PH.D. THESIS

Geometric Phases in Sensing and Control

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ABSTRACT

Title of Dissertation: GEOMETRIC PHASES IN SENSING AND CONTROL

Sean B. Andersson, Doctor of Philosophy, 2003

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In many parameter-dependent systems, varying the parameters along a closed path generates a shift in the system depending only on the path itself and not on the manner in which that path is traversed. This effect is known as a geometric phase. In this thesis we focus on developing techniques to utilize geometric phases as engineering tools in both sensing and control.

We begin by considering systems undergoing an imposed motion. If this motion is adiabatic then its effect on the system can be described by a geometric phase called the Hannay-Berry phase. Direct information about the imposed motion is obtained by measuring the corresponding phase shift. We illustrate this idea with an equal-sided, spring-jointed, four-bar mechanism and then apply the technique to a vibrating ring gyroscope.

In physical systems the imposed motion cannot be truly adiabatic. Using Hamiltonian perturbation theory, we show that the Hannay-Berry phase is the first-order term in a perturbation expansion in the rate of imposed motion. Corrections accounting for the nonadiabatic nature of the imposed motion are then given by carrying the expansion to higher-order. The technique is applied to the vibrating ring gyroscope as an example.

We also consider geometric phases in dissipative systems with symmetry. Given such a system with a parameter-dependent, exponentially asymptotically stable equilibrium point, we define a new connection, termed the Landsberg connection, which captures the effect of a cyclic, adiabatic variation of the parameters. Systems with stable, time-dependent solutions are handled by defining an appropriate dynamic phase. A simple example is developed to illustrate the technique.

Finally we investigate the role of geometric phases in the control of nonholonomic systems with symmetry through an exploration of the $H(3)-$Racer, a two-node, one module $G-$snake on the three-dimensional Heisenberg group. We derive the governing equations for the internal shape of the system and the reconstruction equations relating changes in the shape to the overall motion. The controllability of the system is considered and the effect of various shape changes is explored through simulation.

GEOMETRIC PHASES IN SENSING AND CONTROL

by

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Introduction

1.1 Background

In 1984 Berry published a surprising result concerning quantum systems whose Hamiltonians depend on a set of parameters being adiabatically varied around a closed loop [9]. Using the quantum adiabatic theorem (see, e.g. [40]) and making the assumption that the eigenvalues of the Hamiltonian were isolated he showed that the net change in the phase of the system after completing the loop contains a term dependent solely on the area enclosed by the loop in parameter space. Berry's initial result, now known as Berry's phase, is surprising in its simplicity and has proved to be remarkably effective in understanding a wide variety of physical phenomena. The motivation for the continued interest in this subject is perhaps best captured by the following quote from Shapere and Wilczek [80].

> Examples of geometric phases abound in many areas of physics. Many familiar problems that we do not ordinarily associate with geometric phases may be phrased in terms of them. Often, the result is a clearer understanding of the structure of the problem, and an elegant

expression of its solution.

A classical example of this is the Foucault pendulum, a well-known system in which the rotation of the Earth induces a precession in the swing plane of the pendulum. In the standard analysis of this problem, the precession is understood as being caused by the Coriolis force arising from the moving frame of the pendulum. The rate of rotation of the swing plane is given by $-\Omega_E \cos\alpha$ where Ω_E is the rate of rotation of the Earth and α is the co-latitude. The resulting shift in the swing plane angle after one full rotation of the Earth is $\Delta\theta = -2\pi\cos\alpha$. This phase shift is geometric in nature; it does not depend on the rate of rotation of the Earth but only on the co-latitude of the pendulum system. The phase shift can be explained in purely geometric terms as follows (as in [59]). Consider an orthonormal frame which we wish to parallel transport along the co-latitude line α. Since it is not clear what is meant by parallel transport on a curved surface such as the sphere, we first translate the system to a flat space. To do so, place a cone on the sphere as in Figure 1.1. Flatten the cone by simply cutting and unrolling it as shown in

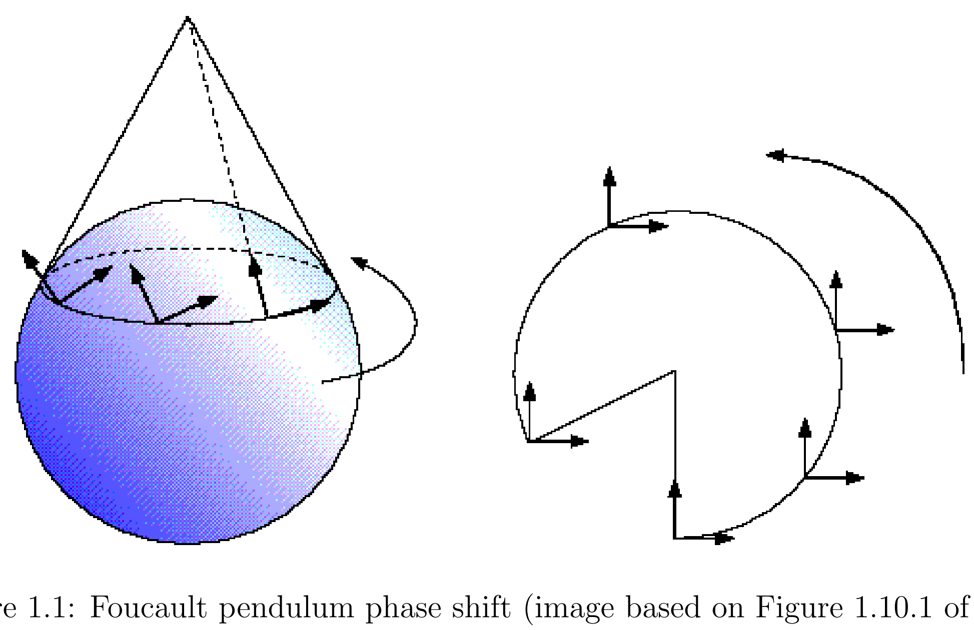

Figure 1.1: Foucault pendulum phase shift (image based on Figure 1.10.1 of [59])

the figure. After parallel translating the frame around the edge of the flattened cone, glue the cut edges of the cone together again and place it back on the sphere. The orthonormal frame is then back at the original position on the sphere but is rotated by $-2\pi \cos \alpha$, equivalent to the phase shift of the Foucault pendulum. A beautiful derivation of the rate of precession of the Foucault pendulum from the point of view of parallel translation using only basic calculus can be found in [68] while in [57] this shift is explicitly described as a geometric phase.

Geometric phases appear in a stunningly wide variety of physical systems. Several examples of effects which are now recognized as geometric phases were described prior to Berry's original paper in 1984, including the work of Pancharatnam on phase shifts in polarized light [72] and of Stone on electronic spin interactions [83]. Following Berry's result, Hannay found an effect analogous to Berry's phase in classical integrable systems [36] and Berry showed that these Hannay angles are the semi-classical limit of Berry's phase [10]. In the quantum setting, the existence of Berry's phase has been verified in various experiments. For example, Tomita and Chiao measured the geometric phase of the rotation of linearly polarized light traveling along a helically wound optical fiber [91] and Suter, Chingas, Harris, and Pines measured Berry's phase in a nuclear magnetic resonance system in the presence of a magnetic field with constant magnitude but varying direction [88]. (For further examples and comments on Berry's phase see the compendium volume of Shapere and Wilczek [80].)

Geometric phases have also been shown to exist in dissipative systems. In [44] Kepler and Kagan consider dynamical systems with limit cycles and show the existence of a geometric shift in the variable parametrizing the limit cycle under the adiabatic transport of a set of parameters (see also [43]). Together with

Epstein they experimentally showed such a shift in a dissipative chemical oscillator system [38]. A similar approach was developed by Ning and Haken and applied to laser dynamics [66]. A more general theory for dissipative systems with abelian symmetries has been developed by Landsberg in [48, 49].

Many researchers have investigated the role of the geometric phase in mechanical systems. In problems of this type, changes in the system's internal (shape) variables lead to changes in the external (group) variables. Examples of this include the work of Shapere and Wilczek on self-propulsion at low Reynolds number [79], of Krishnaprasad on the reorientation of coupled rigid bodes [46], of Bloch, Krishnaprasad, Marsden, and Sánchez de Alvarez on the control of satellites with rotors [16], and of Wisdom on the effect of the intrinsic curvature of space-time on the net translation and rotation of bodies undergoing cyclic shape changes [99]. In [41, 42], Kelly and Murray explore the use of geometric phases in the context of robotic locomotion and in [58] Ostrowski and Marsden explore aspects of motion control based on cyclic evolution of internal variables . Many applications of geometric phases in mechanical systems fall under the general heading of reconstruction phases, a concept treated by Marsden, Montgomery, and Ratiu in [57]. Montgomery has used the ideas of geometric phases together with techniques from optimal control to formulate and address the problem of finding shape changes which achieve a specified change in the group variables while minimizing some quantity such as the expended energy [63].

1.2 Geometric phases: intuition

It is useful to have an intuitive understanding of geometric phases before defining the concept in a rigorous manner. Consider a dynamical system of interest and assume that it depends on a set of parameters. Examples of such parameters include the eigenvalues of the Hamiltonian of a quantum system, the position of the Foucault pendulum in space, the temperature or catalyst concentrations in chemical systems, and the internal shape of a mechanical system for reconstruction phases. As these parameters are varied there is in general a corresponding change in the system. If after some time the parameters are brought back to their original values, it is natural to ask whether the system itself returns to its original configuration or whether there is some net change. If there is some net change and if it depends only on the path followed by the parameters and not on the rate at which that path is traversed then we refer to this as a geometric phase.

Common to many applications of geometric phases is the notion of adiabaticity. The intuitive understanding of adiabaticity is that the changing parameters should not fundamentally alter the natural dynamics of the system but rather should have an effect consistent with the dynamics. If the original system is following a periodic orbit, for example, then the geometric phase would be expected to affect the position along the orbit but not alter the orbit itself. The physical consequences of this notion vary from system to system. For example, in Berry's phase the adiabatic restriction allows one to apply the quantum adiabatic theorem and assume that if the system is begun in a stationary state then at any instant it will be in an eigenstate of the instantaneous Hamiltonian. Simon showed that this allows one to construct a line bundle over the parameter space and endow that bundle with a natural connection[81] . In the classical setting of integrable systems the adiabatic requirement allows us to take the action variables as adiabatic invariants ([36]) and determine a geometric phase effect on the corresponding angle variables. In dissipative systems, adiabaticity requires that the dissipative time scale is much faster than the time scale of the variation of the parameters, thereby allowing the system to relax to the equilibrium state (see [48] and Chapter 5 of this dissertation).

It is important to keep in mind that the adiabatic assumption is not a fundamental requirement for the existence of geometric phases. In reconstruction phases, for example, the geometric terms arise naturally and exist regardless of the rate at which the shape is changed. Similarly, Berry's phase in quantum systems has been extended to the non-adiabatic setting [3] (and comments in [62]).

1.3 Overview

In this thesis we focus on the geometric phase as an engineering tool useful in both sensing and control. We begin in Chapter 2 with a rigorous definition of the geometric phase by first giving a brief overview of fiber bundles and connections. Using the connection, we then define the notion of parallel transport in the bundle. This in turn allows us to describe the effect of parameter variation on the system by defining how to lift the tangent vectors giving the motion of the parameters up to the tangent space of the system. Finally, given a fiber bundle and a connection on that bundle, the geometric phase is the holonomy of the connection. Defined in this way, the geometric phase is an intrinsic geometric object, independent of any particular coordinatization.

Recall the Foucault pendulum example and note that by measuring the shift in the angle of the swing plane one can infer the rotation rate of the Earth. It is precisely this viewpoint we adopt in Chapter 3 where we explore the use of the geometric phase in sensing. Inspired by the Foucault pendulum, we utilize the moving systems approach developed by Marsden, Montgomery, and Ratiu in [57] to model the effect of imposed motion on a system. The resulting geometric phase is termed the Hannay-Berry phase. It is assumed that the imposed motion is adiabatic and the techniques of averaging are used to isolate its effects from those of the natural dynamics. We illustrate the technique on the relatively simple example of a free-floating, equal-sided, four-bar mechanism and then turn to our main application of the theory, the vibrating ring gyroscope. As shown by G.H. Bryan in 1890, an imposed rotation on a vibrating ring results in a precession of the nodal points of vibration in the ring itself [20]. We show this effect is the Hannay-Berry phase and then go on to use the inherently nonlinear nature of the geometric phase approach to calculate a small nonlinear correction to the nodal precession rate.

In practice the imposed motion on a system is not truly adiabatic but only very slow with respect to the natural dynamics. In making the adiabatic approximation higher-order terms in the rate of the imposed motion are discarded. Inspired by work on non-adiabatic corrections to Berry's phase in quantum systems, in Chapter 4 we develop a technique to incorporate effects arising from the slow but non-adiabatic nature of the imposed motion. To do so we use Hamiltonian perturbation theory. The method is illustrated by applying it to the vibrating ring gyroscope.

In Chapter 5 we shift our focus to geometric phases in dissipative systems with symmetry. In this chapter we build upon earlier work by Landsberg [48, 49] and develop a theory for finite-dimensional dissipative systems with symmetry. Adiabaticity once again plays a large role. It is assumed that the system has an exponentially asymptotically stable equilibrium point depending on a parameter which can be controlled. If the parameter is varied adiabatically, the system will stay near the varying equilibrium point at all times. Upon traversing a closed loop in parameter space, the system will return to the original equilibrium point but may experience a shift in the symmetry group. This shift is given by the holonomy in a particular fiber bundle with respect to a connection we term the Landsberg connection. The theory is illustrated through the use of a simple example.

In [13] Bloch, Krishnaprasad, Marsden, and Murray developed a geometric approach to nonholonomic mechanical systems . In this work they defined a new connection called the nonholonomic connection which synthesizes the mechanical connection defined by the kinetic energy of the system and a connection characterizing the nonholonomic constraints. The method allows one to split the system into dynamics on the reduced or shape space and dynamics on the group. The group dynamics consist of a drift term depending on a time-dependent quantity called the nonholonomic momentum and a geometric term given by the horizontal lift of the curve in the shape space with respect to the nonholonomic connection. In Chapter 6 we give a brief review of this approach and then investigate the role of the geometric and drift terms through the use of a detailed example on the three-dimensional Heisenberg group $H(3)$.

We conclude in Chapter 7 with a few brief remarks and future research directions.

1.4 Contributions

There are four main contributions of this thesis. First, we develop a methodology for using geometric phases as a mechanism for sensing imposed motion. To do this we use the moving systems approach of Marsden, Montgomery, and Ratiu to understand the effect of the imposed motion as a geometric phase, under the assumption that the imposed motion is adiabatic. Using this method we are able to re-derive the results of G.H. Bryan on the precession of the nodes of vibration in a rotating, vibrating ring and then extend the results to include (small) nonlinear corrections to the rate of precession of the nodes. In particular we show that the nonlinear terms reduce the rate of precession and thus that the highest sensitivity for these devices is obtained by operating them in the linear regime.

The second contribution is to extend the moving systems approach to account for the non-adiabatic nature of the imposed motion. This is achieved by first showing that the Hannay-Berry phase can be understood as a first-order (in the rate of the imposed motion) correction to the nominal dynamics through the use of Hamiltonian perturbation theory (normal forms). Non-adiabatic corrections are then obtained by taking the perturbation to higher orders.

The third main result is the development of the theory of geometric phases in dissipative systems with symmetry. This is done by describing these systems in terms of the standard framework for geometric phases, i.e. fiber bundles, and then defining a new connection, termed here the Landsberg connection, whose holonomy is the geometric phase. A common feature in dissipative systems with symmetry is the existence of patterns and in many cases these patterns are time dependent (e.g. a traveling wave). To handle this we develop an appropriate definition of the dynamic phase so that the resulting evolution of the system is a combination of the time-independent geometric phase and the time-dependent dynamic phase. While prior work assumed the symmetry group was abelian, our approach is applicable to systems with arbitrary finite-dimensional symmetry groups.

The final contribution is an illustration of geometric phases in nonholonomic systems with symmetry through the exploration of the $H(3)-$Racer. Using the techniques of Bloch, Krishnaprasad, Marsden, and Montgomery [13], we derive the equations of motion for the $H(3)-$Racer and determine the geometric and dynamic phase equations. We show that for this system, the shape space is given by $\mathbb{R}$ and thus, since any closed loop in the phase space has zero winding number, the resulting geometric phase is zero. However, since the symmetry group describes the overall position of the system, one can make sense of the geometric contribution to the group motion due to arbitrary changes in the shape.

Chapter 2

Fiber bundles, connections, and the geometric phase

The natural mathematical framework from which to approach geometric phases is that of connections on fiber bundles. In this chapter we present a review of the necessary background on fiber bundles, connections, and holonomy, both for general Ehresmann connections and for principal connections on principal bundles. Additional references for this material include [15, 32, 67] and references therein.

Throughout the thesis we assume familiarity with differential geometry, geometric mechanics, and Lie groups (see, e.g., [1, 2, 32, 59, 67, 96]).

2.1 Fiber bundles and Ehresmann connections

A **fiber bundle** is defined as follows. Let P, F, and B be manifolds referred to as the **total space** (or **bundle space**), the **fiber**, and the **base space** respectively. Let $\pi : P \to B$ be a surjective submersion. We require that P be locally a product space, that is, for every $b \in B$ there is a neighborhood U of b such that $\pi^{-1}(U)$

is diffeomorphic to $F \times U$. The fiber over b, $\pi^{-1}(b)$, is a diffeomorphic copy of the fiber F for every $b \in B$. See Figure 2.1. The bundle is denoted by the triple

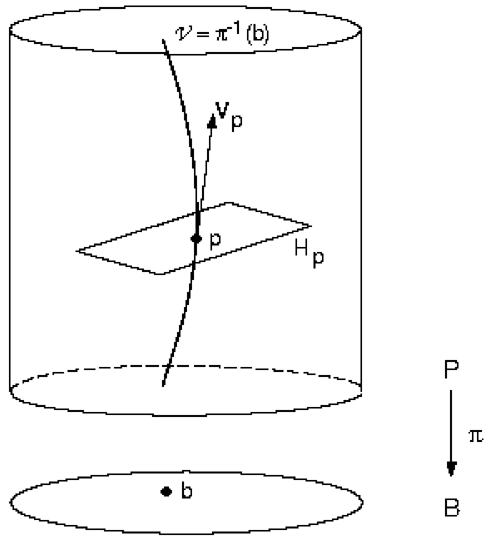

Figure 2.1: Fiber bundle, fiber over b, and splitting of tangent space

(P, F, B) or by the projection map $\pi : P \to B$. If the bundle is globally a product bundle, $P = F \times B$, then it is called a trivial fiber bundle.

Given $p \in P$, there is a natural subspace of T_pP (the tangent space to P at p) called the vertical space at p, denoted by V_p and defined by $V_p \triangleq \ker T_p\pi$. Here $T_p\pi$ is the linearization of the projection map evaluated at p. The union of these subspaces over all p is called the vertical subbundle V, i.e. $V \triangleq \cup_{p \in P} V_p$.

Definition 2.1.1 *An* **Ehresmann connection** *A on P is a vertical valued one-form on P satisfying:*

1. *$A_p : T_pP \to V_p$ is a linear map.*

2. *A_p is a vertical projection. That is, $A_p(v) = v \quad \forall v \in V_p$.*

■

The connection defines a horizontal space $H_p \stackrel{\triangle}{=} \ker A_p$ at each point $p \in P$. The conditions in the definition imply $T_pP = V_p \oplus H_p$ and thus the connection gives us a splitting of the tangent space at each point p into a vertical and a horizontal part (see again Figure 2.1). The union of these subspaces over all p is called the horizontal subbundle, i.e. $H = \cup_{p\in P} H_p$.

Given an Ehresmann connection A, a point $p \in P$, and a tangent vector $w \in T_{\pi(p)}B$, define the horizontal lift of w to T_pP as the unique tangent vector in H_p that projects to w under $T_p\pi$. We call this lift $\mathrm{hor_p}$. The lift of w can be found by

$$\mathrm{hor_p} w = \widetilde{w} - A_p(\widetilde{w}) \tag{2.1}$$

where $\widetilde{w} \in T_pP$ is an arbitrary vector satisfying $T_p\pi(\widetilde{w}) = w$.

Lemma 2.1.2 *The map* $\mathrm{hor_p}$ *is well-defined.*

Proof Let $\widetilde{u}_1, \widetilde{u}_2 \in T_pP$ be tangent vectors such that

$$T_p\pi(\widetilde{u}_1) = T_p\pi(\widetilde{u}_2) = u.$$

Notice that

$$T_p\pi\left(\widetilde{u}_1 - \widetilde{u}_2\right) = T_p\pi\left(\widetilde{u}_1\right) - T_p\pi\left(\widetilde{u}_2\right) = u - u = 0$$

and thus $\widetilde{u}_1 - \widetilde{u}_2$ is a vertical vector. Let

$$\mathrm{hor_p^1}[u] = \widetilde{u}_1 - A_p\widetilde{u}_1 \qquad \text{and} \qquad \mathrm{hor_p^2}[u] = \widetilde{u}_2 - A_p\widetilde{u}_2.$$

Then

$$\begin{aligned}
\mathrm{hor_p^1}[u] - \mathrm{hor_p^2}[u] &= (\widetilde{u}_1 - A_p\widetilde{u}_1) - (\widetilde{u}_2 - A_p\widetilde{u}_2) \\
&= (\widetilde{u}_1 - \widetilde{u}_2) - A_p\left(\widetilde{u}_1 - \widetilde{u}_2\right) \\
&= (\widetilde{u}_1 - \widetilde{u}_2) - (\widetilde{u}_1 - \widetilde{u}_2) = 0.
\end{aligned}$$

■

It should be noted that while here we have defined the Ehresmann connection as a vertical valued one-form and derived the horizontal space and horizontal lift, one can also begin with the definition of the horizontal space or the horizontal lift and define the other two objects. See [57] for details.

With an admitted abuse of notation we define the horizontal part of a tangent vector $X \in T_pP$ with respect to the connection A as

$$\mathrm{hor}X = X - A_p(X). \tag{2.2}$$

The meaning of the map hor should be clear from context.

The **curvature** of a vertical valued one-form A is the vertical valued two-form B defined by

$$B(X,Y) = -A([\![\mathrm{hor}X, \mathrm{hor}Y]\!]) \tag{2.3}$$

where $[\![\cdot,\cdot]\!]$ is the Jacobi-Lie bracket. The curvature is in fact the covariant derivative of the connection form (see [57]).

2.2 Principal fiber bundles and principal connections

Let P be a smooth manifold and let G be a Lie group that acts freely and properly on P on the left. Because the action is free and proper the quotient space P/G is also a manifold (see Proposition 4.1.23 in [1]). A **principal fiber bundle** with structure group G is a fiber bundle $\pi : P \to P/G$ whose fibers are diffeomorphic to the group G. Let $\mathfrak{g}$ denote the Lie algebra associated to G.

Definition 2.2.1 *A* **principal connection** *on the principal bundle* $\pi : P \to P/G$ *is a* $\mathfrak{g}$*-valued one form* $\mathcal{A} : TP \to \mathfrak{g}$ *satisfying:*

1. $\mathcal{A}(\xi_P(p)) = \xi \quad \forall \xi \in \mathfrak{g}$ *and* $p \in P$ *where* $\xi_P(p)$ *is the infinitesimal generator corresponding to* ξ*.*

2. $\mathcal{A}$ *is* Ad$-$*equivariant. That is,*

$$\mathcal{A}(T_p\Phi_g(v_p)) = \mathrm{Ad}_g\mathcal{A}(v_p) \quad \forall v_p \in T_pP \text{ and } g \in G \tag{2.4}$$

where Φ_g *is the action of* G *on* P *and* Ad_g *is the adjoint action of* G *on* $\mathfrak{g}$*.*

■

As in the general fiber bundle setting, the vertical space at p is $V_p \triangleq \ker T_p\pi$. We have also $V_p = T_p\mathrm{Orb}(p)$, the tangent space to the group orbit through p. We define the horizontal space of the connection to be

$$H_p = \{v_p \in T_pP | \mathcal{A}(v_p) = 0\} \tag{2.5}$$

and the horizontal bundle $H = \cup_{p\in P} H_p$. As before this gives a splitting of the tangent space at each p, $T_pP = V_p \oplus H_p$. This splitting defines an Ehresmann connection associated to the principal connection, given by

$$A(v) = (\mathcal{A}(v))_P(p) \tag{2.6}$$

where $(\mathcal{A}(v))_P$ is the infinitesimal generator of the group action corresponding to $\mathcal{A}(v)$. The horizontal part of a vector $X \in T_pP$ is given by

$$\mathrm{hor}X = X - (\mathcal{A}(X))_P(p). \tag{2.7}$$

The **curvature** of $\mathcal{A}$, denoted by $\mathcal{B}$, is

$$\mathcal{B}(X, Y) \triangleq -\mathcal{A}\left([\![\mathrm{hor}X, \mathrm{hor}Y]\!]\right). \tag{2.8}$$

2.3 Parallel transport and holonomy

Given an Ehresmann connection A on a fiber bundle we define parallel transport with respect to A along a curve lifted from the base space in the following way. Let $b(t)$, $t \in [0,1]$, be a piecewise differentiable curve in B. The horizontal lift of $b(t)$ with respect to A is the curve $p(t)$ in P such that $\pi(p(t)) = b(t)$ and that the tangent vector $\frac{dp(t)}{dt}$ is horizontal for each $t \in [0,1]$. We have the following proposition from [57].

Proposition 2.3.1 [57] *Given a curve $b(t)$, $t \in [0,1]$, in B and a point $p_0 \in \pi^{-1}(b(0))$, there exists a unique locally defined horizontal lift $p(t)$ of $b(t)$ to P satisfying $p(0) = p_0$ if P is a locally trivial fiber bundle.*

Proof Since P is locally a trivial bundle we can write locally $p = (b, f)$ and $\dot{p} = (u, v)$ for $b \in B$, $f \in F$, $u \in T_bB$, and $v \in T_fF$. We can then write the connection one form as

$$\begin{aligned} A(b,f)(u,v) &= (0,v) + A(b,f)(u,0) \\ &\triangleq (0, v + \lambda(b,f)u). \end{aligned}$$

(u, v) is horizontal if and only if $A(b,f)(u,v) = 0$ and thus for a horizontal tangent vector $v = -\lambda(b,f)u$. If $b(t)$ is a path in B denote $p(t) = (b(t), f(t))$ where $f(t)$ is the solution to the ordinary differential equation

$$\frac{df}{dt}(t) = -\lambda(b(t), f(t))\dot{b}(t), \quad f(0) = f_0 \text{ where } p_0 = (b_0, f_0). \tag{2.9}$$

Then by local existence and uniqueness for ordinary differential equations this defines $f(t)$ and thus $p(t)$ for small t. If $p(t)$ can be extended for all $t \in [0,1]$ the connection is called *complete*. ∎

Given any curve $b(t)$ in B, $t \in [0, 1]$, and an Ehresmann connection A, the **parallel transport operator** τ_b is defined as

$$\tau_b : \pi^{-1}(b(0)) \to \pi^{-1}(b(1)) \qquad \tau_b(p(0)) = p(1) \tag{2.10}$$

where $p(0) \in \pi^{-1}b(0)$ and $p(t)$ is the horizontal lift of $b(t)$ with respect to A starting at $p(0)$.

By the uniqueness of the horizontal lift, τ_b is a bijection from $\pi^{-1}(b(0))$ to $\pi^{-1}(b(1))$ and by the smooth dependence of solutions of ordinary differential equations on initial conditions it is a diffeomorphism.

Let b_0 be an arbitrary point of B and let C_{b_0} be the set of all closed curves at b_0, that is all $b(t)$ such that $b(0) = b(1) = b_0$. The diffeomorphism of $\pi^{-1}(b_0)$ onto itself given by parallel transport along $b(t)$ is called the **holonomy** of the path $b(t)$. Let Φ_{b_0} be the collection of all parallel transport operators over C_{b_0} and define the group operation as composition. Φ_{b_0} then forms a group, called the holonomy group at b_0. (Assuming B is connected, it is easy to see that Φ_{b_0} and Φ_{b_1} are conjugate for any two $b_0, b_1 \in B$. Thus if B is connected we have simply Φ, the holonomy group of the connection.)

Definition 2.3.2 *Given a bundle $\pi : P \to B$, a connection on the bundle, and a closed curve $b(t)$ in the base space the* **geometric phase** *is the holonomy along the curve $b(t)$.*

The geometric phase can be calculated by solving the ordinary differential equation given in equation (2.9).

2.3.1 Principal bundle setting

If we are in the principal bundle setting then we can identify the holonomy group as a subgroup of the structure group G. To see this we follow the discussion in [102]. Let $\mathcal{A}$ be a principal connection on the principal bundle $\pi : P \to P/G$. Let $b(\cdot) = \{b(t), t \in [0,1]\}$ be a closed curve in the base space and let $p_0 \in \pi^{-1}(b(0))$. Assume $b(\cdot)$ is contained in an open set U of P/G. Let $\sigma : U \to P$ be an arbitrary local section of the bundle and let $p(\cdot) = \{p(t), t \in [0,1]\}$ be the horizontal lift of $b(\cdot)$ with $p(0) = p_0$. Finally, let $g(\cdot) = \{g(t), t \in [0,1]\}$ be a curve in G such that $p(t) = \Phi(g(t), \sigma(b(t)))$ where Φ is the action of G on P. From the chain rule we have

$$\frac{dp(t)}{dt} = T_{\sigma(b(t))}\Phi_{g(t)}[T_{b(t)}\sigma(\dot{b}(t))] + T_{\sigma(b(t))}\Phi_{g(t)}\xi_P(\sigma(b(t))) \qquad (2.11)$$

where $\xi \stackrel{\triangle}{=} g(t)^{-1}\dot{g}$ and ξ_P is the infinitesimal generator corresponding to ξ. Since $p(t)$ is horizontal we have $\mathcal{A}(\dot{p}) = 0$. Applying the connection form to both sides of equation (2.11) we have

$$\begin{aligned} 0 &= \mathcal{A}\left[T_{\sigma(b(t))}\Phi_{g(t)}[T_{b(t)}\sigma(\dot{b}(t))] + T_{\sigma(b(t))}\Phi_{g(t)}\xi_P(\sigma(b(t)))\right] \\ &= \mathrm{Ad}_g\left[\mathcal{A}(T_{b(t)}\sigma(\dot{b}(t))) + \mathcal{A}(\xi_P(\sigma(b(t))))\right] \\ &= \mathrm{Ad}_g\left[(\sigma^*\mathcal{A})(\dot{b}(t)) + \xi\right] \end{aligned}$$

where the second step follows from the equivariance of the connection form and the third from the definition of the pull-back of a map and the fact that the connection maps infinitesimal generators to the corresponding Lie algebra elements. The $\mathfrak{g}$-valued form $\sigma^*\mathcal{A}$ is called the local connection form and is denoted $\mathcal{A}_{loc}$. Thus

$$\xi = -\mathcal{A}_{loc}(\dot{b}(t)). \qquad (2.12)$$

By the definition of ξ we have

$$\dot{g} = g(t)\xi = -g \cdot \mathcal{A}_{loc}(\dot{b}(t)) \qquad (2.13)$$

and the solution of this differential equation at $t = 1$ is the geometric phase.

If G is an abelian group the geometric phase can be represented explicitly as

$$g(1) = \exp\left(-\int_0^1 \mathcal{A}_{loc}(\dot{b}(t))dt\right) = \exp\left(-\int_{\mathcal{D}} \sigma^*\mathcal{B}\right) \tag{2.14}$$

where $\mathcal{D}$ is any surface in the base space with $b(\cdot)$ as the boundary and $\mathcal{B}$ is the curvature form of the connection.

Chapter 3

Geometric phases in sensing

3.1 Introduction

In this chapter we investigate the role of geometric phases in sensing and in particular focus on sensing rotational motion through the use of an effect called the Hannay-Berry phase.

In 1890 G.H. Bryan published a paper on the nature of the beats generated when a vibrating shell is rotated about its central axis [20]. The phenomenon he describes is quite easy to observe; simply take a wine glass, strike it to produce a clear tone, and then rotate it about its stem to produce audible beats. Bryan noticed that these beats are the result of a precession of the nodal points with respect to the shell itself and provided the following reasoning. Consider a ring or cylinder rotating counter-clockwise about its central axis and vibrating with nodes at B,D,F, and H as indicated in the left-side image of Figure 3.1. The material points at A and E are moving towards the center O. This increases their actual angular velocity above that of the imposed rotation and gives them a relative angular acceleration in the direction of rotation as represented by the arrows at

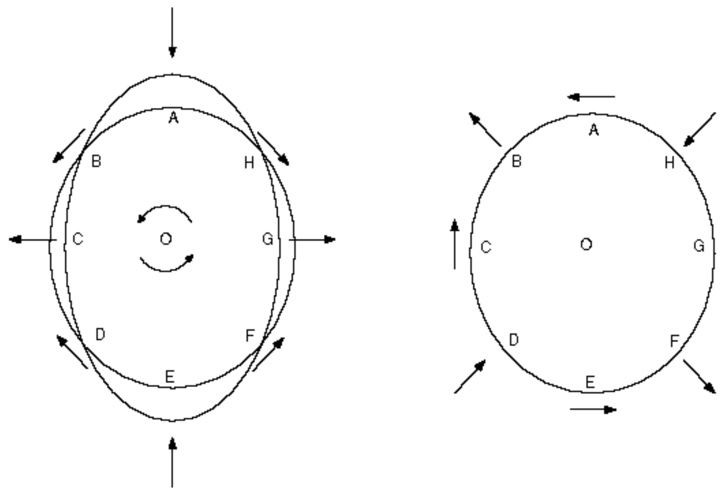

Figure 3.1: Nodal precession in cylinder or ring (Figure from [20])

A and E in the right side image of Figure 3.1. Similarly, the material points at C and G are moving outwards and thus their angular velocity is reduced. Those at B and F are moving with greater total angular velocity than the rest and thus experience a relative outwards acceleration due to a greater centrifugal force. Finally, the material points at D and H are moving with the least angular velocity and thus experience a relative acceleration inwards. Comparing the arrows in the two images of Figure 3.1 reveals that the effect of these relative accelerations is to cause retrograde motion of the nodes relative to the ring. Using classical variational techniques Bryan derived a linearized partial differential equation describing the behavior of the system, found a formula for the rate of precession and discovered that this rate is proportional to the rate of platform rotation.

Due to the immense number of potential applications, research on gyroscopes has been active for many years. Devices have been proposed, analyzed, and produced using a variety of materials and techniques. Because they lack rapidly spinning parts, have low power requirements, and are inherently scalable, vibratory gyroscopes have become particularly popular [75]. One of most successful initial designs was Delco's Hemispherical Resonator Gyroscope (HRG) [53] due to Loper and Lynch, which was able to achieve performance levels equal to the best ring laser gyroscopes. This design, shown diagrammatically in Figure 3.2, consists of a quartz hemispherical resonator supported on a central stem and contained inside an evacuated housing. As predicted by Bryan, the nodal points of the vibration in

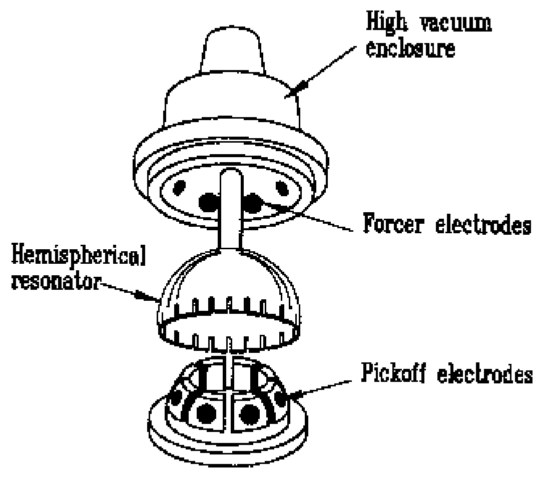

Figure 3.2: Delco HRG (Image from [50])

the hemisphere precess with respect to the shell as the device is rotated. The HRG is driven into elliptical vibration and the resulting precession rate is about 0.3 of that of the imposed rotation. The device operates over a wide temperature range, has high operating acceleration ranges, low acceleration sensitivity, and negligible magnetic sensitivity [54].

Similar ideas were used in the design of a vibrating disc gyroscope by Burdess and Wren [22]. With the explosion of MEMS technology constant innovations are resulting in smaller, cheaper, and more accurate devices. Existing MEMS-based devices include tuning-fork [8] and vibrating-ring designs [7, 76] such as the one shown in Figure 3.3 (provided by Douglas Sparks of Delco Automotive Systems).

Additional designs proposed include a vibrating cylinder [21, 101] and a surface acoustic wave generator [93].

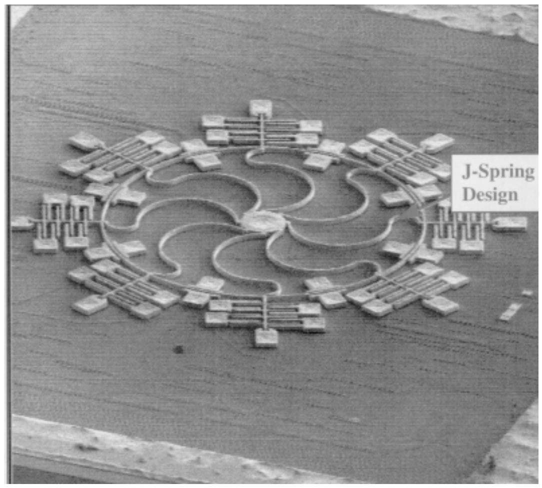

Figure 3.3: MEMS Gyroscope (Image courtesy of Douglas Sparks of Delco Automotive Systems)

These gyroscopes all take advantage of the same physical effect, the Coriolis force arising from the non-inertial character of the rotating frame of the system [50]. Modern analyses are linear in nature and view the Coriolis force as providing a coupling between two vibratory modes of the structure [90]. It is desirable, however, to have a method which, at least in principle, can be extended to a non-linear theory and which provides a unified setting for understanding a variety of systems in which the Coriolis force plays a role. Motivated by these considerations we are led to an approach developed by Marsden, Montgomery, and Ratiu based on modern developments in geometric mechanics. This method is known as the moving systems approach [57]. The technique, descending from the classical work of E. Cartan [23], describes the effect of imposed motion on a system as a geometric phase with respect to a particular Ehresmann connection called the Cartan-Hannay-Berry connection. This geometric phase is called the Hannay-

Berry phase.

While existing techniques have proved to be effective, as evidenced by the various devices constructed from those principles, it is to be expected that a deeper understanding will emerge by appealing to a nonlinear, geometric approach directed at more accurate constitutive models. The fact that Bryan's result for the vibrating ring shows that the nodal precession rate is proportional to the rate of platform rotation is intriguing and suggests that these modern tools may prove useful. In the next section we begin by describing the moving systems approach and defining the Hannay-Berry phase. We illustrate the technique by applying it to a free-floating, spring-jointed, equal-sided four-bar mechanism. We then show how these ideas are useful in sensing by investigating the vibrating ring gyroscope. In so doing we show that the precession of the nodal points of vibration can be understood as a Hannay-Berry phase. Embracing this approach allows us to clearly understand the role of the linearizing assumptions common to earlier analyses of this problem and to calculate the effect of the nonlinear terms by deriving a formula for a correction to the nodal precession rate.

3.2 The Hannay-Berry phase

Let S be a Riemannian manifold and let M be the space of embeddings of a manifold Q into S. We think of S as the ambient space in which Q is being moved and of Q as the configuration space for a system of interest.

Lemma 3.2.1 *A tangent vector u_m to M at m is a map $u_m : Q \to TS$ such that $u_m(q) \in T_{m(q)}S$.*

Proof To prove this lemma we use the notion of a tangent vector as a derivation (see, e.g. [96]). Let

$$\begin{aligned} \psi : [a,b] &\rightarrow M \\ \tau &\mapsto \psi(\tau) \end{aligned} \tag{3.1}$$

be a curve on M such that $\psi(t) = m$ for some $t \in [a,b]$. Let $\mathcal{F}(M,m)$ be the set of all smooth functions $f : M \rightarrow \mathbb{R}$ defined on a neighborhood of m (in the topology of M). The tangent vector at m to M associated to the curve $\psi(\cdot)$ is denoted $\psi_*(t) \in T_m M$ and is defined by its action on partial functions as follows.

$$\psi_*(t) f = \frac{d}{ds} \left(f(\psi(s)) \right) \Big|_{s=t}, \quad \forall f \in \mathcal{F}(M, \psi(t)). \tag{3.2}$$

Now let $g \in \mathcal{F}(S, \psi(t)(q))$, that is g is a function on S defined on a neighborhood of $\psi(t)(q)$. Viewing q as a map of M into S defined by

$$\begin{aligned} q : M &\rightarrow S \\ m &\mapsto m(q) \end{aligned} \tag{3.3}$$

we can define a function on M around $\psi(t)$ by

$$f_q = g \circ q : M \rightarrow \mathbb{R}. \tag{3.4}$$

Then

$$\begin{aligned} \psi_*(t) f_q &= \frac{d}{ds} \left(f_q(\psi(s)) \right) \Big|_{s=t} \\ &= \frac{d}{ds} \left(g \circ q(\psi(t)) \right) \Big|_{s=t} \\ &= \frac{d}{ds} \left(g(\psi(t)(q)) \right) \Big|_{s=t} \\ &= \widetilde{\psi}_*(t) g \end{aligned}$$

where $\widetilde{\psi}_* \in T_{m(q)}S$ is the tangent vector to S at $m(q)$ associated to the curve on S given by $\psi(\cdot)(q)$. From this the lemma follows. ■

Given a tangent vector u_m to M one can construct a tangent vector to T_qQ as follows. Relative to the metric on S orthogonally project the tangent vector $u_m(q)$ to $T_{m(q)}m(Q) \in (T_qm)(T_qQ)$, denote this vector $u_m^T(q)$, and then pull-back $u_m^T(q)$ by $[Tm]^{-1}$ to T_qQ. Using this natural construction, Marsden, Montgomery, and Ratiu define an Ehresmann connection on the product bundle $\pi : Q \times M \to M$ as follows.

Definition 3.2.2 [57] *The* **Cartan connection** *on $\pi : Q \times M \to M$ is given by the vertical valued one-form γ_c defined by*

$$\gamma_c(q, m)(v_q, u_m) = (v_q + ([Tm]^{-1} \circ u_m^T)q, 0). \tag{3.5}$$

■

The Cartan connection induces a connection on $\rho : T^*Q \times M \to M$ as follows.

Definition 3.2.3 [57] *The* **induced Cartan connection** *on $\rho : T^*Q \times M \to M$ is given by the vertical-valued one-form γ_o defined by*

$$\gamma_o(\alpha_q, m)(U_{\alpha_q}, u_m) = \left(U_{\alpha_q} + X_{\mathcal{P}(u_m)}(\alpha_q), 0\right) \tag{3.6}$$

where $\mathcal{P}(u_m)$ is the function defined by

$$\left(\mathcal{P}(u_m)\right)\alpha_q = \alpha_q \cdot ([Tm]^{-1} \circ u_m^T)(q) \tag{3.7}$$

and $X_{\mathcal{P}(u_m)}$ is the Hamiltonian vector field of $\mathcal{P}(u_m)$. ■

To separate the effects of the imposed motion on the system (as defined by the embeddings m_t) from the nominal dynamics (when the imposed motion is zero) we use the ideas of averaging. Abstractly, we assume we are given a left action of a Lie group G on T^*Q. The average of the connection form γ is defined by

$$< \gamma >= \frac{1}{|G|} \int_G g^*(\gamma) dg \tag{3.8}$$

where dg is a left Haar measure and $|G|$ is the total volume of G. From this we have the following definition.

Definition 3.2.4 [57] *The* **Cartan-Hannay-Berry connection** *on* $\rho : T^*Q \times M \to M$ *is given by the vertical-valued one-form* γ *on* $T^*Q \times M$ *defined by*

$$\gamma(\alpha_q, m)(U_{\alpha_q}, u_m) = \left(U_{\alpha_q} + X_{<\mathcal{P}(u_m)>}(\alpha_q), 0\right) \tag{3.9}$$

where $< \cdot >$ *denotes the average with respect to the action of the Lie group* G*.*

■

In [57] Marsden, Montgomery, and Ratiu show that this is an Ehresmann connection. The horizontal lift of a vector field Z on M relative to γ is

$$(hor Z)(\alpha_q, m) = \left(-X_{<\mathcal{P}(Z(m))>}(\alpha_q), Z(m)\right). \tag{3.10}$$

Definition 3.2.5 [57] *The holonomy of the Cartan-Hannay-Berry connection is called the* **Hannay-Berry phase** *for a moving system.* ■

3.2.1 The adiabatic assumption

The Hannay-Berry phase captures the effects of the imposed motion on a system under the assumption that this imposed motion is slow with respect to the nominal dynamics. To better understand this adiabatic assumption, we consider the following system (as in [59]). If a particle in Q is following a curve $q(t)$ and if Q is in turn being moved in the ambient space S by superposing the motion m_t, then the path of the particle in S is given by $m_t(q(t))$. The velocity in S is then

$$T_{q(t)}m_t\dot{q}(t) + \mathcal{Z}_t(m_t(q(t)))$$

where $\mathcal{Z}_t(m_t(q)) = \frac{d}{dt}m_t(q)$ (with q viewed as fixed). The standard Lagrangian is given by the kinetic energy minus the potential energy.

$$L(q,v) = \frac{1}{2}\|T_{q(t)}m_t v + \mathcal{Z}_t(m_t(q))\|^2 - V(q) - U(m_t(q)). \tag{3.11}$$

Here V is a given potential on Q and U is a given potential on S. To compute the associated Hamiltonian we take the Legendre transform. Taking the derivative of L with respect to v in the direction w yields

$$\frac{\partial L}{\partial v}\cdot w \quad = \quad p\cdot w = \langle T_{q(t)}m_t v + \mathcal{Z}_t(m_t(q(t))^T, T_{q(t)}m_t w\rangle_{m_t(q(t))}$$

where $p\cdot w$ is the natural pairing between the covector $p \in T^*_{q(t)}Q$ and the vector $w \in T_{q(t)}Q$, $\langle\cdot,\cdot\rangle_{m_t(q(t))}$ denotes the metric inner product on S at the point $m_t(q(t))$, and ^{T} denotes the orthogonal projection to $Tm_t(Q)$ using the metric of S at $m_t(q(t))$. Q inherits a metric from S such that m_t is an isometry for each t. Thus

$$\begin{aligned} p\cdot w &= \langle v + \left(T_{q(t)}m_t\right)^{-1}\mathcal{Z}_t^T(m_t(q(t))), w\rangle_{q(t)}, \\ \Rightarrow p &= \left(v + \left(T_{q(t)}m_t\right)^{-1}\mathcal{Z}_t^T(m_t(q(t)))\right)^\flat \triangleq (v+Z_t)^\flat \end{aligned} \tag{3.12}$$

where $\flat$ is the map $T_qQ \to T_q^*Q$ defined by

$$z^\flat\cdot w = \langle z, w\rangle_q \quad \forall w \in T_qQ. \tag{3.13}$$

The Hamiltonian is given by

$$\begin{aligned} H(q,p) &= p\cdot v - L(q,v) \\ &= \frac{1}{2}\|p\|^2 - \mathcal{P}(Z_t) - \frac{1}{2}\|\mathcal{Z}_t^\perp\|^2 + V(q) + U(m_t(q)) \end{aligned} \tag{3.14}$$

where $\mathcal{Z}_t^\perp = \mathcal{Z}_t - \mathcal{Z}_t^T$ is the orthogonal complement of $\mathcal{Z}_t$ and $\mathcal{P}(Z_t)$ is the function on T^*Q (defined in equation (3.7)) given by

$$\mathcal{P}(Z_t)(q,p) = p \cdot Z_t(q).$$

Define the nominal Hamiltonian H_0 by setting $\mathcal{Z}_t = 0$ and $U = 0$. The term $\mathcal{P}(Z_t)$ captures what are classically referred to as the Coriolis terms and $\|\mathcal{Z}_t^\perp\|^2$ captures the centrifugal terms.

Recall now that we have a compact Lie group G acting on T^*Q on the left. Assuming the group action leaves the nominal Hamiltonian invariant and applying the corresponding average we obtain

$$< H > (q,p) = \frac{1}{2}\|p\|^2 - < \mathcal{P}(Z_t) > - \frac{1}{2} < \|\mathcal{Z}_t^\perp\|^2 > + V(q) + < U(m_t(q)) > . \tag{3.15}$$

Invoking the adiabatic assumption, we discard $< \|\mathcal{Z}_t^\perp\|^2 >$ since it is small with respect to the other terms in the averaged Hamiltonian. (In Chapter 4 we develop an approach which seeks to account for the fact that the imposed motion, while slow, is not truly adiabatic.) After discarding the centrifugal terms the dynamics of the Hamiltonian system are governed by the Hamiltonian vector field

$$X_{<H>} = X_{H_0} - X_{<\mathcal{P}(Z_t)>} + X_{<U \circ m_t>}. \tag{3.16}$$

The second term captures the effect of the imposed motion in the adiabatic limit and is precisely the term given by the horizontal lift of the vector field $\mathcal{Z}_t$ with respect to the Cartan-Hannay-Berry connection as defined in equation (3.10).

3.2.2 Geometric character of the Hannay-Berry phase

The effect of the vector field $X_{<\mathcal{P}(Z_t)>}$ is geometric in nature. By this we mean that the resulting change in the system is independent of the parametrization of the curve followed in the base space M, i.e. the effect depends only on the loop itself and not on how it is traversed. To see this explicitly recall that the vector field $-X_{<\mathcal{P}(Z_t)>}$ is the horizontal lift of a vector field $\mathcal{Z}_t$ on the base space M to the fiber T^*Q with respect to the Cartan-Hannay-Berry connection and is thus a linear map of $\mathcal{Z}_t$. Denoting points in T^*Q by z, the ordinary differential equation defining the Hannay-Berry phase may be expressed as

$$\frac{dz}{dt} = -X_{<\mathcal{P}(Z_t)>} = D(z)\mathcal{Z}_t.$$

In coordinates, $D(z)$ is a matrix taking tangent vectors on M to tangent vectors on T^*Q. We now change the time parametrization by taking $t \mapsto \tau(t)$ with $\frac{d\tau}{dt}$ strictly positive. Under this new parametrization, the vector field $\mathcal{Z}_t$ is scaled by $\frac{d\tau}{dt}$ and thus

$$\frac{dz}{dt} = \frac{dz}{d\tau}\frac{d\tau}{dt} = D(z)\left(\frac{d\tau}{dt}\right)\mathcal{Z}_\tau.$$

We then have

$$\frac{dz}{d\tau} = D(z)\mathcal{Z}_\tau.$$

The equation defining the Hannay-Berry phase is thus independent of the time parametrization.

3.3 Equal-sided, spring-jointed, four-bar linkage

In this section we apply the moving systems approach to find the Hannay-Berry phase of a free-floating, equal-sided, spring-jointed, four-bar mechanism. Our goal is to determine whether an imposed rotation on this system results in a clearly measurable effect that can be used to sense this imposed motion. We will show that the Hannay-Berry phase in this system is in fact zero and therefore that this mechanism does not constitute a viable device for sensing rotational motion.

The study of the four-bar mechanism has a long history, dating at least back to the work of Grashof in the mid-nineteenth century [35]. (See, e.g., [73] and references therein.) In this work we build on an analysis of the dynamics of four-bar linkages due to Yang and Krishnaprasad [103].

The structure of an equal-sided four-bar mechanism is shown in Figure 3.4. By

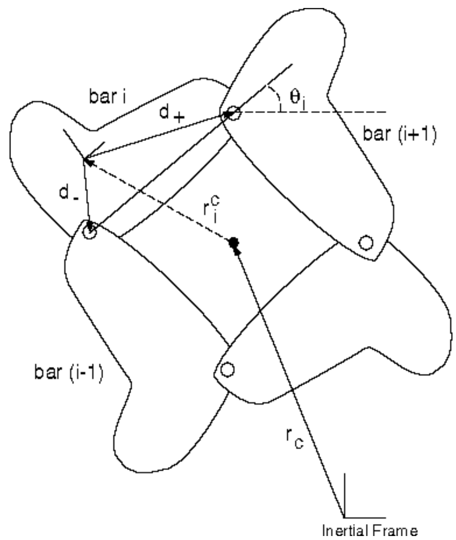

Figure 3.4: Equal-Sided Four-Bar Mechanism

a 'bar' we mean a planar rigid body on which the center of mass and pin joints are arbitrarily located. The identical bars are labeled sequentially from 0 to 3 and on each a body-fixed frame is defined such that its origin is at the body center of mass and the x-axis is parallel to the line connecting the pin joints. The positive direction of the x-axis of the i^{th} bar is defined to be towards the $(i+1)^{th}$ bar for $i = 0, 1, 2, 3$ where we adopt the convention of modulo four addition for subscripts. We define the following:

$\mathbf{d}_+$ the vector from the body center of mass of the i^{th} bar to the pin joint with the $(i+1)^{th}$ bar

$\mathbf{d}_-$ the vector from the body center of mass of the i^{th} bar to the pin joint with the $(i-1)^{th}$ bar

l the length of each bar, given by $\|\mathbf{d}_+ - \mathbf{d}_-\|$

$\mathbf{r}_i^c$ the vector from the system center of mass to the i^{th} body center of mass

$\mathbf{r}_c$ the vector from the origin of the inertial system to the system center of mass

θ_i the angle between the i^{th} bar frame and the inertial frame

$\theta_{i,j}$ the angle $\theta_i - \theta_j$ between the i^{th} and j^{th} bars

I, m the moment of inertia and mass of each bar

From Figure 3.4 we have that

$$r_{i+1}^c = r_i^c + R(\theta_i)\mathbf{d}_+ - R(\theta_{i+1})\mathbf{d}_-, \quad i = 0, 1, 2, 3. \tag{3.17}$$

Here $R(\theta_i)$ is the rotation matrix given by

$$R(\theta_i) = \begin{pmatrix} \cos(\theta_i) & -\sin(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i) \end{pmatrix}.$$

Using these equations recursively we obtain the loop closure constraint

$$F(r) = \sum_{i=0}^{3} R(\theta_i)(\mathbf{d}_+ - \mathbf{d}_-) = 0. \tag{3.18}$$

From [82] we know the configuration space for a free-floating four-link open chain is $\mathcal{R} = \mathbb{R}^2 \times S^1 \times S^1 \times S^1 \times S^1$. The configuration space for a general four-bar mechanism is then $S = \{r \in \mathcal{R} | F(r) = 0\}$. For a mechanism with identical bars, by explicitly requiring that the mechanism not pass through any singularities (joint angles of 0 or π) we can ensure S is a smooth submanifold of $\mathcal{R}$. This is shown in the following lemma.

Lemma 3.3.1 $S = \{r \in \mathcal{R} | F(r) = 0, \theta_{i+1,i} \neq 0, \theta_{i+1,i} \neq \pi\}$ *is a smooth submanifold of* $\mathcal{R}$.

Proof We need only show that 0 is a regular value of F since then S is a smooth submanifold of M. We have:

$$\frac{\partial F}{\partial m} = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial \theta_0} & \frac{\partial F}{\partial \theta_1} & \frac{\partial F}{\partial \theta_2} & \frac{\partial F}{\partial \theta_3} \end{pmatrix}. \tag{3.19}$$

Since $\mathbf{d}_{i,i+1} - \mathbf{d}_{i,i-1}$ is the vector connecting the pin joints on the i^{th} bar we can write

$$R(\theta_i)(\mathbf{d}_{i,i+1} - \mathbf{d}_{i,i-1}) = l \begin{pmatrix} \cos(\theta_i) \\ \sin(\theta_i) \end{pmatrix}.$$

Then

$$\frac{\partial F}{\partial m} = \begin{pmatrix} 0 & 0 & -l\sin(\theta_0) & -l\sin(\theta_1) & -l\sin(\theta_2) & -l\sin(\theta_3) \\ 0 & 0 & l\cos(\theta_0) & l\cos(\theta_1) & l\cos(\theta_2) & l\cos(\theta_3) \end{pmatrix}. \tag{3.20}$$

The nontrivial 2×2 subdeterminants are:

$$\begin{aligned}
-l^2\sin(\theta_0)\cos(\theta_1) + l^2\cos(\theta_0)\sin(\theta_1) &= l^2\sin(\theta_1 - \theta_0) \stackrel{\triangle}{=} g_1(m), \\
-l^2\sin(\theta_0)\cos(\theta_2) + l^2\cos(\theta_0)\sin(\theta_2) &= l^2\sin(\theta_2 - \theta_0) \stackrel{\triangle}{=} g_2(m), \\
-l^2\sin(\theta_0)\cos(\theta_3) + l^2\cos(\theta_0)\sin(\theta_3) &= l^2\sin(\theta_3 - \theta_0) \stackrel{\triangle}{=} g_3(m), \\
-l^2\sin(\theta_1)\cos(\theta_2) + l^2\cos(\theta_1)\sin(\theta_2) &= l^2\sin(\theta_2 - \theta_1) \stackrel{\triangle}{=} g_4(m), \\
-l^2\sin(\theta_1)\cos(\theta_3) + l^2\cos(\theta_1)\sin(\theta_3) &= l^2\sin(\theta_3 - \theta_1) \stackrel{\triangle}{=} g_5(m), \\
-l^2\sin(\theta_2)\cos(\theta_3) + l^2\cos(\theta_2)\sin(\theta_3) &= l^2\sin(\theta_3 - \theta_2) \stackrel{\triangle}{=} g_6(m).
\end{aligned}$$

To ensure that 0 is a regular value of F we must simply ensure that for all possible values of θ_i at least one $g_i(m) \neq 0$. To have $g_i(m) = 0$ for all i we must have

$\theta_{i+1} - \theta_i = \theta_{i+1,i} = 0$ or π for all i. However $\theta_{i+1,i} = 0$ or π is expressly forbidden by the restriction that the mechanism may not achieve any singular configuration.

■

While in the general four-bar mechanism the relations between the angles θ_i can be quite complicated (see, for example, [19] or [102]) they have a particularly simple form for the equal-sided case, namely

$$\theta_2 = \theta_0 + \pi, \qquad \theta_3 = \theta_0 - \pi. \tag{3.21}$$

From this we have the following set of equalities

$$\theta_{32} = \theta_{10}, \quad \theta_{21} = \theta_{03} = \pi - \theta_{10}, \quad \theta_{13} = \theta_{20} = \pi. \tag{3.22}$$

Thus we see that for the free-floating, equal-sided four-bar linkage the configuration is completely specified by the choice of one global angle and one joint angle. We arbitrarily choose θ_0 and θ_{10}. The singular points then correspond to $\theta_{10} = 0$ or π and after removing these points the configuration space is given by $S^1 \times \{(0, \pi) \cup (0, -\pi)\}$. Since the joint angle is not allowed to pass through the singular points we may arbitrarily choose either one of the connected components of this space to describe the configuration of our system with the additional requirement that the initial condition lie in the component we have chosen. Without loss of generality, then, we take $S = S^1 \times (0, \pi)$ as the configuration space of the free-floating equal-sided four-bar mechanism.

As in Yang and Krishnaprasad [103] we assume the inertial observer is placed at the system center of mass. (We can do this because the kinetic energy of the system is invariant under translations in inertial space and the configuration space can then be symplectically reduced by the translation group $\mathbb{R}^2$ as in Sreenath [82].) The total kinetic energy of the system in the center of mass frame is given by

$$T = \frac{1}{2} I \sum_{i=0}^{3} \omega_i^2 + \frac{1}{2} m \sum_{i=0}^{3} \|\dot{r}_i^c\|^2. \tag{3.23}$$

Following [103] we can write this as

$$T = \frac{1}{2} \langle \widetilde{\omega}, \widetilde{M} \widetilde{\omega} \rangle \tag{3.24}$$

where $\widetilde{\omega} = (\omega_0, \omega_1, \omega_2, \omega_3)$ and $\widetilde{M}$ is a 4x4 symmetric matrix whose elements for the equal-sided four-bar are

$$\begin{aligned} \widetilde{M}_{ii} &= I + \frac{3m}{8} \left(\|\mathbf{d}_+\|^2 + \|\mathbf{d}_-\|^2 \right), & (3.25) \\ \widetilde{M}_{i,i+1} &= \frac{m}{8} \left(\langle \mathbf{d}_-, R_{i+1,i} \mathbf{d}_+ \rangle - 3 \langle \mathbf{d}_+, R_{i+1,i} \mathbf{d}_- \rangle \right), & (3.26) \\ \widetilde{M}_{i,i+2} &= -\frac{m}{8} \left(\langle \mathbf{d}_+, R_{i+2,i} \mathbf{d}_+ \rangle + \langle \mathbf{d}_-, R_{i+2,i} \mathbf{d}_- \rangle \right). & (3.27) \end{aligned}$$

Here $R_{i,j} = R(\theta_i - \theta_j)$. From equation (3.21) we have

$$\begin{pmatrix} \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \omega_0 \\ \omega_1 \end{pmatrix} \tag{3.28}$$

where $\omega_i = \dot{\theta}_i$. Define

$$M = (\mathbb{1} \quad \mathbb{1}) \, \widetilde{M} \begin{pmatrix} \mathbb{1} \\ \mathbb{1} \end{pmatrix} \tag{3.29}$$

where $\mathbb{1}$ is the identity matrix. Then

$$T = \frac{1}{2} \langle \begin{pmatrix} \omega_0 \\ \omega_1 \end{pmatrix}, M \begin{pmatrix} \omega_0 \\ \omega_1 \end{pmatrix} \rangle. \tag{3.30}$$

M is symmetric and depends only on the joint angles and thus, given the relations (3.22), depends only on θ_{10}. We would like to express the entries in this matrix in terms of the parameters of the four-bar linkage. From equations (3.22), (3.25),(3.27), and (3.29), we have

$$\begin{aligned} M_{00} &= \widetilde{M}_{00} + \widetilde{M}_{02} + \widetilde{M}_{20} + \widetilde{M}_{22} \\ &= 2(\widetilde{M}_{00} + \widetilde{M}_{02}) \\ &= 2(I + \frac{3m}{8}(\|\mathbf{d}_+\|^2 + \|\mathbf{d}_-\|^2) + \frac{m}{8}(\|\mathbf{d}_+\|^2 + \|\mathbf{d}_-\|^2)) \\ &= 2I + m(\|\mathbf{d}_+\|^2 + \|\mathbf{d}_-\|^2), \qquad (3.31) \\ M_{11} &= \widetilde{M}_{11} + \widetilde{M}_{13} + \widetilde{M}_{31} + \widetilde{M}_{33} \\ &= 2(\widetilde{M}_{11} + \widetilde{M}_{13}) \\ &= 2(\widetilde{M}_{00} + \widetilde{M}_{02}) \\ &= M_{00}. \qquad (3.32) \end{aligned}$$

From equations (3.22), (3.26), and (3.29) we have

$$\begin{aligned} M_{01} &= \widetilde{M}_{10} + \widetilde{M}_{12} + \widetilde{M}_{30} + \widetilde{M}_{32} \\ &= 2(\widetilde{M}_{10} + \widetilde{M}_{12}) \\ &= \frac{m}{4}\left(\langle \mathbf{d}_-, R_{1,0}\mathbf{d}_+\rangle - 3\langle \mathbf{d}_+, R_{1,0}\mathbf{d}_-\rangle\right. \\ &\qquad \left. + \langle \mathbf{d}_-, R_{\pi-\theta_{10}}\mathbf{d}_+\rangle - 3\langle \mathbf{d}_+, R_{\pi-\theta_{10}}\mathbf{d}_-\rangle\right). \qquad (3.33) \end{aligned}$$

Now

$$\begin{aligned} R_{\pi-\theta_{10}} &= \begin{pmatrix} \cos(\pi-\theta_{10}) & -\sin(\pi-\theta_{10}) \\ \sin(\pi-\theta_{10}) & \cos(\pi-\theta_{10}) \end{pmatrix} \\ &= \begin{pmatrix} -\cos(\theta_{10}) & -\sin(\theta_{10}) \\ \sin(\theta_{10}) & -\cos(\theta_{10}) \end{pmatrix} = -R_{0,1}. \qquad (3.34) \end{aligned}$$

Plugging this result into equation (3.33) yields

$$\begin{aligned} M_{01} &= \frac{m}{4}\left(\langle \mathbf{d}_-, R_{1,0}\mathbf{d}_+\rangle - 3\langle \mathbf{d}_+, R_{1,0}\mathbf{d}_-\rangle\right. \\ &\qquad \left. - \langle \mathbf{d}_-, R_{0,1}\mathbf{d}_+\rangle + 3\langle \mathbf{d}_+, R_{0,1}\mathbf{d}_-\rangle\right) \end{aligned}$$

$$= \quad m\left(\langle \mathbf{d}_-, R_{1,0}\mathbf{d}_+\rangle - \langle \mathbf{d}_+, R_{1,0}\mathbf{d}_-\rangle\right)$$

$$= \quad 2m(d_+^1 d_-^2 - d_+^2 d_-^1)\sin(\theta_{10}). \tag{3.35}$$

where $d_\pm^j$ is the j^{th} component of $d_\pm$. Consider the diagram of a single bar in Figure 3.5. We have

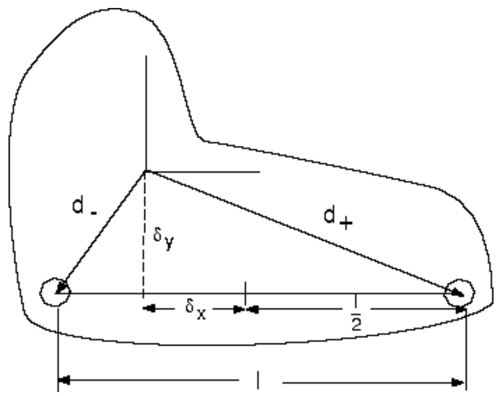

Figure 3.5: Single bar diagram

$$\mathbf{d}_+ = \begin{pmatrix} \frac{l}{2} + \delta_x \\ \delta_y \end{pmatrix} \qquad \mathbf{d}_- = \begin{pmatrix} -(\frac{l}{2} - \delta_x) \\ \delta_y \end{pmatrix}.$$

Thus

$$d_+^1 d_-^2 - d_+^2 d_-^1 = (\frac{l}{2} + \delta_x)(\delta_y) + (\delta_y)(\frac{l}{2} - \delta_x) = l\delta_y.$$

Plugging this into equation (3.35) gives

$$M_{01} = 2ml\delta_y sin(\theta_{10}). \tag{3.36}$$

Since M is symmetric we have $M_{01} = M_{10}$.

We now rewrite the kinetic energy in equation (3.30) as

$$T \quad = \quad \frac{1}{2}\langle \begin{pmatrix} \omega_0 \\ \omega_{10} \end{pmatrix}, \widehat{M}(\theta_{10}) \begin{pmatrix} \omega_0 \\ \omega_{10} \end{pmatrix} \rangle \tag{3.37}$$

where

$$\widehat{M} = \begin{pmatrix} M_{00} + 2M_{10} + M_{11} & M_{11} + M_{10} \\ M_{11} + M_{10} & M_{11} \end{pmatrix}. \tag{3.38}$$

This defines a Riemannian metric K on S given by

$$K(\theta_{10})(X, W) = \langle X, \widehat{M}(\theta_{10})W \rangle, \quad X, W \in T_{(\theta_0, \theta_{10})}S. \tag{3.39}$$

Each joint is equipped with an identical spring. Let the spring potential for each be given by $V_s(\theta_{i+1,i})$, $i = 0, 1, 2, 3$ with V_s twice continuously differentiable. The total potential energy is then

$$\begin{aligned} V(\theta_0, \theta_1, \theta_2, \theta_3) &= \sum_{i=0}^{3} V_s(\theta_{i+1,i}) \\ &= V_s(\theta_{10}) + V_s(\pi - \theta_{10}) + V_s(\theta_{10}) + V_s(\pi - \theta_{10}) \\ &= 2(V_s(\theta_{10}) + V_s(\pi - \theta_{10})) \triangleq V(\theta_{10}). \end{aligned} \tag{3.40}$$

We assume the potential energy is such that there exists $\alpha \in \{0, \pi\}$ such that

$$\left.\frac{\partial V}{\partial \theta_{10}}\right|_{\alpha} = 0, \qquad \left.\frac{\partial^2 V}{\partial \theta_{10}^2}\right|_{\alpha} > 0. \tag{3.41}$$

For convenience we take $V_s(\alpha) = 0$.

The Lagrangian is given by

$$L(\theta_{10}, \omega_{10}) = \frac{1}{2}\langle \begin{pmatrix} \omega_0 \\ \omega_{10} \end{pmatrix}, \widehat{M}(\theta_{10}) \begin{pmatrix} \omega_0 \\ \omega_{10} \end{pmatrix} \rangle - V(\theta_{10}). \tag{3.42}$$

Consider now the following action Φ_g of the Lie group S^1 on S.

$$\Phi_g(\theta_0, \theta_{10}) = (\theta_0 + g, \theta_{10}). \tag{3.43}$$

We have the following lemma.

Lemma 3.3.2 (S, K, V, S^1) *is a simple mechanical system with symmetry where the action of* S^1 *on* S *is given by equation* (3.43). *(For a definition and discussion of simple mechanical systems with symmetry see Section* 4.5 *of* [1].*)*

Proof Immediate since both the kinetic energy given by K and the potential energy V are invariant with respect to the given action. ■

Since the action is both free and proper the reduced space is a manifold. Recall that we defined $S = S^1 \times (0, \pi)$. The reduced (or shape) space is then $Q = (0, \pi)$ with the coordinate θ_{10}. In the language of the moving systems approach, Q is the configuration space and S is the ambient space. To slowly rotate the mechanism set $\theta_0 = \Omega t + \widehat{\theta}_0$ for some fixed initial offset $\widehat{\theta}_0$. Note that θ_0 and $\theta_0 + 2\pi$ are identified. The imposed motion on the four-bar is captured by the parametrized family of embeddings from Q into S given by

$$m_t(\theta_{10}) = \begin{pmatrix} \Omega t + \widehat{\theta}_0 \\ \theta_{10} \end{pmatrix}. \tag{3.44}$$

3.3.1 The nominal dynamics

The nominal dynamics is given by the nominal Lagrangian defined by setting $\Omega = 0$ in equation (3.44). Using this in equation (3.42) we have

$$L^{Nom}(\theta_{10}, \omega_{10}) = \frac{M_{11}}{2}\omega_{10}^2 - V(\theta_{10}). \tag{3.45}$$

The conjugate momentum, defined by the Legendre transform, is given by

$$p_{10} = M_{11}\omega_{10}.$$

The corresponding Hamiltonian is then

$$H^{Nom}(\theta_{10}, p_{10}) = \frac{p_{10}^2}{2M_{11}} + V(\theta_{10}) \tag{3.46}$$

and thus the nominal dynamics is

$$\dot{\omega}_{10} = \frac{p_{10}}{M_{11}}, \quad \dot{p}_{10} = -\frac{\partial V}{\partial \theta_{10}}. \tag{3.47}$$

Existence of periodic solutions in the nominal system

From equations (3.41) and (3.47), we see that there is an equilibrium point at $\theta_{10} = \alpha$ and $p_{10} = 0$. To ensure the existence of a periodic solution we appeal to the following theorem by Weinstein [98], paraphrased from [64].

Theorem 3.3.3 [98] *Consider $H : \mathbb{R}^{2n} \to \mathbb{R}$. If H is twice continuously differentiable near an equilibrium point z and the Hessian matrix at the equilibrium point is positive definite, then for sufficiently small ϵ any energy surface $H(z) = H(0) + \epsilon^2$ contains at least n periodic orbits of the associated Hamiltonian system.* ■

Using this theorem we obtain the following lemma.

Lemma 3.3.4 *If the total energy is sufficiently small, then there exists a periodic orbit around the equilibrium ($\theta_{10} = \alpha, p_{10} = 0$) for the nominal four-bar system defined by the Hamiltonian in equation* (3.46)*.*

Proof The Hessian matrix at the equilibrium of the Hamiltonian for the nominal system is

$$Hess = \left.\begin{pmatrix} \frac{\partial^2 H}{\partial \theta_{10}^2} & 0 \\ 0 & \frac{\partial^2 H}{\partial p_{10}^2} \end{pmatrix}\right|_{(\alpha,0)} = \left.\begin{pmatrix} \frac{\partial^2 V}{\partial \theta_{10}^2} & 0 \\ 0 & M_{11}^{-1} \end{pmatrix}\right|_{(\alpha,0)} . \tag{3.48}$$

Since M is positive definite by its construction, $M_{11} > 0$. By assumption the Hessian of V at α is positive definite. Therefore by Theorem (3.3.3) there is a periodic solution around the equilibrium if the energy is sufficiently small. ■

Action-angle coordinates for the nominal system

Since this is a one-degree-of-freedom system it is integrable and thus there exist action-angle coordinates (J, ϕ) [6]. Let $\Gamma(h)$ be the trajectory in phase space corresponding to the energy h. Then

$$J = \frac{1}{2\pi} \oint_{\Gamma(h)} p_{10} d\theta_{10}. \tag{3.49}$$

The trajectory $\Gamma(h)$ and thus the action depends on the form of $V(\theta_{10})$. We can write in general

$$\begin{aligned} J &= g_1(\theta_{10}, p_{10}), \qquad \theta_{10} = f_1(J, \phi), \\ \phi &= g_2(\theta_{10}, p_{10}), \qquad p_{10} = f_2(J, \phi). \end{aligned} \tag{3.50}$$

The action is a constant of the motion in the nominal system and we can express $H^{Nom}(\theta_{10}, p_{10}) = H^{Nom}(J)$.

3.3.2 The Hannay-Berry phase of the four-bar linkage

From equation (3.44), the velocity vector of the motion in S is

$$\frac{d}{dt}\left(m_t(\theta_{10})\right) = \begin{pmatrix} 0 \\ \omega_{10} \end{pmatrix} + \begin{pmatrix} \Omega \\ 0 \end{pmatrix}$$

and thus the tangent vector we need to project is

$$\mathcal{Z} \stackrel{\triangle}{=} \mathcal{Z}_t(m_t(q(t))) = \begin{pmatrix} \Omega \\ 0 \end{pmatrix}. \tag{3.51}$$

The projection of $\mathcal{Z}$ to $T_{m_t(q)} m_t(Q)$ with respect to the kinetic energy metric on S is given by $\mathcal{Z}^T = \mathcal{Z} - \mathcal{Z}^\perp$ where $\mathcal{Z}^\perp$ satisfies the following orthogonality condition.

$$K(\theta_{10})(\mathcal{Z}^\perp, X) = 0 \quad \forall X \in T_{m_t(q)} m_t(Q).$$

Examining equation (3.44) we see that any tangent vector $X \in T_{m_t(q)} m_t(Q)$ has the form

$$X = \begin{pmatrix} 0 \\ w \end{pmatrix}, \quad w \in T_q Q.$$

Since $\mathcal{Z}^T \in T_{m_t(q)} m_t(Q)$, we have, from $\mathcal{Z}^T = \mathcal{Z} - \mathcal{Z}^\perp$ and equation (3.51),

$$\begin{pmatrix} 0 \\ w_{\mathcal{Z}^T} \end{pmatrix} = \begin{pmatrix} \Omega \\ 0 \end{pmatrix} - \begin{pmatrix} \mathcal{Z}^{\perp_1} \\ \mathcal{Z}^{\perp_2} \end{pmatrix} \tag{3.52}$$

for some $w_{\mathcal{Z}^T} \in T_q Q$. Thus $\mathcal{Z}^{\perp_1} = \Omega$. Applying the orthogonality condition yields

$$\begin{aligned} 0 &= \langle \begin{pmatrix} \Omega \\ \mathcal{Z}^{\perp_2} \end{pmatrix}, \widehat{M} \begin{pmatrix} 0 \\ w \end{pmatrix} \rangle \\ &= w[(M_{11} + M_{10}(\theta_{10}))\Omega + M_{11}\mathcal{Z}^{\perp_2}], \quad \forall w \in T_q Q \end{aligned}$$

and so

$$\mathcal{Z}^\perp = \begin{pmatrix} \Omega \\ -\Omega \left[\frac{M_{10}(\theta_{10}) + M_{11}}{M_{11}} \right] \end{pmatrix}. \tag{3.53}$$

We thus have

$$\mathcal{Z}^T = \begin{pmatrix} 0 \\ \Omega \left[\frac{M_{10}(\theta_{10}) + M_{11}}{M_{11}} \right] \end{pmatrix}. \tag{3.54}$$

The pull-back of $\mathcal{Z}^T$ to $T_q Q$ by $[Tm]^{-1}$ is given by projection onto the second factor. Thus

$$Z \triangleq [Tm]^{-1} \mathcal{Z}^T = \Omega \left[\frac{M_{10}(\theta_{10}) + M_{11}}{M_{11}} \right]. \tag{3.55}$$

The function $\mathcal{P}(Z)$ defining the horizontal lift relative to the induced Cartan connection is then (following equation (3.7))

$$\mathcal{P}(Z)(\theta_{10}, p_{10}) = \Omega \left[\frac{M_{10}(\theta_{10}) + M_{11}}{M_{11}} \right] p_{10} = \Omega \left[\frac{M_{10}(f_1(J,\phi)) + M_{11}}{M_{11}} \right] f_2(J,\phi) \tag{3.56}$$

where we have expressed the function in terms of the action-angle coordinates given by equation (3.50). The flow of the nominal system induces an S^1 action on T^*Q and the average with respect to this action is simply the average over one cycle of the angle coordinate ϕ. Thus the Hamiltonian function defining the lift with respect to the Cartan-Hannay-Berry connection is

$$\begin{aligned} <\mathcal{P}(Z)> &= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{M_{10}(f_1(J,\phi))+M_{11}}{M_{11}}\right)\Omega f_2(J,\phi)d\phi \\ &= \frac{\Omega}{2\pi M_{11}}\int_0^{2\pi}\left(M_{11}+M_{10}(f_1(J,\phi))\right)f_2(J,\phi)d\phi \\ &= \frac{\Omega}{2\pi M_{11}}f_3(J) \end{aligned} \tag{3.57}$$

where we have defined the function

$$f_3(J) = \int_0^{2\pi}\left(M_{11}+M_{10}(f_1(J,\phi))\right)f_2(J,\phi)d\phi. \tag{3.58}$$

The horizontal lift of $\mathcal{Z}_t$ with respect to the Cartan-Hannay-Berry connection is (following equation (3.10))

$$(-X_{<\mathcal{P}(Z_t)>},\Omega) = \left(-\frac{\Omega}{2\pi M_{11}}\frac{\partial f_3(J)}{\partial J}\frac{\partial}{\partial\phi},\Omega\right).$$

Let $T = \frac{2\pi}{\Omega}$ be the time at which we complete one full revolution (imposed motion) of the mechanism. The Hannay-Berry phase is then

$$\Delta\phi = -\frac{\Omega}{2\pi M_{11}}\int_0^T\frac{\partial f_3(J)}{\partial J}dt = -\frac{\Omega T}{2\pi M_{11}}\frac{\partial f_3(J)}{\partial J} = -\frac{1}{M_{11}}\frac{\partial f_3(J)}{\partial J}. \tag{3.59}$$

Assuming the initial conditions are such that the periodic solutions of the nominal system are of small amplitude, we expand $V(\theta_{10})$ about the equilibrium point α.

$$V(\theta_{10}) = \frac{1}{2}\left.\frac{\partial^2 V}{\partial\theta_{10}^2}\right|_\alpha(\theta_{10}-\alpha)^2 + O((\theta_{10}-\alpha)^3)$$

Perform a change of coordinates $\psi_{10} = \theta_{10}-\alpha$ to get

$$V(\psi_{10}) = \frac{1}{2}\left.\frac{\partial^2 V}{\partial\psi_{10}^2}\right|_0\psi_{10}^2 + O((\psi_{10})^3). \tag{3.60}$$

In the small angle limit we take the potential only to second order. The nominal Hamiltonian is then

$$H^{Nom} = \frac{p_{10}^2}{2M_{11}} + \frac{k}{2}\psi_{10}^2 \tag{3.61}$$

where we have made the obvious definition for k. This is the Hamiltonian for a harmonic oscillator. From [6] we know that

$$J = \frac{h}{\omega} \tag{3.62}$$

where $\omega = \sqrt{\frac{k}{M_{11}}}$ is the frequency of oscillation and h is the energy corresponding to the initial conditions. From equations (3.61) and (3.62) we have

$$J = \frac{p_{10}^2 + kM_{11}\psi_{10}^2}{2\sqrt{kM_{11}}}. \tag{3.63}$$

The angle variable is the phase of the oscillation. Therefore

$$\psi_{10} = Acos(\phi), \quad A = \left[\frac{2J}{\sqrt{kM_{11}}}\right]^{\frac{1}{2}} \tag{3.64}$$

and thus

$$\theta_{10} = \alpha + \left[\frac{2J}{\sqrt{kM_{11}}}\right]^{\frac{1}{2}} \cos\phi = f_1(J,\phi). \tag{3.65}$$

From Hamilton's equations for the nominal system of equation (3.61), we have that $\dot{\psi}_{10} = \frac{p_{10}}{M_{11}}$ and therefore, applying equation (3.64), we have

$$p_{10} = -\left[2J\sqrt{kM_{11}}\right]^{\frac{1}{2}} \sin\phi = f_2(J,\phi). \tag{3.66}$$

Using equations (3.65), (3.66), and (3.36) in the function $\mathcal{P}(Z)$ given by equation (3.56), we obtain

$$\mathcal{P}(Z) = -\Omega\left[1 + \frac{2ml\delta_y}{M_{11}} \sin\left(\alpha + \left[\frac{2J}{\sqrt{kM_{11}}}\right]^{\frac{1}{2}} \cos\phi\right)\right]\left[2J\sqrt{kM_{11}}\right]^{\frac{1}{2}} \sin\phi. \tag{3.67}$$

The average of $\mathcal{P}(Z_t)$ over one period of ϕ is then

$$
\begin{aligned}
<\mathcal{P}(Z)> &= -\frac{\Omega\left[2J\sqrt{kM_{11}}\right]^{\frac{1}{2}}}{2\pi}\left[\int_0^{2\pi}\sin\phi d\phi\right. \\
&\qquad \left. + \int_0^{2\pi}\frac{2ml\delta_y}{M_{11}}\sin\left(\alpha+\left[\frac{2J}{\sqrt{kM_{11}}}\right]^{\frac{1}{2}}\cos\phi\right)\sin\phi d\phi\right] \\
&= -\frac{\Omega\sqrt{kM_{11}}}{2\pi}\left[\frac{2ml\delta_y}{M_{11}}\right]\cos\left(\alpha+\left[\frac{2J}{\sqrt{kM_{11}}}\right]^{\frac{1}{2}}\cos\phi\right)\Bigg|_0^{2\pi} \\
&= 0.
\end{aligned}
\tag{3.68}
$$

Thus in the limit of small oscillations of the four-bar linkage (i.e. with linear springs) we have that the Hannay-Berry phase is zero.

3.3.3 Comments on the four-bar results

In this section we have explored the Hannay-Berry phase for a rotating, equal-sided, spring-jointed four-bar mechanism and have shown that in the small oscillation limit the Hannay-Berry phase is zero.

The calculation of action-angle coordinates in general involves solving (analytically) the nominal dynamics and thus is not practical for generic spring potentials. One can, however, investigate the effect of the imposed rotation through numerical simulations. In Figure 3.6 we show a simulation of a four-bar mechanism with a quartic spring potential undergoing an imposed rotation at a rate of 0.001 rad/s, two orders of magnitude smaller than the nominal frequency.

The full dynamics are simulated; i.e. we start from the general Hamiltonian in equation (3.14), show that for the four-bar mechanism we have

$$
H(\theta_{10}, p_{10}) = \frac{p_{10}^2}{2M_{11}} + V(\theta_{10}) - \Omega\left(\frac{M_{10}+M_{11}}{M_{11}}\right)p_{10} - \frac{\Omega^2}{2}\left(\frac{M_{00}^2 - M_{10}^2}{M_{11}}\right)
$$

and then numerically integrate the corresponding system.

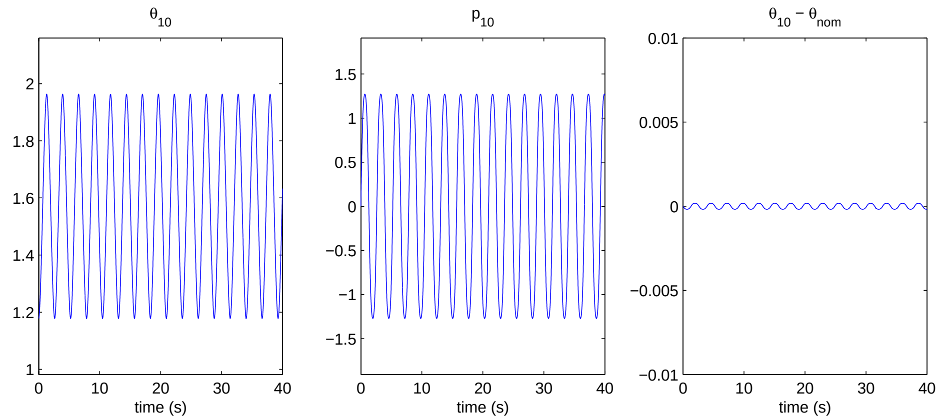

Figure 3.6: Quartic spring potential, $\Omega = 0.001$ rad/s

The first and second plots in Figure 3.6 show the time evolution of the joint angle and the conjugate momentum while the third plot shows the difference between the time evolution of the joint angle and the time evolution of the joint angle in the nominal system, i.e. one in which $\Omega = 0$. From this plot we see that there is a small amplitude, zero average periodic difference in the time evolution of the nominal system and of the true system. The effect is the same for other (small) values of Ω and we thus conclude that the Hannay-Berry phase is zero. From these results we see that the equal-sided, spring-jointed, four-bar mechanism does not constitute a viable mechanism for sensing rotational motion.

As a point of exploration, in Figure 3.7 we show the results for an imposed rotation rate of the same order of magnitude as the frequency of the nominal dynamics. The third plot clearly shows that when the adiabatic assumption is strongly violated then the dynamics are severely affected and the geometric theory proposed here no longer applies. In Chapter 4 we extend the moving systems approach to handle the intermediate case when the imposed motion is slow but no longer assumed to be adiabatic.

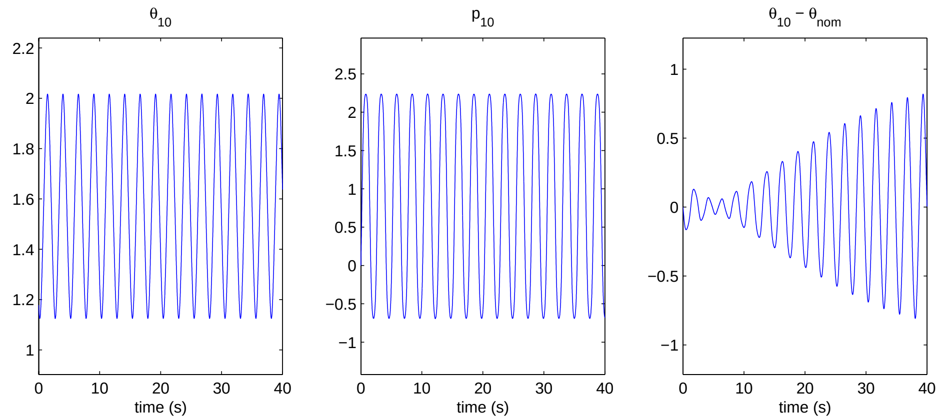

Figure 3.7: Quartic spring potential, $\Omega = 1.0$ rad/s

3.4 The vibrating ring gyroscope

In this section we derive the Hannay-Berry phase for the vibrating ring gyroscope. Using the moving systems approach we first find an explicit formula for the phase shift under linearizing assumptions and show that this result matches that of G.H. Bryan [20]. We then derive a correction term to account for the nonlinear effects arising from the imposed motion. In this work we are interested in the effects of the imposed rotatory motion and as a consequence choose to simplify the analysis of the ring dynamics by assuming the ring has no cross-sectional area. This choice also allows a direct comparison to the results derived by Bryan. A more comprehensive treatment based on the geometrically exact theory of rods could be developed to understand the detailed dynamics of the ring itself (see, for example, [74]).

Consider a thin ring of length L and line density σ. The body is given by $\mathcal{B} = \{b : b \in [0, L]\}$. Let θ be the mapping given by

$$\begin{aligned} \theta : \mathcal{B} &\rightarrow S^1 \\ b &\mapsto \left(\frac{2\pi}{L}\right) b \end{aligned}$$

allowing us to parametrize the ring by $\theta \in [0, 2\pi]$. We define the reference configuration to be a circular ring of radius a centered on an inertial reference frame and $(w(\theta), \gamma(\theta))$ to be the radial and angular deformations from this reference respectively. To maintain integrity of the ring we require $w(0) = w(2\pi)$ and $\gamma(0) = \gamma(2\pi)$. In standard cylindrical coordinates the configuration of the ring is given by $(a + w(\theta), \theta + \gamma(\theta))$. Let $\mathcal{C}$ be the space of all smooth deformations of the ring. (We do not discuss here the explicit infinite dimensional manifold structure for $\mathcal{C}$ and associated structures, although it is standard as in [56]). Since we are interested in imposed rotational movements of the ring (as a sensor), we split $\gamma(\theta) = \psi + \alpha(\theta)$ with $\alpha(0) = \alpha(2\pi)$ where ψ, independent of θ, is a global rotation.

We now use the following argument of Rayleigh [77]. Since the ring is thin the forces resisting bending are small in comparison to those which resist extension. In the limiting case of an infinitely thin ring the flexural vibrations become independent of any extension of the circumference as a whole and one may assume that each part of the circumference retains its natural length throughout the motion. Under this condition we say the ring is *inextensible*. Viewing the deformed ring as a curve in $\mathbb{R}^2$, a point on the curve is given in Cartesian coordinates by

$$\begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} = \begin{pmatrix} (a + w(\theta))\cos(\theta + \gamma(\theta)) \\ (a + w(\theta))\sin(\theta + \gamma(\theta)) \end{pmatrix}. \tag{3.69}$$

Equating the lengths of an arbitrary section of the circumference of the reference configuration to the length of the same section in the deformed configuration yields

$$\begin{aligned} \int_{\theta_1}^{\theta_2} a d\theta &= \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2} d\theta \\ &= \int_{\theta_1}^{\theta_2} \sqrt{(a + w(\theta))^2 \left(1 + \frac{\partial \gamma}{\partial \theta}\right)^2 + \left(\frac{\partial w}{\partial \theta}\right)^2} d\theta. \end{aligned} \tag{3.70}$$

Since θ_1 and θ_2 are arbitrary we have

$$0 = (a + w)^2 \left(1 + \frac{\partial \gamma}{\partial \theta}\right)^2 + \left(\frac{\partial w}{\partial \theta}\right)^2 - a^2$$

$$
\begin{aligned}
&= 2aw + w^2 + 2(a^2 + 2aw + w^2)\frac{\partial \gamma}{\partial \theta} \\
&\quad + (a^2 + 2aw + w^2)\left(\frac{\partial \gamma}{\partial \theta}\right)^2 + \left(\frac{\partial w}{\partial \theta}\right)^2 .
\end{aligned} \tag{3.71}
$$

From here on we assume the deformations are small and so we keep only terms to first order in equation (3.71). The inextensibility condition then requires that

$$
w = -a\frac{\partial \gamma}{\partial \theta} = -a\frac{\partial \alpha}{\partial \theta}. \tag{3.72}
$$

From the above the space $\mathcal{C}$ is given by

$$
\mathcal{C} = \{(\psi, \alpha) | \psi \in S^1, \alpha : S^1 \to S^1, \alpha(0) = \alpha(2\pi), \alpha \; smooth\}.
$$

Any $W \in T_{(\psi,\alpha)}\mathcal{C}$ has the form (with overdot denoting the partial derivative with respect to time)

$$
W = \begin{pmatrix} \dot{\psi} \\ \dot{\alpha}(\theta) \end{pmatrix}. \tag{3.73}
$$

The kinetic energy is easily verified to be

$$
KE = \frac{1}{2}\int_0^{2\pi} \left[\left(1 - \frac{\partial \alpha}{\partial \theta}\right)^2 \left(\dot{\psi}^2 + 2\dot{\psi}\dot{\alpha}(\theta) + \dot{\alpha}^2(\theta)\right) + \left(\frac{\partial \dot{\alpha}}{\partial \theta}\right)^2 \right] \sigma a^3 d\theta \tag{3.74}
$$

where equation (3.72) has been used to express w in terms of α. This defines an inner product on $\mathcal{C}$ given by

$$
\begin{aligned}
(W_1, W_2) &= \int_0^{2\pi} \left[\left(1 - \frac{\partial \alpha}{\partial \theta}\right)^2 \left(\dot{\psi}_1\dot{\psi}_2 + \dot{\psi}_2\dot{\alpha}_1(\theta) + \dot{\psi}_1\dot{\alpha}_2(\theta) + \dot{\alpha}_1(\theta)\dot{\alpha}_2(\theta)\right) \right. \\
&\quad \left. + \frac{\partial \dot{\alpha}_1}{\partial \theta}\frac{\partial \dot{\alpha}_2}{\partial \theta} \right] \sigma a^3 d\theta.
\end{aligned} \tag{3.75}
$$

As in Bryan, we take the potential energy due to the bending of the ring to be proportional to the change in curvature squared of the ring. That is

$$
V = \frac{\beta}{2}\int_0^{2\pi} \left(\kappa_\alpha(\theta) - \kappa_{\alpha\equiv 0}(\theta)\right)^2 a d\theta \tag{3.76}
$$

where $\kappa_\alpha(\theta)$ is the curvature of the ring at the material point θ under the deformation α and β is a material constant. Following [45] we have the following. For a curve $(r(t), \phi(t))$ defined in polar coordinates, the curvature is given by

$$\kappa(t) = \frac{(2\dot{r}^2\dot{\phi} + r\dot{r}\ddot{\phi} - r\ddot{r}\dot{\phi} + r^2\dot{\phi}^3)}{(\dot{r}^2 + r^2\dot{\phi}^2)^{\frac{3}{2}}}. \tag{3.77}$$

In the $\theta-$parametrization, the configuration of the ring under the deformation (α, w) is given by the curve

$$\begin{aligned} r(\theta) &= a + w(\theta), \\ \phi(\theta) &= \theta + \psi + \alpha(\theta). \end{aligned} \tag{3.78}$$

Using equation (3.78) in (3.77) to express the curvature of the ring under the deformation (w, α), we find (keeping only terms to first-order in the numerator and denominator)

$$\kappa_{\alpha,w}(\theta) \approx \frac{(-a\frac{\partial^2 w}{\partial\theta^2} + a^2 + 2aw + 3a^2\frac{\partial\alpha}{\partial\theta})}{\left(a^2 + 3a^2 w + 3a^3\frac{\partial\alpha}{\partial\theta}\right)} \tag{3.79}$$

Writing the curvature in terms of α alone using the inextensibility condition of equation (3.72) yields

$$\kappa_\alpha(\theta) = \frac{\left(\frac{\partial^3\alpha}{\partial\theta^3} + \frac{\partial\alpha}{\partial\theta} + 1\right)}{a}. \tag{3.80}$$

From this we have $\kappa_{\alpha\equiv 0}(\theta) = \frac{1}{a}$. Using these results in the potential energy of equation (3.76) yields

$$V = \frac{\beta}{2a}\int_0^{2\pi}\left[\frac{\partial^3\alpha}{\partial\theta^3} + \frac{\partial\alpha}{\partial\theta}\right]^2 d\theta. \tag{3.81}$$

In Bryan's original work the potential energy included also a term capturing the work done in stretching the ring and the work done against an attracting force which he introduced to separate the effects of the centrifugal force from the remaining terms. These two terms are related by a simple equation involving the rate of the imposed rotation and at the conclusion of his analysis Bryan chooses the attracting force so as to cancel the tension, leaving only the work done in bending. Here we take a simpler approach, similar to Rayleigh, and omit those terms at the outset.

The standard Lagrangian function is defined to be the kinetic minus potential energies and is given here by

$$
\begin{aligned}
L &= \frac{1}{2}\int_0^{2\pi}\left[\left(1-\frac{\partial\alpha}{\partial\theta}\right)^2\left(\dot{\psi}^2+2\dot{\psi}\dot{\alpha}(\theta)+\dot{\alpha}^2(\theta)\right)+\left(\frac{\partial\dot{\alpha}}{\partial\theta}\right)^2\right]\sigma a^3 d\theta \\
&\quad -\frac{\beta}{2a}\int_0^{2\pi}\left[\frac{\partial^3\alpha}{\partial\theta^3}+\frac{\partial\alpha}{\partial\theta}\right]^2 d\theta. \qquad (3.82)
\end{aligned}
$$

Consider now the following action Φ_g of S^1 on $\mathcal{C}$.

$$\Phi_g(\psi,\alpha)=(\psi+g,\alpha). \qquad (3.83)$$

We have the following lemma.

Lemma 3.4.1 $(\mathcal{C},(\cdot,\cdot),V,S^1)$ *is a simple mechanical system with symmetry where the action of* S^1 *on* $\mathcal{C}$ *is given by equation* (3.83).

Proof Immediate since both $(\cdot,\cdot)$ and V are invariant under the given action of S^1 on $\mathcal{C}$. ∎

Since the given action is both free and proper, the reduced space $Q=\mathcal{C}/S^1$ given by

$$Q=\{\alpha : S^1\rightarrow S^1|\alpha(0)=\alpha(2\pi),\alpha\ smooth\}$$

is also a manifold. To fix notation in relation to the general theory presented in Section 3.2, we note that $Q=\mathcal{C}/S^1$ is the configuration space for the ring and $S=\mathcal{C}$ is the ambient space in which Q is moved. To slowly rotate the ring we set

$\psi = \psi_0 + \Omega t$ (identifying $\psi = 0$ with $\psi = 2\pi$) for some small Ω and some fixed initial offset ψ_0 so that the embedding from Q to S is given by

$$m_t(\alpha(\theta)) = (\psi_0 + \Omega t, \alpha(\theta)). \tag{3.84}$$

3.4.1 The nominal dynamics

The nominal dynamics is given by setting $\Omega = 0$ in equation (3.84). Applying this to equation (3.82) yields the nominal Lagrangian.

$$\begin{aligned} L^{Nom}(\alpha, \dot{\alpha}) &= \int_0^{2\pi} \left\{ \frac{\sigma a^3}{2} \left[\left(1 - \frac{\partial \alpha}{\partial \theta}\right)^2 \dot{\alpha}^2 + \left(\frac{\partial \dot{\alpha}}{\partial \theta}\right)^2 \right] \right. \\ &\qquad \left. - \frac{\beta}{2a} \left[\frac{\partial^3 \alpha}{\partial \theta^3} + \frac{\partial \alpha}{\partial \theta} \right]^2 \right\} d\theta \end{aligned} \tag{3.85}$$

The action integral for this Lagrangian is defined to be

$$\mathcal{J}(\alpha, \dot{\alpha}) \triangleq \int_a^b L^{Nom}(\alpha, \dot{\alpha}) dt.$$

The Euler-Lagrange equations for this system are found by applying Hamilton's principle of critical action (see, e.g. [1]) which states that

$$\delta \mathcal{J}(\alpha, \dot{\alpha}) = \delta \int_a^b L^{Nom}(\alpha, \dot{\alpha}) dt = 0$$

for all variations among paths $\eta(t)$ in Q with fixed end-points. Applying the variation yields

$$\begin{aligned} \delta \mathcal{J}(\alpha, \dot{\alpha}) &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathcal{J}(\alpha + \epsilon\eta, \dot{\alpha} + \epsilon\dot{\eta}) \\ &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \int_a^b \int_0^{2\pi} \left\{ \frac{\sigma a^3}{2} \left[\left(1 - \frac{\partial(\alpha + \epsilon\eta)}{\partial \theta}\right)^2 (\dot{\alpha} + \epsilon\dot{\eta})^2 + \left(\frac{\partial(\dot{\alpha} + \epsilon\dot{\eta})}{\partial \theta}\right)^2 \right] \right. \\ &\qquad \left. - \frac{\beta}{2a} \left[\frac{\partial^3(\alpha + \epsilon\eta)}{\partial \theta^3} + \frac{\partial(\alpha + \epsilon\eta)}{\partial \theta} \right]^2 \right\} d\theta dt \end{aligned}$$

$$
\begin{aligned}
= \quad & \int_a^b \int_0^{2\pi} \left\{ \sigma a^3 \left[-\left(1 - \frac{\partial \alpha}{\partial \theta}\right) \dot{\alpha}^2 \frac{\partial \eta}{\partial \theta} + \left(1 - \frac{\partial \alpha}{\partial \theta}\right)^2 \dot{\alpha}\dot{\eta} + \frac{\partial \dot{\alpha}}{\partial \theta}\frac{\partial \dot{\eta}}{\partial \theta} \right] \right. \\
& \left. -\frac{\beta}{a}\left[\frac{\partial^3 \alpha}{\partial \theta^3} + \frac{\partial \alpha}{\partial \theta}\right]\left[\frac{\partial^3 \eta}{\partial \theta^3} + \frac{\partial \eta}{\partial \theta}\right] \right\} d\theta dt.
\end{aligned}
$$

Using integration by parts repeatably on the space variable and the fact that for any element $\xi(\cdot) \in Q$ we have $\xi(0) = \xi(2\pi)$, the variation of the action can be rewritten as

$$
\begin{aligned}
\delta \mathcal{J}(\alpha, \dot{\alpha}) \quad = \quad & \int_a^b \int_0^{2\pi} \left\{ \sigma a^3 \left[2\left(1 - \frac{\partial \alpha}{\partial \theta}\right) \frac{\partial \dot{\alpha}}{\partial \theta}\dot{\alpha}\eta - \frac{\partial^2 \alpha}{\partial \theta^2}\dot{\alpha}^2\eta + \left(1 - \frac{\partial \alpha}{\partial \theta}\right)^2 \dot{\alpha}\dot{\eta} \right. \right. \\
& \left. \left. -\frac{\partial^2 \dot{\alpha}}{\partial \theta^2}\dot{\eta} \right] + \frac{\beta}{a}\left[\frac{\partial^6 \alpha}{\partial \theta^6} + 2\frac{\partial^4 \alpha}{\partial \theta^4} + \frac{\partial^2 \alpha}{\partial \theta^2}\right]\eta \right\} d\theta dt.
\end{aligned}
$$

Using integration by parts once again, this time on the time variable, and utilizing the end point condition on the variations η yields

$$
\begin{aligned}
\delta \mathcal{J}(\alpha, \dot{\alpha}) \quad = \quad & \int_a^b \int_0^{2\pi} \left\{ \sigma a^3 \left[4\left(1 - \frac{\partial \alpha}{\partial \theta}\right) \frac{\partial \dot{\alpha}}{\partial \theta}\dot{\alpha} - \frac{\partial^2 \alpha}{\partial \theta^2}\dot{\alpha}^2 - \left(1 - \frac{\partial \alpha}{\partial \theta}\right)^2 \ddot{\alpha} + \frac{\partial^2 \ddot{\alpha}}{\partial \theta^2} \right] \right. \\
& \left. +\frac{\beta}{a}\left[\frac{\partial^6 \alpha}{\partial \theta^6} + 2\frac{\partial^4 \alpha}{\partial \theta^4} + \frac{\partial^2 \alpha}{\partial \theta^2}\right] \right\} \eta d\theta dt.
\end{aligned}
$$

Since this must equal zero for all variations η we obtain the Euler-Lagrange equations for the nominal system.

$$
\begin{aligned}
0 \quad = \quad & \sigma a^3 \left[\frac{\partial^2 \ddot{\alpha}}{\partial \theta^2} - \frac{\partial^2 \alpha}{\partial \theta^2}\dot{\alpha}^2 + 4\left(1 - \frac{\partial \alpha}{\partial \theta}\right) \frac{\partial \dot{\alpha}}{\partial \theta}\dot{\alpha} - \left(1 - \frac{\partial \alpha}{\partial \theta}\right)^2 \ddot{\alpha} \right] \\
& +\frac{\beta}{a}\left[\frac{\partial^6 \alpha}{\partial \theta^6} + 2\frac{\partial^4 \alpha}{\partial \theta^4} + \frac{\partial^2 \alpha}{\partial \theta^2}\right]. \qquad (3.86)
\end{aligned}
$$

To simplify this difficult nonlinear partial differential equation we use the assumption that the deformations are small and replace the above equation by its linearization, resulting in the following equation of motion for α.

$$
\sigma a^3 \left[\frac{\partial^2 \ddot{\alpha}}{\partial \theta^2} - \ddot{\alpha} \right] + \frac{\beta}{a}\left[\frac{\partial^6 \alpha}{\partial \theta^2} + 2\frac{\partial^4 \alpha}{\partial \theta^4} + \frac{\partial^2 \alpha}{\partial \theta^2}\right] = 0. \qquad (3.87)
$$

Existence and uniqueness of solutions to the linearized nominal dynamics

To establish the existence and uniqueness of solutions to equation (3.87) we will appeal to methods of functional analysis and in particular to the real version of Stone's theorem. (For a more detailed presentation of the methods used here see [4, 56]. Further examples can also be found in [74].)

We first need a few definitions to establish notation. Define the spaces

$$L^p(U, \mathbb{R}^n) = \left\{ u : U \to \mathbb{R}^n | \, u \; smooth, \|u\|_{L^p} = \left(\int_U \|u\|^p dx \right)^{\frac{1}{p}} < \infty \right\} \tag{3.88}$$

for $1 \leq p \leq \infty$ (with $\|u\|_{L^\infty} = \operatorname{ess\,sup}_{x \in U} \|u(x)\|$). Here $\| \cdot \|$ is the standard Euclidean metric on $\mathbb{R}^n$. Define $W^{1,p}(U, \mathbb{R}^n)$ as

$$W^{1,p}(U, \mathbb{R}^n) = \{ u : U \to \mathbb{R}^n | \, u \; smooth, u \in L^p(U, \mathbb{R}^n), Du \in L^p(U, \mathbb{R}^n) \} \tag{3.89}$$

and similarly define the Sobolev spaces $W^{s,p}(U, \mathbb{R}^n)$ for s a positive integer. (Here D is the Frechét derivative.)

Stone's theorem (see, e.g. [104], Section IX.9 or [56], Section 6.2) establishes existence and uniqueness of solutions for linear systems defined by a skew-adjoint operator on a real Hilbert space.

Theorem 3.4.2 (Real version of Stone's Theorem) *Let $\mathcal{A}$ be a skew-adjoint operator on a real Hilbert space (i.e. $\mathcal{A} = -\mathcal{A}^*$). Then $\mathcal{A}$ generates a one-parameter unitary group. Conversely, if $\mathcal{A}$ generates a one-parameter unitary group, then it is skew-adjoint.* ■

We now define the appropriate operator and real Hilbert space for the linearized version of the vibrating ring defined by equation (3.87). Let $k = \frac{\sigma a^4}{\beta}$ and define the operators L and P by

$$L = \left[\frac{\partial^2}{\partial\theta^2} - \mathbb{I}\right], \tag{3.90}$$

$$P = \left[\frac{\partial^6}{\partial\theta^6} + 2\frac{\partial^4}{\partial\theta^4} + \frac{\partial^2}{\partial\theta^2}\right] \tag{3.91}$$

where $\mathbb{I}$ is the identity operator. Equation (3.87) then reads

$$\ddot{\alpha} = -\frac{1}{k}L^{-1}P\alpha. \tag{3.92}$$

Let $x_1 = \alpha$ and $x_2 = \dot{\alpha}$. Then

$$\begin{aligned}\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} &= \begin{pmatrix}0 & \mathbb{I}\\ -\frac{1}{k}L^{-1}P & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}\\ &= \mathcal{A}\begin{pmatrix}x_1\\ x_2\end{pmatrix}\end{aligned} \tag{3.93}$$

which defines the operator $\mathcal{A}$. The corresponding space for this problem is

$$\begin{aligned}\mathcal{H} = \big\{&(x_1, x_2) \in W^{3,2}(S^1, \mathbb{R}^2) \times W^{1,2}(S^1, \mathbb{R}^2) | D^i x_1(0) = D^i x_1(2\pi),\\ &i = 0, 1, 2, 3,\ D^j x_2(0) = D^j x_2(2\pi),\ j = 0, 1\big\}.\end{aligned} \tag{3.94}$$

To make $\mathcal{H}$ into a Hilbert space we endow it with the total energy inner product. Taking the kinetic energy, equation (3.74), and potential energy, equation (3.81), to second order in $\alpha, \dot{\alpha}$, the total energy for the linearized system can be expressed as

$$E_{Lin}^{Nom} = \frac{1}{2}\int_0^{2\pi}\left[k\dot{\alpha}^2 + k\left(\frac{\partial\dot{\alpha}}{\partial\theta}\right)^2 + \left(\frac{\partial^3\alpha}{\partial\theta^3} + \frac{\partial\alpha}{\partial\theta}\right)^2\right]\frac{\beta}{a}d\theta \tag{3.95}$$

and thus the inner product on $\mathcal{H}$ is

$$\langle x, y\rangle = \frac{1}{2}\int_0^{2\pi}\left[kx_2y_2 + k\frac{\partial x_2}{\partial\theta}\frac{\partial y_2}{\partial\theta} + \left(\frac{\partial^3 x_1}{\partial\theta^3} + \frac{\partial x_1}{\partial\theta}\right)\left(\frac{\partial^3 y_1}{\partial\theta^3} + \frac{\partial y_1}{\partial\theta}\right)\right]\frac{\beta}{a}d\theta. \tag{3.96}$$

The domain of definition of $\mathcal{A}$ is

$$\mathcal{D}(\mathcal{A}) = \big\{(x_1, x_2) \in W^{6,2}(S^1, \mathbb{R}^2) \times W^{1,2}(S^1, \mathbb{R}^2) | D^i x_1(0) = D^i x_1(2\pi), \\ i = 0, 1, \ldots, 6,\ D^j x_2(0) = D^j x_2(2\pi),\ j = 0, 1\big\}. \tag{3.97}$$

We have the following theorem.

Theorem 3.4.3 *$\mathcal{A}$ generates a one-parameter unitary group.*

Proof We show that $\mathcal{A}$ is skew-adjoint, i.e. that

$$\langle \mathcal{A}x, y\rangle = -\langle x, \mathcal{A}y\rangle. \tag{3.98}$$

From equations (3.93) and (3.96)

$$\begin{aligned}
\langle \mathcal{A}x, y\rangle &= \frac{1}{2}\int_0^{2\pi} \Big[-y_2 L^{-1} P x_1 - \frac{\partial y_2}{\partial \theta}\frac{\partial}{\partial \theta}\left(L^{-1} P x_1\right) \\
&\quad + \left(\frac{\partial^3 x_2}{\partial \theta^3} + \frac{\partial x_2}{\partial \theta}\right)\left(\frac{\partial^3 y_1}{\partial \theta^3} + \frac{\partial y_1}{\partial \theta}\right)\Big]\frac{\beta}{a} d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \Big[-y_2 L^{-1} P x_1 + y_2 \frac{\partial^2}{\partial \theta^2}\left(L^{-1} P x_1\right) \\
&\quad - x_2 \frac{\partial^6 y_1}{\partial \theta^6} - 2 x_2 \frac{\partial^4 y_1}{\partial \theta^4} - x_2 \frac{\partial^2 y_1}{\partial \theta^2}\Big]\frac{\beta}{a} d\theta \\
&\quad + \Big[-y_2 \frac{\partial}{\partial \theta}\left(L^{-1} P x_1\right)\Big|_0^{2\pi} + \frac{\partial^2 x_2}{\partial \theta^2}\frac{\partial^3 y_1}{\partial \theta^3}\Big|_0^{2\pi} - \frac{\partial x_2}{\partial \theta}\frac{\partial^4 y_1}{\partial \theta^4}\Big|_0^{2\pi} + x_2 \frac{\partial^5 y_1}{\partial \theta^5}\Big|_0^{2\pi} \\
&\quad + 2 x_2 \frac{\partial^3 y_1}{\partial \theta^3}\Big|_0^{2\pi} + \frac{\partial^2 x_2}{\partial \theta^2}\frac{\partial y_1}{\partial \theta}\Big|_0^{2\pi} - \frac{\partial x_2}{\partial \theta}\frac{\partial^2 y_1}{\partial \theta^2}\Big|_0^{2\pi} + x_2 \frac{\partial y_1}{\partial \theta}\Big|_0^{2\pi}\Big]\frac{\beta}{a} \\
&= \frac{1}{2}\int_0^{2\pi} \Big[-y_2 L^{-1} P x_1 + y_2 \frac{\partial^2}{\partial \theta^2}\left(L^{-1} P x_1\right) \\
&\quad - x_2 \left(\frac{\partial^6 y_1}{\partial \theta^6} + 2\frac{\partial^4 y_1}{\partial \theta^4} + \frac{\partial^2 y_1}{\partial \theta^2}\right)\Big]\frac{\beta}{a}
\end{aligned} \tag{3.99}$$

where we have first used integration by parts repeatably and then the boundary conditions on elements of $\mathcal{H}$ in the domain of $\mathcal{A}$. Now

$$\int_0^{2\pi} x_2 \left(\frac{\partial^6 y_1}{\partial \theta^6} + 2\frac{\partial^4 y_1}{\partial \theta^4} + \frac{\partial^2 y_1}{\partial \theta^2}\right)\frac{\beta}{a} d\theta = \int_0^{2\pi} x_2 P y_1 \frac{\beta}{a} d\theta$$

$$
\begin{aligned}
&= \int_0^{2\pi} x_2 L L^{-1} P y_1 \frac{\beta}{a} d\theta \\
&= \int_0^{2\pi} \left[-x_2 L^{-1} P y_1 + x_2 \frac{\partial^2}{\partial \theta^2} \left(L^{-1} P y_1 \right) \right] \frac{\beta}{a} d\theta \\
&= \int_0^{2\pi} \left[-x_2 L^{-1} P y_1 - \frac{\partial x_2}{\partial \theta} \frac{\partial}{\partial \theta} \left(L^{-1} P y_1 \right) \right] \frac{\beta}{a} d\theta - x_2 \frac{\partial}{\partial \theta} \left(L^{-1} P y_1 \right) \Big|_0^{2\pi} \frac{\beta}{a} \\
&= -\int_0^{2\pi} \left[x_2 L^{-1} P y_1 + \frac{\partial x_2}{\partial \theta} \frac{\partial}{\partial \theta} \left(L^{-1} P y_1 \right) \right] \frac{\beta}{a} d\theta. \qquad (3.100)
\end{aligned}
$$

Also

$$
\begin{aligned}
\int_0^{2\pi} \left[y_2 L^{-1} P x_1 - y_2 \frac{\partial^2}{\partial \theta^2} \left(L^{-1} P x_1 \right) \right] \frac{\beta}{a} d\theta &= \int_0^{2\pi} y_2 \left[\mathbb{1} - \frac{\partial^2}{\partial \theta^2} \right] L^{-1} P x_1 \frac{\beta}{a} d\theta \\
&= -\int_0^{2\pi} y_2 L L^{-1} P x_1 \frac{\beta}{a} d\theta \\
&= -\int_0^{2\pi} y_2 \left[\frac{\partial^6 x_1}{\partial \theta^6} + 2 \frac{\partial^4 x_1}{\partial \theta^4} + \frac{\partial^2 x_1}{\partial \theta^2} \right] \frac{\beta}{a} d\theta \\
&= -\int_0^{2\pi} \left[-\frac{\partial^3 x_1}{\partial \theta^3} \frac{\partial^3 y_2}{\partial \theta^3} - \frac{\partial^3 x_1}{\partial \theta^3} \frac{\partial y_2}{\partial \theta} - \frac{\partial x_1}{\partial \theta} \frac{\partial^3 y_2}{\partial \theta^3} - \frac{\partial x_1}{\partial \theta} \frac{\partial y_2}{\partial \theta} \right] \frac{\beta}{a} d\theta \\
&\quad + \left(y_2 \frac{\partial^5 x_1}{\partial \theta^5} \Big|_0^{2\pi} - \frac{\partial y_2}{\partial \theta} \frac{\partial^4 x_1}{\partial \theta^4} \Big|_0^{2\pi} + \frac{\partial^2 y_2}{\partial \theta^2} \frac{\partial^3 x_1}{\partial \theta^3} \Big|_0^{2\pi} + 2\, y_2 \frac{\partial^3 x_1}{\partial \theta^3} \Big|_0^{2\pi} \right. \\
&\quad \left. - \frac{\partial y_2}{\partial \theta} \frac{\partial^2 x_1}{\partial \theta^2} \Big|_0^{2\pi} + \frac{\partial^2 y_2}{\partial \theta^2} \frac{\partial x_1}{\partial \theta} \Big|_0^{2\pi} + y_2 \frac{\partial x_1}{\partial \theta} \Big|_0^{2\pi} \right) \frac{\beta}{a} \\
&= \int_0^{2\pi} \left(\frac{\partial^3 x_1}{\partial \theta^3} + \frac{\partial x_1}{\partial \theta} \right) \left(\frac{\partial^3 y_2}{\partial \theta^3} + \frac{\partial y_2}{\partial \theta} \right) \frac{\beta}{a} d\theta. \qquad (3.101)
\end{aligned}
$$

Using equations (3.100) and (3.101) in equation (3.99) we get

$$
\begin{aligned}
\langle \mathcal{A} x, y \rangle &= \frac{1}{2} \int_0^{2\pi} \left[-\left(\frac{\partial^3 x_1}{\partial \theta^3} + \frac{\partial x_1}{\partial \theta} \right) \left(\frac{\partial^3 y_2}{\partial \theta^3} + \frac{\partial y_2}{\partial \theta} \right) \right. \\
&\qquad \left. + x_2 L^{-1} P y_1 + \frac{\partial x_2}{\partial \theta} \frac{\partial}{\partial \theta} \left(L^{-1} P y_1 \right) \right] \frac{\beta}{a} d\theta \\
&= -\frac{1}{2} \int_0^{2\pi} \left[-x_2 L^{-1} P y_1 - \frac{\partial x_2}{\partial \theta} \frac{\partial}{\partial \theta} \left(L^{-1} P y_1 \right) \right. \\
&\qquad \left. + \left(\frac{\partial^3 x_1}{\partial \theta^3} + \frac{\partial x_1}{\partial \theta} \right) \left(\frac{\partial^3 y_2}{\partial \theta^3} + \frac{\partial y_2}{\partial \theta} \right) \right] \frac{\beta}{a} d\theta \\
&= -\langle x, \mathcal{A} y \rangle. \qquad (3.102)
\end{aligned}
$$

Thus $\mathcal{A}$ is a skew-adjoint operator and by Theorem(3.4.2) it generates a one-parameter unitary group. ∎

Fourier basis

At this stage we state our intention to do all of the calculations associated to the application of the moving systems approach to the vibrating ring problem in a convenient set of coordinates, namely the coefficients of $\alpha, \dot{\alpha}$ expressed in a Fourier basis. We first express the nominal dynamics in these coordinates and in the following section do the holonomy calculations in the same coordinates (after truncation of the Fourier series). In these coordinates α has the form

$$\alpha(\theta) = \sum_{k=1}^{\infty} \left[A_k \cos(k\theta) + B_k \sin(k\theta)\right]. \tag{3.103}$$

The deformation $\alpha(\theta)$ is not allowed to contain any global rotations and so the constant coefficient is set to 0. Inserting this expression for α into the equation of motion (3.87) results in the equation

$$\begin{aligned} 0 \quad = \quad & \sum_{k=1}^{\infty} \Big\{ \sigma a^3 (1+k^2) \left[\ddot{A}_k \cos(k\theta) + \ddot{B}_k \sin(k\theta)\right] \\ & \qquad + \frac{\beta}{a}(k^6 - 2k^4 + k^2)\left[A_k \cos(k\theta) + B_k \sin(k\theta)\right] \Big\}. \end{aligned} \tag{3.104}$$

Collecting terms in $\cos(\theta)$ and $\sin(\theta)$ and setting them separately to zero gives the following set of ordinary differential equations for the Fourier coefficients.

$$\ddot{A}_k = -\frac{\beta}{\sigma a^4}\frac{k^2(k^2-1)^2}{k^2+1} A_k \triangleq -\eta_k^2 A_k, \tag{3.105}$$

$$\ddot{B}_k = -\frac{\beta}{\sigma a^4}\frac{k^2(k^2-1)^2}{k^2+1} A_k \triangleq -\eta_k^2 B_k \tag{3.106}$$

which defines the frequencies η_k. This result is in agreement with a derivation of Rayleigh [77] and defines for each k a pair of uncoupled oscillators with common frequency η_k. The solution to this system is given by

$$A_k(t) = \widehat{A}_k \cos(\eta_k t) + \frac{\widehat{\dot{A}}_k}{\eta_k} \sin(\eta_k t), \quad (3.107)$$

$$B_k(t) = \widehat{B}_k \cos(\eta_k t) + \frac{\widehat{\dot{B}}_k}{\eta_k} \sin(\eta_k t) \quad (3.108)$$

where $\widehat{A}_k$, $\widehat{\dot{A}}_k$, $\widehat{B}_k$, and $\widehat{\dot{B}}_k$ are given by initial conditions. The Hannay-Berry phase is defined as the holonomy on a trivial bundle involving the cotangent space of the system. It will prove useful, then, to have the time evolution of the conjugate momenta for the nominal system. By inserting the Fourier expansion for α into the nominal Lagrangian, equation (3.82), and applying the Legendre transform we obtain

$$p_{A_k} = \frac{\partial L}{\partial \dot{A}_k} = (1+k^2)\sigma a^3 \pi \dot{A}_k, \quad (3.109)$$

$$p_{B_k} = \frac{\partial L}{\partial \dot{B}_k} = (1+k^2)\sigma a^3 \pi \dot{B}_k. \quad (3.110)$$

Thus the solution to the nominal system expressed on the cotangent bundle is given by

$$A_k(t) = \widehat{A}_k \cos(\eta_k t) + \frac{\widehat{p}_{A_k}}{(1+k^2)\sigma a^3 \pi \eta_k} \sin(\eta_k t), \quad (3.111)$$

$$p_{A_k}(t) = -(1+k^2)\sigma a^3 \pi \eta_k \widehat{A}_k \sin(\eta_k t) + \widehat{p}_{A_k} \cos(\eta_k t), \quad (3.112)$$

$$B_k(t) = \widehat{B}_k \cos(\eta_k t) + \frac{\widehat{p}_{B_k}}{(1+k^2)\sigma a^3 \pi \eta_k} \sin(\eta_k t), \quad (3.113)$$

$$p_{B_k}(t) = -(1+k^2)\sigma a^3 \pi \eta_k \widehat{B}_k \sin(\eta_k t) + \widehat{p}_{B_k} \cos(\eta_k t). \quad (3.114)$$

3.4.2 The Hannay-Berry phase of the ring gyroscope

From equation (3.84), the velocity vector of the motion in S is

$$\frac{d}{dt}\left(m_t(\alpha(\theta))\right) = (0, \dot{\alpha}(\theta)) + (\Omega, 0). \quad (3.115)$$

From this we see that the vector field corresponding to the imposed motion is

$$\mathcal{Z}_t(m_t(\alpha(\theta))) = \begin{pmatrix} \Omega \\ 0 \end{pmatrix} \tag{3.116}$$

For ease of notation define $\mathcal{Z} \triangleq \mathcal{Z}_t(m_t(\alpha(\theta)))$. The projection of this tangent vector to $T_{m_t(q)}m_t(Q)$ with respect to the metric of S is given by $\mathcal{Z}^T = \mathcal{Z} - \mathcal{Z}^\perp$ where $(\mathcal{Z}^\perp, X) = 0 \; \forall X \in T_{m_t(q)}m_t(Q)$. From equation (3.73), any vector $X \in T_{m_t(q)}m_t(Q)$ has the form

$$X = \begin{pmatrix} 0 \\ Y \end{pmatrix} \tag{3.117}$$

where $Y \in T_qQ$. Then, from equation (3.116) and the fact that $\mathcal{Z}^T \in T_{m_t(q)}m_t(Q)$, we can write $\mathcal{Z}^T = \mathcal{Z} - \mathcal{Z}^\perp$ as

$$\begin{pmatrix} 0 \\ Y_{\mathcal{Z}^T} \end{pmatrix} = \begin{pmatrix} \Omega \\ 0 \end{pmatrix} - \begin{pmatrix} \mathcal{Z}^{\perp_1} \\ \mathcal{Z}^{\perp_2} \end{pmatrix} \tag{3.118}$$

for some $Y_{\mathcal{Z}^T}, \mathcal{Z}^{\perp_2} \in T_q(Q)$ and $\mathcal{Z}^{\perp_1} \in T_\psi S^1$. From equation (3.118), $\mathcal{Z}^{\perp_1} = \Omega$. Applying equation (3.75), the orthogonality condition states

$$\begin{aligned} 0 &= ((\Omega, \mathcal{Z}^{\perp_2}), (0, Y)) \\ &= \int_0^{2\pi} \left[\left(1 - \frac{\partial \alpha}{\partial \theta}\right)^2 \left(\Omega Y + \mathcal{Z}^{\perp_2} Y\right) + \frac{\partial \mathcal{Z}^{\perp_2}}{\partial \theta} \frac{\partial Y}{\partial \theta} \right] \sigma a^3 d\theta \end{aligned} \tag{3.119}$$

for all $Y \in T_qQ$. In what follows, we express the orthogonality condition of equation (3.119) in the Fourier basis and thus derive an explicit formula for $\mathcal{Z}^{\perp_2}$ (see (3.130) and (3.131) below). Using the Fourier series representation for α, a tangent vector $Y \in T_qQ$ has the form

$$Y = \sum_{k=1}^{\infty} \left[Y_{A_k} \cos(k\theta) + Y_{B_k} \sin(k\theta) \right]. \tag{3.120}$$

With the abbreviations $\mathrm{c}(\psi) = \cos(\psi)$ and $\mathrm{s}(\psi) = \sin(\psi)$ the orthogonality condition is given by

$$
\begin{aligned}
0 \;=\; & \int_0^{2\pi} \Bigg[\left(1 - \sum_{k=1}^{\infty} k \left[B_k \mathrm{c}(k\theta) - A_k \mathrm{s}(k\theta)\right]\right)^2 \left(\Omega \sum_{k=1}^{\infty} \left[Y_{A_k} \mathrm{c}(k\theta) + Y_{B_k} \mathrm{s}(k\theta)\right]\right. \\
& + \left.\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \left[\mathcal{Z}_{A_k}^{\perp_2} \mathrm{c}(k\theta) + \mathcal{Z}_{B_k}^{\perp_2} \mathrm{s}(k\theta)\right] \left[Y_{A_l} \mathrm{c}(l\theta) + Y_{B_l} \mathrm{s}(l\theta)\right]\right) \\
& + \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} kl \left(\mathcal{Z}_{B_k}^{\perp_2} \mathrm{c}(k\theta) - \mathcal{Z}_{A_k}^{\perp_2} \mathrm{s}(k\theta)\right) \left(Y_{B_l} \mathrm{c}(l\theta) - Y_{A_l} \mathrm{s}(l\theta)\right)\Bigg] \sigma a^3 d\theta. \quad (3.121)
\end{aligned}
$$

Using the following identities

$$
\int_0^{2\pi} \mathrm{c}(k\theta)\mathrm{c}(l\theta) d\theta = \int_0^{2\pi} \mathrm{s}(k\theta)\mathrm{s}(l\theta) d\theta = \pi \delta_{kl}, \quad (3.122)
$$

$$
\int_0^{2\pi} \mathrm{c}(k\theta)\mathrm{c}(l\theta)\mathrm{c}(m\theta)\mathrm{c}(n\theta) d\theta = \frac{3\pi}{4}\delta_{klmn}, \quad (3.123)
$$

$$
\int_0^{2\pi} \mathrm{s}(k\theta)\mathrm{s}(l\theta)\mathrm{s}(m\theta)\mathrm{s}(n\theta) d\theta = \frac{3\pi}{4}\delta_{klmn}, \quad (3.124)
$$

$$
\int_0^{2\pi} \mathrm{c}(k\theta)\mathrm{c}(l\theta)\mathrm{s}(m\theta)\mathrm{s}(n\theta) d\theta = \frac{\pi}{4}\delta_{klmn} \quad (3.125)
$$

where

$$
\delta_{kl} = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{otherwise} \end{cases}, \qquad \delta_{klmn} = \begin{cases} 1 & \text{if } k = l = m = n \\ 0 & \text{otherwise} \end{cases} \quad (3.126)
$$

and the fact that all other combinations of sin and cos appearing in equation (3.121) integrate to 0 over $[0, 2\pi]$, the orthogonality condition can be reduced to

$$
\begin{aligned}
0 \;=\; & \sigma a^3 \pi \sum_{k=1}^{\infty} \Bigg[\left(\mathcal{Z}_{A_k}^{\perp_2} + \frac{k^2}{4} A_k^2 \mathcal{Z}_{A_k}^{\perp_2} + \frac{3k^2}{4} B_k^2 \mathcal{Z}_{A_k}^{\perp_2} - \frac{k^2}{2} A_k B_k \mathcal{Z}_{B_k}^{\perp_2} \right. \\
& \left. -2\Omega k B_k + k^2 \mathcal{Z}_{A_k}^{\perp_2} \right) Y_{A_k} \\
& + \left(\mathcal{Z}_{B_k}^{\perp_2} + \frac{3k^2}{4} A_k^2 \mathcal{Z}_{B_k}^{\perp_2} + \frac{k^2}{4} B_k^2 \mathcal{Z}_{B_k}^{\perp_2} - \frac{k^2}{2} A_k B_k \mathcal{Z}_{A_k}^{\perp_2} \right. \\
& \left. +2\Omega k A_k + k^2 \mathcal{Z}_{B_k}^{\perp_2} \right) Y_{B_k} \Bigg]. \quad (3.127)
\end{aligned}
$$

This holds for every $Y \in T_qQ$ and so for every k we have

$$\begin{aligned} 0 &= \mathcal{Z}^{\perp_2}_{A_k} + \frac{k^2}{4}A_k^2\mathcal{Z}^{\perp_2}_{A_k} + \frac{3k^2}{4}B_k^2\mathcal{Z}^{\perp_2}_{A_k} - \frac{k^2}{2}A_kB_k\mathcal{Z}^{\perp_2}_{B_k} \\ &\qquad -2\Omega k B_k + k^2\mathcal{Z}^{\perp_2}_{A_k}, \qquad (3.128) \\ 0 &= \mathcal{Z}^{\perp_2}_{B_k} + \frac{3k^2}{4}A_k^2\mathcal{Z}^{\perp_2}_{B_k} + \frac{k^2}{4}B_k^2\mathcal{Z}^{\perp_2}_{B_k} - \frac{k^2}{2}A_kB_k\mathcal{Z}^{\perp_2}_{A_k} \\ &\qquad +2\Omega k A_k + k^2\mathcal{Z}^{\perp_2}_{B_k}. \qquad (3.129) \end{aligned}$$

Solving these coupled equations for $\mathcal{Z}^{\perp_2}_{A_k}$ and $\mathcal{Z}^{\perp_2}_{B_k}$ yields

$$\mathcal{Z}^{\perp_2}_{A_k} = \Omega\left[\frac{2k(1+k^2+\frac{3k^2}{4}A_k^2+\frac{k^2}{4}B_k^2)B_k - k^3A_k^2B_k}{D_k(A_k,B_k)}\right], \qquad (3.130)$$

$$\mathcal{Z}^{\perp_2}_{B_k} = -\Omega\left[\frac{2k(1+k^2+\frac{k^2}{4}A_k^2+\frac{3k^2}{4}B_k^2)A_k - k^3A_kB_k^2}{D_k(A_k,B_k)}\right] \qquad (3.131)$$

where

$$D_k(A_k,B_k) = (1+k^2+\frac{k^2}{4}A_k^2+\frac{3k^2}{4}B_k^2)(1+k^2+\frac{3k^2}{4}A_k^2+\frac{k^2}{4}B_k^2) - \frac{k^4}{4}A_k^2B_k^2 \qquad (3.132)$$

and so, representing the tangent vector by its coefficients at each k,

$$\mathcal{Z}^{\perp} = \begin{pmatrix} \Omega \\ \left\{\Omega\left[\frac{2k(1+k^2+\frac{3k^2}{4}A_k^2+\frac{k^2}{4}B_k^2)B_k - k^3A_k^2B_k}{D_k(A_k,B_k)}\right]\right\}_{k=1}^{\infty} \\ \left\{-\Omega\left[\frac{2k(1+k^2+\frac{k^2}{4}A_k^2+\frac{3k^2}{4}B_k^2)A_k - k^3A_kB_k^2}{D_k(A_k,B_k)}\right]\right\}_{k=1}^{\infty} \end{pmatrix}. \qquad (3.133)$$

Inserting equation (3.133) into equation (3.118) gives the projection of the tangent vector of the imposed motion onto $T_{m_t(q)}m_t(Q)$ to be

$$\mathcal{Z}^T = \begin{pmatrix} 0 \\ \\ \left\{ -\Omega \left[\frac{2k(1+k^2+\frac{3k^2}{4}A_k^2+\frac{k^2}{4}B_k^2)B_k - k^3 A_k^2 B_k}{D_k(A_k,B_k)} \right] \right\}_{k=1}^{\infty} \\ \\ \left\{ \Omega \left[\frac{2k(1+k^2+\frac{k^2}{4}A_k^2+\frac{3k^2}{4}B_k^2)A_k - k^3 A_k B_k^2}{D_k(A_k,B_k)} \right] \right\}_{k=1}^{\infty} \end{pmatrix}. \tag{3.134}$$

The pull-back of $\mathcal{Z}^T$ to T_qQ by $[Tm]^{-1}$ is given by

$$Z(q) \triangleq [Tm]^{-1}\mathcal{Z}^T = \begin{pmatrix} \left\{ -\Omega \left[\frac{2k(1+k^2+\frac{3k^2}{4}A_k^2+\frac{k^2}{4}B_k^2)B_k - k^3 A_k^2 B_k}{D_k(A_k,B_k)} \right] \right\}_{k=1}^{\infty} \\ \\ \left\{ \Omega \left[\frac{2k(1+k^2+\frac{k^2}{4}A_k^2+\frac{3k^2}{4}B_k^2)A_k - k^3 A_k B_k^2}{D_k(A_k,B_k)} \right] \right\}_{k=1}^{\infty} \end{pmatrix} \tag{3.135}$$

where $Z(q)$ is defined for ease of notation. Recalling that the deformations are assumed to be small, the above expression is expanded in a Taylor series about $A_k = B_k = 0 \; \forall \; k$ and only the first order terms kept. This yields

$$Z(q) = \begin{pmatrix} \left\{ \frac{-2\Omega k}{1+k^2} B_k \right\}_{k=1}^{\infty} \\ \\ \left\{ \frac{2\Omega k}{1+k^2} A_k \right\}_{k=1}^{\infty} \end{pmatrix}. \tag{3.136}$$

Let $q_k = (A_k, B_k)$ so that the coordinates on Q are $q = \{q_k\}_{k=1}^{\infty}$. The conjugate momenta can then be written $p_k = (p_{A_k}, p_{B_k})$. Define

$$Z_k(q_k) \triangleq \frac{2k\Omega}{1+k^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A_k \\ B_k \end{pmatrix} = \frac{2k\Omega}{1+k^2} S(1) q_k \tag{3.137}$$

which also defines the skew symmetric matrix $S(1)$. With this definition the projected vector can be expressed as $Z(q) = \{Z_k(q_k)\}_{k=1}^{\infty}$. To avoid technical difficulties we assume that only N Fourier modes are active where N is some positive, finite number. Using this assumption, the function $\mathcal{P}(Z(q))$ defining the horizontal lift relative to the induced Cartan connection (as in equation (3.7)) is given by

$$\mathcal{P}(Z)(q,p) = p \cdot Z(q) = \sum_{k=1}^{N} \left[\frac{2k\Omega}{1+k^2}\right] p_k \cdot S(1) q_k. \tag{3.138}$$

Define $Q_k = \{(A_k, B_k) \in \mathbb{R}^2\}$ so that $Q = \cup_{k=1}^{N} Q_k$. The configuration space is then the Cartesian product of N copies of $\mathbb{R}^2$ and each coordinate q_k and conjugate momenta p_k can be identified with a vector in $\mathbb{R}^2$. Extend these vectors to $\mathbb{R}^3$ by letting the third coordinate of each be zero. Let $\widehat{x}_{3_k}$ be a unit vector at the origin of the k^{th} copy of $\mathbb{R}^2$ along this third direction. With these identifications, in equation (3.138) we replace $p_k \cdot S(1) q_k$ by $(q_k \times p_k) \cdot \widehat{x}_{3_k}$. Define

$$I_k \triangleq (q_k \times p_k) \cdot \widehat{x}_{3_k} \tag{3.139}$$

so that

$$\mathcal{P}(Z)(q,p) = \sum_{k=1}^{N} \frac{2k}{1+k^2} \Omega I_k. \tag{3.140}$$

Let $\mathcal{F}$ be the subset of C^∞ functions on T^*Q defined by

$$\mathcal{F} = \left\{ f(q,p) | f(q,p) = \sum_{k=1}^{N} a_k f_k(A_k, B_k, p_{A_k}, p_{B_k}), a_k \in \mathbb{R}, f_k \ smooth \right\}. \tag{3.141}$$

Since the time solution to each Fourier mode for the nominal system is periodic, we can define an average on $\mathcal{F}$ with respect to the flow by

$$< f >= \sum_{k=1}^{N} a_k < f_k >_k \tag{3.142}$$

where

$$< f_k >_k = \frac{\eta_k}{2\pi} \int_0^{\frac{2\pi}{\eta_k}} \left(\Phi_t^k\right)^* f_k dt \tag{3.143}$$

where Φ_t^k is the flow corresponding to the k^{th} Fourier mode of the nominal system. From equations (3.139) and (3.140) it is clear that $P(Z) \in \mathcal{F}$. We have the following useful lemma.

Lemma 3.4.4 *I_k is constant along the trajectories of the nominal system.*

Proof

$$\begin{aligned}
\frac{dI_k}{dt} &= \frac{d}{dt}(q_k \times p_k)\cdot \widehat{x}_{3_k} \\
&= \frac{d}{dt}(p_{B_k}A_k - p_{A_k}B_k) \\
&= \dot{p}_{B_k}A_k + p_{B_k}\dot{A}_k - \dot{p}_{A_k}B_k - p_{A_k}\dot{B}_k \\
&= (1+k^2)\sigma a^3\pi\eta_k\left[\ddot{B}_kA_k + \dot{B}_k\dot{A}_k - \ddot{A}_kB_k - \dot{A}_k\dot{B}_k\right] \\
&= (1+k^2)\sigma a^3\pi\eta_k\left[\ddot{B}_kA_k - \ddot{A}_kB_k\right]
\end{aligned}$$

where in the second to last step we have used the definition of the conjugate momenta for the nominal system in equations (3.109,3.110). From equations (3.105,3.106), along the trajectories of the nominal system we have

$$\left[\ddot{B}_kA_k - \ddot{A}_kB_k\right] = -\eta_k^2\left[A_kB_k - A_kB_k\right] = 0. \tag{3.144}$$

■

The average of $\mathcal{P}(Z)$ over the nominal dynamics is then

$$< \mathcal{P}(Z) > (q,p) = \sum_{k=1}^{N}\frac{2k}{1+k^2}\Omega < I_k >_k = \sum_{k=1}^{N}\frac{2k}{1+k^2}\Omega I_k. \tag{3.145}$$

Noticing that this function depends only the coordinates (q_k, p_k) through I_k, we move to the coordinates for the averaged dynamics in phase space defined by $(\phi_k, I_k, \rho_k, p_{\rho_k})$ where $\rho_k =< (A_k^2 + B_k^2) >^{\frac{1}{2}}$. Here ϕ_k is conjugate to I_k and p_{ρ_k} is conjugate to ρ_k.

With these coordinates the horizontal lift of Ω relative to the Cartan-Hannay-Berry connection (as in equation (3.10)) is given by

$$\left(-X_{<\mathcal{P}(Z)>}, \Omega\right) = \left(\left\{-\frac{2k}{1+k^2}\Omega\frac{\partial}{\partial\phi_k}\right\}_{k=1}^{N}, \Omega\right). \tag{3.146}$$

The geometric phase is neatly split into a phase change in each ϕ_k independently. If the loop in M is parametrized by $t \in [0, T]$ where $T = \frac{2\pi}{\Omega}$ is the time to complete one full revolution of the ring, then

$$\Delta\phi_k = -\int_0^T \frac{2k}{1+k^2}\Omega dt = -\frac{2k}{1+k^2}\Omega T = -2\pi \left[\frac{2k}{1+k^2}\right]. \tag{3.147}$$

After one full rotation of the ring each vector q_k has been rotated by the angle $\Delta\phi_k$. In practice one expects to use only one mode under a resonant drive (as in [75, 76]).

Remark 3.4.5 *Note that we perform an average with respect to the flow of the nominal dynamics on a special class of functions as in equation* (3.141)*. This agrees with a group (here S^1) average as in Section* 3.2 *(definition* 3.2.4*) when we restrict the nominal dynamics to a single mode.*

3.4.3 A comparison with the results of Bryan

In [20] Bryan uses classical variational techniques to derive the equations of motion for a thin ring of radius a undergoing a steady rotation about its central axis with angular velocity Ω. His analysis uses two polar coordinate systems, the first fixed in space and the second rotating with angular velocity Ω. If in the undeformed state the coordinate systems are given by (a, ϕ) and (a, θ) we have

$$\phi = \theta + \Omega t \tag{3.148}$$

and θ is constant for any particle of the ring. Let the tangential and radial displacements of a particle of the ring be given by v and w respectively so that the new polar coordinates are $(a+w, \phi+v/a)$ and $(a+w, \theta+v/a)$ in the two systems. The deformations v, w are assumed to be small. The assumption of inextensibility yields

$$w = -\frac{\partial v}{\partial \theta} \tag{3.149}$$

as before. As discussed in the previous section, Bryan includes work done against the tension, $\mathcal{T}$, to stretch the ring and against an attractive force, μ. To match his derivation with the model we have chosen, we set these terms to zero. Taking variations on the total energy and setting them to zero, we find the following equation of motion

$$0 = \ddot{v} - \frac{\partial^2 \ddot{v}}{\partial \theta^2} - 4\Omega \frac{\partial \dot{v}}{\partial \theta} + \Omega^2 \frac{\partial^2 v}{\partial \theta^2} - \frac{\beta}{\sigma a^4} \frac{\partial^2}{\partial \theta^2} \left(\frac{\partial^2}{\partial \theta^2} + 1 \right)^2 v. \tag{3.150}$$

Bryan then assumes that the deformations are of the form

$$v = cos(k\theta + pt). \tag{3.151}$$

Inserting this into the equation of motion yields the following two solutions.

$$v = Acos\left(k\theta + \frac{2k}{1+k^2}\Omega t + \bar{\omega}_k t \right), \tag{3.152}$$

$$v = Acos\left(k\theta + \frac{2k}{1+k^2}\Omega t - \bar{\omega}_k t \right) \tag{3.153}$$

where

$$\begin{aligned} \bar{\omega}_k^2 &= \frac{\beta}{\sigma a^4} \frac{k^2(k^2-1)^2}{1+k^2} - \frac{\Omega^2 k^2 (k^2-3)}{(1+k^2)^2} \\ &= \eta_k^2 - \frac{\Omega^2 k^2 (k^2-3)}{(1+k^2)^2}. \end{aligned} \tag{3.154}$$

Notice that by retaining the terms in Ω^2 there is a slight decrease in the frequency of vibration from η_k. This can be understood as a “softening” of the material and corresponds to the spurious softening that occurs in the theory of rotating rods if the models are linearized prematurely, that is before the effects of external rotation are considered. The geometrically exact theory handles this issue properly and we believe that extending the model of the ring using this theory will prove useful. (We note also that if the attractive force μ introduced by Bryan is kept in the equations of motion and set equal to Ω^2 then the resulting system shows an increase in the frequency of vibration due to the imposed rotation. See [39] for comments on similar ad hoc methods in the theory of rotating beams.)

Assuming the amplitude A in the two solutions is the same the final solution in the fixed frame is given by

$$v = 2Acos(\bar{\omega}_k t)cos\left(k\left[\phi - \frac{k^2-1}{k^2+1}\Omega t\right]\right). \tag{3.155}$$

Bryan then recognizes that this corresponds to an oscillation with $2k$ nodes where the position of the nodes precess in retrograde around the ring with angular velocity

$$\frac{k^2-1}{k^2+1}\Omega. \tag{3.156}$$

If we write this in the rotating system we have

$$\left(\frac{k^2-1}{k^2+1} - 1\right)\Omega = -\frac{2}{k^2+1}\Omega \tag{3.157}$$

and after one rotation the nodes have precessed in the moving frame by

$$-2\pi\left(\frac{2}{1+k^2}\right). \tag{3.158}$$

To compare this to our results in Section 3.4.2 we must first restrict our solution to a single mode so that

$$\alpha(\theta, t) = A_k(t)\cos(k\theta) + B_k(t)\sin(k\theta). \tag{3.159}$$

When (A_k, B_k) is viewed as a vector in $\mathbb{R}^2$, the effect of the geometric phase is seen to be a rotation of this vector about the origin where the counter-clockwise direction is defined to be positive. Using equation (3.147), the rotated vector at the end of one revolution of the ring is given by

$$\begin{pmatrix} A_k(T) \\ B_k(T) \end{pmatrix} = \begin{pmatrix} \cos\left(-2\pi\frac{2k}{1+k^2}\right) & -\sin\left(-2\pi\frac{2k}{1+k^2}\right) \\ \sin\left(-2\pi\frac{2k}{1+k^2}\right) & \cos\left(-2\pi\frac{2k}{1+k^2}\right) \end{pmatrix} \begin{pmatrix} A_k(0) \\ B_k(0) \end{pmatrix}$$

$$= \begin{pmatrix} A_k(0)\cos\left(-2\pi\frac{2k}{1+k^2}\right) - B_k(0)\sin\left(-2\pi\frac{2k}{1+k^2}\right) \\ A_k(0)\sin\left(-2\pi\frac{2k}{1+k^2}\right) + B_k(0)\cos\left(-2\pi\frac{2k}{1+k^2}\right) \end{pmatrix}. \quad (3.160)$$

Inserting this into equation (3.159) and simplifying we get

$$\begin{aligned} \alpha(\theta, T) &= A_k(0)\cos\left(k\left[\theta + 2\pi\frac{2}{1+k^2}\right]\right) \\ &\quad + B_k(0)\sin\left(k\left[\theta + 2\pi\frac{2}{1+k^2}\right]\right) \end{aligned} \quad (3.161)$$

which is course the same solution with the nodes rotated by $-2\pi\left[\frac{2}{1+k^2}\right]$, agreeing with Bryan.

3.4.4 Nonlinear corrections

We now turn to an investigation of corrections to the geometric phase based on the nonlinear terms in the vector field $Z(q)$ given in equation (3.135). It is worth noting that these arise due to the configuration-dependent quadratic form defining the kinetic energy. We proceed by keeping higher-order terms in the Taylor expansion of $Z(q)$. The second-order terms in this expansion can be shown to be zero. To third order the vector field is

$$Z(q) = \begin{pmatrix} \left\{-\Omega\left[\frac{2kB_k}{1+k^2} - \frac{k^2(2+k)A_k^2B_k}{(1+k^2)^2} - \frac{9k^3B_k^3}{(1+k^2)^2}\right]\right\}_{k=1}^N \\ \\ \left\{\Omega\left[\frac{2kA_k}{1+k^2} - \frac{k^2(2+k)A_kB_k^2}{(1+k^2)^2} - \frac{9k^3A_k^3}{(1+k^2)^2}\right]\right\}_{k=1}^N \end{pmatrix}. \quad (3.162)$$

Define the matrix $U(q_k)$ by

$$U(q_k) = \begin{pmatrix} A_k^2 & 0 \\ 0 & B_k^2 \end{pmatrix}. \tag{3.163}$$

With this, equation (3.162) can be written as

$$\begin{aligned} Z(q) &= \left\{ \frac{2k\Omega}{1+k^2} S(1) q_k - \frac{k^2(2+k)\Omega}{(1+k^2)^2} U(q_k) S(1) q_k \right. \\ &\qquad \left. - \frac{9k^3\Omega}{(1+k^2)^2} S(1) U(q_k) q_k \right\}_{k=1}^{N} \end{aligned} \tag{3.164}$$

and the function defining the Hannay-Berry phase is given by

$$\begin{aligned} \mathcal{P}(Z)(q,p) &= \sum_{k=1}^{N} \left[\frac{2k\Omega}{1+k^2} p_k \cdot S(1) q_k - \frac{k^2(2+k)\Omega}{(1+k^2)^2} p_k \cdot U(q_k) S(1) q_k \right. \\ &\qquad \left. - \frac{9k^3\Omega}{(1+k^2)^2} p_k \cdot S(1) U(q_k) q_k \right] \\ &= \sum_{k=1}^{N} \left[\frac{2k\Omega}{1+k^2} I_k - \frac{k^2(2+k)\Omega}{(1+k^2)^2} p_k \cdot U(q_k) S(1) q_k \right. \\ &\qquad \left. - \frac{9k^3\Omega}{(1+k^2)^2} p_k \cdot S(1) U(q_k) q_k \right] \end{aligned} \tag{3.165}$$

where in the second step the definition of I_k from equation (3.139) has been used. To determine the Hannay-Berry phase we need to find the average of equation (3.165) over the nominal dynamics. In Section 3.4.2 we have shown that the first term in square brackets in the sum in equation (3.165) is a constant along trajectories of the nominal system. The second and third terms, however, are not constant and their averages need to be explicitly calculated. For the second term the average is given by

$$\begin{aligned} < p_k \cdot U(q_k) S(1) q_k >_k &= \frac{\eta_k}{2\pi} \int_0^{\frac{2\pi}{\eta_k}} \left[-p_{A_k}(t) A_k^2(t) B_k(t) \right. \\ &\qquad \left. + p_{B_k}(t) A_k(t) B_k^2(t) \right] dt. \end{aligned} \tag{3.166}$$

Using the solution to the nominal system given in equations (3.111–3.114) and making a change of variables in the integration, this becomes

$$
\begin{aligned}
< p_k \cdot U(q_k)S(1)q_k >_k \;&=\; \frac{1}{2\pi}\int_0^{2\pi} \Big\{ \big[(1+k^2)\sigma a^3\pi\eta_k A_k \sin(t) - p_{A_k}\cos(t)\big] \\
&\cdot \left[A_k\cos(t) + \frac{p_{A_k}\sin(t)}{(1+k^2)\sigma a^3\pi\eta_k}\right]^2 \left[B_k\cos(t) + \frac{p_{B_k}\sin(t)}{(1+k^2)\sigma a^3\pi\eta_k}\right] \\
&+ \big[-(1+k^2)\sigma a^3\pi\eta_k B_k\sin(t) + p_{B_k}\cos(t)\big] \\
&\cdot \left[A_k\cos(t) + \frac{p_{A_k}\sin(t)}{(1+k^2)\sigma a^3\pi\eta_k}\right]\left[B_k\cos(t) + \frac{p_{B_k}\sin(t)}{(1+k^2)\sigma a^3\pi\eta_k}\right]^2 \Bigg\}\, dt \qquad (3.167)
\end{aligned}
$$

where, through a standard abuse of notation in averaging, the hats have been dropped on the initial conditions. The integration identities in equations (3.122 – 3.125) can be used to write the above expression as

$$
\begin{aligned}
< p_k \cdot U(q_k)S(1)q_k >_k \;&=\; \frac{1}{8}\Big[\big(A_k^3 p_{B_k} - B_k^3 p_{A_k} - A_k^2 B_k p_{A_k} + A_k B_k^2 p_{B_k}\big) \\
&+ \left(\frac{A_k p_{A_k}^2 p_{B_k} - B_k p_{A_k} p_{B_k}^2 - B_k p_{A_k}^3 + A_k p_{B_k}^3}{((1+k^2)\sigma a^3\pi\eta_k)^2}\right)\Big] \qquad (3.168)
\end{aligned}
$$

Now

$$
\begin{aligned}
A_k^3 p_{B_k} - B_k^3 p_{A_k} - A_k^2 B_k p_{A_k} + A_k B_k^2 p_{B_k} &= (A_k p_{B_k} - B_k p_{A_k})(A_k^2 + B_k^2) \\
&= I_k(A_k^2 + B_k^2) \qquad (3.169)
\end{aligned}
$$

and

$$
\begin{aligned}
A_k p_{A_k}^2 p_{B_k} - B_k p_{A_k} p_{B_k}^2 - B_k p_{A_k}^3 + A_k p_{B_k}^3 &= (A_k p_{B_k} - B_k p_{A_k})(p_{A_k}^2 + p_{B_k}^2) \\
&= I_k(p_{A_k}^2 + p_{B_k}^2) \qquad (3.170)
\end{aligned}
$$

and so equation (3.168) takes the form

$$
< p_k \cdot U(q_k)S(1)q_k >_k = \frac{I_k}{8}\left(A_k^2 + B_k^2 + \frac{p_{A_k}^2 + p_{B_k}^2}{((1+k^2)\sigma a^3\pi\eta_k)^2}\right). \qquad (3.171)
$$

Following the same procedure, the average of the third term is

$$
\begin{aligned}
< p_k \cdot S(1)U(q_k)q_k >_k &= \frac{\eta_k}{2\pi}\int_0^{2\pi}(p_{B_k}(t)A_k^3(t) - p_{A_k}(t)B_k^3(t))dt \\
&= \frac{3}{8}\Big[\big(A_kB_kp_{B_k} - A_k^2B_kp_{A_k} + A_k^3p_{B_k} - B_k^3p_{A_k}\big) \\
&\qquad + \frac{A_kp_{B_k}^3 - B_kp_{A_k}^3 + A_kp_{A_k}^2p_{B_k} - B_kp_{A_k}p_{B_k}^2}{((1+k^2)\sigma a^3\pi\eta_k)^2}\Big] \\
&= \frac{3I_k}{8}\left(A_k^2 + B_k^2 + \frac{p_{A_k}^2 + p_{B_k}^2}{((1+k^2)\sigma a^3\pi\eta_k)^2}\right). \qquad (3.172)
\end{aligned}
$$

Using equations (3.171,3.172), the average of $\mathcal{P}(Z)$ is given by

$$
\begin{aligned}
< \mathcal{P}(Z) > (q,p) &= \sum_{k=1}^{N}\Omega I_k\Bigg[\frac{2k}{1+k^2} \\
&\qquad - \frac{k^2+14k^3}{2(1+k^2)^2}\left(\frac{A_k^2 + B_k^2 + \frac{p_{A_k}^2+p_{B_k}^2}{((1+k^2)\sigma a^3\pi\eta_k)^2}}{2}\right)\Bigg]. \qquad (3.173)
\end{aligned}
$$

From equations (3.111,3.113) we see that the term in parentheses is the average of $(A_k^2 + B_k^2)$ over the nominal dynamics. We move to the averaged coordinates $(\phi_k, I_k, \rho_k, p_{\rho_k})$ as in the comments following equation (3.145). In these coordinates, the average of $\mathcal{P}(Z)$ has the form

$$
< \mathcal{P}(Z) > (q,p) = \sum_{k=1}^{N}\Omega I_k\left[\frac{2k}{1+k^2} - \frac{k^2+14k^3}{2(1+k^2)^2}\rho_k^2\right]. \qquad (3.174)
$$

The lift to third-order of Ω with respect to the Cartan-Hannay-Berry connection is given by

$$
\begin{aligned}
(-X_{<\mathcal{P}(Z)>}, \Omega) &= \Bigg(\Bigg\{-\Omega\left[\frac{2k}{1+k^2} - \frac{k^2+14k^3}{2(1+k^2)^2}\rho_k^2\right]\frac{\partial}{\partial\phi_k} \\
&\qquad - \Omega I_k\rho_k\left[\frac{k^2+14k^3}{(1+k^2)^2}\right]\frac{\partial}{\partial p_{\rho_k}}\Bigg\}_{k=1}^{N}, \Omega\Bigg). \qquad (3.175)
\end{aligned}
$$

From equation (3.175) we see that both I_k and ρ_k are constant. Thus

$$
\begin{aligned}
\Delta\phi_k &= -\int_0^T \Omega\left[\frac{2k}{1+k^2} - \frac{k^2+14k^3}{2(1+k^2)^2}\rho_k^2\right]dt \\
&= -2\pi\left[\frac{2k}{1+k^2} - \frac{k^2+14k^3}{2(1+k^2)^2}\rho_k^2\right]. \qquad (3.176)
\end{aligned}
$$

Notice that the third-order terms act to reduce the rate of nodal rotation and thus the sensitivity of a vibrating ring gyroscope cannot be increased by increasing the amplitude of vibration and using the nonlinear effects.

In contrast to the earlier calculation where we kept in Z only the terms linear in configuration variables, in the present nonlinear setting the imposed rotation causes not only a precession of the nodes of vibration but also a drift in the momentum conjugate to ρ_k. In practical devices the ring is driven into a single mode of oscillation and the imposed rotation sensed by measuring the drift rate of the nodal points of the vibration. Thus the effect of the second term in equation (3.175) will be compensated for by the drive electronics.

It is interesting to ask how large the nonlinear effect on the drift rate of the nodal points of the vibrations is in a typical device. The micromachined ring gyroscope of Putty and Najafi [76] utilizes a ring of radius $a = 500\mu m$ placed into elliptical vibration so that $k = 2$ with a radial deformation amplitude of $0.15\mu m$. From equations (3.72,3.103), the radial deformation for this ring is

$$w(\theta) = 2a\left[A_2 \sin(2\theta) - B_2 \cos(2\theta)\right]. \tag{3.177}$$

Let $t = 0$ to be the time at which the maximum radial deformation is attained and $\theta = 0$ to be the location on the ring of the maximum radial deformation at time $t = 0$. From these definitions we have the initial conditions $\widehat{A}_2 = \widehat{p}_{A_2} = \widehat{p}_{B_2} = 0$ and $\widehat{B}_2 = -\frac{0.15\mu m}{2(500\mu m)}$. Inserting these values into equation (3.176) yields

$$\Delta\phi_2 = -2\pi\left[\frac{4}{5} - 2.61 \times 10^{-8}\right]. \tag{3.178}$$

For the normal operation of this device, then, the nonlinear effects are seven orders of magnitude smaller than the first-order terms.

3.4.5 Comments on the ring gyroscope example

In this section we have explored the rotating, vibrating gyroscope. After first understanding the nominal dynamics we showed that the precession of the nodal points of the vibration due to the imposed rotation can be understood as a Hannay-Berry phase. By linearizing the system we were able to recover the results of Bryan. Using the inherently nonlinear nature of the moving systems approach we then went on to calculate the effect of the imposed rotation on the nodal precession to third order. These calculations show that the nonlinear effects reduce the sensitivity of the device and we therefore conclude that the best performance for these devices is achieved when operating them in the linear regime.

The correction terms we explore in this section are those arising due to the nonlinear character of the vibrations. The analysis still assumes, however, that the imposed motion is adiabatic. We now turn to developing a method to account for the finite but slow rate of rotation.

Chapter 4

Non-adiabatic corrections to the Hannay-Berry phase

4.1 Introduction

In defining the Hannay-Berry phase it is assumed that the imposed motion is adiabatic. In practice, of course, while this motion may be very slow with respect to the nominal dynamics, it is not infinitely slow. In this chapter we seek to account for the effects of the non-adiabatic nature of the motion.

Since Berry's original work on the geometric phase in a quantum system undergoing an adiabatic variation of its parameters [9], various techniques have been proposed to account for the finite rate of change of the parameters in the Hamiltonian. Berry developed an iterative scheme in which the geometric phase at each step is incorporated into the nominal dynamics [11]. Other authors showed that the Berry phase can be viewed as the first-order term in a perturbation expansion of the system and found corrections by carrying the perturbation to higher orders ([84, 28]). More recent work has provided example quantum systems in which the theory has been applied, such as nuclear quadrupole resonance [85], hysteresis loops in manganese acetate crystals [31], and magnetic resonance [33].

A few authors have considered the effect of the finite rate of change of the parameters on the Hannay angles, that is on the geometric phase for classical integrable systems. Bhattacharjee and Sen used a perturbative method [12] that is then compared in [34] by Gjaja and Bhattacharjee to the classical analog of the iterative scheme proposed by Berry.

While the Cartan-Hannay-Berry connection does not come from an adiabatic criterion, there is an underlying assumption of adiabaticity as described earlier. One is immediately led to ask, then, if there is a way to incorporate higher-order corrections which include at least some part of the effect of the neglected terms (see equation (3.15) and comments thereafter). While one can apply classical perturbation techniques to the corresponding vector fields, these methods ignore the underlying geometric structure.

Recall that in defining the Hannay-Berry phase it is assumed there is a Lie group acting on the system with respect to which we can average. Under appropriate assumptions, the nominal dynamics naturally provide such a group action. In that case, as we will show later, the term in the averaged Hamiltonian giving rise to the Hannay-Berry phase Poisson commutes with the nominal Hamiltonian, using the canonical Poisson bracket on the phase space. This leads us to Hamiltonian perturbation theory and Hamiltonian normal forms.

The terms we seek to include are generally described as the centrifugal forces. When not considering the imposed motion as a small perturbation from the nominal system, these terms are often incorporated into an amended potential (see, e.g. [59], Section 8.6). In this setting we have assumed the imposed motion is slow and we wish to develop a technique which takes advantage of this when describing the system dynamics.

We begin in the next section with a brief introduction to Hamiltonian normal form theory and then show that the Hannay-Berry phase can be viewed as a first-order correction to the flow of the nominal system using the Hamiltonian normal forms approach. This allows us to define higher-order approximations which, under appropriate assumptions, take the form of additional corrections. We then illustrate the theory by applying it to the vibrating ring gyroscope.

4.2 Hamiltonian normal form theory

The theory of Hamiltonian normal forms is a generalization of Lie perturbation techniques (see, e.g. [24, 52]) which in turn is built upon the perturbation methods developed by Poincaré and von Ziepel (see [5] for historical comments). In this section we provide a brief description of the theory and refer the reader to [26, 27] for more details and further references.

We first need the notion of a Poisson bracket and a Poisson manifold.

Definition 4.2.1 *A* **Poisson manifold** *is a smooth manifold M together with a $\mathbb{R}$-bilinear map on $C^\infty(M)$*

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

which for all $f, g, h \in C^\infty(M)$ satisfies:

i) Skew symmetry: $\{f, g\} = -\{g, f\}$,

ii) Leibniz identity: $\{f, gh\} = \{f, g\}h + g\{f, h\}$,

iii) Jacobi identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$

■

Consider a Poisson manifold $(M, \{\cdot, \cdot\})$. Let $\mathcal{F}(M)$ be the vector space of formal power series in ϵ with coefficients in $C^\infty(M)$. That is

$$\mathcal{F}(M) = \left\{ f_\epsilon \in C^\infty(M) | f_\epsilon = \sum_{i=0}^{\infty} \epsilon^i f_i, f_i \in C^\infty(M) \right\}. \tag{4.1}$$

Let $\text{ad}_f g = \{g, f\}$ and define $\text{ad}_f^0 g = g$. We then recursively define ad_f^i by

$$\text{ad}_f^i g = \{\text{ad}_f^{i-1} g, f\}. \tag{4.2}$$

We make the following definitions.

Definition 4.2.2 *The* **Lie series** *of f is the formal power series*

$$\phi_\epsilon^f = \exp(\epsilon \text{ad}_f) = \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} \text{ad}_f^i. \tag{4.3}$$

■

ϕ_ϵ^f is the formal flow of the Hamiltonian vector field X_f with ϵ as the time parameter.

Definition 4.2.3 *For $f \in \mathcal{F}(M)$ we say that X_f has* **periodic flow** *if there exists a positive, smooth function T on M such that for all $m \in M$ and for all $g \in \mathcal{F}(M)$ we have that $\left(\left(\phi_T^f \right)^* (g) \right) (m) = g(m)$.* ■

Definition 4.2.4 *Consider $H \in \mathcal{F}(M)$ and suppose X_{H_0} has periodic flow. We say that H is in* **normal form with respect to $\mathbf{H_0}$** *if $\{H_0, H_i\} = 0$ $\forall i \geq 0$ and that H is in* **normal form up to order n with respect to $\mathbf{H_0}$** *if $\{H_0, H_i\} = 0$, $1 \leq i \leq n$.* ■

To bring a Hamiltonian into normal form we will use a formal change of coordinates of the form ϕ_ϵ^f for some appropriate $f \in \mathcal{F}(m)$. The following lemma from [27] shows how the Hamiltonian is modified under such a change of coordinates.

Lemma 4.2.5 [27] *Let $H, f \in \mathcal{F}(M)$. If ϕ_ϵ^f is the flow of X_f then*

$$\left(\phi_\epsilon^f\right)^* H = \exp(\epsilon \mathrm{ad}_f)H. \tag{4.4}$$

■

Using equation (4.3) in equation (4.4) we have

$$\begin{aligned}\left(\phi_\epsilon^f\right)^* H &= \exp(\epsilon \mathrm{ad}_f)H \\ &= \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!}\mathrm{ad}_f^i(\sum_{j=0}^{\infty} \epsilon^j H_j) \\ &= H_0 + \epsilon(H_1 + \mathrm{ad}_f H_0) + \epsilon^2(H_2 + \mathrm{ad}_f H_1 + \frac{1}{2}\mathrm{ad}_f^2 H_0) + O(\epsilon^3). \end{aligned} \tag{4.5}$$

To bring H into first-order normal form, we seek a function $f \in \mathcal{F}(M)$ such that

$$\{H_0, H_1 + ad_f H_0\} = 0. \tag{4.6}$$

To find this function we use the following lemma from [26].

Lemma 4.2.6 [26] *If X_{H_0} has periodic flow on M then*

$$C^\infty(M) = \ker\left(\mathrm{ad}_{H_0}\right) \oplus \mathrm{im}\left(\mathrm{ad}_{H_0}\right). \tag{4.7}$$

Proof We provide a sketch of the proof. Consider a function $g \in C^\infty(M)$. Let $< g >$ be its average over the orbits of H_0, i.e.

$$< g >= \frac{1}{T}\int_0^T \left(\phi_t^{H_0}\right)^* g dt. \tag{4.8}$$

One first shows that $< g >$ is in the kernel of ad_{H_0}, that is $< g >$ Poisson commutes with H_0 (see the proof of Lemma 4.3.1 for the details of this step) and then that $\widetilde{g} = g- < g >$ is in the image of ad_{H_0}. For details see [26]. ■

To put the Hamiltonian into normal form to first-order we proceed as follows. First write

$$H_1 =< H_1 > +(H_1- < H_1 >) \tag{4.9}$$

and then substitute equation (4.9) into equation (4.6). Thus

$$\begin{aligned} 0 &= \{H_0, < H_1 > + (H_1 - < H_1 >) + \mathrm{ad}_f H_0\} \\ &= \{H_0, < H_1 >\} + \{H_0, (H_1 - < H_1 >) + \mathrm{ad}_f H_0\} \\ &= \{H_0, (H_1 - < H_1 >) + \mathrm{ad}_f H_0\} \end{aligned} \tag{4.10}$$

where the last step follows from the fact that $< H_1 > \in \ker(\mathrm{ad}_{H_0})$. We then seek a solution to the *homological equation*

$$\mathrm{ad}_f H_0 = -(H_1 - < H_1 >) \tag{4.11}$$

where f is the unknown function.

Proposition 4.2.7 [26] *The solution to equation* (4.11) *is given by*

$$f = \frac{1}{T}\int_0^T t\left(\phi_t^{H_0}\right)^* (H_1 - < H_1 >)\, dt. \tag{4.12}$$

Proof Let $g = -\mathrm{ad}_f H_0 = \mathrm{ad}_{H_0} f$. This is equivalent to the dynamical system

$$\frac{d}{dt}\left(\phi_t^{H_0}\right)^* f = \left(\phi_t^{H_0}\right)^* g$$

(see, e.g. Proposition 10.2.3 of [59]). We show that $g = H_1 - < H_1 >$ by direct substitution. Thus

$$\begin{aligned} \frac{d}{dt}\left(\phi_t^{H_0}\right)^* f &= \frac{d}{dt}\left(\frac{1}{T}\int_0^T \tau\left(\phi_\tau^{H_0}\right)^* (H_1 - < H_1 >) d\tau\right) \\ &= \frac{1}{T}\frac{d}{dt}\int_0^T \tau\left(\phi_{t+\tau}^{H_0}\right)^* (H_1 - < H_1 >) d\tau \\ &= \frac{1}{T}\frac{d}{dt}\int_t^{t+T} (\sigma - t)\left(\phi_\sigma^{H_0}\right)^* (H_1 - < H_1 >) d\sigma \\ &= \frac{1}{T}\left(T\left(\phi_{t+T}^{H_0}\right)^* (H_1 - < H_1 >) - \int_t^{t+T} \left(\phi_\sigma^{H_0}\right)^* (H_1 - < H_1 >) d\sigma\right) \end{aligned}$$

$$
\begin{aligned}
&= \left(\phi_{t+T}^{H_0}\right)^* (H_1 - < H_1 >) - \frac{1}{T}\int_t^{t+T} \left(\phi_\sigma^{H_0}\right)^* H_1 d\sigma + < H_1 > \\
&= \left(\phi_{t+T}^{H_0}\right)^* (H_1 - < H_1 >) - < H_1 > + < H_1 > \\
&= \left(\phi_t^{H_0}\right)^* (H_1 - < H_1 >).
\end{aligned}
$$

Therefore $g = (H_1 - < H_1 >)$. From this the Proposition follows. ■

With this choice of f, the Hamiltonian in equation (4.4) becomes

$$
\exp(\epsilon \mathrm{ad}_f)H = H_0 + \epsilon < H_1 > + \epsilon^2 \left(H_2 + \mathrm{ad}_f H_1 + \frac{1}{2}\mathrm{ad}_f^2 H_0 \right) + O(\epsilon^3). \qquad (4.13)
$$

Notice that if we wish to bring the Hamiltonian into normal form only up to first-order then there is no need to explicitly calculate the generating function f.

To bring the function into normal form up to second-order we repeat the process, now on the once transformed Hamiltonian. This time we seek a generating function of the form ϵg. Applying the corresponding change of coordinates results in

$$
\begin{aligned}
\exp(\epsilon \mathrm{ad}_{\epsilon g}) \left(\exp(\epsilon \mathrm{ad}_f)H\right) &= \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} \mathrm{ad}_{\epsilon g}^i \left(\exp(\epsilon \mathrm{ad}_f)H\right) \\
= \quad H_0 + \epsilon < H_1 > + \epsilon^2 &\left(H_2 + \mathrm{ad}_f H_1 + \frac{1}{2}\mathrm{ad}_f^2 H_0 + \mathrm{ad}_g H_0 \right) + O(\epsilon^3). \qquad (4.14)
\end{aligned}
$$

The homological equation which needs to be solved is

$$
\begin{aligned}
\mathrm{ad}_g H_0 &= -\left(\left(H_2 + \mathrm{ad}_f H_1 + \frac{1}{2}\mathrm{ad}_f^2 H_0 \right) \right. \\
&\qquad \left. - \left(< H_2 > + < \mathrm{ad}_f H_1 > + < \frac{1}{2}\mathrm{ad}_f^2 H_0 > \right) \right). \qquad (4.15)
\end{aligned}
$$

From Proposition 4.2.7, the solution to this equation is

$$
\begin{aligned}
g &= \frac{1}{T}\int_0^T t \left(\phi_t^{H_0}\right)^* \left[(H_2 - < H_2 >) + (\mathrm{ad}_f H_1 - < \mathrm{ad}_f H_1 >) \right. \\
&\qquad \left. + \frac{1}{2}(\mathrm{ad}_f^2 H_0 - < \mathrm{ad}_f^2 H_0 >) \right] dt. \qquad (4.16)
\end{aligned}
$$

With this choice our transformed Hamiltonian becomes

$$\begin{aligned}\exp(\epsilon \mathrm{ad}_{\epsilon g})\left(\exp(\epsilon \mathrm{ad}_f)H\right) &= H_0 + \epsilon < H_1 > \\ &+\epsilon^2 \left(< H_2 > + < \mathrm{ad}_f H_1 > + \frac{1}{2} < \mathrm{ad}_f^2 H_0 > \right) + O(\epsilon^3). \quad (4.17)\end{aligned}$$

By repeating this process the Hamiltonian can be placed into normal form up to arbitrary order n.

In practice one places the Hamiltonian into normal form up to some desired order n and then drops the higher-order terms. The truncated Hamiltonian gives an approximation to the original system. Since the coefficients of ϵ^i in the Hamiltonian all commute with H_0 for $i = 1, 2, \ldots, n$, the flow of the corresponding Hamiltonian vector field of the higher-order terms also commutes with the flow of the nominal system. Thus for a Hamiltonian in first-order normal form we have

$$\phi_t^{H_0+\epsilon<H_1>}(m) = \phi_t^{\epsilon<H_1>} \circ \phi_t^{H_0}(m), \qquad m \in M \tag{4.18}$$

and the first-order terms give rise naturally to a first-order correcting symplectic map given by the flow of the Hamiltonian system $\epsilon < H_1 >$. For systems in higher-order normal form, however, while the functions at each order do Poisson commute with H_0 they do not in general commute with each other and thus a system in n^{th}-order normal form defines a single n^{th}-order correcting symplectic map.

4.3 Normal forms and the Hannay-Berry phase

In the setting of the moving systems approach the Poisson manifold is T^*Q together with the canonical Poisson bracket defined by

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}, \quad f, g \in C^\infty(M). \tag{4.19}$$

To apply Hamiltonian normal form theory to the moving systems approach we make a few additional assumptions on the Hamiltonian in equation (3.14). We first assume the potential U on S is constant and drop it from the Hamiltonian. Next we assume that $\mathcal{Z}_t(m_t(q))$ can be written in the form

$$\mathcal{Z}_t(m_t(q)) = \epsilon \widehat{\mathcal{Z}}_t(m_t(q)) \tag{4.20}$$

for some parameter ϵ. For example, if M is a Riemannian manifold and $\mathcal{Z}_t$ is a constant magnitude vector field then we may take $\epsilon = \|\mathcal{Z}_t\|$ and $\widehat{\mathcal{Z}_t} = \frac{\mathcal{Z}_t}{\|\mathcal{Z}_t\|}$. If $\|\mathcal{Z}_t\|$ is not constant then one could take ϵ to be the average magnitude of $\|\mathcal{Z}_t\|$ over the loop in M starting at the given initial condition. Note, however, that the form of $\mathcal{Z}_t$ in equation (4.20) is often natural to the problem and in general $\widehat{\mathcal{Z}_t}$ is not a unit vector. Under these assumptions the Hamiltonian, equation (3.14), can be written as

$$H(q,p) = H_0(q,p) + \epsilon H_1(q,p) + \epsilon^2 H_2(q,p) \tag{4.21}$$

where

$$H_0(q,p) = \frac{1}{2}\|p\|^2 + V(q), \tag{4.22}$$

$$H_1(q,p) = -\mathcal{P}(\widehat{Z}_t), \tag{4.23}$$

$$H_2(q,p) = -\frac{1}{2}\|\widehat{\mathcal{Z}}_t^{\perp}\|^2. \tag{4.24}$$

Finally we assume that H_0 has periodic flow with period T. We then have a natural action of S^1 on T^*Q given by $\phi_t^{H_0}$, the flow of X_{H_0}. Let $< \cdot >$ denote the average with respect to this group action, i.e. for a smooth function f on T^*Q we have

$$< f >= \frac{1}{T}\int_0^T \left(\phi_t^{H_0}\right)^* f dt. \tag{4.25}$$

In general, the parameter ϵ captures the rate of the imposed motion on the system. In the adiabatic limit, then, ϵ goes to zero and the terms in ϵ^2 are negligible.

In what follows we are interested in relaxing the adiabatic condition; i.e. we assume that while ϵ is small, the terms in ϵ^2 are not negligible. We begin with the following lemma.

Lemma 4.3.1 *The truncated averaged Hamiltonian defined by*

$$< H >^{(1)} (q,p) = H_0(q,p) + \epsilon < H_1 > (q,p) \tag{4.26}$$

is in first-order normal form.

Proof We show that $< H_1 > \in \ker(\mathrm{ad}_{H_0})$. Let $g = \{< H_1 >, H_0\}$. This is equivalent to the dynamical system

$$\frac{d}{dt} \left(\phi_t^{H_0}\right)^* < H_1 >= \left(\phi_t^{H_0}\right)^* g. \tag{4.27}$$

From this we have

$$\begin{aligned} g(\phi_t^{H_0}(q,p)) &= \frac{d}{dt}\frac{1}{T}\int_0^T H_1(\phi_t^{H_0}(\phi_\tau^{H_0}(q,p)))d\tau \\ &= \frac{1}{T}\frac{d}{dt}\int_t^{t+T} H_1(\phi_\sigma^{H_0}(q,p))d\sigma \\ &= \frac{1}{T}\left[H_1(\phi_{t+T}^{H_0}(q,p)) - H_1(\phi_t^{H_0}(q,p))\right] \\ &= 0 \end{aligned} \tag{4.28}$$

where the last step follows from the periodic flow property of H_0. ∎

Following the comments of Section 4.2, the flow of the system to first-order is given by

$$\phi_t^{H_0-\epsilon<\mathcal{P}(\widehat{Z}_t)>}(q,p) = \phi_t^{-\epsilon<\mathcal{P}(\widehat{Z}_t)>} \circ \phi_t^{H_0}(q,p) \tag{4.29}$$

and the flow of $- < \mathcal{P}(\widehat{Z}_t) >$ defines the correcting symplectic map to first-order. Thus, in the setting where the group action on T^*Q is given by the flow of the nominal dynamics, we interpret the Hannay-Berry phase as arising from this correcting map.

To find a more accurate expression, then, we express the Hamiltonian in normal form to a higher-order before truncating. Let G be the generator of a change of coordinates bringing the original Hamiltonian into first-order normal form (obtained by solving the homological equation). From Proposition 4.2.7 and the form of H_1 in equation (4.23), G is given by

$$G = \frac{1}{T}\int_0^T t\left(\phi_t^{H_0}\right)^* \left[< \mathcal{P}(\widehat{Z}_t) > - \mathcal{P}(\widehat{Z}_t)\right] dt. \tag{4.30}$$

From equation (4.17) and the form of H_2 in equation (4.24), the second-order truncated normal form is

$$\begin{aligned} < H >^{(2)}(q,p) &= H_0(q,p) - \epsilon < \mathcal{P}(\widehat{Z}_t) > - \epsilon^2 \left(\frac{1}{2} < \|\widehat{\mathcal{Z}}_t^{\perp}\|^2 > \right. \\ &\quad \left. + < \mathrm{ad}_G \mathcal{P}(\widehat{Z}_t) > - \frac{1}{2} < \mathrm{ad}_G^2 H_0 > \right). \end{aligned} \tag{4.31}$$

Notice that the terms at second-order in the Hamiltonian account not only for the average effect of the centrifugal force but also include additional terms involving the first-order change of coordinates. The flow of the system to second-order is

$$\begin{aligned} &\phi_t^{H_0 - \epsilon < \mathcal{P}(\widehat{Z}_t) > - \epsilon^2 \left(\frac{1}{2} < \|\widehat{\mathcal{Z}}_t^{\perp}\|^2 > + < \mathrm{ad}_G \mathcal{P}(\widehat{Z}_t) > - \frac{1}{2} < \mathrm{ad}_G^2 H_0 > \right)}(q,p) \\ &= \phi_t^{- \epsilon < \mathcal{P}(\widehat{Z}_t) > - \epsilon^2 \left(\frac{1}{2} < \|\widehat{\mathcal{Z}}_t^{\perp}\|^2 > + < \mathrm{ad}_G \mathcal{P}(\widehat{Z}_t) > - \frac{1}{2} < \mathrm{ad}_G^2 H_0 > \right)} \circ \phi_t^{H_0}(q,p) \end{aligned} \tag{4.32}$$

and thus in general this defines a correcting symplectic map to second-order. If in addition the terms in ϵ Poisson commute with the terms in ϵ^2 then the second-order terms define a second-order correcting symplectic map. In this case the three Hamiltonian systems can be solved independently and their flows composed to obtain the second-order solution. This is captured in the following lemma.

Lemma 4.3.2 *If*

$$\{< \mathcal{P}(\widehat{Z}_t) >, \frac{1}{2} < \|\widehat{\mathcal{Z}}_t^{\perp}\|^2 > + < \mathrm{ad}_G \mathcal{P}(\widehat{Z}_t) > - \frac{1}{2} < \mathrm{ad}_G^2 H_0 >\} = 0 \tag{4.33}$$

then

$$\phi_t^{H_0-\epsilon<\mathcal{P}(\widehat{Z}_t)>-\epsilon^2\left(\frac{1}{2}<\|\widehat{\mathcal{Z}}_t^\perp\|^2>+<\mathrm{ad}_G\mathcal{P}(\widehat{Z}_t)>-\frac{1}{2}<\mathrm{ad}_G^2 H_0>\right)}(q,p)$$
$$= \phi_t^{-\epsilon^2\left(\frac{1}{2}<\|\widehat{\mathcal{Z}}_t^\perp\|^2>+<\mathrm{ad}_G\mathcal{P}(\widehat{Z}_t)>-\frac{1}{2}<\mathrm{ad}_G^2 H_0>\right)} \circ \phi_t^{-\epsilon<\mathcal{P}(\widehat{Z}_t)>} \circ \phi_t^{H_0}(q,p). \quad (4.34)$$

Proof Immediate by the assumption of the Poisson commutativity of the functions. ∎

4.3.1 Time-dependence of non-adiabatic corrections

In Section 3.2.2 we showed the Hannay-Berry phase is a geometric phenomenon by showing the corresponding ordinary differential equation is independent of the time parametrization. We now show that the terms in ϵ^2 in the moving systems Hamiltonian do not result in a geometric effect. Consider equation (4.31). For simplicity assume the generating function for the change of coordinates is $G = 0$ and that $\{< H_1 >, < H_2 >\} = 0$ so that we can calculate the effect on the system from these two terms separately. Denote points in T^*Q by z. Noticing that $X_{<\|\mathcal{Z}^\perp\|^2>}$ is a quadratic form in the vector field $\mathcal{Z}$ on the base space, we define

$$Y(\mathcal{Z}_t, z) = -X_{<\|\mathcal{Z}_t^\perp|^2>} \quad (4.35)$$

where $Y(a\mathcal{Z}_t, z) = a^2 Y(\mathcal{Z}_t, z)$. The corresponding ordinary differential equation is

$$\dot{z} = Y(\mathcal{Z}_t, z).$$

We now change the time parametrization (as in Section 3.2.2) by taking $t \mapsto \tau(t)$ with $\frac{d\tau}{dt}$ strictly positive. Under this parametrization, the vector field $\mathcal{Z}_t$ is scaled by $\frac{d\tau}{dt}$ and thus

$$\frac{dz}{dt} = \frac{dz}{d\tau}\frac{d\tau}{dt} = Y(\frac{d\tau}{dt}\mathcal{Z}_\tau, z) = \left(\frac{d\tau}{dt}\right)^2 Y(\mathcal{Z}_\tau, z).$$

From this we have

$$\frac{dz}{d\tau} = \frac{d\tau}{dt} Y(\mathcal{Z}_\tau, z)$$

which shows the dependence on the time parametrization.

4.4 Non-adiabatic corrections of the ring gyroscope

To illustrate the technique introduced above we now apply it to the vibrating ring gyroscope. To simplify notation we will restrict ourselves to solutions with a single active Fourier mode and work only with the linearized version of the imposed motion vector field. Our goal is to derive a solution to this system to second-order in this simplified, linear setting which can be compared to the solution derived in Section 3.4.3.

4.4.1 Flow map of the nominal dynamics

Consider the nominal dynamics of the vibrating ring as given by equations (3.111 - 3.114). Define

$$\beta_k = \frac{1}{(1+k^2)\sigma a^3 \pi \eta_k}. \tag{4.36}$$

The flow map of the nominal system can then be expressed as

$$\phi_t^{H_0}(q,p) = \begin{pmatrix} \cos(\eta_k t) & \beta_k \sin(\eta_k t) & 0 & 0 \\ -\frac{1}{\beta_k}\sin(\eta_k t) & \cos(\eta_k t) & 0 & 0 \\ 0 & 0 & \cos(\eta_k t) & \beta_k \sin(\eta_k t) \\ 0 & 0 & -\frac{1}{\beta_k}\sin(\eta_k t) & \cos(\eta_k t) \end{pmatrix} \begin{pmatrix} A_k \\ p_{A_k} \\ B_k \\ p_{B_k} \end{pmatrix}. \tag{4.37}$$

4.4.2 Flow map of the first-order correction

From equation (3.137), the linearized tangent vector $Z(q)$ on T_qQ due to the imposed rotation is

$$Z_k(q_k) = \frac{2k\Omega}{1+k^2} S(1) q_k. \tag{4.38}$$

We recognize in Ω the parameter referred to as ϵ above. Define

$$\widehat{Z}(q) = \frac{2k}{1+k^2} S(1) q_k \tag{4.39}$$

so that the function $\mathcal{P}(\widehat{Z})$ is (see equation (3.138))

$$\mathcal{P}(\widehat{Z})(q,p) = \frac{2k}{1+k^2} I_k = \frac{2k}{1+k^2} \left(A_k p_{B_k} - B_k p_{A_k} \right) \tag{4.40}$$

where we have used the definition of I_k from equation (3.139). From equations (4.23) and (4.40) we have

$$H_1(q) = -\frac{2k}{1+k^2} I_k. \tag{4.41}$$

Since I_k is constant along the trajectories of the nominal system, we have $< H_1 > (q,p) = H_1(q,p)$. Inserting this into equation (4.30), we see that the generating function for the change of coordinates is $G = 0$. (Note that if we considered nonlinear terms in H_1, as in Section 3.4.4, then H_1 would not be constant along trajectories and we would have a nontrivial change of coordinates.)

This Hamiltonian system has the simple solution as given in equation (3.147). The corresponding flow map in the coordinates $(A_k, p_{A_k}, B_k, p_{B_k})$ is given by

$$\begin{aligned} \phi_t^{-\Omega<\mathcal{P}(Z_t)>}(q,p) &= \\ &\begin{pmatrix} \cos\left(\frac{-2k\Omega t}{1+k^2}\right) & 0 & -\sin\left(\frac{-2k\Omega t}{1+k^2}\right) & 0 \\ 0 & \cos\left(\frac{-2k\Omega t}{1+k^2}\right) & 0 & -\sin\left(\frac{-2k\Omega t}{1+k^2}\right) \\ \sin\left(\frac{-2k\Omega t}{1+k^2}\right) & 0 & \cos\left(\frac{-2k\Omega t}{1+k^2}\right) & 0 \\ 0 & \sin\left(\frac{-2k\Omega t}{1+k^2}\right) & 0 & \cos\left(\frac{-2k\Omega t}{1+k^2}\right) \end{pmatrix} \begin{pmatrix} A_k \\ p_{A_k} \\ B_k \\ p_{B_k} \end{pmatrix}. \end{aligned} \tag{4.42}$$

4.4.3 Second-order correction

Since the change of coordinates function taking the ring system to first-order normal form is zero, the Hamiltonian in second-order normal form for the ring is given by

$$< H >^{(2)} (q,p) = H_0(q,p) - \Omega < P(\widehat{Z}) > (q,p) - \frac{\Omega^2}{2} < \|\widehat{\mathcal{Z}}^{\perp}\|^2 > \tag{4.43}$$

where H_0, defining the nominal dynamics, has a flow map given by equation (4.37). Linearizing the orthogonal complement of the tangent vector arising from the imposed motion, $\mathcal{Z}^{\perp}$ (equation (3.133)), with respect to the coordinates (A_k, B_k) and restricting to the k^{th} mode we have

$$\mathcal{Z}^{\perp} = \begin{pmatrix} \Omega \\ \left[\frac{2\Omega k}{1+k^2} B_k\right] \\ \left[\frac{-2\Omega k}{1+k^2} A_k\right] \end{pmatrix} = \begin{pmatrix} \Omega \\ \mathcal{Z}^{\perp_2} \end{pmatrix}. \tag{4.44}$$

Using the inner product in equation (3.75), we have

$$\begin{aligned}
\|\mathcal{Z}^{\perp}\|^2 &= ((\Omega, \mathcal{Z}^{\perp_2}), (\Omega, \mathcal{Z}^{\perp_2})) \\
&= \int_0^{2\pi} \left[\left(1 - \frac{\partial \alpha}{\partial \theta}\right)^2 \left(\Omega^2 + 2\Omega \mathcal{Z}^{\perp_2} + \left(\mathcal{Z}^{\perp_2}\right)^2\right) + \left(\frac{\partial \mathcal{Z}^{\perp_2}}{\partial \theta}\right)^2 \right] \sigma a^3 d\theta \\
&= \int_0^{2\pi} \Bigg(\Omega^2 - 2\Omega^2 \frac{\partial \alpha}{\partial \theta} + \Omega^2 \left[\frac{\partial \alpha}{\partial \theta}\right]^2 + 2\Omega \mathcal{Z}^{\perp_2} - 4\Omega \frac{\partial \alpha}{\partial \theta} \mathcal{Z}^{\perp_2} \\
&\qquad +2 \left[\frac{\partial \alpha}{\partial \theta}\right]^2 \mathcal{Z}^{\perp_2} + \left[Z^{\perp_2}\right]^2 - 2\frac{\partial \alpha}{\partial \theta} \left[\mathcal{Z}^{\perp_2}\right]^2 \\
&\qquad + \left[\mathcal{Z}^{\perp_2}\right]^2 \left[\frac{\partial \alpha}{\partial \theta}\right]^2 + \left[\frac{\partial \mathcal{Z}^{\perp_2}}{\partial \theta}\right]^2 \Bigg) \sigma a^3 d\theta.
\end{aligned} \tag{4.45}$$

We now express α in the Fourier basis, as in equation (3.103), and utilize the relations in equations (3.122) - (3.125) concerning the integrals of the various powers of sin and cos over a full period, to reduce the above expression to

$$\|\mathcal{Z}^{\perp}\|^2 = \Omega^2 \sigma a^3 \pi \left[2 + \left(\frac{k^6 - 2k^4 - 3k^2}{(1+k^2)^2} \right) (A_k^2 + B_k^2) \right.$$

$$+\left(\frac{3k^4}{(1+k^2)^2}\right)\left(A_k^4-\frac{2}{3}A_k^2B_k^2+B_k^4\right)\Bigg]. \quad (4.46)$$

Recalling that we are in the small-amplitude vibration regime we drop terms higher than second-order in A_k, B_k to get

$$\begin{aligned}\|\mathcal{Z}^{\perp}\|^2 &= \Omega^2\sigma a^3\pi\left[2+\left(\frac{k^6-2k^4-3k^2}{(1+k^2)^2}\right)(A_k^2+B_k^2)\right]\\ &= \Omega^2\sigma a^3\pi\left[2+\Gamma_k(A_k^2+B_k^2)\right] \quad (4.47)\end{aligned}$$

where we have defined the constant

$$\Gamma_k=\frac{k^6-2k^4-3k^2}{(1+k^2)^2}. \quad (4.48)$$

From equations (4.47) and (4.24) we have (dropping the constant term)

$$H_2(q,p)=-\frac{\sigma a^3\pi\Gamma_k}{2}(A_k^2+B_k^2). \quad (4.49)$$

Averaging over the nominal dynamics of equations (3.111) and (3.113) we have

$$\begin{aligned}< A_k^2 > &= \frac{\eta_k}{2\pi}\int_0^{\frac{2\pi}{\eta_k}}\left(\phi_t^{H_0}\right)^* A_k^2 dt\\ &= \frac{\eta_k}{2\pi}\int_0^{\frac{2\pi}{\eta_k}}\left[A_k\cos(\eta_k t)+\frac{p_{A_k}}{(1+k^2)\sigma a^3\pi\eta_k}\sin(\eta_k t)\right]^2 dt\\ &= \frac{1}{2}\left[A_k^2+\frac{p_{A_k}^2}{[(1+k^2)\sigma a^3\pi\eta_k]^2}\right] \quad (4.50)\end{aligned}$$

and

$$< B_k^2 > = \frac{\eta_k}{2\pi}\int_0^{\frac{2\pi}{\eta_k}}\left(\phi_t^{H_0}\right)^* B_k^2 dt=\frac{1}{2}\left[B_k^2+\frac{p_{B_k}^2}{[(1+k^2)\sigma a^3\pi\eta_k]^2}\right]. \quad (4.51)$$

Using equations (4.50) and (4.51) in (4.49) we have

$$< H_2 > (q,p) = -\frac{\sigma a^3\pi\Gamma_k}{4}\left[A_k^2+B_k^2+\frac{p_{A_k}^2+p_{B_k}^2}{[(1+k^2)\sigma a^3\pi\eta_k]^2}\right] \quad (4.52)$$

We now have the following lemma.

Lemma 4.4.1 $\{< H_1 >, < H_2 >\} = 0.$

Proof

$$\begin{aligned}\{< H_1 >, < H_2 >\} &= \frac{\sigma a^3 \pi \Gamma_k k}{1+k^2}\left[A_k B_k - A_k B_k + \frac{p_{A_k} p_{B_k} - p_{A_k} p_{B_k}}{[(1+k^2)\sigma a^3 \pi \eta_k]^2}\right] \\ &= 0. \end{aligned} \tag{4.53}$$

■

Thus, according to Lemma 4.3.2, the flow of the Hamiltonian vector field associated to $\Omega^2 < H_2 >$ defines a second-order correcting symplectic map for the system. Define

$$\widetilde{\eta}_k^2 = \left(\frac{\Omega^2 \sigma a^3 \pi \Gamma_k}{2}\right)\left(\frac{\Omega^2 \sigma a^3 \pi \Gamma_k}{2[(1+k^2)\sigma a^3 \pi \eta_k]^2}\right) = \left(\frac{\Omega^2 \Gamma_k}{2(1+k^2)\eta_k}\right)^2. \tag{4.54}$$

Using this and the definition of β_k in equation (4.36), $\Omega^2 < H_2 >$ can be written as

$$\Omega^2 < H_2 > (q,p) = -\frac{\widetilde{\eta}_k}{2\beta_k}(A_k^2 + B_k^2) - \frac{\widetilde{\eta}_k \beta_k}{2}(p_{A_k}^2 + p_{B_k}^2).$$

The Hamiltonian vector field associated to this function is given by

$$X_{\Omega^2 < H_2 >} = \begin{pmatrix} -\widetilde{\eta}_k \beta_k p_{A_k} \\ \frac{\widetilde{\eta}_k}{\beta_k} A_k \\ -\widetilde{\eta}_k \beta_k p_{B_k} \\ \frac{\widetilde{\eta}_k}{\beta_k} B_k \end{pmatrix}$$

which defines a pair of uncoupled oscillators. The solution is

$$A_k(t) = \widehat{A}_k \cos(-\widetilde{\eta}_k t) + \beta_k \widehat{p}_{A_k} \sin(-\widetilde{\eta}_k t), \tag{4.55}$$

$$p_{A_k}(t) = -\frac{\widehat{A}_k}{\beta_k} \sin(-\widetilde{\eta}_k t) + \widehat{p}_{A_k} \cos(-\widetilde{\eta}_k t), \tag{4.56}$$

$$B_k(t) = \widehat{B}_k \cos(-\widetilde{\eta}_k t) + \beta_k \widehat{p}_{B_k} \sin(-\widetilde{\eta}_k t), \tag{4.57}$$

$$p_{B_k}(t) = -\frac{\widehat{B}_k}{\beta_k} \sin(-\widetilde{\eta}_k t) + \widehat{p}_{B_k} \cos(-\widetilde{\eta}_k t) \tag{4.58}$$

with the corresponding flow map

$$\phi_t^{\Omega^2<H_2>}(q,p) = \begin{pmatrix} \cos(-\widetilde{\eta}_k t) & \beta_k \sin(-\widetilde{\eta}_k t) & 0 & 0 \\ -\frac{1}{\beta_k}\sin(-\widetilde{\eta}_k t) & \cos(-\widetilde{\eta}_k t) & 0 & 0 \\ 0 & 0 & \cos(-\widetilde{\eta}_k t) & \beta_k \sin(-\widetilde{\eta}_k t) \\ 0 & 0 & -\frac{1}{\beta_k}\sin(-\widetilde{\eta}_k t) & \cos(-\widetilde{\eta}_k t) \end{pmatrix} \begin{pmatrix} A_k \\ p_{A_k} \\ B_k \\ p_{B_k} \end{pmatrix}. \quad (4.59)$$

Since H_0, $< H_1 >$, and $< H_2 >$ all Poisson commute with each other, the flow of the second-order truncated system is given by

$$\begin{aligned} \phi_t^{H_0+\Omega<H_1>+\Omega^2<H_2>}(q,p) &= \phi_t^{\Omega^2<H_2>} \circ \phi_t^{\Omega<H_1>} \circ \phi_t^{H_0}(q,p) \\ &= \phi_t^{\Omega<H_1>} \circ \phi_t^{\Omega^2<H_2>} \circ \phi_t^{H_0}(q,p). \end{aligned} \quad (4.60)$$

From equations (4.37) and (4.59) we have

$$\begin{aligned} \phi_t^{\Omega^2<H_2>} \circ \phi_t^{H_0} &= \begin{pmatrix} \cos(-\widetilde{\eta}_k t) & \beta_k \sin(-\widetilde{\eta}_k t) & 0 & 0 \\ -\frac{1}{\beta_k}\sin(-\widetilde{\eta}_k t) & \cos(-\widetilde{\eta}_k t) & 0 & 0 \\ 0 & 0 & \cos(-\widetilde{\eta}_k t) & \beta_k \sin(-\widetilde{\eta}_k t) \\ 0 & 0 & -\frac{1}{\beta_k}\sin(-\widetilde{\eta}_k t) & \cos(-\widetilde{\eta}_k t) \end{pmatrix} \\ &\quad * \begin{pmatrix} \cos(\eta_k t) & \beta_k \sin(\eta_k t) & 0 & 0 \\ -\frac{1}{\beta_k}\sin(\eta_k t) & \cos(\eta_k t) & 0 & 0 \\ 0 & 0 & \cos(\eta_k t) & \beta_k \sin(\eta_k t) \\ 0 & 0 & -\frac{1}{\beta_k}\sin(\eta_k t) & \cos(\eta_k t) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\omega t) & \beta_k \sin(\omega t) & 0 & 0 \\ -\frac{1}{\beta_k}\sin(\omega t) & \cos(\omega t) & 0 & 0 \\ 0 & 0 & \cos(\omega t) & \beta_k \sin(\omega t) \\ 0 & 0 & -\frac{1}{\beta_k}\sin(\omega t) & \cos(\omega t) \end{pmatrix} \end{aligned} \quad (4.61)$$

where we have defined $\omega = \eta_k - \widetilde{\eta}_k$. From this we see that the terms in Ω^2 appear to soften the material and reduce the frequency of vibration. (This is the usual spurious softening arising from linearizing too early in the modeling process and was seen also in Section 3.4.3. We note once again that we feel it would be interesting to use a geometrically exact model of the ring.)

We now apply the flow map of $\Omega < H_1 >$ to get

$$\begin{aligned}
\phi_t^{\Omega<H_1>} \circ \phi_t^{\Omega^2<H_2>} \circ \phi_t^{H_0} &= \\
&\begin{pmatrix} \cos\left(\frac{-2k\Omega t}{1+k^2}\right) & 0 & -\sin\left(\frac{-2k\Omega t}{1+k^2}\right) & 0 \\ 0 & \cos\left(\frac{-2k\Omega t}{1+k^2}\right) & 0 & -\sin\left(\frac{-2k\Omega t}{1+k^2}\right) \\ \sin\left(\frac{-2k\Omega t}{1+k^2}\right) & 0 & \cos\left(\frac{-2k\Omega t}{1+k^2}\right) & 0 \\ 0 & \sin\left(\frac{-2k\Omega t}{1+k^2}\right) & 0 & \cos\left(\frac{-2k\Omega t}{1+k^2}\right) \end{pmatrix} \\
&* \begin{pmatrix} \cos(\omega t) & \beta_k \sin(\omega t) & 0 & 0 \\ -\frac{1}{\beta_k}\sin(\omega t) & \cos(\omega t) & 0 & 0 \\ 0 & 0 & \cos(\omega t) & \beta_k \sin(\omega t) \\ 0 & 0 & -\frac{1}{\beta_k}\sin(\omega t) & \cos(\omega t) \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} \cos\left(\frac{-2k\Omega t}{1+k^2}\right)\cos(\omega t) & \beta_k \cos\left(\frac{-2k\Omega t}{1+k^2}\right)\sin(\omega t) & -\sin\left(\frac{-2k\Omega t}{1+k^2}\right)\cos(\omega t) & -\beta_k \sin\left(\frac{-2k\Omega t}{1+k^2}\right)\sin(\omega t) \\ -\frac{1}{\beta_k}\cos\left(\frac{-2k\Omega t}{1+k^2}\right)\sin(\omega t) & \cos\left(\frac{-2k\Omega t}{1+k^2}\right)\cos(\omega t) & \frac{1}{\beta_k}\sin\left(\frac{-2k\Omega t}{1+k^2}\right)\sin(\omega t) & -\sin\left(\frac{-2k\Omega t}{1+k^2}\right)\cos(\omega t) \\ \sin\left(\frac{-2k\Omega t}{1+k^2}\right)\cos(\omega t) & \beta_k \sin\left(\frac{-2k\Omega t}{1+k^2}\right)\sin(\omega t) & \cos\left(\frac{-2k\Omega t}{1+k^2}\right)\cos(\omega t) & \beta_k \cos\left(\frac{-2k\Omega t}{1+k^2}\right)\sin(\omega t) \\ -\frac{1}{\beta_k}\sin\left(\frac{-2k\Omega t}{1+k^2}\right)\sin(\omega t) & \sin\left(\frac{-2k\Omega t}{1+k^2}\right)\cos(\omega t) & -\frac{1}{\beta_k}\cos\left(\frac{-2k\Omega t}{1+k^2}\right)\sin(\omega t) & \cos\left(\frac{-2k\Omega t}{1+k^2}\right)\cos(\omega t) \end{pmatrix}.$$

From the above equation the time evolution of (A_k, B_k) given the initial conditions $(\widehat{A}_k, \widehat{B}_k, \widehat{p}_{A_k}, \widehat{p}_{B_k})$ is

$$\begin{aligned}
A_k(t) &= \widehat{A}_k \cos\left(\frac{-2k\Omega t}{1+k^2}\right)\cos(\omega t) + \beta_k \widehat{p}_{A_k} \cos\left(\frac{-2k\Omega t}{1+k^2}\right)\sin(\omega t) \\
&\quad -\widehat{B}_k \sin\left(\frac{-2k\Omega t}{1+k^2}\right)\cos(\omega t) - \beta_k \widehat{p}_{B_k} \sin\left(\frac{-2k\Omega t}{1+k^2}\right)\sin(\omega t), \qquad (4.62) \\
B_k(t) &= \widehat{A}_k \sin\left(\frac{-2k\Omega t}{1+k^2}\right)\cos(\omega t) + \beta_k \widehat{p}_{A_k} \sin\left(\frac{-2k\Omega t}{1+k^2}\right)\sin(\omega t) \\
&\quad +\widehat{B}_k \cos\left(\frac{-2k\Omega t}{1+k^2}\right)\cos(\omega t) + \beta_k \widehat{p}_{B_k} \cos\left(\frac{-2k\Omega t}{1+k^2}\right)\sin(\omega t). \qquad (4.63)
\end{aligned}$$

Inserting the above into equation (3.103) yields the solution of α to be

$$\begin{aligned}\alpha(t) &= A_k(t)\cos(k\theta) + B_k(t)\sin(k\theta)\\ &= \left[\widehat{A}_k \cos\left(k\left[\theta + \frac{2\Omega t}{1+k^2}\right]\right) + \widehat{B}_k \sin\left(k\left[\theta + \frac{2\Omega t}{1+k^2}\right]\right)\right]\cos(\omega t)\\ +\beta_k &\left[\widehat{p}_{A_k} \cos\left(k\left[\theta + \frac{2\Omega t}{1+k^2}\right]\right) + \widehat{p}_{B_k} \sin\left(k\left[\theta + \frac{2\Omega t}{1+k^2}\right]\right)\right]\sin(\omega t). \quad (4.64)\end{aligned}$$

This can be recognized as a ring which is vibrating in mode k at frequency ω whose nodes are rotating at the rate $-\frac{2\Omega}{1+k^2}$.

4.4.4 A comparison with the results of Bryan

Recall from equation (3.155) that the solution to the linearized system undergoing rotation is

$$v = 2A\cos(\bar{\omega}_k t)\cos\left(k\left[\phi - \frac{k^2-1}{k^2+1}\Omega t\right]\right) \quad (4.65)$$

with $\bar{\omega}_k$ given by equation (3.154). Let α_B denote this solution to the angular deformation expressed in the rotating coordinate frame. We have

$$\alpha_B = A\cos(\bar{\omega}_k t)\cos\left(k\left[\phi + \frac{2\Omega t}{1+k^2}\right]\right) \quad (4.66)$$

where, by abuse of notation, we denote the amplitude of α by A. To compare this to the solution to second-order as given by equation (4.64) we first match initial conditions by choosing

$$\widehat{A}_k = A, \quad \widehat{B}_k = 0, \quad \widehat{p}_{A_k} = 0, \quad \widehat{p}_{B_k} = 0. \quad (4.67)$$

We then have the following comparison.

Lemma 4.4.2 $|\alpha - \alpha_B|$ *is* $O(\Omega^2)$ *on the timescale* $\frac{1}{\Omega}$.

Proof

$$
\begin{aligned}
|\alpha - \alpha_B| &= \left| A\cos\left(k\left[\theta + \frac{2\Omega t}{1+k^2}\right]\right)\right| |\cos(\omega_k t) - \cos(\bar{\omega}_k t)| \\
&\leq |A||\cos(\omega_k t) - \cos(\bar{\omega}_k t)| \\
&= |A|\left|\left(1 - \frac{\omega_k^2 t^2}{2} + O(t^3)\right) - \left(1 - \frac{\bar{\omega}_k^2 t^2}{2} + O(t^3)\right)\right| \\
&= |A|\left|(\bar{\omega}_k^2 - \omega_k^2)\frac{t^2}{2} + O(t^3)\right| \\
&= |A|\left|\eta_k^2 - \frac{\Omega^2 k^2(k^2-3)}{(1+k^2)^2} - (\eta_k - \widetilde{\eta}_k)^2\right| t^2 + O(t^3) \\
&= |A|\left|\eta_k^2 - \frac{\Omega^2 k^2(k^2-3)}{(1+k^2)^2} - \eta_k^2 + \frac{\Omega^2 k^2(k^2-3)}{(1+k^2)^2} - \frac{\Omega^4 k^4(k^2-3)^2}{(2[1+k^2]^2\eta_k)^2}\right| t^2 + O(t^3) \\
&= |A|\left(\frac{k^2(k^2-3)}{(1+k^2)^2\eta_k}\right)^2 \Omega^4 t^2 + O(t^3). \qquad (4.68)
\end{aligned}
$$

■

4.4.5 Comments on the corrections to the ring example

In this section we have applied the non-adiabatic corrections to the vibrating ring gyroscope and shown that by incorporating the second-order terms (in the rate of the imposed motion) we are able to recover, to second-order, the shift in the frequency of vibration due to the imposed rotation. While the linearized setting for the ring was chosen so as to enable comparison to the results of Bryan, the approach is directly applicable to systems with nonlinear dynamics.

Chapter 5

Geometric phases in dissipative systems

5.1 Introduction

In a conservative system, phase space volume is preserved under the flow of the system. By contrast, in a dissipative system the volume may decay and as a consequence stable attractors such as exponentially stable equilibria and attracting limit cycles may exist. In this chapter we focus on classical dissipative systems with symmetry and define a framework in which to understand the existence and role of geometric phases in these systems.

The essential idea behind geometric phases in classical dissipative systems is as follows. Consider a system defined by a vector field on a manifold P and assume the vector field is equivariant with respect to a given action of G on P. We then say this system admits G as a symmetry group. Under appropriate assumptions the system can be factored into dynamics on the reduced space, independent of the group variables, and dynamics on the group. In general the group dynamics depends on the variables in the reduced space. Assume further that the reduced dynamics admits an isolated exponentially stable equilibrium point which depends on a parameter. Due to the exponential stability of the equilibrium point, as the parameter is slowly varied we expect the system to remain close to equilibrium and in the adiabatic limit to remain in equilibrium at all times. As the reduced system follows the changing equilibrium, there is a corresponding motion in the group variables. If the parameter is eventually brought back to the original value the reduced system returns to the original equilibrium point. There may, however, be a net shift in the group variables and a component of this shift may depend only on the path followed by the parameter. It is this effect which may be interpreted as a geometric phase.

In physical systems with dissipation it is not uncommon to find the existence of pattern-forming solutions. If these systems exhibit spatial symmetries then any shift of the pattern by the action of the symmetry group will also be a solution. A useful example to have in mind is described by Landsberg in [49]. Consider fluid in an annular container where the relevant symmetry group is $SO(2)$ and suppose we observe some particular stationary wave pattern. Due to the symmetry of the system, this pattern exhibits marginal stability, i.e. rotating the pattern by an arbitrary amount in the direction of the symmetry produces another stable solution. Assume now that the system depends on a parameter which we can control. If the parameter is varied adiabatically then the pattern will slowly deform. After returning the parameter to its original value the initial pattern will be recovered but now, due to the marginal stability, there may be a net rotation. In this case, then, the geometric phase exhibits itself as a spatial shift in the pattern and is described by a net displacement by the symmetry group of the system.

Early work on geometric phases in classical dissipative systems includes that of Kepler and Kagan [44] in which they considered systems with stable limit cycles undergoing an adiabatic variation of a parameter around a closed path. They showed the existence of a geometric shift in the variable parametrizing the limit cycle and expressed this shift as the integral of a two-form over a surface bounded by the loop in parameter space. Together with Epstein they applied these ideas to explore geometric phase shifts in chemical oscillators [38]. In similar but independent work, Ning and Haken explored geometric phase shifts in the context of laser dynamics [66].

Landsberg expanded upon this work and developed a more general theory of geometric phases in classical dissipative systems with symmetry [48, 49]. He considered primarily systems with one-dimensional symmetry groups and developed techniques applicable to general to finite-dimensional abelian groups. In this chapter we build upon that work by allowing arbitrary finite-dimensional symmetry groups. Given a dissipative system with symmetry, we will define an appropriate principal fiber bundle and a connection we term the Landsberg connection. The holonomy of the Landsberg connection then defines the geometric phase.

We begin in the next section by establishing the framework for the problem and defining the Landsberg connection under the condition that the group dynamics are at an equilibrium whenever the reduced dynamics are. This assumption excludes systems which exhibit propagating patterns. These systems, however, are not uncommon (see, e.g., [89]) and in Section 5.3 we introduce the dynamic phase which allows us to cast a system whose group dynamics exhibit non-stationary solutions into the framework of the Landsberg connection. Although Landsberg does describe phases in systems with non-stationary wave patterns through the use of a co-moving reference frame [49], the situation is more complicated when one allows the symmetry group to be non-abelian.

In general, there does not exist a global set of coordinates on a Lie group G. If, however, the group is solvable then we can use the global representation afforded to us by the canonical coordinates of the second kind. In Section 5.4 we define the induced Landsberg connection to describe the geometric phase in this setting, allowing us to determine the phase in terms of these coordinates directly. Finally we present an example to illustrate the proposed method.

5.2 The Landsberg connection

5.2.1 Dissipative systems with symmetries

Let P be a smooth manifold and let $\mathcal{F}(P)$ denote the set of all smooth functions on P. We begin with a few definitions as in [95].

Definition 5.2.1 *Let $h \in \mathcal{F}(P)$. A vector field X on P is called a* **dissipative vector field** *with respect to h in the region $O \subset P$ if*

i) $(X(h))(z) \leq 0$*, for all* $z \in O$*,*

ii) $(X(h))(z) = 0$ *if and only if* $X = 0$ *for all* $z \in O$*.*

■

Thus X is a dissipative vector field with respect to h if h is a non-increasing function along the flow of X and if h is constant along this flow only if the vector field itself vanishes.

Let Φ be a free and proper left action of a matrix Lie group G on P. The reduced space P/G is then also a manifold and we can construct the principal bundle $\pi : P \to P/G$. We define a projectable vector field on this bundle as follows.

Definition 5.2.2 *A vector field X on P is said to be* **projectable** *if for each $h \in \mathcal{F}(P/G)$ there exists an $\widehat{h} \in \mathcal{F}(P/G)$ such that*

$$X(h \circ \pi) = \widehat{h} \circ \pi.$$

The corresponding projected vector field $\widehat{X}$ on the reduced space P/G is defined by

$$\widehat{X}(h) \circ \pi = X(h \circ \pi). \tag{5.1}$$

■

Given a projectable vector field X on P, the reduced dynamics on P/G are defined by the projected vector field $\widehat{X}$. If X is equivariant with respect to a free and proper action of a Lie group G on P then the full dynamics on P can be reconstructed from a solution to the reduced system as described in the following theorem. The proof follows standard reconstruction arguments (see, e.g. [57] or [67]).

Theorem 5.2.3 *Consider a smooth manifold P, a free and proper left action Φ of a Lie group G on P, and the corresponding principal bundle $\pi : P \to P/G$. Let X be a projectable vector field on P which is equivariant with respect to the group action and let $y(t)$ denote the integral curve of the projected vector field $\widehat{X}$ starting from $y_0 \in P/G$ at $t = 0$. Then, given an initial point $p_0 \in \pi^{-1}(y_0)$, there exists a unique curve $p(t)$ in P which projects to $y(t)$ and is an integral curve of X.*

Proof We prove this theorem with an explicit construction of the integral curve $p(t)$. Choose a smooth curve $z(t)$ on P such that $\pi(z(t)) = y(t)$ for every t with

$z(0) = p_0$. Define the curve $p(t) = \Phi_{g(t)} z(t)$. We seek an equation for $g(t)$ such that $p(t)$ is the integral curve for X passing through p_0 at $t = 0$ and thus require that

$$X(p(t)) = X(\Phi_{g(t)} z(t)) = \dot{p} = \left(T_{z(t)} \Phi_{g(t)}\right) \dot{z}(t) + \left(T_{z(t)} \Phi_{g(t)}\right) \xi(t)_P(z(t)) \tag{5.2}$$

where $\xi(t) = g^{-1}(t)\dot{g}(t)$ is a curve on $\mathfrak{g}$ and $\xi(t)_P$ is the corresponding infinitesimal generator at each t. Rearranging this equation yields

$$\begin{aligned}
\dot{z}(t) + \xi(t)_P(z(t)) &= \left(T_{z(t)} \Phi_{g(t)}\right)^{-1} X(\Phi_{g(t)} z(t)) \\
&= \left(T_{\Phi_{g(t)} z(t)} \Phi_{g^{-1}(t)}\right) X(\Phi_{g(t)} z(t)) \\
&= X(\Phi_{g^{-1}(t)}(\Phi_{g(t)} z(t))) \\
&= X(z(t))
\end{aligned} \tag{5.3}$$

where the second-to-last step follows by equivariance of X with respect to the action Φ_g. Solving this equation yields $\xi(t)$. Once ξ is known the group trajectory is determined by

$$\dot{g} = g\xi, \quad g(0) = \mathbb{1} \tag{5.4}$$

and this in turn yields the integral curve $p(t)$. The uniqueness of $p(t)$ is immediate from the uniqueness of integral curves. In addition $p(t)$ projects to $y(t)$ by construction. ∎

A curve $p(\cdot)$ on P can be expressed in a local trivialization of the fiber bundle as a pair $p(t) = (g(t), y(t))$ where $g(\cdot) \in G$ is a curve in G and $y(\cdot) \in P/G$ is a curve in the base space. Thus the curve $p(t)$ starting at $p_0 = (g_0, y_0)$ is locally defined by the system

$$\begin{aligned}
\dot{g} &= g\xi(g, y), \\
\dot{y} &= f(y)
\end{aligned}$$

with $g(0) = g_0$ and $y(0) = y_0$. Here $\xi(\cdot) \in \mathfrak{g}$ is a curve in the Lie algebra. If $p(t)$ is an integral curve of a projectable vector field X then f is the projected vector field on the base space P/G. If X is also equivariant with respect to the group action Φ then the system defining the group variable must be left invariant, i.e. the group equation has the form $\dot{g} = g\xi(y)$. From these considerations, for the remainder of this chapter we will consider systems of the form

$$\begin{aligned} \dot{g} &= g\xi(y, \lambda), \\ \dot{y} &= f(y, \lambda) \end{aligned} \tag{5.5}$$

where we have introduced the parameter $\lambda \in U \subset \mathbb{R}^m$. *We further assume there exists a family of exponentially asymptotically stable equilibria $y^*(\lambda)$, i.e. $f(y^*(\lambda), \lambda) = 0$ for all $\lambda \in U$.* Initially we also assume that $\xi(y^*(\lambda), \lambda) = 0$ for all λ. This condition will be removed when we introduce the dynamic phase in Section 5.3.

5.2.2 An asymptotic analysis

We wish to understand the behavior of system (5.5) as the parameter λ is varied adiabatically. To do so, introduce a time dependence into the parameter by taking $\lambda = \lambda(\tau)$ where $\tau = \epsilon t$, $\epsilon > 0$. We now carry out an asymptotic analysis of the system (see, e.g., [78] or [94] for background material on asymptotic analysis). Begin by assuming y can be expressed as

$$y(t) = y_0(t, \tau) + \epsilon y_1(t, \tau) + \cdots \tag{5.6}$$

with initial condition $y(0) = y^*$. Here we view t and τ as independent variables. From equation (5.6) we have

$$\dot{y} = \frac{\partial y}{\partial t} + \epsilon \frac{\partial y}{\partial \tau} = \frac{\partial y_0}{\partial t} + \epsilon \left[\frac{\partial y_0}{\partial \tau} + \frac{\partial y_1}{\partial t} \right] + O(\epsilon^2). \tag{5.7}$$

Since $\dot{y} = f(y, \lambda)$, we have

$$f(y_0 + \epsilon y_1 + \cdots, \lambda) = \frac{\partial y_0}{\partial t} + \epsilon \left[\frac{\partial y_0}{\partial \tau} + \frac{\partial y_1}{\partial t} \right] + O(\epsilon^2). \tag{5.8}$$

Setting $\epsilon = 0$ yields

$$f(y_0, \lambda) = \frac{\partial y_0}{\partial t}, \quad y_0(0) = y^* \tag{5.9}$$

and therefore $y_0 \equiv y^*$. Now expand f in a Taylor series about the solution $y = y^*$. This gives

$$\begin{aligned} f(y, \lambda) &= f(y^*, \lambda) + (T_{y^*} f)\,(y - y^*) + \cdots \\ &= \epsilon\,(T_{y^*} f)\,y_1 + O(\epsilon^2) \end{aligned} \tag{5.10}$$

where the last step follows from the fact that $f(y^*, \lambda) = 0$ and $y_0 = y^*$. Here $(T_{y^*} f)$ denotes the linearization of f at y^*. Combining equations (5.8) and (5.10) we find

$$\epsilon\,(T_{y^*} f)\,y_1 + O(\epsilon^2) = \epsilon \left[\frac{\partial y^*}{\partial \tau} + \frac{\partial y_1}{\partial t} \right] + O(\epsilon^2). \tag{5.11}$$

At first order in ϵ, then, we have

$$\frac{\partial y^*}{\partial \tau} + \frac{\partial y_1}{\partial t} = (T_{y^*} f)\,y_1 \tag{5.12}$$

and therefore

$$\frac{\partial y_1}{\partial t} = (T_{y^*} f)\,y_1 - \frac{\partial y^*}{\partial \tau}. \tag{5.13}$$

For fixed τ this is a linear ordinary differential equation with constant coefficients and it can thus be solved by the variation of constants formula. Freezing τ, then, we have

$$\begin{aligned} y_1(t) &= e^{(T_{y^*} f)t} y_1(0) - \int_0^t e^{(T_{y^*} f)(t-\sigma)} \frac{\partial y^*}{\partial \tau} d\sigma \\ &= e^{(T_{y^*} f)t} y_1(0) + (T_{y^*} f)^{-1} \left[\frac{\partial y^*}{\partial \tau} - e^{(T_{y^*} f)t} \frac{\partial y^*}{\partial \tau} \right] \\ &= (T_{y^*} f)^{-1} \frac{\partial y^*}{\partial \tau} - (T_{y^*} f)^{-1} e^{(T_{y^*} f)t} \frac{\partial y^*}{\partial \tau} \end{aligned} \tag{5.14}$$

where we have used the fact that since $y(0) = y^* = y_0$ we have $y_i(0) = 0$ for all $i \neq 0$. Since the equilibrium y^* is assumed exponentially stable we know that $(T_{y^*} f)$ is Hurwitz and thus $(T_{y^*} f)^{-1}$ exists. From the Hurwitz property the second term in equation (5.14) decays to zero exponentially. The rate of this decay determines the dissipative time scale of the system. For times long with respect to the dissipative time scale we can neglect the second term in equation (5.14) and thus

$$y(t) \approx y^* + \epsilon \left(T_{y^*} f\right)^{-1} \frac{\partial y^*}{\partial \tau} = y^* + \left(T_{y^*} f\right)^{-1} \frac{\partial y^*}{\partial t}. \tag{5.15}$$

Recalling that y^* depends on time only through its dependence on λ we write

$$y(t) \approx y^* + \left(T_{y^*} f\right)^{-1} \nabla_\lambda y^* \frac{d\lambda}{dt}. \tag{5.16}$$

We now expand the map $\xi(\cdot)$ in a Taylor series around y^* and truncate to first order. This yields

$$\begin{aligned} \xi(y) &\approx \xi(y^*, \lambda) + \left(T_{y^*} \xi\right)(y - y^*) \\ &= \left(T_{y^*} \xi\right) \left(T_{y^*} f\right)^{-1} \nabla_\lambda y^* \frac{d\lambda}{dt} \end{aligned} \tag{5.17}$$

where we have used the assumption that $\xi(y^*, \lambda) = 0$. Define the map $\mathcal{A}_{loc}$ by

$$\begin{aligned} \mathcal{A}_{loc} : T\mathbb{R}^m &\to \mathfrak{g} \\ (\lambda, v) &\mapsto \mathcal{A}_{loc}(\lambda)(v) = \left(\left(T_{y^*} \xi\right) \left(T_{y^*} f\right)^{-1} \nabla_\lambda y^*\right) v. \end{aligned} \tag{5.18}$$

Notice that this map is linear in the tangent vector v.

5.2.3 The Landsberg connection

Consider the principal bundle $G \times U \to U$. Let $\widetilde{\Phi}$ be the action of G on $G \times U$ defined by

$$\widetilde{\Phi} : G \times (G \times U) \to G \times U$$

$$(h,(g,\lambda) \quad \mapsto \quad (hg,\lambda). \tag{5.19}$$

The infinitesimal generator corresponding to an element $\eta \in \mathfrak{g}$ is given by

$$\eta_{G\times U} = \left.\frac{d}{ds}\right|_{s=0} (\exp(s\eta)g,\lambda) = (\eta g, 0). \tag{5.20}$$

(We note that the infinitesimal generator of a left action is *right* invariant. For further comments see, e.g. Example 4.1.25 of [1].)

The map $\mathcal{A}_{loc}$ defines a principal connection on the principal bundle $\pi : G\times U \to U$ as follows.

Definition 5.2.4 *The* **Landsberg connection** *on $\pi : G\times U \to U$ is the $\mathfrak{g}$-valued one-form given by*

$$\mathcal{A}_L(g,\lambda)(\dot{g},\dot{\lambda}) = \mathrm{Ad}_g\left(g^{-1}\dot{g} - \mathcal{A}_{loc}(\lambda)\dot{\lambda}\right). \tag{5.21}$$

■

We have the following proposition.

Proposition 5.2.5 *$\mathcal{A}_L$ is principal connection on $G \times U \to U$.*

Proof We need to show that for $\eta \in \mathfrak{g}$, $\mathcal{A}_L(g,\lambda)(\eta_{G\times U}) = \eta$ and that $\mathcal{A}_L$ is Ad$-$equivariant. We have

$$\begin{aligned} \mathcal{A}_L(g,\lambda)(\eta_{G\times U}) &= \mathcal{A}_L(g,\lambda)(\eta g, 0) \\ &= \mathrm{Ad}_g(g^{-1}\eta g) = \eta. \end{aligned}$$

For Ad-equivariance we have

$$\begin{aligned} \mathcal{A}_L(hg,\lambda)(h\dot{g},\dot{\lambda}) &= \mathrm{Ad}_{hg}\left((hg)^{-1}\, h\dot{g} - \mathcal{A}_{loc}(\lambda)\dot{\lambda}\right) \\ &= \mathrm{Ad}_{hg}\left(g^{-1}\dot{g} - \mathcal{A}_{loc}(\lambda)\dot{\lambda}\right) \\ &= \mathrm{Ad}_h\left(\mathrm{Ad}_g\left(g^{-1}\dot{g} - \mathcal{A}_{loc}(\lambda)\dot{\lambda}\right)\right) \\ &= \mathrm{Ad}_h\left(\mathcal{A}_L(g,\lambda)(\dot{g},\dot{\lambda})\right). \end{aligned}$$

■

The geometric phase equation resulting from the Landsberg connection is

$$\dot{g} = g\mathcal{A}_{loc}(\lambda)\dot{\lambda}. \tag{5.22}$$

For a dissipative system with symmetry, system (5.5), the geometric phase corresponding to an adiabatic variation of the parameter λ around a given closed loop parametrized by $s \in [0, 1]$ is the solution to equation (5.22) at the time $s = 1$.

5.3 The dynamic phase

Consider once again the system in (5.5). We would like to remove the restriction that $\xi(y^*, \lambda) = 0$ by expressing g as the product of a geometric component, g_{gp}, and a dynamic component, g_{dp}. Since G is not necessarily an abelian group, there are two ways to combine g_{gp} and g_{dp}, namely

$$g = g_{gp}g_{dp}, \tag{5.23}$$

$$g = g_{dp}g_{gp}. \tag{5.24}$$

Here g_{gp}, the geometric phase, is intended to capture the component of the group evolution which is geometric in nature and it should therefore be amenable to the treatment in Section 5.2. In particular the defining system for g_{gp} should be at equilibrium when the reduced dynamics are. On the other hand, g_{dp}, the dynamic phase, should capture the dynamics of g when y is at equilibrium. Naively we would take

$$\dot{g}_{dp} = g_{dp}\xi(y^*, \lambda), \tag{5.25}$$

$$\dot{g}_{gp} = g_{gp}(\xi(y, \lambda) - \xi(y^*, \lambda)) \stackrel{\triangle}{=} g_{gp}\xi_{gp}(y, \lambda) \tag{5.26}$$

with the initial conditions $g_{dp}(0) = \mathbb{1}$ and $g_{gp}(0) = \mathbb{1}$. The dynamic phase g_{dp} then exactly captures the dynamics when y is at equilibrium and g_{gp} satisfies the conditions in the previous section. These definitions, however, do not necessarily combine to give the full dynamics in equation (5.5).

We thus have four options; we can choose to combine the components either as in equation (5.23) or as in (5.24) and in each case we can either define g_{gp} through (5.26) and derive the resulting equation for g_{dp} or define g_{dp} through equation (5.25) and derive the resulting equation for g_{gp}. We consider each in turn.

Case 1: Equation (5.23) and equation (5.26)

Here we fix the form of g_{gp} as in equation (5.26) and the form of g as in equation (5.23). Taking the time derivative of equation (5.23) we have

$$\dot{g} = \dot{g}_{gp} g_{dp} + g_{gp} \dot{g}_{dp}. \tag{5.27}$$

Solving for the dynamic phase equation from equation (5.27) using equations (5.26) and (5.5) yields

$$\begin{aligned}
\dot{g}_{dp} &= g_{gp}^{-1} \left(\dot{g} - \dot{g}_{gp} g_{dp} \right) \\
&= g_{gp}^{-1} \left(g\xi(y,\lambda) - g_{gp} \left[\xi(y,\lambda) - \xi(y^*,\lambda) \right] g_{dp} \right) \\
&= g_{gp}^{-1} \left(g_{gp} g_{dp} \xi(y,\lambda) - g_{gp} \xi(y,\lambda) g_{dp} + g_{gp} \xi(y^*,\lambda) g_{dp} \right) \\
&= g_{dp} \xi(y,\lambda) - \xi(y,\lambda) g_{dp} + \xi(y^*,\lambda) g_{dp} \\
&= g_{dp} \xi(y^*,\lambda) + g_{dp} \left(\xi(y,\lambda) - \xi(y^*,\lambda) \right) + \left(\xi(y^*,\lambda) - \xi(y,\lambda) \right) g_{dp} \\
&= g_{dp} \xi(y^*,\lambda) + g_{dp} \xi_{gp}(y,\lambda) - \xi_{gp}(y,\lambda) g_{dp}.
\end{aligned} \tag{5.28}$$

This equation is similar to our naive expression in equation (5.25) but contains additional y-dependent terms. This dependence on the dynamics of y will in general greatly complicate finding a solution to equation (5.28). Since as ϵ approaches zero, ξ_{gp} also approaches zero, one might expect to get a good approximation by discarding the y-dependent terms in equation (5.28). This is not true, however, since the small size of $\xi_{gp}(y)$ is compensated for by the increase in time over which the parameter variation takes place and thus the final two terms cannot be neglected. ■

Case 2: Equation (5.23) and equation (5.25)

We once again choose the form of g as in (5.23) but now fix g_{dp} as in equation (5.25). Solving for g_{gp} from equation (5.27) yields

$$\begin{aligned}
\dot{g}_{gp} &= (\dot{g} - g_{gp}\dot{g}_{dp})\, g_{dp}^{-1} \\
&= (g\xi(y,\lambda) - g_{gp}g_{dp}\xi(y^*,\lambda))\, g_{dp}^{-1} \\
&= (g_{gp}g_{dp}\xi(y,\lambda) - g_{gp}g_{dp}\xi(y^*,\lambda))\, g_{dp}^{-1} \\
&= g_{gp}\mathrm{Ad}_{g_{dp}}(\xi_{gp}(y,\lambda)) \stackrel{\triangle}{=} g_{gp}\widehat{\xi}_{gp}(y,\lambda).
\end{aligned} \tag{5.29}$$

Since $\xi_{gp}(y^*) = 0$, this equation for g_{gp} meets the condition that it be at equilibrium when the reduced system is and the technique of Section 5.2 can be applied using $\widehat{\xi}_{gp}$. To do so, one must first solve equation (5.25) for g_{dp} as an explicit function of time to determine $\widehat{\xi}_{gp}$. The resulting expression for $\mathcal{A}_{loc}$ (see equation (5.18)) will then depend on time through g_{dp}. We therefore no longer expect the resulting adiabatic approximation for g_{gp} to be geometric and we discard this case.

■

Case 3: Equation (5.24) and equation (5.26)

In this case we once again fix the equation for g_{gp} as in equation (5.26) but combine g_{dp} and g_{gp} as in equation (5.24). From equation (5.24) we have

$$\dot{g} = \dot{g}_{dp} g_{gp} + g_{dp} \dot{g}_{gp}. \tag{5.30}$$

Solving for the dynamic phase equation yields

$$\begin{aligned} \dot{g}_{dp} &= (\dot{g} - g_{dp}\dot{g}_{gp})\, g_{gp}^{-1} \\ &= (g\xi(y,\lambda) - g_{dp}g_{gp}\,(\xi(y,\lambda) - \xi(y^*,\lambda)))\, g_{gp}^{-1} \\ &= (g_{dp}g_{gp}\xi(y,\lambda) - g_{dp}g_{gp}\xi(y,\lambda) + g_{dp}g_{gp}\xi(y^*,\lambda))\, g_{gp}^{-1} \\ &= g_{dp}\mathrm{Ad}_{g_{gp}}\,(\xi(y^*,\lambda))\,. \end{aligned} \tag{5.31}$$

To solve equation (5.31) we must first solve for g_{gp} as a function of time. In the adiabatic approximation the equations for the geometric phase are defined by the Landsberg connection where now in the definition of $\mathcal{A}_{loc}$ in equation (5.18) we take ξ to be ξ_{gp}. The resulting solution for g_{gp} can be used in equation (5.31). By using this approximation for g_{gp} we then obtain an approximation to g_{dp}. The errors so introduced go to zero in the adiabatic limit. ■

Case 4: Equation (5.24) and equation (5.25)

Finally, we again use the ordering $g = g_{dp}g_{gp}$ but fix g_{dp} as in equation (5.25). Using equation (5.30) to solve for the geometric phase equation, we obtain

$$\begin{aligned} \dot{g}_{gp} &= g_{dp}^{-1}\,(\dot{g} - \dot{g}_{dp}g_{gp}) \\ &= g_{dp}^{-1}\,(g_{dp}g_{gp}\xi(y,\lambda) - g_{dp}\xi(y^*,\lambda)g_{gp}) \\ &= g_{gp}\xi(y,\lambda) - \xi(y^*,\lambda)g_{gp}. \end{aligned} \tag{5.32}$$

In the nonabelian setting, equation (5.32) does not satisfy the condition that $\dot{g}_{gp} = 0$ when $y = y^*$ and therefore does not allow the perturbation theoretic approach of Section 5.2. We therefore discard this case. ∎

Of these options, case three is the natural choice. We thus replace the system defined by equation (5.5) by

$$\begin{aligned} \dot{g}_{gp} &= g_{gp}\xi_{gp}(y, \lambda), \\ \dot{y} &= f(y, \lambda), \\ \dot{g}_{dp} &= g_{dp}\mathrm{Ad}_{g_{gp}}\left(\xi(y^*, \lambda)\right). \end{aligned} \tag{5.33}$$

The first two equations can be treated as in Section 5.2 to yield the geometric phase. The third equation can then be solved and the group trajectory reconstructed from $g = g_{dp}g_{gp}$.

Remark 5.3.1 *The geometric phase in a dissipative system with symmetry dependent on a parameter undergoing an adiabatic evolution is by definition the holonomy of the Landsberg connection with respect to the closed loop in parameter space. From equation* (5.33) *we see that in general to solve for the dynamic phase we must know the entire time evolution of the geometric phase as the parameter is varied, not simply the value at the completion of the loop. One should keep in mind, however, that in general the character of the equilibrium point in the full system is changing as the parameter is varied. If, for example, the equilibrium corresponds to some pattern in the full system then for each parameter value this pattern may be different. It does not in general make sense, then, to consider the value of* g_{gp} *at intermediate values along the loop as a phase shift. In special cases, such as if the group describes the state of a physical property such as position or orientation of the system, the comparison at different equilibrium points may be valid.*

5.3.1 The abelian case

The following theorem states that if the symmetry group is abelian, the four cases above are all equivalent to the naive choice in equations (5.25) and (5.26).

Theorem 5.3.2 *If G is an abelian group then the dynamical systems in cases one through four are equivalent to the system in equations* (5.25) *and* (5.26)*.*

Proof We need to show that all four cases reduce to the equations

$$\begin{aligned}\dot{g}_{gp} &= g_{gp}\xi_{gp}(y,\lambda),\\ \dot{g}_{dp} &= g_{dp}\xi(y^*,\lambda).\end{aligned}$$

In cases (1) and (3) we have $\dot{g}_{gp} = g_{gp}\xi_{gp}(y,\lambda)$. Furthermore, in case (1), from equation (5.28) we have

$$\begin{aligned}\dot{g}_{dp} &= g_{dp}\xi(y^*,\lambda) + g_{dp}\xi_{gp} - \xi_{gp}g_{dp}\\ &= g_{dp}\xi(y^*,\lambda) + g_{dp}\xi_{gp} - g_{dp}\xi_{gp}\\ &= g_{dp}\xi(y^*,\lambda).\end{aligned} \tag{5.34}$$

In case (3), from equation (5.31) we have

$$\begin{aligned}\dot{g}_{dp} &= g_{dp}\mathrm{Ad}_{g_{gp}}\left(\xi(y^*,\lambda)\right)\\ &= g_{dp}g_{gp}\xi(y^*,\lambda)g_{gp}^{-1}\\ &= g_{dp}\xi(y^*,\lambda).\end{aligned} \tag{5.35}$$

In cases (2) and (4) we have $\dot{g}_{dp} = g_{dp}\xi(y^*,\lambda)$. In case (2), from equation (5.29), we have

$$\begin{aligned}\dot{g}_{gp} &= g_{gp}\mathrm{Ad}_{g_{dp}}\left(\xi_{gp}(y,\lambda)\right)\\ &= g_{gp}g_{dp}\xi_{gp}(y,\lambda)g_{dp}^{-1}\\ &= g_{gp}\xi_{gp}(y,\lambda).\end{aligned} \tag{5.36}$$

Finally, from equation (5.32), we have

$$\begin{aligned} \dot{g}_{gp} &= g_{gp}\xi(y,\lambda) - \xi(y^*,\lambda)g_{gp} \\ &= g_{gp}\xi_{gp}(y,\lambda). \end{aligned} \quad (5.37)$$

■

Thus the situation is greatly simplified when the symmetry group is abelian. In particular the dynamic and geometric phases may be solved for independently.

Remark 5.3.3 *We note that if the symmetry group is abelian, the geometric phase can be calculated using either the line integral of the connection around the closed loop or the area integral of the curvature form as in equation* (2.14).

5.4 The induced Landsberg connection

In general the Lie group G will not have a global set of coordinates. If, however, the associated Lie algebra is solvable then a theorem of Wei and Norman [97] states that there is a global representation given by a product of exponentials involving a set of parameters known as the Wei-Norman parameters determined by a set of ordinary differential equations. These equations are solvable by quadrature.

It is usually simpler to deal with the equations in Cartesian space defining the Wei-Norman parameters than it is to handle the group equations (as in equation (5.33)) on the manifold G, both analytically and numerically. Because of this, in practice one usually performs calculations in terms of these parameters directly. In this section we therefore develop the geometric and dynamic phases in dissipative systems with symmetry in terms of the Wei-Norman parameters.

5.4.1 Canonical coordinates of the second kind

Consider a left-invariant dynamical system on an $n-$dimensional matrix Lie group G defined by

$$\dot{g} = (T_e L_g)\xi = g\xi \tag{5.38}$$

where $\xi(\cdot)$ is a curve in $\mathfrak{g}$. Let $\mathfrak{g}^*$ be the dual space to $\mathfrak{g}$, i.e. the space of linear functionals from $\mathfrak{g}$ into $\mathbb{R}$. Let $\{\mathcal{A}_i, i = 1, \ldots, n\}$ be a basis for $\mathfrak{g}$ and let $\{\mathcal{A}_i^\flat, i = 1, \ldots, n\}$ be the basis on $\mathfrak{g}^*$ dual to the $\mathcal{A}_i$ basis, that is

$$\mathcal{A}_i^\flat(\mathcal{A}_j) = \delta_j^i, \qquad i, j = 1, \ldots, n \tag{5.39}$$

where δ_j^i is the Kronecker delta symbol. Then the curve $\xi(\cdot) \subset \mathfrak{g}$ can be represented as

$$\xi(t) = \sum_{i=1}^{n} \xi_i(t)\mathcal{A}_i = \sum_{i=1}^{n} \mathcal{A}_i^\flat(\xi(t))\mathcal{A}_i. \tag{5.40}$$

We have the following theorem.

Theorem 5.4.1 [97] *Let $g(t)$ be the solution to the left-invariant dynamical system given by equation (5.38) with $g(0) = \mathbb{1}$. Then there exists $t_0 > 0$ such that for $|t| < t_0$, $g(t)$ can be expressed in the form*

$$g(t) = e^{\gamma_1(t)\mathcal{A}_1} e^{\gamma_2(t)\mathcal{A}_2} \cdots e^{\gamma_n(t)\mathcal{A}_n} \tag{5.41}$$

where $e^{\gamma_i \mathcal{A}_i} = \exp(\gamma_i \mathcal{A}_i)$ is the exponential map. The Wei-Norman parameters $\gamma = (\gamma_1, \cdots \gamma_n)$ satisfy

$$\begin{pmatrix} \dot{\gamma}_1 \\ \vdots \\ \dot{\gamma}_n \end{pmatrix} = M(\gamma) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \qquad \text{for } |t| < t_0 \tag{5.42}$$

where $\gamma(0) = 0$ and $M(\gamma)$ is a real-analytic, matrix valued function of γ. If $\mathfrak{g}$ is solvable then there exists a basis of $\mathfrak{g}$ and an ordering of that basis for which (5.41) is global. In that case equation (5.42) can be integrated by quadrature. ■

Thus when the symmetry group of the dissipative system is solvable, we can replace the system on G with the system defined by equation (5.42). We can also do this for a general symmetry group if we can guarantee we remain for all time in a neighborhood of the identity where the representation holds. The geometric phase is, however, generically a global phenomenon and the group dynamics are not inherently restricted to a neighborhood of the identity. In what follows, then, we consider only systems whose symmetries are solvable groups, though in principle the techniques are applicable in general under appropriate restrictions.

5.4.2 The induced Landsberg connection

Consider once again the system in equation (5.5) and assume that $\xi(y^*, \lambda) = 0$. Assume further that the Lie algebra $\mathfrak{g}$ is solvable. Let $\{\mathcal{A}_i\}_{i=1}^n$ be a basis for $\mathfrak{g}$ such that the Wei-Norman coordinates are a global representation. From Theorem 5.4.1 we can replace system (5.5) with

$$\begin{aligned} \dot{\gamma} &= M(\gamma)\xi(y, \lambda), \\ \dot{y} &= f(y, \lambda). \end{aligned} \tag{5.43}$$

To describe the behavior of this system as the parameter is adiabatically varied we proceed as follows. Construct the product bundle $\widetilde{\pi} : \mathbb{R}^n \times U \to U$ and make the following definition.

Definition 5.4.2 *The* **induced Landsberg connection** A_L *is the vertical valued one-form on the product bundle* $\widetilde{\pi} : \mathbb{R}^n \times U \to U$ *defined by*

$$A_L(\gamma, \lambda)(v_\gamma, v_\lambda) = \left(v_\gamma - M(\gamma)(T_{y^*}\xi)(T_{y^*}f)^{-1}(\nabla_\lambda y^*)v_\lambda, 0\right). \tag{5.44}$$

■

Here $T_{y^*}\xi$ is the linearization of the map ξ and is given by

$$T_{y^*}\xi = \left.\begin{pmatrix} \frac{\partial \xi_1}{\partial y} \\ \vdots \\ \frac{\partial \xi_n}{\partial y} \end{pmatrix}\right|_{y^*} \tag{5.45}$$

where $\xi_i = \mathcal{A}_i^\flat(\xi)$. We have the following proposition.

Proposition 5.4.3 *A_L is an Ehresmann connection on $\widetilde{\pi} : \mathbb{R}^n \times U \to U$.*

Proof We need only show that A_L is a vertical projection. Since the bundle is globally a product bundle, the vertical vectors at the point (γ, λ) have the form $(v_\gamma, 0)$ for $v_\gamma \in T_\gamma\mathbb{R}^n$. Then

$$A_L(\gamma, \lambda)(v_\gamma, 0) = (v_\gamma, 0). \tag{5.46}$$

■

The horizontal subspace at (γ, λ) of this connection is

$$H_{(\gamma,\lambda)} = ker A_L(\gamma, \lambda) = \left\{(M(\gamma)(T_{y^*}\xi)(T_{y^*}f)^{-1}(\nabla_\lambda y^*)v_\lambda, v_\lambda) | v_\lambda \in T_\lambda\mathbb{R}^m\right\}. \tag{5.47}$$

The horizontal lift of a vector field X on the base space to the total space is

$$hor_{A_L}(X)(\gamma, \lambda) = \left(M(\gamma)(T_{y^*}\xi)(T_{y^*}f)^{-1}(\nabla_\lambda y^*)X(\lambda), X(\lambda)\right). \tag{5.48}$$

5.4.3 The dynamic phase

To remove the assumption that $\xi(y^*, \lambda) = 0$ we use the results of Section 5.3 but now express the dynamic phase equations using the product of exponentials solution and define the dynamic phase in terms of the corresponding Wei-Norman parameters directly.

The differential equation defining the Wei-Norman parameters for the dynamic phase is, from Theorem 5.4.1,

$$\dot{\gamma}_{dp} = M(\gamma_{dp})\xi_{dp} \tag{5.49}$$

where, from equation (5.33),

$$\xi_{dp} = \mathrm{Ad}_{g_{gp}}(\xi(y^*, \lambda)). \tag{5.50}$$

We would like to express ξ_{dp} in terms of the basis $\{\mathcal{A}_i\}_{i=1}^n$. We have

$$\begin{aligned} \xi_{dp}(y^*) &= \mathrm{Ad}_{g_{gp}}(\xi(y^*, \lambda)) \\ &= \mathrm{Ad}_{g_{gp}}(\sum_{i=1}^{n} \xi_i(y^*)\mathcal{A}_i) \\ &= \sum_{i=1}^{n} \xi_i(y^*)\mathrm{Ad}_{g_{gp}}(\mathcal{A}_i). \end{aligned} \tag{5.51}$$

For a given element $g \in G$ define the constants α_g^{ij} by

$$\mathrm{Ad}_g(\mathcal{A}_i) = \sum_{j=1}^{n} \alpha_g^{ij}\mathcal{A}_j. \tag{5.52}$$

Then

$$\begin{aligned} \xi_{dp}(y^*) &= \sum_{i=1}^{n} \xi_i(y^*)\mathrm{Ad}_{g_{gp}}(\mathcal{A}_i) \\ &= \sum_{i=1}^{n} \xi_i(y^*) \sum_{j=1}^{n} \alpha_{g_{gp}}^{ij}\mathcal{A}_j \\ &= \sum_{j=1}^{n} \left[\sum_{i=1}^{n} \alpha_{g_{gp}}^{ij}\xi_i(y^*) \right] \mathcal{A}_j \end{aligned} \tag{5.53}$$

and so

$$(\xi_{dp})_i(y^*) = \sum_{j=1}^{n} \alpha_{g_{gp}}^{ji}\xi_j(y^*). \tag{5.54}$$

The dynamic phase is then given by inserting equation (5.54) into equation (5.49), solving for the Wei-Norman parameters for the dynamic phase, and using the result in equation (5.41).

5.5 Example

In this section we illustrate the techniques developed in this chapter with an example containing both a geometric and a dynamic phase. We will take $G = SE(2)$, the group of rigid rotations and translations in the plane, as our symmetry group and couple it to a damped, driven harmonic oscillator. We choose the following basis for the Lie algebra $se(2)$.

$$\mathcal{A}_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathcal{A}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.55}$$

Consider the equation for a damped harmonic oscillator with forcing where the natural frequency and the driving force are parameter-dependent.

$$\ddot{x} + k\dot{x} + \omega^2(\lambda_1)x = f(\lambda_2) \tag{5.56}$$

Let $y = (x, \dot{x})$ and write the system in state space form.

$$\begin{aligned} \dot{y} &= \begin{pmatrix} 0 & \omega(\lambda_1) \\ -\omega(\lambda_1) & -k \end{pmatrix} y + \begin{pmatrix} 0 \\ f(\lambda_2) \end{pmatrix} \\ &= Ay + b \end{aligned} \tag{5.57}$$

which defines the Hurwitz matrix $A(\lambda)$ and the vector $b(\lambda)$. The example system we consider is given by

$$\begin{aligned} \dot{g} &= g\left(\mathcal{A}_1 y_1 + \mathcal{A}_2 y_2\right), \\ \dot{y} &= A(\lambda)y + b(\lambda) \end{aligned} \tag{5.58}$$

with $g \in SE(2)$ and $\lambda \in \mathbb{R}^2$. The parameter-dependent equilibrium point for the harmonic oscillator is

$$y^*(\lambda) = -A^{-1}(\lambda)b(\lambda)$$

$$= -\begin{pmatrix} -\frac{k}{\omega^2(\lambda_1)} & -\frac{1}{\omega(\lambda_1)} \\ \frac{1}{\omega(\lambda_1)} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ f(\lambda_2) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{f(\lambda_2)}{\omega(\lambda_1)} \\ 0 \end{pmatrix}. \tag{5.59}$$

At this equilibrium point we have

$$\xi(y^*, \lambda) = \mathcal{A}_1 \frac{f(\lambda_2)}{\omega(\lambda_1)} \tag{5.60}$$

and so the group dynamics are not stationary when the dynamics on the reduced space are at equilibrium. Following the technique outlined in Section 5.3, we replace the system in equation (5.58) by the system

$$\begin{aligned} \dot{g}_{gp} &= g_{gp}\left(\mathcal{A}_1(y_1 - y_1^*) + \mathcal{A}_2 y_2\right), \\ \dot{y} &= A(\lambda)y + b(\lambda) \end{aligned} \tag{5.61}$$

together with

$$\dot{g}_{dp} = g_{dp} \mathrm{Ad}_{g_{gp}} \left(\mathcal{A}_1 \frac{f(\lambda_2)}{\omega(\lambda_1)} \right). \tag{5.62}$$

The natural frequency and the driving force are taken to have the following forms.

$$\begin{aligned} \omega(\lambda_1) &= \bar{\omega} + \lambda_1, \\ f(\lambda_2) &= \tfrac{\lambda_2^2}{2}. \end{aligned} \tag{5.63}$$

From this we have

$$\nabla_\lambda y^* = \begin{pmatrix} \frac{-\lambda_2^2}{2(\bar{\omega}+\lambda_1)^2} & \frac{\lambda_2}{(\bar{\omega}+\lambda_1)} \\ 0 & 0 \end{pmatrix} \tag{5.64}$$

5.5.1 Dissipation rate in the linear system

To determine the rate of convergence to the equilibrium in the linear system we first shift the coordinates by defining $z = y - y^*$. We then have

$$\dot{z} = \dot{y}$$

$$
\begin{aligned}
&= A(y + y^* - y^*) + b \\
&= Az + Ay^* + b \\
&= Az. \qquad (5.65)
\end{aligned}
$$

The eigenvalues of the Hurwitz matrix A are given by

$$\mu_{l,s} = \frac{-k \pm \sqrt{k^2 - 4\omega^2}}{2}. \qquad (5.66)$$

Letting $\|\cdot\|$ denote the standard Euclidean norm on $\mathbb{R}^2$ we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|z\|^2 &= \frac{1}{2}\left(\dot{z}^T z + z^T \dot{z}\right) \\
&= z^T A z \\
&\leq Re(\mu_l)\|z\|^2 \stackrel{\triangle}{=} -\zeta\|z\|^2 \qquad (5.67)
\end{aligned}
$$

where $\zeta = |Re(\mu_l)|$ is the absolute value of the largest (least negative) real part of the eigenvalues of A. Therefore

$$\|z\|^2 \leq e^{-\zeta t}\|z(0)\|^2. \qquad (5.68)$$

From this we see that the rate of dissipation is given by the largest real part of the eigenvalues of the matrix A. From equation (5.66), the system will be critically damped if ω and k are chosen so as to satisfy $k = 2\omega$. We thus take $\bar{\omega} = \frac{k}{2}$. If the variations in λ_1 are small in amplitude with respect to $\bar{\omega}$ the system will remain nearly critically damped at all times.

5.5.2 The geometric phase equations

Since $SE(2)$ is solvable we know from Theorem 5.4.1 that the ordinary differential equations defining the Wei-Norman parameters are solvable by quadrature. The following proposition, paraphrased from [92], explicitly gives this solution.

Proposition 5.5.1 [92] *Let $G = SE(2)$ and let $g(\cdot)$ be a solution to the left-invariant system on G given by equation* (5.38) *starting from the identity. Then there exists a global representation of the curve $g(\cdot)$ of the form* (5.41) *with $M(\gamma)$ given by*

$$M(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ \gamma_3 & 1 & 0 \\ -\gamma_2 & 0 & 1 \end{pmatrix} \tag{5.69}$$

and the Wei-Norman parameters $(\gamma_1, \gamma_2, \gamma_3)$ are solvable by quadratures. With $g(0) = \mathbb{I}$ we have $\gamma_i(0) = 0$, $i = 1, 2, 3$ and the solution is given by

$$\gamma_1(t) = \int_0^t \xi_1(\tau) d\tau, \tag{5.70}$$

$$\begin{aligned} \gamma_2(t) &= \int_0^t \xi_2(t) \cos\left(\int_\tau^t \xi_1(\sigma) d\sigma\right) d\tau \\ &\quad + \int_0^t \xi_3(\tau) \sin\left(\int_\tau^t \xi_1(\sigma) d\sigma\right) d\tau, \end{aligned} \tag{5.71}$$

$$\begin{aligned} \gamma_3(t) &= -\int_0^t \xi_2(\tau) \sin\left(\int_\tau^t \xi_1(\sigma) d\sigma\right) d\tau \\ &\quad + \int_0^t \xi_3(\tau) \cos\left(\int_\tau^t \xi_1(\sigma) d\sigma\right) d\tau. \end{aligned} \tag{5.72}$$

■

Consider a curve $\lambda(\cdot) \in \mathbb{R}^2$ parametrized by t. From equations (5.48), (5.61), (5.64), and (5.69), the geometric phase equation for the Wei-Norman parameters is

$$\dot{\gamma} = M(\gamma) \begin{pmatrix} \frac{k\lambda_2^2}{2(\bar{\omega}+\lambda_1)^4} \dot{\lambda}_1 - \frac{k\lambda_2}{(\bar{\omega}+\lambda_1)^3} \dot{\lambda}_2 \\ \frac{-\lambda_2^2}{2(\bar{\omega}+\lambda_1)^3} \dot{\lambda}_1 + \frac{\lambda_2}{(\bar{\omega}+\lambda_1)^2} \dot{\lambda}_2 \\ 0 \end{pmatrix}. \tag{5.73}$$

From equations (5.70)-(5.72) the solution is given by

$$\gamma_1(t) = \int_0^t \left[\frac{k\lambda_2^2(\tau)}{2(\bar{\omega} + \lambda_1(\tau))^4} \dot{\lambda}_1(\tau) - \frac{k\lambda_2(\tau)}{(\bar{\omega} + \lambda_1(\tau))^3} \dot{\lambda}_2(\tau) \right] d\tau, \tag{5.74}$$

$$\gamma_2(t) = \int_0^t \left[\frac{-\lambda_2^2(\tau)}{2(\bar{\omega}+\lambda_1(\tau))^3}\dot{\lambda}_1(\tau) + \frac{\lambda_2(\tau)}{(\bar{\omega}+\lambda_1(\tau))^2}\dot{\lambda}_2(\tau)\right]$$
$$\cdot \cos\left(\int_\tau^t \left[\frac{k\lambda_2^2(\sigma)}{2(\bar{\omega}+\lambda_1(\sigma))^4}\dot{\lambda}_1(\sigma) - \frac{k\lambda_2(\sigma)}{(\bar{\omega}+\lambda_1(\sigma))^3}\dot{\lambda}_2(\sigma)\right] d\sigma\right) d\tau, \quad (5.75)$$
$$\gamma_3(t) = -\int_0^t \left[\frac{-\lambda_2^2(\tau)}{2(\bar{\omega}+\lambda_1(\tau))^3}\dot{\lambda}_1(\tau) + \frac{\lambda_2(\tau)}{(\bar{\omega}+\lambda_1(\tau))^2}\dot{\lambda}_2(\tau)\right]$$
$$\cdot \sin\left(\int_\tau^t \left[\frac{k\lambda_2^2(\sigma)}{2(\bar{\omega}+\lambda_1(\sigma))^4}\dot{\lambda}_1(\sigma) - \frac{k\lambda_2(\sigma)}{(\bar{\omega}+\lambda_1(\sigma))^3}\dot{\lambda}_2(\sigma)\right] d\sigma\right) d\tau. \quad (5.76)$$

If the loop is completed at time $t = T$, the geometric phase is given by $\gamma(T)$.

To see that the geometric phase is not necessarily trivial for all loops, consider equation (5.74) and a smooth path $\mathcal{C}$ in parameter space. The integral over the closed path can then be written as

$$\gamma_1(T) = \oint_{\mathcal{C}} \left[\frac{k\lambda_2^2}{2(\bar{\omega}+\lambda_1)^4}d\lambda_1 - \frac{k\lambda_2}{(\bar{\omega}+\lambda_1)^3}d\lambda_2\right] \quad (5.77)$$
$$= \int_{\mathcal{D}} \frac{2k\lambda_2}{(\bar{\omega}+\lambda_1)^4} d\lambda_1 \wedge d\lambda_2 \quad (5.78)$$

where $\mathcal{D}$ is any surface in parameter space bounded by $\mathcal{C}$. We see then that the one-form in the integral of equation (5.77) is not exact.

5.5.3 The dynamic phase equations

It can be shown (see, e.g. [92]) that the Wei-Norman representation for an element $g \in SE(2)$ with the basis for $se(2)$ given by equation (5.55) is

$$g = e^{\gamma_1 \mathcal{A}_1} e^{\gamma_2 \mathcal{A}_2} e^{\gamma_3 \mathcal{A}_3} = \begin{pmatrix} \cos\gamma_1 & -\sin\gamma_1 & \gamma_2\cos\gamma_1 - \gamma_3\sin\gamma_1 \\ \sin\gamma_1 & \cos\gamma_1 & \gamma_2\sin\gamma_1 + \gamma_3\cos\gamma_1 \\ 0 & 0 & 1 \end{pmatrix} \quad (5.79)$$
$$= \begin{pmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{pmatrix} \quad (5.80)$$

where in the second equation we have shown the usual form of elements of $SE(2)$ as rigid motions in the plane. We then have the following proposition.

Proposition 5.5.2 *For a given element $g \in SE(2)$ given by equation* (5.79) *the constants α_g^{ij} are*

$$\begin{array}{llllll} \alpha_g^{11} = 1, & \alpha_g^{12} = \gamma_2 \sin\gamma_1 + \gamma_3 \cos\gamma_1, & \alpha_g^{13} = -\gamma_2 \cos\gamma_1 + \gamma_3\gamma_1, \\ \alpha_g^{21} = 0, & \alpha_g^{22} = \cos\gamma_1, & \alpha_g^{23} = \sin\gamma_1, \\ \alpha_g^{31} = 0, & \alpha_g^{32} = -\sin\gamma_1, & \alpha_g^{33} = \cos\gamma_1. \end{array} \tag{5.81}$$

If g is given by equation (5.80) *the constants take the form*

$$\begin{array}{lll} \alpha_g^{11} = 1, & \alpha_g^{12} = y, & \alpha_g^{13} = -x, \\ \alpha_g^{21} = 0, & \alpha_g^{22} = \cos\phi, & \alpha_g^{23} = \sin\phi, \\ \alpha_g^{31} = 0, & \alpha_g^{32} = -\sin\phi, & \alpha_g^{33} = \cos\phi. \end{array} \tag{5.82}$$

Proof We prove the form in equation (5.82). We have

$$\begin{aligned} \mathrm{Ad}_g\mathcal{A}_1 &= g\mathcal{A}_1 g^{-1} = \begin{pmatrix} 0 & -1 & y \\ 1 & 0 & -x \\ 0 & 0 & 0 \end{pmatrix} = \mathcal{A}_1 + \mathcal{A}_2 y - \mathcal{A}_3 x. \\ \mathrm{Ad}_g\mathcal{A}_2 &= g\mathcal{A}_2 g^{-1} = \begin{pmatrix} 0 & 0 & \cos\phi \\ 0 & 0 & \sin\phi \\ 0 & 0 & 0 \end{pmatrix} = \mathcal{A}_2 \cos\phi + \mathcal{A}_3 \sin\phi. \\ \mathrm{Ad}_g\mathcal{A}_3 &= g\mathcal{A}_3 g^{-1} = \begin{pmatrix} 0 & 0 & -\sin\phi \\ 0 & 0 & \cos\phi \\ 0 & 0 & 0 \end{pmatrix} = -\mathcal{A}_2 \sin\phi + \mathcal{A}_3 \cos\phi. \end{aligned}$$

This establishes equation (5.82). Equation (5.81) then follows from equations (5.79) and (5.80). ∎

From equations (5.62), (5.54), and (5.81) we have

$$\begin{aligned}\xi_{dp} &= \mathcal{A}_1\frac{f(\lambda_2)}{\omega(\lambda_1)} + \mathcal{A}_2\frac{f(\lambda_2)}{\omega(\lambda_1)}(\gamma_2(t)\sin(\gamma_1(t)) + \gamma_3(t)\cos(\gamma_1(t))) \\ &\quad -\mathcal{A}_3\frac{f(\lambda_2)}{\omega(\lambda_1)}(\gamma_2(t)\sin(\gamma_1(t)) + \gamma_3(t)\cos(\gamma_1(t))) \qquad (5.83)\end{aligned}$$

where the γ_i are the Wei-Norman parameters for the geometric phase, equations (5.74 - 5.76).

5.5.4 An elliptical loop

We now choose to vary the parameter along a closed loop given by

$$\lambda_1 = a\cos\theta, \quad \lambda_2 = b\sin\theta \qquad (5.84)$$

for $\theta \in [0, 2\pi]$. Using these in equation (5.77) we have

$$\begin{aligned}\gamma_1(2\pi) &= \int_0^{2\pi}\left[\frac{-ab^2k\sin^3\theta}{2(\bar{\omega}+a\cos\theta)^4} - \frac{kb^2\sin\theta\cos\theta}{(\bar{\omega}+a\cos\theta)^3}\right]d\theta \\ &= -kb^2\int_0^{2\pi}\left[\frac{a(1-\cos^2\theta)}{2(\bar{\omega}+a\cos\theta)^4} + \frac{\cos\theta}{(\bar{\omega}+a\cos\theta)^3}\right]\sin\theta d\theta. \qquad (5.85)\end{aligned}$$

Making the substitution $u = \cos\theta$ we have

$$\gamma_1(2\pi) = -kb^2\int_1^1\left[\frac{a(1-u^2)}{2(\bar{\omega}+au)^4} + \frac{u}{(\bar{\omega}+au)^3}\right]du = 0 \qquad (5.86)$$

and thus for this loop the geometric phase in γ_1 is zero. To determine the geometric phase in γ_2 and γ_3 we need the evolution of γ_1 around the loop. Once again making the substitution $u = \cos\theta$ we obtain the equation

$$\begin{aligned}\gamma_1(\theta) &= -kb^2\int_1^{\cos(\theta)}\left[\frac{a(1-u^2)}{2(\bar{\omega}+au)^4} + \frac{u}{(\bar{\omega}+au)^3}\right]du \\ &= -kb^2\left[\int_1^{\cos(\theta)}\frac{a}{2(\bar{\omega}+au)^4}du - \frac{u^2}{6(\bar{\omega}+au)^3}\bigg|_1^{\cos(\theta)} - \frac{u}{6a(\bar{\omega}+au)^2}\bigg|_1^{\cos(\theta)}\right. \\ &\quad \left. + \int_1^{\cos(\theta)}\frac{1}{6a(\bar{\omega}+au)^2}du - \frac{u}{2a(\bar{\omega}+au)^2}\bigg|_1^{\cos(\theta)}\right.\end{aligned}$$

$$+\int_1^{\cos(\theta)} \frac{1}{2a(\bar{\omega}+au)^2}du\Bigg]$$

$$= \frac{kb^2}{6}\left[\frac{\cos^2(\theta)-1}{(\bar{\omega}+a\cos(\theta))^3}+\frac{2}{a(\bar{\omega}+a)^2}-\frac{2\cos(\theta)}{a(\bar{\omega}+a\cos(\theta))^2}\right.$$
$$\left.+\frac{2}{a^2(\bar{\omega}+a)}-\frac{2}{a^2(\bar{\omega}+a\cos(\theta))}\right]. \quad (5.87)$$

We note that γ_1 depends on θ only through $\cos(\theta)$ and write $\gamma_1(\theta) = \gamma_1(\cos(\theta))$. Using the solution for γ_1 in equation (5.75) together with the equations for the parameter variation in equation (5.84) we have

$$\gamma_2(2\pi) = \int_0^{2\pi}\left[\frac{ab^2\sin^3\theta}{2(\bar{\omega}+a\cos\theta)^4}+\frac{b^2\sin\theta\cos\theta}{(\bar{\omega}+a\cos\theta)^3}\right]\cos\left(\int_\theta^{2\pi}\gamma_1(\cos(\psi))d\psi\right)d\theta. \quad (5.88)$$

With the substitution $u = \cos\theta$ this can be written as

$$\gamma_2(2\pi) = -\int_1^1\left[\frac{ab^2(1-u^2)}{2(\bar{\omega}+au)^4}+\frac{b^2u}{(\bar{\omega}+au)^3}\right]\cos\left(-\int_{\cos^{-1}u}^1\frac{\gamma_1(v)}{\sin(\cos^{-1}v)}dv\right)du = 0. \quad (5.89)$$

Similarly, from equation (5.76),

$$\gamma_2(2\pi) = \int_0^{2\pi}\left[\frac{ab^2\sin^3\theta}{2(\bar{\omega}+a\cos\theta)^4}+\frac{b^2\sin\theta\cos\theta}{(\bar{\omega}+a\cos\theta)^3}\right]\sin\left(\int_\theta^{2\pi}\gamma_1(\cos(\psi))d\psi\right)d\theta \quad (5.90)$$

and with the substitution $u = \cos\theta$ this becomes

$$\gamma_3(2\pi) = -\int_1^1\left[\frac{ab^2(1-u^2)}{2(\bar{\omega}+au)^4}+\frac{b^2u}{(\bar{\omega}+au)^3}\right]\sin\left(-\int_{\cos^{-1}u}^1\frac{\gamma_1(v)}{\sin(\cos^{-1}v)}dv\right)du = 0. \quad (5.91)$$

Thus for the elliptical loop defined by equation (5.84) the geometric phase is zero. However g_{gp} is not identically zero around the loop. To solve for the dynamic phase analytically we need both γ_2 and γ_3 as functions of θ. Due to the complexity of the equations we turn to numerical simulation.

To vary the parameter we set $\theta = \frac{2\pi}{T}t$ where T should be taken so as to satisfy the adiabatic condition. In the simulations that follow we choose $\bar{\omega} = 100$ and, to ensure the linear system will be critically damped when $\lambda_1 = 0$, we take $k = 200$. From equation (5.66) we have, at critical damping, $\mu_l = \frac{k}{2} = 100$. The adiabatic criterion will be met if $\frac{2\pi}{T} << \mu_l$. We then choose T such that $T >> \frac{2\pi}{100}$.

In Figures 5.1 and 5.2 we show the loop in parameter space and the evolution of the linear system for the choices $\bar{\omega} = 100$, $k = 200$, $T = 10$, $a = 50$, and $b = 25$.

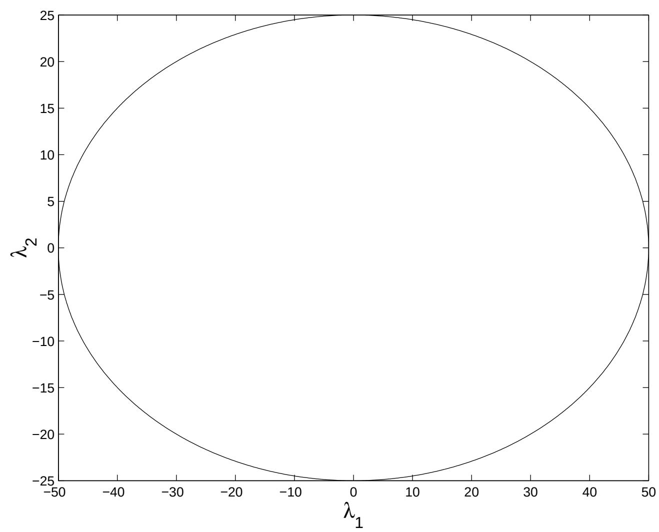

Figure 5.1: Elliptical parameter loop with $a = 50, b = 25$

From Figure 5.2 we see that y_2 is close to zero at all times and therefore the system remains close to equilibrium throughout the parameter variation. Note that the amplitude of the variation of λ_1 is on the order of $\bar{\omega}$ so the system actually strays far from the critical damping condition.

In Figures 5.3, 5.4, 5.5 we show the evolution in the group under the full system (labeled as 'true evolution'), the evolution of the geometric and dynamic phase, and the evolution of the reconstructed system using the standard $SE(2)$ coordinates of (ϕ, x, y) as calculated from the Wei-Norman coordinates (see equations (5.79) and (5.80)).

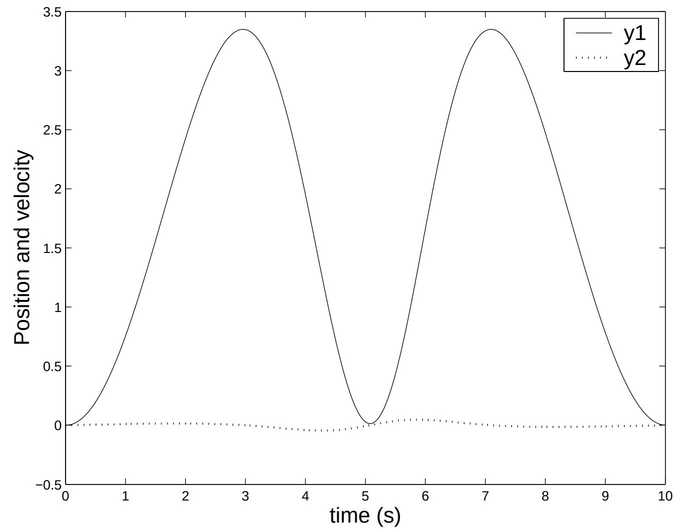

Figure 5.2: Evolution of linear system with $a = 50$, $b = 25$, and $T = 10$

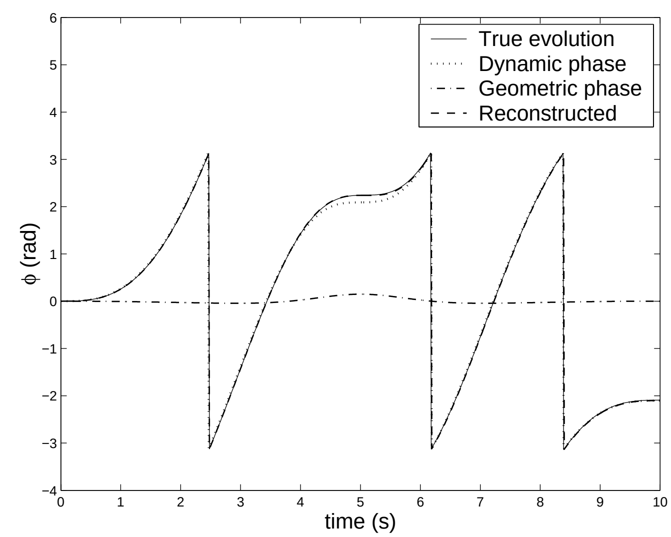

Figure 5.3: Evolution of ϕ with $a = 50$, $b = 25$, and $T = 10$

From the figures we see, as expected, that the geometric phase is zero. We see also that the trajectory of the dynamic phase is not even qualitatively similar to the full system while the reconstructed trajectory is both similar in structure and a good approximation of the true group trajectory. Since the geometric phase is zero, the dynamic phase at the end time is equal to the reconstructed phase. The

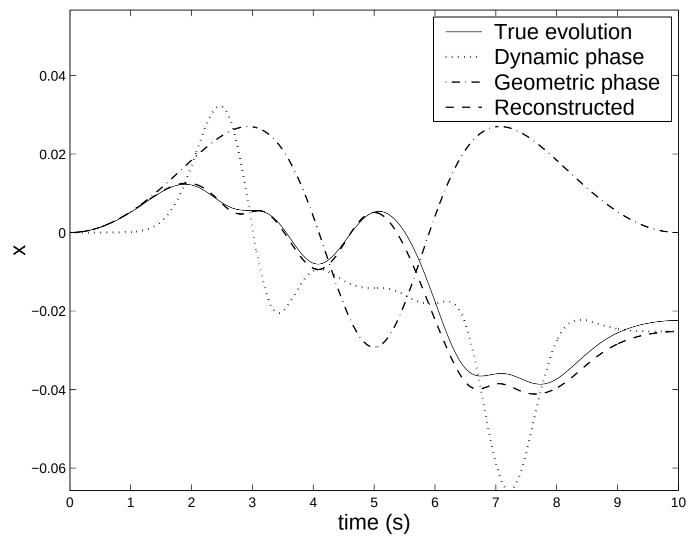

Figure 5.4: Evolution of x with $a = 50$, $b = 25$, and $T = 10$

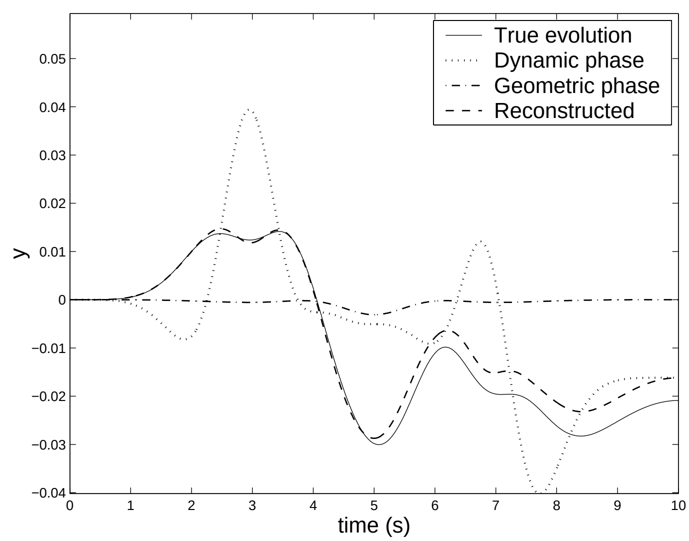

Figure 5.5: Evolution of y with $a = 50$, $b = 25$, and $T = 10$

remaining error is due to the fact that it is the evolution of the geometric phase, i.e. the adiabatic approximation to g_{gp} through the use of the induced Landsberg connection, that is used in the equation for the dynamic phase, equation (5.62). We expect, then, that as the rate of parameter variation is decreased, the dynamic phase will better approximate the true system. In Figures 5.6, 5.7 and 5.8 we show

(ϕ, x, y) when $T = 50$.

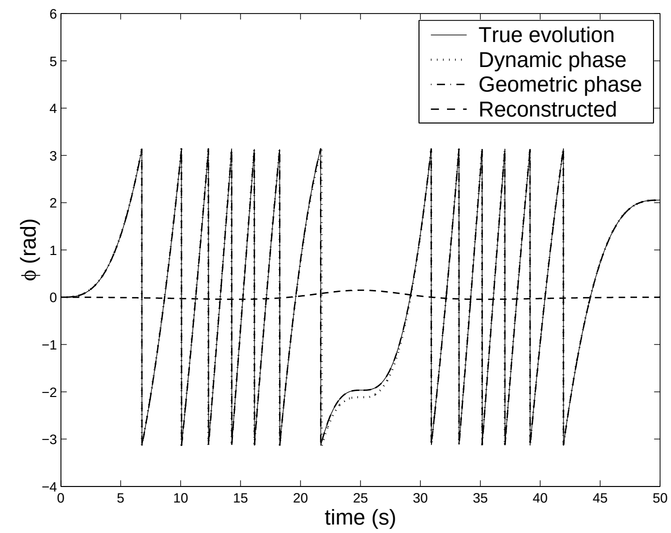

Figure 5.6: Evolution of ϕ with $a = 50$, $b = 25$, and $T = 50$

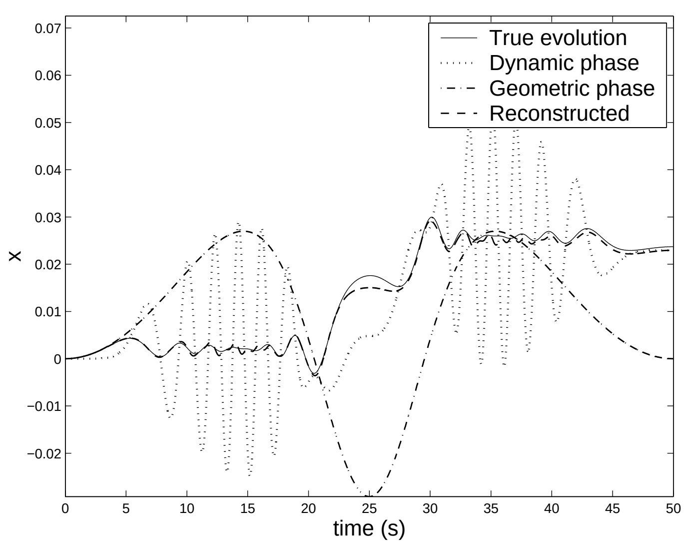

Figure 5.7: Evolution of x with $a = 50$, $b = 25$, and $T = 50$

The figures show that indeed the final dynamic phase value is much closer to the actual position in the group but its trajectory, while better than the previous simulation, is still a poor predictor of the true dynamics throughout the parameter

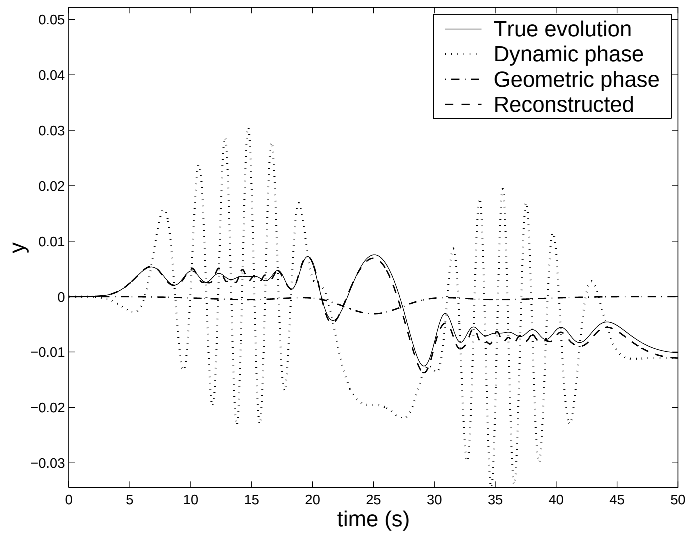

Figure 5.8: Evolution of y with $a = 50$, $b = 25$, and $T = 50$

cycle. The reconstructed trajectory, on the other hand, is very close to the true trajectory. Notice also that the geometric phase trajectory when $T = 50$ is simply a time-scaled version of the trajectory when $T = 10$; this is a direct reflection of the fact that this phase is a geometric quantity.

As a point of interest, we illustrate in Figure 5.9 the trajectory in the (x, y) coordinates of the true system, the reconstructed system, and a reconstructed system based on the naive equations, (5.25) and (5.26). This figure clearly shows that the trajectory determined from the naive equations is very different from the true system and the non-abelian nature of the system cannot be ignored even in the adiabatic limit.

5.5.5 Other loops

We now present the results of numerical simulations of two other loops in parameter space, first a figure-eight and then a square loop.

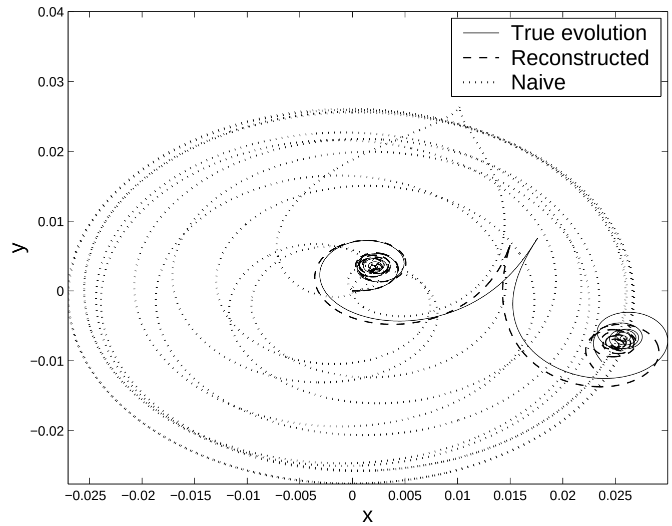

Figure 5.9: True, reconstructed, and naive planar trajectories

Figure eight loop

Consider the loop defined by

$$\lambda_1(\theta) = a\cos(\theta), \qquad \lambda_2 = b\cos(\theta)\sin(\theta). \tag{5.92}$$

As before, set $\theta = \frac{2\pi}{T}t$ and choose $a = 50$, $b = 25$, and $T = 10$. The loop in parameter space is shown in Figure 5.10 and the evolution of the linear system during the parameter variation is shown in Figure 5.11. We see that for $T = 10$ the system remains close to equilibrium at all times and we thus expect the adiabatic approximation to be a good one.

The evolution of the group variables as described by the full system, the geometric phase equation, the dynamic phase equation, and the reconstructed system, are shown in Figures 5.12, 5.13, and 5.14.

The path of the system under this loop is quite different than under the elliptical loop. From the figures we see that at the completion of the loop, the geometric phase is zero.

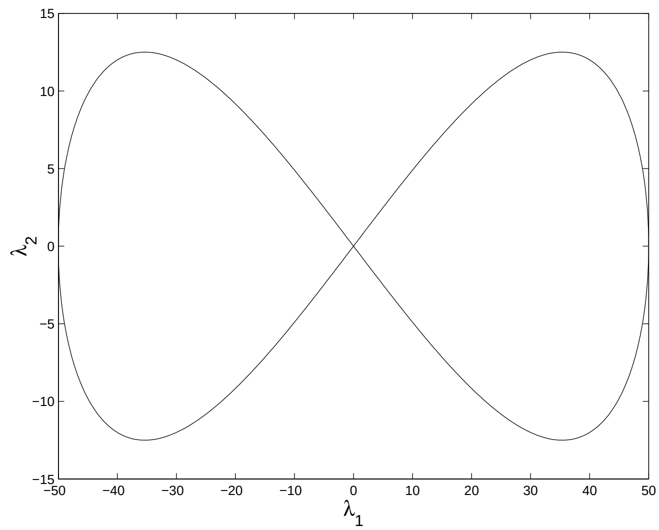

Figure 5.10: Figure-eight loop with $a = 50$, $b = 25$

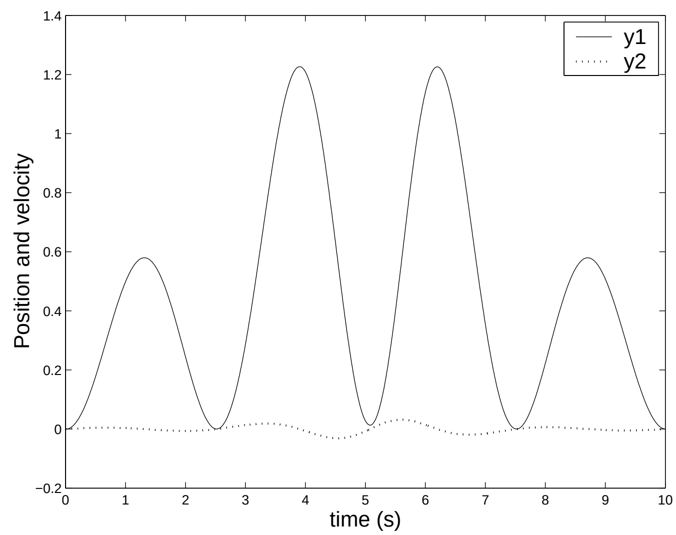

Figure 5.11: Evolution of linear system under figure-eight loop

Rectangular loop

Consider now the following loop.

$$\lambda_1 = \begin{cases} \frac{4a}{T}t, & 0 \le t < \frac{T}{4} \\ a, & \frac{T}{4} \le t < \frac{T}{2} \\ a - \frac{4a}{T}\left(t - \frac{T}{2}\right), & \frac{T}{2} \le t < \frac{3T}{4} \\ 0, & \frac{3T}{4} \le t \le T \end{cases}, \qquad \lambda_2 = \begin{cases} 0, & 0 \le t < \frac{T}{4} \\ \frac{4b}{T}\left(t - \frac{T}{4}\right), & \frac{T}{4} \le t < \frac{T}{2} \\ b, & \frac{T}{2} \le t < \frac{3T}{4} \\ b - \frac{4b}{T}\left(t - \frac{3T}{4}\right), & \frac{3T}{4} \le t \le T \end{cases}. \tag{5.93}$$

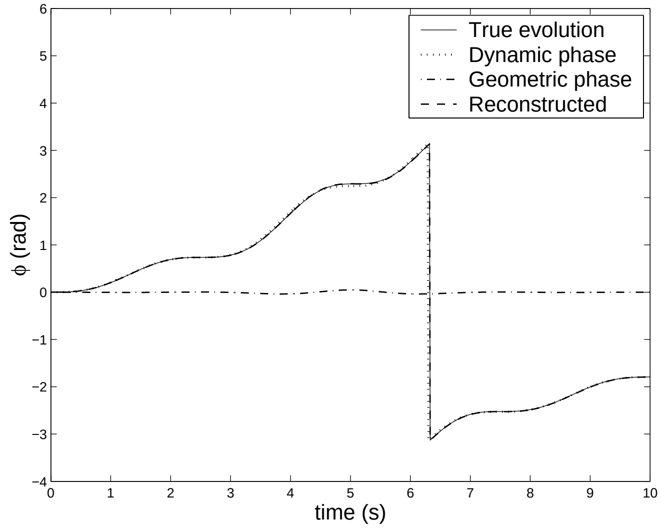

Figure 5.12: Evolution of ϕ under figure-eight loop

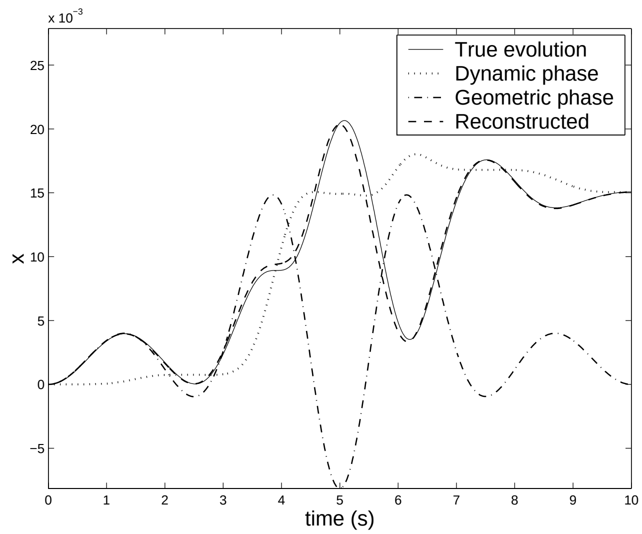

Figure 5.13: Evolution of x under figure-eight loop

We once again choose $a = 50$, $b = 25$, and $T = 10$. The loop in parameter space and the evolution of the linear system are shown in Figures 5.15 and 5.16 respectively.

From Figure 5.16 we see that the system once again remains close to equilibrium. The evolution of the full system in the group variables and of g_{dp}, g_{gp}, and

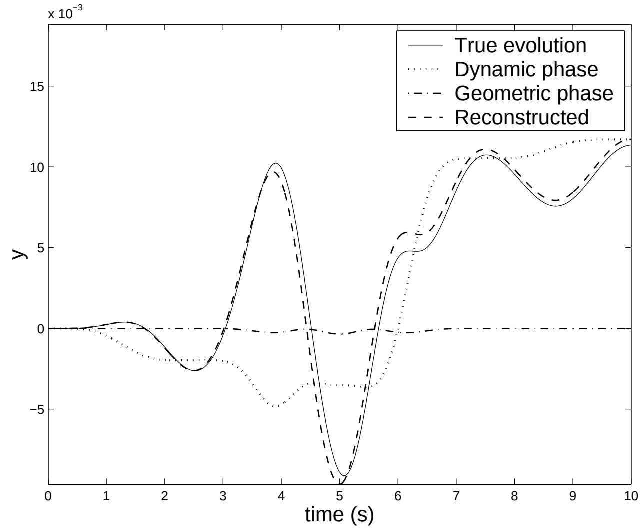

Figure 5.14: Evolution of y under figure-eight loop

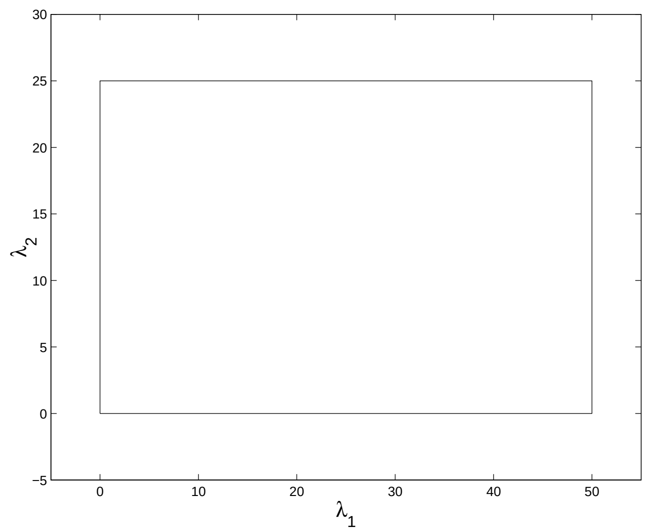

Figure 5.15: Rectangular loop with $a = 50$ and $b = 25$

the reconstructed system are shown in Figures 5.17, 5.18, 5.19.

For this loop there is a nonzero geometric phase in the group. To see this more explicitly we show the evolution of the geometric phase in Figures 5.20, 5.21, 5.22.

Finally, as a point of interest, we consider the evolution of the system when

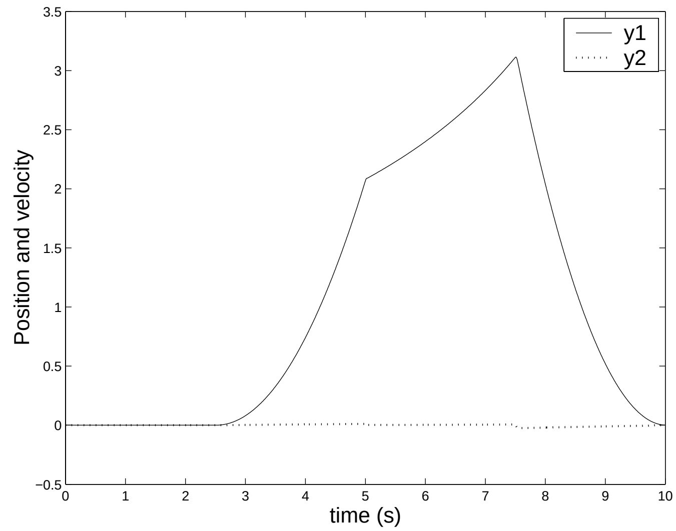

Figure 5.16: Evolution of linear system under rectangular loop

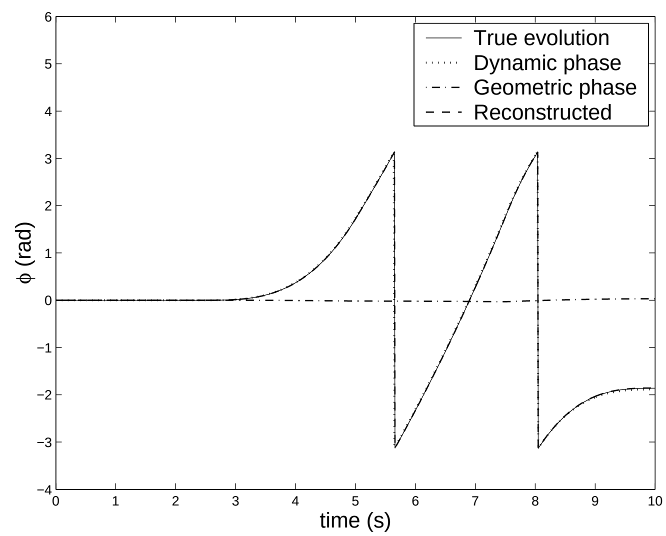

Figure 5.17: Evolution of ϕ under rectangular loop

the adiabatic criterion is not satisfied. We expect, therefore, that the techniques developed in this chapter will no longer be applicable. We choose the same rectangular loop but reduce the rate of dissipation by setting $\bar{\omega} = 10$ and $k = 20$, and increase the rate of change of the parameter by choosing $T = 1$. The evolution of the linear system is shown in Figure 5.23.

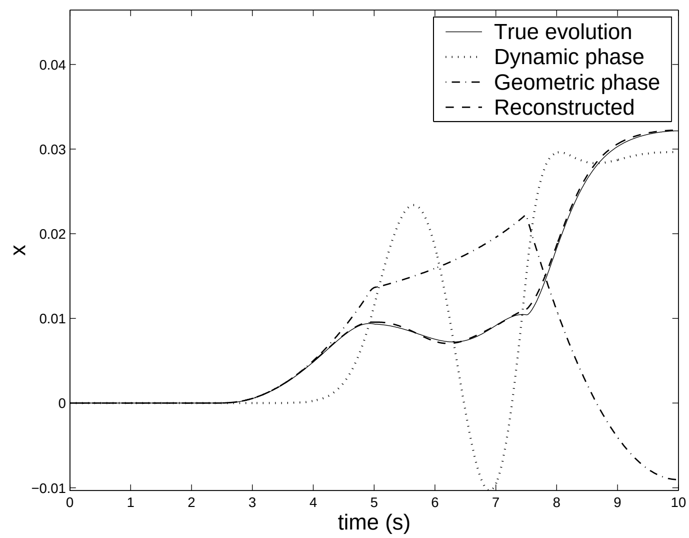

Figure 5.18: Evolution of x under rectangular loop

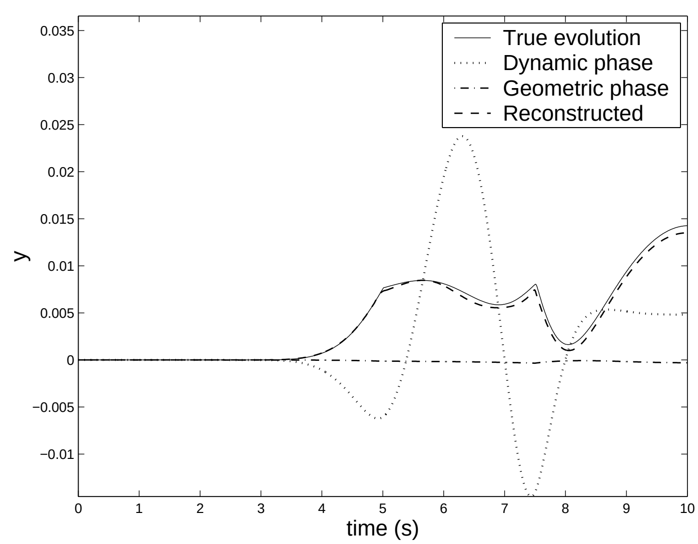

Figure 5.19: Evolution of y under rectangular loop

The system is clearly not at equilibrium for a large portion of the cycle as evidenced by the large values of y_2. The evolution of the group variables of the full system and of the dynamic, geometric, and reconstructed phases are shown in Figure 5.24, 5.25, 5.26. As expected the reconstructed system is no longer a good approximation of the true system and in fact no longer even exhibits the same

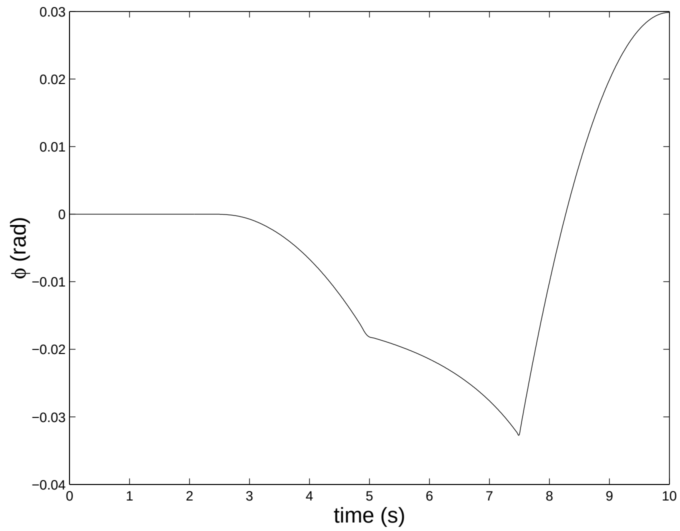

Figure 5.20: Evolution of ϕ_{gp} under rectangular loop

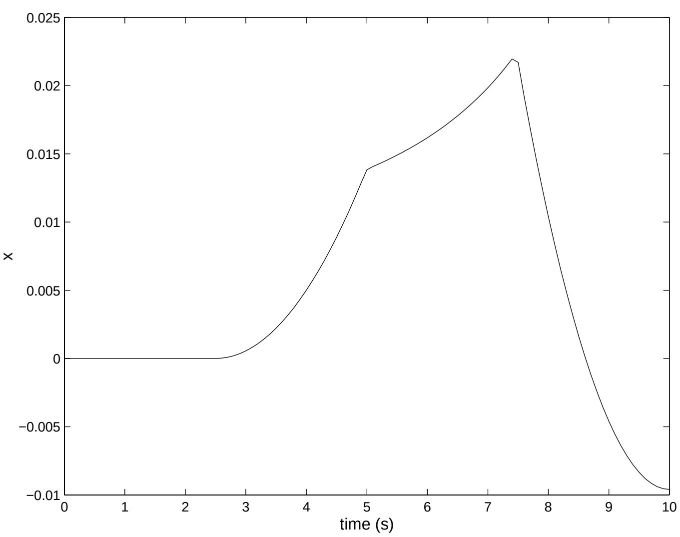

Figure 5.21: Evolution of x_{gp} under rectangular loop

general character.

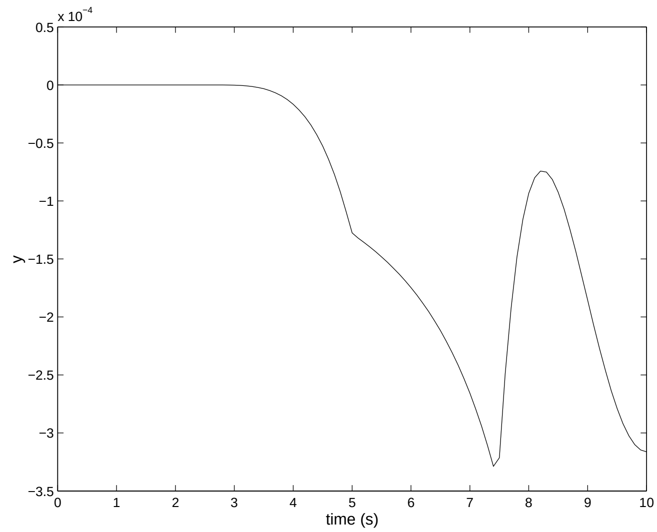

Figure 5.22: Evolution of y_{gp} under rectangular loop

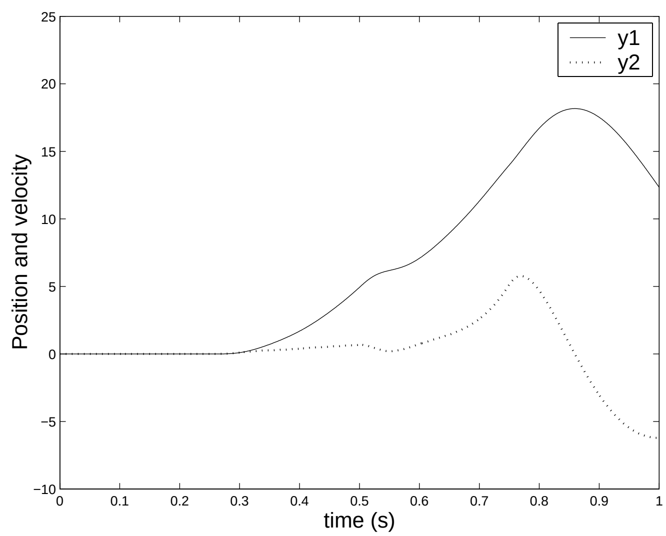

Figure 5.23: Non-adiabatic variation: linear system evolution

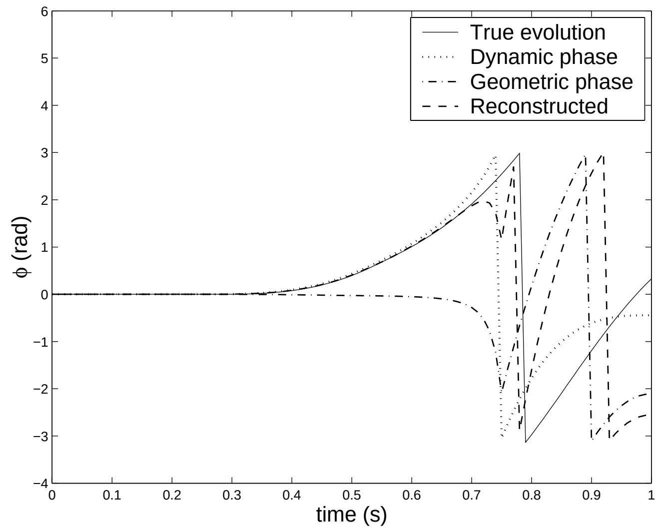

Figure 5.24: Non-adiabatic variation: evolution of ϕ

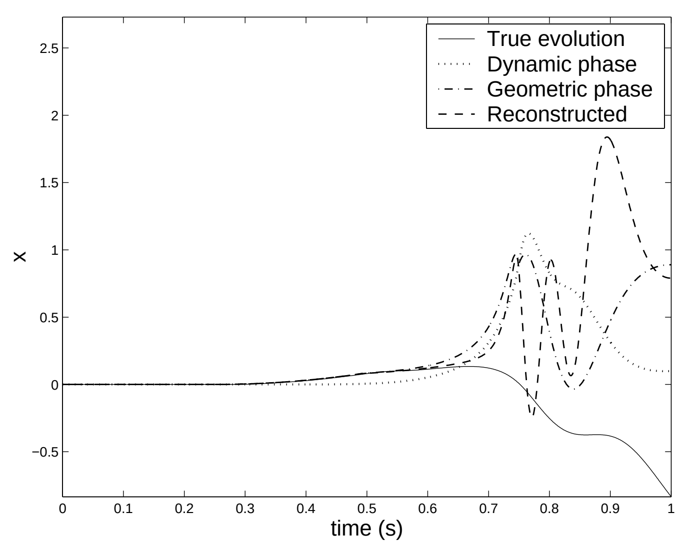

Figure 5.25: Non-adiabatic variation: evolution of x

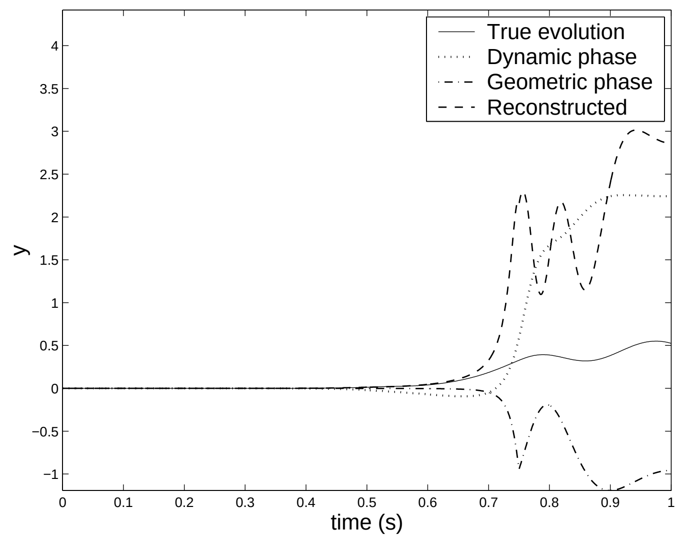

Figure 5.26: Non-adiabatic variation: evolution of y

Chapter 6

Geometric phases in nonholonomic systems with symmetry

6.1 Introduction

In this chapter we focus on investigating the role geometric phases play in the control of nonholonomic systems with symmetry. To do this we will use a methodology developed by Bloch, Krishnaprasad, Marsden, and Murray in [13] in which the nonholonomic constraints are modeled by using an Ehresmann connection . These systems are naturally described by a principal bundle structure in which the group variables describe the overall position or attitude of the system and the base space variables describe the internal configuration or “shape” of the system. In many cases, through proper choice of feedback control laws, the shape variables may be viewed as the controls for the system.

Among the contributions of [13] is a generalization of Noether’s theorem to nonholonomic systems through the use of a quantity known as the nonholonomic momentum (for a description of Noether's theorem see, e.g. [1]). Due to the constraints this momentum is no longer conserved but does evolve according to a particular equation. Using the nonholonomic momentum and a connection called the nonholonomic connection, the evolution of the group variables can be reconstructed from a path in the base space. The reconstruction equations contain a drift term involving the nonholonomic momentum and a geometric term dependent only on the path followed in the base (shape) space. Since the nonholonomic connection arises naturally through a synthesis of the constraints and the mechanical connection (which captures the momentum terms), there is no notion of adiabaticity required in order to establish the existence of a geometric phase. In addition, since the fiber over each element in the base space is identified with the group, it makes sense to discuss not just the geometric phase (defined only for closed loops in the base space) but the geometric shift in the group as the result of following any path in the base.

The techniques introduced in [13] have proved to be useful in understanding a variety of systems including the Snakeboard, a modified version of the skateboard consisting of a two sets of independently rotating wheel pairs connected by a rigid cross-brace [51, 69, 70] and the Roller Racer, a child's toy patented in 1972 by W.E. Hendricks and shown in Figure 6.1. Modeling the system as a two-node, one-module $SE(2)-$snake, the dynamics of this system have been analyzed using the techniques of nonholonomic systems with symmetry in [47, 92]. $G-$snakes were introduced in [92] to describe systems consisting of identical, linked units each of which has a configuration space given by a copy of the Lie group G and is subject to the same set of nonholonomic constraints.

Figure 6.1: The Roller Racer

Physically the Roller Racer is a particularly simple mechanical system. The equations describing the evolution of the nonholonomic momentum and the group variables, however, are quite complicated. Motivated by this, we will derive the governing equations for the $H(3)-$Racer, the two-node, one-module $G-$snake on the three-dimensional Heisenberg group. This system exhibits many of the same properties as the Roller Racer but with significantly simpler equations describing the evolution of the nonholonomic momentum and group dynamics.

Our desire to explore the role of geometric phases in the control of nonholonomic mechanical systems with symmetry is motivated by various general properties of the effect. For example, geometric phases are robust to control (actuator) noise in the following sense. Recall that in general the geometric phase is related to the area enclosed by a loop in the base space and thus zero mean noise in the controls can be expected to average out as the loop is traversed. It is precisely this feature which has led various researchers to propose the notion of holonomic quantum computing in which geometric phase shifts are used to build universal quantum computing gates (see, e.g., [29, 30, 71]). In addition, since the geometric phase is dependent only on the path followed in the base space, the effect is robust to time scaling. In this setting this means that the shapes through which the system must be carried as the curve is traversed do not need to be achieved at precise times, making the control of the shape an easier task.

In the next section we present an overview of the approach to nonholonomic systems with symmetry developed in [13]. To simplify notation, we will assume the symmetry group is a matrix Lie group. Following this overview we use the ideas of the dynamic phase introduced in Chapter 5 to separate the geometric and dynamic effects of following a shape trajectory. We then develop the $H(3)-$Racer and explore the the effect of shape variations on the overall dynamics.

6.2 Nonholonomic systems with symmetry

The approach developed in [13] is targeted towards systems described by a Lagrangian function $L : TQ \to \mathbb{R}$ and subject to a set of nonholonomic constraints. In this section we present a brief overview of the method.

6.2.1 The constraint distribution

We assume the constraints are kinematic and are described by a distribution $\mathcal{D} = \cup_q \mathcal{D}_q \subset T_q Q$. A curve $q(t) \in Q$ is said to satisfy the constraints if $\dot{q}(t) \in \mathcal{D}_{q(t)}$ for all t. In general $\mathcal{D}$ is a nonintegrable distribution; we say then that the constraints are nonholonomic. The distribution is generally given as the null space of a set of 1-forms (referred to as a Pfaffian system).

In some cases the constraints are affine in nature; for example a ball on a rotating turntable where the rotational velocity of the turntable represents the affine part. These constraints are captured by assuming there is a given vector field V_0 on Q and requiring that $\dot{q}(t) - V_0(q(t)) \in \mathcal{D}_{q(t)}$.

The distribution $\mathcal{D}$ defines an Ehresmann connection A by declaring that $\mathcal{D}$ is the horizontal subbundle for the connection. In this case the constraint equations can be expressed as

$$A(q) \cdot \dot{q} = 0 \tag{6.1}$$

if the constraints are linear and

$$A(q) \cdot \dot{q} = A(q) \cdot V_0(q) \tag{6.2}$$

if the constraints are affine.

6.2.2 Systems with symmetry

We now assume we are given a Lie group G and a free and proper left action Φ of G on Q. The group orbit through q is denoted $\mathrm{Orb}(q) := \{\Phi_g(q) | g \in G\}$.

Since the action is free and proper, we can construct the principal fiber bundle $\pi : Q \to M = Q/G$. The base space is referred to as the shape space. Recall that the vertical space at the point q is defined to be the kernel of the map $T_q\pi$ and in the principal bundle setting it is given by the set of infinitesimal generators of the group action at the point q, that is

$$ker T_q \pi = \{\xi_Q(q) | \xi \in \mathfrak{g}\} \tag{6.3}$$

and thus the vertical space at q is the tangent space to the group orbit through q.

We assume that the Lagrangian and the distribution $\mathcal{D}$ are invariant under the lifted action, i.e. that $(T\Phi_g)^* L = L$ and $(T_q \Phi_g) \mathcal{D}_q = \mathcal{D}_{\Phi_g(q)}$.

6.2.3 The nonholonomic momentum

In general the tangent space to the group orbit through q intersects the constraint distribution at q nontrivially. We have the following definitions.

Definition 6.2.1 [13] *The intersection of the tangent space to the group orbit through the point $q \in Q$ and the constraint distribution at this point is denoted $\mathcal{S}_q$. We let the union of these spaces over $q \in Q$ be denoted $\mathcal{S}$. Thus,*

$$\mathcal{S}_q = \mathcal{D}_q \cap T_q(Orb(q)). \tag{6.4}$$

Definition 6.2.2 [13] *For each $q \in Q$ define the vector subspace $\mathfrak{g}^q$ to be the set of Lie algebra elements in $\mathfrak{g}$ whose infinitesimal generators evaluated at q lie in $\mathcal{S}_q$:*

$$\mathfrak{g}^q = \{\xi \in \mathfrak{g} | \xi_Q(q) \in \mathcal{S}_q\}.$$

The corresponding bundle over Q whose fiber at the point q is given by $\mathfrak{g}^q$ is denoted $\mathfrak{g}^{\mathcal{D}}$.

Definition 6.2.3 [13] *The* **nonholonomic momentum map** *J^{nhc} is the bundle map taking TQ to the bundle $(\mathfrak{g}^{\mathcal{D}})^*$ whose fiber over the point q is the dual of the vector space $\mathfrak{g}^q$ that is defined by*

$$< J^{nhc}(v_q), \xi >= \frac{\partial L}{\partial \dot{q}^i} (\xi_Q)^i \tag{6.5}$$

where summation over i is understood. For notational convenience we will often write the left hand side of this equation as $J^{nhc}(\xi)$.

When the momentum map is paired with a section in this way we will refer to it simply as the momentum and write $p = J^{nhc}(\xi)$.

In the classical Noether theorem, the presence of the symmetry leads to a momentum map whose value is constant along the trajectories of the system. The nonholonomic momentum map may be viewed as giving the components of the usual momentum map which lie along the symmetry directions that are consistent with the constraints. As a result of these constraints the momentum is no longer conserved but rather is subject to a nontrivial equation as given by the following theorem.

Theorem 6.2.4 [13] *Assume that the Lagrangian is invariant under the lifted action of G on TQ and that ξ^q is a section of the bundle $\mathfrak{g}^{\mathcal{D}}$. Then any solution of the Lagrange d'Alembert equations for a nonholonomic system must satisfy, in addition to the given kinematic constraints, the momentum equation:*

$$\frac{d}{dt}\left(J^{nhc}(\xi^{q(t)})\right) = \frac{\partial L}{\partial \dot{q}^i}\left[\frac{d}{dt}(\xi^{q(t)})\right]_Q^i. \tag{6.6}$$

6.2.4 The nonholonomic connection

We now make two additional assumptions on the system. First, we assume that there is a $G-$invariant metric on the configuration space, usually given by the kinetic energy of the system. Second, we assume the constraints and the orbit directions span the entire tangent space to the configuration space at each point $q \in Q$:

$$\mathcal{D}_q + T_q(\mathrm{Orb}(q)) = T_qQ. \tag{6.7}$$

This is known as the principal case. With these assumptions, the momentum equation augments the constraints to provide a connection on $Q \to Q/G$. To define this connection we first need the notion of the locked inertia tensor in the nonholonomic setting.

Definition 6.2.5 [13] *The* **locked inertia tensor** $\mathbb{I}(q) : \mathfrak{g}^{\mathcal{D}} \to (\mathfrak{g}^D)^*$ *is defined by*

$$< \mathbb{I}(q)\xi, \eta > = \ll \xi_Q(q), \eta_Q(q) \gg \tag{6.8}$$

where $\ll \cdot, \cdot \gg$ is the kinetic energy inner product.

Define a map $A^{sym} : TQ \to \mathcal{S}$ by

$$A^{sym}(v_q) = \left(\mathbb{I}^{-1}(q) J^{nhc}(v_q)\right)_Q . \qquad (6.9)$$

This map is equivariant and is a projection onto $\mathcal{S}_q$. Now choose $\mathcal{U}_q \subset T_q(\mathrm{Orb}(q))$ such that $T_q(\mathrm{Orb}(q)) = \mathcal{S}_q \oplus \mathcal{U}_q$. This splitting of subspaces is shown in Figure 6.2. Let $A^{kin} : T_qQ \to \mathcal{U}_q$ be a $\mathcal{U}_q$−valued form projecting $\mathcal{U}_q$ onto itself and mapping

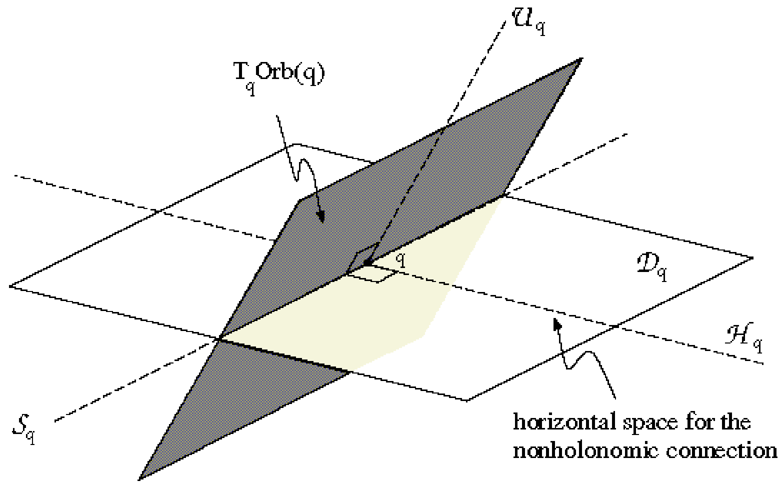

Figure 6.2: Subspace definitions for the nonholonomic connection (From [13])

$\mathcal{D}_q$ to zero; for example, it can be given by orthogonal projection with respect to the kinetic energy metric. The constraints plus momentum equation can then be written as

$$\begin{aligned} A^{kin}(q) \cdot \dot{q} &= 0, \\ A^{sym}(q) \cdot \dot{q} &= \left(\mathbb{I}^{-1}(q)p\right)_Q \end{aligned}$$

where $p \in (\mathfrak{g}^{\mathcal{D}})^*$ is the nonholonomic momentum. We have the following definition.

Definition 6.2.6 [13] *In the principal case, under the assumption that the Lagrangian is of the form kinetic minus potential energies, the* **nonholonomic connection** $\mathcal{A}$ *is the connection on the principal bundle $Q \to Q/G$ whose horizontal* *space at the point $q \in Q$ is given by the orthogonal complement to the space $\mathcal{S}_q$ within the space $\mathcal{D}_q$; see Figure* 6.2.

In [13] it is shown that the nonholonomic connection is a principal connection. The overall motion of the system then satisfies

$$\mathcal{A}(q)\dot{q} = \mathbb{I}^{-1}(q)J^{nhc}(\dot{q}). \tag{6.10}$$

6.2.5 The system equations

In [13] Bloch, Krishnaprasad, Marsden, and Murray show that the equations of motion on the reduced space and the reconstruction equations are of the form

$$\begin{aligned} \dot{g} &= g\left(-\mathcal{A}_{loc}(r)\dot{r} + \mathbb{I}_{loc}^{-1}(r)p\right), && (6.11) \\ \dot{p} &= \dot{r}^T\alpha(r)\dot{r} + \dot{r}^T\beta(r)p + p^T\gamma(r)p, && (6.12) \\ M(r)\ddot{r} &= -C(r,\dot{r}) + N(r,\dot{r},p) + \tau && (6.13) \end{aligned}$$

where g denotes the group element, p the nonholonomic momentum, and r the shape variables. Here $\mathcal{A}_{loc}$ is the local form of the nonholonomic connection and $\mathbb{I}_{loc}$ is the local form of the locked inertia tensor.

In the last equation, $M(r)$ is the mass matrix of the system, C, the Coriolis term, is quadratic in $\dot{r}$, and N is quadratic in $\dot{r}$ and p. The variable τ represents the conservative and external forces (controls) applied to the system. If we assume the dynamics on the shape space are controllable so that arbitrary trajectories may be followed we can replace the system on the reduced space by

$$\ddot{r} = u. \tag{6.14}$$

The reconstruction process is as follows. Given an initial condition and a path $r(t)$ in the base space (i.e. a solution to equation (6.13)), we first integrate the momentum equation to determine $p(t)$ for all time and then use $r(t)$ and $p(t)$ to determine the motion in the group.

6.3 The geometric and dynamic phase

Consider the dynamical system in the group variables in a nonholonomic system with symmetry, equation (6.11). We wish to separate the geometric effect, that is the portion of the evolution in the group variables which depends only on the path followed in the base space, from the time-dependent part of the evolution. We thus define the geometric phase g_{gp} by the equation

$$\dot{g}_{gp} = -g_{gp}\mathcal{A}_{loc}(r)\dot{r}. \tag{6.15}$$

As discussed in the introduction to this chapter, we are interested in g_{gp} not only at the completion of the loop in the base space but along the loop as well.

Following the techniques developed in Section 5.3 we define the dynamic phase by

$$\dot{g}_{dp} = g_{dp}\mathrm{Ad}_{g_{gp}}\mathbb{I}_{loc}^{-1}(r)p \tag{6.16}$$

so that $g = g_{dp}g_{gp}$. The dynamic phase defined in this way captures all of the dynamics which are dependent on the time parametrization. It is influenced by the reduced dynamics through both the geometric phase and through the evolution of the nonholonomic momentum. Note that this differs from the discussion in [13] in which the geometric and dynamic phase terms are defined infinitesimally as arising from the Lie algebra elements $-\mathcal{A}_{loc}(r)\dot{r}$ and $\mathbb{I}_{loc}^{-1}(r)p$ respectively.

6.4 The $H(3)-$Racer

The Roller Racer consists of two platforms hinged together at a point. Each platform has a configuration space given by $SE(2)$ and is subject to a no-sliding constraint. Analogously, a general Racer system is constructed as follows. One begins with two copies of a three dimensional Lie group G. The configuration space $Q \subset G \times G$ of the Racer is defined by a pair of independent holonomic constraints (mimicking the hinge of the Roller Racer). A "no-slip" constraint is then introduced on each copy of G by requiring that the Lie algebra elements defining the group velocities reside in a two-dimensional subspace of the full Lie algebra.

In the remainder of this chapter we focus on the $H(3)-$Racer, a two-node, one-module $G-$snake on the three-dimensional Heisenberg group. This group is the collection of matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

where a, b, and c are real numbers. The associated Lie algebra is denoted $\mathfrak{g} = h(3)$. We choose the following basis for $h(3)$.

$$\mathcal{A}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (6.17)$$

Let $\mathcal{A}_i^\flat$ be defined by $\mathcal{A}_i^\flat(\mathcal{A}_j) = \delta_i^j$ so that $\left\{\mathcal{A}_i^\flat\right\}_{i=1}^3$ is a basis for the dual space $h^*(3)$. We now construct a two-node, one module $H(3)-$snake as follows. Let the configuration of node i be given by

$$g_i = \begin{pmatrix} 1 & a_i & b_i \\ 0 & 1 & c_i \\ 0 & 0 & 1 \end{pmatrix}, \quad i = 1, 2 \tag{6.18}$$

and define

$$g_{12} = g_1^{-1} g_2 = \begin{pmatrix} 1 & a_2 - a_1 & b_2 - b_1 - a_1(c_2 - c_1) \\ 0 & 1 & c_2 - c_1 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.19}$$

so that g_{12} is the relative configuration between the two nodes. We set

$$a_{12} = a_2 - a_1, \tag{6.20}$$

$$b_{12} = b_2 - b_1 - a_1(c_2 - c_1), \tag{6.21}$$

$$c_{12} = c_2 - c_1. \tag{6.22}$$

To make this system into a Racer we introduce two holonomic constraints

$$\begin{aligned} F_1(a_{12}, b_{12}, c_{12}) &= 0, \\ F_2(a_{12}, b_{12}, c_{12}) &= 0 \end{aligned}$$

such that

$$\left. \frac{\partial(F_1, F_2)}{\partial(a_{12}, b_{12})} \right|_{g_{12}} \neq 0 \quad \forall g_{12} \in H(3)$$

and thus by the implicit function theorem we can solve for a_{12}, b_{12} in terms of c_{12}. For simplicity we choose, with an admitted abuse of notation,

$$a_{12} = F_1(c_{12}), \quad b_{12} = F_2(c_{12}). \tag{6.23}$$

The configuration space of the $H(3)-$Racer is then $Q = H(3) \times \mathbb{R}$ with coordinates given by (a_1, b_1, c_1, c_{12}).

6.4.1 The constraint distribution

Consider the basis for $h(3)$ given in equation (6.17). We have the following bracket relations

$$[\mathcal{A}_1, \mathcal{A}_2] = \mathcal{A}_3, \quad [\mathcal{A}_1, \mathcal{A}_3] = 0, \quad [\mathcal{A}_2, \mathcal{A}_3] = 0.$$

Thus the two-dimensional subspace $h = span\{\mathcal{A}_1, \mathcal{A}_2\}$ generates the entire algebra by Lie bracketing. We impose a nonholonomic constraint on each node by restricting to the subspace h as follows. For a given $\xi \in h(3)$ define

$$\xi_a = \mathcal{A}_1^\flat(\xi), \quad \xi_c = \mathcal{A}_2^\flat(\xi), \quad \xi_b = \mathcal{A}_3^\flat(\xi). \tag{6.24}$$

Then a left-invariant system on $H(3)$ has the form

$$\dot{g} = g\xi = g\left(\xi_a \mathcal{A}_1 + \xi_c \mathcal{A}_2 + \xi_b \mathcal{A}_3\right).$$

To restrict to the subspace h we require that $\xi_b = 0$. From $\dot{g} = g\xi$ we have

$$\begin{pmatrix} 0 & \dot{a} & \dot{b} \\ 0 & 0 & \dot{c} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \xi_a & \xi_b + a\xi_c \\ 0 & 0 & \xi_c \\ 0 & 0 & 0 \end{pmatrix}$$

and thus

$$\begin{aligned} \xi_a &= \dot{a}, \\ \xi_c &= \dot{c}, \\ \xi_b &= \dot{b} - a\dot{c}. \end{aligned}$$

The nonholonomic constraints for each node are then given by

$$\xi_b^1 = \dot{b}_1 - a_1\dot{c}_1 = 0, \tag{6.25}$$

$$\begin{aligned} \xi_b^2 &= \dot{b}_2 - a_2\dot{c}_2 \\ &= \left(\frac{\partial F_2}{\partial c_{12}} - F_1\right)\dot{c}_{12} + c_{12}\dot{a}_1 + \dot{b}_1 - (F_1 + a_1)\dot{c}_1 = 0 \end{aligned} \tag{6.26}$$

where in the last expression we have used equations (6.20) - (6.22) together with the holonomic constraints for a_{12} and b_{12} in equation (6.23) to express (a_2, b_2, c_2) in terms of the relative configuration c_{12}. The constraint one-forms are then

$$\omega_q^1 = db_1 - a_1 dc_1, \tag{6.27}$$

$$\omega_q^2 = \left(\frac{\partial F_2}{\partial c_{12}} - F_1\right) dc_{12} + db_1 + c_{12} da_1 - (F_1 + a_1) dc_1. \tag{6.28}$$

The constraint distribution is defined by

$$\mathcal{D} = \mathrm{Ker}(\omega_q^1) \cap \mathrm{Ker}(\omega_q^2).$$

The following proposition asserts that under an appropriate assumption this distribution is two-dimensional.

Proposition 6.4.1 *Assume $F_1(0) \neq 0$. Then ω_q^1 and ω_q^2 are linearly independent $\forall q \in Q$.*

Proof If ω_q^1 and ω_q^2 are linearly dependent for some q then there exists a k such that $k\omega_q^1 = \omega_q^2$. Then

$$k(db_1 - a_1 dc_1) = \left(\frac{\partial F_2}{\partial c_{12}} - F_1\right) dc_{12} + db_1 + c_{12} da_1 - (F_1 + a_1) dc_1. \tag{6.29}$$

Rearranging we get

$$(k-1)db_1 - (ka_1 - F_1 - a_1)dc_1 - c_{12} da_1 - \left(\frac{\partial F_2}{\partial c_{12}} - F_1\right) dc_{12} = 0$$

and so we must have all the following met simultaneously

$$k = 1, \tag{6.30}$$

$$a_1(k-1) - F_1 = 0, \tag{6.31}$$

$$c_{12} = 0, \tag{6.32}$$

$$\frac{\partial F_2}{\partial c_{12}} - F_1 = 0. \tag{6.33}$$

Combining equations (6.30), (6.31), and (6.32) we see that in order to have linear dependence we must have $F_1(0) = 0$. This contradicts our assumption and therefore the one-forms are linearly independent. ■

The condition $F_1(0) \neq 0$ is the analog of the nonzero offset condition for the Roller Racer (see page 352 in [47]). The following proposition gives us a basis for $\mathcal{D}_q$.

Proposition 6.4.2 *Assume that $F_1(c_{12}) \neq 0$ for all c_{12}. Then $\mathcal{D}_q$ is spanned by*

$$\xi_Q^1 = \frac{a_1}{F_1}\left(\frac{\partial F_2}{\partial c_{12}} - F_1\right)\frac{\partial}{\partial b_1} + \frac{1}{F_1}\left(\frac{\partial F_2}{\partial c_{12}} - F_1\right)\frac{\partial}{\partial c_1} + \frac{\partial}{\partial c_{12}}, \qquad (6.34)$$

$$\xi_Q^2 = \frac{\partial}{\partial a_1} + \frac{a_1 c_{12}}{F_1}\frac{\partial}{\partial b_1} + \frac{c_{12}}{F_1}\frac{\partial}{\partial c_1}. \qquad (6.35)$$

Proof Let

$$X = x_1\frac{\partial}{\partial a_1} + x_2\frac{\partial}{\partial b_1} + x_3\frac{\partial}{\partial c_1} + x_4\frac{\partial}{\partial c_{12}}.$$

Then if $X \in \mathcal{D}_q$ we must have

$$\omega_q^1(X) = x_2 - a_1 x_3 = 0, \qquad (6.36)$$

$$\omega_q^2(X) = \left(\frac{\partial F_2}{\partial c_{12}} - F_1\right)x_4 + x_2 + c_{12}x_1 - (F_1 + a_1)x_3 = 0. \qquad (6.37)$$

Use equation (6.36) in equation (6.37) to get

$$\left(\frac{\partial F_2}{\partial c_{12}} - F_1\right)x_4 - F_1 x_3 + c_{12}x_1 = 0.$$

Upon rearranging we have

$$x_3 = \frac{1}{F_1}\left[\left(\frac{\partial F_2}{\partial c_{12}} - F_1\right)x_4 + c_{12}x_1\right].$$

Choosing $x_1 = 0, x_4 = 1$ yields

$$x_3 = \frac{1}{F_1}\left(\frac{\partial F_2}{\partial c_{12}} - F_1\right), \quad x_2 = \frac{a_1}{F_1}\left(\frac{\partial F_2}{\partial c_{12}} - F_1\right).$$

Alternatively choosing $x_1 = 1, x_4 = 0$ yields

$$x_3 = \frac{c_{12}}{F_1}, \quad x_2 = \frac{a_1 c_{12}}{F_1}.$$

These choices yield the following pair of tangent vectors.

$$\begin{aligned}
\xi_Q^1 &= \frac{a_1}{F_1}\left(\frac{\partial F_2}{\partial c_{12}} - F_1\right)\frac{\partial}{\partial b_1} + \frac{1}{F_1}\left(\frac{\partial F_2}{\partial c_{12}} - F_1\right)\frac{\partial}{\partial c_1} + \frac{\partial}{\partial c_{12}}, \\
\xi_Q^2 &= \frac{\partial}{\partial a_1} + \frac{a_1 c_{12}}{F_1}\frac{\partial}{\partial b_1} + \frac{c_{12}}{F_1}\frac{\partial}{\partial c_1}.
\end{aligned}$$

ξ_Q^1 and ξ_Q^2 are clearly linearly independent for all q. ■

The assumption that $F_1(c_{12}) \neq 0$ for all c_{12} requires that a_{12} has the same sign for all values of c_{12}. This restriction is similar in spirit to the constraint on the $SE(2)-$Roller Racer in which the joint angle between the two platforms is restricted to a subset of S^1 by the mechanism assembly.

6.4.2 Symmetry of the $H(3)$-Racer

Consider the following action of $H(3)$ on Q.

$$\begin{aligned}
\Phi : G \times Q &\to Q \\
(g, (g_1, c_{12})) &\mapsto (gg_1, c_{12}) \\
(a, b, c, (a_1, b_1, c_1, c_{12})) &\mapsto (a_1 + a, b_1 + b + ac_1, c_1 + c, c_{12}) \qquad (6.38)
\end{aligned}$$

Since $H(3)$ is a matrix Lie group, the exponential map is simply the matrix exponential, that is, for $\xi \in h(3)$,

$$\exp(t\xi) = \begin{pmatrix} 1 & \xi_a t & \xi_b t + \frac{1}{2}\xi_a \xi_c t^2 \\ 0 & 1 & \xi_c t \\ 0 & 0 & 1 \end{pmatrix}.$$

The infinitesimal generator of the given action corresponding to an element $\xi \in h(3)$ is

$$\begin{aligned}\xi_Q(q) &= \left.\frac{d}{dt}\right|_{t=0} \Phi_{\exp(t\xi)}(q) \\ &= \left.\frac{d}{dt}\right|_{t=0} (a_1 + \xi_a t, b_1 + \xi_b t + \frac{1}{2}\xi_a \xi_c t^2 + \xi_a t c_1, c_1 + \xi_c t, c_{12}) \\ &= (\xi_a, \xi_b + \xi_a c_1, \xi_c, 0). \end{aligned} \tag{6.39}$$

Recall that $T_q\mathrm{Orb}(q) = \{\xi_Q(q) | \xi \in \mathfrak{g}\}$. From equation (6.39), then, we have

$$T_q\mathrm{Orb}(q) = \mathrm{Span}\{\frac{\partial}{\partial a_1}, \frac{\partial}{\partial b_1}, \frac{\partial}{\partial c_1}\}. \tag{6.40}$$

From the expression for the basis of $\mathcal{D}_q$ in Proposition 6.4.2 and the basis for $T_q\mathrm{Orb}(q)$ in (6.40) we see that at any $q \in Q$ we have $T_qQ = \mathcal{D} + T_q\mathrm{Orb}(q)$. Define $\mathcal{S}_q = \mathcal{D}_q \cap T_q\mathrm{Orb}(q)$. Since this intersection is nontrivial we are in the principal case as defined in [13]. The following proposition establishes a basis for $\mathcal{S}_q$.

Proposition 6.4.3 *$\mathcal{S}_q$ is given by*

$$\mathcal{S}_q = \mathit{Span}\{\xi_Q^q\} \text{ where } \xi_Q^q = \frac{\partial}{\partial a_1} + \frac{a_1 c_{12}}{F_1}\frac{\partial}{\partial b_1} + \frac{c_{12}}{F_1}\frac{\partial}{\partial c_1}. \tag{6.41}$$

Proof Let $X_q \in \mathcal{S}_q$. Then $X_q \in \mathcal{D}_q$ and so

$$X_q = u_1 \xi_Q^1 + u_2 \xi_Q^2.$$

Similarly $X_q \in T_q\mathrm{Orb}(q)$ and so

$$X_q = v_1 \frac{\partial}{\partial a_1} + v_2 \frac{\partial}{\partial b_1} + v_3 \frac{\partial}{\partial c_1}.$$

Equating these two expressions, using the expressions for ξ_Q^i from Proposition 6.4.2, and rearranging gives

$$\begin{aligned} 0 &= (u_2 - v_1)\frac{\partial}{\partial a_1} + \left(\frac{a_1}{F_1}\left[\frac{\partial F_2}{\partial c_{12}} - F_1\right] u_1 + \frac{a_1 c_{12}}{F_1} u_2 - v_2\right)\frac{\partial}{\partial b_1} \\ &\quad + \left(\frac{1}{F_1}\left[\frac{\partial F_2}{\partial c_{12}} - F_1\right] u_1 + \frac{c_{12}}{F_1} u_2 - v_3\right)\frac{\partial}{\partial c_1} + u_1 \frac{\partial}{\partial c_{12}}. \end{aligned}$$

For this equation to hold, each coefficient must be zero. The resulting set of equations can be expressed as the following linear system

$$\begin{pmatrix} 0 & 1 & -1 & 0 & 0 \\ \frac{a_1}{F_1}\left[\frac{\partial F_2}{\partial c_{12}} - F_1\right] & \frac{a_1 c_{12}}{F_1} & 0 & -1 & 0 \\ \frac{1}{F_1}\left[\frac{\partial F_2}{\partial c_{12}} - F_1\right] & \frac{c_{12}}{F_1} & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{6.42}$$

The matrix is full rank and thus $\mathcal{S}_q$ is one-dimensional. To satisfy the system of equations we have

$$\begin{aligned} u_1 &= 0, \\ v_1 &= u_2, \\ v_2 &= \frac{a_1}{F_1}\left(\frac{\partial F_2}{\partial c_{12}} - F_1\right) u_1 + \frac{a_1 c_{12}}{F_1} u_2 = \frac{a_1 c_{12}}{F_1} u_2, \\ v_3 &= \frac{1}{F_1}\left(\frac{\partial F_2}{\partial c_{12}} - F_1\right) u_1 + \frac{c_{12}}{F_1} u_2 = \frac{c_{12}}{F_1} u_2. \end{aligned}$$

Thus

$$X_q = \left[\frac{\partial}{\partial a_1} + \frac{a_1 c_{12}}{F_1}\frac{\partial}{\partial b_1} + \frac{c_{12}}{F_1}\frac{\partial}{\partial c_1}\right] u_2$$

for arbitrary u_2. ■

We note in passing that the basis element for $\mathcal{S}_q$ is equal to the second basis vector for $\mathcal{D}_q$, that is $\xi_Q^q = \xi_Q^2$.

Since $\xi_Q^q \in T_q\mathrm{Orb}(q)$, it is an infinitesimal generator corresponding to some $\xi^q \in h(3)$. The following proposition gives us that Lie algebra element.

Proposition 6.4.4 *The Lie algebra element $\xi^q \in h(3)$ corresponding to the infinitesimal generator ξ_Q^q is*

$$\xi^q = \mathcal{A}_1 + \frac{c_{12}}{F_1}\mathcal{A}_2 + \left(\frac{a_1 c_{12}}{F_1} - c_1\right)\mathcal{A}_3. \tag{6.43}$$

Proof Comparing equation (6.39) to equation (6.41) we have that

$$\xi_a^q = 1, \quad \xi_b^q + \xi_a^q c_1 = \frac{a_1 c_{12}}{F_1}, \quad \xi_c^q = \frac{c_{12}}{F_1}.$$

■

Finally, we establish that the constraint distribution is invariant under the group action Φ.

Proposition 6.4.5 *The non-holonomic constraints in equations* (6.25,6.26) *are invariant under the action* Φ *given in equation* (6.38).

Proof Let $q = (a_1, b_1, c_1, c_{12})$. Under the action of Φ we have $q \mapsto \bar{q} = (\bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{c}_{12})$ with

$$\left.\begin{array}{rcl} \bar{a}_1 &=& a_1 + a, \\ \bar{b}_1 &=& b_1 + b + ac_1, \\ \bar{c}_1 &=& c_1 + c, \\ \bar{c}_{12} &=& c_{12}, \end{array}\right\} \Rightarrow \begin{array}{rcl} \dot{\bar{a}}_1 &=& \dot{a}_1, \\ \dot{\bar{b}}_1 &=& \dot{b}_1 + a\dot{c}_1, \\ \dot{\bar{c}}_1 &=& \dot{c}_1, \\ \dot{\bar{c}}_{12} &=& \dot{c}_{12}. \end{array}$$

Using these in the first constraint, equation (6.25), we have

$$\dot{\bar{b}}_1 - \bar{a}_1\dot{\bar{c}}_1 = \dot{b}_1 + a\dot{c}_1 - (a_1 + a)\dot{c}_1 = \dot{b}_1 - a_1\dot{c}_1.$$

From the second constraint, equation (6.26), we have

$$\begin{aligned} &\left(\frac{\partial F_2(\bar{c}_{12})}{\partial \bar{c}_{12}} - F_1(\bar{c}_{12})\right)\dot{\bar{c}}_{12} + \dot{\bar{b}}_1 + \dot{\bar{a}}_1\bar{c}_{12} - (F_1(\bar{c}_{12}) + \bar{a}_1)\dot{\bar{c}}_1 \\ &\quad = \left(\frac{\partial F_2(c_{12})}{\partial c_{12}} - F_1(c_{12})\right)\dot{c}_{12} + \dot{b}_1 + a\dot{c}_1 + \dot{a}_1 c_{12} - (F_1(c_{12}) + a_1 + a)\dot{c}_1 \\ &\quad = \left(\frac{\partial F_2(c_{12})}{\partial c_{12}} - F_1(c_{12})\right)\dot{c}_{12} + \dot{b}_1 + \dot{a}_1 c_{12} - (F_1(c_{12}) + a_1)\dot{c}_1. \end{aligned}$$

■

6.4.3 An invariant inner product on T_qQ

We will define a Lagrangian on TQ which is invariant under the group action Φ by first defining an invariant metric. Given the basis for $h(3)$ in equation (6.17), an element $\xi \in h(3)$ may be expressed as $\xi = \xi_a \mathcal{A}_1 + \xi_c \mathcal{A}_2 + \xi_b \mathcal{A}_3$. We can then identify ξ with the triple (ξ_a, ξ_b, ξ_c) and in the following we will denote both the matrix form and the triple as ξ with the meaning of the notation clear from the context.

Consider now a left-invariant system on $H(3)$ defined by $\dot{g} = g\xi$ and assume there is a symmetric, positive definite inner product on $h(3)$ given by $\mathbf{K} : h(3) \times h(3) \to \mathbb{R}$ where

$$\mathbf{K}(\xi_1, \xi_2) = \xi_1^T K \xi_2$$

with K a positive definite, symmetric 3×3 matrix. Here ^{T} denotes transpose. We then define a left-invariant inner product, $\bar{M}$, on $TH(3)$ by

$$\bar{M}_g(v_1, v_2) = \mathbf{K}(g^{-1}v_1, g^{-1}v_2), \quad v_1, v_2 \in T_gH(3).$$

Define an inner product on TQ, the tangent bundle of the $H(3)-$Racer, as follows. First consider two copies of $H(3)$ and let $\mathbf{K}_1, \mathbf{K}_2$ be inner products on each copy of the Lie algebra $h(3)$ respectively. Define an inner product, $\widetilde{M}$, on $T(H(3) \times H(3))$ by

$$\widetilde{M}_{g_1,g_2}((v_1, v_2), (u_1, u_2)) = \mathbf{K}_1(g_1^{-1}v_1, g_1^{-1}u_1) + \mathbf{K}_2(g_2^{-1}v_2, g_2^{-1}u_2)$$

where $v_1, u_1 \in T_{g_1}H(3)$ and $v_2, u_2 \in T_{g_2}H(3)$. Consider now $v \in T_qQ$ given by $v = (v^{a_1}, v^{b_1}, v^{c_1}, v^{c_{12}})$. Define

$$\begin{aligned} v_{g_1} &= (v^{a_1}, v^{b_1}, v^{c_1}), \\ v_{g_{12}} &= \left(\frac{\partial F_1}{\partial c_{12}} v^{c_{12}}, \frac{\partial F_2}{\partial c_{12}} v^{c_{12}}, v^{c_{12}} \right) \end{aligned}$$

where in the second equation we have used the holonomic constraints in equation (6.23). Solving equation (6.19) for g_2 yields $g_2 = g_1 g_{12}$ and thus

$$\dot{g}_2 = \dot{g}_1 g_{12} + g_1 \dot{g}_{12}.$$

We can now define an inner product on $T_q Q$ by

$$\begin{aligned}
M_q(v_1, v_2) &= \mathbf{K}_1(g_1^{-1} v_{1_{g_1}}, g_1^{-1} v_{2_{g_1}}) \\
&\quad + \mathbf{K}_2(g_{12}^{-1} g_1^{-1}[v_{1_{g_1}} g_{12} + g_1 v_{1_{g_{12}}}], g_{12}^{-1} g_1^{-1}[v_{2_{g_1}} g_{12} + g_1 v_{2_{g_{12}}}]) \\
&= \mathbf{K}_1(g_1^{-1} v_{1_{g_1}}, g_1^{-1} v_{2_{g_1}}) \\
&\quad + \mathbf{K}_2(\mathrm{Ad}_{g_{12}^{-1}}[g_1^{-1} v_{1_{g_1}} + v_{1_{g_{12}}} g_{12}^{-1}], \mathrm{Ad}_{g_{12}^{-1}}[g_1^{-1} v_{2_{g_1}} + v_{2_{g_{12}}} g_{12}^{-1}]). \quad (6.44)
\end{aligned}$$

We have the following proposition.

Proposition 6.4.6 *M is invariant under the action Φ_g defined in equation* (6.38).

Proof

$$\begin{aligned}
M_{\Phi_g(q)}(T_q\Phi_g v_1, T_q\Phi_g v_2) &= \mathbf{K}_1((gg_1)^{-1} g v_{1_{g_1}}, (gg_1)^{-1} g v_{2_{g_1}}) \\
&\quad + \mathbf{K}_2(\mathrm{Ad}_{g_{12}^{-1}}[(gg_1)^{-1} g v_{1_{g_1}} + v_{1_{g_{12}}} g_{12}^{-1}], \mathrm{Ad}_{g_{12}^{-1}}[(gg_1)^{-1} g v_{2_{g_1}} + v_{2_{g_{12}}} g_{12}^{-1}]) \\
&= \mathbf{K}_1(g_1^{-1} v_{1_{g_1}}, g_1^{-1} v_{2_{g_1}}) \\
&\quad + \mathbf{K}_2(\mathrm{Ad}_{g_{12}^{-1}}[g_1^{-1} v_{1_{g_1}} + v_{1_{g_{12}}} g_{12}^{-1}], \mathrm{Ad}_{g_{12}^{-1}}[g_1^{-1} v_{2_{g_1}} + v_{2_{g_{12}}} g_{12}^{-1}]) \\
&= M_1(v_1, v_2).
\end{aligned}$$

■

A simple but lengthy calculation shows that the inner product may be expressed as (for $v_1, v_2 \in T_q Q$)

$$M_q(v_1, v_2) = v_1^T M(q) v_2 \quad (6.45)$$

where $M(q)$ is a symmetric, positive definite 4×4 matrix defined by

$$
\begin{aligned}
m_{11} &= \left(k_{1_{11}} + k_{2_{11}} + c_{12}^2 k_{2_{22}} + 2c_{12}k_{2_{12}}\right), \\
m_{22} &= \left(k_{1_{22}} + k_{2_{22}}\right), \\
m_{33} &= \left(a_1^2 k_{1_{22}} + k_{1_{33}} - 2a_1 k_{1_{23}} + (F_1 + a_1)^2 k_{2_{22}} + k_{2_{33}} - 2(F_1 + a_1)k_{2_{23}}\right), \\
m_{44} &= \left(\left[\frac{\partial F_1}{\partial c_{12}}\right]^2 k_{2_{11}} + \left[\frac{\partial F_2}{\partial c_{12}} - F_1\right]^2 k_{2_{22}} + k_{2_{33}} + 2\frac{\partial F_1}{\partial c_{12}}\left[\frac{\partial F_2}{\partial c_{12}} - F_1\right] k_{2_{12}}\right. \\
&\quad \left. + 2\frac{\partial F_1}{\partial c_{12}} k_{2_{12}} + 2\left[\frac{\partial F_2}{\partial c_{12}} - F_1\right] k_{2_{23}}\right), \\
m_{12} &= \left(k_{1_{12}} + c_{12}k_{2_{22}} + k_{2_{12}}\right), \\
m_{13} &= \left(k_{1_{13}} - a_1 k_{1_{12}} - c_{12}\left[F_1 + a_1\right] k_{2_{22}} - \left[F_1 + a_1\right] k_{2_{12}} + k_{2_{13}} + c_{12}k_{2_{23}}\right), \\
m_{14} &= \left(\frac{\partial F_1}{\partial c_{12}} k_{2_{11}} + c_{12}\left[\frac{\partial F_2}{\partial c_{12}} - F_1\right] k_{2_{22}} + \left[\frac{\partial F_2}{\partial c_{12}} - F_1\right] k_{2_{12}}\right. \\
&\quad \left. + c_{12}\frac{\partial F_1}{\partial c_{12}} k_{2_{12}} + k_{2_{13}} + c_{12}k_{2_{23}}\right), \\
m_{23} &= \left(k_{1_{23}} - a_1 k_{1_{22}} - [F_1 + a_1]k_{2_{22}} + k_{2_{23}}\right), \\
m_{24} &= \left(\left[\frac{\partial F_2}{\partial c_{12}} - F_1\right] k_{2_{22}} + \frac{\partial F_1}{\partial c_{12}} k_{2_{12}} + k_{2_{23}}\right), \\
m_{34} &= \left(k_{2_{33}} - \left[\frac{\partial F_2}{\partial c_{12}} - F_1\right]\left[F_1 + a_1\right] k_{2_{22}} - \frac{\partial F_1}{\partial c_{12}}\left[F_1 + a_1\right] k_{2_{12}}\right. \\
&\quad \left. + \frac{\partial F_1}{\partial c_{12}} k_{2_{13}} + \left[\frac{\partial F_2}{\partial c_{12}} - 2F_1 - a_1\right] k_{2_{23}}\right)
\end{aligned}
$$

where $k_{l_{ij}}$ is the ij^{th} element of the matrix K_l.

6.4.4 A Lagrangian and the Lagrange-D'Alembert equations of motion

Given the inner product on T_qQ in equation (6.44), we take as a Lagrangian the function $L : TQ \to \mathbb{R}$ defined by

$$L(q, \dot{q}) = \frac{1}{2} M_q(\dot{q}, \dot{q}). \tag{6.46}$$

The explicit form of this function depends on the holonomic constraints in equation (6.23) and the inner products defined by $\mathbf{K}_1, \mathbf{K}_2$. For simplicity, in the remainder of this chapter we take

$$K_1 = K_2 = k\mathbb{1}, \qquad F_1(c_{12}) = l_1, \qquad F_2(c_{12}) = l_2 \tag{6.47}$$

for some positive constants k, l_1, l_2. The Lagrangian in equation (6.46) is then given by

$$\begin{aligned} L(q,\dot{q}) \quad = \quad & \frac{k}{2}\Big[(2+c_{12}^2)\dot{a}_1^2 + 2\dot{b}_1^2 + (2+a_1^2+[l_1+a_1]^2)\dot{c}_1^2 + (1+l_1^2)\dot{c}_{12}^2 + 2c_{12}\dot{a}_1\dot{b}_1 \\ & -2c_{12}(l_1+a_1)\dot{a}_1\dot{c}_1 - 2c_{12}l_1\dot{a}_1\dot{c}_{12} - 2(l_1+2a_1)\dot{b}_1\dot{c}_1 - 2l_1\dot{b}_1\dot{c}_{12} \\ & +2(1+l_1[l_1+a_1])\dot{c}_1\dot{c}_{12}\Big]. \end{aligned} \tag{6.48}$$

To derive the equations of motion we use the Lagrange-D'Alembert principle for constrained systems (see, e.g. [15] or [102]) which in this setting states that

$$\left[\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}(q,v) - \frac{\partial L}{\partial q}(q,v)\right]\cdot u = \alpha \cdot u \tag{6.49}$$

for $v, u \in \mathcal{D}_q \subset T_qQ$. Here u is an arbitrary test tangent vector satisfying the constraints and α is a vector of one-forms representing the applied forces.

Proposition 6.4.7 *The Lagrange-D'Alembert equations of motion for the $H(3)-$Racer are*

$$2k\left[1+\frac{c_{12}^2}{l_1^2}\right]\ddot{a}_1 - k\frac{c_{12}}{l_1}\ddot{c}_{12} + 2k\frac{c_{12}}{l_1^2}\dot{a}_1\dot{c}_{12} \quad = \quad \alpha_1 + \frac{a_1c_{12}}{l_1}\alpha_2 + c_{12}l_1\alpha_3 \tag{6.50}$$

$$\frac{-2kc_{12}}{l_1}\ddot{a}_1 + 2k\ddot{c}_{12} - 2\frac{k}{l_1}\dot{a}_1\dot{c}_{12} \quad = \quad \alpha_4 - a_1\alpha_2 - \alpha_3 \tag{6.51}$$

together with the constraint equations in (6.25) *and* (6.26).

Proof From equation (6.48) we have

$$\frac{\partial L}{\partial \dot q} = \begin{pmatrix} k[(2+c_{12}^2)\dot a_1 + c_{12}\dot b_1 - c_{12}(l_1+a_1)\dot c_1 - c_{12}l_1\dot c_{12}] \\ k[2\dot b_1 + c_{12}\dot a_1 - (l_1+2a_1)\dot c_1 - l_1\dot c_{12}] \\ k[(2+a_1^2+[l_1+a_1]^2)\dot c_1 - c_{12}(l_1+a_1)\dot a_1 \\ \qquad -(l_1+2a_1)\dot b_1 + (1+l_1[l_1+a_1])\dot c_{12}] \\ k[(1+l_1^2)\dot c_{12} - c_{12}l_1\dot a_1 - l_1\dot b_1 + (1+l_1[l_1+a_1])\dot c_1] \end{pmatrix} \tag{6.52}$$

and

$$\frac{\partial L}{\partial q} = \begin{pmatrix} k\left[(l_1+2a_1)\dot c_1^2 - c_{12}\dot a_1\dot c_1 - 2\dot b_1\dot c_1 + l_1\dot c_1\dot c_{12}\right] \\ 0 \\ 0 \\ k\left[c_{12}\dot a_1^2 + \dot a_1\dot b_1 - (l_1+a_1)\dot a_1\dot c_1 - l_1\dot a_1\dot c_{12}\right] \end{pmatrix}. \tag{6.53}$$

Inserting these into equation (6.49) we have

$$\begin{aligned} & k\left[(2+c_{12}^2)\dot v_{a_1} + c_{12}\dot v_{b_1} - c_{12}(l_1+a_1)\dot v_{c_1} - l_1c_{12}\dot v_{c_{12}} - c_{12}v_{a_1}v_{c_1}\right. \\ & \qquad +2c_{12}v_{a_1}v_{c_{12}} + v_{b_1}v_{c_{12}} - (l_1+a_1)v_{c_1}v_{c_{12}} - l_1v_{c_{12}}^2 - (l_1+2a_1)v_{c_1}^2 \\ & \qquad \left.+c_{12}v_{a_1}v_{c_1} + 2v_{b_1}v_{c_1} - l_1v_{c_1}v_{c_{12}}\right] u_{a_1} \\ & +k\left[c_{12}\dot v_{a_1} + 2\dot v_{b_1} - (l_1+2a_1)\dot v_{c_1} - l_1\dot v_{c_{12}} - 2v_{a_1}v_{c_1} + v_{a_1}v_{c_{12}}\right] u_{b_1} \\ & +k\left[-c_{12}(l_1+a_1)\dot v_{a_1} - (l_1+2a_1)\dot v_{b_1} + (2+a_1^2+[l_1+a_1]^2)\dot v_{c_1}\right. \\ & \qquad +(1+l_1[l_1+a_1])\dot v_{c_{12}} - 2v_{a_1}v_{b_1} + 2(l_1+2a_1)v_{a_1}v_{c_1} - a_1v_{a_1}v_{c_{12}} \\ & \qquad \left.-c_{12}v_{a_1}^2\right] u_{c_1} \\ & +k\left[-l_1c_{12}\dot v_{a_1} - l_1\dot v_{b_1} + (1+l_1[l_1+a_1])\dot v_{c_1} + (1+l_1^2)\dot v_{c_{12}} + l_1v_{a_1}v_{c_1}\right. \\ & \qquad \left.-l_1v_{a_1}v_{c_{12}} - c_{12}v_{a_1}^2 - v_{a_1}v_{b_1} + (l_1+a_1)v_{a_1}v_{c_1} + l_1v_{a_1}v_{c_{12}}\right] u_{c_{12}} \\ = \;& \alpha_1u_{a_1} + \alpha_2u_{b_1} + \alpha_3u_{c_1} + \alpha_4u_{c_{12}}. \end{aligned} \tag{6.54}$$

Since v and u lie in $\mathcal{D}_q$ they must satisfy the nonholonomic constraints in equations (6.25) and (6.26). Therefore

$$v_{b_1} = a_1 v_{c_1}, \qquad c_{12} v_{a_1} = l_1(v_{c_1} + v_{c_{12}})$$

and

$$u_{b_1} = a_1 u_{c_1}, \qquad c_{12} u_{a_1} = l_1(u_{c_1} + u_{c_{12}}).$$

Differentiating the nonholonomic constraints on v yields

$$\dot{v}_{b_1} = a_1 \dot{v}_{c_1} + v_{a_1} v_{c_1}, \qquad c_{12}\dot{v}_{a_1} + v_{a_1} v_{c_{12}} = l_1(\dot{v}_{c_1} + \dot{v}_{c_{12}}).$$

Inserting these constraint equations into equation (6.54) and simplifying yields

$$\begin{aligned}\left(\alpha_1 + \frac{a_1 c_{12}}{l_1}\alpha_2 + \frac{c_{12}}{l_1}\alpha_3\right) u_{a_1} + (\alpha_4 - a_1\alpha_2 - \alpha_3) u_{c_{12}} & \\ = \quad & k u_{a_1}\left[\left(2 + 2\frac{c_{12}^2}{l_1^2}\right)\dot{v}_{a_1} - \frac{c_{12}}{l_1}\dot{v}_{c_{12}} + 2\frac{c_{12}}{l_1^2} v_{a_1} v_{c_{12}}\right] \\ & -2k u_{c_{12}}\left[\frac{c_{12}}{l_1}\dot{v}_{a_1} - \dot{v}_{c_{12}} + \frac{1}{l_1} v_{a_1} v_{c_{12}}\right].\end{aligned}$$

Since u is arbitrary, the Lagrange-D'Alembert equations of motion are

$$\begin{aligned}2k\left[1 + \frac{c_{12}^2}{l_1^2}\right]\ddot{a}_1 - k\frac{c_{12}}{l_1}\ddot{c}_{12} + 2k\frac{c_{12}}{l_1^2}\dot{a}_1\dot{c}_{12} &= \alpha_1 + \frac{a_1 c_{12}}{l_1}\alpha_2 + \frac{c_{12}}{l_1}\alpha_3, \\ \frac{-2k c_{12}}{l_1}\ddot{a}_1 + 2k\ddot{c}_{12} - 2\frac{k}{l_1}\dot{a}_1\dot{c}_{12} &= \alpha_4 - a_1\alpha_2 - \alpha_3.\end{aligned}$$

■

These equations do not easily yield insight in the behavior of the $H(3)-$Racer. The purpose of the method developed in [13] is to separate these dynamics into the shape dynamics, the nonholonomic momentum, and the group dynamics and in this way to understand the effect of shape changes on the group dynamics. In the following sections we carry out this approach for the $H(3)-$Racer.

6.4.5 The nonholonomic momentum

Recall the definition of the nonholonomic momentum in equation (6.5). Inserting equations (6.52) and (6.41) into (6.5) and simplifying we have

$$p = k\left[2\dot{a}_1 + \frac{2c_{12}}{l_1}\dot{c}_1 + \frac{c_{12}}{l_1}\dot{c}_{12}\right]. \tag{6.55}$$

Using the nonholonomic constraints in equations (6.25) and (6.26) together with the particular forms of the holonomic constraints from equation (6.47), p can be rewritten as

$$p = 2k\left[\left(\frac{l_1^2 + c_{12}^2}{l_1^2}\right)\dot{a}_1 - \frac{c_{12}}{2l_1}\dot{c}_{12}\right]. \tag{6.56}$$

To reconstruct the dynamics in the group from a path in the shape space, we need the momentum expressed purely as a function of the shape variable c_{12}. This is determined by the momentum equation. We have the following proposition.

Proposition 6.4.8 *The momentum equation for the $H(3)-$Racer is*

$$\dot{p} = \left[\frac{c_{12}}{(l_1^2 + c_{12}^2)}\right]\dot{c}_{12}p - \left[\frac{kl_1}{(l_1^2 + c_{12}^2)}\right]\dot{c}_{12}^2. \tag{6.57}$$

Proof From the form of ξ^q in equation (6.43) and the particular choice for the holonomic constraints in equation (6.47) we have

$$\left[\frac{d}{dt}\xi^q\right]_Q = \begin{pmatrix} 0 \\ \frac{\dot{a}_1 c_{12} + a_1\dot{c}_{12}}{l_1} - \dot{c}_1 \\ \frac{\dot{c}_{12}}{l_1} \\ 0 \end{pmatrix}$$

where we have used the form of the infinitesimal generator in equation (6.39). Using the above in the general momentum equation, (6.6), we have

$$\dot{p} = \frac{\partial L}{\partial \dot{q}^i}\left[\frac{d}{dt}\left(\xi^{q(t)}\right)\right]_Q^i$$

$$= \quad k\left[\frac{2c_{12}}{l_1}\dot{a}_1\dot{b}_1 - \dot{b}_1\dot{c}_{12} - 2\dot{b}_1\dot{c}_1 + \frac{c_{12}^2}{l_1}\dot{a}_1^2 - 2c_{12}\dot{a}_1\dot{c}_{12} - 2c_{12}\dot{a}_1\dot{c}_1 - 2\frac{a_1c_{12}}{l_1}\dot{a}_1\dot{c}_1 \right.$$
$$\left. +2\left(\frac{1}{l_1} + l_1 + a_1\right)\dot{c}_1\dot{c}_{12} + (l_1 + 2a_1)\dot{c}_1^2 + \left(\frac{1}{l_1} + l_1\right)\dot{c}_{12}^2\right].$$

Using the nonholonomic constraints, equations (6.25) and (6.26), this can be simplified to

$$\dot{p} = k\left[\frac{2c_{12}}{l_1^2}\dot{a}_1\dot{c}_{12} - \frac{1}{l_1}\dot{c}_{12}^2\right].$$

Solving equation (6.56) for $\dot{a}_1$ and inserting the result into the previous equation yields

$$\begin{aligned}\dot{p} &= k\left[\frac{2c_{12}\dot{c}_{12}}{l_1^2}\left(\frac{l_1^2 p}{2k(l_1^2 + c_{12}^2)} + \frac{l_1c_{12}\dot{c}_{12}}{2(l_1^2 + c_{12}^2)}\right) - \frac{1}{l_1}\dot{c}_{12}^2\right] \\ &= \left[\frac{c_{12}}{(l_1^2 + c_{12}^2)}\right]\dot{c}_{12}p - \left[\frac{kl_1}{(l_1^2 + c_{12}^2)}\right]\dot{c}_{12}^2.\end{aligned}$$

■

The following proposition gives the solution to the momentum equation.

Proposition 6.4.9 *The solution to the momentum equation is*

$$p(t) = \left[\frac{l_1^2 + c_{12}^2(t)}{l_1^2 + c_{12}^2(t_0)}\right]^{\frac{1}{2}} p(t_0) - kl_1\sqrt{l_1^2 + c_{12}^2(t)}\int_{t_0}^{t}\frac{\dot{c}_{12}^2(\tau)}{(l_1^2 + c_{12}^2(\tau))^{\frac{3}{2}}}d\tau. \qquad (6.58)$$

Proof The momentum equation in (6.57) is a scalar, linear, time-varying ordinary differential equation and thus the general solution is given by the variation of constants formula

$$p(t) = \Phi(t, t_0)p(t_0) - \int_{t_0}^{t}\Phi(t,\tau)\frac{kl_1\dot{c}_{12}^2(\tau)}{(l_1^2 + c_{12}^2(\tau))}d\tau$$

where the transition matrix satisfies

$$\frac{d}{dt}\Phi(t, t_0) = \left[\frac{c_{12}(t)\dot{c}_{12}(t)}{(l_1^2 + c_{12}^2(t))}\right]\Phi(t, t_0).$$

Solving this equation for Φ yields

$$\Phi(t,t_0) = \exp\left(\int_{t_0}^{t} \frac{c_{12}(t)\dot{c}_{12}(t)}{(l_1^2 + c_{12}^2(t))} dt\right) = \exp\left(\int_{c_{12}(t_0)}^{c_{12}(t)} \frac{c_{12}}{(l_1^2 + c_{12}^2)} dc_{12}\right).$$

The integral in the exponential evaluates to

$$\int_{c_{12}(t_0)}^{c_{12}(t)} \frac{c_{12}}{(l_1^2 + c_{12}^2)} dc_{12} = \frac{1}{2}\ln(l_1^2 + c_{12}^2)\Big|_{c_{12}(t_0)}^{c_{12}(t)} = \ln\left[\frac{l_1^2 + c_{12}^2(t)}{l_1^2 + c_{12}^2(t_0)}\right]^{\frac{1}{2}}$$

and therefore

$$\Phi(t,t_0) = \left[\frac{l_1^2 + c_{12}^2(t)}{l_1^2 + c_{12}^2(t_0)}\right]^{\frac{1}{2}}.$$

The form of $p(t)$ in equation (6.58) then follows. ■

From this proposition we have the following corollary.

Corollary 6.4.10 *Assume $l_1 > 0$. If $p(t_0) \leq 0$ then $p(t) \leq 0$ for all $t \geq t_0$.*

6.4.6 The reduced Euler-Lagrange equations

In general, the equations of motion for the reduced system (the shape dynamics together with the nonholonomic momentum equation) are determined through Lagrangian reduction (see [13]). Here we obtain the reduced shape equations by directly manipulating the overall Lagrange-D'Alembert equations of motion in (6.50) and (6.51).

Proposition 6.4.11 *Assume there are no applied forces in the group directions. Then the equation of motion for the shape variable c_{12} is given by*

$$\ddot{c}_{12} = \left[\frac{l_1^3 p \dot{c}_{12}}{k(2l_1^4 + 3l_1^2 c_{12}^2 + c_{12}^4)}\right] + \left[\frac{2l_1^2 c_{12}\dot{c}_{12}^2}{(2l_1^4 + 3l_1^2 c_{12}^2 + c_{12}^4)}\right] + \left[\frac{(l_1^2 + c_{12}^2)}{k(2l_1^2 + c_{12}^2)}\right]\alpha_4. \quad (6.59)$$

Proof Rearranging equations (6.50) and (6.51) yields

$$\begin{aligned} \ddot{a}_1 &= \left[\frac{l_1 c_{12}}{2(l_1^2 + c_{12}^2)}\right]\ddot{c}_{12} - \left[\frac{c_{12}}{l_1^2 + c_{12}^2}\right]\dot{a}_1\dot{c}_{12} + \left[\frac{l_1(l_1\alpha_1 + a_1 c_{12}\alpha_2 + c_{12}\alpha_3)}{2k(l_1^2 + c_{12}^2)}\right], \\ \ddot{c}_{12} &= \frac{c_{12}}{l_1}\ddot{a}_1 + \frac{1}{l_1}\dot{a}_1\dot{c}_{12} + \frac{1}{2k}(\alpha_4 - a_1\alpha_2 - \alpha_3). \end{aligned}$$

Inserting the first of these into the second and simplifying we find

$$\ddot{c}_{12} = \left[\frac{2l_1}{2l_1^2 + c_{12}^2}\right] \dot{a}_1 \dot{c}_{12} + \left[\frac{(l_1 c_{12}\alpha_1 - l_1^2(a_1\alpha_2 + \alpha_3) + (l_1^2 + c_{12}^2)\alpha_4)}{k(2l_1^2 + c_{12}^2)}\right].$$

Solving equation (6.56) for $\dot{a}_1$ and using the resulting expression in the above equation yields

$$\begin{aligned}\ddot{c}_{12} &= \left[\frac{l_1^3}{k(2l_1^4 + 3l_1^2 c_{12}^2 + c_{12}^4)}\right] p\dot{c}_{12} + \left[\frac{2l_1^2 c_{12}}{(2l_1^4 + 3l_1^2 c_{12}^2 + c_{12}^4)}\right] \dot{c}_{12}^2 \\ &\quad + \left[\frac{(l_1 c_{12}\alpha_1 - l_1^2(a_1\alpha_2 + \alpha_3) + (l_1^2 + c_{12}^2)\alpha_4)}{k(2l_1^2 + c_{12}^2)}\right].\end{aligned}$$

Assuming there are no applied forces in the group directions, we set $\alpha_1 = \alpha_2 = \alpha_3 = 0$. This yields equation (6.59). ■

We now introduce a static state feedback linearizing control law for the reduced dynamics.

Proposition 6.4.12 *Under the control law*

$$\alpha_4 = \left[\frac{k(2l_1^2 + c_{12}^2)}{(l_1^2 + c_{12}^2)}\right] \left[u - \frac{l_1^3 p\dot{c}_{12}}{k(2l_1^4 + 3l_1^2 c_{12}^4)} - \frac{2l_1^2 c_{12}\dot{c}_{12}^2}{2l_1^4 + 3l_1^2 c_{12}^2 + c_{12}^4}\right] \tag{6.60}$$

the reduced dynamics has the form

$$\ddot{c}_{12} = u, \tag{6.61}$$

$$\dot{p} = \left[\frac{c_{12}}{(l_1^2 + c_{12}^2)}\right] \dot{c}_{12} p - \left[\frac{kl_1}{(l_1^2 + c_{12}^2)}\right] \dot{c}_{12}^2. \tag{6.62}$$

Proof The momentum equation was found previously in Proposition 6.4.8. The form of the shape dynamics is immediate upon inserting the control law of equation (6.60) into equation (6.59). ■

From equation (6.61) we see that so long as the nonholonomic momentum can be measured, the shape is fully controllable.

6.4.7 The nonholonomic connection and reconstruction

Let $\mathcal{H}_q$ be the orthogonal complement to $\mathcal{S}_q$ in $\mathcal{D}_q$ with respect to the metric M defined in equation (6.44). From Definition 6.2.6, $\mathcal{H}_q$ is the horizontal space for the nonholonomic connection. The following propositions describe this connection for the $H(3)-$Racer.

Proposition 6.4.13 *The space $\mathcal{H}_q$ is given by*

$$\mathcal{H}_q = span\left\{\frac{\partial}{\partial a_1} - \left[\frac{2a_1 l_1^2 + a_1 c_{12}^2}{l_1 c_{12}}\right]\frac{\partial}{\partial b_1} - \left[\frac{2l_1^2 + c_{12}^2}{l_1 c_{12}}\right]\frac{\partial}{\partial c_1} + \left[\frac{2(l_1^2 + c_{12}^2)}{l_1 c_{12}}\right]\frac{\partial}{\partial c_{12}}\right\}. \tag{6.63}$$

Proof Since $dim(\mathcal{D}_q) = 2$, $dim(\mathcal{S}_q) = 1$, and $\mathcal{D}_q = \mathcal{S}_q \oplus \mathcal{H}_q$, we have that $dim(\mathcal{H}_q) = 1$. Let $\xi^q_{\mathcal{H}} \in \mathcal{H}_q \subset \mathcal{D}_q$. Since $\xi^q_{\mathcal{H}}$ is in $\mathcal{D}_q$ we can write $\xi^q_{\mathcal{H}} = u_1\xi^1_Q + u_2\xi^2_Q$. Using the expressions for ξ^1_Q, ξ^2_Q from Proposition (6.4.2) we have

$$\xi^q_H = u_2\frac{\partial}{\partial a_1} + \left[\frac{a_1 c_{12}}{l_1}u_2 - a_1 u_1\right]\frac{\partial}{\partial b_1} + \left[\frac{c_{12}}{l_1}u_2 - u_1\right]\frac{\partial}{\partial c_1} + u_1\frac{\partial}{\partial c_{12}}. \tag{6.64}$$

Since $\mathcal{H}_q$ is the orthogonal complement to $\mathcal{S}_q$, ξ^q_H must be orthogonal to every element of $\mathcal{S}_q$ and therefore we must have $M_q(\xi^q_{\mathcal{H}}, \xi^q_Q) = 0$. This yields

$$\begin{aligned} 0 \;=\; & k(2 + c_{12}^2)u_2 + 2k\left[\frac{a_1 c_{12}}{l_1}u_2 - a_1 u_1\right]\left[\frac{a_1 c_{12}}{l_1}\right] \\ & + k(2 + a_1^2 + (l_1 + a_1)^2)\left[\frac{c_{12}}{l_1}u_2 - u_1\right]\left[\frac{c_{12}}{l_1}\right] + kc_{12}\left[2\frac{a_1 c_{12}}{l_1}u_2 - a_1 u_1\right] \\ & - k(l_1 + a_1)c_{12}\left[2\frac{c_{12}}{l_1}u_2 - u_1\right] - kl_1 c_{12} u_1 - 2k(l_1 + 2a_1)\left[\frac{c_{12}}{l_1}u_1 - u_1\right]\frac{a_1 c_{12}}{l_1} \\ & - kl_1\frac{a_1 c_{12}}{l_1}u_1 + k(1 + l_1(l_1 + a_1))\frac{c_{12}}{l_1}u_1. \end{aligned}$$

This expression simplifies to

$$u_1 = \left[\frac{2(l_1^2 + c_{12}^2)}{l_1 c_{12}}\right]u_2$$

and therefore

$$\xi_H^q = \left[\frac{\partial}{\partial a_1} - \left(\frac{2a_1 l_1^2 + a_1 c_{12}^2}{l_1 c_{12}}\right)\frac{\partial}{\partial b_1} - \left(\frac{2l_1^2 + c_{12}^2}{l_1 c_{12}}\right)\frac{\partial}{\partial c_1} + \left(\frac{2(l_1^2 + c_{12}^2)}{l_1 c_{12}}\right)\frac{\partial}{\partial c_{12}}\right] u_2.$$

From this equation (6.63) follows. ■

Using the definition of the horizontal space for a connection, we can write down the corresponding connection form.

Proposition 6.4.14 *The $h(3)-$valued one-form, $\mathcal{A}$, for the nonholonomic connection is*

$$\mathcal{A} = \left[da_1 - \frac{l_1 c_{12}}{2(l_1^2 + c_{12}^2)} dc_{12}\right]\mathcal{A}_1 + \left[dc_1 + \frac{2l_1^2 + c_{12}^2}{2(l_1^2 + c_{12}^2)} dc_{12}\right]\mathcal{A}_2 + \left[db_1 - c_1 da_1 + \left(\frac{l_1 c_1 c_{12}}{2(l_1^2 + c_{12}^2)} + \frac{2l_1^2 a_1 + a_1 c_{12}^2}{2(l_1^2 + c_{12}^2)}\right) dc_{12}\right]\mathcal{A}_3. \quad (6.65)$$

Proof Given the basis for $h(3)$ in equation (6.17), $\mathcal{A}$ has the general form $\mathcal{A} = \beta_1 \mathcal{A}_1 + \beta_2 \mathcal{A}_2 + \beta_3 \mathcal{A}_3$ where $\beta_i \in T_q^* Q$. Since $\mathcal{A}$ is a principal connection it must map infinitesimal generators to the corresponding Lie algebra elements. Thus, given $\xi \in h(3)$, we have

$$\begin{aligned}
\xi_a \mathcal{A}_1 + \xi_c \mathcal{A}_2 + \xi_b \mathcal{A}_3 &= \mathcal{A}(\xi_Q) \\
&= \mathcal{A}\left(\xi_a \frac{\partial}{\partial a_1} + [\xi_b + \xi_a c_1]\frac{\partial}{\partial b_1} + \xi_c \frac{\partial}{\partial c_1}\right) \\
&= \beta_1 \left(\xi_a \frac{\partial}{\partial a_1} + [\xi_b + \xi_a c_1]\frac{\partial}{\partial b_1} + \xi_c \frac{\partial}{\partial c_1}\right)\mathcal{A}_1 \\
&\quad + \beta_2 \left(\xi_a \frac{\partial}{\partial a_1} + [\xi_b + \xi_a c_1]\frac{\partial}{\partial b_1} + \xi_c \frac{\partial}{\partial c_1}\right)\mathcal{A}_2 \\
&\quad + \beta_3 \left(\xi_a \frac{\partial}{\partial a_1} + [\xi_b + \xi_a c_1]\frac{\partial}{\partial b_1} + \xi_c \frac{\partial}{\partial c_1}\right)\mathcal{A}_3. \quad (6.66)
\end{aligned}$$

Let $(da_1, db_1, dc_1, dc_{12})$ be the basis for $T_q^* Q$ dual to $(\frac{\partial}{\partial a_1}, \frac{\partial}{\partial b_1}, \frac{\partial}{\partial c_1}, \frac{\partial}{\partial c_{12}})$. Any one-form $\beta \in T_q^* Q$ may then be expressed as

$$\beta = \beta_{a_1} da_1 + \beta_{b_1} db_1 + \beta_{c_1} dc_1 + \beta_{c_{12}} dc_{12}$$

for $\beta_{a_1}, \beta_{b_1}, \beta_{c_1}, \beta_{c_{12}} \in \mathbb{R}$. Using this form for each β_i and equating coefficients of each $\mathcal{A}_i$ on each side of equation (6.66) leads to the following three equations.

$$\begin{aligned}
\beta_{1a_1}\xi_a + \beta_{1b_1}[\xi_b + \xi_a c_1] + \beta_{1c_1}\xi_c &= \xi_a, \\
\beta_{2a_1}\xi_a + \beta_{2b_1}[\xi_b + \xi_a c_1] + \beta_{2c_1}\xi_c &= \xi_c, \\
\beta_{3a_1}\xi_a + \beta_{3b_1}[\xi_b + \xi_a c_1] + \beta_{3c_1}\xi_c &= \xi_b
\end{aligned}$$

and therefore

$$\begin{aligned}
\beta_1 &= da_1 + \gamma_1 dc_{12}, \\
\beta_2 &= dc_1 + \gamma_2 dc_{12}, \\
\beta_3 &= db_1 - c_1 da_1 + \gamma_3 dc_{12}
\end{aligned}$$

where the γ_i are yet to be determined. To find them, recall that by definition of the horizontal space for a connection, $\mathcal{A}(X) = 0$ for every X in $\mathcal{H}_q$. Letting $\mathcal{A}$ act on the basis tangent vector for $\mathcal{H}_q$ in equation (6.63) yields

$$\begin{aligned}
0 = & \left[1 + \gamma_1 \frac{2(l_1^2 + c_{12}^2)}{l_1 c_{12}}\right] \mathcal{A}_1 + \left[-\frac{2l_1^2 + c_{12}^2}{l_1 c_{12}} + \gamma_2 \frac{2(l_1^2 + c_{12}^2)}{l_1 c_{12}}\right] \mathcal{A}_2 \\
& + \left[-\frac{2l_1^2 a_1 + c_{12}^2 a_1}{l_1 c_{12}} - c_1 + \gamma_2 \frac{2(l_1^2 + c_{12}^2)}{l_1 c_{12}}\right] \mathcal{A}_3
\end{aligned}$$

and thus

$$\begin{aligned}
\gamma_1 &= -\frac{l_1 c_{12}}{2(l_1^2 + c_{12}^2)}, \\
\gamma_2 &= \frac{2l_1^2 + c_{12}^2}{2(l_1^2 + c_{12}^2)}, \\
\gamma_3 &= \frac{l_1 c_{12}}{2(l_1^2 + c_{12}^2)} \left[c_1 + \frac{2l_1^2 a_1 + a_1 c_{12}^2}{l_1 c_{12}}\right]
\end{aligned}$$

and from this the connection form in equation (6.65) follows. ■

It is often useful to express the connection in its local form as in the following proposition.

Proposition 6.4.15 *The local form of the principal connection $\mathcal{A}$ is given by*

$$\mathcal{A}_{loc}(c_{12}) = \left[-\frac{l_1 c_{12}}{2(l_1^2+c_{12}^2)}dc_{12}\right]\mathcal{A}_1 + \left[\frac{2l_1^2+c_{12}^2}{2(l_1^2+c_{12}^2)}dc_{12}\right]\mathcal{A}_2. \tag{6.67}$$

Proof Recall that the local form of a connection is defined by $\mathcal{A}_{loc} = \sigma^*\mathcal{A}$ where σ is an arbitrary section of the total space. Let $\sigma(c_{12}) = (0,0,0,c_{12})$. Then

$$\begin{aligned}\mathcal{A}_{loc}(c_{12})\dot{c}_{12} &= (\sigma^*\mathcal{A})(c_{12})\dot{c}_{12} \\ &= \mathcal{A}(\sigma(c_{12}))(T_{c_{12}}\sigma)\dot{c}_{12} \\ &= \left[-\frac{l_1 c_{12}}{2(l_1^2+c_{12}^2)}\dot{c}_{12}\right]\mathcal{A}_1 + \left[\frac{2l_1^2+c_{12}^2}{2(l_1^2+c_{12}^2)}\dot{c}_{12}\right]\mathcal{A}_2.\end{aligned}$$

■

Recall from equation (6.10) that the system satisfies

$$\mathcal{A}(q)\dot{q} = \mathbb{I}^{-1}(q)J^{nhc}(\dot{q})$$

where $\mathbb{I}$ is the locked inertia tensor (Definition 6.2.5). We now establish the form of the nonholonomic momentum map.

Lemma 6.4.16 *The nonholonomic momentum map has the form*

$$J^{nhc}(\dot{q}) = \frac{pl_1^2}{2k(l_1^2+c_{12}^2)}\mathbb{I}(q)(\xi^q) \tag{6.68}$$

where p is the nonholonomic momentum.

Proof Let $(\xi^q)^*$ denote the basis for $(\mathfrak{g}^q)^*$ dual to ξ^q. By definition, J^{nhc} maps TQ to $(\mathfrak{g}^{\mathcal{D}})^*$. Therefore $J^{nhc}(\dot{q}) \in (\mathfrak{g}^q)^*$ and it may be written as $\beta(\xi^q)^*$ for some β. Letting $J^{nhc}(\dot{q})$ act on ξ^q yields

$$< J^{nhc}(\dot{q}), \xi^q >= \beta < (\xi^q)^*, \xi^q >= \beta$$

since $(\xi^q)^*$ is dual to ξ^q. The left hand side of this is the definition of the nonholonomic momentum p and so $\beta = p$.

Consider now the locked inertia tensor, $\mathbb{I} : \mathfrak{g}^{\mathcal{D}} \to (\mathfrak{g}^{\mathcal{D}})^*$. This is a one-to-one and onto map and therefore $\mathbb{I}(q)(\xi^q)$ is also basis for $(\mathfrak{g}^q)^*$. Thus $(\xi^q)^* = \alpha\mathbb{I}(q)(\xi^q)$ for some α. Then

$$1 =< (\xi^q)^*, \xi^q >=< \alpha\mathbb{I}(q)(\xi^q), \xi^q >= \alpha M(q)(\xi^q_Q, \xi^q_Q) = \alpha\left(\frac{2k(l_1^2 + c_{12}^2)}{l_1^2}\right)$$

where the last step follows from the form of $M(q)$ with the particular choices for $K_1, K_2, F_1(c_{12})$, and $F_2(c_{12})$ in equation (6.47) and the form of ξ_q^Q in equation (6.41). Thus

$$(\xi^q)^* = \frac{l_1^2}{2k(l_1^2 + c_{12}^2)}\mathbb{I}(q)(\xi^q).$$

From this the lemma follows. ∎

With this lemma we can determine the reconstruction equations.

Proposition 6.4.17 *The reconstruction equations for the group variables are*

$$\dot{g}_1 = g_1\left(-\mathcal{A}_{loc}(c_{12})\dot{c}_{12} + \mathbb{I}_{loc}^{-1}(c_{12})p\right) \tag{6.69}$$

where $\mathcal{A}_{loc}$ *is given in equation* (6.67) *and* $\mathbb{I}_{loc}^{-1}$ *is given by*

$$\mathbb{I}_{loc}^{-1} = \frac{l_1^2}{2k(l_1^2 + c_{12}^2)}\mathcal{A}_1 + \frac{l_1 c_{12}}{2k(l_1^2 + c_{12}^2)}\mathcal{A}_2. \tag{6.70}$$

Proof Starting from equation (6.10) we have

$$\begin{aligned} \mathbb{I}^{-1}(q)J^{nhc}(\dot{q}) &= \mathcal{A}(q)\dot{q} \\ &= \mathrm{Ad}_{g_1}(g_1^{-1}\dot{g} + \mathcal{A}_{loc}(c_{12})\dot{c}_{12}) \end{aligned}$$

where the last step follows from the definition of the local form of a connection. From this we have

$$\dot{g}_1 = g_1\left(-\mathcal{A}_{loc}(c_{12})\dot{c}_{12} + \mathrm{Ad}_{g_1^{-1}}\mathbb{I}^{-1}(q)J^{nhc}(\dot{q})\right).$$

Using Lemma 6.4.16 we have

$$\begin{aligned}\mathrm{Ad}_{g_1^{-1}}\mathbb{I}^{-1}(q)J^{nhc}(\dot{q}) &= \frac{pl_1^2}{2k(l_1^2+c_{12}^2)}\mathrm{Ad}_{g_1^{-1}}\mathbb{I}^{-1}\mathbb{I}(q)\xi^q \\ &= \frac{pl_1^2}{2k(l_1^2+c_{12}^2)}g_1^{-1}\xi^q g_1 \\ &= \frac{pl_1^2}{2k(l_1^2+c_{12}^2)}\left(\mathcal{A}_1+\frac{c_{12}}{l_1}\mathcal{A}_2\right)\end{aligned}$$

where the last step follows from the form of g_1 in equation (6.18) and the form of ξ^q in equation (6.43). Define $\mathbb{I}_{loc}^{-1}$ as in equation (6.70). The reconstruction equations in (6.69) then follow. ■

Corollary 6.4.18 *The reconstruction equations may also be expressed as*

$$\dot{a}_1 = \left[\frac{l_1^2}{2k(l_1^2+c_{12}^2)}\right]p + \left[\frac{l_1c_{12}}{2(l_1^2+c_{12}^2)}\right]\dot{c}_{12}, \tag{6.71}$$

$$\dot{b}_1 = \left[\frac{l_1a_1c_{12}}{2k(l_1^2+c_{12}^2)}\right]p - \left[\frac{a_1(2l_1^2+c_{12}^2)}{2(l_1^2+c_{12}^2)}\right]\dot{c}_{12}, \tag{6.72}$$

$$\dot{c}_1 = \left[\frac{l_1c_{12}}{2k(l_1^2+c_{12}^2)}\right]p - \left[\frac{2l_1^2+c_{12}^2}{2(l_1^2+c_{12}^2)}\right]\dot{c}_{12}. \tag{6.73}$$

6.4.8 Controllability of the $H(3)-$Racer

In this section we consider the controllability of the $H(3)-$Racer. We consider separately the reduced dynamics, that is the dynamics of the shape variable together with the nonholonomic momentum, and the full dynamics, that is the reduced dynamics together with the reconstruction equations. In each case we can express the system as an affine control system of the form

$$\dot{z} = f(z) + \sum_{j=1}^{m} g_j(z)u_j \tag{6.74}$$

where $z \in M^n$, a manifold of dimension n. We begin with a few definitions for systems of this type.

Definition 6.4.19 [65] *The* **reachable set** *from a point* z_0 *at time* T *is given by*

$$R^V(z_0, T) \triangleq \{z \in M | \exists u : [0, T] \to U \text{ such that } z(t) \in V, z(0) = z_0, z(T) = z\}$$

where U *is the set of admissible inputs and* V *is a neighborhood of* z_0.

Denote the set of all points reachable from z_0 *within time* T *by* $R_T^V(z_0)$. *That is*

$$R_T^V(z_0) \triangleq \bigcup_{t \leq T} R^V(z_0, t).$$

■

Definition 6.4.20 [65] *The system* (6.74) *is* **locally accessible from** z_0 *if for any neighborhood* V *of* z_0 *and for all* $T > 0$, $R_T^V(z_0)$ *contains a non-empty open set and* **locally accessible** *if it is locally accessible from every* $z \in M$. *It is* **locally strongly accessible from** z_0 *if for any neighborhood* V *of* z_0 *and for any* T *sufficiently small,* $R^V(z_0, T)$ *contains a non-empty open set and* **locally strongly accessible** *if it is locally strongly accessible from every* $z \in M$. ■

Definition 6.4.21 [65] *The* **accessibility algebra**, $\mathcal{C}$, *is the smallest subalgebra of* $\mathcal{X}(M)$, *the Lie algebra of smooth vector fields on* M, *containing* $f, g_1, \cdots, g_m$. *The* **accessibility distribution** *is* $C \triangleq span\{X | X \in \mathcal{C}\}$. ■

Definition 6.4.22 [65] *The* **strong accessibility algebra**, $\mathcal{C}_0$ *is the smallest subalgebra of* $\mathcal{X}(M)$ *containing* $g_1, \cdots, g_m$ *which is invariant under the drift vector field* f, *i.e.* $[f, X] \in \mathcal{C}_0$ $\forall X \in \mathcal{C}_0$. *The* **strong accessibility distribution** *is* $C_0 \triangleq span\{X | X \in \mathcal{C}_0\}$. ■

The following theorem establishes the local (strong) accessibility of a system under an assumption known as the rank condition.

Theorem 6.4.23 [65] *If* $dim(C(z_0)) = n$ *then the system is locally accessible at* z_0*. If* $dim(C_0(z_0)) = n$ *then the system is locally strongly accessible at* z_0*.* ■

For systems with non-zero drift, accessibility does not imply controllability. We have the following definition and theorem due to Sussman.

Definition 6.4.24 [87] *The system* (6.74) *is said to be* **small time locally controllable (STLC) from** z_0 *if for any* $T > 0$*,* z_0 *is an interior point of* $R_T^V(z_0)$*, i.e. an entire neighborhood of* z_0 *is reachable for arbitrarily small time.* ■

Proposition 6.4.25 [86] *Consider system* (6.74) *with* $m = 1$*, i.e. with a single input. Assume that* $|u| \leq 1$*,* $f(z_0) = 0$*, and* $g(z_0) \neq 0$ *for some* $z_0 \in M$*. If* $[g, [g, f]](z_0)$ *does not belong to span*$\{\mathrm{ad}_f^j g(z_0), j = 0, 1, \cdots\}$ *then the system is not STLC from* z_0*.* ■

Finally, from [65], we have the following theorem about static feedback linearizability.

Theorem 6.4.26 [65] *Consider system* (6.74)*. Assume that the strong accessibility rank condition holds at* z_0*. This system is static feedback linearizable if and only if the distributions* $D_1, \cdots, D_n$ *defined by*

$$D_k(z) = span\{\mathrm{ad}_f^r g_1(z), \cdots, \mathrm{ad}_f^r g_m(z) | r = 0, 1, \cdots, k-1\}, k = 1, 2, \cdots$$

are all involutive and constant dimensional in a neighborhood of z_0*.*

Assume further that this is a single-input system. Then the system is static feedback linearizable around z_0 *if and only if* $dim(D_n(z_0)) = n$ *and* D_{n-1} *is involutive around* z_0*.* ■

We now turn to the controllability of the reduced dynamics, given by the momentum equation in (6.57) and the shape dynamics. Using the feedback control law in equation (6.60) we have that the reduced system is

$$\begin{aligned}\ddot{c}_{12} &= u,\\ \dot{p} &= \left[\frac{c_{12}}{(l_1^2+c_{12}^2)}\right]\dot{c}_{12}p - \left[\frac{kl_1}{(l_1^2+c_{12}^2)}\right]\dot{c}_{12}^2.\end{aligned}$$

Let $z = (p, c_{12}, \dot{c}_{12})$ and define

$$f(z) \triangleq \begin{pmatrix} \frac{pc_{12}\dot{c}_{12} - kl_1\dot{c}_{12}^2}{l_1^2+c_{12}^2} \\ \dot{c}_{12} \\ 0 \end{pmatrix}, \qquad g(z) \triangleq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{6.75}$$

Then the reduced dynamics can be expressed as

$$\dot{z} = f(z) + g(z)u. \tag{6.76}$$

The equilibria of this system are given by $z_e = (p_e, c_{12_e}, 0)$ for any p_e, c_{12_e}. The following proposition establishes the accessibility properties for the reduced system.

Proposition 6.4.27 *The reduced dynamics, system* (6.76)*, are locally accessible and locally strongly accessible from any equilibrium point.*

Proof Consider the following brackets

$$[f,g](z_e) = \left.\begin{pmatrix} \frac{pc_{12} - 2kl_1\dot{c}_{12}}{l_1^2+c_{12}^2} \\ 1 \\ 0 \end{pmatrix}\right|_{z_e} = \begin{pmatrix} \frac{p_e c_{12e}}{l_1^2+c_{12e}^2} \\ 1 \\ 0 \end{pmatrix},$$

$$[[f,g],g](z_e) = \left.\begin{pmatrix} \frac{-2kl_1}{l_1^2+c_{12}^2} \\ 0 \\ 0 \end{pmatrix}\right|_{z_e} = \begin{pmatrix} \frac{-2kl_1}{l_1^2+c_{12e}^2} \\ 0 \\ 0 \end{pmatrix}.$$

Then $dim(span\{g(z_e), [f, g](z_e), [[f, g], g](z_e)\}) = 3$. This set of vector fields is in both $\mathcal{C}$ and $\mathcal{C}_0$ and therefore, from Theorem 6.4.23, the reduced system is both locally accessible and locally strongly accessible from the equilibrium points. ■

The following two propositions establish that the reduced dynamics are neither STLC nor static feedback linearizable around the equilibrium points.

Proposition 6.4.28 *The reduced dynamics, system* (6.76)*, are not STLC from any equilibrium point.*

Proof We have that

$$(\mathrm{ad}_f^2 g)(z_e) = \left.\begin{pmatrix} \frac{-3kl_1c_{12}\dot{c}_{12}^2}{(l_1^2+c_{12}^2)^2} \\ 0 \\ 0 \end{pmatrix}\right|_{z_e} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Similarly it can be shown that $(\mathrm{ad}_f^j g)(z_e) = 0$. Thus

$$span\{\mathrm{ad}_f^j g(z_e), j = 0, 1, \cdots\} = span\{g(z_e), [f, g](z_e)\}.$$

From the form of $[[f, g], g]$ (in the proof of Proposition 6.4.27) we have that $[[f, g], g](z_e) \notin span\{g(z_e), [f, g](z_e)\}$. From Proposition 6.4.25, then, the system is not STLC from any equilibrium point. ■

Proposition 6.4.29 *The reduced dynamics are not static state feedback linearizable at the equilibrium points.*

Proof We have that

$$D_n = span\{g, [f, g], [f, [f, g]]\} = span\{g, [f, g]\}. \tag{6.77}$$

Since $dim(D_n) = 2$ and the strong accessibility condition holds, the result is immediate from Theorem 6.4.26. ■

We now consider the full dynamics given by the reduced system together with the reconstruction equations in (6.71-6.73). Define $x = (a_1, b_1, c_1, p, c_{12}, \dot{c}_{12})$ and let

$$f(x) = \begin{pmatrix} \frac{l_1^2 p + k l_1 c_{12} \dot{c}_{12}}{2k(l_1^2 + c_{12}^2)} \\ \frac{l_1 a_1 p c_{12} - k(2l_1^2 + c_{12}^2) a_1 \dot{c}_{12}}{2k(l_1^2 + c_{12}^2)} \\ \frac{l_1 p c_{12} - k(2l_1^2 + c_{12}^2)\dot{c}_{12}}{2k(l_1^2 + c_{12}^2)} \\ \frac{p c_{12} \dot{c}_{12} - k l_1 \dot{c}_{12}^2}{l_1^2 + c_{12}^2} \\ \dot{c}_{12} \\ 0 \end{pmatrix}, \qquad g(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{6.78}$$

The system is then given by $\dot{x} = f(x) + g(x)u$ and the equilibria are $x_e = (a_{1_e}, b_{1_e}, c_{1_e}, 0, c_{12_e}, 0)$. The following propositions establish the accessibility and the lack of STLC for the full system.

Proposition 6.4.30 *The full dynamics are locally accessible and locally strongly accessible from the equilibrium points.*

Proof Consider the following brackets (calculated using Mathematica) at the equilibrium points.

$$[f, g](x_e) = \begin{pmatrix} \frac{c_{12_e} l_1}{2(c_{12_e}^2 + l_1^2)} \\ -\frac{a_{1_e}(c_{12_e}^2 + 2l_1^2)}{2(c_{12_e}^2 + l_1^2)} \\ -\frac{c_{12_e}^2 + 2l_1^2}{2(c_{12_e}^2 + l_1^2)} \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad [[f, g], g](x_e) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{2kl_1}{c_{12_e}^2 + l_1^2} \\ 0 \\ 0 \end{pmatrix},$$

$$[[[f,g],f],g](x_e) = \begin{pmatrix} \frac{l_1^3}{(c_{12_e}^2 + l_1^2)^2} \\ \frac{a_{1_e} c_{12_e} l_1^2}{(c_{12_e}^{+} l_1^2)^2} \\ \frac{c_{12_e} l_1^2}{(c_{12_e}^2 + l_1^2)^2} \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$[[[f,g],g],[[f,g],f]](x_e) = \begin{pmatrix} \frac{c_{12_e} l_1^3}{(c_{12_e}^2 + l_1^2)^3} \\ -\frac{l_1^3 (c_{12_e}^2 + l_1(a_{1_e} + l_1))}{(c_{12_e}^2 + l_1^2)^3} \\ -\frac{l_1^4}{(c_{12_e}^2 + l_1^2)^3} \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$[[[[f,g],f],[f,g]],[[f,g],g]](x_e) = \begin{pmatrix} \frac{2c_{12_e}^2 l_1^3 - l_1^5}{(c_{12_e}^2 + l_1^2)^4} \\ -\frac{3c_{12_e} l_1^3 (c_{12_e}^2 + l_1(2a_{1_e} + l_1))}{2(c_{12_e}^2 + l_1^2)^4} \\ -\frac{3c_{12_e} l_1^4}{(c_{12_e}^2 + l_1^2)^4} \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Form a 6×6 matrix whose columns are the above vector fields together with $g(x_e)$. This matrix can be shown to have rank six and therefore the vector fields are linearly independent. Thus

$$dim(span\{g(x_e), [f,g](x_e), [[f,g],g](x_e), [[[f,g],f],g](x_e), \\ [[[f,g],g],[[f,g],f]](x_e), [[[[f,g],f],[f,g]],[[f,g],g]](x_e)\}) = 6.$$

These vector fields are in both $\mathcal{C}$ and $\mathcal{C}_0$ and thus from Theorem 6.4.23 the system is both locally accessible and locally strongly accessible from the equilibrium points. ■

Proposition 6.4.31 *The full dynamics are not STLC.*

Proof Immediate from Proposition 6.4.28. ■

Proposition 6.4.32 *The full dynamics are not static feedback linearizable.*

Proof Consider the distribution $D_n(x_e) = span\{\mathrm{ad}_f^r(g)(x_e)|r = 0, 1, \cdots, 5\}$. Using Mathematica we calculate that $\mathrm{ad}_f^r(g)(x_e) = 0,\ r = 2, 3, 4, 5$. Therefore $dim(D_n(x_e)) = 2$ and by Theorem 6.4.26 the system is not static feedback linearizable. ■

These same properties, namely that both the reduced dynamics and the full dynamics are locally accessible and locally strongly accessible without being STLC or static feedback linearizable, are shared with the Roller Racer (see Section 5 of [47]).

6.4.9 The geometric phase

From equations (6.15) and (6.67), the geometric phase for the $H(3)-$Racer is defined by

$$\dot{g}_{gp} = -g_{gp}\left[\frac{-l_1 c_{12}}{2(l_1^2 + c_{12}^2)}\dot{c}_{12}\mathcal{A}_1 + \frac{2l_1^2 + c_{12}^2}{2(l_1^2 + c_{12}^2)}\dot{c}_{12}\mathcal{A}_2\right]. \tag{6.79}$$

In the following we calculate the geometric phase equations explicitly using the Wei-Norman product of exponentials representation. We have the following proposition from [92].

Proposition 6.4.33 [92] *Consider a left-invariant system on $H(3)$ given by*

$$\dot{g} = g\left[\xi_1 \mathcal{A}_1 + \xi_2 \mathcal{A}_2 + \xi_3 \mathcal{A}_3\right].$$

Then $g \in H(3)$ has a global Wei-Norman representation given by

$$g = e^{\gamma_1 \mathcal{A}_1} e^{\gamma_2 \mathcal{A}_2} e^{\gamma_3 \mathcal{A}_3} = \begin{pmatrix} 1 & \gamma_1 & \gamma_1\gamma_2 + \gamma_3 \\ 0 & 1 & \gamma_2 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.80}$$

where the Wei-Norman parameters satisfy

$$\begin{pmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\gamma_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}.$$

These equations are solvable by quadrature:

$$\gamma_1(t) = \gamma_1(0) + \int_0^t \xi_1(\tau) d\tau, \tag{6.81}$$

$$\gamma_2(t) = \gamma_2(0) + \int_0^t \xi_2(\tau) d\tau, \tag{6.82}$$

$$\gamma_3(t) = \gamma_3(0) - \int_0^t \gamma_2(\tau)\xi_1(\tau) d\tau + \int_0^t \xi_3(\tau) d\tau. \tag{6.83}$$

■

By left-invariance we may assume that the initial condition in the group is $g(0) = \mathbb{1}$. In this case the initial conditions for the Wei-Norman parameters are given by $\gamma_1(0) = \gamma_2(0) = \gamma_3(0) = 0$. From equation (6.79) we have that for the geometric phase equation

$$\xi_1 = \frac{l_1 c_{12} \dot{c}_{12}}{2(l_1^2 + c_{12}^2)}, \qquad \xi_2 = \frac{-(2l_1^2 + c_{12}^2)\dot{c}_{12}}{2(l_1^2 + c_{12}^2)}, \qquad \xi_3 = 0. \tag{6.84}$$

Applying Proposition 6.4.33 yields the following.

Proposition 6.4.34 *Given a curve $c_{12}(t)$ starting at $c_{12}(0)$ and an initial condition in the group given by $g(0) = \mathbb{I}$, the Wei-Norman parameters for the geometric phase of the $H(3)-$Racer are given by*

$$
\begin{aligned}
\gamma_1(t) &= \frac{l_1}{4}\ln\left(\frac{l_1^2 + c_{12}^2(t)}{l_1^2 + c_{12}^2(0)}\right),\\
\gamma_2(t) &= -\frac{1}{2}\left[c_{12}(t) - c_{12}(0) + l_1\left(\arctan\left[\frac{c_{12}(t)}{l_1}\right] - \arctan\left[\frac{c_{12}(0)}{l_1}\right]\right)\right],\\
\gamma_3(t) &= -\frac{l_1}{4}\left[c_{12}(t) - c_{12}(0) + l_1\left(\arctan\left[\frac{c_{12}(t)}{l_1}\right] - \arctan\left[\frac{c_{12}(0)}{l_1}\right]\right)\right.\\
&\quad -\frac{1}{2}\left(c_{12}(0) + l_1\arctan\left[\frac{c_{12}(0)}{l_1}\right]\right)\ln\left(\frac{l_1^2 + c_{12}^2(t)}{l_1^2 + c_{12}^2(0)}\right)\\
&\quad +\frac{l_1}{2}\left(\arctan\left[\frac{c_{12}(t)}{l_1}\right]\ln\left[\frac{l_1^2 + c_{12}^2(t)}{l_1^2}\right] - \arctan\left[\frac{c_{12}(0)}{l_1}\right]\ln\left[\frac{l_1^2 + c_{12}^2(0)}{l_1^2}\right]\right.\\
&\qquad \left.\left. - l_1^2\int_{c_{12}(0)}^{c_{12}(t)}\frac{\ln\left[\frac{l_1^2 + c_{12}^2}{l_1^2}\right]}{l_1^2 + c_{12}^2}dc_{12}\right)\right].
\end{aligned}
$$

Proof Consider first γ_1. From equations (6.81) and (6.84) we have

$$
\begin{aligned}
\gamma_1(t) &= \int_0^t \frac{l_1 c_{12}(\tau)\dot{c}_{12}(\tau)}{2(l_1^2 + c_{12}^2(\tau))}d\tau\\
&= \frac{l_1}{2}\int_{c_{12}(0)}^{c_{12}(t)}\frac{c_{12}}{l_1^2 + c_{12}^2}dc_{12} = \frac{l_1}{4}\ln\left(\frac{l_1^2 + c_{12}^2(t)}{l_1^2 + c_{12}^2(0)}\right) \qquad (6.85)
\end{aligned}
$$

which is the form of γ_1 in the Proposition. For γ_2 we begin with equations (6.82) and (6.84). We have

$$
\begin{aligned}
\gamma_2(t) &= -\int_0^t \frac{(2l_1^2 + c_{12}^2(\tau))\dot{c}_{12}(\tau)}{2(l_1^2 + c_{12}^2(\tau))}d\tau = -\int_{c_{12}(0)}^{c_{12}(t)}\frac{2l_1^2 + c_{12}^2}{2(l_1^2 + c_{12}^2)}dc_{12}\\
&= -\left[l_1^2\int_{c_{12}(0)}^{c_{12}(t)}\frac{dc_{12}}{l_1^2 + c_{12}^2} + \frac{1}{2}\int_{c_{12}(0)}^{c_{12}(t)}\frac{c_{12}^2}{l_1^2 + c_{12}^2}dc_{12}\right]. \qquad (6.86)
\end{aligned}
$$

Solving the first integral in equation (6.86) yields

$$
\begin{aligned}
\int_{c_{12}(0)}^{c_{12}(t)}\frac{dc_{12}}{l_1^2 + c_{12}^2} &= \frac{1}{l_1^2}\int_{c_{12}(0)}^{c_{12}(t)}\frac{dc_{12}}{1 + \left(\frac{c_{12}}{l_1}\right)^2} = \frac{1}{l_1}\arctan\left(\frac{c_{12}}{l_1}\right)\Bigg|_{c_{12}(0)}^{c_{12}(t)}\\
&= \frac{1}{l_1}\left[\arctan\left(\frac{c_{12}(t)}{l_1}\right) - \arctan\left(\frac{c_{12}(0)}{l_1}\right)\right]. \qquad (6.87)
\end{aligned}
$$

The second integral is given by

$$\begin{aligned}\int_{c_{12}(0)}^{c_{12}(t)} \frac{c_{12}^2}{l_1^2 + c_{12}^2} dc_{12} &= c_{12} - l_1 \arctan\left(\frac{c_{12}}{l_1}\right)\Bigg|_{c_{12}(0)}^{c_{12}(t)} \\ &= c_{12}(t) - c_{12}(0) - l_1 \left[\arctan\left(\frac{c_{12}(t)}{l_1}\right) - \arctan\left(\frac{c_{12}(0)}{l_1}\right)\right]. \quad (6.88)\end{aligned}$$

Inserting equations (6.87) and (6.88) into equation (6.86) yields the form of γ_2 in the Proposition. Finally equations (6.83) and (6.84), and the solution for $\gamma_2(t)$ we have

$$\begin{aligned}\gamma_3(t) &= -\frac{1}{2}\int_0^t \frac{l_1 c_{12}(\tau)\dot{c}_{12}(\tau)}{2(l_1^2 + c_{12}^2(\tau))}\left[c_{12}(\tau) - c_{12}(0)\right. \\ &\quad \left. + l_1\left(\arctan\left[\frac{c_{12}(\tau)}{l_1}\right] - \arctan\left[\frac{c_{12}(0)}{l_1}\right]\right)\right] d\tau \\ &= -\frac{1}{2}\int_{c_{12}(0)}^{c_{12}(t)} \frac{l_1 c_{12}}{2(l_1^2 + c_{12}^2)}\left[c_{12} - c_{12}(0)\right. \\ &\quad \left. + l_1\left(\arctan\left[\frac{c_{12}}{l_1}\right] - \arctan\left[\frac{c_{12}(0)}{l_1}\right]\right)\right] dc_{12} \\ &= -\frac{l_1}{4}\left[\int_{c_{12}(0)}^{c_{12}(t)} \frac{c_{12}^2}{l_1^2 + c_{12}^2} dc_{12}\right. \\ &\quad - \left(c_{12}(0) + l_1 \arctan\left[\frac{c_{12}(0)}{l_1}\right]\right)\int_{c_{12}(0)}^{c_{12}(t)} \frac{c_{12}}{l_1^2 + c_{12}^2} dc_{12} \\ &\quad \left. + l_1 \int_{c_{12}(0)}^{c_{12}(t)} \frac{c_{12}\arctan\left[\frac{c_{12}}{l_1}\right]}{l_1^2 + c_{12}^2} dc_{12}\right]. \quad (6.89)\end{aligned}$$

The first of these integrals is given by equation (6.88) and the second by equation (6.85). The third integral is expanded using integration by parts.

$$\begin{aligned}\int_{c_{12}(0)}^{c_{12}(t)} \frac{c_{12}\arctan\left(\frac{c_{12}}{l_1}\right)}{l_1^2 + c_{12}^2} dc_{12} &= \frac{1}{2}\arctan\left(\frac{c_{12}}{l_1}\right)\ln\left(\frac{l_1^2 + c_{12}^2}{l_1^2}\right)\Bigg|_{c_{12}(0)}^{c_{12}(t)} \\ &\quad - \frac{l_1^2}{2}\int_{c_{12}(0)}^{c_{12}(t)} \frac{\ln\left(\frac{l_1^2 + c_{12}^2}{l_1^2}\right)}{l_1^2 + c_{12}^2} dc_{12}. \quad (6.90)\end{aligned}$$

Inserting equations (6.88), (6.85), and (6.90) into equation (6.89) yields the form of γ_3 in the Proposition. ∎

From Proposition 6.4.34 we have the following corollary which establishes that the geometric phase at the end of any closed path is zero.

Corollary 6.4.35 *Consider a curve in the shape space given by* $c_{12}(t)$*. If there exists a* T *such that* $c_{12}(T) = c_{12}(0)$ *then* $g_{gp}(T) = \mathbb{I}$*.*

We note that this result is a consequence of the topology of the shape space of the $H(3)-$Racer. This space is the real line and thus the only way to follow a closed path is to initially trace a curve in one direction and then retrace the same curve in the opposite direction. The shape space of the Roller Racer, however, is S^1 and since the circle admits non-exact forms it is possible to generate a non-trivial geometric phase (by going fully around the circle). In practice, however, there is an assembly constraint on the joint of the Roller Racer and one cannot follow full loops in joint space.

6.4.10 Simulations

In this section we present simulations of the dynamics of the $H(3)-$Racer for particular trajectories of $c_{12}(t)$. In each case a control law u was chosen and the shape trajectory determined by numerically integrating equation (6.61). The nonholonomic momentum was found by numerically integrating the momentum equation, (6.57). The group trajectory and the geometric phase were determined by numerically solving the ordinary differential equations for the Wei-Norman parameters as in Proposition 6.4.33 and then using equation (6.80) to determine the corresponding group variables. All numerical integrations were performed using standard ordinary differential equation solvers in Matlab. In the simulations we selected $l_1 = l_2 = 1$ for the holonomic constraints and $k = 1$ for the metric.

6.4.11 Wiggle

We first consider the evolution of the system when the shape is controlled to follow a sinusoidal path about zero. To effect this motion we set $u = -\omega^2 c_{12}$ with ω arbitrarily set to π and choose as initial conditions the values $c_{12}(0) = 0$, $\dot{c}_{12}(0) = 1$. The shape trajectory and shape velocity are shown in Figure 6.3.

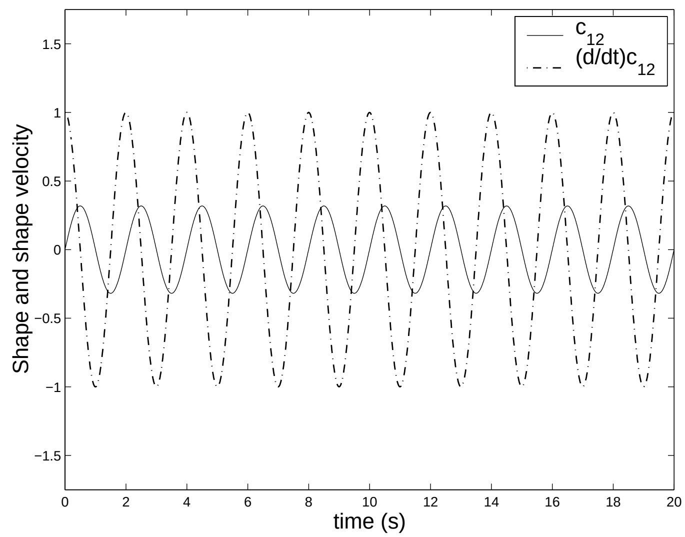

Figure 6.3: Shape and shape velocity with $u = -\omega^2 c_{12}$

An initial condition of $p(0) = 0$ was chosen for the nonholonomic momentum and the resulting trajectory is shown in Figure 6.4. In accordance with Corollary 6.4.10, the nonholonomic momentum is nonpositive. While it is not monotonically decreasing, it is decreasing on average and after each cycle of the shape there is a net increase in the magnitude of the nonholonomic momentum, indicating that energy is being pumped into the system.

Consider the group reconstruction equations (6.71 - 6.73). From equation (6.71) we see that the direction of drift in a_1 is determined by the sign of the nonholonomic momentum. Furthermore, if the magnitude of $c_{12}\dot{c}_{12}$ is small with respect to the nonholonomic momentum then the drift term in the equation for $\dot{a}_1$ will

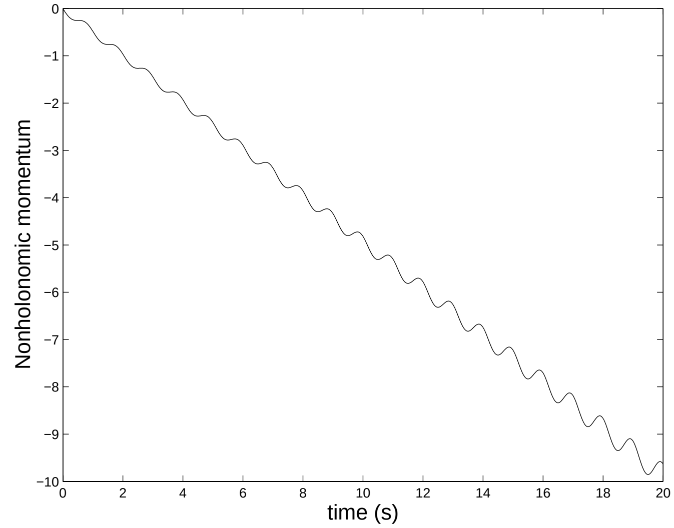

Figure 6.4: Nonholonomic momentum with $u = -\omega^2 c_{12}$

dominate. Thus, since the nonholonomic momentum is nonpositive and on average is decreasing without bound when the shape follows a sinusoidal trajectory, a_1 will also decrease when the shape varies periodically about zero, except possibly during a short initial period when the magnitude of the drift and geometric terms are comparable. On the other hand, the drift terms in the equations for $\dot{b}_1$ and $\dot{c}_1$ are scaled by c_{12}. Once again we expect the drift terms to dominate in the long run and thus, since c_{12} is undergoing an oscillatory motion, we expect an oscillation in b_1 and c_1 as well. In Figure 6.5 we show the reconstructed group motion for this simulation with initial condition set to the group identity. We see that as expected a_1 trends downward while b_1 and c_1 oscillate. Due to the increasing magnitude of the nonholonomic momentum overtime, the rate of change of the group variables is also increasing.

Since c_{12} is following a closed curve, from Corollary 6.4.35 we expect the geometric phase at the completion of each cycle of c_{12} to be zero. The time evolution of the geometric phase is shown in Figure 6.6 and, as expected, each variable returns

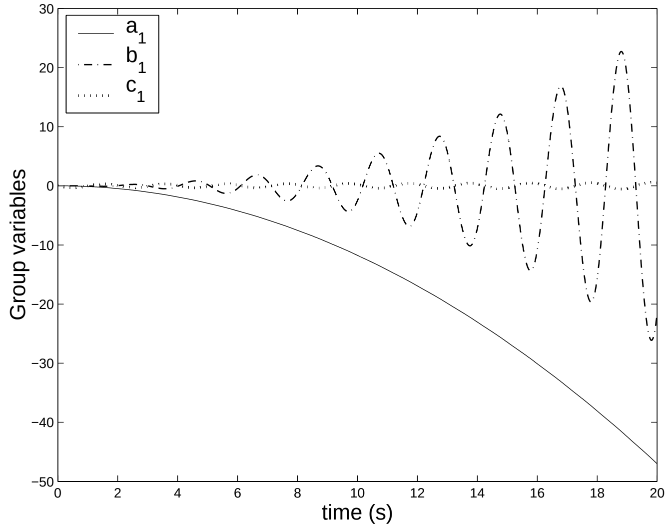

Figure 6.5: Group motion with $u = -\omega^2 c_{12}$

to zero at the end of each cycle of c_{12}.

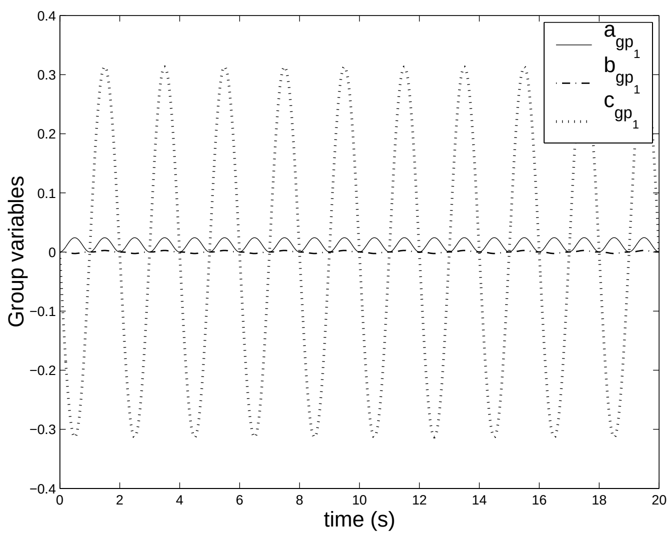

Figure 6.6: Geometric phase with $u = -\omega^2 c_{12}$

These results should be compared with the analogous case for the Roller Racer in Figures 6 and 7 of [47]. In that system, a sinusoidal oscillation of the joint angle between the two platforms about the configuration in which the platforms are aligned generates a translation along the initial line of orientation of the Roller Racer and an oscillation about that line. This motion is very similar to that of of the $H(3)-$Racer shown in Figure 6.5 in which the system translates along the a_1 direction.

6.4.12 Offset wiggle

We once again consider an oscillation in the shape variable c_{12} but now choose a nonzero mean by choosing the control $u = -\omega^2(c_{12} - \bar{c})$ and the initial conditions $c_{12}(0) = 0$, $\dot{c}_{12}(0) = \bar{c}_{12}$. The resulting shape trajectory with $\bar{c} = 0.2$ is shown in Figure 6.7.

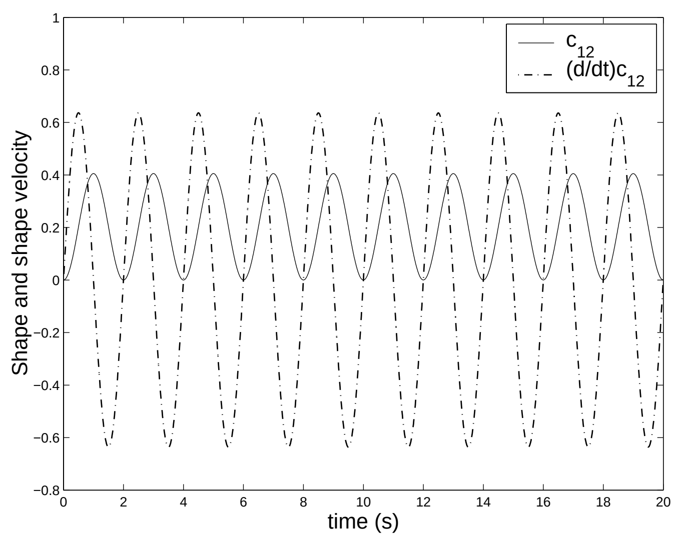

Figure 6.7: Shape and shape velocity for offset wiggle with $\bar{c} = 0.2$

The nonholonomic momentum, shown in Figure 6.8, has a more complex shape than when the shape follows a zero mean oscillation but as before there is a net decrease in its value at the end of each cycle of c_{12}.

In Figure 6.9 we show the evolution of the group variables under this shape variation. As in the zero-mean case, a_1 monotonically decreases. Now, however,

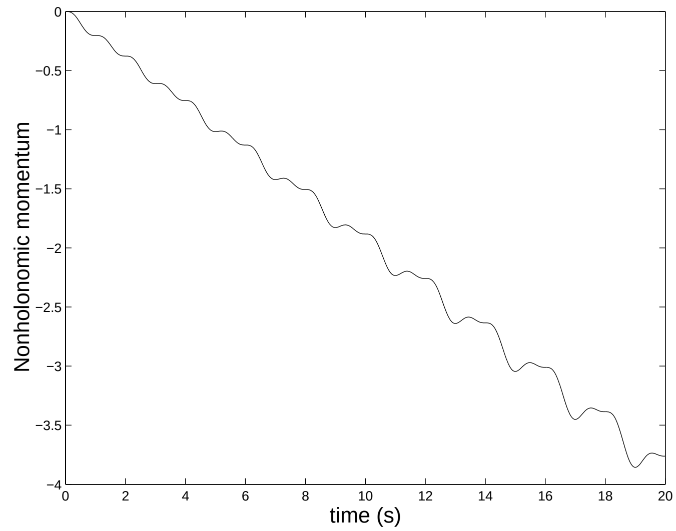

Figure 6.8: Nonholonomic momentum for offset wiggle with $\bar{c} = 0.2$

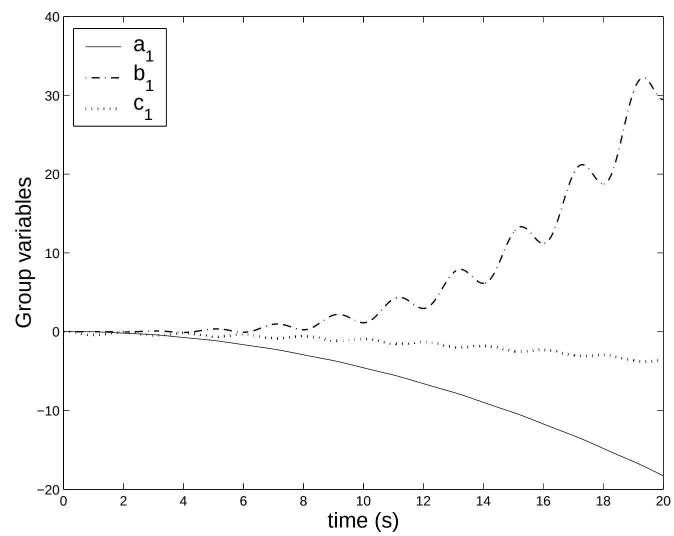

Figure 6.9: Group trajectory with $\bar{c} = 0.2$

since c_{12} is nonnegative at all times, b_1 and c_1 no longer oscillate around zero. In addition the drift term for $\dot{b}_1$ is scaled by a_1 and thus, since a_1 is always negative, b_1 grows with the sign opposite to c_1.

Due to the scaling by c_{12} of the drift terms in the equations for $\dot{b}_1$ and $\dot{c}_1$, the direction of growth of b_1, c_1 can be reversed by changing the sign of c_{12}. In Figure

6.10 we show the group trajectory when we select $\bar{c}_{12} = -0.2$.

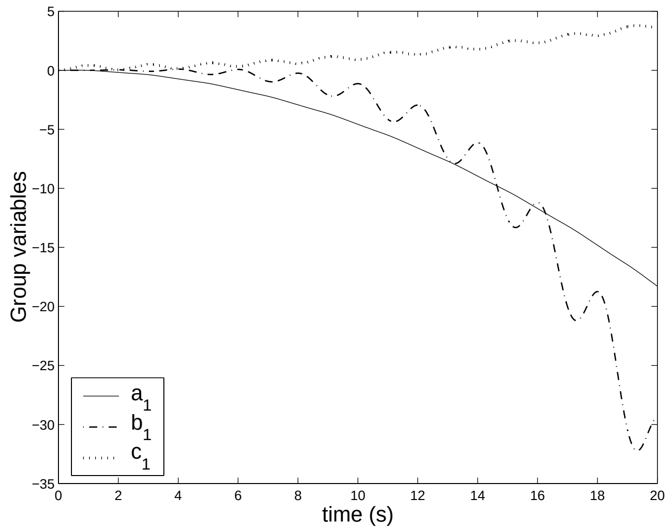

Figure 6.10: Group trajectory with $\bar{c}_{12} = -0.2$

6.4.13 Constant momentum shape variation

Consider the nonholonomic momentum equation in 6.57. Setting $\dot{p} = 0$ we obtain the following condition on $\dot{c}_{12}$ to ensure the nonholonomic momentum remains constant while the shape is varied.

$$\dot{c}_{12}(kl_1\dot{c}_{12} - c_{12}p) = 0. \tag{6.91}$$

Thus either $\dot{c}_{12} = 0$, in which case the evolution in the group is purely due to drift, or $\dot{c}_{12} = \frac{pc_{12}}{kl_1}$. From this we see that if the nonholonomic momentum is nonzero then the following nontrivial evolution of the shape will keep p constant.

$$c_{12}(t) = c_{12}(0)e^{\frac{pt}{kl_1}}. \tag{6.92}$$

To realize this trajectory we set $u = \left(\frac{p}{kl_1}\right)^2 c_{12}$ and choose $p(0) = -1$ and $c_{12}(0) = 1$. The resulting evolution of the shape and the shape velocity are shown in Figure 6.11.

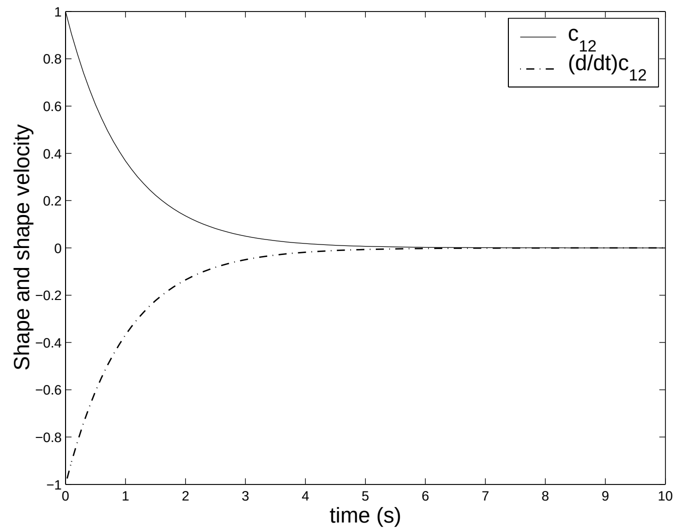

Figure 6.11: Shape and shape velocity for constant nonholonomic momentum

In Figure 6.12 we see that the nonholonomic momentum is indeed constant during the evolution of the shape.

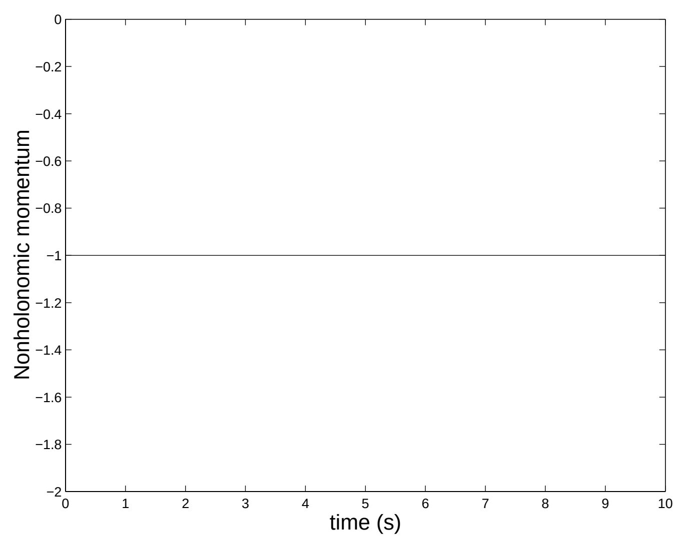

Figure 6.12: Constant nonholonomic momentum

The group trajectory, starting from the identity, is shown in Figure 6.13. Since the shape and shape velocity asymptotically approach zero, the rate of change of both b_1 and c_1 approach zero. a_1, on the other hand, continues to be driven by the nonzero nonholonomic momentum.

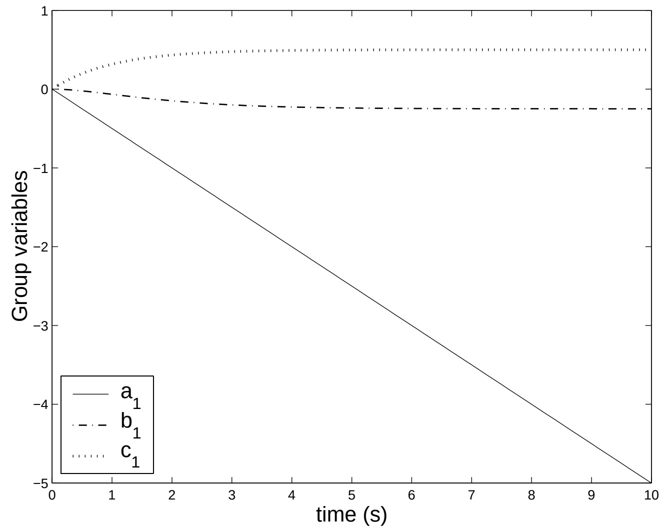

Figure 6.13: Reconstructed group motion for constant nonholonomic momentum

Chapter 7

Conclusions and future research directions

7.1 Conclusions

In this thesis we have developed techniques for applying geometric phases as engineering tools in sensing and control. In Chapter 3 we presented the moving systems approach of Marsden, Montgomery, and Ratiu, and showed how the Hannay-Berry phase can be used to sense the external motion imposed on a system. The technique was first illustrated through the example of a free-floating, equal-sided, spring-jointed, rotating four-bar mechanism and the Hannay-Berry phase due to an imposed rotation was found to be zero. We then applied the method to the vibrating ring gyroscope and showed that the precession of the nodal points of vibration is the Hannay-Berry phase. Using the inherently nonlinear nature of the moving systems approach, we derived nonlinear corrections to the rate of precession to third order. These corrections were shown to have a sign opposite to that of the first-order term and thus reduce the sensitivity of the ring gyroscope. From this we concluded that the linear regime is the appropriate setting in which to operate these devices.

To apply the moving systems approach it is assumed that the imposed motion is adiabatic. In physical systems, while this motion may be very slow with respect to the natural dynamics, it cannot be truly adiabatic. In Chapter 4, then, we developed nonadiabatic corrections to the moving systems approach by applying Hamiltonian perturbation theory and showing that the Hannay-Berry phase can be viewed as a first-order correction. The non-adiabatic corrections are then given by finding higher-order perturbation terms. We applied the technique to the vibrating ring gyroscope to illustrate its use.

In Chapter 5 we turned our attention to the theory of geometric phases in dissipative systems with symmetry. Earlier work by Landsberg considered dissipative systems with abelian symmetry groups and an isolated exponentially stable equilibrium point dependent on a parameter. He showed that if the parameter is slowly varied along a closed loop then there is a corresponding shift in the symmetry direction which is geometric in nature. We first recast this work into the standard framework for geometric phases by defining an appropriate fiber bundle and a connection, termed the Landsberg connection, on that bundle. Using this approach we were able to consider arbitrary finite-dimensional symmetry groups and showed the existence of a geometric phase in these systems, manifesting itself as a shift in the group variables. We also developed a definition of the dynamic phase which allowed us to consider the effect of the adiabatic variation of the parameter in dissipative systems with symmetry which exhibit stable, time dependent solutions. The theory was illustrated through the simple example of a linear, damped harmonic oscillator coupled to the group of rigid rotations and translations in the plane.

In Chapter 6 we considered the role of geometric phases in nonholonomic systems with symmetry through the detailed example of the $H(3)-$Racer, the two-node, one module $G-$snake on the three dimensional Heisenberg group. Applying the theory developed by Bloch, Krishnaprasad, Marsden, and Montgomery in [13], we derived the reduced equations describing the evolution of the shape of the $H(3)-$Racer under application of a control torque and the evolution of the nonholonomic momentum due to the corresponding shape trajectory. Using the nonholonomic connection, we determined the reconstruction equations giving the group dynamics induced by the shape changes and the nonholonomic momentum and showed that while the geometric phase along a path in the shape space is not necessarily trivial, for the $H(3)-$Racer it is zero at the completion of any closed loop due to the topology of its shape space. We then explored the effect of several shape trajectories through simulation.

7.2 Future research directions

There are several avenues for future research building upon the results of this thesis. For sensing, it would be interesting to apply the techniques developed in Chapters 3 and 4 to other systems, in particular to structures that have evolved in biological systems. For example, the blowfly has evolved a pair of sense organs known as halteres, tiny club-shaped organs which beat out of phase with the wings during flight. The external rotation of the fly induces a strain at the point of attachment of the haltere with the body, providing a measurement of the rate of rotation [25]. Mechanical structures based on this idea have been constructed for the micromechanical flying insect project at the University of California at

Berkeley [100]. It would be interesting to determine if the effect used for sensing in these structures is purely a geometric phase.

In the development of the ring gyroscope example, the linearized equations for the nominal dynamics are used. As discussed in Chapters 3 and 4, this results in a spurious softening effect due to the second-order terms in the rate of imposed rotation. The use of geometrically exact models in rod theory correctly accounts for the effects of the centrifugal force on a rotating rod and it would be interesting to apply the moving systems approach together with the nonadiabatic corrections to a geometrically exact model of the ring.

It would also be intriguing to develop a physical example for the theory of geometric phases in dissipative systems with symmetry developed in Chapter 5 of this thesis. One possible source for such an example is [14] in which Bloch, Krishnaprasad, Marsden, and Ratiu consider Euler-Poincaé systems obtained by reduction of Lagrangian systems with symmetry and show that adding nonlinear dissipative in the form of the double bracket equations of Brockett [17, 18] preserves the structure of the system. Furthermore, in many pattern forming systems, the governing equations are partial differential equations rather than the ordinary differential equations considered in this thesis. The theory developed in this thesis should be extended to include these systems. It would then be intriguing to investigate how to use the geometric phase in a pattern-forming system to control the pattern along the symmetry directions.

The $H(3)-$Racer and the Roller Racer share many features in common. In particular, in both systems the dynamics are locally accessible and locally strongly accessible but are not small-time locally controllable. The development of control algorithms for systems of this type is still an open question. One possible approach is to find control laws which yield effective, basic motions from which others may be built. This is an idea captured through the use of a motion description language such as MDLe [55, 37]. Similar ideas can be found in recent work by Ostrowski and McIsaac [61, 60] in which they develop a hierarchical approach to controlling underactuated dynamical systems.

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