# TECHNICAL RESEARCH REPORT

## A Parallel Manipulator with Only Translational Degrees of Freedom

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# A Parallel Manipulator with Only Translational Degrees of Freedom

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**Abstract**

This paper presents a novel three degree of freedom parallel manipulator that employs only revolute joints and constrains the manipulator output to translational motion. Closed-form solutions are developed for both the inverse and forward kinematics. It is shown that the inverse kinematics problem has up to four real solutions, and the forward kinematics problem has up to 16 real solutions.

## Introduction

The Stewart platform has been studied extensively (Stewart, 1965; Hunt, 1983; Griffis and Duffy, 1989; Innocenti and Parenti-Castelli, 1993 and 1990; and Nanua et al., 1990). Other variations of the Stewart platform have also been proposed. Kohli (1988) presented six degree-of-freedom (DOF) parallel manipulators that utilize base-mounted rotary-linear actuators; Hudgens and Tesar (1988) introduced a six-DOF parallel micromanipulator; Pierrot, et al. (1990) developed a high-speed six-DOF parallel manipulator; and recently Tahmasebi and Tsai (1994, 1994a, 1994b) conceived of a six-DOF parallel minimanipulator with three inextensible limbs. However, most of the six-DOF parallel manipulators studied to date consist of six limbs which connect a moving platform to a base by spherical joints. These six-limbed manipulators suffer the following disadvantages:

1. Their direct kinematics are difficult to solve.

2. Position and orientation of the moving platform are coupled.

3. Their workspace is relatively small.

4. Spherical joints are difficult to manufacture and are generally imprecise.

Note that the only six-limbed, six-DOF parallel manipulators for which closed-form direct kinematic solutions have been reported in the literature are special forms of the Stewart platform (Nanua, 1990; Grffis and Duffy, 1989; Innocenti and Parenti-Castelli, 1990; Lin, et al., 1994; and Zhang and Song, 1994). In these special forms, pairs of spherical joints may present design and manufacturing problems. As to the general Stewart platform, researchers have to resort to numerical techniques for the solutions. Innocenti and Parenti-Castelli (1993) developed an exhaustive mono-dimensional search algorithm to find the direct kinematics solutions of the general Stewart platform. Raghavan (1993) applied the continuation method and showed that the general Stewart platform has 40 direct kinematics solutions.

To overcome the above shortcomings, a parallel manipulator was invented by L.W. Tsai (1995) with three translational degrees of freedom that has the following advantages:

1. It has closed-form direct and inverse kinematics solutions.

2. Position and orientation of the moving platform are uncoupled.

3. The workspace is larger than that of a general Stewart platform.

4. The construction uses only revolute joints, as opposed to the less precise and more expensive spherical joints.

# Description of the Manipulator

A schematic of the manipulator being considered is shown in Fig. 1, where the stationary platform is labeled 0 and the moving platform is labeled 16. Three identical limbs connect the moving platform to the stationary platform. Each limb consists of an upper arm and a lower arm. The lower arms are labeled 1, 2, and 3. Each upper arm is a planar four-bar parallelogram: links 4, 7, 10, and 13 for the first limb; 5, 8, 11, and 14 for the second limb; and 6, 9, 12, and 15 for the third limb. For each limb, the upper and lower arms, and the two platforms are connected by three parallel revolute joints at axes A, B, and E

Figure 1: Schematic of the three-DOF manipulator.

as shown in Fig. 1. The axes of these revolute joints are perpendicular to the axes of the four-bar parallelogram for each limb. There is also a small offset between the axes of B and C, and between the axes of D and E. A special configuration of the mechanism, where the offsets are 0 such that the axes intersect, was presented in another paper by Tsai et al. (1996). Revolute joints are used for the parallelograms as opposed to ball joints as used in a similar manipulator by Pierrot et al. (1990). Only a portion of the workspace in the 3 DOF Delta robot developed by Pierrot et al. (1990) exhibits a pure translational characteristic. The mechanism shown in Fig. 1 has a pure translational characteristic within its entire workspace.

A reference frame (XYZ) is attached to the fixed base at point O, located at the center of the fixed platform. Another coordinate system (UVW) is attached to the fixed base at A for each leg, such that $\bar{u}$ is perpendicular to the axis of rotation of the joint at A and at an angle $\phi$ from the x axis. The angle $\phi_i$ for the $i^{th}$ leg is a parameter of the manipulator design and remains constant. The $i^{th}$ leg of the manipulator is shown in Fig. 2. The vector $\bar{p}$ is the position vector of point P in the (XYZ) coordinate frame, where P is attached at the center of the moving platform. The angle $\theta_{1i}$ is measured from $\bar{u}$ to $\overline{AB}$. The angle $\theta_{2i}$ is defined from the $\bar{u}$ direction to $\overline{BC}$. The angle $\theta_{3i}$ is defined by the angle from the $\bar{v}$ direction to $\overline{CD}$. The moving platform remains parallel to the fixed platform from the constraints of any two legs. The link lengths are also shown in Fig. 2.

For this paper, $\theta_{11}, \theta_{12}$, and $\theta_{13}$ are considered the actuated joints. Other combinations of actuated joints are also possible, but actuating $\theta_{11}, \theta_{12}$, and $\theta_{13}$ offers the advantage of attaching each of the actuators to ground.

Considering the manipulator mobility, let $F$ be the degrees of freedom, $n$ the number of links, $j$ the number of joints, $f_i$ the degrees of freedom associated with the $i^{th}$ joint, and $\lambda = 6$, the motion parameter. Then, the degrees of freedom of a mechanism is generally governed by the following mobility equation:

$$F = \lambda(n - j - 1) + \sum_i f_i \tag{1}$$

For the manipulator shown in Fig. 1, $n = 17, j = 21$, and $f_i = 1$ for $i = 1, 2, ..., 21$. Applying Eq. (1) to the manipulator produces: $F = 6(17-21-1)+21 = -9$. Hence, the manipulator is an overconstrained mechanism. However, due to the arrangement of the links and joints, many of the constraints imposed by the joints are redundant and the resulting mechanism does have three translational degrees of freedom. The redundant constraints are a product of the three revolute joints at A, B, and E (as shown in Fig. 2) having parallel axes. As a result, any single limb constrains rotation about the z and u axes. Hence, the combination of any two limbs constrains rotation about the x, y, and z axes. This leaves the mechanism

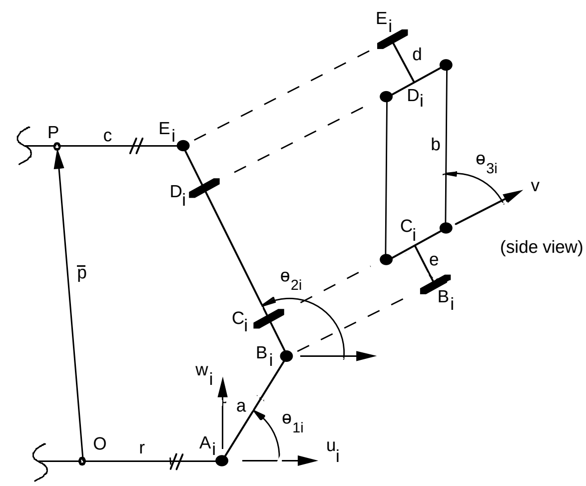


Figure 2: Depiction of the joint angles and link lengths for leg $i$.

with three translational degrees of freedom and forces the moving platform to remain in the same orientation at all times. This unique characteristic is useful in many applications such as an X-Y-Z positioning device. A hybrid serial-parallel manipulator can also be constructed by mounting a wrist mechanism onto the moving platform.

# Inverse Kinematics

The objective of the inverse kinematics is to develop a set-valued function $f^{-1} : \bar{p} \rightarrow \bar{\theta}$, where $\bar{\theta}$ is the vector consisting of the nine joint angles and $\bar{p}$ is the position vector of point $P$ in the (XYZ) frame, $\bar{p} = [p_x, p_y, p_z]^T$. The following transformation expresses the position of $P$ in the (UVW) cordinate frame attached at point $A$ for leg $i$:

$$\tilde{p}_i = \begin{bmatrix} \cos(\phi_i) & \sin(\phi_i) & 0 \\ -\sin(\phi_i) & \cos(\phi_i) & 0 \\ 0 & 0 & 1 \end{bmatrix} \bar{p}_i + \begin{bmatrix} -r \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

where $\tilde{p}_i = [p_{ui}, p_{vi}, p_{wi}]^T$ and $\bar{r} = \overline{OA}$. Expressions for $p_{ui}, p_{vi}$, and $p_{wi}$ are given by:

$$
\begin{aligned}
p_{ui} &= a\cos(\theta_{1i}) - c \\
&\quad + [d + e + b\sin(\theta_{3i})]\cos(\theta_{2i}) && (3)\\
p_{vi} &= b\cos(\theta_{3i}) && (4)\\
p_{wi} &= a\sin(\theta_{1i}) + [d + e + b\sin(\theta_{3i})]\sin(\theta_{2i}). && (5)
\end{aligned}
$$

Two solutions are immediately found for $\theta_{3i}$ from Eq. (4):

$$\theta_{3i} = \pm\arccos\left(\frac{p_{vi}}{b}\right) \tag{6}$$

With $\theta_{3i}$ known, an equation with $\theta_{1i}$ as the only unknown is generated by isolating the $\theta_{2i}$ terms in equations (3) and (5) and then summing the squares of those two equations so that $\theta_{2i}$ is eliminated with the application of the Pythagorean relationship:

$$
\begin{aligned}
(p_{ui} \quad + \quad & c)^2 + p_{wi}^2 + a^2 - 2a(p_{ui} + c)\cos(\theta_{1i}) - 2ap_{wi}\sin(\theta_{1i}) \\
&= (d + e)^2 + 2(d + e)b\sin(\theta_{3i}) + b^2\sin(\theta_{3i})^2. && (7)
\end{aligned}
$$

To transform Eq. (7) into a polynomial expression, a half-angle tangent is defined as:

$$t_{1i} = \tan\left(\frac{\theta_{1i}}{2}\right), \tag{8}$$

producing the following relationships:

$$\sin(\theta_{1i}) = \tfrac{2t_{1i}}{1+t_{1i}^2} \quad \text{and} \quad \cos(\theta_{1i}) = \tfrac{1-t_{1i}^2}{1+t_{1i}^2} \quad . \tag{9}$$

The half-angle substitution is applied to Eq. (7), and simplified to produce:

$$l_{2i}t_{1i}^2 + l_{1i}t_{1i} + l_{0i} = 0 \tag{10}$$

where:

$$
\begin{aligned}
l_{0i} &= p_{wi}^2 + p_{ui}^2 + 2cp_{ui} - 2ap_{ui} - b^2\sin(\theta_{3i})^2 \\
&\quad -2be\sin(\theta_{3i}) - 2bd\sin(\theta_{3i}) - 2de - 2ac \\
&\quad +a^2 + c^2 - d^2 - e^2 \\
l_{1i} &= -4ap_{wi} \\
l_{2i} &= p_{wi}^2 + p_{ui}^2 + 2cp_{ui} + 2ap_{ui} - b^2\sin(\theta_{3i})^2 \\
&\quad -2be\sin(\theta_{3i}) - 2bd\sin(\theta_{3i}) - 2de + 2ac \\
&\quad +a^2 + c^2 - d^2 - e^2
\end{aligned}
$$

Equation (10) can be solved for $t_{1i}$, producing two possible values for $\theta_{1i}$ for each of the two solutions found for $\theta_{3i}$. With $\theta_{1i}$ and $\theta_{3i}$ known, $\theta_{2i}$ is found by backsubstitution into equations (3) and (5). Hence, for a given position of the moving platform, there are four possible configurations for each leg.

## Forward Kinematics

For the forward kinematic analysis, the joint angles $\theta_{11}$, $\theta_{12}$, and $\theta_{13}$ are considered the input angles. Given these joint angles, a set valued map $f : \bar{\theta_{1i}} \rightarrow \bar{p}$ is developed.

For each leg, three equations for the position of point P in the (XYZ) coordinate frame are written by substituting equation (3), (4), and (5) into equation (2):

$$\begin{aligned} p_x \quad & \cos(\phi_i) + p_y \sin(\phi_i) - a\cos(\theta_{1i}) - r + c \\ & - [d + e + b\sin(\theta_{3i})]\cos(\theta_{2i}) = 0 \qquad (11) \\ p_y \quad & \cos(\phi_i) - p_x \sin(\phi_i) - b\cos(\theta_{3i}) = 0 \qquad (12) \\ p_z \quad & -a\sin(\theta_{1i}) \\ & - [d + e + b\sin(\theta_{3i})]\sin(\theta_{2i}) = 0 \qquad (13) \end{aligned}$$

for $i = 1, 2$, and 3. Equations (11), (12), and (13) for $i = 1, 2$, and 3, form a system of 9 equations in 9 unknowns ($p_x, p_y, p_z, \theta_{21}, \theta_{22}, \theta_{23}, \theta_{31}, \theta_{32}$, and $\theta_{33}$). The solution of this set of equations, represents the solution of the forward kinematics problem. To determine the solution, equations (11), (12), and (13) are manipulated to produce two equations in the tangent of the half-angle of $\theta_{31}$ and $\theta_{32}$. These two equations are then solved using the dailytic elimination method. The algebra required to achieve this solution is what follows.

Letting $\phi_1 = 0$, an expression for $p_y$ is found by rewriting equation (12) for $i = 1$:

$$p_y = b\cos(\theta_{31}). \qquad (14)$$

An expression for $p_x$ is developed by substituting (14) into equation (12) for $i = 2$:

$$p_x = \frac{b}{\sin(\phi_2)}\left[\cos(\phi_2)\cos(\theta_{31}) - \cos(\theta_{32})\right] \qquad (15)$$

An expression without $\theta_{2i}$ is generated by isolating the $\theta_{2i}$ terms in equations (11) and (13) and then summing the squares of those two equations along with the square of equation (12) so that $\theta_{2i}$ is eliminated with the application of the Pythagorean relationship:

$$
\begin{aligned}
& p_x^2 + p_y^2 + p_z^2 - 2a\sin(\theta_{1i})p_z \\
& \quad +2\left[c - r - a\cos(\theta_{1i})\right]\left[p_x\cos(\phi_i) + p_y\sin(\phi_i)\right] \\
& \quad +a^2 + (r-c)^2 - 2a(c-r)\cos(\theta_{1i}) \\
& \quad -b^2 - (d+e)^2 - 2b(d+e)\sin(\theta_{3i}) = 0
\end{aligned} \tag{16}
$$

for $i = 1, 2$, and 3.

An equation that is linear in $p_x,\ p_y,\ p_z,\ \sin(\theta_{31})$, and $\sin(\theta_{32})$ is generated by subtracting Eq. (16) for $i = 1$ from Eq. (16) for $i = 2$:

$$
\begin{aligned}
& k_1 p_x + k_2 p_y + k_3 p_z + k_4 \sin(\theta_{31}) \\
& \qquad + k_5 \sin(\theta 32) + k_6 = 0
\end{aligned} \tag{17}
$$

where the constants are defined in the appendix. Similarly, an equation that is linear in $p_x,\ p_y,\ p_z,\ \sin(\theta_{31})$, and $\sin(\theta_{33})$ is generated by subtracting Eq. (16) for $i = 1$ from Eq. (16) for $i = 3$:

$$
\begin{aligned}
& k_7 p_x + k_8 p_y + k_9 p_z + k_{10} \sin(\theta_{31}) \\
& \qquad + k_{11} \sin(\theta 33) + k_{12} = 0
\end{aligned} \tag{18}
$$

An expression for $p_z$ is generated by substituting Eqs. (14) and (15) into Eq. (17), and solving for $p_z$:

$$
\begin{aligned}
p_z &= k_{13}\sin(\theta_{31}) + k_{14}\cos(\theta_{31}) + k_{15}\sin(\theta_{32}) \\
&+ k_{16}\cos(\theta_{32}) + k_{17},
\end{aligned} \tag{19}
$$

Substituting the expressions for $p_x,\ p_y$, and $p_z$ into Eq. (18) produces an equation that is linear in $\sin(\theta_{31}),\ \cos(\theta_{31}),\ \sin(\theta_{32}),\ \cos(\theta_{32})$, and $\sin(\theta_{33})$:

$$
\begin{aligned}
k_{18} \quad & \sin(\theta_{31}) + k_{19}\cos(\theta_{31}) + k_{20}\sin(\theta_{32}) \\
& + k_{21}\cos(\theta_{32}) + k_{22}\sin(\theta_{33}) + k_{23} = 0
\end{aligned} \tag{20}
$$

Substituting $p_x$ and $p_y$ into Eq. (12) for $i = 3$ generates another equation in only $\theta_{31},\ \theta_{32}$, and $\theta_{33}$:

$$
\begin{aligned}
& \sin(\phi_2 - \phi_3)\cos(\theta_{31}) + \sin(\phi_3)\cos(\theta_{32}) \\
& - \quad \sin(\phi_2)\cos(\theta_{33}) = 0
\end{aligned} \tag{21}
$$

A third equation in $\theta_{31}$ and $\theta_{32}$ is created by substituting the expressions for $p_x$, $p_y$, and $p_z$ into Eq. (16) for $i = 1$:

$$\begin{aligned}
&k_{24}\cos^2(\theta_{31}) + k_{25}\sin^2(\theta_{31}) + k_{26}\cos^2(\theta_{32}) \\
&+k_{27}\sin^2(\theta_{32}) + k_{28}\cos(\theta_{31})\sin(\theta_{31}) \\
&+k_{29}\cos(\theta_{31})\cos(\theta_{32}) + k_{30}\cos(\theta_{31})\sin(\theta_{32}) \\
&+k_{31}\sin(\theta_{31})\cos(\theta_{32}) + k_{32}\sin(\theta_{31})\sin(\theta_{32}) \\
&+k_{33}\cos(\theta_{32})\sin(\theta_{32}) + k_{34}\cos(\theta_{31}) + k_{35}\sin(\theta_{31}) \\
&+k_{36}\cos(\theta_{32}) + k_{37}\sin(\theta_{32}) + k_{38} = 0
\end{aligned} \tag{22}$$

This leaves three equations (20), (21), and (22) in three unknowns ($\theta_{31}$, $\theta_{32}$, and $\theta_{33}$). Solving Eqs. (20) and (21) for $\sin(\theta_{33})$ and $\cos(\theta_{33})$ respectively, and then substituting these expressions into the Pythagorean relationship, $\sin^2(\theta_{33}) + \cos^2(\theta_{33}) = 1$, yields:

$$\begin{aligned}
&k_{39}\cos^2(\theta_{31}) + k_{40}\sin^2(\theta_{31}) + k_{41}\cos^2(\theta_{32}) \\
&+k_{42}\sin^2(\theta_{32}) + k_{43}\cos(\theta_{31})\sin(\theta_{31}) \\
&+k_{44}\cos(\theta_{31})\cos(\theta_{32}) + k_{45}\cos(\theta_{31})\sin(\theta_{32}) \\
&+k_{46}\sin(\theta_{31})\cos(\theta_{32}) + k_{47}\sin(\theta_{31})\sin(\theta_{32}) \\
&+k_{48}\cos(\theta_{32})\sin(\theta_{32}) + k_{49}\cos(\theta_{31}) + k_{50}\sin(\theta_{31}) \\
&+k_{51}\cos(\theta_{32}) + k_{52}\sin(\theta_{32}) + k_{53} = 0
\end{aligned} \tag{23}$$

Considering equations (22) and (23) along with the relationship $\sin(\theta_{3i})^2 + \cos(\theta_{3i})^2 = 1$ as functions of four independent variables, $\sin(\theta_{31})$, $\cos(\theta_{31})$, $\sin(\theta_{32})$, and $\cos(\theta_{32})$, the system of equations can be considered as four equations in four unknowns. The total degree of the system is 16. Therefore, there are at most 16 solutions.

Equations (22) and (23) are transformed into polynomials by applying the half-angle tangent relationships (9) and multiplying by $[(1 + t_{31}^2)^2(1 + t_{32}^2)^2]$ to clear the denominators, which is only valid if $(1 + t_{31}^2) \neq 0$ and $(1 + t_{32}^2) \neq 0$. Hence, any solution with values of $t_{31} = \pm i$ or $t_{32} = \pm i$ must be discarded from the final solution. The polynomials are expressed with $t_{31}$ suppressed as shown:

$$g_1 t_{32}^4 + g_2 t_{32}^3 + g_3 t_{32}^2 + g_4 t_{32} + g_5 = 0 \tag{24}$$
$$g_6 t_{32}^4 + g_7 t_{32}^3 + g_8 t_{32}^2 + g_9 t_{32} + g_{10} = 0, \tag{25}$$

where:

$$g_1 = k_{54} t_{31}^4 + k_{55} t_{31}^3 + k_{56} t_{31}^2 + k_{57} t_{31} + k_{58}$$

$$
\begin{aligned}
g_2 &= k_{59}t_{31}^4 + k_{60}t_{31}^3 + k_{61}t_{31}^2 + k_{62}t_{31} + k_{63} \\
g_3 &= k_{64}t_{31}^4 + k_{65}t_{31}^3 + k_{66}t_{31}^2 + k_{67}t_{31} + k_{68} \\
g_4 &= k_{69}t_{31}^4 + k_{70}t_{31}^3 + k_{71}t_{31}^2 + k_{72}t_{31} + k_{73} \\
g_5 &= k_{74}t_{31}^4 + k_{75}t_{31}^3 + k_{76}t_{31}^2 + k_{77}t_{31} + k_{78} \\
g_6 &= k_{79}t_{31}^4 + k_{80}t_{31}^3 + k_{81}t_{31}^2 + k_{82}t_{31} + k_{83} \\
g_7 &= k_{84}t_{31}^4 + k_{85}t_{31}^3 + k_{86}t_{31}^2 + k_{87}t_{31} + k_{88} \\
g_8 &= k_{89}t_{31}^4 + k_{90}t_{31}^3 + k_{91}t_{31}^2 + k_{92}t_{31} + k_{93} \\
g_9 &= k_{94}t_{31}^4 + k_{95}t_{31}^3 + k_{96}t_{31}^2 + k_{97}t_{31} + k_{98} \\
g_{10} &= k_{99}t_{31}^4 + k_{100}t_{31}^3 + k_{101}t_{31}^2 + k_{102}t_{31} + k_{103}
\end{aligned}
$$

The dialytic elimination method is applied to Eqs. (24) and (25), producing the following matrix equation:

$$
\begin{bmatrix}
g_5 & g_4 & g_3 & g_2 & g_1 & 0 & 0 & 0 \\
g_{10} & g_9 & g_8 & g_7 & g_6 & 0 & 0 & 0 \\
0 & g_5 & g_4 & g_3 & g_2 & g_1 & 0 & 0 \\
0 & g_{10} & g_9 & g_8 & g_7 & g_6 & 0 & 0 \\
0 & 0 & g_5 & g_4 & g_3 & g_2 & g_1 & 0 \\
0 & 0 & g_{10} & g_9 & g_8 & g_7 & g_6 & 0 \\
0 & 0 & 0 & g_5 & g_4 & g_3 & g_2 & g_1 \\
0 & 0 & 0 & g_{10} & g_9 & g_8 & g_7 & g_6
\end{bmatrix}
\begin{bmatrix}
1 \\ t_{32} \\ t_{32}^2 \\ t_{32}^3 \\ t_{32}^4 \\ t_{32}^5 \\ t_{32}^6 \\ t_{32}^7
\end{bmatrix}
= 0 \quad . \tag{26}
$$

For a nontrivial solution to exist for Eq. (26), the determinant of the square matrix must equal 0. This produces a $32^{nd}$ degree polynomial in $t_{31}$. This equation can be solved for $t_{31}$, and then the values of $p_x$, $p_y$, and $p_z$ are determined by back-substitution. Of the 32 solutions generated by this method, 16 of the solutions are extraneous which can be shown by checking the 32 solutions against equations (11), (12), and (13). These extraneous solutions occur when $(t_{31}^2 + 1) = 0$.

# Numerical Example

As an example of the forward kinematics solution, let the manipulator parameters be:

$$
\begin{aligned}
a &= r = 4 \\
b &= 5 \\
c &= 3 \\
d &= e = 1 \\
\phi_1 &= 0 \text{ deg} \\
\phi_2 &= 120 \text{ deg} \\
\phi_3 &= 240 \text{ deg}
\end{aligned}
$$

Let the input angles be:

$$
\begin{aligned}
\theta_{11} &= 10 \text{ deg} \\
\theta_{12} &= 45 \text{ deg} \\
\theta_{13} &= 35 \text{ deg}
\end{aligned}
$$

The $32^{nd}$ degree polynomial that results from the determinant of the square matrix in Eq. (26) is:

$$
\begin{aligned}
& t_{31}^{32} - 0.0430t_{31}^{31} - 4.4974t_{31}^{30} - 0.2538t_{31}^{29} \\
& -10.1363t_{31}^{28} + 0.4756t_{31}^{27} + 42.7144t_{31}^{26} + 3.3519t_{31}^{25} \\
& +67.1691t_{31}^{24} + 1.0562t_{31}^{23} - 136.9453t_{31}^{22} - 11.8407t_{31}^{21} \\
& -225.2437t_{31}^{20} - 12.0375t_{31}^{19} + 190.0160t_{31}^{18} + 14.5724t_{31}^{17} \\
& +385.7121t_{31}^{16} + 25.2153t_{31}^{15} - 99.1512t_{31}^{14} - 2.6413t_{31}^{13} \\
& -344.7164t_{31}^{12} - 20.5185t_{31}^{11} - 15.6474t_{31}^{10} - 6.0458t_{31}^{9} \\
& +153.2054t_{31}^{8} + 6.3137t_{31}^{7} + 28.1032t_{31}^{6} + 3.1571t_{31}^{5} \\
& -29.4723t_{31}^{4} - 0.5473t_{31}^{3} - 4.6271t_{31}^{2} - 0.3855t_{31} \\
& +2.4473 = 0
\end{aligned}
$$

The roots of this polynomial are:

$$
\begin{array}{l}
-1.89 \pm 0.03i \\
-1.11 \\
-1.05 \pm 0.05i \\
-1.01 \\
-0.60 \pm 0.04i \\
0.00 \pm 1.00\ i \ \ \text{(multiplicity 8)} \\
0.55 \\
0.60 \\
0.96 \\
1.04 \pm 0.03i \\
1.16 \\
1.82 \\
2.07
\end{array}
$$

There are eight real roots. So, for the given input angles there are eight possible poses for this manipulator. The 16 extraneous solutions are determined by checking the solutions against the condition that $t_{31} \neq \pm i$, as imposed during the formulation of eqs. (24) and (25).

As an example of a real solution, consider the root $t_{31} = 1.82$, with $\theta_{31} = 122$ deg. The angle $\theta_{32}$ is found by backsubstituting $t_{31}$ into Eqs. (24) and (25), yielding $t_{32} = 0.60$ and in turn $\theta_{32} = 62$ deg. With $\theta_{31}$ and $\theta_{32}$ known, $p_x$ and $p_y$ can be solved for directly from Eqs. (14) and (15). In this example, $p_x = -1.19$ and $p_y = -2.67$. Equation (16) can then be used to solve for $p_z = -0.37$, completing the forward kinematics solution.

# Conclusion

In this paper, a novel parallel manipulator with three translational degrees of freedom is presented. The general design of the manipulator is discussed, along with the mobility that results from the unique link and joint configuration of the manipulator. Closed-formed solutions for both the forward and inverse kinematics are also developed. These solutions demonstrate that in general, there are sixteen possible poses for the forward kinematics, and four possible poses for each leg for the inverse kinematics.

# Appendix

$$
\begin{aligned}
k_1 &= 2\left[c - r - a\cos(\theta_{11})\right] - 2\cos(\phi_2)\left[c - r - a\cos(\theta_{12})\right] \\
k_2 &= 2\sin(\phi_2)\left[r - c + a\cos(\theta_{12})\right] \\
k_3 &= 2a\left[\sin(\theta_{12}) - \sin(\theta_{11})\right] \\
k_4 &= -2b(d+e) \\
k_5 &= 2b(d+e) \\
k_6 &= 2a(r-c)\left[\cos(\theta_{11}) - \cos(\theta_{12})\right] \\
k_7 &= 2\left[c - r - a\cos(\theta_{11})\right] - 2\cos(\phi_3)\left[c - r - a\cos(\theta_{13})\right] \\
k_8 &= 2\sin(\phi_3)\left[r - c + a\cos(\theta_{13})\right] \\
k_9 &= 2a\left[\sin(\theta_{13}) - \sin(\theta_{11})\right] \\
k_{10} &= -2b(d+e) \\
k_{11} &= 2b(d+e) \\
k_{12} &= 2a(r-c)\left[\cos(\theta_{11}) - \cos(\theta_{13})\right] \\
k_{13} &= -\frac{k_4}{k_3} \\
k_{14} &= -\frac{k_2 b\sin(\phi_2) + k_1 b\cos(\phi_2)}{k_3\sin(\phi_2)} \\
k_{15} &= -\frac{k_5}{k_3} \\
k_{16} &= \frac{k_1 b}{k_3\sin(\phi_2)} \\
k_{17} &= -\frac{k_6}{k_3} \\
k_{18} &= k_{10} + k_9 k_{13} \\
k_{19} &= k_8 b + \frac{k_7 b\cos(\phi_2)}{\sin(\phi_2)} + k_9 k_{14} \\
k_{20} &= k_9 k_{15} \\
k_{21} &= k_9 k_{16} - \frac{k_7 b}{\sin(\phi_2)} \\
k_{22} &= k_5 \\
k_{23} &= k_{12} + k_9 k_{17} \\
k_{24} &= b^2 + k_{14}^2\sin^2(\phi_2)
\end{aligned}
$$

$$
\begin{aligned}
k_{25} &= k_{13}^2 \sin^2(\phi_2) \\
k_{26} &= b^2 + k_{16}^2 \sin^2(\phi_2) \\
k_{27} &= k_{15}^2 \sin^2(\phi 2) \\
k_{28} &= 2k_{13}k_{14} \sin^2(\phi_2) \\
k_{29} &= 2k_{14}k_{16} \sin^2(\phi_2) - 2b^2 \cos(\phi_2) \\
k_{30} &= 2k_{14}k_{15} \sin^2(\phi_2) \\
k_{31} &= 2k_{13}k_{16} \sin^2(\phi_2) \\
k_{32} &= 2k_{13}k_{15} \sin^2(\phi_2) \\
k_{33} &= 2k_{15}k_{16} \sin^2(\phi_2) \\
k_{34} &= 2bc \cos(\phi_2) \sin(\phi_2) - 2ab \cos(\phi_2) \sin(\phi_2) \cos(\theta_{11}) \\
&\quad -2br \cos(\phi_2) \sin(\phi_2) + 2k_{14}k_{17} \sin^2(\phi_2) \\
&\quad -2ak_{14} \sin^2(\phi_2) \sin(\theta_{11}) \\
k_{35} &= -2bd \sin^2(\phi_2) - 2be \sin^2(\phi_2) + 2k_{13}k_{17} \sin^2(\phi_2) \\
&\quad -2ak_{13} \sin^2(\phi_2) \sin(\theta_{11}) \\
k_{36} &= -2bc \sin(\phi_2) + 2ab \cos(\theta_{11}) \sin(\phi_2) + 2br \sin(\phi_2) \\
&\quad +2k_{16}k_{17} \sin^2(\phi_2) - 2ak_{16} \sin^2(\phi_2) \sin(\theta_{11}) \\
k_{37} &= 2k_{15}k_{17} \sin^2(\phi_2) - 2ak_{15} \sin^2(\phi_2) \sin(\theta_{11}) \\
k_{38} &= \sin^2(\phi_2)(a^2 - b^2 + c^2 - 2ac \cos(\theta_{11}) \\
&\quad -d^2 - 2de - e^2 + k_{17}^2 - 2cr + 2ar \cos(\theta_{11}) \\
&\quad +r^2 - 2ak_{17} \sin(\theta_{11})) \\
k_{39} &= k_{19}^2 \sin^2(\phi_2) + k_{22}^2 \sin^2(\phi_2 - \phi_3) \\
k_{40} &= k_{18}^2 \sin^2(\phi_2) \\
k_{41} &= k_{21}^2 \sin^2(\phi_2) + k_{22}^2 \sin^2(\phi_3) \\
k_{42} &= k_{20}^2 \sin^2(\phi_2) \\
k_{43} &= 2k_{18}k_{19} \sin^2(\phi_2) \\
k_{44} &= 2k_{19}k_{21} \sin^2(\phi_2) + 2k_{22}^2 \sin(\phi_2 - \phi_3) \sin(\phi_3) \\
k_{45} &= 2k_{19}k_{20} \sin^2(\phi_2) \\
k_{46} &= 2k_{18}k_{21} \sin^2(\phi_2) \\
k_{47} &= 2k_{18}k_{20} \sin^2(\phi_2) \\
k_{48} &= 2k_{20}k_{21} \sin^2(\phi_2)
\end{aligned}
$$

$$
\begin{aligned}
k_{49} &= 2k_{19}k_{23}\sin^2(\phi_2) \\
k_{50} &= 2k_{18}k_{23}\sin^2(\phi_2) \\
k_{51} &= 2k_{21}k_{23}\sin^2(\phi_2) \\
k_{52} &= 2k_{20}k_{23}\sin^2(\phi_2) \\
k_{53} &= \sin^2(\phi_2)\left[k_{23}^2 - k_{22}^2\right] \\
k_{54} &= k_{24} + k_{26} + k_{29} - k_{34} - k_{36} + k_{38} \\
k_{55} &= -2k_{28} - 2k_{31} + 2k_{35} \\
k_{56} &= -2k_{24} + 4k_{25} + 2k_{26} - 2k_{36} + 2k_{38} \\
k_{57} &= 2k_{28} - 2k_{31} + 2k_{35} \\
k_{58} &= k_{24} + k_{26} - k_{29} + k_{34} - k_{36} + k_{38} \\
k_{59} &= -2k_{30} - 2k_{33} + 2k_{37} \\
k_{60} &= 4k_{32} \\
k_{61} &= -4k_{33} + 4k_{37} \\
k_{62} &= 4k_{32} \\
k_{63} &= 2k_{30} - 2k_{33} + 2k_{37} \\
k_{64} &= 2k_{24} - 2k_{26} + 4k_{27} - 2k_{34} + 2k_{38} \\
k_{65} &= -4k_{28} + 4k_{35} \\
k_{66} &= -4k_{24} + 8k_{25} - 4k_{26} + 8k_{27} + 4k_{38} \\
k_{67} &= 4k_{28} + 4k_{35} \\
k_{68} &= 2k_{24} - 2k_{26} + 4k_{27} + 2k_{34} + 2k_{38} \\
k_{69} &= -2k_{30} + 2k_{33} + 2k_{37} \\
k_{70} &= 4k_{32} \\
k_{71} &= 4k_{33} + 4k_{37} \\
k_{72} &= 4k_{32} \\
k_{73} &= 2k_{30} + 2k_{33} + 2k_{37} \\
k_{74} &= k_{24} + k_{26} - k_{29} - k_{34} + k_{36} + k_{38} \\
k_{75} &= -2k_{28} + 2k_{31} + 2k_{35} \\
k_{76} &= -2k_{24} + 4k_{25} + 2k_{26} + 2k_{36} + 2k_{38} \\
k_{77} &= 2k_{28} + 2k_{31} + 2k_{35} \\
k_{78} &= k_{24} + k_{26} + k_{29} + k_{34} + k_{36} + k_{38}
\end{aligned}
$$

$$
\begin{aligned}
k_{79} &= k_{39} + k_{41} + k_{44} - k_{49} - k_{51} + k_{53} \\
k_{80} &= -2k_{43} - 2k_{46} + 2k_{50} \\
k_{81} &= -2k_{39} + 4k_{40} + 2k_{41} - 2k_{51} + 2k_{53} \\
k_{82} &= 2k_{43} - 2k_{46} + 2k_{50} \\
k_{83} &= k_{39} + k_{41} - k_{44} + k_{49} - k_{51} + k_{53} \\
k_{84} &= -2k_{45} - 2k_{48} + 2k_{52} \\
k_{85} &= 4k_{47} \\
k_{86} &= -4k_{48} + 4k_{52} \\
k_{87} &= 4k_{47} \\
k_{88} &= 2k_{45} - 2k_{48} + 2k_{52} \\
k_{89} &= 2k_{39} - 2k_{41} + 4k_{42} - 2k_{49} + 2k_{53} \\
k_{90} &= -4k_{43} + 4k_{50} \\
k_{91} &= -4k_{39} + 8k_{40} - 4k_{41} + 8k_{42} + 4k_{53} \\
k_{92} &= 4k_{43} + 4k_{50} \\
k_{93} &= 2k_{39} - 2k_{41} + 4k_{42} + 2k_{49} + 2k_{53} \\
k_{94} &= -2k_{45} + 2k_{48} + 2k_{52} \\
k_{95} &= 4k_{47} \\
k_{96} &= 4k_{48} + 4k_{52} \\
k_{97} &= 4k_{47} \\
k_{98} &= 2k_{45} + 2k_{48} + 2k_{52} \\
k_{99} &= k_{39} + k_{41} - k_{44} - k_{49} + k_{51} + k_{53} \\
k_{100} &= -2k_{43} + 2k_{46} + 2k_{50} \\
k_{101} &= -2k_{39} + 4k_{40} + 2k_{41} + 2k_{51} + 2k_{53} \\
k_{102} &= 2k_{43} + 2k_{46} + 2k_{50} \\
k_{103} &= k_{39} + k_{41} + k_{44} + k_{49} + k_{51} + k_{53}
\end{aligned}
$$

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