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Orientation Control of a Satellite

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Abstract

In this paper we consider the problem of controlling the orientation of a satellite with thrusters which provide two control torques. We derive open loop steering control laws for steering the satellite from one orientation to another. Then we develop closed loop regulation control laws. For a "symmetric" satellite, (i.e. one with two moments of inertia equal) we stabilize a five dimensional subset of the states. The regulator for the asymmetric satellite, which is controllable, renders an arbitrary configuration asymptotically stable. A novel choice of "Listing" coordinates for $SO(3)$ proves to be useful in deriving these control laws.

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1 Introduction

The problem of controlling the attitude of a rigid body has a long and rich history in control. Early work includes the work of Meyer [12], Brockett [2], Baillieul [1] and Crouch [8], on controllability and stabilization. Controllability was discussed for models of the satellite with fewer than three inputs both in [1, 8], but work on stabilization by Meyer [12] and others [27] considered the fully actuated satellite. Our paper focuses on the control of a satellite with only two input torques. Control laws stabilizing a subset of the states of the system of a satellite with two inputs can be found in [3].

Despite the wealth of fine work, some interesting questions about stabilization remain open and are worth reconsidering in the light of recent work on the control of nonholonomic systems [22, 24, 26]. Recent results of Coron [6] indicate that it is possible to find time-periodic and smooth control laws which stabilize the origin. One of our contributions is the construction of such stabilizing control laws. Indeed, there has been renewed interest in this direction, see for example [5, 20]. The contributions of this paper are:

1. a set of open–loop planners for steering a satellite with two inputs,
2. a novel set of coordinates for the orientation of the satellite related to Listing's law and the Hopf fibration,
3. and, finally, control laws which stabilize the angular velocities and orientation of the satellite.

The control laws derived by us are smooth time varying control laws. It has been conjectured, however, that somewhat faster convergence for the regulators may be attainable by use of non-smooth feedback control laws (see [17]). After we announced [23] our results, two other papers have presented different control laws for stabilization of the asymmetric satellite. The one by Morin, Samson, Pomet amd Jiang [15] employ techniques close to those described in this paper, with different choices of output function. The paper of Coron [7] uses different techniques. He transforms the system into a locally quadratic form and using a dilation technique demonstrates Lyapunov stability.

We consider a rigid body model of the satellite with thrusters, providing input torques about the first two principal axes. This assumption is without loss of generality: what follows is easily modified for other locations of the thrusters. The satellite model obeys the Euler equations for the rigid body. The moments of inertia through the unit axes e_i, $i = 1, 2, 3$ will be denoted I_i and the input torques are denoted τ_1, τ_2. Euler's equations for the model of the satellite are given by:

$$\begin{aligned} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} &= \begin{bmatrix} \frac{I_2-I_3}{I_1}\omega_2\omega_3 \\ \frac{I_3-I_1}{I_2}\omega_3\omega_1 \\ \frac{I_1-I_2}{I_3}\omega_1\omega_2 \end{bmatrix} + \begin{bmatrix} \frac{\tau_1}{I_1} \\ \frac{\tau_2}{I_2} \\ 0 \end{bmatrix} \\ \dot{R} &= R\hat{\omega} \end{aligned}$$

where $\hat{\omega}$ refers to the skew-symmetric matrix given by

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} .$$

The state of the satellite is characterized by the orientation matrix $R \in SO(3)$, and the angular velocity vector $\omega \in \Re^3$. $SO(3)$ denotes the group of rotations on $\Re^3$. As noted in the literature, the dynamics simplify after the following input transformation,

$$\begin{aligned}\tau_1 &= I_1\left(u_1 + \frac{I_3 - I_2}{I_1}\omega_2\omega_3\right)\\ \tau_2 &= I_2\left(u_2 + \frac{I_1 - I_3}{I_2}\omega_3\omega_1\right)\end{aligned}$$

to

$$\begin{aligned}\begin{bmatrix}\dot{\omega}_1\\ \dot{\omega}_2\\ \dot{\omega}_3\end{bmatrix} &= \begin{bmatrix}u_1\\ u_2\\ \alpha\omega_1\omega_2\end{bmatrix}\\ \dot{R} &= R\hat{\omega} \qquad (1)\end{aligned}$$

The constant $\alpha := \frac{I_1 - I_2}{I_3}$. The analysis of the system divides into two cases: the *symmetric satellite*, with $I_1 = I_2$, i.e. $\alpha = 0$, and the *asymmetric satellite* with $\alpha \neq 0$. A control law for pointing the symmetric satellite, that is, stabilizing 4 of the 6 states, may be found in [20]. (The four variables that are stabilized are Re_3, a point on the two-sphere, and ω_1, ω_2. An alternative to this law is rederived in Section 3.1 as a starting point for our final goal.)

The organization of the paper is as follows: In Section 2, we present the open–loop path planning algorithms for steering the satellite between a given initial and final orientation. Separate solutions are given for the symmetric and asymmetric satellite. The symmetric and asymmetric satellite are qualitatively different, because in the symmetric case it is impossible to control the angular velocity about the symmetry axes. In Section 3, we construct feedback controls for stabilizing the satellite. The method of exact input–output linearization is employed, using a clever choice of the output functions and coordinates. Finally, in the second part of this section we derive the stabilizing control law, and prove that it works using a Lyapunov argument. Once again, the symmetric and asymmetric cases must be treated separately. Finally, we provide simulation results for both cases.

2 Open–Loop Steering

In this section, we focus on the problem of *constructively* generating control laws for steering the satellite of (1) from an initial orientation $R_i \in SO(3)$, initial angular velocity $\omega_i \in \Re^3$ at time 0 to a final orientation $R_f \in SO(3)$ and final angular velocity $\omega_f \in \Re^3$ at time T. We solve the problem at first for a symmetric satellite, that is $I_1 = I_2$, with $\omega_3(0) = 0$. Owing to the symmetry, ω_3 is constant for all control laws and so we will need to choose $\omega_3(T) = 0$, as well.

Proposition 1 (Steering the Symmetric Satellite) *Consider the dynamics of the satellite on* $SO(3) \times \Re^3$ *described by equation (1) with* $I_1 = I_2$*, that is* $\alpha = 0$ *and* $\omega_3 \equiv 0$*. Given* $R_i, R_f \in SO(3)$*, an initial angular velocity* ω_i *and a time* $T > 0$*, there exists a piecewise constant control law,* $u(\cdot)$ *defined on* $[0, T]$*, which steers the symmetric satellite from* R_i, ω_i *at time 0 to* $R_f, \omega_f = 0$ *at time* T*.*

Proof: The proof is constructive and involves an algorithm. The function $\text{atan2}(y, x) = \arg(x + iy)$.

1. *System Halt,* $(0, \frac{T}{5})$.
   For $\frac{T}{5}$ seconds apply the controls:
   $[u_1, u_2] = [\frac{-5\omega_1(0)}{T}, \frac{-5\omega_2(0)}{T}]$.
   The system halt maneuver brings the possibly non-zero initial ω_1, ω_2 to zero.

2. *Pointing*, $(\frac{T}{5}, \frac{2T}{5})$.
   Define θ, ϕ by performing the following computation:
   $x_d = R(\frac{T}{5})^{-1} R_f e_3$
   $\theta = \text{atan2}(-x_d(2), x_d(3))$.
   $\phi = \arcsin(x_d(1))$, with $\phi \in [0, \pi]$.

   Apply each of the following control laws for $\frac{T}{20}$ seconds:

   $[u_1, u_2] = [\theta \left(\frac{20}{T}\right)^2, 0]$

   $[u_1, u_2] = [-\theta \left(\frac{20}{T}\right)^2, 0]$

   $[u_1, u_2] = [0, \phi \left(\frac{20}{T}\right)^2]$

   $[u_1, u_2] = [0, -\phi \left(\frac{20}{T}\right)^2]$

   The maneuver is composed of two rotations:

$$R(\frac{2T}{5}) \quad = \quad R(\frac{T}{5}) \ \text{Exp}(\theta \hat{e}_1) \ \text{Exp}(\phi \hat{e}_2). \tag{2}$$

   The angles θ, ϕ have been chosen so that:

$$R^{-1}(\frac{T}{5}) R_f e_3 = \ \text{Exp}(\theta \hat{e}_1) \ \text{Exp}(\phi \hat{e}_2) e_3. \tag{3}$$

   By using relation (2) in (3), we find $R_f e_3 = R(\frac{2T}{5}) e_3$. We are now pointing in the desired direction, $R_f e_3$.

3. *Measurement*, $(\frac{2T}{5})$.
   Compute: $R_d = R^{-1}(\frac{2T}{5}) R_f$.
   $\psi = \text{atan2}(-R_d(1,2), R_d(1,1))$.
   If $\psi = 0$, STOP.

   In the next step, the residual rotation error is computed. Because $R(\frac{2T}{5}) e_3 = R_f e_3$ we see that $R_d e_3 = e_3$. This implies that $R_d = \ \text{Exp}(\psi \hat{e}_3)$ and further, $\psi = \text{atan2}(-R_d(1,2), R_d(1,1))$. If $\psi = 0$ we have achieved the final orientation, otherwise the rest of the algorithm must be executed.

4. *Outward Leg*, $(\frac{2T}{5}, \frac{3T}{5})$.

   Apply each for $\frac{T}{10}$ seconds:

   $[u_1, u_2] = [\frac{\pi}{2} \left(\frac{10}{T}\right)^2, 0]$

   $[u_1, u_2] = [-\frac{\pi}{2} \left(\frac{10}{T}\right)^2, 0]$

5. *Correction*, $(\frac{3T}{5}, \frac{4T}{5})$.

   Apply each for $\frac{T}{10}$ seconds:

   $[u_1, u_2] = [0, \psi \left(\frac{10}{T}\right)^2]$

   $[u_1, u_2] = [0, -\psi \left(\frac{10}{T}\right)^2]$

6. *Return Leg* $(\frac{4T}{5}, T)$.

   Apply each for $\frac{T}{10}$ seconds:
   $[u_1, u_2] = [-\frac{\pi}{2}\left(\frac{10}{T}\right)^2, 0]$
   $[u_1, u_2] = [\frac{\pi}{2}\left(\frac{10}{T}\right)^2, 0]$

   To see the justification of these last three steps, examine the product $\text{Exp}(\frac{\pi}{2}\hat{e}_1)\ \text{Exp}(\psi\hat{e}_2)\ \text{Exp}(-\frac{\pi}{2}\hat{e}_1)$. Recall that $R\ \text{Exp}(\hat{a})R^{-1}$ is equal to $\text{Exp}\left(\widehat{Ra}\right)$. Thus, the net motion of the last three steps is $\text{Exp}(\psi\ \widehat{\text{Exp}(\frac{\pi}{2}\hat{e}_1)e_2}) = \text{Exp}(\psi\hat{e}_3)R_d$.

Remarks

1. *Area Form.* If we set $\omega_3 = 0$ and think of the remaining angular velocities ω_1, ω_2 as the controls then the resulting system on $SO(3)$ is equivalent to the equations of parallel transport for a unit vector tangent to the two-sphere S^2. Consequently, it follows from the Gauss-Bonnet theorem that the rotation about the e_3 axis induced by the last three steps is equal to the area enclosed by the path traced by $R(t)e_3$ on S^2. For a detailed proof of this fact, see [21, 25]. This suggests that the dynamics of the system is related to an area form which should have a simple expression in the right set of coordinates. This expression is found in §3.1.

2. *Regulator from the Planner.* There is little hope of converting this algorithm into a regulator since large deviations are needed to correct for small errors in rotation about e_3. The resulting regulator would not be stable in the sense of Lyapunov. We can, however, linearize the control system about the trajectory and construct a feedback control law (see [26] for details) which will stabilize to the open loop path in the face of noise and small modeling errors.

3. *Symmetric satellite with $\omega_3 \neq 0$:* Since ω_3 is constant, it is useful to solve the problem for $R_r(t) = R(t)Exp(-\omega_3 t\hat{e}_3)$. The dynamics in this new set of coordinates is given by:

$$\begin{aligned} \dot{R}_r(t) &= R_r\hat{\omega}_r \\ \omega_r(t) &= \begin{bmatrix} \cos(\omega_3 t) & -\sin(\omega_3 t) & 0 \\ \sin(\omega_3 t) & \cos(\omega_3 t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ 0 \end{bmatrix} \end{aligned}$$

   This is verified by differentiating R_r. Now defining $u = [u_1, u_2]^T$, and $v = [v_1, v_2]^T$, set

$$u = \begin{bmatrix} \omega_2\omega_3 \\ -\omega_1\omega_3 \end{bmatrix} + \begin{bmatrix} \cos(\omega_3 t) & \sin(\omega_3 t) \\ -\sin(\omega_3 t) & \cos(\omega_3 t) \end{bmatrix} v$$

   It may now be verified that under this input transformation $\dot{\omega}_1^r = v_1$ and $\dot{\omega}_2^r = v_2$. An open–loop planner, similar to the one found in Proposition 1, may now be used.

4. *Asymmetric Satellite.* Notice that at each point after system halt in the algorithm of Proposition 1, either ω_1 or ω_2 is zero. This ensures that $\dot{\omega}_3$ remains zero. Therefore, as long as the algorithm of Proposition 1 is given an asymmetric satellite with $\omega_3 = 0$, it will work as designed. An initial non-zero ω_3 can be removed by ramping up and down both ω_1 and ω_2 the proper amounts.

3 Feedback Control

In the following section, we apply geometric nonlinear control to capture the dynamics of four of the six states of either the symmetric or asymmetric satellite. Exact linearization methods require coordinates and so impose Euclidean structure on the fundamentally non-Euclidean space $SO(3)\times\Re^3$. It is not possible to avoid coordinate singularities since $SO(3)\times\Re^3$ is not $\Re^6$. However, a judicious choice of output functions confines the singularities to an S^1 subset of $SO(3)$. This is achieved through the use of a non-standard coordinate chart related to Listing's law [11] of eye movement. This chart also appeared in the attitude control of a model of a falling cat with a no-twist joint [14]. It is closely related to the Hopf fibration $SO(3) \to S^2$ which takes R to Re_3. In this section, controllers stabilizing the additional states of the zero dynamics as well as the four linear states are presented.

3.1 Input-Output Linearization

In this subsection we input-output linearize [10] the system (1) for both the symmetric and asymmetric satellite. A natural choice of output is the pointing direction $R(t)e_3$, a variable vector confined to the surface of the unit sphere S^2. The compicated control dynamics involving ω_3 and rotations about e_3 do not appear in these outputs and their derivatives. We use a stereographic parameterization of S^2, which has only one singularity. Our region of interest will be the identity matrix which projects to the north pole of the sphere. Consequently, we choose the stereographic projection to have its singularity at the south pole. The resulting output functions are

$$\begin{aligned} h_1(R,\omega) &= \frac{e_1^T Re_3}{1+e_3^T Re_3} \\ h_2(R,\omega) &= \frac{e_2^T Re_3}{1+e_3^T Re_3}. \end{aligned} \tag{4}$$

We will show that the outputs have vector relative degree 2, 2 [10] – that is to say, the controls are recaptured upon taking two derivatives of the h_i. Two states that are rendered unobservable by feedback can be expressed explicitly by noting that for all $R \in SO(3)$ of the form $R = \operatorname{Exp}(\psi\hat{e}_3)$, h_1 and h_2 are zero. It follows that the h_i do not capture the ψ, and $\dot{\psi} = \omega_3$ dynamics.

Proposition 2 (Input–Output Linearization) *Given the control system for a satellite on $SO(3)\times\Re^3$ described by equation (1), the outputs $h(R,\omega)$ of equation (4) have vector relative degree 2, 2 for all $\|h\| < \infty$.*

Proof: Differentiating the outputs for $i = 1, 2$, we get

$$\dot{h}_i = \frac{e_i^T R\hat{\omega}e_3}{1+e_3^T Re_3} - \frac{e_i^T Re_3 e_3^T R\hat{\omega}e_3}{(1+e_3^T Re_3)^2} \quad ,$$

which may be written

$$\begin{bmatrix} \dot{h}_1 \\ \dot{h}_2 \end{bmatrix} = A(R)\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \quad . \tag{5}$$

If we denote $r_{ij} = e_i^T R e_j$, the ij^{th} element of the matrix R, the matrix $A(R)$ is

$$A(R) \;=\; \frac{-1}{(1+r_{33})^2}\begin{bmatrix} (1+r_{33})r_{12} & -(1+r_{33})r_{11}+r_{13}r_{31} \\ (1+r_{33})r_{22}-r_{23}r_{32} & -(1+r_{33})r_{21}+r_{23}r_{31} \end{bmatrix} .$$

Note that these equations contain no explicit dependence on ω_3. The determinant of the matrix $A(R)$, the so-called decoupling matrix [10], is

$$\det(A(R)) \;=\; \frac{1}{(1+r_{33})^2} .$$

The determinant is bounded for every h_1, h_2 since $r_{33} = -1$ corresponds to the south pole of S^2. In terms of the outputs h_1, h_2, the determinant[1] of the decoupling matrix is $\frac{1}{4}(1+h_1^2+h_2^2)^2$, thus the chosen outputs thus have vector relative degree 2, 2.

The second derivative is

$$\begin{bmatrix} \ddot{h}_1 \\ \ddot{h}_2 \end{bmatrix} \;=\; f_0(\omega, R) + A(R)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} .$$

The choice of inputs $u = A^{-1}(R)\,(v - f_0(\omega, R))$ input-output linearizes the system:

$$\begin{aligned} \ddot{h}_1 &= v_1 \\ \ddot{h}_2 &= v_2 \quad . \end{aligned} \tag{6}$$

If the outputs were zero, then $h_1 = \dot{h}_1 = h_2 = \dot{h}_2 = 0$. By equation (5), $\omega_1 = \omega_2 = 0$, thus from equation (1) we see that $\dot{\omega}_3 = 0$. The system will be rotating at the constant rate ω_3 about the e_3 axis. Thus, the zero dynamics manifold is TS^1. Furthermore, the trajectories are bounded for all time. To solve for the zero dynamics, we require a coordinate chart for $SO(3)$ which will keep track of this rotation ψ about e_3 as measured in the body frame.

Euler angles, although a favorite of the literature [16, 3, 8], are not advisable for this system. The $x-y-z$ Euler angle chart, when projected down to the S^2, is not symmetric. Any Euler angle chart which is symmetric, for example the $z-y-z$, is singular at the north pole. Listing, a psychologist studying the movement of the eye, noted that the eye moves in a way which minimally twists the optic nerve. Listing's law [9, 11] describes the subset of $SO(3)$ swept out by such an eye. This subset is $\Re P^2 \subset \Re P^3 = SO(3)$. The same subset of $SO(3)$ describes the configuration space of a cat which does not twist its spinal cord [13]. It can be thought of as the image under the exponential map of a two dimensional subspace of $so(3)$. We will use polar coordinates on $\Re^2$ in order to parameterize the subspace. Thus $R_\ell(\theta,\phi) = \text{Exp}(\phi(\cos(\theta)\widehat{e_1 + \sin}(\theta)e_2))$ for any $\theta, \phi \in T^2$. The formula for $R_\ell(\theta,\phi)$ is

$$\begin{bmatrix} \cos^2(\theta)(1-\cos(\phi))+\cos(\phi) & \cos(\theta)\sin(\theta)(1-\cos(\phi)) & \sin(\phi)\sin(\theta) \\ \cos(\theta)\sin(\theta)(1-\cos(\phi)) & \sin^2(\theta)(1-\cos(\phi))+\cos(\phi) & -\sin(\phi)\cos(\theta) \\ -\sin(\phi)\sin(\theta) & \sin(\phi)\cos(\theta) & \cos(\phi) \end{bmatrix} .$$

[1] In the standard Euler angle parameterization, the determinant is $\frac{1}{(1+\cos(\theta)\cos(\phi))^2}$.

The complete chart for $SO(3)$ is $R(\theta,\phi,\psi) = R_\ell(\theta,\phi)\ \mathrm{Exp}(\psi\hat{e}_3)$. It may be verified that the equation for the decoupling matrix has the following pleasing form in these coordinates.

$$A(R) = \frac{(1+h_1^2+h_2^2)}{2}\begin{bmatrix} -\sin(\psi) & -\cos(\psi) \\ \cos(\psi) & -\sin(\psi) \end{bmatrix}$$

Proposition 3 (Zero Dynamics) *Consider the control system of the satellite of equation (1) with the input-output linearizing control law of Proposition 2. The zero dynamics for these outputs, parametrized by ψ and ω_3, are given by*

$$\begin{aligned} \dot{\psi} &= \frac{2}{(1+h_1^2+h_2^2)}(h_2\dot{h}_1 - h_1\dot{h}_2) + \omega_3 \\ \dot{\omega}_3 &= \frac{-2\alpha}{(1+h_1^2+h_2^2)^2}\left(\sin(2\psi)(\dot{h}_2^2 - \dot{h}_1^2) + 2\cos(2\psi)\dot{h}_1\dot{h}_2\right) \end{aligned}$$

where ψ is measured as in the Listing parameterization.

Proof: The derivative of ω_3 may be computed directly using equation (5). To compute the derivative of ψ, differentiate the Listing coordinate chart $R(\theta,\phi,\psi)$.

$$\begin{aligned} \dot{R} = R\hat{\omega} &= R_\ell\hat{\omega}^\ell\ \mathrm{Exp}(\psi\hat{e}_3) + \dot{\psi}R\hat{e}_3 \\ &= R(\ \widehat{\mathrm{Exp}(-\psi\hat{e}_3)\hat{\omega}^\ell}) + \dot{\psi}R\hat{e}_3 \end{aligned}$$

This implies

$$\omega_3 = \omega_3^\ell + \dot{\psi} \quad .$$

The quantity ω_3^ℓ may be computed directly from the derivative of the matrix $R_\ell(\phi,\theta)$. The equation for $\dot{\psi}$ is

$$\dot{\psi} = \omega_3 - (1-\cos(\phi))\dot{\theta} \quad .$$

The angles ϕ, θ and their derivatives depend only on h and $\dot{h}$. It may be verified that these dynamics are

$$\begin{aligned} \dot{\psi} &= \frac{2}{(1+h_1^2+h_2^2)}(h_2\dot{h}_1 - h_1\dot{h}_2) + \omega_3 \\ \dot{\omega}_3 &= \frac{-2\alpha}{(1+h_1^2+h_2^2)^2}\left(\sin(2\psi)(\dot{h}_2^2 - \dot{h}_1^2) + 2\cos(2\psi)\dot{h}_1\dot{h}_2\right) \quad . \end{aligned} \tag{7}$$

Remarks

1. *Stabilization Strategy*
   The four states $h_1, h_2, \dot{h}_1$ and $\dot{h}_2$ constitute a controllable linear system and are therefore easy to stabilize to a point. The challenge rests with the remaining dynamics, ω_3 and ψ. It is impossible to affect the dynamics of ω_3 in the symmetric ($\alpha = 0$) case so we are forced to consider the symmetric and asymmetric cases separately. The controllers we propose in the next two subsection are composite: a linear system controller for the h variables will be perturbed by a high order control law to stabilize what would normally be the zero dynamics.

2. *Use of Euler Angles*
   Suppose we use $x-y-z$ Euler angles instead of this nonstandard chart. We set $R(\theta,\phi,\psi) = \operatorname{Exp}(\theta\hat{e}_1)\ \operatorname{Exp}(\phi\hat{e}_2)\ \operatorname{Exp}(\psi\hat{e}_3)$. It may be verified that the zero dynamics take on the deceptively simple form:

$$\dot{\psi} = -\sin(\phi)\dot{\theta} + \omega_3$$

   This essentially says that ψ evolves according to area swept out on the two-sphere whose spherical coordinates are (ϕ,θ)[2]. Because this is such a special equation, one might expect that it would have a simple expression in h_1, h_2 coordinates. However, this differential equation is ill defined at the equator of S^2. The singularity and the asymmetry are artifacts of the coordinate chart we have chosen for $SO(3)$.

3. *Use of Quaternions*
   Hamilton's quaternions parameterize $SO(3)$ in a way which avoids singularities (see for example [16]). Specifically, the unit quaternions form a Lie group which is diffeomorphic to the three-sphere S^3 inside the space $\Re^4$ of all quaternions, and this group comes with a canonical double covering map $S^3 \to SO(3)$. One composes this double cover with our map $R \to Re_3$ to form the standard Hopf fibration $S^3 \to S^2$. The splitting of the orientation space of our two-torque satellite into a 'direction' $Re_3 \in S^2$ and a rotational angle ψ about e_3, when pulled up to the quaternions, corresponds to the base and fiber of the standard Hopf fibration.

3.2 The Regulator for the Symmetric Satellite

We can control at most five states of the symmetric satellite since ω_3 is a constant for all inputs. We will consider the case $\omega_3 \equiv 0$ here. The case of $\omega_3 \neq 0$ is similar. The notion of a saturation function is useful (see, for example, [18]). This is a C^3 function $\rho_\epsilon(\cdot) : \Re \to \Re$ chosen so that $\rho_\epsilon(x) = x$ for $|x| < \epsilon$ and $\rho_\epsilon(x) = \epsilon\ \ sgn(x)$ for $|x| > \epsilon$. Here ϵ is a small positive number. Note that $\frac{\partial \rho_\epsilon}{\partial \psi}(\psi) = 0$ for all $|\psi| > \epsilon$. The new output functions use these saturation functions:

$$y = h + \begin{bmatrix} \rho_\epsilon^2(\psi)\cos(t) \\ \rho_\epsilon(\psi)\sin(t) \end{bmatrix}.$$

It may be verified that the second derivative of the output y is

$$\ddot{y} = B(h,\dot{h},\psi)v + f_1(h,\dot{h},\psi) \quad .$$

For sufficiently small ϵ, the matrix $B(\cdot)$ is invertible for all $h, \dot{h}$, and ψ. With this notation, the matrix B becomes

$$B(h,\dot{h},\psi) = \begin{bmatrix} 1 + \frac{4\rho_\epsilon(\psi)\frac{\partial \rho_\epsilon}{\partial \psi}(\psi)\cos(t)h_2}{1+h_1^2+h_2^2} & \frac{-4\rho_\epsilon(\psi)\frac{\partial \rho_\epsilon}{\partial \psi}(\psi)\cos(t)h_1}{1+h_1^2+h_2^2} \\ \frac{2\frac{\partial \rho_\epsilon}{\partial \psi}(\psi)\sin(t)h_2}{1+h_1^2+h_2^2} & 1 - \frac{2\frac{\partial \rho_\epsilon}{\partial \psi}(\psi)\sin(t)h_1}{1+h_1^2+h_2^2} \end{bmatrix}.$$

Its determinant is

$$\det(B) = 1 + \frac{4\rho_\epsilon(\psi)\frac{\partial \rho_\epsilon}{\partial \psi}(\psi)\cos(t)h_2}{1+h_1^2+h_2^2} - \frac{2\frac{\partial \rho_\epsilon}{\partial \psi}(\psi)\sin(t)h_1}{1+h_1^2+h_2^2} \quad . \tag{8}$$

Thanks to the higher order terms in the denominators of the second and third terms, this determinant is bounded away from zero by a careful selection of $\rho_\epsilon(\cdot)$ and ϵ.

[2] $d(-\sin\phi d\theta) = -\cos\phi d\phi \wedge d\theta$ is the area form on the two-sphere.

Proposition 4 (Regulator for the Symmetric Satellite) *Consider the control system of the symmetric satellite given by equation (1) with $\alpha = 0$ and $\omega_3 = 0$, Then, all smooth control laws $v(R, \omega, t)$ of the form $v = -B^{-1}(h, \dot{h}, \psi)\left(f_1(h, \dot{h}, \psi) + k_1 y + k_2 \dot{y}\right)$ with $k_1, k_2 > 0$ render the equilibrium point $R = I, \omega = 0$ asymptotically stable for $\epsilon > 0$ chosen sufficiently small, .*

Proof: From the definition of y, it follows that $(h, \dot{h}, \psi) \to (y, \dot{y}, \psi)$ is an invertable change of coordinates. In the y-coordinates we have that $\ddot{y} = -k_1 y - k_2 \dot{y}$, with $k_i > 0$. This implies that $y, \dot{y}$ are exponentially stable. Now apply center manifold theory [4]. The derivative of ψ is higher order in y, so ψ and t parameterize the center manifold[3]. The variables y are exponentially stable with no higher order perturbation so on the center manifold we have $y = \dot{y} = 0$. To finish the proof, we have to solve for the dynamics on the center manifold. Using $y = \dot{y} = 0$, we may solve for h and $\dot{h}$. To reduce the burden of the notation, define $b(\psi, t) = h - y$; thus, restricted to the center manifold

$$\dot{h} = \frac{\partial b}{\partial t} + \frac{\partial b}{\partial \psi}\dot{\psi} \tag{9}$$

Substituting for $\dot{h}$, we find that:

$$\dot{\psi}\left(1 - \frac{\rho_\epsilon^2(\psi)\frac{\partial \rho_\epsilon}{\partial \psi}(\psi)\sin(2t)}{1 + h_1^2 + h_2^2}\right) = \frac{-2\rho_\epsilon^3(\psi)}{1 + h_1^2 + h_2^2}$$

For small enough ϵ, the term $1 - \frac{\rho_\epsilon^2(\psi)\frac{\partial \rho_\epsilon}{\partial \psi}(\psi)\sin(2t)}{1+h_1^2+h_2^2}$ is greater than zero. Thus, the ψ dynamics are asymptotically stable. By the center manifold theorem, the equilibrium point $(R, \omega) = (I, 0)$ is stable.

Figure 1 shows simulations of the trajectory of the system in the h_1, h_2 space and resulting time evolution of the coordinate ψ.

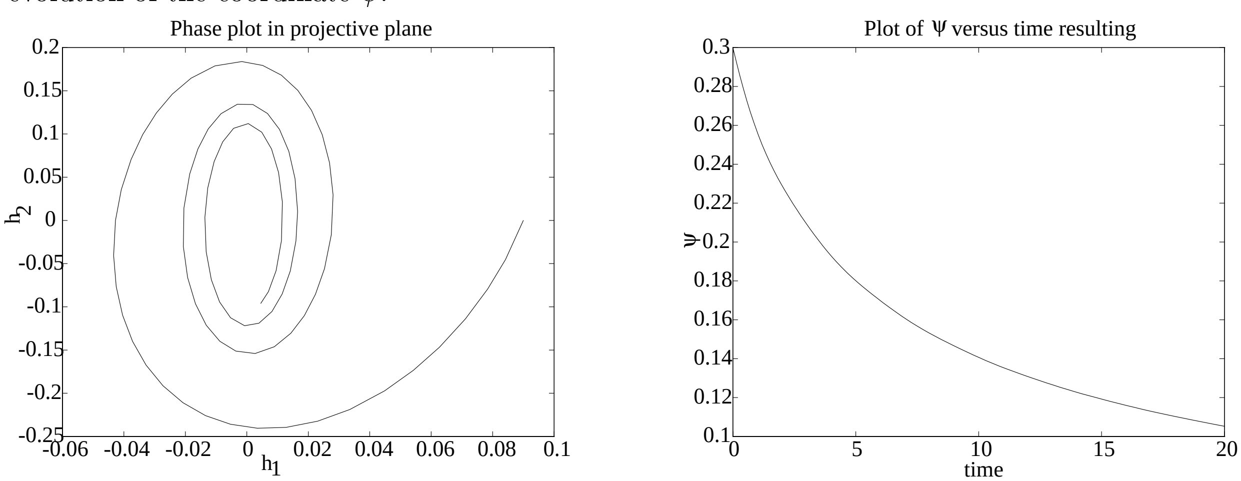

Figure 1. The trajectory in the phase plot of h_1, h_2 on the left spirals into the origin as the error along the "fiber direction" ψ is reduced. The resulting time plot of ψ is on the right.

[3]The version of the theorem that we use is for periodic time-varying systems, see for example, [22].

3.3 The Asymmetric Satellite Regulator

In this sub-section we present the regulator control law for the asymmetric satellite. To begin with we rewrite the model in a more convenient form. Write

$$S = S(\psi) = \begin{bmatrix} -\sin(\psi) & -\cos(\psi) \\ \cos(\psi) & -\sin(\psi) \end{bmatrix}$$

and

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

Recall that $h, u \in \Re^2$. The control system is

$$\begin{aligned} \dot{h} &= \frac{(1 + h_1^2 + h_2^2)}{2} S(\psi)\omega \\ \dot{\omega} &= u \\ \dot{\omega}_3 &= \alpha\omega_1\omega_2 \\ \dot{\psi} &= \omega_3 + \frac{2}{1 + h_1^2 + h_2^2}\left(h_2\dot{h}_1 - h_1\dot{h}_2\right) \end{aligned} \tag{10}$$

Note that

$$S^{-1} = \begin{bmatrix} -\sin(\psi) & \cos(\psi) \\ -\cos(\psi) & -\sin(\psi) \end{bmatrix}$$

Now we define the new outputs y by:

$$y = \omega + S(\psi)^{-1}h + \ell(\psi, \omega_3, t) \tag{11}$$

where

$$\ell = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} = \begin{bmatrix} (\psi + \omega_3)\cos(t) \\ -k_1(\psi + \omega_3)^2\cos(t) \end{bmatrix} \tag{12}$$

(We are now working locally near $\psi = 0$ so we may think of ψ as a real instead of as an angular coordinate. The reader may replace the occurrences of $\psi + \omega_3$ in the definition of ℓ with any function which agrees with $\psi + \omega_3$ up to terms of order 3. In particular, this function can be taken to be periodic in ψ.) Differentiate the output to find:

$$\dot{y} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + f_2(h, y, \psi, \omega_3, t)$$

Proposition 5 (Regulator for the Asymmetric Satellite) *Consider the satellite with dynamics described by equation (1) with $\alpha \neq 0$. All smooth control laws $u(R, \omega, t)$ of the form $u = -(k_0 y + f_2(h, y, \psi, \omega_3, t))$ with y as defined in (11), $k_0 > 0$, $sgn(k_1) = sgn(\alpha)$ and $|k_1| > 0.4$, locally asymptotically stabilize the equilibrium point $R = I, \omega = 0$.*

Proof: We begin by writing the dynamical system resulting from our control law in terms of the variables (h, y, ω_3, ψ). The system has a time-periodic center manifold [19, 22] passing through the origin which we approximate. We express the dynamics on this center manifold to the appropriate order. Then we use the method of averaging together with a Lyapunov function to prove stability.

The control law $u = -(k_0 y + f_2(h, y, \psi, \omega_3, t))$ leads to closed loop dynamics given by:

$$\begin{aligned}
\dot{h} &= \frac{(1+h_1^2+h_2^2)}{2}\Big(-h + S(\psi)^{-1}(y-\ell)\Big) \\
\dot{y} &= -k_0 y \\
\dot{\omega}_3 &= \alpha\left(y_1 - (\psi+\omega_3)\cos(t) + \sin(\psi)h_1 - \cos(\psi)h_2\right) \\
&\quad \Big(y_2 + k_1(\psi+\omega_3)^2\cos(t) + \cos(\psi)h_1 + \sin(\psi)h_2\Big) \\
\dot{\psi} &= \omega_3 + \frac{2}{1+h_1^2+h_2^2}\Big(h_2\dot{h}_1 - h_1\dot{h}_2\Big)
\end{aligned} \tag{13}$$

The linearization of this differential equation is

$$\begin{aligned}
\dot{h} &= -h + S(0)^{-1}(y - (\ell_1(\psi,\omega_3,t),0)^T) \\
\dot{y} &= -k_0 y \\
\dot{\omega}_3 &= 0 \\
\dot{\psi} &= \omega_3
\end{aligned} \tag{14}$$

It follows that the h, y directions belong to the stable eigen-directions, and that the ψ, ω_3 plane corresponds to the (generalized) eigenspace of the 0 eigenvalue. Also the subspace $y = 0$ is preserved by the dynamics. It follows that there is a time dependent center manifold passing through 0 and contained in the subspace $y = 0$. To calculate it we will use a coordinate transformation $\tilde{h} = h + \mu(\psi, \omega_3, t)$, to eliminate the time-varying dependence found in the equation for $\dot{h}_2$. In terms of these new variables, the linearized $\tilde{h}$ dynamics will be $\dot{\tilde{h}} = -\tilde{h}$. Consequently, the periodic time-varying, center manifold [19, 22] is parameterized by (ψ, ω_3, t) and can be expressed in the form $y = 0$ and $\tilde{h} = \Pi(\psi, \omega_3, t)$.

We now solve for the lower order terms of h on the center manifold. Since $y \equiv 0$ on the center manifold, we have (see (11)):

$$\omega = -\ell - S^{-1}h \tag{15}$$

We know that $\dot{h} = A(h, \psi)\omega$ with $A = \frac{1+h_1^2+h_2^2}{2}S$. Consequently

$$\dot{h} = \frac{1+h_1^2+h_2^2}{2}S\omega$$

which we expand out:

$$\begin{aligned}
\dot{h}_1 &= \frac{1+h_1^2+h_2^2}{2}\left(-\sin(\psi)\omega_1 - \cos(\psi)\omega_2\right) \\
\dot{h}_2 &= \frac{1+h_1^2+h_2^2}{2}\left(\cos(\psi)\omega_1 - \sin(\psi)\omega_2\right) \quad .
\end{aligned} \tag{16}$$

By substituting (15) into (16) we compute the the dynamics of h,

$$\begin{aligned}
\dot{h}_1 &= \frac{1+h_1^2+h_2^2}{2}\Big(-h_1 + \sin(\psi)(\psi+\omega_3)\cos(t) - k_1\cos(\psi)(\psi+\omega_3)^2\cos(t)\Big) \\
\dot{h}_2 &= \frac{1+h_1^2+h_2^2}{2}\Big(-h_2 - \cos(\psi)(\psi+\omega_3)\cos(t) - k_1\sin(\psi)(\psi+\omega_3)^2\cos(t)\Big) \quad .
\end{aligned}$$

Now we solve for the coordinates $\tilde{h}$. We require that the dynamics of $\tilde{h}_1$ have no linear time-varying terms and that the nonlinear terms of $\tilde{h}_2$ dynamics be strictly third order. The resulting center manifold equation $\tilde{h}_i = \Pi_i(t, \psi, \omega_3)$, $i = 1, 2$, will be of order two and three respectively. The general form of such a transformation is:

$$\begin{aligned} \tilde{h}_1 &= h_1 + (\gamma_{11}\psi^2 + \gamma_{12}\psi\omega_3 + \gamma_{13}\omega_3^2)\cos(t) + (\gamma_{14}\psi^2 + \gamma_{15}\psi\omega_3 + \gamma_{16}\omega_3^2)\sin(t) \\ \tilde{h}_2 &= h_2 + (\gamma_{21}\psi + \gamma_{22}\omega_3)\cos(t) + (\gamma_{23}\psi + \gamma_{24}\omega_3)\sin(t) \end{aligned} \tag{17}$$

Solving for the vectors $\gamma_1 = (\gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{14})^T \in \Re^4$, $\gamma_2 = (\gamma_{21}, \gamma_{22}, \gamma_{23}, \gamma_{24}, \gamma_{25}, \gamma_{26})^T \in \Re^6$ we get, for given k_1,

$$\begin{aligned} \gamma_1 &= (0.2, 0.44, 0.4, 0.08)^T \\ \gamma_2 &= (-0.2 + 0.2k_1, -0.68 + 0.88k_1, 0.464 - 0.024k_1, \\ &\quad -0.4 + 0.4k_1, 0.24 + 0.16k_1, 0.448 - 0.368k_1)^T \quad . \end{aligned} \tag{18}$$

Knowing the center manifold to second order, we can express the dynamics of ψ and ω_3 on the center manifold to third order. We will denote time independent polynomials by $P_i(\cdot)$, time-average zero terms by $O_i(t, \cdot)$, and remainder terms by $R_i(t, \cdot)$. The functions O_i, R_i are time periodic.

$$\begin{aligned} \dot{\psi} &= \omega_3\,(1 + P_1(\psi, \omega_3)) + O_1(t, \psi, \omega_3) + R_1(t, \psi, \omega_3) \\ \dot{\omega}_3 &= P_2(\psi, \omega_3) + O_2(t, \psi, \omega_3) + R_2(t, \psi, \omega_3) \end{aligned}$$

The terms O_i are third order, while the terms R_i are fourth or higher order. The polynomials P_1 and P_2 are

$$\begin{aligned} P_1(\psi, \omega_3) &= \gamma_{31}\omega_3^2 + \gamma_{32}\omega_3\psi + \gamma_{33}\psi^2 \\ P_2(\psi, \omega_3) &= \gamma_{41}\omega_3^3 + \gamma_{42}\omega_3^2\psi + \gamma_{43}\omega_3\psi^2 + \gamma_{44}\psi^3 \quad , \end{aligned} \tag{19}$$

with the vectors $\gamma_3 \in \Re^3$ and $\gamma_4 \in \Re^4$ being

$$\begin{aligned} \gamma_3 &= (k_1 - 1)\,(0.116, 0.56, 0.06)^T \\ \gamma_4 &= -k_1\alpha\left(0.148208 + \frac{0.01792}{k_1}, 0.4576 + \frac{0.1696}{k_1}, 0.656 + \frac{0.16}{k_1}, 0.4\right)^T \quad . \end{aligned} \tag{20}$$

We apply an averaging transformation [4], $(\psi, \omega_3) = x_1 + \Psi(x_1, t)$ where $x_1 \in \Re^2$ and where Ψ is third order in x_1 and is time-average zero. The linearization of such a transformation at $x_1 = 0$ is full rank hence a local diffeomorphism. The dynamics of x_1, computed by differentiation, are then,

$$\dot{x}_1 = \begin{bmatrix} x_{12}\,(1 + P_1(x_1)) + O_1(x_1, t) + \frac{\partial \Psi_1}{\partial t} - x_{12}\frac{\partial \Psi_1}{\partial x_{11}} + \Psi_2 + R_{11}(x_1, t) \\ P_2(x_1) + O_2(x_1, t) + \frac{\partial \Psi_2}{\partial t} - x_{12}\frac{\partial \Psi_2}{\partial x_{11}} + R_{12}(x_1, t) \end{bmatrix} \tag{21}$$

where $R_{jk}(x_1, t)$ is at least order four and x_{12} denotes the second component of the vector x_1. In the standard averaging theory, one sets

$$\frac{\partial \Psi}{\partial t} = -\begin{bmatrix} O_1(x_1, t) \\ O_2(x_1, t) \end{bmatrix} \tag{22}$$

resulting in a time periodic, time average zero Ψ and removing lower order time dependent terms. In our case, averaging will have to be repeatedly applied to obtain this result. With the choice of Ψ satisfying (22), we have

$$\dot{x}_1 = \begin{bmatrix} x_{12}\left(1 + P_1(x_1)\right) + O_{11}(x_1, t) \\ P_2(x_1) + x_{12} O_{12}(x_1, t) \end{bmatrix} + R_1(x_1, t)$$

where $R_1(x_1, t)$ is of order four and higher, $O_{11}(x_1, t)$ is third order and is time-average zero, and $O_{12}(x_1, t)$ is second order and time-average zero. The i^{th} iteration of averaging has the form, $x_{i-1} = x_i + \Psi(x_i, t)$. It maybe verified that the dynamics of the i^{th} iteration are given by

$$\dot{x}_i = \begin{bmatrix} x_{i2}\left(1 + P_1(x_i)\right) + x_{i2}^{i-1} O_{i1}(x_1, t) \\ P_2(x_1) + x_{i2}^{i} O_{i2}(x_i, t) \end{bmatrix} + R_i(x_i, t)$$

with $R_i(x_i, t)$ fourth and higher order, $O_{i1}(x_i, t)$ order $4 - i$ for $i \in (1, 4)$, and $O_{i2}(x_i, t)$ order $3 - i$ for $i \in (1, 3)$. At the fourth iteration, third order time dependent terms drop out of the equation for $\dot{x}_{i2}$ and in the fifth iteration third order time dependent terms drop out of the equation for $\dot{x}_{i1}$.

Relabel the state variables of the fifth interation by $z \in \Re^2$. In (23) below, the remainder terms, written as $R_i(t, z)$, are time periodic terms of order 4 and higher. The polynomials P_1, P_2 are as defined earlier.

$$\begin{aligned} \dot{z}_1 &= z_2\left(1 + P_1(z)\right) + R_3(t, z) \\ \dot{z}_2 &= P_2(z) + R_4(t, z) \end{aligned} \tag{23}$$

The following Lyapunov function is locally positive definite if $\alpha k_1 > 0$ and β_0 is sufficiently small.

$$\begin{aligned} V(t, z) &= 0.1\alpha k_1 z_1^4 + 0.5 z_2^2 + \alpha k_1 \beta_0 z_1^3 z_2 + \\ &\quad z_1^4 z_2 g_1(t) + \beta_2 z_1^5 + z_1^5 z_2 g_2(t) + \beta_4 z_1^6 \end{aligned} \tag{24}$$

The functions $g_i(t)$ are periodic and time average zero. They will be specified later, as will the constants β_i. We now compute the derivative of the Lyapunov function, keeping track of terms up to order six. The order four terms of $\dot{V}$ are:

$$z_2 P_2(z) + 0.4\alpha k_1 z_1^3 z_2 + 3\alpha k_1 \beta_0 z_1^2 z_2^2 \quad .$$

The Lyapunov function has been chosen so that terms involving $z_1^3 z_2$ cancel. Thus the fourth order term may be written as $z_2^2 P_3(z)$, where

$$P_3(z) = \gamma_{51} z_2^2 + \gamma_{52} z_2 z_1 + \gamma_{53} z_1^2 \tag{25}$$

is a quadratic with coefficients:

$$\gamma_5 = -\alpha k_1 \left(0.148208, 0.4576 + \frac{0.1696}{k_1}, 0.7064 + \frac{0.16}{k_1} - 3\beta_0\right)^T \quad . \tag{26}$$

The order five terms are

$$z_2 R_3(t, z) + z_1^4 z_2 \frac{\partial g_1(t)}{\partial t} + 5\beta_2 z_1^4 z_2 + 3 z_1^3 z_2^2 g_1(t).$$

Due to the periodic nature of the vector field, the order 5 part of the first term may be written $z_2^2 R_5(t, z) + z_2 z_1^4 \left(\kappa_{10} + \kappa_{11}(t)\right)$, where $\kappa_{11}(t)$ is periodic with time average zero and $R_5(t, z)$ is order

3. This calculation does not take into account the 6^{th} order terms in $z_2 R_3(t,z)$. For this reason we have to consider it again amongst the sixth order terms. Choose $\beta_2 = -\frac{\kappa_{10}}{5}$ so as to cancel the constant (averaged) part and choose $g_1(t) = \int_0^t \kappa_{11}(\tau)d\tau$ so as to cancel the time dependent average zero part of the the $z_2 z_1^4$ coefficient of $z_2 R_3$. Consequently, the order 5 terms are now of the form $z_2^2 \tilde{R}_3(t,z)$.

The order six terms are as follows.

$$3\alpha k_1 \beta_0 z_1^2 z_2^2 P_1(z) + \alpha k_1 \beta_0 z_1^3 P_2(z) + 6\beta_4 z_1^5 z_2$$
$$+5 z_1^4 z_2^2 g_2(t) + z_1^5 z_2 \frac{\partial g_2(t)}{\partial t} + z_2 R_6(t,z) \quad .$$

As with the order four terms, the periodic nature of the vector field implies that it may be written $z_2^2 R_4(t,z) + z_2 z_1^5(\kappa_{20} + \kappa_{21}(t))$. As before, β_4 and $g_2(t)$ can be chosen as to cancel all the $z_1^5 z_2$ term, leaving the 6^{th} order terms in the form $-0.4(\alpha k_1)^2 \beta_0 z_1^6 + z_2^2 R_7(t,z)$. All other terms are order 7 or higher. The derivative of the Lyapunov function is now

$$\dot{V} \;=\; z_2^2 \left(P_3(z) + R_8(t,z)\right) - 0.4(\alpha k_1)^2 \beta_0 z_1^6 + R_9(t,z) \quad . \tag{27}$$

We must check that this is negative definite for z small. The constant β_0 is chosen arbitrarily small and positive. The polynomial $P_3(z)$ is quadratic. $R_8(t,z)$ is cubic and higher in z and R_9 is seventh order and higher. So it suffices to show that the quadratic $P_3(z)$ is negative definite. The coefficients of z_1^2 and z_2^2 occurring in P_3 are both negative, since α and k_1 are taken positive. So it remains to show that the discriminant $D = b^2 - 4ac = \gamma_{52}^2 - 4\gamma_{51}\gamma_{61}$ of P_3 is negative for appropriate β_0, k_1. This is easily seen from the numerical expression for the γ_{5i}. (In fact, $D = (\alpha k_1)^2$ times, approximately $-.18 - .01/k_1 + .02/k_1^2 - 1.7\beta_0$ which is negative for all $1/k_1$ and β_0 sufficiently small.) The precise calculation of the critical gain 0.4 is left to the interested reader.

Figure 2 shows simulations of the control law for the asymmetric satellite.

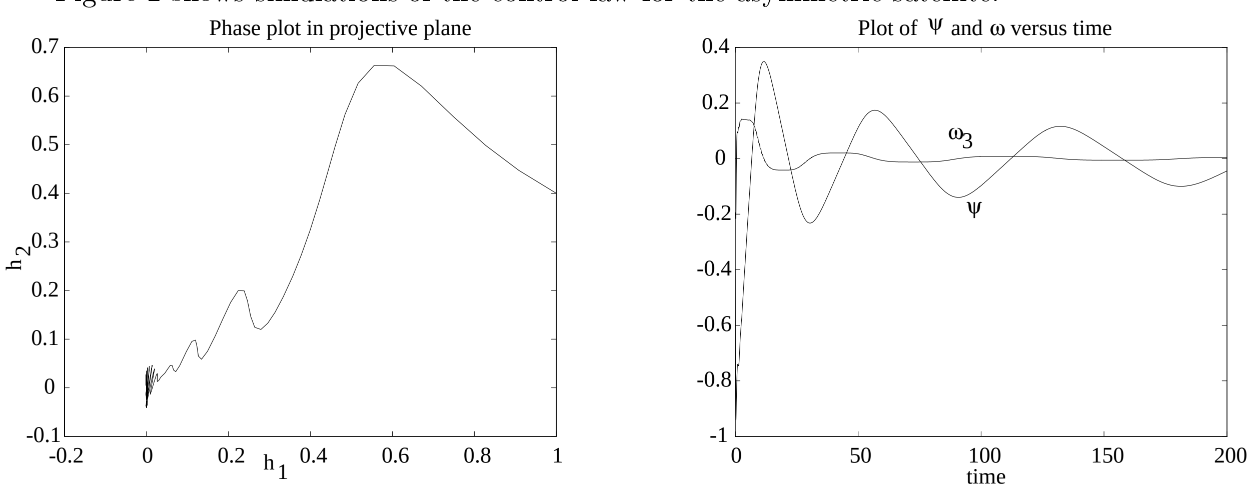

Figure 2. After an initial transient wherein the large error in h is corrected, note that the trajectory in the phase plot of h_1, h_2 on the left wobbles back and forth about the origin, unlike the symmetric regulator phase plane trajectory. The plot on the right shows the resulting evolution of ψ and ω, which converge at a slower than linear rate.

4 Conclusion

Our main contribution is the construction of stabilizing control laws for regulation of both the symmetric and asymmetric satellite. We also contribute both open loop control laws for large motions of the satellite and a natural set of coordinates for the satellite related to both the falling cat and eye-movement. All control laws are smooth. Possible future work includes the investigation of non-smooth control laws in order to improve convergence rates.

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