ABSTRACT

Title of dissertation: LONG TIME STABILITY OF ROTATIONAL EULER DYNAMICS

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We study the stabilizing effect of rotational forcing in the nonlinear setting of two-dimensional shallow-water and Euler equations. We prove that when rotational force dominates the pressure, it prolongs the life-span of smooth solutions for $t \lesssim 1 + \ln(\delta^{-1})$ where $\delta \ll 1$ is the ratio of the (inverse of) squared Froude number measuring the amplitude of pressure, relative to the (inverse of) Rossby number, measuring the dominant rotational force. The strong rotation also imposes certain periodicity to the flow in the sense that there exists a "nearby" periodic-in-time approximation of the exact solution. In the opposite regime of large δ's, the flow is dispersive so that the divergence field substantially decays in finite time and therefore periodicity is not retained.

LONG TIME STABILITY OF ROTATIONAL EULER DYNAMICS

by

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Table of Contents

Chapter 1

Introduction

We are concerned here with a family of 2D compressible rotating Eulerian equations such as the Rotational Shallow Water (RSW) equations. Our main result shows that the rotational forcing, commonly referred to as the Coriolis forcing in Geophysics, has a stabilizing effect within a scaling regime that is rotationally dominant. In other words, breakdown of classical solutions – namely blow-up of the velocity gradients – is postponed until after $O(|\ln(\delta)|)$, the single parameter $\delta(\ll 1)$ closely related to the *relative* dominance of the rotational forcing. The underlying flow is shown to exhibit the so called "approximate periodicity" which characterizes the flow with a time-periodic approximation that is $O(\delta)$ apart measured in terms of certain Sobolev norms.

To get a flavor of our main theorem, consider the following RSW equations

$$\partial_t h + \mathbf{u}\cdot\nabla h + (\frac{1}{\sigma} + h)\nabla\cdot\mathbf{u} = 0,$$
$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\sigma}\nabla h - \frac{1}{\tau}\mathbf{u}^{\perp} = 0,$$

where the unknowns are height h and 2D velocity $\mathbf{u}$ defined on a 2D torus $\mathbb{T}^2$. Parameters σ and τ respectively denote the Froude number and the Rossby number in Geophysics ([39]). Define

$$\delta = \frac{\tau}{\sigma^2}$$

measuring the relative strength of the rotational forcing. Consider a rotationally dominant regime $\delta \ll 1$. Then, for a large set of subcritical data in $H^m(\mathbb{T}^2)$, there exists a globally smooth, $2\pi\tau$-periodic approximation $(h_2(t,\cdot), \mathbf{u}_2(t,\cdot))$ such that,

$$\|h(t,\cdot) - h_2(t,\cdot)\|_{H^{m-3}} + \|\mathbf{u}(t,\cdot) - \mathbf{u}_2(t,\cdot)\|_{H^{m-3}} \lesssim \frac{e^{C_0 t}\delta}{(1 - e^{C_0 t}\delta)^2},$$

for $t \lesssim t_\delta := \ln(\delta^{-1})$ where constant C_0 only depends on m and the size of the initial data. In particular, when $\delta \downarrow 0$, the life span of classical RSW solution $t_\delta \uparrow \infty$. We comment that the formal notation "approximate periodicity" emphasizes on the existence of a periodic approximation $(h_2, \mathbf{u}_2)$ nearby the actual flow $(h, \mathbf{u})$ with the error up-bounded by $O(\delta) \ll 1$ for sufficiently long time.

This result is understood in the context of previous results:

1. In the absence of rotation, classical solutions lose C^1 smoothness in finite time for generic initial data. This is the well-known shock theory for nonlinear hyperbolic systems, starting from Peter Lax's celebrated book [30].

2. In the mere presence of rotation as the only external forcing mechanism, Liu and Tadmor has shown in [33] that the solution either blows up in finite time or stays smooth (and time-periodic) for global time, depending on whether or not the initial data cross a non-trivial *critical threshold*. As a corollary, the solution stays smooth for global time if the rotational forcing is "stronger" than a $O(1)$ critical value that only depends on the initial data.

3. For the RSW equations, when the rotational forcing is of the same order as the pressure gradient forcing – commonly known as the "geostrophic balance", the underlying flow is approximately dictated by the so called "quasi-geostrophic equations" upon which fast gravity waves are superimposed ([13]).

A natural question hinted by these results is that: how does the relative significance of rotational forcing affect the behavior of underlying flows? One postulates from the previous results that the stronger the rotation, the longer the flow stays stable away from breakdown. To this end, we introduce the new parameter δ to measure the relative significance of rotational forcing over other forcing such as pressure forcing. The scaling regime of interest, $\delta \ll 1$, corresponds to the relative dominance of rotation.

However, the main difficulties in dealing with compressible Euler equations rise from two factors and their interaction: nonlinearity in the advection terms and singularity in the forcing terms. Nonlinear advection terms such as $\mathbf{u}\nabla\mathbf{u}$ are ubiquitous in Eulerian models of fluids. Fully understanding its mechanism in various settings has drawn enormous attention. On the other hand, singular parameters that serve as scales of external forces plays an essential role in determining properties of underlying flows. It enriches flow structures, which in turn increases the problem's complexity.

We use novel approaches to tackle these difficulties and to gain insightful knowledge of rotationally dominant flows (that is when $\delta \ll 1$). Two successive approximations are constructed. We start with the purely rotational case $\delta = 0$ and show global stability and periodicity under mild assumptions. Then we linearize the system around the first approximation and make careful adjustment so that this second approximate system admits global and periodic solutions as well. Finally we use energy methods (given in the preliminary Chapter 2) to estimate the error introduced by the second approximation. The parameter δ appears naturally in this process. It is the smallness of δ that prolongs the life span of classical solutions to $O(\ln \delta)$ as well as imposes approximate periodicity to the dynamics. The RSW equations, simple yet typical, are analyzed in Chapter 3 with detail proofs. Then, in Chapter 4, we give natural extension of our result to general Euler flows governed by e.g. isentropic gas equations and ideal gasdynamics equations so long as they are equipped with corresponding energy principles. Our methodology and results remain valid in various other settings. In Chapter 5, we will discuss parameter regimes that are complementary to $\delta \ll 1$. We use Strichartz estimate to study the time decay of solutions' L^∞ norm (in $\mathbb{R}^2$ domain), which turns out to be fast enough to contradict periodicity if δ is large. Thus, δ is the critical parameter that characterizes the relative significance of two competing dynamics: dispersion vs periodicity. Some possible future works will be outlined in Chapter 6.

The rest of this chapter provides an overview of this thesis regarding both its content and structure. In Section 1.1, I will give the formulation of our problem with relevant scaling parameters; in Section 1.2, I will sketch the proof of our main results; then Section 1.3 will regard our results in a broader mathematical and geophysical contexts.

1.1 Formulations and Definitions

We'll start with the basic physical laws, describe their formulation and derive the nondimensional version which is more adapted for mathematical analysis.

§. Three Fluid Models and Their Pressure Laws. The Eulerian formulation of fluids such as air and water is widely used in science and engineering,

$$\partial_t \mathbf{u} + \mathbf{u}\nabla\cdot\mathbf{u} + \mathbf{F}[\mathbf{u}] = 0 \tag{1.1}$$

where the velocity field $\mathbf{u} := (u_1, u_2, ...)^T$ depends on time variable t and spatial variables $\mathbf{x} := (x_1, x_2, ...)$. The operator ∇ takes the spatial gradients. The external forcing term $\mathbf{F}[\mathbf{u}]$, distinguishing one model from another, is a formal generalization which may depend on $\mathbf{u}$, its derivatives and quantities driven by $\mathbf{u}$.

The rotational models in our study will be mainly 2D, that is, $\mathbf{u} = (u_1, u_2)^T$. Thus the rotational forcing is written as

$$fJu \text{ where } J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Here scalar f denotes effective frequency of the rotation – commonly referred to as the Coriolis frequency in Geophysics – and J is called the rotational matrix since

$$\|J\mathbf{u}\| = \|\mathbf{u}\| \text{ and } J\mathbf{u}\cdot\mathbf{u} = 0.$$

Strictly speaking, one should use $-J$ for motions in the Southern Hemisphere and, thanks to symmetry, our analysis still remains valid in this case.

The rotational forcing appears widely in fluid modeling in areas such as Geophysics (e.g. [39]) and Magnetohydrodynamics (e.g. [11]). In large scale geophysical motions, the Coriolis force plays one of the essential roles that determine the nature of weather system. For instance, it forces a full 3D flow to behave closely as a 2D one (stratification), which is rigorously argued in [9]. In our paper, rotational forcing is further explored for 2D models.

We will then specify the other forcing terms in $\mathbf{F}[\mathbf{u}]$ of (1.1) as pressure forcing so that,

$$\mathbf{F}[\mathbf{u}] = \nabla \breve{p} - fJ\mathbf{u}$$

where $\breve{p}$ – up to a density factor – denotes the pressure that depends on various quantities (the dependence relation called "pressure law"). A typical example is the Rotational Shallow Water (RSW) equations

$$\partial_t h + \nabla \cdot (h\mathbf{u}) = 0 \tag{1.2a}$$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + g\nabla h = fJ\mathbf{u} \tag{1.2b}$$

where g is the gravitational constant and scalar variable $h = h(t, x, y)$ denotes the height. This system models a thin layer of homogeneous fluid moving in vertical columns (the so called Taylor column [39]) driven by the gravitational force $g\nabla h$ and the rotational force $fJ\mathbf{u}$ which are given in the momentum equations (1.2b). Notice h denotes the height of these moving columns. Therefore the pressure law $\breve{p} = gh$ corresponds to, up to a density factor, the hydro pressure asserted by the fluid column of height h. Equation (1.2a) simply follows the conservation of mass.

Remark 1.1 *If the RSW equations (1.2) are derived as the vertical mean of the flow of an isentropic atmosphere with a free upper boundary, the pressure law* $\breve{p} = gh$ *has to be replaced by* $\breve{p} = gh^{\frac{\gamma-1}{\gamma}}$ *where* γ *is the gas constant. If the flow is restricted between rigid upper and lower boundaries, the pressure law becomes* $\breve{p} = gh^{\gamma-1}$ *(consult e.g. [10]). These cases are covered by the next model – the isentropic gas equations.*

The isentropic model assumes that pressure depends on density only. In the special case of polytropic gas, the pressure obeys the following *gamma*-power law,

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{1.3a}$$

$$\partial_t \mathbf{u} + \mathbf{u} \nabla \cdot \mathbf{u} + \rho^{-1} \nabla \hat{p}(\rho) = f J \mathbf{u} \tag{1.3b}$$

The physical pressure $\hat{p} = \hat{p}(\rho)$ in the momentum equations (1.3b) is given by

$$\hat{p} := A \rho^{\gamma},$$

where A, γ are gas-specific constants. The special case $\gamma = 2, A = g/2$ corresponds the RSW equations. Thus, our discussion on the RSW system facilitates a natural generalization to isentropic gas in 2D.

In the third model of ideal gasdynamics, the pressure depends on both density and entropy. The system is equipped with an additional balance law for the entropy S,

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1.4a}$$

$$\partial_t \mathbf{u} + \mathbf{u} \nabla \cdot \mathbf{u} + \rho^{-1} \nabla \hat{p}(\rho, S) = f J \mathbf{u}, \tag{1.4b}$$

$$\partial_t S + \mathbf{u} \cdot \nabla S = 0. \tag{1.4c}$$

Here the physical pressure law $\hat{p} = \hat{p}(\rho, S)$ takes various forms, among which polytropic gas obeys

$$S = \ln(\hat{p} \rho^{-\gamma}) \quad \leftrightarrow \quad \hat{p} = e^{S} \rho^{\gamma}.$$

Thus, the isentropic model is a special case with constant entropy S.

§. Nondimensional Form and Scaling Parameters. We aim to make the above formulation independent of physical units – the so called nondimensional form. First for the RSW system (1.2), we introduce the horizontal length scale L, the velocity scale U, the back ground height H, the height perturbation scale D. Then the derived time scale L/U comes up as a natural choice. Rewrite the variables according to the following rules,

$$h(t,\mathbf{x}) \to H + Dh(\underline{t}\frac{L}{U}, \underline{\mathbf{x}}L), \quad \mathbf{u}(t,\mathbf{x}) \to \underline{\mathbf{u}}(\underline{t}\frac{L}{U}, \underline{\mathbf{x}}L),$$

and rewrite derivatives according to

$$\nabla \to \frac{1}{L}\nabla_{\underline{\mathbf{x}}} \quad \partial_t \to \frac{U}{L}\partial_{\underline{t}}.$$

Then discarding all the underlines, we arrive at a nondimensional RSW system,

$$\partial_t h + \mathbf{u}\cdot\nabla h + \left(\frac{H}{D} + h\right)\nabla\cdot\mathbf{u} = 0, \tag{1.5a}$$

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{gD}{U^2}\nabla h - \frac{fL}{U}\mathbf{u}^{\perp} = 0. \tag{1.5b}$$

We are here concerned with the scaling regime where compressibility in (1.5a) and the pressure gradient in (1.5b) are of the same order, $\frac{gD}{U^2} \approx \frac{H}{D}$ (see e.g. [34]). Thus, we arrive at the very RSW system of our study,

$$\partial_t h + \mathbf{u}\cdot\nabla h + (\frac{1}{\sigma} + h)\nabla\cdot\mathbf{u} \;=\; 0 \tag{1.6a}$$

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\sigma}\nabla h - \frac{1}{\tau}J\mathbf{u} \;=\; 0 \tag{1.6b}$$

Here the scaling parameters, defined as,

$$\tau := \frac{U}{fL}, \quad \sigma := \frac{U}{\sqrt{gH}} \tag{1.7}$$

are respectively the Rossby number measuring the (inverse) rotational forcing and the Froude number measuring the (inverse) pressure forcing. More precisely they measure the ratio of the inertia force (U) relative to these two forces ([39]).

For the isentropic gas equations (1.3), we obtain a similar form

$$\partial_t\rho + \mathbf{u}\cdot\nabla\rho + (\frac{1}{\sigma} + \rho)\nabla\cdot\mathbf{u} = 0 \tag{1.8a}$$

$$\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\sigma^2}\nabla\hat{p} - \frac{1}{\tau}J\mathbf{u} = 0 \tag{1.8b}$$

where σ should be understood as the Mach number, measuring the ratio of fluid velocity to sound speed. Here the *rescaled* pressure variable $\hat{p}$ satisfies $1 + \sigma\hat{p} = (1 + \sigma\rho)^{\gamma-1}$.

The ideal gasdynamics equations will be discussed at the end of the next section.

1.2 Main Results and Method of Iterative Approximations.

To trace their long-time behavior, we approximate the RSW equations (1.6) with the following successive iterations,

$$\partial_t h_j + \mathbf{u}_{j-1}\cdot\nabla h_j + \left(\frac{1}{\sigma} + h_j\right)\nabla\cdot\mathbf{u}_{j-1} = 0, \qquad j = 2, 3, \ldots \tag{1.9a}$$

$$\partial_t\mathbf{u}_j + \mathbf{u}_j\cdot\nabla\mathbf{u}_j + \frac{1}{\sigma}\nabla h_j - \frac{1}{\tau}J\mathbf{u}_j = 0, \qquad j = 1, 2, \ldots, \tag{1.9b}$$

subject to initial conditions, $h_j(0,\cdot) = h_1(\cdot)$ and $\mathbf{u}_j(0,\cdot) = \mathbf{u}_0(\cdot)$. Observe that, given j, (1.9) are only weakly coupled through the dependence of $\mathbf{u}_j$ on h_j, so that we only need to specify the initial height h_1. For $\sigma \gg \tau$, the momentum equations (1.5b) are "approximately decoupled" from the mass equation (1.5a) since rotational forcing is substantially dominant over pressure forcing. Therefore a first approximation of constant height function will enforce this decoupling, serving as the starting point of the above iterative scheme,

$$h_1 \equiv \text{constant}.$$

This, in turn, leads to the first approximate velocity field, $\mathbf{u}_1$, satisfying the pressureless equations,

$$\partial_t \mathbf{u}_1 + \mathbf{u}_1 \cdot \nabla \mathbf{u}_1 - \frac{1}{\tau} J \mathbf{u}_1 = 0, \quad \mathbf{u}_1(0, \cdot) = \mathbf{u}_0(\cdot). \tag{1.10}$$

Liu and Tadmor [33] have shown that there is a "large set" of so-called sub-critical initial configurations $\mathbf{u}_0$, for which the pressureless equations (1.10) admit global smooth solutions. Moreover, the pressureless velocity $\mathbf{u}_1(t, \cdot)$ is in fact $2\pi\tau$-periodic in time. The regularity of $\mathbf{u}_1$ is discussed in Section 3.1.

Having the pressureless solution, $(h_1 \equiv \text{constant}, \mathbf{u}_1)$ as a first approximation for the RSW solution $(h, \mathbf{u})$, in Section 3.2 we introduce an improved approximation of the RSW equations, $(h_2, \mathbf{u}_2)$, which solves an "adapted" version of the second iteration ($j = 2$) of (1.9). This improved approximation satisfies a specific linearization of the RSW equations around the pressureless velocity $\mathbf{u}_1$, with only a one-way coupling between the momentum and the mass equations. Building on the regularity and periodicity of the pressureless velocity $\mathbf{u}_1$, we show that the solution of this linearized system subject to sub-critical initial data $(h_0, \mathbf{u}_0)$, is globally smooth; in fact, both $h_2(t, \cdot)$ and $\mathbf{u}_2(t, \cdot)$ retain $2\pi\tau$-periodicity in time.

Next, we turn to estimate the deviation between the solution of the linearized RSW system, $(h_2, \mathbf{u}_2)$, and the solution of the full RSW system, $(h, \mathbf{u})$. To this end, we introduce a new non-dimensional parameter

$$\delta := \frac{\tau}{\sigma^2} = \frac{gH}{fLU},$$

measuring the *relative* strength of rotation vs. the pressure forcing and we assume that rotation is the dominant forcing in the sense that $\delta \ll 1$. Using the standard energy method we show in theorem 3.3 that, starting with H^m initial data, the RSW solution $\big(h(t,\cdot), \mathbf{u}(t,\cdot)\big)$ remains sufficiently close to $\big(h_2(t,\cdot), \mathbf{u}_2(t,\cdot)\big)$ in the sense that,

$$\|h(t,\cdot) - h_2(t,\cdot)\|_{H^{m-3}} + \|\mathbf{u}(t,\cdot) - \mathbf{u}_2(t,\cdot)\|_{H^{m-3}} \lesssim \frac{e^{C_0 t}\delta}{(1 - e^{C_0 t}\delta)^2}, \tag{1.11}$$

where constant $C_0 = \hat{C}_0(m, |\nabla \mathbf{u}_0|_\infty, |h_0|_\infty) \cdot \|\mathbf{u}_0, h_0\|_m$. In particular, we conclude that for a large set of sub-critical initial data, the RSW equations (1.6) admit smooth, "approximately periodic" solutions for large time, $t \leq t_\delta := 1 + \ln(\delta^{-1})$, in the rotationally dominant regime $\delta \ll 1$. Here, we introduce,

Definition 1.1 *A function $f(t,\cdot)$ is δ-approximately periodic regarding certain norm $\|\cdot\|$ over time period $[t_1, t_2]$ if there exist a periodic approximation $\tilde{f}(t,\cdot)$ such that*

$$\max_{t \in [t_1, t_2]} \|f(t,\cdot) - \tilde{f}(t,\cdot)\| \leq C(t)\delta,$$

for some bounded function $C(t)$.

We comment that this notation emphasizes on the existence of a periodic approximation $(h_2, \mathbf{u}_2)$ nearby the actual flow $(h, \mathbf{u})$ with the error up-bounded by $O(\delta) \ll 1$ for sufficiently long time. Therefore, strong rotation stabilizes the flow by imposing approximate periodicity to the flow, which in turn postpones finite time breakdown of classical solutions to long time.

We generalize our result to Euler systems describing e.g. the isentropic gas dynamics and ideal gas dynamics in Chapter 4. We regard these two systems as successive generalizations of the RSW system under the framework,

$$\partial_t \rho + \nabla\cdot(\rho \mathbf{u}) = 0 \tag{1.12a}$$

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla \mathbf{u} + \rho^{-1}\nabla \hat{p}(\rho, S) = fJ\mathbf{u} \tag{1.12b}$$

$$\partial_t S + \mathbf{u}\cdot\nabla S = 0, \tag{1.12c}$$

For the ideal gas dynamics, the pressure law is given by $\hat{p} := A\rho^\gamma e^S$ where A, γ are two gas-specific physical constants. The isentropic gas equations correspond to $S \equiv$ constant and thus, the entropy equation (1.12c) becomes redundant. One more specification of setting $A = g, \gamma = 2$ yields the RSW equation (with height h playing the same role as density ρ). Due to such analogue, these systems can all be symmetrized by introducing a "normalized" pressure function,

$$p := C\hat{p}^{\frac{\gamma-1}{2\gamma}}(\rho, S),$$

and thus, replacing the density equation by a pressure equation,

$$\partial_t p + \mathbf{u}\cdot\nabla p + Cp\nabla\cdot\mathbf{u} = 0. \tag{1.12d}$$

We then nondemsionalize the above system (1.12b), (1.12c) and (1.12d) into

$$\begin{aligned} \partial_t p + \mathbf{u}\cdot\nabla p + \frac{\gamma-1}{2}\left(\frac{1}{\sigma} + p\right)\nabla\cdot\mathbf{u} &= 0 \\ \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{\gamma-1}{2}\left(\frac{1}{\sigma} + p\right)e^{\sigma S}\nabla p &= \frac{1}{\tau}J\mathbf{u} \\ \partial_t S + \mathbf{u}\cdot\nabla S &= 0 \end{aligned}$$

Our methodology, independent of the pressure law, still applies to the above system. In particular, our first approximation, the pressureless system, remains the same as in (1.10) since it ignores any effect of pressure. We then obtain the second approximation $(p_2, \mathbf{u}_2, S_2)$ (or $(p_2, \mathbf{u}_2)$ in the isentropic case) from a specific linearization around the pressureless velocity $\mathbf{u}_1$. The similar mechanism governing h, p and S as passive scalars transported by $\mathbf{u}$ allows us to apply the same regularity and periodicity argument for $(p_2, \mathbf{u}_2, S_2)$ as in the RSW case. The energy estimate, however, needs careful modification for the ideal gas equations due to additional nonlinearity in such terms as $(\frac{1}{\sigma} + p)e^{\sigma S}\nabla p$. Finally, we conclude in Theorem 4.1 and Theorem 4.2 that, in the rotationally dominant regime $\delta \ll 1$, the globally smooth, $2\pi\tau$-periodic approximation $(p_2, \mathbf{u}_2, S_2)$ stays "close" to the exact solution for long time in the sense that, starting with H^m data, the following estimate holds true for time $t \lesssim 1 + \ln \delta^{-1}$,

$$\|p(t,\cdot) - p_2(t,\cdot)\|_{m-3} + \|\mathbf{u}(t,\cdot) - \mathbf{u}_2(t,\cdot)\|_{m-3} + \|S(t,\cdot) - S_2(t,\cdot)\|_{m-3} < \frac{e^{C_0 t}\delta}{1 - e^{C_0 t}\delta}.$$

1.3 Mathematical and Geophysical Significance

Our results confirm the stabilization effect of rotation in the nonlinear setting, when it interacts with the slow components of the system, which otherwise tends to destabilize the dynamics. The study of such interaction is essential to the understanding of rotating dynamics, primarily to geophysical flows. We can mention only few works from the vast literature available on this topic, and we refer the reader to the recent book of Chemin et. al., [6] and the references therein, for a state-of-the art of the mathematical theory for rapidly rotating flows. Embid and Majda [13, 14] studied the singular limit of RSW equations under two regimes $\tau^{-1} \sim \sigma^{-1} \uparrow \infty$ and $\tau^{-1} \sim \mathcal{O}(1)$, $\sigma^{-1} \uparrow \infty$. Extensions to more general skew-symmetric perturbations can be found in the work of Gallagher, e.g. [16]. The series of works of Babin, Mahalov and Nicolaenko, consult [2, 3, 4] and references therein, establish long term stability effects of the rapidly rotating 3D Euler, Navier-Stokes and primitive equations. On the geophysical side, Zeitlin, Reznik and Ben Jelloul give categorization and asymptotic analysis of various scaling regimes in the context of Rossby adjustment ([47, 48]). There are also asymptotic results of RSW on bounded domains (e.g. [21]).

Along another direction, the dispersive nature of fast acoustic waves in the *whole space* also stabilizes compressible Euler flows. Strichartz proved in [43] the decay properties of linear wave equations. These decay properties also yields smoothing effects, which have been the beginning of a long series of works, in e.g. [7, 24, 1, 19, 42]. Consult [6] for particular discussion in the geophysical equations. In our setting of periodic domains, however, the dispersive mechanism is not available and nevertheless, as shown in Section 3.4 for the whole plane $\mathbb{R}^2$, strong rotation "entraps" the flow so locally that even fast pressure waves are kept from escaping to farfield. We also note by passing the local smoothing effects of high order dispersive terms (e.g. [8]) in such cases as the KdV equations and Schrödinger's equations but generally not available for the wave equations. Another stabilizing mechanism absent from our model is the viscosity. Consult the work of e.g. Lions and Masmoudi in [31, 35] and references therein.

We comment here that the approach employed in the above literature relies on identifying the limiting system as $\tau \downarrow 0$, which filters out fast scales. The full system is then approximated to a first order, by this slowly evolving limiting system. A rigorous mathematical foundation along these lines was developed by Schochet [41], which can be traced back to the earlier works of Klainerman and Majda [26, 27] (see also [28], [44]). The key point was the separation of (linear) fast oscillations from the slow scales. The novelty of our approach, inspired by the critical threshold phenomena [33], is to adapt the rapidly oscillating and fully *nonlinear* pressureless system as a first approximation and then consider the full system as a perturbation of this fast scale. This enables us to preserve both slow and fast dynamics, and especially, the rotation-induced time periodicity.

We also note that the smallness of the new parameter δ in our result corresponds the *relative* dominant of rotation, that is, $\frac{1}{\tau} \gg \frac{1}{\sigma^2}$. Previous literature, on the other hand, deals with *absolute* singularity in the form of $\frac{1}{\tau} \sim \frac{1}{\sigma} \gg O(1)$ or $\frac{1}{\sigma} \gg O(1) \sim \frac{1}{\tau}$. The missing regime between these cases is $\frac{1}{\sigma^2} \lesssim \frac{1}{\tau} \lesssim 1$, which we have not fully explored. Partial results shown in Chapter 5, however, confirms that approximate periodicity in the small δ case becomes invalid if δ is large. In such sense, our result in the regime of $\delta \ll 1$ is optimal.

Geophysically our result is supported by observations of the so called "near inertial oscillation" (NIO) in oceanography (e.g. [46]). These NIOs are mostly seen after a storm blows over the oceans. They exhibit almost periodic dynamics with a period consistent with the Coriolis force and stay stable for about 20 days which is a long time scale relative to many oceanic processes such as the storm itself. This observation agrees with our theoretical result regarding the stability and periodicity of RSW solutions. Written in terms of physical parameters, our result requires $\delta = \dfrac{gH}{fLU} \ll 1$ which is consistent with the fact that NIOs are triggered when storms pass by (large U's) and only a thin layer of the oceans is reactive (small aspect ratio H/L). An even more interesting argument is that cyclonic storms on the Northern Hemisphere rotates counterclockwise, generating negative vorticity (i.e. $\partial_y u^{(1)} - \partial_x u^{(2)} < 0$) which is a preferred scenario in favor of the critical threshold condition (3.4). We also note by passing that the rotationally dominant regime differs from the geostrophically balanced regime where Rossby adjustment (consult e.g. the textbook of E. Kalnay [22] and references therein) leads to fast dispersion that would quickly destroy the abovementioned circular pattern of motions.

Outside the Earth, the giant and fast rotating Jovian planets provide parameter settings that suits our result even better. For instance, the Jupiter rotates two times fast as the earth ([17]) and the scale of motions is also much larger – the Great Red Spot alone can contain 2 to 3 Earth's diameters. Therefore, hypothetically, the large scale motion on these planets should be more stable and less chaotic (not excluding turbulence over smaller scales). In fact, consistent flow patterns have been long observed in the Jupiter atmosphere, including the famous long lasting Great Red Spot.

Chapter 2

Preliminary: Energy Method

In this chapter, we will give some review of the standard energy methods for symmetric (more generally, symmetrizable) hyperbolic PDEs in the form of

$$A_0(\mathbf{v})\partial_t \mathbf{v} + \sum_{j=1}^{d} A_j(\mathbf{v})\partial_{x_j}\mathbf{v} = 0. \tag{2.1}$$

Here and below, we use $\mathbf{v} = \mathbf{v}(t, x_1, ..., x_d) \in R^N$ to denote the unknown variables and use $N \times N$ matrices $A_0, A_1, ..., A_d$ to denote the coefficients that depend on the unknowns $\mathbf{v}$. We are concerned with symmetric A_j's, in particular symmetric positive definite A_0.

The energy method has been extensively used to study PDE solutions – consult e.g. the textbooks [37], [15]. It was originally (and is still largely) based on the establishment of the Sobolev inequality and its generalizations, e.g. the Gagliardo-Nirenberg inequality. The basic methodology is to obtain estimates on functional norms of quantities that are linked by the PDE. In the center of such analysis lies the interplay among the structure of the PDE, the choice of functional norms and various embedding properties of these norms.

In this writing, we will use the H^m Sobolev norms of periodic functions defined on torus domain. The L^2 nature of such argument is the very reason for its name, "energy" method. In a more specific setting of hyperbolic PDE system (2.1), our goal is to upper-bound the energy growth rate $\frac{d}{dt}\|\mathbf{v}(t,\cdot)\|_m$ using the PDE relation so that the H^m regularity of $\mathbf{v}$ readily follows from solving a Grownwell inequality. To this end, the basic Sobolev type inequalities regarding H^m are introduced in section 1. Then in section 2, we will specifically explore the symmetric structure of PDE system (2.1) and together with the embedding theorems, give a general framework of energy arguments.

2.1 The H^m Norm and Sobolev Type Inequalities

Here and below, let Ω be $\mathbb{R}^2$, $\mathbb{T}^2$ or a bounded domain in $\mathbb{R}^2$.

Definition 2.1 *The (spatial) Sobolev norm* $\|\cdot\|_m$ *of function* $\mathbf{v} : \Omega \mapsto R^N$ *is*

$$\|\mathbf{v}\|_m := \sqrt{\sum_{|k|=0}^{m} \langle \mathbf{v}, \mathbf{v} \rangle_k},$$

where the inner product $\langle \cdot, \cdot \rangle_k$ *is defined as usual,*

$$\langle \mathbf{v}, \mathbf{w} \rangle_k := \int_\Omega D^k \mathbf{v}(\mathbf{x}) \cdot D^k \mathbf{w}(\mathbf{x}) \, d\mathbf{x}.$$

We use multi-index $k := (k_1, ..., k_d)$ *with* $|k| := k_1 + ... + k_d$ *for the spatial derivative so that* $D^k \mathbf{v} = \partial_{x_1}^{k_1} ... \partial_{x_d}^{k_d} \mathbf{v}$.

We use the same notation for time-dependent $\mathbf{v}(t, \cdot)$ and sometimes use a compact notation $\|\mathbf{v}\|_m := \|\mathbf{v}(t, \cdot)\|_m$.

The basic Sobolev inequality ([37], [15]) states that, for $\mathbf{v}(\cdot)$ defined on Ω with periodic or vanishing boundary condition,

$$\|\mathbf{v}\|_{L^{p^*}} \lesssim \|\nabla \mathbf{v}\|_{L^p} \text{ for } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{d} \geq 0. \tag{2.2}$$

As a corollary, the Sobolev embedding theorem claims that,

$$|\mathbf{v}|_\infty \lesssim \|\mathbf{v}\|_m \text{ for } m > \frac{d}{2} \tag{2.3}$$

which can also be proved using e.g. the Fourier representation of $\mathbf{v}$. Combining (2.2) with the popular Holder inequality

$$\|\mathbf{v}\mathbf{w}\|_{L^1} \leq \|\mathbf{v}\|_{L^p}\|\mathbf{w}\|_{L^{\frac{p}{p-1}}},$$

one can prove the Gagliardo-Nirenberg inequality,

$$\|\mathbf{v}\|_{L^r} \lesssim \|\nabla\mathbf{v}\|_{L^p}^{\theta}\|\mathbf{v}\|_{L^q}^{1-\theta},$$

for suitable r, p, q, θ. We will, however, solely use a corollary regarding products of two functions (which is the main form of nonlinearity in our study),

Proposition 2.1 *Consider two functions* $\mathbf{v}, \mathbf{w} \in H^m(\Omega)$ *with periodic or vanishing boundary condition. Then for any* $|k| \leq m$*,*

$$\begin{aligned} \|D^k(\mathbf{v}\mathbf{w})\|_0 &\lesssim \|\mathbf{v}\|_m|\mathbf{w}|_\infty + |\mathbf{v}|_\infty\|\mathbf{w}\|_m, \\ \|D^k(\mathbf{v}\mathbf{w}) - \mathbf{v}D^k\mathbf{w}\|_0 &\lesssim \|\nabla\mathbf{v}\|_{m-1}|\mathbf{w}|_\infty + |\nabla\mathbf{v}|_\infty\|\mathbf{w}\|_{m-1}. \end{aligned}$$

Combining the first inequality with the Sobolev embedding theorem in (2.3), we conclude that for any $m > \frac{d}{2}$, $H^m(\mathbb{T}^2)$ is an algebra, in particular, it is closed under multiplication, $\|\mathbf{v}\mathbf{w}\|_m \lesssim \|\mathbf{v}\|_m\|\mathbf{w}\|_m$. The second inequality gives estimates on commutator terms in the form of $[D^k, \mathbf{v}]\mathbf{w} := D^k(\mathbf{v}\mathbf{w}) - \mathbf{v}D^k\mathbf{w}$.

2.2 Energy Method for Symmetric Hyperbolic PDEs: A Framework

We now turn to the PDE system (2.1) subject to initial data $\mathbf{v}_0$ and periodic boundary condition, that is, $\Omega = \mathbb{T}^2$. Our discussion still remains valid for unbounded or bounded domain with vanishing boundary condition. Let's start with taking inner product $\langle\cdot,\cdot\rangle_k$ of $\mathbf{v}$ with (2.1) and sum it over $|k| \leq m$, giving rise to *nonlinear* terms in the form

$$\langle A_0(\mathbf{v})\partial_t \mathbf{v}, \mathbf{v}\rangle_k \text{ and } \langle A_j(\mathbf{v})\partial_{x_j}\mathbf{v}, \mathbf{v}\rangle_k,$$

which introduces higher order derivatives such as $D^k\nabla\mathbf{v}$. This section is devoted to obtain nonlinear estimates on these high order derivatives upon which a local-in-time existence result is based.

§. Nonlinear estimate on spatial derivatives. We rewrite a typical term $\langle A(\mathbf{v})\partial_x \mathbf{v}, \mathbf{v}\rangle_k$ as

$$\begin{aligned}
\langle D^k(A\partial_x\mathbf{v}), D^k\mathbf{v}\rangle &= \langle AD^k\partial_x\mathbf{v}, D^k\mathbf{v}\rangle + \langle [D^k, A]\partial_x v, D^k\mathbf{v}\rangle \\
&= \langle D^k\partial_x\mathbf{v}, AD^k\mathbf{v}\rangle + I \qquad\qquad \text{(by the symmetry of } A) \\
&= -\langle D^k\mathbf{v}, \partial_x(AD^k\mathbf{v})\rangle + I \qquad \text{(by the skew symmetry of } \partial_x) \\
&= -\langle D^k\mathbf{v}, AD^k\partial_x\mathbf{v}\rangle - \langle D^k\mathbf{v}, [\partial_x, A]D^k\mathbf{v}\rangle + I \qquad ....(*) \\
&= -\langle D^k\mathbf{v}, D^k(A\partial_x\mathbf{v})\rangle - \langle D^k\mathbf{v}, [A, D^k]\partial_x\rangle + II + I
\end{aligned} \tag{2.4}$$

$$\Rightarrow \quad \begin{aligned}
\langle A\partial_x\mathbf{v}, \mathbf{v}\rangle_k &= \tfrac{1}{2}(I + II + III) \\
&\lesssim (|\nabla_x A|_\infty \|\nabla_x \mathbf{v}\|_{m-1} + |\nabla_x\mathbf{v}|_\infty\|\nabla_x A\|_{m-1})\|\mathbf{v}\|_m,
\end{aligned} \tag{2.5}$$

where we used Proposition 2.1 to obtain sharp estimates on the commutator terms I, II, III so that no derivatives higher than m enter the upper bound. Combining this estimate with the Sobolev embedding theorem 2.3, we arrive at

Proposition 2.2 *Consider* $\mathbf{v} \in H^m(\mathbb{T}^2)$ *with* $m > \frac{d}{2} + 1$. *Assume* $A \in H^m(\mathbb{T}^2)$ *is* *a matrix-valued function that not necessarily depends on* $\mathbf{v}$*. Then for any* $|k| \leq m$*,*

$$|\langle A\partial_{x_j}\mathbf{v}, \mathbf{v}\rangle_k| \lesssim \|A\|_m \|\mathbf{v}\|_m^2.$$

This provides a way of *closure* in the sense that the $m+1$-th order derivatives of $\mathbf{v}$ are up-bounded in terms of its H^m norms.

§. Nonlinear estimates on time derivatives. Now, we turn to a more difficult term $\langle A_0(\mathbf{v})\partial_t \mathbf{v}, \mathbf{v}\rangle_k$. We replace in (2.4) every ∂_x with ∂_t and proceed as before, that is, use the symmetry of A_0 and use commutator terms to control the highest derivatives. However, we can not use the skew-symmetry of ∂_x. Instead, there will be an extra term $\partial_t \langle A_0 D^k \mathbf{v}, D^k \mathbf{v}\rangle$ in the line (*). In short, we get an estimate similar to (2.5) with this extra term and with all ∇_x replaced with $\nabla_{t,x}$

$$\frac{d}{dt}\langle A_0 D^k \mathbf{v}, D^k \mathbf{v}\rangle - 2\langle A_0 \partial_t \mathbf{v}, \mathbf{v}\rangle_k \lesssim (|\nabla_{t,x} A_0|_\infty \|\nabla_{t,x}\mathbf{v}\|_{m-1} + |\nabla_{t,x}\mathbf{v}|_\infty \|\nabla_{t,x} A_0\|_{m-1})\|\mathbf{v}\|_m.$$

Since $\partial_t \mathbf{v}$ is given in the PDE system (2.1) in terms of A_j, $\mathbf{v}$ and $\partial_x \mathbf{v}$, its H^{m-1} (with $m > \frac{d}{2} + 1$) norm will be up-bounded by products of H^{m-1} norms of these terms – notice H^{m-1} is an algebra. Thus, we arrive that

Proposition 2.3 *Consider a solution to the system (2.1)* $\mathbf{v} \in H^m(\mathbb{T}^2)$ *with* $m < \frac{d}{2} + 1$*. Assume* $A_0, A_1, ..., A_d$ *are matrix-valued functions such that* $\nabla_{t,x} A_j \in H^{m-1}(\mathbb{T}^2)$*. Then for any* $|k| \leq m$*,*

$$\frac{d}{dt}\langle A_0 D^k \mathbf{v}, D^k \mathbf{v}\rangle \lesssim \langle A_0 \partial_t \mathbf{v}, \mathbf{v}\rangle_k + \left(\sum_{j=1,d} \|\nabla_x A_j\|_{m-1}\right) \|\nabla_{t,x} A_0\|_{m-1} \|\mathbf{v}\|_m^2.$$

§. Local existence of classical solutions. The above estimates imply classical results of finite time existence for symmetric hyperbolic systems. For simplicity, we consider (2.1) with constant A_0 so that $\nabla_{t,x} A_0$ vanishes. We also assume $A_j(\mathbf{v})$ depends linearly on $\mathbf{v}$ so that $\nabla A_j \propto \nabla \mathbf{v}$. Taking inner products $\sum_{|k|=0}^{m} \langle \cdot, \cdot \rangle_k$ of (2.1) with $\mathbf{v}$ and applying Property 2.2, we arrive at,

Proposition 2.4 *The symmetric hyperbolic PDE system (2.1) with constant A_0 and linear $A_j(\mathbf{v})$ satisfies an energy inequality for $\|v\|_m = \|\mathbf{v}(t, \cdot)\|_m$ (with $m > \frac{d}{2} + 1$),*

$$\frac{d}{dt} \|\mathbf{v}\|_m \lesssim \|\mathbf{v}\|_m^2, \tag{2.6}$$

and thus, starting with H^m initial data $\mathbf{v}_0 := \mathbf{v}(0, \cdot) \in H^m(\mathbb{T}^2)$, the solution $\mathbf{v}$ stay smooth – that is, in $H^m(\mathbb{T}^2)$ – for a finite time $O\left(\frac{1}{\|\mathbf{v}_0\|_m}\right)$.

The more complicated case of variable A_0 and nonlinear A_j's follows the same line. One should add regularity assumptions on $\nabla_{\mathbf{v}} A_j(\mathbf{v})$ $(j = 0, 1, ..., d)$ for estimates on $\nabla_{t,x} A_j(\mathbf{v})$. The additional nonlinearity in turn adds more growth on the energy inequality than just quadratic in (2.6),

$$\frac{d}{dt} \|\mathbf{v}\|_m \leq P(\|v\|_m),$$

where $P(\cdot)$ is a positive and continuous function, typically a polynomial function. The finite time existence result then readily follows.

Chapter 3

Rotationally Dominant Shallow Water Equations

– Long Time Stability and Approximate Periodicity

We will now give the detail proof of the main result (1.11) on rotationally dominant shallow water equations

$$\partial_t h + \mathbf{u}\cdot\nabla h + (\frac{1}{\sigma} + h)\nabla\cdot\mathbf{u} = 0 \tag{3.1a}$$

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\sigma}\nabla h - \frac{1}{\tau}J\mathbf{u} = 0 \tag{3.1b}$$

in the rotationally dominant regime $\delta = \frac{\tau}{\sigma^2} \ll 1$. Recall that in Section 1.2, a series of successive iterations (1.9) are suggested to serve as approximation of the RSW system. For the reader's convenience, they are duplicated here,

$$\partial_t h_j + \mathbf{u}_{j-1}\cdot\nabla h_j + (\frac{1}{\sigma} + h_j)\nabla\cdot\mathbf{u}_{j-1} = 0, \quad j = 2, 3, ... \tag{3.2a}$$

$$\partial_t \mathbf{u}_j + \mathbf{u}_j\cdot\nabla\mathbf{u}_j + \frac{1}{\sigma}\nabla h_j - \frac{1}{\tau}\mathbf{u}_j^\perp = 0, \quad j = 1, 2, ... \tag{3.2b}$$

Therefore we will start with the first approximation ($j = 1$) – the pressureless system – in Section 3.1, proving the Critical Threshold theorem of Liu and Tadmor in [33]. A new argument is used, however, to derive higher order regularity results on $\mathbf{u}_1$. Then the second approximation (that is, $j = 2$) is carried out in Section 3.2 with careful adaptation of the original form in (3.2) so that the resulting approximate RSW solution $(h_2, \mathbf{u}_2)$ is $2\pi\tau$-periodic and globally smooth while still staying close to the exact solution. Section 3.3 completes the proof by employing the energy method to estimate the error $(h, \mathbf{u}) - (h_2, \mathbf{u}_2)$.

Remark 3.1 *We point out that the above approximate equation (3.2b) for $j = 2$ will be not be exactly solved. In other words, the actual approximation $\mathbf{u}_2$, committing abuse of notation, solves an adapted system (see (3.18b) below) that is "close" to (3.2b).*

Notations. Here and below, $\|\cdot\|_m$ denotes the usual H^m-Sobolev norm over the 2D torus $\mathbb{T}^2$. And $|\cdot|_\infty$ denotes the L^∞ norm. We abbreviate $a \lesssim_m b$ for $a \leq cb$ whenever the constant c only depends on the dimension m. We let $\hat{C}_0$ denote constants that depend (nonlinearly) on m as well as the initial data $|h_0|_\infty$ and $|\nabla \mathbf{u}_0|_\infty$. Then a universal constant $C_0 := \hat{C}_0 \cdot \|(h_0, \mathbf{u}_0)\|_m$ will be used for estimates on Sobolev regularity.

3.1 First Approximation: Pressureless System

The first iteration in (3.2) solely consists of a momentum equation while implicitly setting $h_1 \equiv$ constant. The approximate equation (3.2b) for $j = 1$ therefore contains no pressure term and is decoupled from the mass equation,

$$\partial_t \mathbf{u}_1 + \mathbf{u}_1 \cdot \nabla \mathbf{u}_1 - \frac{1}{\tau} J \mathbf{u}_1 = 0. \tag{3.3}$$

Liu and Tadmor studied the global regularity of (3.3) in [33] and proved

Theorem 3.1 *The pressureless system (3.3) admits C^1 solution for all time if and only if the initial data satisfy the critical threshold condition,*

$$\tau \omega_0(x) + \frac{\tau^2}{2} \eta_0^2(x) < 1, \quad \textit{for all } x \in \mathbb{T}^2. \tag{3.4}$$

Here, $\omega_0 = \partial_y u_0^{(1)} - \partial_x u_0^{(2)}$ is the initial vorticity and $\eta_0 := \lambda_1 - \lambda_2$ is the (possibly complex-valued) spectral gap associated with the eigenvalues of gradient matrix $\nabla \mathbf{u}_0$. Moreover, these globally smooth solutions $\mathbf{u}_1(t,\cdot)$ are $2\pi\tau$-periodic in time.

The authors gave two different proofs of (3.4) in [33]. One was based on the spectral dynamics of $\lambda_j(\nabla \mathbf{u}_1)$. Another was based on the flow map associated with (3.3). Here we will note yet another version of the latter, from which not only the C^1 regularity but also the Sobolev H^m regularity will readily follow.

Taking gradient on the pressureless equation (3.3), we obtain a Riccatti-type equation satisfied by the gradient matrix $M := \nabla \mathbf{u}_1$,

$$M' + M^2 = \frac{1}{\tau} J M. \tag{3.5}$$

Here $\{\cdot\}' := \partial_t + \mathbf{u}_1 \cdot \nabla$ denotes differentiation along the particle trajectories

$$\Gamma_0 := \{(t, \mathbf{x}) \,|\, \dot{\mathbf{x}}(t) = \mathbf{u}_1(t, \mathbf{x}(t)),\, \mathbf{x}(t_0) = \mathbf{x}_0\}. \tag{3.6}$$

Starting with $M_0 = M(t_0, \mathbf{x}_0)$, the solution of the M equation (3.5) along the corresponding trajectory Γ_0 is given by

$$M = e^{tJ/\tau} \left(I + \frac{1}{\tau} J \left(I - e^{tJ/\tau} \right) M_0 \right)^{-1} M_0, \tag{3.7}$$

as long as the matrix $I + \frac{1}{\tau} J \left(I - e^{tJ/\tau} \right) M_0$ is invertible. A straightforward calculation shows the determinant of this matrix is given by $1 - \tau\omega_0 - \frac{\tau^2}{2}\eta_0^2$ and (3.4) follows.

The periodicity of the pressureless velocity $\mathbf{u}_1$ also relies on the Lagrangian dynamics of (3.3): $\mathbf{u}_1' = \frac{J}{\tau}\mathbf{u}_1$ along Γ_0 as defined in (3.6), yielding $\mathbf{u}_1(t, \mathbf{x}(t)) =$

$e^{tJ/\tau}\mathbf{u}_0(\mathbf{x}_0)$ and therefore $\mathbf{u}_1$ is $2\pi\tau$-periodic along Γ_0, that is $\mathbf{u}_1(t+2\pi\tau, \mathbf{x}(t+2\pi\tau)) = \mathbf{u}_1(t, \mathbf{x}(t))$. On the other hand, we integrate $\mathbf{x}' = \mathbf{u}_1$ to find,

$$\mathbf{x}(2\pi\tau) = \mathbf{x}(0), \tag{3.8}$$

namely, the trajectories are also $2\pi\tau$-periodic in the sense that they come back to their initial position at time $t = 2\pi\tau$. Combining these facts, we conclude that $\mathbf{u}_1(t, \cdot)$ is $2\pi\tau$-periodic in time.

Since the C^1 regularity of $\mathbf{u}_1$ is equivalent to the boundedness of its gradient matrix M, it is worthwhile to expand the expression in (3.7) and compute its L^∞ norm. Based on the Cayley-Hamilton theorem, we have,

$$\max_{t,x} |\nabla \mathbf{u}_1| = \max_{t,x} |M| = \max_{t,x} \left| \frac{\text{polynomial}(\tau, e^{tJ/\tau}, \nabla \mathbf{u}_0)}{(1 - \tau\omega_0 - \frac{\tau^2}{2}\eta_0^2)_+} \right|. \tag{3.9}$$

This leads to the following corollary on the criticality of the Rossby number τ.

Corollary 3.1 *Consider the pressureless system (3.3) subject to initial data* $\mathbf{u}_1(0, \cdot) = \mathbf{u}_0(\cdot)$. *Then there exists a critical value* $\tau_c = \tau_c(\nabla \mathbf{u}_0)$ *such that*

$$|\nabla \mathbf{u}_1|_\infty \leq \hat{C}_0 \textit{ for } \tau \in (0, \tau_c), \tag{3.10}$$

where $\hat{C}_0$ *only depends on the initial gradient matrix* $\nabla \mathbf{u}_0$.

Proof. A simply continuity argument shows there exists a value $\tau_c := \tau_c(\omega_0, \eta_0^2)$ such that

$$1 - \tau\omega_0 - \frac{\tau^2}{2}\eta_0^2 > \frac{1}{2} \text{ for } \tau \in (0, \tau_c).$$

This estimate, combined with (3.9), immediately proves the corollary with constant $\hat{C}_0 := 2 \cdot \text{Polynomial}(\tau_c, \nabla \mathbf{u}_0)$. $\square$

Thus we find another weaker yet more intuitive characterization of the critical threshold condition (3.4), that is, stronger rotation tends to stabilize the flow more since $\frac{1}{\tau}$ indicates the magnitude of the rotational forcing. Observe, nevertheless, that the critical threshold τ_c *need not* be small. In fact, regarding $1-\tau\omega_0-\frac{\tau^2}{2}\eta_0$ as a quadrature of τ, one finds, using the same idea of the above proof, that

$$\tau_c = +\infty \text{ iff } \eta_0^2 < 0,\, \omega_0 < \sqrt{-2\eta_0^2},$$

which essentially corresponds to rotational initial data. We shall always limit ourselves, however, to a *finite* value of τ_c.

The C^1 regularity of $\mathbf{u}_1$, obtained *without using energy method*, provides a closure to its H^m regularity estimates under the framework of the standard energy method described in Chapter 2. Noticing the rotational term $\frac{1}{\tau}\mathbf{u}_1^{\perp}$ on the RHS of the pressureless system (3.3) does not contribute to the energy growth of $\mathbf{u}_1$, we apply estimate 2.5 to (3.3),

$$\frac{d}{dt}\|\mathbf{u}_1(t,\cdot)\|_m \le C_m|\nabla\mathbf{u}_1(t,\cdot)|_\infty\|\mathbf{u}_1(t,\cdot)\|_m.$$

Since $\mathbf{u}_1$ is $2\pi\tau$-periodic in time, it suffices to consider its energy growth over $t \in [0, 2\pi\tau)$. Solving the above Grownwell inequality for $t < 2\pi\tau$ and combining it with Corollary 3.1, we arrive at the next corollary,

Corollary 3.2 *Consider the pressureless system (3.3). Let $m \ge 0$ be a fixed integer. There exists a critical value τ_c such that we have, uniformly in time,*

$$\|\mathbf{u}_1(t,\cdot)\|_m \le e^{2\pi\tau\hat{C}_0}\|\mathbf{u}_0\|_m \le e^{2\pi\tau_c\hat{C}_0}\|\mathbf{u}_0\|_m = C_0 \quad \textit{for all } \tau \in (0,\tau_c). \tag{3.11}$$

3.2 Second Approximation: Linearized System

Once we established the global properties of the pressureless velocity $\mathbf{u}_1$, it is then used as the starting point for the second iteration (3.2) with $j = 2$.

§. The Approximate Height h_1. The approximate height equation (3.2a) with $j = 2$ is given by

$$\partial_t h_2 + \mathbf{u}_1 \cdot \nabla h_2 + (\frac{1}{\sigma} + h_2) \nabla \cdot \mathbf{u}_1 = 0. \tag{3.12}$$

Its formality suggests us to study functions transported by $\mathbf{u}_1$. Thus we give the following lemma regarding the periodicity of such functions which is essentially determined by the periodicity of $\mathbf{u}_1$.

Lemma 3.1 *Let scalar function $w(t, \cdot)$ be governed by*

$$\partial_t w + \nabla \cdot (w \mathbf{u}_1) = 0, \tag{3.13}$$

where $\mathbf{u}_1(t, \cdot)$ is a globally smooth, $2\pi\tau$-periodic solution of (3.3). Then $w(t, \cdot)$ is $2\pi\tau$-periodic in time.

Proof. Let $\phi := \nabla \times \mathbf{u}_1 + \frac{1}{\tau}$ denote the so called relative vorticity. By (3.3), it satisfies the same equation as w, namely,

$$\partial_t \phi + \nabla \cdot (\phi \mathbf{u}_1) = 0.$$

Coupled with (3.13), it is easy to show that the ratio $\frac{w}{\phi}$ satisfies a transport equation

$$(\partial_t + \mathbf{u}_1 \cdot \nabla) \frac{w}{\phi} = 0,$$

which in turn implies that $\frac{w}{\phi}$ remains constant along the trajectories Γ_0 in (3.6). We invoke the same argument used for the periodicity of $\mathbf{u}_1$, in particular, the periodicity of Γ_0 in (3.8). Hence $\frac{w}{\phi}(t,\cdot)$ is also $2\pi\tau$-periodic . The conclusion follows from the periodicity of $\mathbf{u}_1(t,\cdot)$ and thus the periodicity of $\phi(t,\cdot)$. □

Equipped with this lemma, we conclude,

Theorem 3.2 *Consider the mass equation (3.12) on a 2D torus, $\mathbb{T}^2$, linearized around the pressureless velocity field $\mathbf{u}_1$ and subject to sub-critical initial data $(h_0, \mathbf{u}_0) \in H^m(\mathbb{T}^2)$ with $m > 5$. It admits a globally smooth solutions, $h_2(t,\cdot) \in H^{m-1}(\mathbb{T}^2)$ which is $2\pi\tau$-periodic in time, and the following upper bounds hold uniform in time,*

$$|h_2(t,\cdot)|_\infty \le \hat{C}_0\left(1+\frac{\tau}{\sigma}\right), \tag{3.14a}$$

$$\|h_2(t,\cdot)\|_{m-1} \le C_0\left(1+\frac{\tau}{\sigma}\right). \tag{3.14b}$$

Proof. Apply Lemma 3.1 with $w := \sigma^{-1}+h_2$ to (3.12) to conclude that h_2 is also $2\pi\tau$-periodic. We turn to the examine the regularity of h_2. First, its L^∞ bound (3.14a) is studied using the Maximum principle for (linear) Hyperbolic systems ([20]) which yields an inequality for $|h_2|_\infty = |h_2(t,\cdot)|_\infty$,

$$\frac{d}{dt}|h_2|_\infty \le |\nabla\cdot\mathbf{u}_1|_\infty(\sigma^{-1}+|h_2|_\infty).$$

Combined with the L^∞ estimate of $\nabla\mathbf{u}_1$ in (3.10), this Grownwell inequality implies

$$|h_2|_\infty \le e^{\hat{C}_0 t}|h_0|_\infty + \frac{1}{\sigma}\left(e^{\hat{C}_0 t}-1\right).$$

Just like before, due to the $2\pi\tau$-periodicity of h_2 and the subcritical condition $\tau \le \tau_c$, we replace the first t with τ_c and the second t with $2\pi\tau$ in the above estimate and therefore conclude (3.14a).

For the H^{m-1} estimate (3.14b), we use the energy method – in particular, estimate (2.5) – to obtain a similar inequality for $|h_2|_{m-1} = |h_2(t,\cdot)|_{m-1}$,

$$\frac{d}{dt}\|h_2\|_{m-1} \lesssim_m |\nabla \mathbf{u}_1|_\infty \|h_2\|_{m-1} + \left(\frac{1}{\sigma} + |h_2|_\infty\right) \|\mathbf{u}_1\|_m.$$

Applying the estimates on $\mathbf{u}_1$ in (3.10), (3.11) and the L^∞ estimate on h_2 in (3.14a), we find the above inequality shares a similar form as the previous one. Thus the estimate (3.14b) follows by the same periodicity and subcriticality argument as for (3.14a). $\square$

§. The Approximate Velocity Field $\mathbf{u}_2$. To continue with the second approximation, we turn to the approximate momentum equation (3.2b) with $j = 2$.

$$\partial_t \mathbf{u}_2 + \mathbf{u}_2 \cdot \nabla \mathbf{u}_2 + \frac{1}{\sigma} \nabla h_2 - \frac{1}{\tau} \mathbf{u}_2 = 0. \tag{3.15}$$

The following *splitting* approach will lead to a simplified linearization of (3.15) which is "close" to (3.15) and still maintains the nature of our methodology. The idea is to treat the nonlinear term and the pressure term in (3.15) separately, resulting in two systems for $\tilde{\mathbf{v}} \approx \mathbf{u}_2$ and $\hat{\mathbf{v}} \approx \mathbf{u}_2$,

$$\partial_t \tilde{\mathbf{v}} + \tilde{\mathbf{v}} \nabla \cdot \tilde{\mathbf{v}} - \frac{1}{\tau} J \tilde{\mathbf{v}} = 0, \tag{3.16a}$$

$$\partial_t \hat{\mathbf{v}} + \frac{1}{\sigma} \nabla h_2 - \frac{1}{\tau} J \hat{\mathbf{v}} = 0, \tag{3.16b}$$

subject to the same initial data $\tilde{\mathbf{v}}(0,\cdot) = \hat{\mathbf{v}}(0,\cdot) = \mathbf{u}_0(\cdot)$.

The first system (3.16a), ignoring the pressure term, is identified as the pressureless system (3.3) and therefore is solved as

$$\tilde{\mathbf{v}} = \mathbf{u}_1,$$

while the second system (3.16b), ignoring the nonlinear advection term, is solved using the Duhamel's principle,

$$\begin{aligned}\hat{\mathbf{v}}(t,\cdot) &= e^{tJ/\tau}\Big(\mathbf{u}_0(\cdot) - \textstyle\int_0^t \frac{e^{-sJ/\sigma}}{\sigma}\nabla h_2(s,\cdot)\,ds\Big)\\ &\approx e^{tJ/\tau}\Big(\mathbf{u}_0(\cdot) - \textstyle\int_0^t \frac{e^{-sJ/\sigma}}{\sigma}\nabla h_2(t,\cdot)\,ds\Big)\\ &= e^{tJ/\tau}\mathbf{u}_0(\cdot) + \tfrac{\tau}{\sigma}J(I - e^{tJ/\tau})\nabla h_2(t,\cdot).\end{aligned}$$

Here, we make an approximation by replacing $h_2(s,\cdot)$ with $h_2(t,\cdot)$ in the integrand, which introduces an error of order τ, taking into account the $2\pi\tau$ period of $h(t,\cdot)$.

Now synthesizing the two solutions listed above, we make a correction to $\hat{\mathbf{v}}$ by replacing $e^{tJ/\tau}\mathbf{u}_0$ with $\mathbf{u}_1$. This gives the very form of our approximate velocity field $\mathbf{u}_2$ (with tolerable abuse of notations)

$$\mathbf{u}_2 := \mathbf{u}_1 + \frac{\tau}{\sigma}J(I - e^{tJ/\tau})\nabla h_2(t,\cdot). \tag{3.18a}$$

A straightforward computation shows that this velocity field, $\mathbf{u}_2$, satisfies the following approximate momentum equation,

$$\partial_t \mathbf{u}_2 + \mathbf{u}_1\cdot\nabla\mathbf{u}_2 + \frac{1}{\sigma}\nabla h_2 - \frac{1}{\tau}\mathbf{u}_2^{\perp} = R \tag{3.18b}$$

$$\begin{aligned}\text{where}\quad R &:= \tfrac{\tau}{\sigma}J(I - e^{tJ/\tau})(\partial_t + \mathbf{u}_1\cdot\nabla)\nabla h_2(t,\cdot)\\ \text{(by (3.12))}\qquad &= -\tfrac{\tau}{\sigma}J(I - e^{tJ/\tau})\left[(\nabla\mathbf{u}_1)^{\top}\nabla h_2 + \nabla((\tfrac{1}{\sigma} + h_2)\nabla\cdot\mathbf{u}_1)\right].\end{aligned} \tag{3.18c}$$

The Sobolev regularity and $2\pi\tau$-periodicity of $h_2(t,\cdot)$ lead to similar properties of $\mathbf{u}_2$.

Corollary 3.3 *Consider the velocity field $\mathbf{u}_2$ in (3.18b) subject to sub-critical initial data $(h_0,\mathbf{u}_0) \in H^m(\mathbb{T}^2)$ with $m > 5$. Then, $\mathbf{u}_2(t,\cdot)$ is a $2\pi\tau$-periodic in time, and the following upper bound, uniformly in time, holds,*

$$\|\mathbf{u}_2 - \mathbf{u}_1\|_{m-2} \le C_0\frac{\tau}{\sigma}\left(1 + \frac{\tau}{\sigma}\right). \tag{3.19a}$$

In particular, since $\|\mathbf{u}_1\|_m \le C_0$ *for subcritical* τ*, we conclude that* $\mathbf{u}_2(t,\cdot)$ *has the Sobolev regularity,*

$$\|\mathbf{u}_2\|_{m-2} \le C_0 \left(1 + \frac{\tau}{\sigma} + \frac{\tau^2}{\sigma^2}\right). \tag{3.19b}$$

Remark 3.2 *The estimate (3.19a) actually suggests a small difference* $\sim O(\frac{\tau}{\sigma}) \ll 1$ *between* $\mathbf{u}_1$ *and* $\mathbf{u}_2$ *in terms of their spatial regularity. The improvement of* $\mathbf{u}_2$ *over* $\mathbf{u}_1$*, however, should be understood in terms of their time derivatives. By (3.18a),*

$$\partial_t \mathbf{u}_2 - \partial_t \mathbf{u}_1 = \frac{1}{\sigma} J \nabla h_2 + ...$$

whose leading term of order $O(\frac{1}{\sigma})$ *is not bounded in the scaling regime of our study. In other words, control of time derivatives plays an essential role in the presence of fast oscillation brought by singular forcing such as pressure forcing* $\frac{1}{\sigma}\nabla h$*. This is the subject of so called "bounded derivative method" (see [5] and references there in).*

We close this section by noting that the second iteration led to an approximate RSW system linearized around the pressureless velocity field, $\mathbf{u}_1$, (3.12),(3.18b), which governs our improved, $2\pi\tau$-periodic approximation, $(h_2(t,\cdot), \mathbf{u}_2(t,\cdot)) \in H^{m-1}(\mathbb{T}^2) \times H^{m-2}(\mathbb{T}^2)$.

3.3 Error Estimate: Energy Method

How close is $(h_2(t,\cdot), \mathbf{u}_2(t,\cdot))$ to the exact solution $(h(t,\cdot), \mathbf{u}(t,\cdot))$? Below we shall show that their distance, measured in $H^{m-3}(\mathbb{T}^2)$, does not exceed $\frac{e^{C_0 t}\delta}{(1-e^{C_0 t}\delta)^2}$. Thus, for sufficiently small δ, the RSW solution $(h, \mathbf{u})$ is *"almost periodic"* which in turn implies its long time stability. This is the content of our main result.

Theorem 3.3 *Consider the rotational shallow water (RSW) equations on a fixed 2D torus,*

$$\partial_t h + \mathbf{u}\cdot\nabla h + \left(\frac{1}{\sigma} + h\right)\nabla\cdot\mathbf{u} = 0 \tag{3.20a}$$

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\sigma}\nabla h - \frac{1}{\tau}J\mathbf{u} = 0 \tag{3.20b}$$

subject to initial data $(h_0, \mathbf{u}_0) \in H^m(\mathbb{T}^2)$ *with* $m > 5$ *and* $\alpha_0 := \min(1+\sigma h_0(\cdot)) > 0$. *Let*

$$\delta = \frac{\tau}{\sigma^2}$$

denote the ratio between the Rossby number τ *and the squared Froude number* σ, *and assume the subcritical condition* $\tau \leq \tau_c$ *so that (3.4) holds. Assume* $\sigma \leq 1$ *for substantial amount of pressure forcing in (3.20b). Then, there exists a constant* C_0, *depending only on* m, τ_c, α_0 *and in particular depending linearly on* $\|(h_0, \mathbf{u}_0)\|_m$, *such that the RSW equations admit a smooth, "*δ*-approximately periodic" solution in the sense that there exists a near-by* $2\pi\tau$*-periodic solution,* $(h_2(t,\cdot), \mathbf{u}_2(t,\cdot))$ *such that*

$$\|p(t,\cdot) - p_2(t,\cdot)\|_{m-3} + \|\mathbf{u}(t,\cdot) - \mathbf{u}_2(t,\cdot)\|_{m-3} \leq \frac{e^{C_0 t}\delta}{1 - e^{C_0 t}\delta}, \tag{3.21}$$

where p *is the "normalized height" such that* $1+\frac{1}{2}\sigma p = \sqrt{1+\sigma h}$ *and correspondingly* p_2 *satisfies* $1 + \frac{1}{2}\sigma p_2 = \sqrt{1+\sigma h_2}$.

It follows that the life span of the RSW solution, $t \lesssim t_\delta := 1 + \ln(\delta^{-1})$ *is prolonged due to the rapid rotation* $\delta \ll 1$, *and in particular, it tends to infinity when* $\delta \downarrow 0$.

Corollary 3.4 *The estimate on the actual height function* h *follows by applying the Gagliardo-Nirenberg inequality in Proposition 2.1 to* $h - h_2 = p(1+\frac{\sigma}{4}p) - p_2(1+\frac{\sigma}{4}p_2) =$

$(p-p_2)(1+\frac{\sigma}{4}(p-p_2)+\frac{\sigma}{2}p_2)$,

$$\|h(t,\cdot)-h_2(t,\cdot)\|_{m-3}\lesssim \frac{e^{C_0t}\delta}{(1-e^{C_0t}\delta)^2}.$$

Proof of Theorem 3.3. We compare the solution of the RSW system (3.20a),(3.20b) with the solution, $(h_2,\mathbf{u}_2)$, of approximate RSW system (3.12),(3.18b). To this end, we rewrite the latter in the equivalent form,

$$\begin{aligned}
\partial_t h_2+\mathbf{u}_2\cdot\nabla h_2+\left(\frac{1}{\sigma}+h_2\right)\nabla\cdot\mathbf{u}_2 &= (\mathbf{u}_2-\mathbf{u}_1)\cdot\nabla h_2+ \\
&\quad \left(\frac{1}{\sigma}+h_2\right)\nabla\cdot(\mathbf{u}_2-\mathbf{u}_1) \qquad (3.22a)\\
\partial_t\mathbf{u}_2+\mathbf{u}_2\cdot\nabla\mathbf{u}_2+\frac{1}{\sigma}\nabla h_2-\frac{1}{\tau}J\mathbf{u}_2 &= (\mathbf{u}_2-\mathbf{u}_1)\cdot\nabla\mathbf{u}_2+R. \qquad (3.22b)
\end{aligned}$$

The approximate system differs from the exact one, (3.20a),(3.20b), in the residuals on the RHS of (3.22a),(3.22b). We will show that they have an amplitude of order δ. In particular, the comparison in the rotationally dominant regime, $\delta\ll 1$ leads to a long-time existence of a smooth RSW solution, "nearby" the time-periodic solution $(h_2,\mathbf{u}_2)$. To show that $(h_2,\mathbf{u}_2)$ is indeed an approximate solution for the RSW equations, we proceed as follows.

We first *symmetrize* the both systems so that we can employ the standard energy method for nonlinear hyperbolic PDEs. To this end, We set the new variable ("normalized height") p such that $1+\frac{1}{2}\sigma p=\sqrt{1+\sigma h}$. Compressing notations with $\mathbf{U}:=(p,\mathbf{u})^\top$, we transform (3.20a),(3.20b) into the *symmetric hyperbolic* quasilinear system

$$\partial_t\mathbf{U}+B(\mathbf{U},\nabla\mathbf{U})+K[\mathbf{U}]=0. \qquad (3.23)$$

Here $B(\mathbf{F}, \nabla\mathbf{G}) := A_1(\mathbf{F})\mathbf{G}_x + A_2(\mathbf{F})\mathbf{G}_y$ where A_1, A_2 are bounded linear functions with values being symmetric matrices, and $K[\mathbf{F}]$ is a skew-symmetric linear operator so that $\langle K[\mathbf{F}], \mathbf{F}\rangle = 0$. By Proposition 2.4 that uses standard energy arguments, e.g. [23], (or the more recent treatments in [26], [29]), the symmetry form of (3.23) yields an exact RSW solution $\mathbf{U}$, which stays smooth for finite time $t \lesssim 1$. The essence of our main theorem is that for small δ's, rotation prolongs the life span of classical solutions up to $t \sim \mathcal{O}(\ln\delta^{-1})$. To this end, we symmetrize the approximate system (3.22a), (3.22b), using a new variable p_2 such that $1 + \frac{1}{2}\sigma p_2 = \sqrt{1+\sigma h_2}$. Compressing notation with $\mathbf{U}_2 := (p_2, \mathbf{u}_2)^\top$, we have

$$\partial_t\mathbf{U}_2 + B(\mathbf{U}_2, \nabla\mathbf{U}_2) + K(\mathbf{U}_2) = \mathbf{R} \tag{3.24}$$

where the residual $\mathbf{R}$ is given by

$$\mathbf{R} := \begin{pmatrix} (\mathbf{u}_2 - \mathbf{u}_1)\cdot\nabla p_2 & + & \left(\frac{2}{\sigma} + p_2\right)\nabla\cdot(\mathbf{u}_2 - \mathbf{u}_1) \\ (\mathbf{u}_2 - \mathbf{u}_1)\cdot\nabla\mathbf{u}_2 & - & R \end{pmatrix},$$

with R defined in (3.18c). We will show $\mathbf{R}$ is small which in turn, using the symmetry of (3.23) and (3.24), implies that $\|\mathbf{U} - \mathbf{U}_2\|_{m-3}$ is equally small. Indeed, thanks to the fact that $H^{m-3}(\mathbb{T})$ is an algebra for $m > 5$, every term in the above expression is up-bounded in H^{m-3} by the quadratic products of $\|\mathbf{u}_1\|_m, \|p_2\|_{m-1}, \|\mathbf{u}_2\|_{m-2}, \|\mathbf{u}_2 - \mathbf{u}_1\|_{m-2}$ up to a factor of $O(1+\frac{1}{\sigma})$. We use previous results on the Sobolev regularity of $\mathbf{u}_1, \mathbf{u}_2$ in Theorem 3.2, Corollary 3.3 and for p_2, we use the non-vacuum condition, $1 + \sigma h_0 \geq \alpha_0 > 0$, to find that $1 + \sigma h_2$ remains uniformly bounded from below, and by standard arguments (carried out in Appendix A) $\|p_2\|_{m-2} \leq C_0(1 + \tau/\sigma)$.

Summing up, the residual $\mathbf{R}$ does not exceed

$$\|\mathbf{R}\|_{m-3} \leq C_0^2\left(\delta + \frac{\tau}{\sigma} + ... + \frac{\tau^4}{\sigma^4}\right) \leq C_0^2\delta, \tag{3.25}$$

under sub-critical assumption $\tau \in (0, \tau_c)$ and scaling assumptions $\delta < 1$, $\sigma < 1$.

We now claim that the same upper bound holds for the error $\mathbf{W} := \mathbf{U}_2 - \mathbf{U}$ up to a factor increasing with time. Indeed, subtracting (3.23) from (3.24), we find the error equation

$$\partial_t\mathbf{W} + B(\mathbf{W}, \nabla\mathbf{W}) + K[\mathbf{W}] = -B(\mathbf{U}_2, \nabla\mathbf{W}) - B(\mathbf{W}, \nabla\mathbf{U}_2) + \mathbf{R}.$$

By the standard energy method using integration by parts and Sobolev inequalities while utilizing the symmetric structure of B and the skew-symmetry of K, we arrive at

$$\frac{d}{dt}\|\mathbf{W}\|_{m-3}^2 \lesssim_m \|\mathbf{W}\|_{m-3}^3 + \|\mathbf{U}_2\|_{m-2}\|\mathbf{W}\|_{m-3}^2 + \|\mathbf{R}\|_{m-3}\|\mathbf{W}\|_{m-3}.$$

Using the regularity estimates of $\mathbf{U}_2 = (p_2, \mathbf{u}_2)^\top$ and the upper bounds on $\mathbf{R}$ in (3.25), we end up with an energy inequality for $\|\mathbf{W}(t,\cdot)\|_{m-3}$,

$$\frac{d}{dt}\|\mathbf{W}\|_{m-3} \lesssim_m \|\mathbf{W}\|_{m-3}^2 + C_0\|\mathbf{W}\|_{m-3} + C_0^2\delta, \quad \|\mathbf{W}(0,\cdot)\|_{m-3} = 0.$$

A straightforward integration of the forced Ricatti equation (consult for example, [32, §5]), shows that the error $\|\mathbf{W}\|_{m-3}$ does not exceed

$$\|\mathbf{U}(t,\cdot) - \mathbf{U}_2(t,\cdot)\|_{m-3} \leq \frac{e^{C_0t}\delta}{1 - e^{C_0t}\delta}. \tag{3.26}$$

In particular, the RSW equations admits an "almost periodic" $H^{m-3}(\mathbb{T}^2)$-smooth solutions for $t \leq \frac{1}{C_0}\ln\frac{1}{\delta}$ for $\delta \ll 1$. $\square$

3.4 Extension to Other Domains

Torus domains (that is period-in-space) are less likely to resemble the reality as is the entire space $\mathbb{R}^2$ or bounded domain $\Omega \in \mathbb{R}^2$, which leads to distinguishable methodology in existing literature. In the rotationally dominant regime, however, we make the following observation: the approximate solution $(h_2, \mathbf{u}_2)$ is $2\pi\tau$-periodic and stays local in space without causing significant boundary effects or dispersive effects. The exact solution $(h, \mathbf{u})$, being $O(\delta) \ll 1$ away, is therefore "entrapped" around this approximate solution with little sensitivity to boundary conditions. To be more precise, we state the following corollary for the case of $\mathbb{R}^2$ with compactly supported initial data while leaving open the more subtle case of bounded domain $\Omega \in \mathbb{R}^2$.

Corollary 3.5 *Consider the RSW equations (3.1) subject to Cauchy initial data* $\mathbf{u}_0 \in H^m(\mathbb{R}^2)$ *with* $m > 5$. *Suppose* $\mathbf{u}_0$ *is compactly supported. Then Theorem 3.3 holds true. In particular, the exact RSW solution stays close to a globally smooth,* $2\pi\tau$*-periodic approximation* $(h_2, \mathbf{u}_2)$ *such that, for* $t \lesssim \ln(\delta^{-1})$,

$$\|h(t,\cdot) - h_2(t,\cdot)\|_{m-3} + \|\mathbf{u}(t,\cdot) - \mathbf{u}_2(t,\cdot)\|_{m-3} \lesssim \frac{e^{C_0 t}\delta}{(1 - e^{C_0 t}\delta)^2}. \tag{3.27}$$

The proof follows exactly the same line as in previous sections since all the Sobolev-Gagliardo-Nirenberg inequalities we have used are still valid for $\mathbb{R}^2$ and so is the energy method. The only addition work is to verify the vanishing boundary condition for the exact and approximate solutions. Clearly, it suffices to show finite speed of wave propagation since the initial data is compactly supported. This is true according to the standard hyperbolic theory. Indeed, using that fact that waves propagate at speeds $\mathbf{u} - \frac{1}{\sigma}, \mathbf{u}, \mathbf{u} + \frac{1}{\sigma}$ and the maximum principle for $\mathbf{u}$, we finish the proof.

Remark 3.3 *The H^m estimate (3.27) implies that there is only $O(\frac{e^{C_0 t}\delta}{(1-e^{C_0 t}\delta)^2})$ amount of energy in the exact solution $(h, \mathbf{u})$ leaking out of supp$(h_2, \mathbf{u}_2)$, which is only $O(\tau)$ away from supp$(h_0, \mathbf{u}_0)$ due to its $2\pi\tau$ periodicity. This phenomenon differs from the standard hyperbolic theory, that is, local waves propagate at one of the speeds $\mathbf{u} - \frac{1}{\sigma}, \mathbf{u}, \mathbf{u} + \frac{1}{\sigma}$ and "carry away" substantial amount of energy over a fairly short time scale $\frac{1}{\sigma}$.*

Chapter 4

Rotational Euler Equations with General Pressure Laws

We extend the method in last chapter to general Euler equations, in particular, the isentropic gas equations and ideal gas equations. As described in Section 1.2, these two systems are successive generalizations of the RSW equations and share similar formality in such a way that our method of iterative approximation remains effective for these general systems. In particular, the first approximation, the pressureless system in terms of $\mathbf{u}_1$, is exactly the same due to it independence of any pressure law. The second approximation, regarded as a specific linearization around the pressureless velocity $\mathbf{u}_1$, extends naturally to general Euler systems. In particular, we take advantage of the similar role of entropy S and density ρ (or h in the RSW system) as passive scalars driven by $\mathbf{u}$. One major complication, however, is the energy method for the ideal gasdynamics due to increased nonlinearity in the equations. Careful modification is carried out in Section 4.2 and we arrive at the same conclusion of long time existence of approximately periodic solutions.

4.1 Isentropic Gas Dynamics

In this section we extend the main theorem 3.3 to rotational 2D Euler equations for isentropic gas,

$$\partial_t \rho + \mathbf{u}\cdot\nabla\rho + \left(\frac{1}{\sigma} + \rho\right)\nabla\cdot\mathbf{u} = 0 \tag{4.1a}$$

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\sigma^2}\nabla(1+\sigma\rho)^{\gamma-1} - \frac{1}{\tau}J\mathbf{u} = 0 \tag{4.1b}$$

where the Mach number σ plays the same role as the Froude number in the RSW equation. In order to utilize the technique developed in the previous chapter, we introduce a new variable h so that it satisfies $1+\sigma h = (1+\sigma\rho)^{\gamma-1}$, so that the new variables (h, u) satisfy

$$\partial_t h + \mathbf{u}\cdot\nabla h + (\gamma-1)\left(\frac{1}{\sigma} + h\right)\nabla\cdot\mathbf{u} = 0 \tag{4.2a}$$

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\sigma}\nabla h - \frac{1}{\tau}J\mathbf{u} = 0. \tag{4.2b}$$

This is an analog to the RSW equations (3.20a),(3.20b) except for the additional factor $(\gamma-1)$ in the mass equation (4.2a). We can therefore duplicate the steps which led to Theorem 3.3 to obtain a long time existence for the rotational Euler equations (4.2a),(4.2b). We proceed as follows.

An approximate solution is constructed in two steps. First, we use the $2\pi\tau$-periodic pressureless solution, $(h_1 \equiv 0, \mathbf{u}_1(t,\cdot))$ for sub-critical initial data, $(h_0, \mathbf{u}_0)$. Second, we construct a $2\pi\tau$-periodic solution $(h_2(t,\cdot), \mathbf{u}_2(t\cdot))$ as the solution to an approximate system of the isentropic equations, *linearized* around the pressureless velocity $\mathbf{u}_1$,

$$\partial_t h_2 + \mathbf{u}_1 \cdot \nabla h_2 + (\gamma - 1)\left(\frac{1}{\sigma} + h_2\right)\nabla \cdot \mathbf{u}_1 = 0,$$
$$\mathbf{u}_2 := \mathbf{u}_1 + \frac{\tau}{\sigma} J\left(I - e^{tJ/\tau}\right)\nabla h_2(t, \cdot).$$

In the final step, we compare $(h, \mathbf{u})$ with the $2\pi\tau$-periodic approximate solution, $(h_2, \mathbf{u}_2)$. To this end, we symmetrize the corresponding systems using $\mathbf{U} = (p, \mathbf{u})^\top$ with the normalized density function p satisfying $1 + \frac{1}{2}\sqrt{\frac{1}{\gamma-1}}\sigma p = \sqrt{1 + \sigma h}$. Similarly, the approximate system is symmetrized with the variables $\mathbf{U}_2 = (p_2, \mathbf{u}_2)$ where $1 + \frac{1}{2}\sqrt{\frac{1}{\gamma-1}}\sigma p_2 = \sqrt{1 + \sigma h_2}$. We conclude

Theorem 4.1 *Consider the rotational isentropic equations on a fixed 2D torus, (4.1) subject to initial data $(\rho_0, \mathbf{u}_0) \in H^m(\mathbb{T}^2)$ with $m > 5$ and $\alpha_0 := \min(1 + \sigma\rho_0(\cdot)) > 0$.*

Let

$$\delta = \frac{\tau}{\sigma^2}$$

denote the ratio between the Rossby and the squared Mach numbers, and assume the subcritical condition $\tau \leq \tau_c$ so that (3.4) holds. Assume $\sigma < 1$ for substantial amount of pressure in (4.1b). Then, there exists a constant C_0, depending only on m, $\|(\rho_0, \mathbf{u}_0)\|_m$, τ_c, such that the RSW equations admit a smooth, "approximate periodic" solution in the sense that there exists a near-by $2\pi\tau$-periodic solution, $(\rho_2(t, \cdot), \mathbf{u}_2(t, \cdot))$ such that

$$\|p(t, \cdot) - p_2(t, \cdot)\|_{m-3} + \|\mathbf{u}(t, \cdot) - \mathbf{u}_2(t, \cdot)\|_{m-3} \leq \frac{e^{C_0 t}\delta}{1 - e^{C_0 t}\delta} \tag{4.3}$$

where p is the normalized density function satisfying $1 + \sigma p = (1 + \sigma\rho)^{\frac{\gamma-1}{2}}$ and p_2 results from the same normalization for ρ_2.

It follows that the life span of the isentropic solution, $t \lesssim t_\delta := 1 + \ln(\delta^{-1})$ is prolonged due to the rapid rotation $\delta \ll 1$, and in particular, it tends to infinity when $\delta \to 0$.

4.2 Ideal Gasdynamics

We turn our attention to the full Euler equations in a 2D torus,

$$\begin{aligned} \partial_t \rho + \nabla\cdot(\rho\mathbf{u}) &= 0 \\ \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \rho^{-1}\nabla\hat{p}(\rho, S) &= fJ\mathbf{u} \\ \partial_t S + \mathbf{u}\cdot\nabla S &= 0, \end{aligned}$$

where the pressure law is given as a function of the density, ρ and the specific entropy S, $\hat{p}(\rho, S) := \rho^\gamma e^S$. It can be symmetrized by defining a new variable – the "normalized" pressure function,

$$p := \frac{\sqrt{\gamma}}{\gamma - 1}\hat{p}^{\frac{\gamma-1}{2\gamma}}$$

and by replacing the density equation (4.4a) by a (normalized) pressure equation so that the above system is recast into an equivalent and symmetric form,

$$\begin{aligned} e^S\partial_t p + e^S\mathbf{u}\cdot\nabla p + C_\gamma e^S p\nabla\cdot\mathbf{u} &= 0 \\ \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + C_\gamma e^S p\nabla p &= fJu \\ \partial_t S + \mathbf{u}\cdot\nabla S &= 0 \end{aligned}$$

where constant $C_\gamma := \frac{\gamma-1}{2}$. Here, it is the exponential function e^S and triple products such as $e^S p\nabla p$ that make the ideal gas system a nontrivial generalization of the RSW and isentropic gas equations.

We then proceed to the nondimensional form by substitution,

$$\mathbf{u} \to \mathrm{U}u', \quad p \to \mathrm{P}(1+\sigma p'), \quad S = \ln(p\rho^{-\gamma}) \to \ln(\mathrm{P}\mathrm{R}^{-\gamma}) + \sigma S'$$

After discarding all the primes, we arrive at a nondimensional system

$$\begin{aligned}
e^{\sigma S}\partial_t p + e^{\sigma S}\mathbf{u}\cdot\nabla p + C_\gamma\left(\frac{e^{\sigma S}-1}{\sigma} + e^{\sigma S}p\right)\nabla\cdot\mathbf{u} &= -C_\gamma\frac{1}{\sigma}\nabla\cdot\mathbf{u} && (4.5a)\\
\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + C_\gamma\left(\frac{e^{\sigma S}-1}{\sigma} + e^{\sigma S}p\right)\nabla p &= -C_\gamma\frac{1}{\sigma}\nabla p + \frac{1}{\tau}Ju && (4.5b)\\
\partial_t S + \mathbf{u}\cdot\nabla S &= 0, && (4.5c)
\end{aligned}$$

where σ is the Mach number and τ is the Rossby number. With abbreviated notation $\mathbf{U} := (p, \mathbf{u}, S)^\top$, the above system is written in a compact form,

$$A_0(S)\partial_t\mathbf{U} + A_1(\mathbf{U})\partial_x\mathbf{U} + A_2(\mathbf{U})\partial_y\mathbf{U} = K[\mathbf{U}]. \tag{4.6}$$

Here, $A_i (i = 0, 1, 2)$ are symmetric-matrix-valued functions, *nonlinear* in $\mathbf{U}$ and in particular A_0 is always positive definite. The linear operator K is skew-symmetric such that $\langle K[\mathbf{U}], \mathbf{U}\rangle = 0$.

Two successive approximations are then constructed based on the iterations (1.9), starting with $j = 1$,

$$\begin{aligned}
p_1 &\equiv \text{constant}\\
\partial_t\mathbf{u}_1 + \mathbf{u}_1\cdot\nabla\mathbf{u}_1 &= \frac{1}{\tau}J\mathbf{u}_1\\
S_1 &\equiv \text{constant.}
\end{aligned}$$

Identified as the pressureless solution, $\mathbf{u}_1$ is used to linearize the system, resulting in the following approximation

$$\partial_t p_2 + \mathbf{u}_1 \cdot \nabla p_2 + C_\gamma p_2 \nabla \cdot \mathbf{u}_2 = -C_\gamma \frac{1}{\sigma} \nabla \cdot \mathbf{u}_2 \tag{4.7a}$$

$$\mathbf{u}_2 - \mathbf{u}_1 = \frac{\tau}{\sigma} J(I - e^{tJ/\tau}) C_\gamma e^{\sigma S_2}(1 + \sigma p_2) \nabla p_2 \tag{4.7b}$$

$$\partial_t S_2 + \mathbf{u}_1 \cdot \nabla S_2 = 0 \tag{4.7c}$$

The $2\pi\tau$-periodicity and global regularity of $\mathbf{U}_2 := (p_2, \mathbf{u}_2, S_2)^\top$ comes from the same line of argument as for the RSW equations in Section 3.2 together with the following nonlinear estimate for exponential functions such as $e^{\sigma S}$,

$$\begin{aligned} \|e^f - 1\|_m &= \| \textstyle\sum_{j=1}^{\infty} \frac{f^j}{j!} \|_m \\ &\lesssim_m \textstyle\sum_{j=1}^{\infty} \frac{(C_m |f|_\infty)^{j-1}}{j!} \|f\|_m \\ &= \frac{e^{C_m |f|_\infty} - 1}{C_m |f|_\infty} \|f\|_m \end{aligned}$$

where we recursively apply the Gagliardo-Nirenberg inequality to such terms as $\|f^j\|_m$. Notice the entropy variable (both the exact and approximate ones) always satisfies a transport equation and therefore is conserved along particle trajectories, which implies that the L^∞ norm of the entropy variable is an invariant. Thus, we arrive at an estimate

$$\|e^{\sigma S} - 1\|_m \leq \sigma \hat{C}_0 \|S\|_m. \tag{4.8}$$

And the same type of estimate holds true for S_2.

Finally, we subtract the approximate system (4.7) from the exact system (4.6), arriving at an error equation for $\mathbf{W} := \mathbf{U} - \mathbf{U}_2$ that shares the form as for the RSW system in Section 3.3, except that $A_i(\mathbf{U}) - A_i(\mathbf{U}_2) \neq A_i(\mathbf{U} - \mathbf{U}_2)$ due to nonlinearity which is essentially quadratic in the sense that,

Proposition 4.1 *Fix any number $n > 2$. Then, for any $\mathbf{U}, \mathbf{U}_2 \in H^n(\mathbb{T}^2)$ that satisfy*

$$\|A_0(\mathbf{U}) - A_0(\mathbf{U}_2)\|_n \lesssim \|S - S_2\|_n,$$

$$\|A_i(\mathbf{U}) - A_i(\mathbf{U}_2)\|_n \lesssim \|\mathbf{U} - \mathbf{U}_2\|_n^2 + \|\mathbf{U} - \mathbf{U}_2\|_n, \qquad i = 1, 2,$$

Proof. Consider a typical term of A_i, e.g. $e^{\sigma S}p$. Applying (4.8) together with Gagliardo-Nirenberg inequality to $e^{\sigma S} - e^{\sigma S_2} = e^{\sigma S_2}(e^{\sigma(S-S_2)-1})$, we can show $\|e^{\sigma S} - e^{\sigma S_2}\|_n \lesssim \|S - S_2\|_n$. The estimate on $\|e^{\sigma S}p - e^{\sigma S_2}p_2\|_n$ then follows by applying identity $ab - a_2b_2 = (a-a_2)(b-b_2) + (a-a_2)b_2 + a_2(b-b_2)$ together with the triangle inequality and the G-N inequality. Here regularity of S_2 and p_2 is a priori known. □

Yet another nonlinearity comes from $\nabla_{t,x}A_0$ – consult Proposition 2.3. Formally, we have $\nabla_{t,x}A_0 \sim \sigma e^{\sigma S}\nabla_{t,x}S \sim \sigma e^{\sigma S}(\mathbf{u}\cdot\nabla_x S + \nabla_x S)$ which manifests as two more multiplications in the energy estimate,

$$\frac{d}{dt}\|\mathbf{W}\|_{m-3} \lesssim \|\mathbf{W}\|_{m-3}^5 + ... + \|\mathbf{W}\|_{m-3} + \delta, \qquad \|\mathbf{W}(0,\cdot)\|_{m-3} = 0,$$

for which the solution has the same (asymptotic) behavior as for the quadratic Riccati equations derived in the previous sections.

Theorem 4.2 *Consider the (symmetrized) rotational Euler equations on a fixed 2D torus (4.5) subject to initial data $(p_0, \mathbf{u}_0, S_0) \in H^m(\mathbb{T}^2)$ with $m > 5$.*

Let

$$\delta = \frac{\tau}{\sigma^2}$$

denote the ratio between the Rossby and the squared Mach numbers, and assume the subcritical condition $\tau \le \tau_c$ *so that (3.4) holds. Assume* $\sigma < 1$ *for substantial amount of pressure forcing in (4.5b).*

Then, there exists a constant C_0*, depending only on* m, $\|(p_0, \mathbf{u}_0, S_0)\|_m$, τ_c*, such that the ideal gas equations admit a smooth, "approximate periodic" solution in the sense that there exists a near-by* $2\pi\tau$*-periodic solution,* $(p_2(t,\cdot), \mathbf{u}_2(t,\cdot), S_2(t,\cdot))$ *such that*

$$\|p(t,\cdot) - p_2(t,\cdot)\|_{m-3} + \|\mathbf{u}(t,\cdot) - \mathbf{u}_2(t,\cdot)\|_{m-3} + \|S(t,\cdot) - S_2(t,\cdot)\|_{m-3} \le \frac{e^{C_0 t}\delta}{1 - e^{C_0 t}\delta}. \tag{4.9}$$

It follows that the life span of the ideal gas solution, $t \lesssim t_\delta := 1 + \ln(\delta^{-1})$ *is prolonged due to the rapid rotation* $\delta \ll 1$*, and in particular, it tends to infinity when* $\delta \downarrow 0$*.*

Chapter 5

Significance of δ: Periodicity vs. Dispersion

Is the new parameter $\delta = \dfrac{\tau}{\sigma^2}$ a random choice just for convenience of analysis or is it mathematically significant? In this chapter, we will discuss the role of δ in determining how fast the flow disperses away to farfield versus how close the flow retains its time-periodicity. To be precise, we show that, in the $\mathbb{R}^2$ domain, the divergence field $\nabla\cdot \mathbf{u}(t,x)$ decays as

$$|\nabla\cdot \mathbf{u}| \lesssim \begin{cases} \sqrt{1/\delta}\, t^{-3/4}, & \text{for} \quad \tau \leq \sigma \\ \sqrt{\sigma}\, t^{-3/4}, & \text{for} \quad \tau \geq \sigma \end{cases}$$

with an additional nonlinear growth of order $O(e^t)$ over time scale $t \lesssim O(1)$. Therefore, when both δ and σ^{-1} approaches infinity, the divergence disperses so fast that any approximate periodicity is not retained. This asymptotic regime corresponds large pressure ($\sigma^{-1} \gg 1$) and less dominant rotation $\delta \gg 1$. We remark that rewriting the above estimate in the form

$$|\nabla\cdot \mathbf{u}| \lesssim \frac{\sigma}{\sqrt{\min\{\tau,\sigma\}}}\, t^{-3/4}$$

reveals the fact that the dispersive property is determined by the winner of the competition between rotation ($\sim \frac{1}{\tau}$) and pressure ($\sim \frac{1}{\sigma}$). In such a sense, our new parameter δ serves a critical parameter that indicates whether the dynamics is periodic or dispersive.

In the center of our analysis is Strichartz type estimate in the spirit of [43] regarding decay properties of linear wave equations in unbounded domains. The book of Chemin etc. [6] reexamines this approach in the context of geophysical equations. The starting point is the linearized RSW equations

$$\partial_t h + \frac{1}{\sigma}\nabla\cdot\mathbf{u} = 0, \tag{5.1a}$$

$$\partial_t \mathbf{u} + \frac{1}{\sigma}\nabla h - \frac{1}{\tau}J\mathbf{u} = 0. \tag{5.1b}$$

Straightforward calculation shows that the divergence $f := \nabla\cdot\mathbf{u}$ satisfies the Klein-Gordon equation

$$\partial_{tt} f - \frac{1}{\sigma^2}\Delta f + \frac{1}{\tau^2} f = 0, \tag{5.2}$$

subject to initial data $f|_{t=0} = \nabla\cdot\mathbf{u}_0$, $\partial_t f|_{t=0} = -\frac{1}{\sigma}\Delta h_0 + \frac{1}{\tau}\nabla\times\mathbf{u}_0$. This equation shows two competing mechanism: wave propagator $\partial_{tt} - \dfrac{1}{\sigma^2}\Delta$ and harmonic oscillator $\partial_{tt} + \dfrac{1}{\tau^2}$. Regarding the K-G equation (5.2) as a perturbation of harmonic oscillation $\partial_{tt} f + \dfrac{1}{\tau^2} f = 0$, we use Fourier transform to obtain their dispersion relations,

$$\varpi(\xi) = \pm\sqrt{\frac{1}{\tau^2} + \frac{1}{\sigma^2}|\xi|^2} \quad \text{and} \quad \varpi_0(\xi) = \pm\frac{1}{\tau}, \tag{5.3}$$

which implies the difference of their solution phases, increasing with time, is

$$t\varpi - t\varpi_0 = \frac{t}{\tau}\left(\sqrt{1 + \frac{\sigma^2}{\tau^2}|\xi|^2} - 1\right) \sim t\,|\xi|\,\frac{\tau}{\sigma^2}.$$

This hand-waving argument suggests that it is necessary to require $\delta = \frac{\tau}{\sigma^2} \ll 1$ instead of $\frac{\tau}{\sigma} \ll 1$ for the harmonic oscillator to dominate the wave dynamics. This intuition is also related to $\nabla_\xi \varpi$ whose behavior essentially decides the L^∞ decay rate of the K-G equation. Indeed, the previous computation amounts to estimating,

$$t\varpi(\xi) - t\varpi_0(\xi) = t(\varpi(\xi) - \varpi(0)) \sim t|\xi|\nabla_\xi\varpi(\xi)$$

This chapter is organized as following. In Section 5.1, we will use Strichartz estimate to derive the L^∞ decay rate for $f(t, \cdot)$ in the linear K-G equation (5.2) in $\mathbb{R}^2$ domain. There are existing work of Klainerman [25] for the $\mathbb{R}^3$ case and Hömander [20] for the harder case of $\mathbb{R}^2$. Our argument below, however, takes a different perspective on various regimes of physical parameters τ, σ and δ upon which the nature of underlying flow hinges. We then discuss in Section 5.2 the fully nonlinear RSW equations using energy estimate on nonlinear terms which are $O(1)$ in finite time.

We comment here that the decay of divergence field corresponds to the "Rossby adjustment" in geophysical sciences. It has been long established that the atmosphere and ocean are in a permanent process of filtering out compressible waves and approaching the state of geostrophic balance (consult e.g. the textbooks of [39], [22]). Our analytical result shows that the parameter δ plays the very role of determining how fast the dynamics converges to the incompressible, geostrophic balance. We note by passing that the drastic effects of nonlinearity, especially in the long term, shall not be underestimated – consult Zeitlin, etc. [47, 48] for multiple time scale analysis of nonlinear geostrophic adjustment.

5.1 Linear Theory: Strichartz Estimate

Let $\hat{f}(t, \xi)$ denote the Fourier form of $f(t, x)$. Consider a wave propagator associated with phase function $\phi(\xi, x/t)$ so that, starting with initial data $f_0(\cdot)$, the wave solution is given by

$$f(t,x) = \fint_{\mathbb{R}^2} e^{it\phi(\xi,x/t)}\,\hat{f}_0(\xi)\,d\xi \tag{5.4}$$

where $\fint$ denotes integral with a constant prefactor. The phase function for the K-G equation is

$$\phi^{\pm}(\xi, x/t) := \pm\varpi(\xi) - \frac{x}{t} = \pm\sqrt{\frac{1}{\tau^2} + \frac{1}{\sigma^2}|\xi|^2} - \xi\cdot\frac{x}{t} \tag{5.5}$$

with the dispersion relation $\varpi(\xi)$ given in (5.3). Observe there are two phase functions with opposite signs. The corresponding amplitude functions $\hat{f}_0^{\pm}(\xi)$ are determined from initial data f_0 and $\partial_t f_0$ by

$$\hat{f}_0^{\pm}(\xi) = \frac{1}{2}\left(\hat{f}_0(\xi) \pm \frac{\widehat{\partial_t f}_0(\xi)}{i\varpi(\xi)}\right). \tag{5.6}$$

The following lemma gives estimates on growth of $\hat{f}_0^{\pm}(\xi)$ at farfield in terms of the RSW initial data $(h_0, \mathbf{u}_0)$.

Lemma 5.1 *Consider $\hat{f}_0^{\pm}(\xi)$ in (5.6) with $f_0(x), \partial_t f_0(x)$ given under (5.2). Then for any integer $\alpha > 0$,*

$$\left\| |\xi|^{\alpha} \hat{f}_0^{\pm} \right\|_{L^\infty} \lesssim \|(h_0, \mathbf{u}_0)\|_{W^{1,\alpha+1}}$$

$$\left\| |\xi|^{\alpha} \nabla_\xi \hat{f}_0^{\pm} \right\|_{L^\infty} \lesssim \|(h_0, \mathbf{u}_0, xh_0, x\mathbf{u}_0)\|_{W^{1,\alpha+1}}$$

Here and below, the hidden constant behind "$\lesssim$" is independent of parameter τ, σ and initial data.

Proof. By (5.6), it suffices to prove these bounds for $\hat{f}_0$ and $\widehat{\partial_t f_0}/\varpi$. Observe from $f_0(x) = \nabla\cdot\mathbf{u}_0(x)$, $\partial_t f_0(x) = -\frac{1}{\sigma}\Delta h_0(x) + \frac{1}{\tau}\nabla\times\mathbf{u}_0(x)$ that

$$\hat{f}_0(\xi) = \xi\cdot\hat{\mathbf{u}}_0, \qquad \widehat{\partial_t f_0} = \frac{1}{\sigma}|\xi|^2\hat{h}_0 + \frac{i}{\tau}\xi\times\hat{\mathbf{u}}_0$$

and in particular,

$$|\widehat{\partial_t f_0}| \lesssim \varpi \cdot |\xi \widehat{(h_0, \mathbf{u}_0)}|.$$

Upon using standard Fourier argument, we arrive at the first estimate regarding $\hat{f}_0^{\pm}$.

For the case of $\nabla_\xi \hat{f}_0^{\pm}$, we differentiate to get

$$\nabla_\xi \hat{f}_0 = \hat{\mathbf{u}}_0 + \xi \cdot \nabla_\xi \hat{\mathbf{u}}_0, \qquad \nabla_\xi \frac{\widehat{\partial_t f_0}}{\varpi} = \frac{\nabla_\xi \widehat{\partial_t f_0}}{\varpi} - \frac{\nabla_x i\varpi}{\varpi} \frac{\widehat{\partial_t f_0}}{\varpi}.$$

It is easy to check that

$$|\nabla_\xi \widehat{\partial_t f_0}| \lesssim \varpi \cdot (|\widehat{(h_0, \mathbf{u}_0)}| + |\xi \nabla_\xi \widehat{(h_0, \mathbf{u}_0)}|)$$

and

$$\left| \frac{\nabla_\xi \varpi}{\varpi} \right| \lesssim 1.$$

The second estimate then readily follows. $\square$

Therefore, we consider $\hat{f}_0 = \hat{f}_0^{+}$ in the following argument unless specified otherwise.

§. Integration by Parts. We aim to obtain L^∞ bound on f given in the form (5.4). Fast oscillation, caused by $e^{it\phi}$ for large t, effectively leads to cancelation in L^∞ type of estimates – consider a heuristic example $\left| \int_0^1 e^{iNx} f(x)\, dx \right| \lesssim \frac{1}{N} |f'|_\infty$. To this end, we apply integration by parts to (5.4) with the help of identity $\frac{-iS \cdot \nabla_\xi}{|S|^2 t} e^{it\phi} = e^{it\phi}$ where the phase gradient $S = S(\xi, x/t) := \nabla_\xi \phi(\xi, x/t)$,

$$\begin{aligned}
f(t,x) &= \fint_{\mathbb{R}^2} e^{it\phi} \hat{f}_0 \, d\xi \\
&= \fint_{\mathbb{R}^2} \left[\frac{1 - iS \cdot \nabla_\xi}{1 + |S|^2 t} e^{it\phi} \right] \hat{f}_0 \, d\xi \\
&= \fint_{\mathbb{R}^2} e^{it\phi} \left[\frac{\hat{f}_0}{1 + |S|^2 t} + \frac{i \nabla \cdot (S \hat{f}_0)}{1 + |S|^2 t} - \frac{i(S \cdot \nabla |S|^2) t \hat{f}_0}{(1 + |S|^2 t)^2} \right] d\xi
\end{aligned}$$

Here, there is no contribution from boundary integral under mild growth conditions, e.g. $S \sim |\xi|$ and $\hat{f}_0 \lesssim |\xi|^{-1}$ as $|\xi| \to \infty$. Upon a straightforward estimation on the right hand side, we arrive at the following lemma,

Lemma 5.2 *Consider a wave solution $f(t,x)$ give in (5.4). Define the phase gradient*

$$S(\xi, x/t) := \nabla_\xi \phi(\xi, x/t).$$

Assume growth condition $S \sim |\xi|$ and $\hat{f}_0 \lesssim |\xi|^{-1}$ as $|\xi| \to \infty$. Then,

$$|f(t,x)| \lesssim \left\| \frac{\hat{f}_0}{1+|S|^2 t} \right\|_{L^1} + \left\| \frac{|\nabla_\xi S| \hat{f}_0}{1+|S|^2 t} \right\|_{L^1} + \left\| \frac{|S| \nabla_\xi \hat{f}_0}{1+|S|^2 t} \right\|_{L^1}.$$

Here, the RHS depends on t and x/t. This lemma suggests that the time decay of $f(t,x)$ be associated with the behavior of $|S(\xi, x/t)|$, in particular, the stationary set in which $|S(\xi, x/t)| = 0$.

§. L^∞ Estimate of linear K-G solution. Now we turn to the specific case of K-G equation (5.2) in 2D with phase function given in (5.5). The phase gradient is therefore given by

$$S(\xi, t) = \frac{\sigma^{-2}\xi}{\sqrt{\tau^{-2} + \sigma^{-2}|\xi|^2}} - \frac{x}{t} = a(|\xi|)\xi - \frac{x}{t} \tag{5.7}$$

where the auxiliary function $a(r)$ is given by

$$a(r) := \frac{\sigma^{-2}}{\sqrt{\tau^{-2} + \sigma^{-2} r^2}}$$

Multiplying $\min\{\tau, \sigma\} / \min\{\tau, \sigma\}$ to the RHS gives

Lemma 5.3

$$a(r) = \frac{\sigma^{-2} \min\{\tau, \sigma\}}{\sqrt{\min^2\{1, \sigma/\tau\} + \min^2\{\tau/\sigma, 1\} r^2}} \geq \frac{\kappa}{\sqrt{1+r^2}}$$

where the auxiliary parameter

$$\kappa := \min\{\delta, \sigma^{-1}\} = \begin{cases} \delta, & \text{if} \quad \tau \le \sigma \\ \\ \sigma^{-1}, & \text{if} \quad \tau \ge \sigma \end{cases}$$

This lemma reveals the role of δ in the strong rotation regime $\tau \le \sigma$. On the other hand, when pressure is stronger, i.e. $\tau \ge \sigma$, the key parameter turns to σ^{-1}.

We then compute $\nabla_\xi S$,

$$\nabla_\xi S(\xi, x/t) = a(|\xi|)I - \sigma^2 a^3(|\xi|)\xi \otimes \xi.$$

Since $|\xi \otimes \xi| \lesssim |\xi|^2 \lesssim \sigma^{-2}a^{-2}$, we have estimate

$$|\nabla_\xi S(\xi, x/t)| \lesssim a(|\xi|). \tag{5.8}$$

Lemma 5.2 and the formality of $S, \nabla_\xi S$ in (5.7), (5.8) suggests us to use polar representation. To this end, we introduce the following lemma regarding integration with respect to angular variable.

Lemma 5.4 *Consider S given by (5.7) in polar coordinates, $S = S(r, \theta, x/t)$. Then*

$$\int_0^{2\pi} \frac{1}{1 + |S|^2 t}\, d\theta \lesssim 1/\sqrt{1 + 2t(a^2r^2 + z^2) + t^2(a^2r^2 - z^2)^2},$$

where $z := |x/t|$.

Proof. Assume $x/t = (z, 0)$ for simplicity since the following proof is invariant under rotation. Then rewrite $|S|^2$ as

$$|S|^2 = (a\, r\cos\theta - z)^2 + (a\, r\sin\theta)^2$$

$$= a^2r^2 + z^2 - 2a\, rz\cos\theta$$

and therefore the integral $\int_0^{2\pi} \frac{1}{1+|S|^2 t}\, dt$ is of the elliptic type and can be calculated using e.g. Mathematica (an online version [36]). Indeed, it is easy to verify that

$$\int \frac{1}{\alpha+\beta\sin\theta}\, d\theta = \frac{2}{\sqrt{\alpha^2-\beta^2}} \arctan\left(\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\tan(\theta/2)\right),$$

where arctan function is always bounded by $\pi/2$. □

We remark that this lemma indicates the time decay of angular integral at rate $\sim O(1/t)$ that degenerates to $O(1/\sqrt{t})$ when $ar = z \leftrightarrow a\xi = x/t \leftrightarrow |S| = 0$, namely, in the stationary set.

We will now prove the main result of this chapter,

Theorem 5.1 *Consider the linear RSW equations (5.1) subject to compactly supported, $H^m(\mathbb{R}^2)$ initial data with $m > 5$. Let*

$$\kappa := \min\{\delta, \sigma^{-1}\} = \frac{1}{\sigma^2}\min\{\tau, \sigma\}.$$

Assume $\kappa \geq 1$ for substantial amount of pressure ($\sigma^{-1} \geq 1$) and avoidance of dominant rotation ($\delta \not< 1$). Then the divergence field satisfies estimate,

$$|\nabla\cdot\mathbf{u}(t,x)| \lesssim \frac{1}{\sqrt{\kappa}}\left(\frac{1}{t^{1/2}} + \frac{1}{t^{3/4}}\right)$$

for $t \geq 2/\kappa$, $|x/t| \leq \sqrt{\kappa}/2$. In particular, when $\kappa \uparrow \infty$, the divergence field $\nabla\cdot\mathbf{u}(t,x)$ approaches zero at any fixed time-space point (t,x) (for $t > 0$).

Proof. Combining Lemma 5.2, 5.4 with the definition of S in (5.7) and estimate on $\nabla_\xi S$ in (5.8), it suffices to consider

$$|f(t,x)| \lesssim \int_0^\infty \frac{1+a+ar+z}{\sqrt{1+2t(a^2r^2+z^2)+t^2(a^2r^2-z^2)^2}} g(r)\, r\, dr \qquad (5.9)$$

where $z = |x/t|$ and $g(r) := \max_\theta\{|\hat{f}_0(r,\theta)| + |\nabla_\xi \hat{f}_0(r,\theta)|\}$.

We then use the standard decomposition technique to split the above integral into frequency intervals of r. The choice of these intervals relies on the stationary set of S which, after angular integration, translates to $a^2r^2 = z^2$ as suggested by the leading term on the denominator of the integrand in (5.9). Since by assumption $z \le \sqrt{\kappa}/2$, it is reasonable to consider a low frequency intervals $I_L := [0, 1/\sqrt{\kappa}]$ and a high frequency $I_H := (1/\sqrt{\kappa}, \infty)$. The latter interval is high enough so that by Lemma 5.3 and the scaling assumption $\kappa \ge 1$,

$$r \in I_H \rightsquigarrow a^2r^2 \ge \frac{\kappa^2}{1+r^2} r^2 > \frac{\kappa^2}{1+1/\kappa}\frac{1}{\kappa} \ge \frac{\kappa}{2} \ge 2z^2, \tag{5.10}$$

namely I_H is far away from the stationary set $\{S = 0\}$.

On the low frequency interval $I_L = [0, 1/\sqrt{\kappa}]$. The shortness of this interval allows us not to rely on the possibly degenerate leading term on the denominator of the integrand in (5.9). Thus, we discard this (positive) leading term,

$$\begin{aligned}
&\int_{I_L} \frac{1+a+ar+z}{\sqrt{1+2t(a^2r^2+z^2)+t^2(a^2r^2-z^2)^2}}\, g(r)\, r\, dr \\
&\le \int_{I_L} \frac{1+a+ar+z}{\sqrt{1+2t(a^2r^2+z^2)}}\, g(r) r\, dr \\
&\lesssim \int_{I_L} \frac{1+a+ar+z}{1+\sqrt{t}(ar+z)}\, g(r) r\, dr \\
&= \int_{I_L} \left\{ \frac{1}{1+\sqrt{t}(ar+z)} + \frac{1}{a^{-1}+\sqrt{t}(r+a^{-1}z)} + \frac{1}{(ar+z)^{-1}+\sqrt{t}} \right\} g(r) r\, dr \\
&< \int_{I_L} \left\{ \frac{1}{\sqrt{t}ar} + \frac{1}{\sqrt{t}r} + \frac{1}{\sqrt{t}} \right\} g(r) r\, dr \\
&\lesssim \frac{1}{\sqrt{\kappa t}} \|g\|_{L^\infty}.
\end{aligned}$$

In the last step we used Lemma 5.3 together with scaling assumption $\kappa \ge 1$ and the range of $I_L = [0, 1/\sqrt{\kappa}]$.

On the high frequency interval $I_H = (1/\sqrt{\kappa}, \infty)$. As we argued in (5.10), the leading term $t^2(a^2r^2 - z^2)^2$ does not vanish and is bounded from below by $t^2(a^2r^2/2)^2$. Therefore,

$$\begin{aligned}
&\int_{I_H} \frac{1+a+ar+z}{\sqrt{1+2t(a^2r^2+z^2)+t^2(a^2r^2-z^2)^2}}\, g(r)\, r\, dr \\
\lesssim &\int_{I_H} \frac{1+a+ar+z}{1+ta^2r^2}\, g(r)\, r\, dr \\
=: &\int_{I_H} F(a(r), r) g(r) r\, dr.
\end{aligned}$$

Considering F as a function of *two independent variables*, we will show $F(a,r) \leq F(\kappa/\sqrt{1+r^2}, r)$. Indeed, since Lemma 5.3 claims that $a \geq \kappa/\sqrt{1+r^2}$, it suffices to prove $\partial_a F(a,r) \leq 0$ which leads to proving

$$\frac{\partial}{\partial a} \frac{a}{1+ta^2r^2} = \frac{1-ta^2r^2}{(1+ta^2r^2)^2} \leq 0.$$

This is true due to the assumption $t \geq 2/\kappa$ and the estimate (5.10). We then carry on previous computation

$$\begin{aligned}
\int_{I_H} F(a(r), r) g(r) r\, dr &\leq \int_{I_H} F(\kappa/\sqrt{1+r^2}, r) \\
&= \int_{I_H} \frac{(1+z)(1+r^2) + \kappa\sqrt{1+r^2}(1+r)}{1+r^2+\kappa^2 tr^2}\, g(r) r\, dr \\
&\lesssim \int_{I_H} \frac{\kappa(1+r^2)}{1+\kappa^2 tr^2}\, r g(r)\, dr \\
&\leq \left\| \frac{r}{[1+\kappa^2 tr^2]^{5/4}} \right\|_{L^1} \cdot \|\kappa(1+r^2) g(r) [1+\kappa^2 tr^2]^{1/4}\|_{L^\infty} \\
&\lesssim \frac{1}{\kappa^2 t} \|\kappa(1+r^2) g(r) [1+\kappa^2 tr^2]^{1/4}\|_{L^\infty} \\
&\lesssim \frac{1}{\kappa^{1/2} t^{3/4}} \|(1+r^2)^{5/4} g(r)\|_{L^\infty},
\end{aligned}$$

where we repeatedly use scaling assumptions $\kappa \geq 1$, $t \geq 2/\kappa$, $z \leq \sqrt{\kappa}/2$ to simplify the computation.

Combining these two intervals, we arrive at an estimate for $f(t, x)$,

$$|f(t,x)| \lesssim \frac{1}{\sqrt{\kappa}} \left(\frac{1}{t^{1/2}} + \frac{1}{t^{3/4}} \right) \|(1+r^2)^{5/4} g(r)\|_{L^\infty}.$$

In the final step, we apply Lemma 5.1 to estimate growth of $g(r) = \max_\theta\{|\hat{f}_0(r,\theta)| + |\nabla_\xi \hat{f}_0(r,\theta)|\}$, which turns out to be bounded by the $W^{1,4}$ norm of $(h_0, \mathbf{u}_0)$ and $(xh_0, x\mathbf{u}_0)$. Since the initial data is compactly supported and in $H^m(\mathbb{R}^2)$ $(m > 5)$, a simple Sobolev estimate shows that $\|(1+r^2)^{5/4} g(r)\|_{L^\infty}$ is bounded. Therefore, we reach the conclusion. $\square$

We remark here that the above proof emphasizes on effects of parameter regimes and therefore does not yield the usual $O(1/t)$ decay rate for large time ([20]).

5.2 Nonlinear System

We now turn to the fully nonlinear RSW (with a different notation from previous chapters)

$$\partial_t h_1 + \frac{1}{\sigma} \nabla\cdot \mathbf{u}_1 = -\nabla\cdot(h_1 \mathbf{u}_1) \tag{5.11a}$$

$$\partial_t \mathbf{u}_1 + \frac{1}{\sigma} h_1 - \frac{1}{\tau} J\mathbf{u}_1 = -\mathbf{u}_1 \cdot \nabla \mathbf{u}_1 \tag{5.11b}$$

By the standard energy method (e.g. Proposition 2.4), starting with H^m $(m > 2)$ initial data, $\|(h_1(t,\cdot), \mathbf{u}_1(t,\cdot)\|_{H^m})$ stays bounded for finite time. Thus, the nonlinear terms on the RHS of the above system stays bounded for finite time also. We then regard it as a perturbed system of the linear RSW equations (5.1) and take the difference, resulting as a linear system with an $O(1)$ RHS for finite time. By the energy method for linear system, we obtain

Corollary 5.1 *Consider the nonlinear RSW equations (5.11). Under the same assumptions as for Theorem 5.1, there exists a constant* C *such that starting with smooth initial data* $\|(h_0, \mathbf{u}_0)\|_{H^m} \leq C$,

$$|\nabla\cdot \mathbf{u}_1(t,x)| \lesssim \frac{1}{\sqrt{\kappa}} \frac{1}{t^{3/4}} + t$$

for $t \in [2/\kappa, 1]$ *and* $|x/t| < \sqrt{\kappa}/2$. *In particular,* $|\nabla\cdot \mathbf{u}_1(2/\sqrt{\kappa}, 0)| \lesssim 1/\kappa^{1/8} \downarrow 0$ *when* $\kappa \uparrow \infty$.

Chapter 6

Future Works

There are several open questions raised in this thesis. One regards effects of boundary on rotationally dominant flows. Because strong rotation restraints most of the energy locally, we conjecture that the flow stays almost local even in the presence of boundary and thus our main result on long time existence of approximately periodic solutions still remains valid. Another question is on the dispersive phenomenon on periodic domains. Even though energy can not escape to farfield in periodic settings, dispersion in short time may still happen provided the support of initial data is localized enough.

For inertial oscillations governed by the pressureless system, there are motions other than circular motion in various settings. For instance, background flow may lead to elliptic oscillations. Earth geometry plays an important role especially in large scales ([38]). Periodicity, nevertheless, appears in many cases. The question is therefore: can we extend our methodology if the pressureless flow is non-circular but periodic? The averaging methods ([40]), originally applied to dynamic systems, provide possible approaches to deal with general periodic or near periodic dynamics.

In Chapter 5, we discussed the dispersion phenomenon of linear RSW equations but didn't fully explore the nonlinear case. Given existing results on fast wave averaging/filtering in the regime $\tau \sim \sigma \downarrow 0$ and $\tau \sim O(1), \sigma \downarrow 0$ for fully nonlinear systems, can we obtain similar asymptotic results for the $\delta \uparrow \infty$ case? In geophysical terms, how does the *nonlinear* Rossby adjustment affect the flow dynamics when $\delta \gg 1$ rather than $\sigma^{-1} \gg 1$? It will also be interesting to study the limiting dynamics in the large δ regime.

Chapter A

Appendix. Staying away from vaccum

We will show the following proposition on the new variable p_2 defined in Section 3.3.

Proposition A.1 *Let p_2 satisfies*

$$1 + \frac{1}{2}\sigma p_2 = \sqrt{1 + \sigma h_2} \tag{A.1}$$

where h_2 is defined as in (3.12), that is,

$$\partial_t h_2 + \mathbf{u}_1 \cdot \nabla h_2 + \left(\frac{1}{\sigma} + h_2 \right) \nabla \cdot \mathbf{u}_1 = 0 \tag{A.2}$$

subject to initial data $h_2(0, \cdot) = h_0(\cdot)$ that satisfies the non-vacumm condition $1 + \sigma h_0(\cdot) \geq \alpha_0 > 0$. Then,

$$|p_2|_\infty \leq \hat{C}_0 \left(1 + \frac{\tau}{\sigma} \right),$$
$$\|p_2\|_n \leq C_0 \left(1 + \frac{\tau}{\sigma} \right).$$

The proof of this proposition follows two steps. First, we show that the L^∞ and H^n norms of $p_2(0, \cdot)$ are dominated by $h_2(0, \cdot)$ due to the non-vacumm condition. Second, we derive the equation for p_2 and obtain regularity estimates using similar techniques from Section 3.2.

Step 1. For simplicity, we use $p := p_2(0, \cdot)$ and $h := h_2(0, \cdot)$.

Solving (A.1) and differentiation yield

$$p = \frac{2h}{\sqrt{1 + \sigma h} + 1}, \quad \nabla p = \frac{\nabla h}{\sqrt{1 + \sigma h}}.$$

Apparently $|p|_\infty \leq |h|_\infty$.

The above identities, together with the non-vacumm condition imply

$$\|p\|_1 \leq 2\|h\|_1 \quad \text{and} \quad |\nabla p|_{L^\infty} \leq \frac{|\nabla h|_{L^\infty}}{\sqrt{\alpha_0}}.$$

For higher derivatives of p, we use the following recursive relation. Rewrite (A.1) as $p + \frac{1}{4}\sigma p^2 = h$ and then take the k-th derivative on both sides

$$D^k p + \frac{1}{4}\sigma 2pD^k p + \frac{1}{4}\sigma \left(D^k(q^2) - 2pD^k p\right) = D^k h$$

so that taking L^2 norm of this equation yields

$$I - II := \left\|(1 + \frac{1}{2}\sigma p)D^k p\right\|_0 - \frac{1}{4}\sigma \left\|D^k(q^2) - 2pD^k p\right\|_0 \leq \|D^k h\|_0.$$

Furthermore, we find $I \geq \sqrt{\alpha_0}\|D^k p\|_0$ by (A.1) and the non-vacumm condition. We also find $II \lesssim_n |\nabla p|_\infty \|p\|_{|k|-1}$ by Gargliado-Nirenberg inequalities. Thus we arrive at a recursive relation

$$\|p\|_{|k|} \leq \hat{C}_0(\|p\|_{|k|-1} + \|h\|_{|k|})$$

which implies that the H^n norm of $p_2(0,\cdot) = p$ is dominated by $\|h_2(0,\cdot)\|_n = \|h\|_n$.

Step 2. We derive an equation for p_2 using relation (A.1) and equation (A.2),

$$\partial_t p_2 + 2\mathbf{u}_1 \cdot \nabla p_2 + \left(\frac{1}{\sigma} + p_2\right) \nabla \cdot \mathbf{u}_1 = 0.$$

This equation resembles the formality of the approximate mass equation (3.12) for h_2 and thus we apply similar technique to arrive at the same regularity estimate for p_2,

$$\begin{aligned} |p_2(t,\cdot)|_\infty &\leq \hat{C}_0\left(1 + \frac{\tau}{\sigma}\right), \\ \|p_2(t,\cdot)\|_n &\leq C_0\left(1 + \frac{\tau}{\sigma}\right). \end{aligned}$$

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