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Universal Duality in Conic Convex Optimization

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Abstract

Given a primal-dual pair of linear programs, it is known that if their optimal values are viewed as lying on the extended real line, then the duality gap is zero, unless both problems are infeasible, in which case the optimal values are $+\infty$ and $-\infty$. In contrast, for optimization problems over nonpolyhedral convex cones, a nonzero duality gap can exist even in the case where the primal and dual problems are both feasible.

For a pair of dual conic convex programs, we provide simple conditions on the "constraint matrices" and cone under which the duality gap is zero for *every* choice of linear objective function and "right-hand-side". We refer to this property as "universal duality". Our conditions possess the following properties: (i) they are necessary and sufficient, in the sense that if (and only if) they do not hold, the duality gap is nonzero for some linear objective function and "right-hand-side"; (ii) they are metrically and topologically generic; and (iii) they can be verified by solving a single conic convex program. As a side result, we also show that the feasible sets of a primal conic convex program and its dual cannot both be bounded, unless they are both empty, and we relate this to universal duality.

Keywords. Conic convex optimization, constraint qualification, duality gap, universal duality, generic property.

1 Introduction and background

It is well known that if a linear program and its Lagrangian dual are both feasible, then strong duality holds for that pair of problems. That is, there is a zero duality gap, and both (finite) optimal values are attained. A key to proving this result is Farkas' Lemma.

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It is also well known that for nonpolyhedral convex cones, simple generalizations of Farkas' Lemma do not necessarily hold. (However an "asymptotic" Farkas Lemma does hold; see e.g., [2].) In fact there exist conic convex programs that admit a finite nonzero duality gap; see [7, 11] for examples. (Throughout this paper, we will refer to conic convex programs as conic programs; convexity will always be assumed.) The reason for the failure of simple extensions of Farkas' Lemma is the potential nonclosedness of the linear image of a closed convex cone. (When the cone is polyhedral, as is the case in linear optimization, its linear image is always closed.) Conditions under which closedness is guaranteed to occur can be found, e.g., in [9], and the references therein.

As a consequence of this failure, in optimization over nonpolyhedral convex cones, a regularity condition is usually assumed in order to guarantee a zero duality gap. An example of such a condition is the generalized Slater constraint qualification. A sufficient condition for strong duality of a pair of dual conic programs is that both problems satisfy the generalized Slater constraint qualification. If this constraint qualification holds for only one of the two problems, then a zero duality gap still results, but the optimal values need not both be attained. Further results on duality in linear and nonlinear programming can be found in, e.g., [8].

In some contexts, one wishes to study a family of optimization problems parameterized by their objective function or "right-hand-side". For example, in a network optimization problem, it may be the case that the network "structure" remains fixed, but say, the arc costs or arc capacities vary. Under such circumstances, it would be desirable for the "constraint matrices" (corresponding to the network "structure") to be such that the duality gap of the pair of Lagrangian dual problems is always zero, regardless of the objective function or "right-hand-side" (which may correspond to arc costs or arc capacities).

In this work, motivated by such considerations, we give necessary and sufficient conditions on the "constraint matrices" and cone that ensure, for every linear objective function and "right-hand-side", the satisfaction of the generalized Slater constraint qualification for a conic program or its dual. Therefore our conditions are sufficient to ensure a zero duality gap for every choice of linear objective function and "right-hand-side". We show that they are also necessary. We refer to our characterization as "universal duality", and show that it holds generically in a metric as well as a topological sense. Finally, we show that universal duality—which gives information about an infinite family of conic programs—can be verified by solving a single conic program with essentially the same size and structure as that of the original "primal" problem.

The layout of this paper is as follows. In Section 2, we define notation and give some results from convex analysis that will be used later in the paper. In Section 3, we give simple necessary and sufficient conditions for universal duality to hold. In that section, we also show that the feasible sets of a pair of dual conic programs cannot both be bounded (unless they are both empty), and we relate this result to universal duality. We show in Section 4 that universal duality is a generic property, and in Section 5 that universal duality for a pair of dual conic programs can be verified by solving a single conic program. Finally, in Section 6 we state some conclusions.

2 Notation and preliminaries

Given a set $S \subseteq \mathcal{R}^n$, we will write $\mathrm{ri}(S)$, $\mathrm{int}(S)$, and $\mathrm{cl}(S)$ to denote its relative interior, interior, and closure, respectively. We endow $\mathcal{R}^n$ with the inner product $\langle \cdot, \cdot \rangle$, which induces a vector norm and a corresponding operator norm, both denoted by $\| \cdot \|$. The dual of S is given by $S^* := \{x \in \mathcal{R}^n \mid \langle x, y \rangle \geq 0 \; \forall \, y \in S\}$, and the orthogonal complement of S is given by $S^{\perp} := \{x \in \mathcal{R}^n \mid \langle x, y \rangle = 0 \; \forall \, y \in S\}$. The (Minkowski) sum of two sets S_1 and S_2 is given by $S_1 + S_2 = \{x_1 + x_2 \mid x_1 \in S_1, \, x_2 \in S_2\}$, and the set difference is given by $S_1 \setminus S_2 = \{x \in S_1 \mid x \notin S_2\}$. The adjoint of a linear map A is denoted by A^*. We denote the space of linear maps from $\mathcal{R}^n$ to $\mathcal{R}^m$ by $\mathcal{R}^{m \times n}$, and denote by I_n the identity map from $\mathcal{R}^n$ to $\mathcal{R}^n$, and the identity matrix of order n. When a linear map $A : \mathcal{R}^n \to \mathcal{R}^m$ is said to be *onto*, it is to be understood that A is onto $\mathcal{R}^m$. As a matrix, A has full row rank. We say that A is *one-to-one* if every point in its range has at most one point in its preimage. As a matrix, such an A has full column rank.

A *cone* $K \subseteq \mathcal{R}^n$ is a nonempty set satisfying $\lambda K \subseteq K$ for all nonnegative λ. If K also satisfies $K + K \subseteq K$, then K is a convex cone. The dual of any set is a closed convex cone. A cone whose interior is nonempty is said to be *solid*. If K contains no nontrivial subspace, i.e., its lineality space $K \cap -K$ is the origin, then K is said to be *pointed*. A cone that is closed, convex, solid, and pointed, is said to be *full*.[1] A pointed convex cone K induces a partial ordering $\succeq_K$, where $x \succeq_K y$ is defined by $x - y \in K$. If K is also solid, we write $x \succ_K y$ to mean $x - y \in \mathrm{int}(K)$. For any solid convex cone K, $K + \mathrm{int}(K) = \mathrm{int}(K)$. Therefore $x_1 \succeq_K y_1$ and $x_2 \succ_K y_2$ implies that $x_1 + x_2 \succ_K y_1 + y_2$. We will use the standard convention that the infimum (supremum) of an empty set is $+\infty$ $(-\infty)$, and the infimum (supremum) of a subset of the real line unbounded from below (above) is $-\infty$ $(+\infty)$. The kernel and range of a finite dimensional linear map A will be denoted by $\mathcal{N}(A)$ and $\mathrm{Range}(A)$ respectively. We will write $A(S) := \{Ax \mid x \in S\}$ to denote the image of a set S under a linear map A. The following results from convex analysis will be used in the sequel.

Lemma 2.1 (e.g., [2, 12]). *Let $A : \mathcal{R}^n \to \mathcal{R}^m$ be a linear map. Let S, S_1, and S_2 be convex subsets of $\mathcal{R}^n$. Then:*
(a) [12, Theorem 6.3] $\mathrm{ri}(\mathrm{cl}(S)) = \mathrm{ri}(S)$ *and* $\mathrm{cl}(\mathrm{ri}(S)) = \mathrm{cl}(S)$*;*
(b) [12, Corollary 6.6.2] $\mathrm{ri}(S_1) + \mathrm{ri}(S_2) = \mathrm{ri}(S_1 + S_2)$*;*
(c) [12, Theorem 6.6] $\mathrm{ri}(A(S)) = A(\mathrm{ri}(S))$*;*
(d) [2, Corollary 2.1] *If S is a convex cone, then $S^{**} = \mathrm{cl}(S)$.* (This result is sometimes called the *bipolar* theorem.) *In particular, if S is also closed, then $S^{**} = S$;*
(e) [2, Corollary 2.2] *If S_1 and S_2 are closed convex cones, then* $\mathrm{cl}(S_1^* + S_2^*) = (S_1 \cap S_2)^*$*;*
(f) [2, Theorem 2.3] *If S is a closed convex cone, then S is pointed if and only if its dual (cone) S^* is solid.* (It follows that S is solid if and only if S^* is pointed.)

[1] Some authors call such a cone *proper*.

3 Universal duality in conic optimization

Consider the following "primal" conic program and its Lagrangian dual:

$$v_P = \inf_x \{\langle f, x\rangle \mid Ax = b,\ Cx \succeq_K d\}, \tag{1}$$

$$v_D = \sup_{y,w} \{\langle b, y\rangle + \langle d, w\rangle \mid A^*y + C^*w = f,\ w \succeq_{K^*} 0\}. \tag{2}$$

Here $A : \mathcal{R}^n \to \mathcal{R}^m$ and $C : \mathcal{R}^n \to \mathcal{R}^p$ are linear maps, and $K \subset \mathcal{R}^p$ is a closed convex cone. Note that three different inner products (on $\mathcal{R}^n$, $\mathcal{R}^m$, and $\mathcal{R}^p$) are used in (1)–(2). The form of (1)–(2) is a generalization of the so-called "standard form" of a conic program (where C is the identity map, $d = 0$, and K is a full cone). We consider the standard form at the end of Section 3.1. The primal formulation (1) can be found in, e.g., [3, Section 4.6.1]. The following two assumptions will be in effect throughout this paper.

Assumption 3.1. *The equality constraints $Ax = b$ and "inequality" constraints $Cx \succeq_K d$ in* (1) *are nonvacuous, i.e., $m, p > 0$. (Of course, it is assumed also that $n > 0$.)*

In Remarks 3.12 and 5.4, we discuss the cases where $m = 0$ or $p = 0$.

Assumption 3.2. *K is a full cone.*

It follows from Lemma 2.1(f) that the closed convex cone K^* is also full.

We now state a well known relation between v_P and v_D, and a definition from [6].[2]

Lemma 3.3. *Weak duality holds for* (1)–(2)*, viz., $v_P \geq v_D$.*

Definition 3.4. *The primal problem* (1) *is said to be* strongly feasible[3] *if $\{x \mid Ax = b,\ Cx \succ_K d\}$ is nonempty. Its dual* (2) *is said to be* strongly feasible *if $\{(y, w) \mid A^*y + C^*w = f,\ w \succ_{K^*} 0\}$ is nonempty.*

The following two results are also well known; c.f. [12, Theorem 30.4].

Lemma 3.5. *If $A, C, K, b, d,$ and f are such that the set of optimal primal solutions or the set of optimal dual solutions is nonempty and bounded, then $v_P = v_D$.*

Lemma 3.6. *Fix $A, C,$ and K. If for some b and d,* (1) *is strongly feasible, then for every f, $v_P = v_D$. If for some f,* (2) *is strongly feasible, then for every b and d, $v_P = v_D$.*

Lemma 3.6 gives conditions under which a zero duality gap occurs for a family of conic problems parameterized by the linear objective function of the primal or dual. We now investigate conditions under which a zero duality gap occurs for every linear objective function *and* every right-hand-side of (1)–(2).

[2] The conic formulation used in [6] is more general than ours.

[3] As it is defined here, strong feasibility is equivalent to the generalized Slater constraint qualification. Some authors refer to strong feasibility as *strict* feasibility.

3.1 Universal duality

We now introduce the main concept of this paper.

Definition 3.7. *Given linear maps $A : \mathcal{R}^n \to \mathcal{R}^m$ and $C : \mathcal{R}^n \to \mathcal{R}^p$, and a full cone $K \subset \mathcal{R}^p$, we say that* universal duality *holds for the triple (A, C, K) if for all choices of b, d, and f, $v_P = v_D$ holds in* (1)–(2). *(A common value of $+\infty$ or $-\infty$ is permitted.)*

The following result is the cornerstone of the relationship between universal duality and feasibility of (1)–(2).

Lemma 3.8. *Consider the following statements.*
(a) *For every b and d,* (1) *is feasible.*
(b) *For every b and d,* (1) *is strongly feasible.*
(c) *For every f,* (2) *is feasible.*
(d) *For every f,* (2) *is strongly feasible.*
(e) *Universal duality holds for (A, C, K).*
We have (a)⇔(b), (c)⇔(d), *and* (e) *holds if and only if at least one of* (a)–(d) *holds.*

Proof. (a)⇔(b) It is clear that (b)⇒(a). To show the converse, suppose that A and C are such that (1) is feasible for every choice of b and d. Let $t \succ_K 0$. (Such a t exists, due to K being solid.) Then

$$\{x \mid Ax = b,\ Cx \succeq_K d\} \subseteq \{x \mid Ax = b,\ Cx \succ_K d - t\} \tag{3}$$

for every b and d. By assumption, the first set in (3) is nonempty for every b and d, so the second set in (3) is also nonempty for every b and d, and hence statement (b) holds.

(c)⇔(d) It is clear that (d)⇒(c). To show the converse, suppose that A and C are such that (2) is feasible for every choice of f. That is, $\text{Range}(A^*) + C^*(K^*) = \mathcal{R}^n$. Hence $\text{ri}(\text{Range}(A^*) + C^*(K^*)) = \mathcal{R}^n$. It follows from Lemma 2.1(b),(c) that $\text{Range}(A^*) + C^*(\text{ri}(K^*)) = \mathcal{R}^n$, which is strong feasibility of (2) for every f.

We now show that (e) holds if and only if at least one of (a)–(d) holds.
(⇒) Suppose universal duality holds for (A, C, K), but it does not hold that (1) is feasible for all choices of b and d. Then there exist vectors $b = \bar{b}$ and $d = \bar{d}$ such that (1) is infeasible, i.e., $v_P = \infty$ for $b = \bar{b}$, $d = \bar{d}$, and every f. So universal duality implies that $v_D = \infty$ for $b = \bar{b}$, $d = \bar{d}$, and every f, which proves that (2) is feasible for all f. We conclude that (e) implies (a) or (c).
(⇐) It follows from Lemma 3.6 that (b)⇒(e) and (d)⇒(e), and we have proved that (a)⇒(b) and (c)⇒(d). ■

Given A, C, and K in (1)–(2), it will be convenient to use the notation

$$\boxed{S_{\rm o}(C, K) := \{x \mid Cx \succ_K 0\}, \qquad S_{\rm c}(C, K) := \{x \mid Cx \succeq_K 0\}.}$$

(The subscripts o and c remind the reader that $S_{\rm o}(C, K)$ is open and $S_{\rm c}(C, K)$ is closed.) It will also be convenient to define the following conditions:

Property $\mathrm{P_o}(A, C, K)$:	$\mathcal{N}(A) \cap S_\mathrm{o}(C, K)$ is nonempty, and A is onto;
Property $\mathrm{P_c}(A, C, K)$:	$\mathcal{N}(A) \cap S_\mathrm{c}(C, K) = \{0\}$.

Note that properties $\mathrm{P_o}(A, C, K)$ and $\mathrm{P_c}(A, C, K)$ are mutually exclusive. We will see in Theorem 3.15 that property $\mathrm{P_o}(A, C, K)$ (resp. $\mathrm{P_c}(A, C, K)$) holds if and only if the feasible set of (2) (resp. (1)) is bounded for every right-hand-side.

The following result will be used.

Lemma 3.9. *Let $C : \mathcal{R}^n \to \mathcal{R}^p$ be a linear map and let $K \subset \mathcal{R}^p$ be a full cone. If $S_\mathrm{o}(C, K)$ is nonempty, then $S_\mathrm{o}(C, K) = \mathrm{int}(S_\mathrm{c}(C, K))$.*

Proof. Suppose that C and K are such that $S_\mathrm{o}(C, K)$ is nonempty, and let $S_\mathrm{o} := S_\mathrm{o}(C, K)$ and $S_\mathrm{c} := S_\mathrm{c}(C, K)$. Clearly $S_\mathrm{o} = \mathrm{int}(S_\mathrm{o}) \subseteq \mathrm{int}(S_\mathrm{c})$, so it suffices to show that $\mathrm{int}(S_\mathrm{c}) \subseteq S_\mathrm{o}$. To prove this, let $x_\mathrm{c} \in \mathrm{int}(S_\mathrm{c})$ and $x_\mathrm{o} \in S_\mathrm{o}$. Then there exists $\alpha > 0$ such that $x_\mathrm{c} - \alpha x_\mathrm{o} \in S_\mathrm{c}$, i.e., $C(x_\mathrm{c} - \alpha x_\mathrm{o}) \succeq_K 0$. Since $S_\mathrm{o} \cup \{0\}$ is a cone, then $\alpha x_\mathrm{o} \in S_\mathrm{o}$, i.e., $C(\alpha x_\mathrm{o}) \succ_K 0$. Since $x_\mathrm{c} = (x_\mathrm{c} - \alpha x_\mathrm{o}) + \alpha x_\mathrm{o}$, it follows that $C x_\mathrm{c} \succ_K 0$, i.e., $x_\mathrm{c} \in S_\mathrm{o}$. ■

The following result will allow us to characterize universal duality more explicitly than was done in Lemma 3.8.

Lemma 3.10. *Given A, C, and K:*
(a) *Property $\mathrm{P_o}(A, C, K)$ holds if and only if (1) is strongly feasible for every b and d;*
(b) *Property $\mathrm{P_c}(A, C, K)$ holds if and only if (2) is strongly feasible for every f.*

Proof. (a) Suppose that property $\mathrm{P_o}(A, C, K)$ holds, and let $x \in \mathcal{N}(A) \cap S_\mathrm{o}(C, K)$. Since A is onto, then for every b, there exists $x_0 \in \mathcal{R}^n$ such that $Ax_0 = b$. Since $x \in S_\mathrm{o}(C, K)$, then for every d, and for $\alpha > 0$ sufficiently large,

$$C(x_0 + \alpha x) - d = \alpha\big(Cx + \frac{1}{\alpha}(Cx_0 - d)\big) \in \mathrm{int}(K).$$

Since $x \in \mathcal{N}(A)$, then $A(x_0 + \alpha x) = b$, so (1) is strongly feasible for every b and d.

To show the converse, assume that for every b and d, (1) is strongly feasible. Clearly A must then be onto. Moreover, strong feasibility of (1) for $b = 0$ and $d = 0$ means that $\mathcal{N}(A) \cap S_\mathrm{o}(C, K)$ is nonempty. Hence property $\mathrm{P_o}(A, C, K)$ holds.

(b) Let $T := C^*(K^*)$. We claim that $S_\mathrm{c}(C, K)^* = \mathrm{cl}(T)$. Indeed,

$$\begin{aligned} T^* &= \{x \mid \langle x, y\rangle \geq 0 \;\forall\, y \in C^*(K^*)\} \\ &= \{x \mid \langle x, C^*w\rangle \geq 0 \;\forall\, w \in K^*\} \\ &= \{x \mid \langle Cx, w\rangle \geq 0 \;\forall\, w \in K^*\} \\ &= \{x \mid Cx \in K\} \\ &= S_\mathrm{c}(C, K), \end{aligned}$$

and the claim follows from Lemma 2.1(d). In view of Lemma 2.1(a),(c),

$$\mathrm{ri}(S_\mathrm{c}(C, K)^*) = \mathrm{ri}(T) = C^*(\mathrm{ri}(K^*)). \tag{4}$$

Now since $\mathcal{N}(A)$ and $S_c(C,K)$ are closed convex cones, we have from Lemma 2.1(e) that

$$(\mathcal{N}(A) \cap S_c(C,K))^* = \mathrm{cl}(\mathrm{Range}(A^*) + S_c(C,K)^*). \quad (5)$$

Using Lemma 2.1(a),(b), and (c), it follows from (4)–(5) that

$$\mathrm{ri}((\mathcal{N}(A) \cap S_c(C,K))^*) = \mathrm{ri}(\mathrm{Range}(A^*) + S_c(C,K)^*) = \mathrm{Range}(A^*) + C^*(\mathrm{ri}(K^*)). \quad (6)$$

Since $\{0\}^* = \mathcal{R}^n$, it follows from (6) that property $\mathrm{P_c}(A,C,K)$ holds if and only if $\mathcal{R}^n = \mathrm{Range}(A^*) + C^*(\mathrm{ri}(K^*))$, which is strong feasibility of (2) for every f. ∎

The above leads to the main result of this section.

Theorem 3.11 (Universal duality for conic optimization). *Universal duality holds for (A,C,K) if and only if either property $\mathrm{P_o}(A,C,K)$ or property $\mathrm{P_c}(A,C,K)$ holds.*

Proof. Follows from Lemmas 3.8 and 3.10. ∎

Remark 3.12. *Theorem 3.11 still applies when $m = 0$ or $p = 0$, under appropriate conventions. We will adopt the convention that if $m = 0$, then A is onto and $\mathcal{N}(A) = \mathcal{R}^n$. Properties $\mathrm{P_o}(A,C,K)$ and $\mathrm{P_c}(A,C,K)$ then become property $\mathrm{P'_o}(C,K)$: $S_o(C,K)$ is nonempty, and property $\mathrm{P'_c}(C,K)$: $S_c(C,K) = \{0\}$, respectively. Properties $\mathrm{P'_o}(C,K)$ and $\mathrm{P'_c}(C,K)$ are mutually exclusive. Further, we will adopt the convention that if $p = 0$, then $K = \mathcal{R}^p = \{0\}$, and $S_o(C,K) = S_c(C,K) = \mathcal{R}^n$. Properties $\mathrm{P_o}(A,C,K)$ and $\mathrm{P_c}(A,C,K)$ then become property $\mathrm{P''_o}(A)$: A is onto, and property $\mathrm{P''_c}(A)$: A is one-to-one, respectively. If $p = 0$ and A is invertible (so that $m = n$), then clearly properties $\mathrm{P''_o}(A)$ and $\mathrm{P''_c}(A)$ both hold. Otherwise both properties are mutually exclusive. Under these conventions, Theorem 3.11 holds when $m = 0$ or $p = 0$, with properties $\mathrm{P_o}(A,C,K)$ and $\mathrm{P_c}(A,C,K)$ replaced by their primed versions defined above.*

Figure 1 shows three possible geometrical positions for the sets $\mathcal{N}(A)$ and $S_c(C,K)$, for an instance in which $S_o(C,K)$ is nonempty.

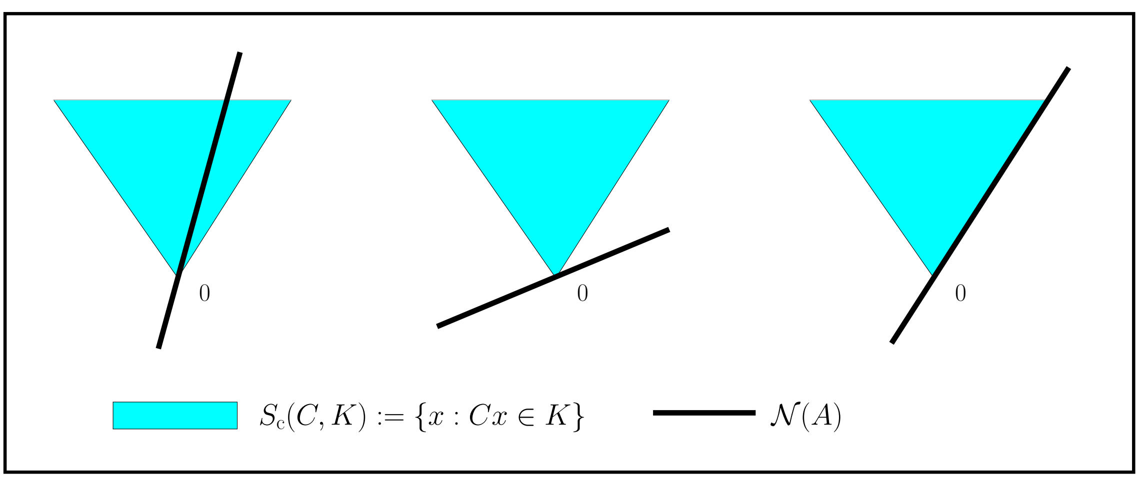

Figure 1: *The first and second plots illustrate how A, C, and K can be such that properties* $\mathrm{P_o}(A, C, K)$ *and* $\mathrm{P_c}(A, C, K)$ *hold respectively. The third plot shows how both properties can fail to hold.*

To conclude this subsection, we specialize the universal duality characterizations we have obtained for (1)–(2) to the case where our primal-dual pair of conic problems is in "standard form":

$$\inf_{x} \{\langle f, x\rangle \mid Ax = b,\ x \succeq_K 0\}, \tag{7}$$

$$\sup_{y,w} \{\langle b, y\rangle \mid A^*y + w = f,\ w \succeq_{K^*} 0\}. \tag{8}$$

(The quantities in (7)–(8) are the same as those defined for (1)–(2). In particular, K is still a full cone.) If (7)–(8) admit a zero duality gap for every b and f, we shall say that *homogeneous universal duality* holds for (A, K). It is clear that universal duality for (A, I_n, K) implies homogeneous universal duality for (A, K). It turns out that the converse also holds.

Theorem 3.13. *Universal duality holds for* (A, I_n, K) *if and only if homogeneous universal duality holds for* (A, K).

Proof. It is enough to prove the (contrapositive of the) "if" direction of the theorem. Suppose that universal duality does not hold for (A, I_n, K). Then for some b, d and f, (1)–(2) with $C = I_n$ exhibits a nonzero duality gap. Now consider (1)–(2) with $C = I_n$, and let $\hat{x} = x - d$ and $\hat{b} = b - Ad$. Then the primal equality constraints become $A\hat{x} = \hat{b}$, and the primal objective function becomes $\langle f, \hat{x}\rangle + \langle f, d\rangle$. Noting that $w = f - A^*y$, we can write the dual objective function as $\langle \hat{b}, y\rangle + \langle f, d\rangle$. So (1) and (2) take the form of (7) and (8) respectively, except for the addition of a common constant term $\langle f, d\rangle$ in each objective function. Hence homogeneous universal duality does not hold for (A, K). ■

Corollary 3.14. *Homogeneous universal duality holds for* (A, K) *if and only if either*
(i) $\mathcal{N}(A) \cap \operatorname{int}(K)$ *is nonempty, and* A *is onto, or*
(ii) $\mathcal{N}(A) \cap K = \{0\}$.

3.2 Universal duality and the boundedness of primal and dual feasible sets

We now prove a result that is of interest in its own right. It shows that the feasible sets of (1) and (2) cannot both be bounded (unless they are both empty). One implication is that when solving a conic program, we cannot bound the size of the primal and dual feasible sets a priori. This result has been proved for linear programs in [14, Corollary II.15], and it is stated in [10, p. 660] that the result holds for semidefinite programs. To the authors' knowledge, such a result for conic programs does not exist in the literature. In fact the result is a simple corollary of a stronger result that connects the boundedness of the primal and dual feasible sets to properties $\mathrm{P_o}(A, C, K)$ and $\mathrm{P_c}(A, C, K)$. We show this connection below, and then relate these results to universal duality via Theorem 3.11.

Denote the feasible sets of (1) and (2) by $\mathcal{F}_P$ and $\mathcal{F}_D$ respectively:

$$\begin{aligned}\mathcal{F}_P &:= \{x \mid Ax = b,\ Cx \succeq_K d\},\\ \mathcal{F}_D &:= \{(y,w) \mid A^*y + C^*w = f,\ w \succeq_{K^*} 0\}.\end{aligned}$$

The following two results link properties $\mathrm{P_o}(A,C,K)$ and $\mathrm{P_c}(A,C,K)$ to the boundedness of $\mathcal{F}_P$ and $\mathcal{F}_D$.

Theorem 3.15. (i) $\mathrm{P_c}(A,C,K)$ *fails if and only if* $\mathcal{F}_P$ *is unbounded for every* b *and* d *for which it is nonempty.*
(ii) $\mathrm{P_o}(A,C,K)$ *fails if and only if* $\mathcal{F}_D$ *is unbounded for every* f *for which it is nonempty.*

Proof. We will use the fact that a nonempty closed convex set is unbounded if and only if it contains a (nonzero) recession direction. (This follows from [12, Theorem 8.4].)
(i) Suppose $\mathrm{P_c}(A,C,K)$ fails. Then there exists $x \neq 0$ such that

$$Ax = 0 \quad \text{and} \quad Cx \succeq_K 0,$$

and this is a recession direction for $\mathcal{F}_D$ whenever that set is nonempty. To prove the converse, let $(b,d) = (0,0)$. Then $\mathcal{F}_P$ is nonempty, and if it is also unbounded, then $\mathcal{F}_P$ contains a recession direction. Hence $\mathrm{P_c}(A,C,K)$ fails.

(ii) We will prove the contrapositive of the statement given in the theorem. From Lemma 3.10, we see that property $\mathrm{P_o}(A,C,K)$ holds if and only if $v_P < \infty$ for every b, d, and f. We now show that $v_P < \infty$ for every b, d, and f, is equivalent to $v_D < \infty$ for every b, d, and f. The forward implication is clear from weak duality. To prove the reverse implication, suppose that $v_D < \infty$ for every b, d, and f. Then for $f = 0$, the nonempty set $\mathcal{F}_D$ is bounded. So for every b and d, and $f = 0$, it follows from Lemma 3.5 that $v_P = v_D$, and hence $v_P < \infty$. Since this implies that (1) is feasible for every b and d, and the feasibility of (1) is independent of f, we have shown that $v_P < \infty$ for every b, d, and f.

Now $v_D < \infty$ for every b, d, and f, is equivalent to the nonexistence of a recession direction for $\mathcal{F}_D$. So we have proved that property $\mathrm{P_o}(A,C,K)$ fails if and only if $\mathcal{F}_D$ contains a recession direction whenever it is nonempty. ■

Theorem 3.16. (a) *If property* $\mathrm{P_o}(A,C,K)$ *holds, then for every* b, d, *and* f, $\mathcal{F}_P$ *is unbounded and* $\mathcal{F}_D$ *is bounded (and possibly empty).*
(b) *If property* $\mathrm{P_c}(A,C,K)$ *holds, then for every* b, d, *and* f, $\mathcal{F}_D$ *is unbounded and* $\mathcal{F}_P$ *is bounded (and possibly empty).*
(c) *If properties* $\mathrm{P_o}(A,C,K)$ *and* $\mathrm{P_c}(A,C,K)$ *both fail, then for every* b, d, *and* f, *both* $\mathcal{F}_P$ *and* $\mathcal{F}_D$ *are unbounded or empty.*

Proof. To prove (a), observe that the nonemptiness of $\mathcal{F}_P$ for every b and d is guaranteed by Lemma 3.10(a). Since properties $\mathrm{P_o}(A,C,K)$ and $\mathrm{P_c}(A,C,K)$ are mutually exclusive, it must be the case that $\mathrm{P_c}(A,C,K)$ fails. It follows from Theorem 3.15(i) that for every b and d, $\mathcal{F}_P$ is unbounded. The contrapositive of Theorem 3.15(ii) implies that $\mathcal{F}_D$ is nonempty and bounded for some f. It follows that for every f, $\mathcal{F}_D$ contains no recession direction, and is therefore bounded (though possibly empty). This concludes the proof of statement (a). Statement (b) is proved similarly. Statement (c) follows immediately from Theorem 3.15. ■

Corollary 3.17. *Given $A, C, K, b, d,$ and f, if $\mathcal{F}_P$ and $\mathcal{F}_D$ are not both empty, then at least one of these sets is unbounded.*

Finally, we relate the boundedness of $\mathcal{F}_P$ and $\mathcal{F}_D$ to universal duality.

Theorem 3.18. *Fix A, C, and K.* (a) *If for some b, d, and f, either $\mathcal{F}_P$ or $\mathcal{F}_D$ is nonempty and bounded, then universal duality holds for (A, C, K).*
(b) *If universal duality holds for (A, C, K), then one of $\mathcal{F}_P$ and $\mathcal{F}_D$ is unbounded for every b, d, and f, and the other is bounded (and possibly empty) for every b, d, and f.*

Proof. (a) If either $\mathcal{F}_P$ or $\mathcal{F}_D$ is nonempty and bounded, then it follows from Theorem 3.15 that either property $\mathrm{P_c}(A, C, K)$ or $\mathrm{P_o}(A, C, K)$ holds respectively. So by Theorem 3.11, universal duality holds for (A, C, K).
(b) If universal duality holds for (A, C, K), then exactly one of properties $\mathrm{P_o}(A, C, K)$ and $\mathrm{P_c}(A, C, K)$ holds. Hence either statement (a) or (b) in Theorem 3.16 holds. ■

It is of interest to compare statement (a) in Theorem 3.18 with the result that for a pair of dual conic programs, if the set of primal or dual *optimal* solutions is nonempty and bounded, then a zero duality gap results (Lemma 3.5).

4 Generic properties of universal duality

On a Euclidean space X, we can speak of a *metrically generic* property, which holds at "almost all" points in X, or a *topologically generic* property, which holds on a *residual set* in X. Here, "almost all" is in the sense of Lebesgue measure, and a residual set is the complement of a countable union of *nowhere dense* sets. (A nowhere dense set is a set whose closure has empty interior.)[4] We will take X to be $\mathcal{R}^{m\times n} \times \mathcal{R}^{p\times n}$, since this is the domain of the pair of linear maps (A, C). We will show that for a fixed full cone K, universal duality for (A, C, K) is both metrically generic and topologically generic on X. In fact we prove a stronger result than topological genericness: universal duality holds on an *open dense* set of pairs (A, C).

4.1 Metric genericness of universal duality

In showing that universal duality is metrically generic, we will use several lemmas. The first two results are well known; the first follows from Fubini's theorem (see e.g., [5, p. 147, Theorem A]).

Lemma 4.1. *A Lebesgue measurable set $W \subseteq \mathcal{R}^m \times \mathcal{R}^n$ has zero Lebesgue measure if and only if the set $\{x \in \mathcal{R}^m \mid (x, y) \in W\}$ has zero Lebesgue measure for Lebesgue almost every $y \in \mathcal{R}^n$.*

Lemma 4.2. *The set of matrices in $\mathcal{R}^{m\times n}$ containing a square singular submatrix has zero Lebesgue measure. In particular, the set of rank deficient matrices in $\mathcal{R}^{m\times n}$ has zero Lebesgue measure.*

[4]Neither type of genericness is implied by the other. The terminology "topologically generic" and "metrically generic" can be found in [16].

The next lemma is proved in the appendix.

Lemma 4.3. *Let $S \subseteq \mathcal{R}^n$ be a solid closed convex cone, and let p be a positive integer. Then the sets*

$$\mathcal{M}_1 := \{M \in \mathcal{R}^{p\times n} \mid \mathcal{N}(M) \cap \mathrm{int}(S) \text{ is empty, and } \mathcal{N}(M) \cap S \neq \{0\}\}$$

and

$$\mathcal{M}_2 := \{M \in \mathcal{R}^{n\times p} \mid \mathrm{Range}(M) \cap \mathrm{int}(S) \text{ is empty, and } \mathrm{Range}(M) \cap S \neq \{0\}\}$$

have zero Lebesgue measure.

To prove that universal duality is a metrically generic property, we will show that for a fixed full cone K, the set of matrix pairs (A, C) such that properties $\mathrm{P_o}(A, C, K)$ and $\mathrm{P_c}(A, C, K)$ both fail, has measure zero. (See Theorem 3.11.)

Theorem 4.4. *Universal duality is metrically generic. Specifically, given a full cone K, the set of pairs (A, C) such that universal duality fails to hold for (A, C, K) has zero Lebesgue measure in $\mathcal{R}^{m\times n} \times \mathcal{R}^{p\times n}$.*

Proof. Let T be the set of pairs (A, C) such that universal duality fails to hold for (A, C, K). We consider the two cases $m \geq n$ and $m < n$.

First, suppose that $m \geq n$. Then by Theorem 3.11 we have

$$\begin{aligned} T &\subseteq \{(A, C) \mid \text{property } \mathrm{P_c}(A, C, K) \text{ fails}\} \\ &= \{(A, C) \mid \mathcal{N}(A) \cap S_\mathrm{c}(C, K) \neq \{0\}\ \} \\ &\subseteq \{A \in \mathcal{R}^{m\times n} \mid \mathcal{N}(A) \neq \{0\}\ \} \times \mathcal{R}^{p\times n} \\ &= \{A \in \mathcal{R}^{m\times n} \mid \mathrm{rank}(A) < n\} \times \mathcal{R}^{p\times n}. \end{aligned}$$

It follows from Lemma 4.2 that $\{A \in \mathcal{R}^{m\times n} \mid \mathrm{rank}(A) < n\}$ has zero Lebesgue measure, and then from Lemma 4.1 that T has zero Lebesgue measure.

Suppose now $m < n$. Consider the following conditions on A and C:

(i) $\mathcal{N}(A) \cap S_\mathrm{o}(C, K)$ is empty, (ii) $\mathcal{N}(A) \cap S_\mathrm{c}(C, K) \neq \{0\}$.

Noting the relationship between these conditions and properties $\mathrm{P_o}(A, C, K)$ and $\mathrm{P_c}(A, C, K)$, we see that Theorem 3.11 implies

$$T \subseteq \{(A, C) \mid A \text{ is not onto}\} \bigcup \{(A, C) \mid \text{(i) and (ii) hold}\}. \tag{9}$$

The first set on the right-hand-side of (9) has zero Lebesgue measure by Lemmas 4.1 and 4.2. We now proceed to show the same for the second set. So define

$$T_1 := \{(A, C) \mid \text{(i) and (ii) hold}\}.$$

We can write $T_1 = T_2 \cup T_3$, where

$$T_2 := T_1 \bigcap \{(A,C) \mid S_o(C,K) \text{ is empty}\}, \qquad T_3 := T_1 \bigcap \{(A,C) \mid S_o(C,K) \text{ is nonempty}\}.$$

It suffices to show that T_2 and T_3 have zero Lebesgue measure. In view of Lemmas 4.1 and 4.2, we can restrict our attention to matrices C (and A) having full rank. If $\text{rank}(C) = p$, then $\text{Range}(C) = \mathcal{R}^p$, so that $S_o(C,K)$ is nonempty. Therefore we can assume that any C such that $(A,C) \in T_2$ satisfies $\text{rank}(C) = n < p$. Now

$$\begin{aligned} T_2 &= \{(A,C) \mid S_o(C,K) \text{ is empty, and (ii) holds}\} \\ &\subseteq \{(A,C) \mid S_o(C,K) \text{ is empty, and } S_c(C,K) \neq \{0\}\} \\ &= \{(A,C) \mid \text{Range}(C) \cap \text{int}(K) \text{ is empty, and } \text{Range}(C) \cap K \neq \{0\}\}, \end{aligned}$$

where the last equality holds due to C having full column rank. (This condition implies that $S_c(C,K) \neq \{0\}$ if and only if $\text{Range}(C) \cap K \neq \{0\}$.) It follows from Lemmas 4.3 and 4.1 that T_2 has zero Lebesgue measure. Now in view of Lemma 3.9, any C satisfying $(A,C) \in T_3$ will also satisfy $S_o(C,K) = \text{int}(S_c(C,K))$, and hence $S_c(C,K)$ will be solid. So

$$T_3 \subseteq \{(A,C) \mid S_c(C,K) \text{ is solid}, \mathcal{N}(A) \cap \text{int}(S_c(C,K)) \text{ is empty}, \mathcal{N}(A) \cap S_c(C,K) \neq \{0\}\}.$$

It follows from Lemmas 4.3 and 4.1 that T_3 has zero Lebesgue measure. ■

4.2 Topological genericness of universal duality

Theorem 4.5. *Universal duality is topologically generic. In fact, given a full cone K, the set of pairs (A,C) for which universal duality holds for (A,C,K) is open and dense in $\mathcal{R}^{m\times n} \times \mathcal{R}^{p\times n}$.*

Proof. The complement of a set having zero Lebesgue measure is dense. (If not, then that set would contain an open hypercube, which must have positive measure.) So Theorem 4.4 implies that the set of pairs (A,C) such that universal duality holds for (A,C,K) is dense in $\mathcal{R}^{m\times n} \times \mathcal{R}^{p\times n}$. We now show that:
(a) The set of pairs (A,C) such that property $\text{P}_o(A,C,K)$ holds is open in $\mathcal{R}^{m\times n} \times \mathcal{R}^{p\times n}$; and
(b) The set of pairs (A,C) such that property $\text{P}_c(A,C,K)$ holds is open in $\mathcal{R}^{m\times n} \times \mathcal{R}^{p\times n}$.

To prove (a), suppose that (A,C) is such that property $\text{P}_o(A,C,K)$ holds. When $m > n$, A cannot be onto, so it must be the case that $m \leq n$. Further, if $m = n$, then $\mathcal{N}(A) \cap S_o(C,K)$ is empty whenever A is onto, so property $\text{P}_o(A,C,K)$ fails to hold. Hence $m < n$. Now let $\{(A^i, C^i)\}_i$ be an infinite sequence such that $(A^i, C^i) \to (A,C)$. Since the set of full rank matrices is open, then A^i is onto for i large enough. So it is enough to show that $\mathcal{N}(A^i) \cap S_o(C^i, K)$ is nonempty for i large enough. Let $x \in \mathcal{N}(A) \cap S_o(C,K)$, and let x^i be the orthogonal projection of x onto $\mathcal{N}(A^i)$. Then $\lim_{i\to\infty} x^i = x$. Now writing $C^i x^i - Cx = C^i(x^i - x) + (C^i - C)x$, we have

$$\|C^i x^i - Cx\| \leq \|C^i\| \|x^i - x\| + \|C^i - C\| \|x\|. \tag{10}$$

As $i \to \infty$, the right-hand-side of (10), and hence the left-hand-side, tends to zero. It follows from $Cx \succ_K 0$ and the openness of $\text{S}_o(C,K)$, that $C^i x^i \succ_K 0$ for i large enough.

That is, $x^i \in \mathcal{N}(A^i) \cap \mathrm{S_o}(C^i, K)$ for such i. This proves statement (a).

To prove (b), let S be the set of pairs (A, C) such that property $\mathrm{P_c}(A, C, K)$ holds. Proceeding by contradiction, we suppose that S is not open in $\mathcal{R}^{m\times n} \times \mathcal{R}^{p\times n}$. Then for some $(A, C) \in S$, there exists a sequence $\{(A^i, C^i)\}_i$ with $(A^i, C^i) \notin S$ for all i, but $(A^i, C^i) \to (A, C)$. So for each i, there exists a nonzero $x^i \in \mathcal{N}(A^i) \cap S_\mathrm{c}(C^i, K)$. Now set $y^i = x^i/\|x^i\|$, so that $\|y^i\| = 1$, $A^i y^i = 0$ and $C^i y^i \succeq_K 0$ for all i. Since $\{y^i\}$ is a bounded sequence, it contains a convergent subsequence. Passing to such a subsequence if necessary, we conclude that there exists a limit point $y \neq 0$. Since K is closed, $y \in \mathcal{N}(A) \cap S_\mathrm{c}(C, K)$, so that $(A, C) \notin S$—a contradiction. ■

5 Verifying universal duality

We show that universal duality for (A, C, K) can be checked by solving a single conic program with essentially the same size and "structure" as that in (1). We first prove two lemmas.

Lemma 5.1. *Let S be any set in Euclidean space such that S^* (defined with respect to the inner product $\langle\cdot,\cdot\rangle$) has a nonempty interior. Then for any $y \in S$ and $z \in \mathrm{int}(S^*)$, $\langle y, z\rangle \leq 0$ implies that $y = 0$.*

Proof. Let $y \in S$ and $z \in \mathrm{int}(S^*)$ be such that $\langle y, z\rangle \leq 0$. Choose $\epsilon > 0$ such that $z - \epsilon y \in \mathrm{int}(S^*)$. Then

$$0 \leq \langle y, z - \epsilon y\rangle = \langle y, z\rangle - \epsilon\langle y, y\rangle \leq -\epsilon\langle y, y\rangle.$$

It follows that $y = 0$. ■

Lemma 5.2. *If $K_0 \subset \mathcal{R}^p$ is a full cone, then $\mathrm{int}(K_0) \cap \mathrm{int}(K_0^*)$ is nonempty.*

Proof. Since K_0, hence K_0^*, is a full cone, there exists $0 \neq \bar{d} \in \mathcal{R}^p$ such that both K_0 and K_0^* lie in the upper half space $H := \{x \mid \langle \bar{d}, x\rangle \geq 0\}$ of the hyperplane $\langle \bar{d}, x\rangle = 0$, and $K_0 \cap H = K_0^* \cap H = \{0\}$. (If no such $\bar{d}$ exists, then there exists $\ell \neq 0$ such that $\ell \in K_0$ but $-\ell \in K_0^*$. So $\langle \ell, -\ell\rangle \geq 0$, which is impossible.) Now $K_0 \cap H = \{0\}$ implies that $\bar{d} \in \{d \mid \langle d, y\rangle > 0 \ \forall\, y \in K_0 \setminus \{0\}\,\} = \mathrm{int}(K_0^*)$, and $K_0^* \cap H = \{0\}$ implies that $\bar{d} \in \{d \mid \langle d, y\rangle > 0 \ \forall\, y \in K_0^* \setminus \{0\}\,\} = \mathrm{int}(K_0)$. Hence $\bar{d} \in \mathrm{int}(K_0) \cap \mathrm{int}(K_0^*)$. ■

We now show how properties $\mathrm{P_o}(A, C, K)$ and $\mathrm{P_c}(A, C, K)$, and hence universal duality, can be verified by solving a single conic program.

Theorem 5.3. *Let $e \in \mathrm{int}(K) \cap \mathrm{int}(K^*)$. Universal duality for (A, C, K) can be verified by solving the conic program*

$$\bar{r} = \sup_{x,\, r} \{r \mid Ax = 0,\ Cx \succeq_K re,\ \langle Cx, e\rangle = 1\}, \qquad (11)$$

where the inner product $\langle \cdot, \cdot \rangle$ is that defined on $\mathcal{R}^p$ in (2). Specifically,[5]
(a) *Property* $\mathrm{P_o}(A, C, K)$ *holds if and only if* $\bar{r} > 0$ *and* A *is onto;*
(b) *Property* $\mathrm{P_c}(A, C, K)$ *holds if and only if* $\bar{r} < 0$ *and* $\mathcal{N}(A) \cap \mathcal{N}(C) = \{0\}$.

Proof. We first show that (a) holds. It suffices to show that $\mathcal{N}(A) \cap S_\mathrm{o}(C, K)$ is nonempty if and only if $\bar{r} > 0$.
($\Rightarrow$) The nonemptiness of $\mathcal{N}(A) \cap S_\mathrm{o}(C, K)$ implies that there exists $\tilde{x}$ such that $A\tilde{x} = 0$ and $C\tilde{x} \succ_K 0$. Hence $C\tilde{x} - \tilde{r}e \succ_K 0$ for some $\tilde{r} > 0$ sufficiently small. Since $C\tilde{x} - \tilde{r}e \in K$ and $e \in K^*$, we have $k := \langle C\tilde{x}, e\rangle = \langle C\tilde{x} - \tilde{r}e, e\rangle + \tilde{r}\langle e, e\rangle > 0$. Hence $(\tilde{x}/k, \tilde{r}/k)$ is feasible for (11), and $\bar{r} \geq \tilde{r}/k > 0$.
($\Leftarrow$) Suppose $\bar{r} > 0$. So there exists $\tilde{r} > 0$ and $\tilde{x}$ such that $A\tilde{x} = 0$ and $C\tilde{x} \succeq_K \tilde{r}e \succ_K 0$. It follows that $\mathcal{N}(A) \cap S_\mathrm{o}(C, K)$ is nonempty.

We now prove statement (b).
($\Rightarrow$) Suppose that $\mathrm{P_c}(A, C, K)$ holds. Since $\mathcal{N}(A) \cap \mathcal{N}(C)$ is a linear subspace contained in $\mathcal{N}(A) \cap S_\mathrm{c}(C, K) = \{0\}$, then $\mathcal{N}(A) \cap \mathcal{N}(C) = \{0\}$. It remains to prove that $\bar{r} < 0$.

If (x, r) with $r \geq 0$ satisfies the constraints $Ax = 0$ and $Cx \succeq_K re (\succeq_K 0)$ in (11), then $x \in \mathcal{N}(A) \cap S_\mathrm{c}(C, K)$. Since $\mathrm{P_c}(A, C, K)$ holds, we must have $x = 0$, but this violates the constraint $\langle Cx, e\rangle = 1$. Hence every pair (x, r) with $r \geq 0$ is infeasible for (11). It follows that $\bar{r} \leq 0$. We now rule out the case $\bar{r} = 0$.

If (11) is infeasible, there is nothing to prove, so suppose that (11) is feasible for $(\hat{x}, \hat{r})$ with $\hat{r} < 0$. Consider the set T of feasible points (x, r) satisfying $\hat{r} \leq r \leq 0$. Suppose there exists a recession direction $(d_x, d_r) \in \mathcal{R}^n \times \mathcal{R}$ for T. Since r is bounded in T, $d_r = 0$, and d_x satisfies $Ad_x = 0$, $Cd_x \succeq_K 0$, and $\langle Cd_x, e\rangle = 0$. Since $e \succ_{K^*} 0$, then by Lemma 5.1, the last two conditions imply that $Cd_x = 0$. So $d_x \in \mathcal{N}(A) \cap \mathcal{N}(C)$, which was shown to be trivial. Hence T is nonempty and bounded. It is clear that the feasible set of (11), and hence T, is closed. So $\bar{r}$, the supremum of a linear function over a compact set, is achieved. Since (x, r) is infeasible for every $r \geq 0$, it follows that $\bar{r} < 0$.

($\Leftarrow$) If $\bar{r} < 0$, then $r = 0$ is infeasible for (11), so there does not exist an x such that $Ax = 0$, $Cx \succeq_K 0$, and $\langle Cx, e\rangle > 0$. That is, any x satisfying $Ax = 0$ and $Cx \succeq_K 0$ must also satisfy $\langle Cx, e\rangle \leq 0$, which implies $Cx = 0$ by Lemma 5.1, since $e \succ_{K^*} 0$. In other words, $\mathcal{N}(A) \cap S_\mathrm{c}(C, K) = \mathcal{N}(A) \cap \mathcal{N}(C)$. Since $\mathcal{N}(A) \cap \mathcal{N}(C) = \{0\}$, property $\mathrm{P_c}(A, C, K)$ holds. ■

Remark 5.4. *If* $p = 0$, *then Remark 3.12—with properties* $\mathrm{P_o}(A, C, K)$ *and* $\mathrm{P_c}(A, C, K)$ *replaced by* $\mathrm{P''_o}(A)$ *and* $\mathrm{P''_c}(A)$*—tells us that universal duality for a pair of dual problems containing linear equality constraints only, occurs if and only if* A *is onto or one-to-one. Of course, there is no need to solve a conic program to verify whether* A *satisfies these conditions. If* $m = 0$, *then Theorem 5.3—with properties* $\mathrm{P_o}(A, C, K)$ *and* $\mathrm{P_c}(A, C, K)$ *replaced by* $\mathrm{P'_o}(C, K)$ *and* $\mathrm{P'_c}(C, K)$*—holds under the convention specified in Remark 3.12.*

[5] The set of instances for which $\bar{r} < 0$ includes those for which (11) is infeasible, i.e., $\bar{r} = -\infty$. In contrast, it is not possible for A, C, and K to be such that $\bar{r} = +\infty$. In fact the constraints in (11) imply that $\bar{r} \leq 1/\|e\|^2$.

6 Conclusions

Given a pair of dual conic convex problems, we introduced the concept of universal duality, which is said to hold if a zero duality gap occurs for every linear objective function and "right-hand-side". We showed that in conic convex optimization, there exist simple necessary and sufficient conditions on the "constraint matrices" and cone that guarantee universal duality. We also gave a relationship between universal duality for conic optimization and boundedness of the primal and dual feasible sets. In connection to this, we showed that the feasible sets of a pair of dual conic programs cannot both be bounded (unless they are both empty). In addition, universal duality holds Lebesgue almost always, and holds on an open, dense set of "constraint matrices". Finally, it was shown that universal duality can be verified by solving a single conic optimization problem.

One application of universal duality lies in duality theory for certain minimax problems whose variables are constrained to lie in a convex set. It is possible to use the universal duality framework we have formulated to determine conditions under which we have "inf-sup = sup-inf" for every "right-hand-side" and bilinear objective function. This will form the basis of future work.

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A Appendix: Proof of Lemma 4.3

Our aim is to show that the two sets $\mathcal{M}_1$ and $\mathcal{M}_2$ have zero Lebesgue measure. The set $\mathcal{M}_1$ is closely related to the set of p-dimensional linear subspaces L of $\mathcal{R}^n$ for which $L \cap \operatorname{int}(S)$ is empty, and $L \cap S \neq \{0\}$. (A similar correspondence holds for the set $\mathcal{M}_2$.) We exploit this correspondence by invoking a deep theorem on convex bodies in [15, p. 93]. (The result there was first stated in [17].) This result is adapted as Lemma A.1 below, which concerns the Hausdorff measure of a particular subset of $G(n, p)$—the metric space of p-dimensional linear subspaces of $\mathcal{R}^n$.[6] Before stating this result, we discuss Hausdorff measure on metric spaces, and the distance function (i.e., metric) we will associate with the metric space $G(n, p)$.

Given a metric space (X, ρ), where ρ is the distance function, and given $t \geq 0$, the t-dimensional *Hausdorff measure* of $T \subseteq X$ is defined by

$$\mathcal{H}^t_\rho(T) := \lim_{\delta \to 0} \inf \left\{ \sum_i d_\rho(U_i)^t \;\middle|\; \{U_i\} \text{ is a } \delta\text{-cover of } T \right\}. \tag{A-1}$$

(The limit in (A-1) always exists, though its value may be infinite.) Here d_ρ is the diameter function

$$d_\rho(T) := \sup \{\rho(x, y) \mid x, y \in T\}, \tag{A-2}$$

[6]The set $G(n, p)$ together with a specified "differentiable structure", is known as the *Grassmann manifold.* We will not explicitly use any of the topological properties of this manifold however.

and a δ-*cover* of T is a countable collection of sets $\{U_i\}$ satisfying $T \subseteq \bigcup_i U_i$ and $0 < d_\rho(U_i) \leq \delta$ for each i.

Suppose now that t is a positive integer. It can be shown that on a t-dimensional Euclidean space endowed with the usual Euclidean distance function, the associated t-dimensional Hausdorff measure of a set $T \subseteq \mathcal{R}^t$ is a constant multiple of the Lebesgue measure $\mathcal{L}(T)$ of T in $\mathcal{R}^t$; see e.g., [13, Theorem 30]. In particular, $T \subset \mathcal{R}^t$ has zero t-dimensional Hausdorff measure if and only if $\mathcal{L}(T) = 0$.

For any positive integers n and p with $n > p$, the Hausdorff measure on $G(n,p)$ referred to throughout this appendix will be that associated with the "arc-length" distance function ρ, which is the distance function induced by the unique (to scale) "rotation-invariant Riemannian metric" on $G(n,p)$. It is pointed out in [1, Section 3] that this distance function can be expressed as the two-norm of the vector of "principal" or "canonical" angles between linear subspaces. See also [4, p. 337].

In the following lemma, an q-dimensional affine subspace $L \subset \mathcal{R}^n$ with $1 \leq q \leq n-1$, is said to *support* a nonempty closed convex set S, if L is contained in a supporting hyperplane of S, and $L \cap S$ is nonempty.

Lemma A.1. *Let $S \subseteq \mathcal{R}^n$ be a closed convex cone, q be an integer satisfying $1 \leq q \leq n-1$, and $\ell = q(n-q)$. The set of linear subspaces lying in $G(n,q)$ that support S, and that contain a ray of S, has zero ℓ-dimensional Hausdorff measure.*[7]

Proof. From the theorem in [15, p. 93], the result holds when S is a *convex body*, i.e., S is nonempty, compact, and convex. Now let S' be the intersection of the closed convex cone S with some convex body containing the origin in its interior. Clearly, S' is a convex body, and any linear subspace that supports S will also support S'. Since the result holds when S is replaced by S', it also holds for S itself. ■

We now state a useful result that specializes [13, Theorem 29].

Lemma A.2. *Let (X,μ) and (Y,ν) be metric spaces, and $T \subseteq X$. Let $f : T \to Y$ be a Lipschitz mapping, viz., there exists a constant $k > 0$ independent of x_1 and x_2 such that*

$$\nu(f(x_1) - f(x_2)) \leq k\mu(x_1, x_2) \quad \forall\, x_1, x_2 \in T.$$

Then for any $r \geq 0$,

$$\mathcal{H}^r_\nu(f(T)) \leq k\mathcal{H}^r_\mu(T).$$

In particular, if $Y = \mathcal{R}^r$ and T is such that $\mathcal{H}^r_\mu(T) = 0$, then $\mathcal{L}(f(T)) = 0$.

Our final preliminary result shows that if $T \subset G(n, n-q)$ has zero $q(n-q)$-dimensional Hausdorff measure, then the set of matrices whose nullspace or range is T has zero Lebesgue measure.

[7] A stronger result is stated in [15]. In particular, the set of linear subspaces in the lemma has σ-finite $(\ell-1)$-dimensional Hausdorff measure. A set having σ-finite measure can be written as a countable union of sets having finite measure. It is worth noting that ℓ is both the "topological dimension" and the "Hausdorff dimension" of the entire metric space $G(n,q)$.

Lemma A.3. *Let n and q with $n > q$ be positive integers, and $\ell = q(n-q)$. Let $T \subset G(n, n-q)$ be such that $\mathcal{H}^{\ell}_{\rho}(T) = 0$. Then the set $\{A \in \mathcal{R}^{q\times n} \mid \mathcal{N}(A) \in T\}$ has zero Lebesgue measure. Dually, if $T \subset G(n,q)$ is such that $\mathcal{H}^{\ell}_{\rho}(T) = 0$, then $\{A \in \mathcal{R}^{n\times q} \mid \mathrm{Range}(A) \in T\}$ has zero Lebesgue measure.*

Proof. Let

$$U := \{\mathcal{N}([I_q \;\; B]) \text{ for some } B \in \mathcal{R}^{q\times(n-q)}\} \subset G(n, n-q),$$

and let $\tilde{U}$ denote the complement of U. The set $\{A \in \mathcal{R}^{q\times n} \mid \mathcal{N}(A) \in \tilde{U}\}$ is the set of matrices in $\mathcal{R}^{q\times n}$ whose leading square full-dimensional submatrix is singular. By Lemma 4.2, this set has zero Lebesgue measure, and therefore so does $\{A \in \mathcal{R}^{q\times n} \mid \mathcal{N}(A) \in T \cap \tilde{U}\}$. To complete the proof of the first claim of the lemma, it therefore suffices to show that $\{A \in \mathcal{R}^{q\times n} \mid \mathcal{N}(A) \in T \cap U\}$ has zero Lebesgue measure.

We proceed by first defining the map $\phi : U \to \mathcal{R}^{q\times(n-q)}$ by $\mathcal{N}([I_q \;\; B]) \mapsto B$.[8] Let

$$U_i := \{L \in U \mid \|\phi(L)\| \le i\}$$

for each positive integer i. (Here $\|\cdot\|$ is an operator norm on $\mathcal{R}^{q\times(n-q)}$.) It can be verified that the restriction of ϕ to each U_i is Lipschitz continuous with respect to the arc-length distance function ρ on U_i and the metric induced by the operator norm $\|\cdot\|$ on $\mathcal{R}^{q\times(n-q)}$. Since $\mathcal{H}^{\ell}_{\rho}(T \cap U_i) \le \mathcal{H}^{\ell}_{\rho}(T) = 0$ for each i, and the range of ϕ has dimension ℓ, it follows from Lemma A.2 that $\mathcal{L}(\phi(T \cap U_i)) = 0$ for each i.

Now let $\mathrm{GL_q}$ denote the set of square nonsingular matrices of order q with real entries.[9] Define the map $g : \mathrm{GL_q} \times \mathcal{R}^{q\times(n-q)} \to \mathcal{R}^{q\times n}$ by $(M, B) \mapsto M[I_q \;\; B]$, and let $V := \mathrm{GL_q} \times \phi(T \cap U)$. It can be verified that

$$g(V) = \{A \in \mathcal{R}^{q\times n} \mid \mathcal{N}(A) \in T \cap U\},$$

so it remains to show that $\mathcal{L}(g(V)) = 0$.

Now define $\mathrm{GL}_{\mathrm{q},i} := \{M \in \mathrm{GL_q} \mid \|M\| \le i\}$ and $V_i := \mathrm{GL}_{\mathrm{q},i} \times \phi(T \cap U_i)$ for positive integers i. It is clear that the restriction of g to each V_i is Lipschitz continuous. Since $\mathcal{L}(\phi(T \cap U_i)) = 0$ for each i, it follows from Lemma 4.1 that $\mathcal{L}(V_i) = 0$ for each i. Now the domain and range of g are of the same dimension qn, so it follows from Lemma A.2 that $\mathcal{L}(g(V_i)) = 0$ for each i. Finally, since $V = \bigcup_{i=1}^{\infty} V_i$ is a countable union, we have

$$\mathcal{L}(g(V)) = \mathcal{L}\big(g\big(\bigcup_i V_i\big)\big) = \mathcal{L}\big(\bigcup_i g(V_i)\big) \le \sum_i \mathcal{L}(g(V_i)) = 0.$$

The dual statement is proved similarly, using

$$U := \left\{\mathrm{Range}\left(\begin{bmatrix} I_q \\ B \end{bmatrix}\right) \text{ for some } B \in \mathcal{R}^{(n-q)\times q}\right\} \subset G(n,q),$$

[8] To see that ϕ is a single-valued mapping, suppose that $B_1, B_2 \in R^{q\times(n-q)}$ are such that $\phi^{-1}(B_1) = \phi^{-1}(B_2)$, i.e., $\mathcal{N}([I_q \;\; B_1]) = \mathcal{N}([I_q \;\; B_2])$. Then there exists a nonsingular matrix $M \in \mathcal{R}^{q\times q}$ such that $[I_q \;\; B_1] = M[I_q \;\; B_2]$. It follows that $M = I_q$ and $B_1 = B_2$. The map ϕ is one of the canonical "chart mappings" that give the Grassmann manifold its "differentiable structure".

[9] Typically, $\mathrm{GL_q}$ is used to denote the general linear group of order q over $\mathcal{R}$, equipped with matrix multiplication. In a slight abuse of notation, we use $\mathrm{GL_q}$ to denote the *set* of matrices in this group.

and the maps $\phi : U \to \mathcal{R}^{(n-q)\times q}$ defined by Range $\left(\begin{bmatrix} I_q \\ B \end{bmatrix}\right) \mapsto B$, and $g : \mathrm{GL_q} \times \mathcal{R}^{(n-q)\times q} \to \mathcal{R}^{n\times q}$ defined by $(M, B) \mapsto \begin{bmatrix} I_q \\ B \end{bmatrix} M$. ■

With these results in hand, we now complete the proof of Lemma 4.3.

Proof. If $p \geq n$, then Lemma 4.1 implies that the sets $\{M \in \mathcal{R}^{p\times n} \mid \mathcal{N}(M) \neq \{0\}\}$ and $\{M \in \mathcal{R}^{n\times p} \mid \mathrm{Range}(M) \neq \mathcal{R}^n\}$ have zero Lebesgue measure. So for the purposes of proving that the sets $\mathcal{M}_1$ and $\mathcal{M}_2$ have zero Lebesgue measure, we can assume that $1 \leq p \leq n-1$. Define

$$\hat{G}(n, q, S) := \{L \in G(n, q) \mid L \cap \mathrm{int}(S) \text{ is empty, and } L \cap S \neq \{0\}\}.$$

for $q = p$ or $n-p$. Since S is a solid closed convex cone, every linear subspace $L \in \hat{G}(n, p, S)$ supports S, and since $L \cap S$ is a cone, then $L \cap S \neq \{0\}$ implies that L contains a ray of S. It follows from Lemma A.1 that $\mathcal{H}^{\ell}_{\rho}(\hat{G}(n, p, S)) = 0$. Hence from Lemma A.3, the sets $\{M \in \mathcal{R}^{p\times n} \mid \mathcal{N}(M) \in \hat{G}(n, n-p, S)\}$ and $\{M \in \mathcal{R}^{n\times p} \mid \mathrm{Range}(M) \in \hat{G}(n, p, S)\}$ have zero Lebesgue measure. It follows that $\mathcal{M}_1$ and $\mathcal{M}_2$ also have zero Lebesgue measure. ■

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