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Risk-Sensitive Probability for Markov Chains

by Vahid R. Ramezani and Steven I. Marcus

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Vahid Reza Ramezani† and Steven I. Marcus‡
Department of Electrical and Computer Engineering
and Institute for Systems Research
University of Maryland
College Park, MD 20742

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Abstract

The probability distribution of a Markov chain is viewed as the information state of an additive optimization problem. This optimization problem is then generalized to a product form whose information state gives rise to a generalized notion of probability distribution for Markov chains. The evolution and the asymptotic behavior of this generalized or "risk-sensitive" probability distribution is studied in this paper and a conjecture is proposed regarding the asymptotic periodicity of risk-sensitive probability. The relation between a set of simultaneous non-linear equations and the set of periodic attractors is analyzed.

Keywords: Markov Chains, Risk-sensitive estimation, Asymptotic periodicity.

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†Institute for Systems Research and the Department of Electrical and Computer Engineering University of Maryland College Park, MD 20742. 301-405-2942 rvahid@isr.umd.edu

‡Institute for Systems Research and the Department of Electrical and Computer Engineering University of Maryland College Park, MD 20742. 301-405-7589 marcus@isr.umd.edu (communicating author).

1 Introduction

It is well known that the probability distribution of an ergodic Markov chain is asymptotically stationary, independent of the initial probability distribution, and that the stationary distribution is the solution to a fixed point problem [5]. This probability distribution can be viewed as the information state for an estimation problem arising from the Maximum A Posterior Probability Estimator (MAP) estimation of the Markov chain for which no observation is available.

Risk-sensitive filters [7]-[13] take into account the "higher order" moments of the estimation error. Roughly speaking, this follows from the analytic property of the exponential $e^x = \sum_{k=0}^{\infty} x^k/k!$ so that if Ψ stands for the sum of the error functions over some interval of time then

$$E[exp(\gamma\Psi)] = E[1 + \gamma\Psi + (\gamma)^2(\Psi)^2/2 + ...].$$

Thus, at the expense of the mean error cost, the higher order moments are included in the minimization of the expected cost, reducing the "risk" of large deviations and increasing our "confidence" in the estimator. The parameter $\gamma > 0$ controls the extent to which the higher order moments are included. In particular, the first order approximation, $\gamma \to 0$, $E[exp(\gamma\Psi] \cong 1 + \gamma E\Psi$, indicates that the original minimization of the sum criterion or the risk-neutral problem is recovered as the small risk limit of the exponential criterion.

Another point of view is that the exponential function has the unique algebraic property of converting the sum into a product. In this paper we show that a notion of probability for Markov chains follows from this point of view which due to its connection to risk-sensitive filters, will be termed "risk-sensitive probability (RS-probability)". We consider an estimation problem of the states of a Markov chain in which the cost has a product structure. We assume no observation is available and that the initial probability distribution is known. We will define the RS-probability of a Markov chain as an information state for this estimation problem whose evolution is governed by a non-liear operator. The asymptotic behavior of RS-probability appears to be periodic. Asymptotic periodicity has been reported to emerge from random perturbations of dynamical systems governed by *constrictive* Markov integral operators [3][4]. In our case, the Markov operator is given by a matrix; the perturbation has a simple non-linear structure and the attractors can be explicitly calculated.

In Section 2, we view the probability distribution of a Markov chain as the information state of an additive optimization problem. RS-probability for Markov chains are introduced in section 3. We show that its evolution is governed by an operator (denoted by F^γ) which can be viewed as a generalization of the usual linear Markov operator. The asymptotic behavior of this operator is studied in section 3 and a conjecture is proposed. Under mild conditions, it appears that RS-probability is asymptotically periodic. This periodic behavior is governed by a set of simultaneous quadratic equations.

2 Probability as an information state

In [2],[1] we studied the exponential (risk-sensitive) criterion for the estimation of HMM's and introduced risk-sensitive filter banks.

The probability distribution of a Markov chain, knowing only initial distribution, determines the most "likely state" in the sense of MAP. In the context of Hidden Markov Models (HMM), the problem can be viewed as that of "pure prediction"; i.e., an HMM whose states are entirely hidden.

Define a Hidden Markov Model as a five tuple $< \mathbf{X}, \mathbf{Y}, \mathbf{X}, \mathcal{A}, Q >$; here $\mathcal{A}$ is the transition matrix, $\mathbf{Y} = \{1, 2, ..., N_{\mathbf{Y}}\}$ is the set of observations and $\mathbf{X} = \{1, 2, ..., N_{\mathbf{X}}\}$ is the finite set of (internal) states as well as the set of estimates or decisions. In addition, we have that $Q := [q_{x,y}]$ is the $N_{\mathbf{X}} \times N_{\mathbf{Y}}$ state/observation matrix, i.e., $q_{x,y}$ is the probability of observing y when the state is x. We consider the following information pattern. At decision epoch t, the system is in the (unobservable) state $X_t = i$ and the corresponding observation Y_t is gathered, such that

$$P(Y_t = j | X_t = i) = q_{i,j}. \tag{1}$$

The estimators V_t are functions of observations $(Y_0, ...Y_t)$ and are chosen according to some specified criterion. Consider a sequence of finite dimensional random variables X_t and the corresponding observations Y_t defined on the common probability space $(\mathbf{\Omega}, M, \mathbf{P})$. Let $\hat{X}_t$ be a Borel measurable function of the filtration generated by observations up to Y_t denoted by $\mathcal{Y}_t$. The Maximum A Posteriori Probability (MAP) estimator is defined recursively; given $\hat{X}_0, ..., \hat{X}_{t-1}$, $\hat{X}_t$ is chosen such that the following sum is minimized:

$$E[\sum_{i=0}^{t} \rho(X_i, \hat{X}_i)], \tag{2}$$

where

$$\rho(u, v) = \begin{cases} 0 & \text{if } u = v; \\ 1 & otherwise, \end{cases}$$

The usual definition of MAP as the argument with the greatest probability given the observation follows from the above [6]. The solution is well known; we need to define recursively an *information state*

$$\sigma_{t+1} = N_{\mathbf{Y}} \cdot \overline{Q}(Y_{t+1}) \mathcal{A}^T \cdot \sigma_t, \tag{3}$$

where $\overline{Q}(y) := diag(q_{i,y})$, $\mathcal{A}^T$ denotes the transpose of the matrix $\mathcal{A}$. σ_0 is set equal to $N_{\mathbf{Y}} \cdot \overline{Q}(Y_0) p_0$, where p_0 is the initial distribution of the state and is assumed to be known.

When no observation is available, it is easy to see that $N_{\mathbf{Y}} \cdot \overline{Q}(Y_t) = I$, where I is the identity matrix. Thus, the information state for the prediction case evolves according to $\sigma_{t+1} = \mathcal{A}^T \cdot \sigma_t$ which when normalized is simply the probability distribution of the chain. This "prediction" optimization problem for a multiplicative cost will be considered next.

3 RS-Probability for Markov chains

With the notation of the previous section, given $\hat{X}_0, ..., \hat{X}_{t-1}$, define $\hat{X}_t$ recursively as the estimator which minimizes the product

$$E[\prod_{i=0}^{t} \rho^*(X_i, \hat{X}_i)] \tag{4}$$

$$\rho^*(u,v) = \begin{cases} 1 & \text{if } u = v; \\ r = e^{\gamma} & otherwise. \end{cases}$$

Associate with each $i \in X$, a unit vector in $R^N_{\mathbf{X}}$ whose i^{th} component is 1. Assume that no observation is available and that the initial probability distribution is given.

Theorem 1: The estimator which minimizes (4) is given by

$$\hat{X}_t = argmax_{\ i \in \mathbf{S_X}} < U_t \ , \ e_i >,$$

where U_t evolves according to

$$U_{t+1} = \mathcal{A}^T \cdot H\{diag\ (\ \ exp(\gamma < e_{\underset{i}{argmax}\ U^i_t} \ \ , \ \ \ \ e_j >) \ \ ) \cdot U_t\} =: F^{\gamma}(U_t), \tag{5}$$

and $H(X) = \frac{X}{\sum_i (X_i)}$ and $U_0 = p_0$.

Proof: See [2].

The operator F^{γ} can be viewed as a non-linear generalization of the linear operator $\mathcal{A}^T$. It can be shown that that this operator plays a similar role in the estimation of risk-sensitive MAP for HMM's as the operator $\mathcal{A}^T$ in the risk-neutral case. The purpose of this paper is to compare the asymptotic behavior of F^{γ} and $\mathcal{A}^T$.

It is well known that under primitivity of the matrix $\mathcal{A}$, the dynamical system defined by

$$p_{n+1} = \mathcal{A}^T p_n, \tag{6}$$

for every choice of the initial probability distribution p_0, converges to p^* which satisfies $\mathcal{A}^T p^* = p^*$[5].

Definition: A Cycle of RS-Probability (CRP) is a finite set of probabilities $\{v^1, ..., v^m\}$ such that $F^{\gamma}(v^i) = v^{i+1}$ with $F^{\gamma}(v^m) = v^1$; m is called the period of the CRP.

We pose the following conjecture:

Conjecture: Let the stochastic matrix $\mathcal{A}$ be primitive. Then, for every choice of the initial probability distribution p_0, the dynamical system

$$U_{t+1} = F^{\gamma}(U_t) \tag{7}$$

is asymptotically periodic, i.e., U_t approaches a CRP as $t \to \infty$ satisfying the equations

$$F^{\gamma}(v^1) = v^2, F^{\gamma}(v^2) = v^3, ..., F^{\gamma}(v^m) = v^1. \tag{8}$$

The condition $F^{\gamma}(v^1) = v^2, F^{\gamma}(v^2) = v^3, ..., F^{\gamma}(v^m) = v^1$ can be considered a generalization of the equation $\mathcal{A}^T p^* = p^*$. It is not difficult to show that in general, the equations are quadratic. Note that we do not exclude the case $m = 1$; the CRP only has one element and thus F^{γ} is asymptotically stationary. Next, we report a number of other properties of F^{γ}.

Property 1: Dependence of the asymptotic behavior on the initial condition. The asymptotic behavior of F^{γ} may depend on the initial conditions. That is, depending on the initial condition a different CRP may emerge. Let $\mathcal{A}$ be given by

$$\mathcal{A} = \begin{bmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{bmatrix} \qquad e^{\gamma} = 100. \tag{9}$$

Let the initial condition be given by (u_1, u_2). There are two different CRP's depending on the initial conditions:

$$F^{\gamma}(u) = u = \begin{bmatrix} 0.594 \\ 0.405 \end{bmatrix} \; if \;\; u_1 \geq u_2 \tag{10}$$

$$F^{\gamma}(v) = v = \begin{bmatrix} 0.214 \\ 0.785 \end{bmatrix} \; if \;\; u_2 > u_1. \tag{11}$$

When is the asymptotic behavior independent of the initial condition? We believe this depends on the relation between the diagonal and off-diagonal elements of $\mathcal{A}$. For example, consider the matrix

$$\mathcal{A} = \begin{bmatrix} 0.6 & 0.4 \\ 0.25 & 0.75 \end{bmatrix} \qquad e^{\gamma} = 10. \tag{12}$$

The CRP, for every initial condition, has two elements

$$CRP : (v^1, v^2) \quad F^{\gamma}(v^1) = v^2 \;\; F^{\gamma}(v^2) = v^1. \tag{13}$$

$$v^1 = \begin{bmatrix} 0.283 \\ 0.716 \end{bmatrix} \qquad v^2 = \begin{bmatrix} 0.534 \\ 0.465 \end{bmatrix}. \tag{14}$$

It appears that when the diagonal elements "dominate" the off-diagonal elements, the asymptotic behavior is independent of the initial condition. We have carried out a thorough investigation for 6×6 stochastic matrices and lower dimensions, but we suspect the property holds in higher dimensions. One could reason that large diagonal elements indicate a more "stable" dynamical system compared to the case with high "cross-flow" among the states. The non-linear perturbation of our dynamical system with higher levels of cross-flow tends to "split" the stationary attractor. Understanding the precise behavior is an open problem. But, below we describe some special cases.

Property 2: Dependence of the period on γ. Our simulations show that for small values of γ the period is 1; i.e., F^{γ} is asymptotically stationary. As γ increases periodic behavior may emerge; based on simulation of the examples we have studied, the period tends to increase with increasing γ but then decrease for large values. So, the most complex behavior occurs for the mid-range values of γ. Consider

$$\mathcal{A} = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}, \tag{15}$$

and let m be the period. Our simulations show that the period m of the CRP's depends on the choice of γ; our simulations results in the pairs $(e^{\gamma}\ , m)$:(2.1,1) (2.7,1) (2.9,1) (3,1) (3.01,7) (3.1,5) (3.3,4) (3.9,3) (10,2) (21,2). We can see that even in two dimensions, the behavior of F^{γ} is complex.

When does the periodic behavior emerge? The fixed point problem provides the answer. If the fix point problem $F^{\gamma}(u) = u$ does not have a solution satisfying $0 \leq u \leq 1$, the asymptotic behavior cannot be stationary. For two dimensions, the equation $F^{\gamma}(u) = u = (u_1, u_2)^T$ is easy to write. Assume $u_1 > u_2$ (for the case $u_2 > u_1$, we transpose 1 and 2).

$$\mathcal{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \tag{16}$$

and recall that $u_1 + u_2 = 1$. This yields

$$(e^{\gamma} - 1)u_1^2 + u_1(a_{11} - e^{\gamma}a_{21} - e^{\gamma}) + a_{21}e^{\gamma} = 0 \quad u_1 \geq u_2 \tag{17}$$

$$(e^{\gamma} - 1)u_2^2 + u_1(a_{22} - e^{\gamma}a_{12} - e^{\gamma}) + a_{12}e^{\gamma} = 0 \quad u_2 > u_1 \tag{18}$$

First, note that when $\gamma = 0$, we have

$$u_1(a_{12} + a_{21}) = a_{21} \tag{19}$$

which is linear and is the fixed point problem $\mathcal{A}^T(u) = u$. For the above example, the roots of the equation resulting from the assumption $u_2 > u_1$ are greater than one for all ranges of $e^{\gamma} > 1$. Thus, stationarity requires that a solution to

$$(e^{\gamma} - 1)u_1^2 + u_1(0.8 - e^{\gamma}0.4 - e^{\gamma}) + 0.4e^{\gamma} = 0 \quad u_2 < u_1 \tag{20}$$

exist. One solution turns out to be greater than one. The other solution is plotted vs. $r = e^{\gamma}$ in Figure 1. The condition $u_2 < u_1$ fails for $e^{\gamma} > 3$. Thus for $e^{\gamma} > 3$ no stationary solution can exist. If the conjecture is correct, the periodic behavior must emerge, which is exactly what we observed above. Based on the examples we have studied, this is a general property of F^{γ} in two dimensions when diagonal elements "dominate".

Let $a_{11} > a_{12}$ and $a_{22} > a_{21}$. Also, assume without loss of generality, that $a_{11} > a_{22}$. For the stationary solution to exist as we showed above, (17) must have a solution. Let $\Delta = a_{11} - e^{\gamma}a_{21} - e^{\gamma}$. For small values of γ, the probability solution of (17) $(0 \leq u_1 \leq 1)$ turns out to be

$$\frac{-\Delta - \sqrt{\Delta^2 - 4a_{21}e^{\gamma}(e^{\gamma} - 1)}}{2(e^{\gamma} - 1)}, \tag{21}$$

and as $u_2 < u_1$ implies $1/2 < u_1$, we must have

$$\frac{-\Delta - \sqrt{\Delta^2 - 4a_{21}e^{\gamma}(e^{\gamma} - 1)}}{2(e^{\gamma} - 1)} > 1/2,$$

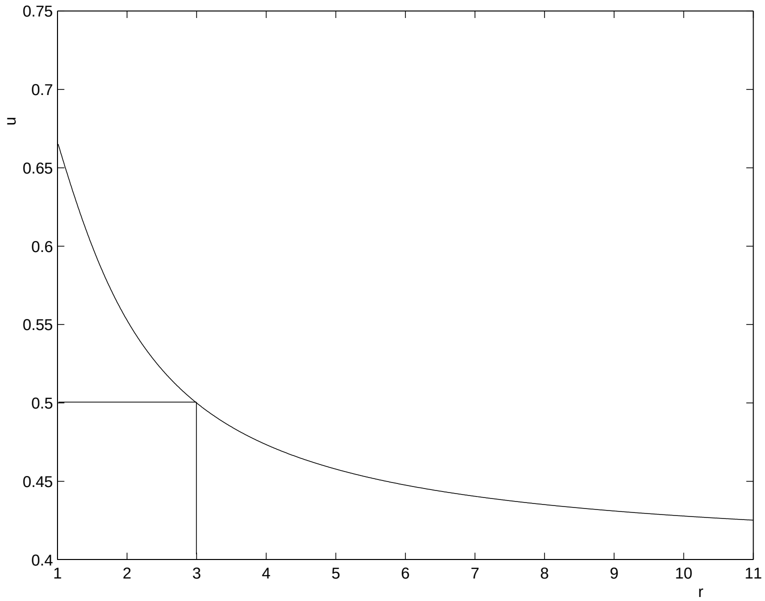

Figure 1: The emergence of periodicity.

which after some simple algebra implies

$$e^{\gamma} < \frac{2a_{11} - 1}{1 - 2a_{21}}. \tag{22}$$

If we plug in $a_{11} = 0.8$ and $a_{21} = 0.4$, we get $e^{\gamma} < 3$. If the conjecture is true, periods must appear for $e^{\gamma} > 3$. At $e^{\gamma} = \frac{2a_{11}-1}{1-2a_{21}}$, we get $u_1 = u_2 = 1/2$ which can be shown to be an acceptable stationary solution; hence $\frac{2a_{11}-1}{1-2a_{21}}$ is a sharp threshold. Our computations have been consistent with this result. For the case $a_{11} < a_{22}$, we obtain

$$e^{\gamma} < \frac{2a_{22} - 1}{1 - 2a_{12}}. \tag{23}$$

Writing $a_{ii} = 1/2 + \epsilon$ and $a_{ji} = 1/2 - \delta$, both results can be written as

$$e^{\gamma} < \frac{\epsilon}{\delta}. \tag{24}$$

(24) is a measure of sensitivity to risk.

Periodicity seems persistent; once the periodic solutions emerge, increasing e^{γ} does not seem to bring back the stationary behavior. In two dimensions for large values of e^{γ}, an interesting classification is possible. Given that the conjecture hold, an obvious sufficient condition for periodicity would be for the roots of (17) and (18) to be complex:

$$(a_{11} - e^{\gamma}a_{21} - e^{\gamma})^2 - 4(e^{\gamma} - 1)a_{21}e^{\gamma} < 0 \tag{25}$$

$$(a_{22} - e^{\gamma}a_{12} - e^{\gamma})^2 - 4(e^{\gamma} - 1)a_{12}e^{\gamma} < 0 \tag{26}$$

	period	attractors
$a_{ii} < a_{ij}$ $i \neq j$	1	2
$a_{ii} > a_{ij}$ $i \neq j$	2	2
$a_{ii} < a_{ij}$ $a_{jj} > a_{ji}$ $i \neq j$	1	1

Figure 2: The classification in two dimensions.

But, further inspection shows for sufficiently large values of e^{γ}, the inequalities give

$$e^{2\gamma}(1 - a_{21})^2 < 0 \tag{27}$$

$$e^{2\gamma}(1 - a_{12})^2 < 0 \tag{28}$$

which are clearly false and so real roots exist. Other relations can be exploited to show that these roots are unacceptable and hence demonstrate the existence of periodic attractors as we will show next. Consider the case $e^{\gamma} >> a_{ij}, 0 < a_{ij} < 1$. Then, the fixed point problem (17) can be written as

$$e^{\gamma}u_1^2 - e^{\gamma}(1 + a_{21})u_1 + a_{21}e^{\gamma} = 0 \tag{29}$$

$$u_1^2 - u_1(1 + a_{21}) + a_{21} = (u_1 - 1)(u_1 - a_{21}). \tag{30}$$

The solutions turn out to be $(1, 0)$ and (a_{21}, a_{22}). $(1, 0)$ can be ruled out by the assumption $0 < a_{ij} < 1$. The assumption $u_1 < u_2$, (18), leads by transposition to solutions $(1, 0)$ and (a_{11}, a_{12}). Thus, if we assume $a_{11} > a_{12}$ and $a_{22} > a_{21}$, both solutions can be ruled out; for large values of e^{γ} the fixed point value problem with period one (the stationary case) does not have a solution and the period must be 2 or more.

If we assume $a_{21} > a_{22}$ and $a_{12} > a_{11}$ then both (a_{21}, a_{22}) and (a_{11}, a_{12}) are acceptable solutions. This was the case in (10) and (11); our computations show that there are in fact two stationary solutions close to (a_{21}, a_{22}) and (a_{12}, a_{11}) depending on the initial conditions. Likewise, we can use the simultaneous quadratic equations to classify all the attractors in two dimensions which emerge with increasing e^{γ}. For the matrix $\mathcal{A}$ given by (15), we see that this behavior is already emergent at about $e^{\gamma} = 10$. Figure 2 shows the classification. It is possible to write down these equations in higher dimensions as simultaneous quadratic equations parameterized by e^{γ}. Classifying the solutions of these equations is an interesting open problem.

Property 3: Dependence of the period and and the periodic attractors on transition probabilities. The dependence of the period on transition probabilities is shown next. Let

ϵ	0.1	0.01	0.008	0.006	0.004	0.002	0.001	0.0002
Period	4	4	29	21	17	78	430	682

Table 1: The periodic behavior for (31)

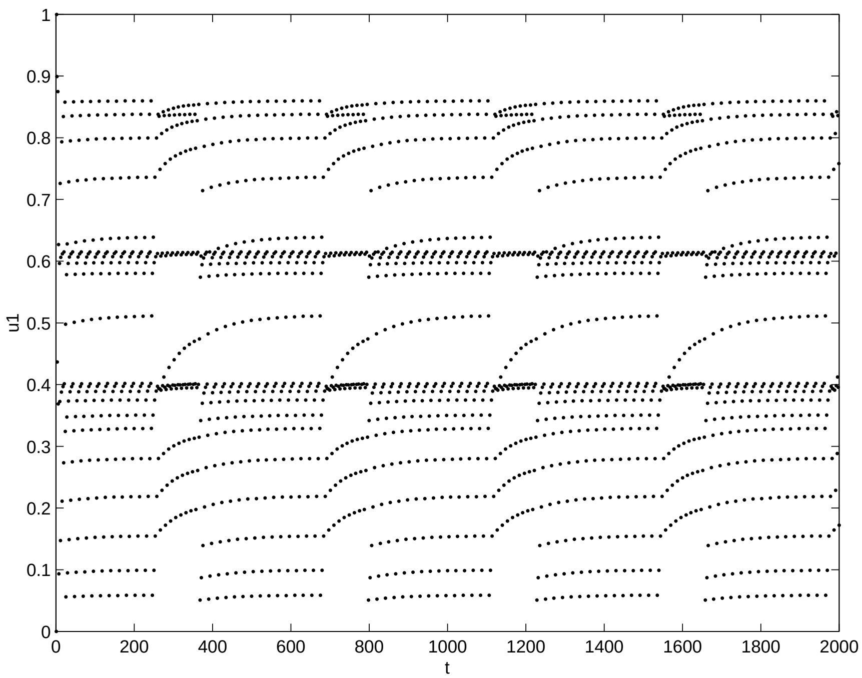

Figure 3: The first component of RS-probability vs. time for $\epsilon = 0.001$.

$$\mathcal{A} = \begin{bmatrix} 0.9-\epsilon & 0.1 & \epsilon \\ 0.4 & 0.6 & 0.0 \\ 0.0 & \epsilon & 1.0-\epsilon \end{bmatrix}, \tag{31}$$

and $e^{\gamma} = 101$. The CRP's appear to be independent of the initial conditions but the period can depend strongly on ϵ as Table 1 shows. Figure 3 shows the values of the first component of the RS-probability vs. time for $\epsilon = 0.001$. (There are 2000 data points and hence some apparent overlaps). In Table 1, we see that as $\epsilon \to \infty$, the period increases rapidly. One possible explanation is that ϵ controls the mixing properties of (31); the matrix $\mathcal{A}$ is primitive only for strictly positive values of ϵ and as ϵ approaches zero, (31) "approaches" a non-mixing dynamical system and hence its stationary behavior becomes less "stable".

The result suggests that the asymptotic behavior of non-primitive stochastic matrices under risk-sensitivity is interesting and merits investigation.

	Cost	evolution	Asym.	attractors
Probability	additive	linear	asym. stationarity	$A^{T}p=p$ Independent of initial conditions
RS−Probability	multiplicative	non−linear	asym. periodicity	$F^{\gamma}v1=v2$, $F^{\gamma}v2=v3$.... $F^{\gamma}vm=v1$ Dependent on initial conditions

Figure 4: Comparing F^{γ} and $\mathcal{A}^{\mathcal{T}}$.

4 Conclusions

The risk-sensitive estimation of HMM's gives rise to a notion of probability for Markov chains arising from a non-linear generalization of the linear operator $\mathcal{A}^{T}$, where $\mathcal{A}$ is a row-stochastic primitive matrix. This operator, denoted by F^{γ} in this paper, has a number of properties summarized in the table above. There is an interesting relation between the asymptotic behavior of F^{γ} and a set of simultaneous non-linear equations parameterized by e^{γ} determining the periodic solutions. We provided some description of this relation for the two-dimensional case in this paper. We have posed a series of open problems which are the subject of our further research.

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