \underbrace{\overset{\Diamond}{\min_{(\mathbf{Q},\mathbf{V})}} \mathcal{L}\left(\mathbf{Q}, \mathbf{V}, \mathbf{H}\right)}_{\text{primal problem}} \right] \tag{50}$$

$$\mathbf{H} \stackrel{def}{=} \{\mathbf{h}_{i,j}\}_{i=1, j \in \mathbb{O}(i)}^{i=\Delta+\Omega} \tag{51}$$

$$\mathbf{h}_{i,j} \geq 0, \;\; (i,j) \in \bigcup_{i=1}^{\Delta+\Omega} \{i\} \times \mathbb{O}(i) \tag{52}$$

where $\Diamond$ indicates that the minimum is taken subject to constraints (44) and (46), while $\mathcal{L}$ is the following Lagrangean[7]:

$$\mathcal{L}\left(\mathbf{Q}, \mathbf{V}, \mathbf{H}\right) \stackrel{def}{=} \underbrace{\mathcal{L}_1\left(\{\mathbf{P}_{i \to j}\}_{i=1, j \in \mathbb{O}(i)}^{i=\Delta+\Omega}, \mathbf{V}, \mathbf{H}\right)}_{\text{power allocation}} + \underbrace{\mathcal{L}_2\left(\{\mathbf{x}_{i+\Delta}, \mathbf{y}_{i+\Delta}\}_{i=1}^{\Omega}, \mathbf{H}\right)}_{\text{node placement}} \tag{53}$$

where

$$\mathcal{L}_1\left(\{\mathbf{P}_{i \to j}\}_{i=1, j \in \mathbb{O}(i)}^{i=\Delta+\Omega}, \mathbf{V}, \mathbf{H}\right) \stackrel{def}{=} \tilde{\mathcal{U}}(\mathbf{V}) + \sum_{i=1}^{\Delta+\Omega} \sum_{j \in \mathbb{O}(i)} \mathbf{h}_{i,j}\left[\mathbf{P}_{i \to j} - \mathbf{v}_{i \to j} + 10 \log \varphi\right] \tag{54}$$

$$\mathcal{L}_2\left(\{\mathbf{x}_{i+\Delta}, \mathbf{y}_{i+\Delta}\}_{i=1}^{\Omega}, \mathbf{H}\right) \stackrel{def}{=} \sum_{i=1}^{\Delta+\Omega} \sum_{j \in \mathbb{O}(i)} \mathbf{h}_{i,j} 10 \alpha \mathbf{e}_{i,j} \tag{55}$$

By inspection, we find that the primal problem in (50) can be decomposed into two independent subproblems, as indicated in the following Remark:

Remark 7.2: (**Layering**) The Lagrangian (53) comprises the two additive terms given by $\mathcal{L}_1$ and $\mathcal{L}_2$. As such, given any choice of $\mathbf{H}$, the primal problem in (50) can be recast as follows:

$$\overset{\Diamond}{\min_{(\mathbf{Q},\mathbf{V})}} \mathcal{L}\left(\mathbf{Q}, \mathbf{V}, \mathbf{H}\right) = \underbrace{\overset{\Diamond}{\min_{\left(\{\mathbf{P}_{i \to j}\}_{i=1, j \in \mathbb{O}(i)}^{i=\Delta+\Omega}, \mathbf{V}\right)}} \mathcal{L}_1\left(\{\mathbf{P}_{i \to j}\}_{i=1, j \in \mathbb{O}(i)}^{i=\Delta+\Omega}, \mathbf{V}, \mathbf{H}\right)}_{\text{Power allocation subproblem}} + \underbrace{\min_{\{\mathbf{x}_{i+\Delta}, \mathbf{y}_{i+\Delta}\}_{i=1}^{\Omega}} \mathcal{L}_2\left(\{\mathbf{x}_{i+\Delta}, \mathbf{y}_{i+\Delta}\}_{i=1}^{\Omega}, \mathbf{H}\right)}_{\text{Node placement subproblem}} \tag{56}$$

[7]Notice that in (50), the minimization signaled with $\Diamond$ is not subject to (45). Here we use Lagrange multipliers $\mathbf{H}$ to impose constraint (45)

where $\Diamond$ indicates that the minimum is taken subject to constraints (44) and (46). Notice that, given $\mathbf{H}$, the first term in the right hand side of (56) involves only the received powers $\{\mathbf{P}_{i \to j}\}_{i=1, j \in \mathbb{O}(i)}^{i=\Delta+\Omega}$ and the supplemental variables $\mathbf{V}$, while the second term addresses node placement. We denote these independent subproblems as power allocation and node placement, respectively.

Proof of Proposition 7.1 Since Problems 7.1 and 7.2 are equivalent, we only need to prove that Problem 7.2 can be re-cast as in (50)-(52).

We start by noticing that the inequalities in (45) can be written in linear form as $10 \log \varphi + \mathbf{P}_{i \to k} + 10 \alpha \mathbf{e}_{i,k} - \mathbf{v}_{i \to k} \leq 0$. Relaxing these linear constraints by introducing the multipliers $\mathbf{H}$, leads to the following Lagrangian for Problem 7.2, which is identical to the Lagrangian in (53):

$$\mathcal{L}(\mathbf{Q}, \mathbf{V}, \mathbf{H}) = \tilde{\mathcal{U}}(\mathbf{Q}) + \sum_{i=1}^{\Delta+\Omega} \sum_{j \in \mathbb{O}(i)} \mathbf{h}_{i,j}\left[\mathbf{P}_{i \to j} - \mathbf{v}_{i \to j} + 10 \log \varphi + 10 \alpha \mathbf{e}_{i,j}\right] \tag{57}$$

In fact, using the Lagrangian defined in (57), we conclude that the following optimization problem is equivalent to (47):

$$(\mathbf{Q}^*, \mathbf{V}^*) = \arg \overset{\Diamond}{\min_{(\mathbf{Q},\mathbf{V})}} \left[\max_{\mathbf{H}} \mathcal{L}(\mathbf{Q}, \mathbf{V}, \mathbf{H})\right] \tag{58}$$

where $\Diamond$ indicates that the minimum is taken subject to constraints (44) and (46). Notice that the sole role of the Lagrange multipliers $\mathbf{H}$ is to impose (45).

Now we just have to prove that (58) is equivalent to (50). We proceed by noticing that $\mathcal{L}(\mathbf{Q}, \mathbf{V}, \mathbf{H})$ is convex in $(\mathbf{Q}, \mathbf{V})$ and that it is concave in $\mathbf{H}$. In addition, the constraints (44) and (46) are also convex in $(\mathbf{Q}, \mathbf{V})$, as we have already proved in Theorem 4.2. Hence, from basic minmax theory [23], we know that the $\min$ and the $\max$ in (58) can be exchanged, leading to the conclusion that (58) is equivalent to (50). □

B. A primal-dual iterative solution to Problems 7.1 and 7.2

In this subsection, we use a subgradient technique to develop an iterative solution to Problem 7.1. More specifically, we now propose a primal-dual iteration for solving Problem 7.1 globally.

In the description of our primal-dual algorithm, we will use the following notation:

- We use k to denote an integer counter that is increased at every step of the iterative algorithm.
- Given the parameters defining Problems 7.1 and 7.2, our iterative algorithm will use the following sequences of iterated solutions:

$$\mathbf{Q}^*(k) \stackrel{def}{=} \left(\{\mathbf{P}_{i \to j}^*(k)\}_{i=1, j \in \mathbb{O}(i)}^{\Delta+\Omega}, \{\mathbf{x}_i^*(k), \mathbf{y}_i^*(k)\}_{i=\Delta}^{\Delta+\Omega}\right)$$

$$\mathbf{V}^*(k) \stackrel{def}{=} \{\mathbf{v}_{i \to j}^*(k)\}_{i=1, j \in \mathbb{O}(i)}^{\Delta+\Omega}$$

  where, for each k, $\mathbf{P}_{i \to j}^*(k)$, $(\mathbf{x}_i^*(k), \mathbf{y}_i^*(k))$ and $\mathbf{v}_{i \to j}^*(k)$ are the solutions of the iterative algorithm correspoding to the kth iteration. These sequences of optimization variables are the iterative analogous to the optimization variables defined in Section VII-A. On the same token, we also define the following sequence of Lagrange multipliers:

$$\mathbf{H}(k) \stackrel{def}{=} \{\mathbf{h}_{i,j}(k)\}_{i=1, j\in\mathbb{O}(i)}^{\Delta+\Omega}$$

We can now specify the following iterative algorithm:

Algorithm 1: **(Primal-Dual iteratitive solution to Problems 7.1 and 7.2)** For the design parameters specifying Problems 7.1 and 7.2, consider the following coupled iterative equations, where k is a non negative integer counter:

(Initialization) Initialize $\mathbf{H}(0)$ as $\mathbf{h}_{i,j}(0) = 0$

(Primal step)

$$(\mathbf{Q}^*(k+1), \mathbf{V}^*(k+1)) = \arg \overset{\diamondsuit}{\min_{(\mathbf{Q},\mathbf{V})}} \mathcal{L}\left(\mathbf{Q}, \mathbf{V}, \mathbf{H}(k)\right) \quad (59)$$

where $\diamondsuit$ indicates that, for each pair (i,j) in the set $\bigcup_{i=1}^{\Delta+\Omega}\{i\}\times\mathbb{O}(i)$, the minimum is taken subject to the following constraints:

$$\mathbf{S}_{i\to j} \geq \mathcal{R}^{-1}(\varrho_{i\to j}) \quad (60)$$

$$\sum_{j\in\mathbb{O}(i)} 10^{0.1\mathbf{v}_{i\to j}} \leq 10^{0.1\Psi_i} \quad (61)$$

(Price update) Update the elements of $\mathbf{H}(k)$ via the following iteration:

$$\mathbf{h}_{i,j}(k+1) = [\varepsilon(k)\mathbf{w}_{i,j}^*(k) + \mathbf{h}_{i,j}(k)]^+, \quad (i,j) \in \bigcup_{i=1}^{\Delta+\Omega}\{i\}\times\mathbb{O}(i) \quad (62)$$

$$\mathbf{w}_{i,j}^*(k) \stackrel{def}{=} 10\log\varphi + \mathbf{P}_{i\to j}^*(k) + 10\alpha\mathbf{e}_{i,j}^*(k) - \mathbf{v}_{i\to j}^*(k) \quad (63)$$

where for any real s, $[s]^+ \stackrel{def}{=} \max\{s, 0\}$ and $\{\varepsilon(k)\}_{k=0}^{\infty}$ is a positive, square summable but not summable sequence.

In the following proposition, we prove that Algorithm 1 generates to a sequence of primal and dual variables that converges to an optimal solution of Problems 7.1 and 7.2 . In section VII-C, we will describe numerically efficient ways of computing and implementing the Algorithm.

Proposition 7.2: Let the parameters defining Problem 7.1 be given, and assume that the Problem has a feasible set with non-empty interior (i.e., there exists a strictly feasible solution in the constraint set). Consider that $\{\mathbf{Q}^*(k)\}_{k=0}^{\infty}$ is obtained via Algorithm 1. The following holds:

$$\lim_{k\to\infty} \mathcal{U}(\mathbf{Q}^*(k)) = \mathcal{U}(\mathbf{Q}^*) \quad (64)$$

$$\mathbf{S}_{i\to j}^*(k) \geq \mathcal{R}^{-1}(\varrho_{i\to j}),\ (i,j) \in \bigcup_{i=1}^{\Delta+\Omega}\{i\}\times\mathbb{O}(i),\ k \geq 0 \quad (65)$$

$$\limsup_{k\to\infty} \mathbf{P}_i^{total,*}(k) \leq \Psi_i,\ i \in \{1,\ldots,\Delta+\Omega\} \quad (66)$$

where $\mathcal{U}$ and $\mathbf{Q}^*$ are the cost and an optimal solution of Problem 7.1, respectively, while $\mathbf{P}_i^{total,*}(k)$ is given by:

$$\mathbf{P}_i^{total,*}(k) \stackrel{def}{=} \sum_{j\in\mathbb{O}(i)} \phi 10^{0.1\mathbf{P}_{i\to j}^*(k) + \alpha\mathbf{e}_{i,j}^*(k)}$$

Proof: We outline the proof here, which follows a standard procedure, such as in [26]. The convexity of Problem 7.1 was shown in section IV-A for a more general case. In addition, from our assumption that (for the given parameters) Problem 7.1 has a strictly feasible point in the constraint set, we conclude that Slater's condition holds. Since strong duality holds for convex problems under Slater's condition, finding the optimal solution of Problem 7.1 is equivalent to solving the associated dual maximization. The method described via Algorithm 1 is a subgradient iterative solution to the aforementioned dual maximization problem. Notice that (62) describes the update of the dual variables towards the sub-gradient direction[8]. Indeed, from (53)-(55), we conclude that $\mathbf{w}_{i,j}^*(k)$ in (63) is the gradient of $\mathcal{L}\left(\mathbf{Q}^*(k), \mathbf{V}^*(k), \mathbf{H}(k)\right)$ with respect to $\mathbf{h}_{i,j}(k)$. If the sequence $\varepsilon(k)$ is suitably chosen, e.g., any positive square summable but not summable sequence, then the convergence of this subgradient method is guaranteed [29]. Hence, the above algorithm converges to a solution that is optimal both in the primal and dual domains. It only remains to check that (65) and (66) hold. That (65) holds follows from the fact that (60) is imposed at every step of the iteration. Now notice that convergence of $\mathbf{h}_{i,j}(k)$ implies that $\limsup_{k\to\infty}\mathbf{w}_{i,j}(k) \leq 0$. Hence, from the aforementioned fact, (63) and (61), we can also conclude (66). ■

C. Computation and implementation of the primal step of Algorithm 1

While the price update in Algorithm 1 can be implemented via elementary operations, the computation of the primal step deserves further attention. In this section, we discuss how the structure of the primal step of Algorithm 1 can be exploited for efficient implementation.

From Remark 7.2, we conclude that the primal step of Algorithm 1 admits a decomposition in two layers that can be computed via the independent minimization of (54) and (55). For the remainder of this section, we will investigate the minimization of the two terms in (56), denoted as power allocation and node placement subproblems.

1) Power allocation subproblem, solution and computational considerations: Let all parameters needed in the definition of Problem 7.1 be given along with Lagrange multipliers $\mathbf{H}$, for simplicity of exposition, we start by re-writing the power allocation subproblem in (56) as follows:

$$\left(\{\mathbf{P}_{i\to j}^*(k+1)\}_{i=1,j\in\mathbb{O}(i)}^{i=\Delta+\Omega}, \mathbf{V}^*(k+1)\right) \stackrel{def}{=} \arg \overset{\diamondsuit}{\min_{\left(\{\mathbf{P}_{i\to j}\}_{i=1,j\in\mathbb{O}(i)}^{i=\Delta+\Omega}, \mathbf{V}\right)}} \mathcal{L}_1\left(\{\mathbf{P}_{i\to j}\}_{i=1,j\in\mathbb{O}(i)}^{i=\Delta+\Omega}, \mathbf{V}, \mathbf{H}(k)\right) \quad (67)$$

[8]We have used the supremum rule for subgradients in which $\partial\mathcal{L}(\mathbf{Q}^*, \mathbf{V}^*, \mathbf{H}) \subseteq \partial\sup_{(\mathbf{Q},\mathbf{V})}\mathcal{L}(\mathbf{Q}, \mathbf{V}, \mathbf{H})$, where $(\mathbf{Q}^*, \mathbf{V}^*) = \arg\min_{(\mathbf{Q},\mathbf{V})}[\mathcal{L}(\mathbf{Q}, \mathbf{V}, \mathbf{H})]$ and ∂ denotes the subgradient set, with respect to $\mathbf{H}$. See [27, lemma 3.1.11]

where $\diamondsuit$ indicates that, for each pair (i,j) in the set $\bigcup_{i=1}^{\Delta+\Omega}\{i\}\times\mathbb{O}(i)$, the minimum is taken subject to constraints (44) and (46), repeated below for convenience:

$$\mathbf{S}_{i\to j} \geq \mathcal{R}^{-1}(\varrho_{i\to j}), i\in\{1,\dots,\Delta+\Omega\}, j\in\mathbb{O}(i) \quad (68)$$

$$\sum_{j\in\mathbb{O}(i)} 10^{0.1\mathbf{v}_{i\to j}} \leq 10^{0.1\Psi_i}, i\in\{1,\dots,\Delta+\Omega\} \quad (69)$$

We proceed by noticing that $\mathcal{L}_1$, defined in (54), can be re-written as:

$$\mathcal{L}_1\left(\{\mathbf{P}_{i\to j}\}_{i=1,j\in\mathbb{O}(i)}^{i=\Delta+\Omega}, \mathbf{V}, \mathbf{H}(k)\right) = \mathcal{L}_{1,1}\left(\{\mathbf{P}_{i\to j}\}_{i=1,j\in\mathbb{O}(i)}^{i=\Delta+\Omega}, \mathbf{H}(k)\right) + \mathcal{L}_{1,2}\left(\mathbf{V},\mathbf{H}(k)\right) \quad (70)$$

where $\mathcal{L}_{1,1}$ and $\mathcal{L}_{1,2}$ are given by:

$$\mathcal{L}_{1,1}\left(\{\mathbf{P}_{i\to j}\}_{i=1,j\in\mathbb{O}(i)}^{i=\Delta+\Omega}, \mathbf{H}(k)\right) \stackrel{def}{=} \sum_{j=1}^{\Delta+\Omega} \mathcal{L}_{1,1,j}\left(\{\mathbf{P}_{i\to j}, \mathbf{h}_{i,j}(k)\}_{i\in\mathbb{I}(j)}\right),$$
$$\mathcal{L}_{1,1,j}\left(\{\mathbf{P}_{i\to j}, \mathbf{h}_{i,j}(k)\}_{i\in\mathbb{I}(j)}\right) \stackrel{def}{=} \sum_{i\in\mathbb{I}(j)} \mathbf{P}_{i\to j}\mathbf{h}_{ij}(k) \quad (71)$$

$$\mathcal{L}_{1,2}\left(\mathbf{V},\mathbf{H}(k)\right) \stackrel{def}{=} \sum_{i=1}^{\Delta+\Omega} \mathcal{L}_{1,2,i}\left(\{\mathbf{v}_{i\to j}, \mathbf{h}_{i,j}(k)\}_{j\in\mathbb{O}(i)}\right),$$
$$\mathcal{L}_{1,2,i}\left(\{\mathbf{v}_{i\to j}, \mathbf{h}_{i,j}(k)\}_{j\in\mathbb{O}(i)}\right) \stackrel{def}{=} \sum_{j\in\mathbb{O}(i)} \varphi_i 10^{0.1\mathbf{v}_{i\to j}} - \mathbf{h}_{ij}(k)\mathbf{v}_{i\to j} \quad (72)$$

In (71), $\mathbb{I}(j)$ denotes the set of transmitter nodes that have links to destination node j.

Remark 7.3: Inspection of (70)-(72) indicates that the optimal power allocation of (67)-(69) can be equivalently obtained from:

$$\{\mathbf{P}^*_{i\to j}(k+1)\}_{i\in\mathbb{I}(j)} = \arg\min_{\{\mathbf{P}_{i\to j}\}_{i\in\mathbb{I}(j)}} \mathcal{L}_{1,1,j}\left(\{\mathbf{P}_{i\to j}, \mathbf{h}_{i,j}(k)\}_{i\in\mathbb{I}(j)}\right), \quad j\in\{1,\dots,\Delta+\Omega\} \quad (73)$$

subject to:

$$\mathbf{S}_{i\to j} \geq \mathcal{R}^{-1}(\varrho_{i\to j}), (j,i)\in \bigcup_{j\in\{1,\dots,\Delta+\Omega\}} \{j\}\times\mathbb{I}(j) \quad (74)$$

Remark 7.4: In addition, $\mathbf{V}^*(k+1)$ resulting from (67)-(69) can be computed as:

$$\{\mathbf{v}^*_{i\to j}(k+1)\}_{j\in\mathbb{O}(i)} = \arg\min_{\{\mathbf{v}_{i\to j}\}_{j\in\mathbb{O}(i)}} \mathcal{L}_{1,2,i}\left(\{\mathbf{v}_{i,j}, \mathbf{h}_{i,j}(k)\}_{j\in\mathbb{O}(i)}\right), \quad i\in\{1,\dots,\Delta+\Omega\} \quad (75)$$

subject to:

$$\sum_{j\in\mathbb{O}(i)} 10^{0.1\mathbf{v}_{i\to j}} \leq 10^{0.1\Psi_i}, i\in\{1,\dots,\Delta+\Omega\} \quad (76)$$

Remark 7.5: (**Computational considerations**) Inspection of Remarks 7.3 and 7.4 leads to the following conclusions:

- The sequence of power allocation solutions at all nodes $\mathbf{P}^*_{i\to j}(k)$ and the auxiliary variables $\mathbf{v}^*_{i\to j}(k)$ can be computed separately via de cost and constraint pairs (73)-(74) and (75)-(76), respectively.
- The optimization problem in (73)-(74) is a geometric program with respect to $\{\mathbf{P}_{i\to j}\}_{i\in\mathbb{I}(j)}$.
- The optimization specified in (75)-(76) has an analytic solution given in Appendix A.

2) Distributed placement: In order to complete the discussion on the computation of the primal step of Algorithm 1, it remains to address the solution of the node placement sub-problem as defined in Remark 7.2. More precisely, given a positive integer k and all parameters needed in the definition of Problem 7.1, along with Lagrange multipliers $\mathbf{H}(k)$, we want to solve the following optimization problem:

$$\{\mathbf{x}^*_{i+\Delta}(k+1), \mathbf{y}^*_{i+\Delta}(k+1)\}_{i=1}^{\Omega} = \arg\min_{\{\mathbf{x}_{i+\Delta},\mathbf{y}_{i+\Delta}\}_{i=1}^{\Omega}} \sum_{i=1}^{\Delta+\Omega}\sum_{j\in\mathbb{O}(i)} \mathbf{h}_{i,j}(k)10\alpha\mathbf{e}_{i,j} \quad (77)$$

where the distances $\mathbf{e}_{i,j}$ are a function of the placements $\{\mathbf{x}_{i+\Delta},\mathbf{y}_{i+\Delta}\}_{i=1}^{\Omega}$. Now notice that (77) is an unconstrained convex problem because the Euclidean distances are convex functions of the positions of the nodes. In Appendix B we use a standard sub-gradient descent method to obtain a distributed algorithm for solving (77).

Remark 7.6: **(Information pattern required for Algorithm 1)** It follows from our analysis that implementation of Algorithm 1 can be done in a distributed way, where each node i only needs to communicate with its neighbors, i.e., nodes in the set $\mathbb{O}(i)\bigcup\mathbb{I}(i)$. More specifically, computation of $\{\mathbf{P}^*_{i\to j}(k+1)\}_{i\in\mathbb{I}(j)}$ in (73)-(74) can be done at the destination node j and the resulting value communicated back to nodes in the set $\mathbb{I}(j)$. In a similar fashion, $\{\mathbf{v}^*_{i\to j}(k+1)\}_{j\in\mathbb{O}(i)}$ in (75)-(76) can be computed at the transmitter node i and the result conveyed to nodes in the set $\mathbb{O}(i)$. In addition, as described in Appendix B, computation of the node placement sub-problem can be done at each node i, where one only requires knowledge of the direction of the vector directed from i towards nodes in the set $\mathbb{I}(i)\bigcup\mathbb{O}(i)$.

VIII. SIMULATIONS

In this section, we provide a numeric simulation of the method proposed in Section VII.

Example 8.1: We consider four fixed nodes ($\Delta = 4$), labeled as 1 through 4, and three mobile nodes ($\Omega = 3$), labeled A, B and C.In this example, we constrain the solution to satisfy the connectivity graph specified in Table 8.1, where each link is required to have 1 bit/sec while the channel gain to the sum of interference coefficients is 15dB and both the channel gain and interference coefficients are generated according to log-normal fading. The variance of the noise at the receiver is assumed to be unit and we also used (1) with $\alpha = 1, \phi = 0.04$ as of the propagation loss model. The objective is to minimize the total power consumption in

T \ R	4	A	B	C
1		X		
2		X		
3			X	
4				
A			X	X
B		X		X
C	X			

TABLE I

TABLE DESCRIBING THE CONNECTIVITY GRAPH IMPOSED AS A CONSTRAINT IN EXAMPLE 8.1. HERE T AND R REPRESENT TRANSMITTER AND RECEIVER AND X INDICATES THE PRESENCE OF A LINK. FOR INSTANCE, THERE IS A LINK FROM NODE 1 TO NODE A AND A BI-DIRECTIONAL LINK BETWEEN NODES A AND B. ALL EXISTING LINKS ARE REQUIRED TO SATISFY A MINIMUM BIT RATE OF 1 BIT/SEC.

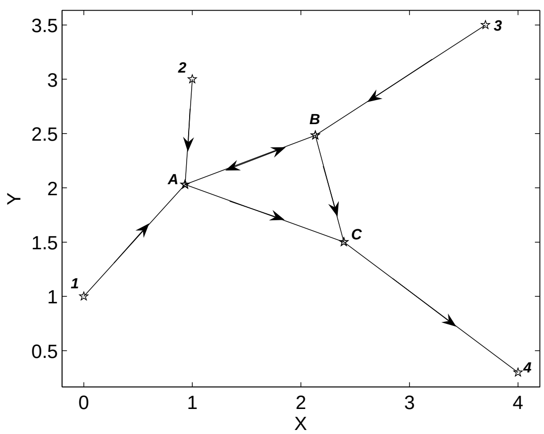

Fig. 1. Optimal positions of the mobile nodes A, B *and* C.

the network. We use our proposed distributed algorithm in section VII to find the optimal power of the transmitters and the position of the mobile nodes. Figures 1- 3 display the optimal position of the mobile nodes, the power allocations in the transmitter side of the links and the dual variables, during our iterative algorithm until they converge.

Other simulations that can be found in [30], where we have compared the speed of convergence in high-interference

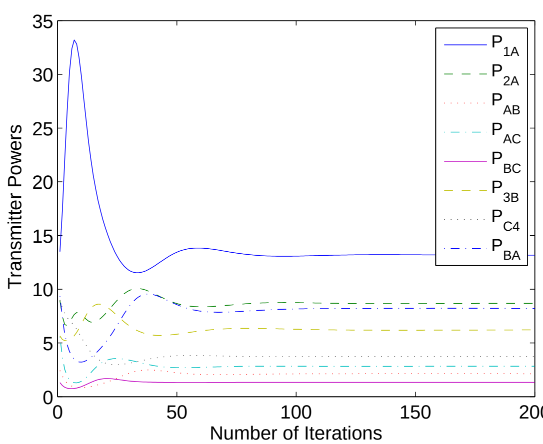

Fig. 2. Evolution of values of the power variables relative to the iterative solution of Example 8.1.

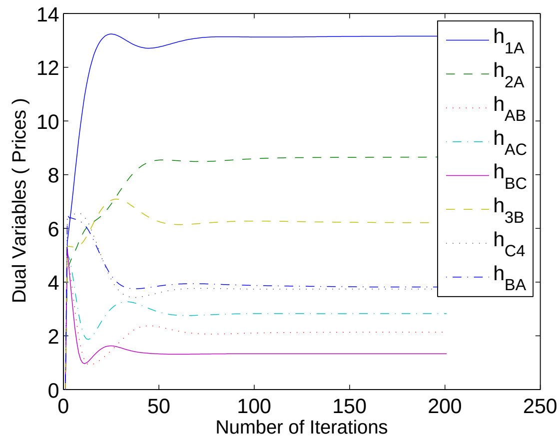

Fig. 3. Evolution of values of the dual variables relative to the iterative solution of Example 8.1.

and low-interference scenarios.. We also illustrate situations in which the optimization paradigm is not feasible and the shadow prices do not converge.

IX. GENERALIZED PROPAGATION LOSS MODEL

The complexity of signal propagation makes it difficult to obtain a single model for characterizing propagation loss across a range of different environments. Therefore, there are several models that have been developed over the years to predict path-loss in different wireless environments. What we discussed here in this paper was the case where the pass-loss function[9] can be expressed as an exponentially decreasing function of the distance between source and receiver, i.e $P_L = \varphi 10^{\alpha\ \mathbf{d}}$. In section (II-C1) we mentioned some motivations for such path-loss function, specially in urban and sub sea environment. Though, in this extension, we want to emphasize on general path-loss function $P_L(d)$ that is modeling the variation of received signal power over distance.

According to what we showed in section IV-A, under the assumption of exponential path-loss function, the general node placement problem is a convex optimization problem. The main difficulty of general path-loss functions is that the convexity of the problem doesn't hold anymore for arbitrary models, necessarily. In other words, the main watershed between convexity and non-convexity in placement optimization of the wireless nodes is the model that we use for propagation loss. Our main goal in this subsection is to mitigate the disadvantages of non-convexity by introducing an iterative method that is a convex approximation of the original problem in each step.

Based on what we have done earlier in this paper, we know that our node placement sub-problem is in a nice convex form if we adopt the exponential path-loss function. This underlying

[9] What we mean by *path-loss function*(P_L) for the communication channel is the ratio of the transmitted power to received power(Note that in this ratio the powers are expressed in W). Generally, path-loss is a function of transmitter-receiver distance, transmission frequency, transmitter and receiver antenna gain and also the environment (all are assumed to be fixed, except the distance that includes in our optimization variables.)

convexity, leaded us to cast the problem as standard GP in section VI and later was the substructure of our decentralized algorithm in section VII. The method that we want to propose in this subsection for solving the placement optimization problem with general path-loss function, starts by solving a problem in which we approximate the path-loss functions for each link with an exponential function. After that, in each iteration, we try to find the constants φ and α, for each link, in a way that the exponential function $\varphi 10^{\alpha \mathbf{d}}$, becomes the first order approximation of the actual path-loss function around the optimal distances that we have found in the previous iteration. Note that, since we are using exponential path-loss function in each iteration, we have to solve a convex optimization problem in each step which is possible with fast algorithms such as interior point methods or primal dual algorithms[25]. It is not hard to see that the converging optimum solution that we would find by this iterative method is also a local optimum for the original problem with arbitrary path-loss function. Meanwhile, there is no guarantee for the convergence of the algorithm or finding the global optimum, since the overall problem is not convex anymore.

The idea of solving the problem for generalized path-loss models iteratively, can also be implemented by our distributed algorithm that is proposed in section VII. More specifically, each link can update it's path loss model parameters, φ and α, during the iterations. This update can be done by measuring the separation between the transmitter and the receiver and trying to fit the exponential path-loss model, $\varphi 10^{\alpha \mathbf{d}}$, to the first order approximation of the actual path-loss function around the measured distance.

In order to check the efficiency of the proposed method, we repeat our simulation in section VIII with a new path-loss function of the following form [19]:

$$P_L(d) = K(d/d_0)^{\gamma} \tag{78}$$

where d_0 is a reference distance for the antenna far-field, γ is the path loss fall off exponent which is typically a number between 2 and 6 and K is a constant that encompass the transmission frequency and the gain of the antennas. We use $d_0 = 0.5$, $\gamma = 4$ and $K = 1$ for our simulation. Also, we assume the same bit rate constraints in all of the links and the objective is still minimizing the total power consumption in the network with a decentralized method. Figures 4-6 illustrates the optimal position of the mobile nodes, and parameters φ and α for different links during the iterations. This figure also shows the trajectory of the mobile nodes during the algorithm until it converges to the optimum position. Note that in this case the convergence happens after approximately two hundred iterations which is roughly two times slower in comparison with the previous case where we assumed pure exponential path-loss model(see Figures 1- 3).

What we proposed in this extension was a simple approach for solving the placement optimization problems where we have general path-loss model. Although we show the convergence of the algorithm for a specific path-loss model by simulation, there is still a lack of proof for convergence and even global optimality of the solution for general case and needs further studies.

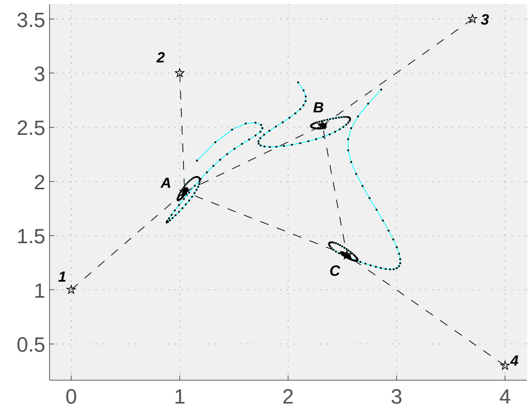

Fig. 4. Optimal positions of the mobile nodes A, B and C.

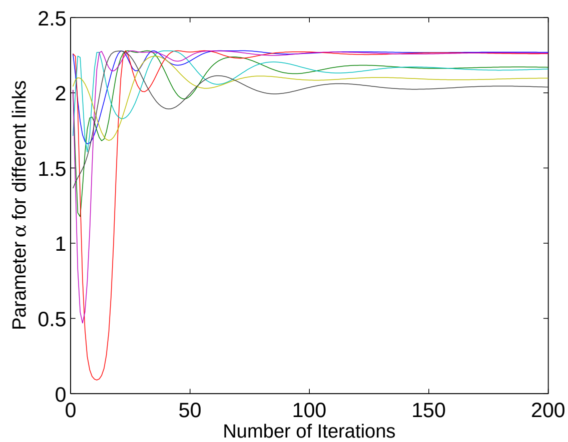

Fig. 5. Parameter α for different links.

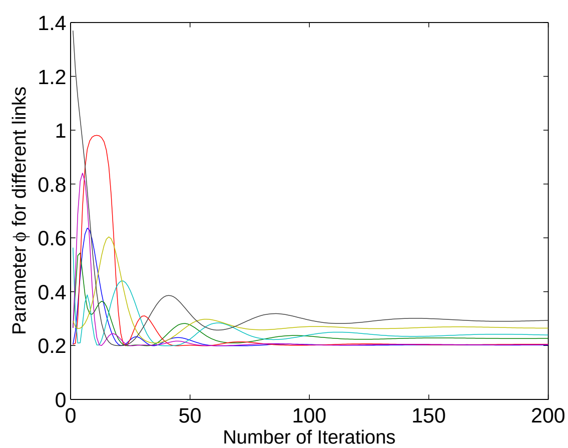

Fig. 6. Parameter φ for different links.

X. Conclusions and open problems

In this paper, we proposed a paradigm for the optimal design of wireless networks, with respect to power allocation and the placement of nodes in a Cartesian plane. We consider the optimization of network-centric figures of merit, which are functions of the transmission power and of the signal to interference ratio. In the high signal to interference regime, we can also adopt constraints and cost functions that involve rate-regions or throughput and under the "no dominant interferer" assumption we can also include outage probabilities. Under the assumption of exponential path loss, we show that our paradigm is convex and that it admits an approximate solution via geometric programming. We also provide a case study that illustrates how certain instance of our paradigm can be optimized, with no approximation, via a primal-dual iterative algorithm. This solution also leads to a layering in the primal step, where the power allocation and the node placement can be optimized independently. The following problems require further investigation, and we believe that this paper might be a stepping stone towards their solution:

- It is important to handle cost and the constraints that are functions of the transmission rate, while lifting the high signal to interference assumption. In the paper [2], this problem was solved with respect to power allocation.
- It is also important to obtain the solution to our paradigm for the case where the path loss is polynomial.

Acknowledgment The authors would like to thank Ananthram Swami (ARL) for suggesting the investigation of this problem in the presence of exponential attenuation.

Appendix A
Analytic Solution of (75)-(76)

Here we want to find the optimal solution of (75)-(76), repeated below for convenience:

$$\{\mathbf{v}^*_{i\to j}(k+1)\}_{j\in\mathbb{O}(i)} = \min_{\{\mathbf{v}_{i\to j}\}_{j\in\mathbb{O}(i)}} \sum_{j\in\mathbb{O}(i)} \varphi_i 10^{0.1\mathbf{v}_{i\to j}} - \mathbf{v}_{i\to j}\mathbf{h}_{ij}(k) \quad (79)$$

subject to the following constraints:

$$\sum_{j\in\mathbb{O}(i)} 10^{0.1\mathbf{v}_{i\to j}} \le 10^{0.1\Psi_i} \quad (80)$$

First note that , neglecting the constraint (80), the global solution that minimizes (79) is given by $\mathbf{v}^*_{i\to j}(k+1) = 10\log\frac{10\ \mathbf{h}_{ij}(k)}{\ln(10)\varphi_i}$. It is clear that if such an unconstrained optimal solution satisfies (80) then it is the optimal solution of (79). Now consider the case where the unconstrained solution does not satisfy (80). In this case the level-set of the cost function at the optimal point should be tangent to the feasible set at optimal point, i.e., the following equalities must hold:

$$\frac{\varphi_i\ 10^{0.1\mathbf{v}^*_{i\to j}(k+1)} - \mathbf{h}_{ij}(k)}{10^{0.1\mathbf{v}^*_{i\to j}(k+1)}} = K_i \quad , for\ \forall j\in\mathbb{O}(i) \quad (81)$$

$$\sum_{j\in\mathbb{O}(i)} 10^{0.1\mathbf{v}^*_{i\to k}(k+1)} = 10^{0.1\Psi_i} \quad (82)$$

where K_i are real constants. That equation (81) holds follows from the fact that the gradient of the level-set of the cost function should have the same direction as the gradient of the boundary of the feasible set at optimal point and (82) is the fact that the optimum is in the boundary of the feasible set. [10] Solving (81) and (82) simultaneously, we can conclude that $10^{0.1\mathbf{v}^*_{i\to j}}$, which is actually the optimum value of the maximum power that should be allocated in each iteration, is $\frac{\mathbf{h}_{ij}}{\sum_{j\in\mathbb{O}(i)}\mathbf{h}_{ij}}10^{0.1P_i^{max}}$. This result has also an interesting interpretation in dual domain that says the maximum power which should be allocated in each iteration to the links is proportional to the prices $\mathbf{h}_{ij}$.

Appendix B
Subgradient Method for Node Placement Module

In this Appendix, we use a standard gradient descent method to illustrate how (77) can be solved via an algorithm that only needs that each node exchanges information with its neighbors. For convenience, we repeat (77) below:

$$\{\mathbf{x}^*_{i+\Delta}(k+1), \mathbf{y}^*_{i+\Delta}(k+1)\}_{i=1}^{\Omega} = \arg\min_{\{\mathbf{x}_{i+\Delta},\mathbf{y}_{i+\Delta}\}_{i=1}^{\Omega}} \sum_{i=1}^{\Delta+\Omega}\sum_{j\in\mathbb{O}(i)} \mathbf{h}_{i,j}(k)10\alpha\mathbf{e}_{i,j} \quad (83)$$

where the distances $\mathbf{e}_{i,j}$ are a function of the placements $\{\mathbf{x}_{i+\Delta}, \mathbf{y}_{i+\Delta}\}_{i=1}^{\Omega}$.

Consider the following iteration, where t is an integer counter[11] :

$$(\tilde{\mathbf{x}}_i, \tilde{\mathbf{y}}_i)(k, t+1) = (\tilde{\mathbf{x}}_i, \tilde{\mathbf{y}}_i)(k, t) + \varepsilon(t) \sum_{j\in\{\mathbb{O}(i)\bigcup\mathbb{I}(i)\}} \mathbf{h}_{ij}(k)\overrightarrow{\theta_{ij}}(k,t) \quad (84)$$

where $\{\varepsilon(t)\}$ is an appropriate[12] step size sequence and $\overrightarrow{\theta_{ij}}(t)$ is a unit vector pointing in the direction from node i towards node j, defined to be:

$$\overrightarrow{\theta_{ij}}(k,t) = \delta_{ij}(k,t)(\tilde{\mathbf{x}}_j(k,t) - \tilde{\mathbf{x}}_i(k,t), \tilde{\mathbf{y}}_j(k,t) - \tilde{\mathbf{y}}_i(k,t)) \quad (85)$$

$$\delta(k,t) \overset{def}{=} \begin{cases} \frac{1}{\tilde{\mathbf{e}}_{i,j}(k,t)} & \text{if } \tilde{\mathbf{e}}_{i,j}(k,t) \neq 0 \\ 0 & Otherwise \end{cases} \quad (86)$$

where the distances $\tilde{\mathbf{e}}_{i,j}(k,t)$ are a function of the placements $\{\tilde{\mathbf{x}}_{i+\Delta}(k,t), \tilde{\mathbf{y}}_{i+\Delta}(k,t)\}_{i=1}^{\Omega}$.

In the subsequent analysis, we show that the subgradient iteration specified by (84)-(85) converges to the optimal solution of (77), i.e., the following limit holds:

$$\{\mathbf{x}^*_{i+\Delta}(k+1), \mathbf{y}^*_{i+\Delta}(k+1)\}_{i=1}^{\Omega} = \lim_{t\to\infty}\{\tilde{\mathbf{x}}_{i+\Delta}(k,t), \tilde{\mathbf{y}}_{i+\Delta}(k,t)\}_{i=1}^{\Omega} \quad (87)$$

That we need to work with subgradients of the cost, as opposed to the gradient alone, is a consequence of the fact that the

[10] Note that these equalities can also be derived from the KKT conditions.

[11] The counter t should not be confused with k in (77). In fact, the optimization in (77), or equivalently (83), needs to be re-solved each time k is increased by one time step.

[12] e.g. any square summable but not summable sequence

Euclidean distance is not differentiable at the origin. In order to show that (87) holds, we start by defining the subgradient for a nonsmooth convex function and we proceed by utilizing a few lemmas to compute the subgradient of the cost in (83).

Definition B.1: Let f be a convex function. A vector g is called a *subgradient* of function f at point $z_0 \in dom f$ if for any $z \in dom f$ we have

$$f(z) \geq f(z_0) + \langle g, z - z_0 \rangle. \tag{88}$$

Where $\langle ., . \rangle$ is a dot product. The set of all subgradients of f at z_0, $\partial f(z_0)$, is called *subdifferential* of function f at point z_0.

Recall that the optimization variables in this problem is the position of the mobile nodes $\{\mathbf{x}_{i+\Delta}(k+1), \mathbf{y}_{i+\Delta}(k+1)\}_{i=1}^{\Omega}$. In order to show that the iterations specified by (84)-(85) is simply moving in the opposite direction of subgradient of the cost function in (83), i.e it is a subgradient decent method, we need to calculate this subgradient. The cost function is nothing but a weighted sum of the Euclidean distances. Hence we need to calculate the subgradient for a weighted sum of the Euclidean distances which is not difficult by using the following lemmas.

Lemma B.1: Let $\mathbf{e}_{i,j}$ be the Euclidean distance function given by (1). Suppose $i \geq \Delta + 1, j \leq \Delta$, hence $\mathbf{e}_{i,j}$ is merely a function of $(\mathbf{x}_i, \mathbf{y}_i)$. For this case we have:

$$\partial \mathbf{e}_{i,j}(\mathbf{x}_i, \mathbf{y}_i) = \begin{cases} \frac{(\mathbf{x}_i - \chi_j, \mathbf{y}_i - \gamma_j)}{\sqrt{(\mathbf{x}_i - \chi_k)^2 + (\mathbf{y}_i - \gamma_k)^2}} & (\mathbf{x}_i, \mathbf{y}_i) \neq (\chi_j, \gamma_j) \\ B_2(0,1) & Otherwise \end{cases} \tag{89}$$

Where $B_2(0,1) = \{(x,y) \mid \sqrt{x^2+y^2} \leq 1\}$

Proof: see [27, example 3.1.5]. ■

Note that, according to the definition of $\overrightarrow{\theta_{ij}}$ in (85), we have $\overrightarrow{\theta_{ij}} \in \partial \mathbf{e}_{i,j}(\mathbf{x}_i, \mathbf{y}_i)$, i.e. $\overrightarrow{\theta_{ij}}$ is a valid subgradient for Euclidean norm function $\mathbf{e}_{i,j}$. We can derive similar expressions for the cases when $\{j \geq \Delta + 1, i \leq \Delta\}$ or $\{j \geq \Delta + 1, i \geq \Delta + 1\}$.

Lemma B.2: Let $f_1(z)$ and $f_2(z)$ be convex functions with same domain and $h_1 \geq 0,\ h_2 \geq 0$. For the function $f(z) = h_1 f_1(z) + h_2 f_2(z)$ we have:

$$\partial f(z) = h_1 \partial f_1(z) + h_2 \partial f_2(z) \tag{90}$$

Proof: The proof is in [27, page 131]. It is also trivial to generalize this result for a weighted sum of more than two functions. ■

Now we can conclude that the proposed iterations specified by (84)-(85) is a subgradient descent method, and the update formula (84) is just moving in the opposite direction of subgradient of the cost function. Choosing appropriate steps sizes $\{\varepsilon(t)\}$, e.g. any square summable but not summable sequence, it has been proved in [27, section 3.2.3] that our subgradient decent method will converge to the optimum solution of (77).

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