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Output Constraint Softening for SISO Model Predictive Control

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Abstract

The presence of constraints in the on-line optimization problem solved by Model Predictive Control algorithms results in a nonlinear control system, even if the plant and model dynamics are linear. This is the case both for physical constraints, like saturation constraints, as well for performance or safety constraints on outputs or other variables of the process. Performance constraints can usually be softened by allowing violation if necessary. This is advisable, as hard constraints can lead to stability problems. The determination of the necessary degree of softening is usually a trial-and-error matter. This paper utilizes a theoretical framework that allows to relate hard as well as soft constraints to closed-loop stability. We focus on the special case of output constraints for single-input single-output systems and develop a non-conservative condition. This condition allows the determination of the appropriate amount of softening either numerically or via a suitable Nyquist plot.

1. Introduction

A major attraction of Model Predictive Control is the ability to include constraints in the control problem formulation by simply carrying out an optimization subject to satisfaction of these constraints. The standard formulation in the literature is to list constraints on inputs, outputs and possibly other variables, and handle these constraints as hard ones, i.e., constraints that have to be satisfied before any objective is optimized. The presence of these constraints, however, complicates the question of stability. Closed-loop stability cannot be assumed simply because the on-line optimization finds a solution. Furthermore, the constraints of the on-line optimization, even if they are not physical constraints, result in a nonlinear closed-loop system in spite of the fact that the process dynamics are usually assumed to be linear. Zafiriou (1990) suggested a framework that allows the translation of robust stability of the constrained, and therefore nonlinear, closed-loop system into robustness conditions for a set of linear systems.

In this paper we focus on the case where output constraints are imposed over the prediction horizon. Although the idea behind the development in the paper can be applied to the general multi-input multi-output case (Zafiriou and Chiou, 1992), we will limit the discussion here to the single-input single-output case, since in this case a non-conservative and simple to use condition can be obtained. Zafiriou and Marchal (1991) showed in detail how hard output constraints can result in very aggressive controllers. Ricker *et al.*(1989) suggested that softening such constraints may help avoid these problems. Since not all constraints can be softened, as is the case, e.g., for saturation constraints, one needs a framework that can deal with a mix of hard and soft constraints. Zafiriou (1991) extended his original framework to include the effect of softening on closed-loop stability. In that paper a conservative sufficient closed-loop contraction condition was developed for the SISO case. In this paper we use a different approach that results in a nonconservative condition.

2. Closed-loop Stability

An impulse response model is used to describe the process:

$$\tilde{y}(k) = H_1 u(k-1) + H_2 u(k-2) + \ldots + H_N u(k-N) \quad (1)$$

where $\tilde{y}$ is the model output, u is the input and N the truncation number, i.e., it is assumed that $H_i = 0$ for $i > N$. The plant is assumed to be open-loop stable. Other types of models can also be used, e.g., step response models (Garcia and Morshedi, 1986) or state space descriptions (Li *et al.*, 1989; Ricker, 1990). The z-transfer function, $\tilde{p}^*(z)$, describing the process model is related to (1) through

$$\tilde{p}^*(z) = \sum_{i=1}^{N} H_i z^{-i} \quad (2)$$

At sampling point k, the following optimization is carried out on-line:

$$\min_{\Delta u(k),\ldots,\Delta u(k+M-1)} \sum_{l=1}^{P} [e(k+l)^2 + D^2 \Delta u(k+l-1)^2] \quad (3)$$

The minimization of the objective function is carried out over the values of $\Delta u(k)$, $\Delta u(k+1)$,..., $\Delta u(k+M-1)$, where M is a specified parameter. The minimization is subject to possible hard constraints on the inputs u, their rate of change Δu, the outputs y and other process variables

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usually referred to as associated variables. The details on the formulation of the optimization problem can be found in Prett and Garcia (1988). After the problem is solved on-line at k, only the optimal value for the first input $\Delta u(k)$ is implemented and the problem is solved again at $k+1$. The optimal $u(k)$ depends on the tuning parameters of the optimization problem, the current output measurement $y(k)$ and the past inputs $u(k-1),\ldots, u(k-N)$ that are involved in the model output prediction. Let f describe the $u(k)$ that is obtained by adding $u(k-1)$ to the $\Delta u(k)$ that is the result of the optimization solved at sampling point k :

$$u(k) = f(y(k), u(k-1), \ldots, u(k-N); r_P(k), d(k)) \qquad (4)$$

where $r_P(k)$ includes all the values of the reference signal (setpoint) during the prediction horizon from $k+1$ to $k+P$ and $d(k)$ is the disturbance effect at the output at k.

For linear model dynamics, Zafiriou (1990) showed that the *constrained* MPC is *piece-wise linear*, meaning that the dynamics of MPC for a certain constraint set J_i active, are those of a discrete linear controller. This linear controller, denoted $c_{J_i}(z)$, depends explicitly only on J_i; it depends only implicitly on the past and current values of the plant inputs and outputs. These values together with external inputs (setpoints, disturbances) determine the J_i that corresponds to a sampling point. However, if at different sampling points the Quadratic Program (QP) solution results in the same J_i, the MPC dynamics at those points are those of the *same* linear controller. The expression for the c_{J_i} is given by:

$$c_{J_i}(z) = \frac{-(\nabla_y f)_{J_i}}{1-(\nabla_{x_1} f)_{J_i} z^{-1} - \ldots - (\nabla_{x_N} f)_{J_i} z^{-N}} \qquad (5)$$

where x_j are the states of the system, defined as:

$$x_j(k) = u(k-j), \quad 1 \le j \le N \qquad (6)$$

A necessary condition for the closed-loop operator mapping the states of the system (plant + controller) from one sampling point to the next, is that each of these linear controllers gives a closed-loop stable system. Note that the contraction property implies closed-loop stability. For more details and discussion the reader is referred to Zafiriou (1990).

In this paper we consider the case of output constraints only. These are defined over a future prediction horizon:

$$y_L \le y(k+l) \le y_U, \quad w_b \le l \le w_e \qquad (7)$$

where y_L, y_U are the lower and upper limits respectively. In Zafiriou and Marchal (1991) the expressions for the c_{J_i} are given for special cases of combinations of points in the horizon, at which the hard constraints may become active at the optimum of the on-line optimization. It is also shown that for many important cases, the corresponding c_{J_i} result in an unstable closed-loop system, regardless of the values of the tuning parameters of the objective function. In such cases the only option is to soften the constraints by allowing violation by an amount ϵ. In the formulation here, the same violation variable $\epsilon \ge 0$ is used for all the points in the constraint window. Hence the output constraints are softened to be:

$$y_L - \epsilon \le y(k+l) \le y_U + \epsilon, \quad w_b \le l \le w_e \qquad (8)$$

The term $W^2\epsilon^2$ is added to the objective function, where W is the weight that determines the extent of softening. For $W = \infty$ we get hard constraints. $W = 0$ corresponds to completely removing the constraints. For a nonzero finite W, and when the on-line QP results in a nonzero ϵ, then at the optimum for at least one of the points in the constraint window, say for $N_a \in [w_b, w_e]$, we will have $y(k+N_a) = y_U+\epsilon$ or $y(k+N_a) = y_L-\epsilon$. Otherwise a smaller ϵ would reduce the objective function, while still satisfying the constraints. This point is the one for which satisfaction of the constraint presents the greatest difficulty.

We will consider the case $M = 1$, which is the usual choice for the unconstrained case, but which for the case of hard output constraints was shown to be a risky one in Zafiriou and Marchal (1991). Let the subscripts u and h correspond to the unconstrained and hard constrained cases, respectively, and f_u, f_h the result of the MPC optimization for these cases as defined in (4). Then by carrying out the computations it can be shown that when the constraint is softened, we have for the coefficients of the c_{J_i} (from (5)) that corresponds to the softened constraint at $k+N_a$:

$$\nabla_{x_j} f_s = \frac{1}{1+G^{-1}S_{N_a}^2 W^2}\nabla_{x_j} f_u + \frac{G^{-1}S_{N_a}^2 W^2}{1+G^{-1}S_{N_a}^2 W^2}\nabla_{x_j} f_h \qquad (9)$$

for $j = 1,\ldots,N$ and also for ∇_y, where the subscript s stands for soft. S_{N_a} is the value of the open-loop unit-step response of the process model at the N_a sampling point. G is the Hessian of the quadratic objective defined in (3). Note that both f_u and f_s correspond to the constraint at $k+N_a$. The closed-loop poles for each of these cases are the roots of the characteristic polynomials obtained from the numerator of $(1+\tilde{p}c_{J_i})$:

$$r_{J_i} = 1-(\psi_1)_{J_i} z^{-1} - \ldots - (\psi_N)_{J_i} z^{-N}, \quad 1 \le j \le N \qquad (10)$$

where

$$(\psi_j)_{J_i} = (\nabla_{x_j} f)_{J_i} + (\nabla_y f)_{J_i} H_j, \quad 1 \le j \le N \qquad (11)$$

We know that for the unconstrained case (Garcia and Morari, 1982) a value $M = 1$ combined with a large horizon P will result in a stable control system. (The use of the D weight also helps.) Let us assume that this has been accomplished and therefore the closed-loop characteristic polynomial $r_u(z)$ has roots inside the unit circle:

$$r_u(z) = 1 - \psi_{1,u} z^{-1} - \ldots - \psi_{N,u} z^{-N}$$

On the other hand, $r_h(z)$ is often unstable, and for $M = 1$ the parameters of the objective function have no effect of it:

$$r_h(z) = 1 - \psi_{1,h}z^{-1} - \ldots - \psi_{N,h}z^{-N}$$

Softening of the constraint allows us to tune W for stability. Define

$$\delta_{N_a} = \frac{G^{-1}S_{N_a}^2 W^2}{1 + G^{-1}S_{N_a}^2 W^2} \tag{12}$$

Note that $\delta_{N_a} = 0$ for $W = 0$ (unconstrained);
$\delta_{N_a} \to 1$ for $W \to \infty$ (hard constraint). Equation (9) can be re-written as:

$$\nabla f_s = (1 - \delta_{N_a})\nabla f_u + \delta_{N_a}\nabla f_h$$

Combined with (11) yields:

$$\psi_{j,s} = (1 - \delta_{N_a})\psi_{j,u} + \delta_{N_a}\psi_{j,h}$$

We can now use (10) to obtain:

$$\begin{aligned} r_s(z) = \quad & (1 - \delta_{N_a})r_u(z) + \delta_{N_a}r_h(z) \\ & = r_u(z) + \delta_{N_a}(r_h(z) - r_u(z)) \end{aligned}$$

We can assume at this point that the tuning parameters of the on-line objective function have been selected to yield a stable unconstrained system ($r_u(z)$). However, $r_h(z)$ may be unstable and therefore the appropriate degree of softening, defined by the value of W, can be obtained by answering the question of which is the largest δ_{N_a} (≤ 1) for which $r_s(z)$ is stable. Let $z = e^{i\theta}$. As δ_{N_a} increases from zero, if $r_h(z)$ is unstable, eventually $r_u(e^{i\theta}) + \delta_{N_a}(r_h(e^{i\theta}) - r_u(e^{i\theta})) = 0$ for some θ. This is the largest δ_{N_a} that we can accept from this analysis:

$$\delta_{N_a} = \frac{r_u(e^{i\theta})}{r_u(e^{i\theta}) - r_h(e^{i\theta})}$$

This value can be computed from a Nyquist plot of $\frac{r_u}{r_u - r_h}$. If $r_h(z)$ unstable, $\frac{r_u}{r_u - r_h}$ will cross the real axis at least once between 0 and 1. The smallest such value is the largest δ_{N_a} for which the nominal system remains stable. (12) then yields the largest W. The dependence on N_a comes form the dependence of r_h on the point in the constraint window, for which the constraint may be predicted active at the optimum of the optimization. Each point in the window will result in a different value for W. The smallest value should be used. From Zafiriou and Marchal (1991) we know:

$$r_h(z) = 1 - \frac{H_{N_a+1}}{S_{N_a}}z^{-1} - \ldots - \frac{H_N}{S_{N_a}}z^{-N+N_a}$$

Alternatively, W can be directly obtained from a different Nyquist plot. By substituting (12) into $r_u(e^{i\theta}) + \delta_{N_a}(r_h(e^{i\theta}) - r_u(e^{i\theta})) = 0$ we obtain

$$r_u(e^{i\theta}) + G^{-1}S_{N_a}^2 W^2 r_h(e^{i\theta}) = 0$$

which yields

$$W^2 = -\frac{1}{G^{-1}S_{N_a}^2}\frac{r_u(e^{i\theta})}{r_h(e^{i\theta})} \tag{13}$$

A Nyquist plot of $\frac{r_u}{r_h}$ is obtained. If it does not intersect the negative real axis, any W is acceptable, and therefore the corresponding constraint can be made hard. Otherwise the intersect nearest 0 gives W from (13).

3. Example

The example is the top SISO part of the heavy oil fractionator defined in the Shell Standard Control Problem (Prett and Garcia, 1988). The output is the top end point concentration and the input is the top draw. It is used to demonstrate how the technique described in the previous section can be used for computing the largest softening weight W. This is a case where the use of a hard constraint for the output prediction on the first point in the window (after the time delay) gives rise to instability of the control system. The process model is:

$$\frac{4.05e^{-27s}}{50s + 1}$$

For a sampling time of 4 minutes, the discrete model is:

$$\frac{0.0802(z + 2.8828)z^{-7}}{z - 0.92312}$$

Notice that there is an unstable zero in the discrete model. It becomes an unstable root of $r_h(z)$, when the hard constraint is set on the first possible point after the time delay:

$$-0.5 \leq y(k+7) \leq 0.5$$

This root at -2.8828 cannot be moved via tuning of the MPC parameters, as long as the constraint remains a hard one. As a result instability occurs during operation for which the constraint becomes active at the optimum of the optimization. This is illustrated by simulating the response to step output disturbances. The simulations (figures 1 and 2) show that enlarging the prediction horizon is useless for stabilizing the control sytem. The tuning parameters for the simulations are: $D = 0$, $M = 1, P = 8$ (figure 1), $P = 60$ (figure 2). The unconstrained simulations with the same tuning parameters are shown in figures 3, 4.

We now proceed with softening the constraint at $k + 7$ ($N_a = 7$). The weight W is determined with the method developed in the previous section. The largest values are obtained from the Nyquist plots in figures 7 and 8. For $P = 8$, we get $W = 3.6$ and for $P = 60$, $W = 290$. The simulation with a soft constraint at $k+7$ for the case $P = 8$, $W = 3.6$ is shown in figure 5. The control system is closed-loop stable and the constraint bounds are essentially satisfied. For the case with $P = 60$, $W = 290$, the simulation is shown in figure 6. The control system is stable and its performance with respect to constraint satisfaction is clearly better than that of the unconstrained case, shown in figure 4.

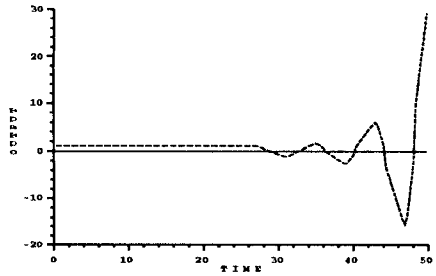

Figure 1 (a): The output of the hard constraint simulation with $P = 8$

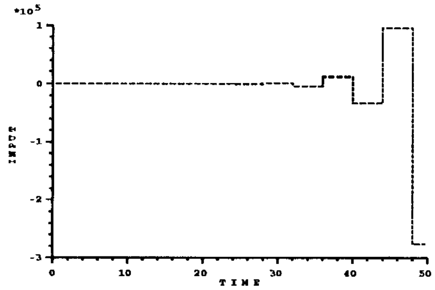

Figure 1 (b): The input of the hard constraint simulation with $P = 8$

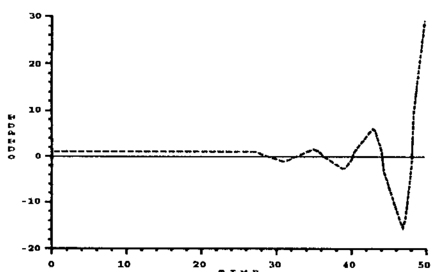

Figure 2 (a): The output of the hard constraint simulation with $P = 60$

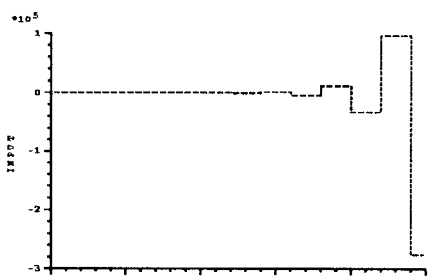

Figure 2 (b): The input of the hard constraint simulation with $P = 60$

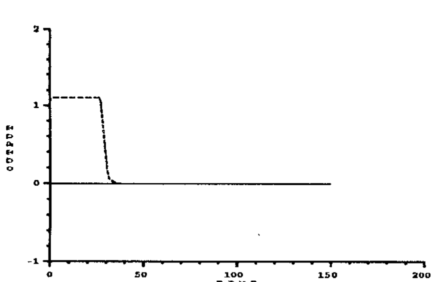

Figure 3 (a): The output of the unconstrained simulation with $P = 8$

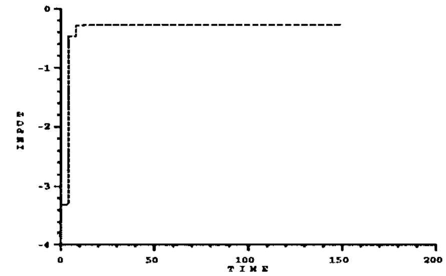

Figure 3 (b): The input of the unconstrained simulation with $P = 8$

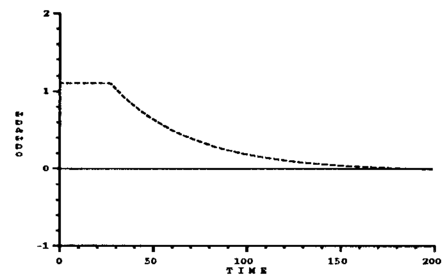

Figure 4 (a): The output of the unconstrained simulation with $P = 60$

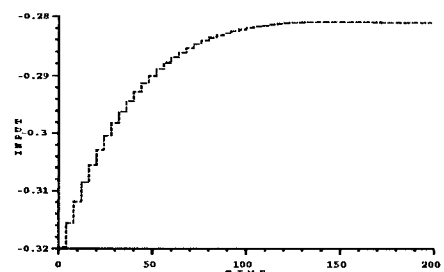

Figure 4 (b): The input of the unconstrained simulation with $P = 60$

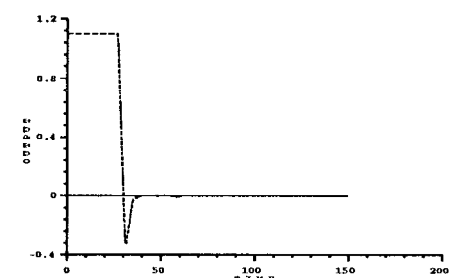

Figure 5 (a): The output of the soft constraint simulation with $P = 8,\ W = 3.6$

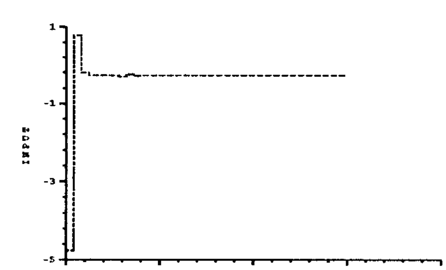

Figure 5 (b): The input of the soft constraint simulation with $P = 8,\ W = 3.6$

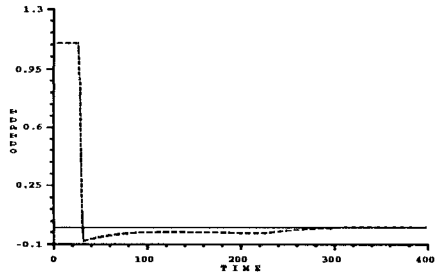

Figure 6 (a): The output of the soft constraint simulation with $P = 60,\ W = 290$

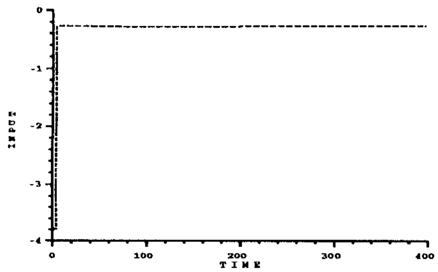

Figure 6 (b): The input of the soft constraint simulation with $P = 60,\ W = 290$

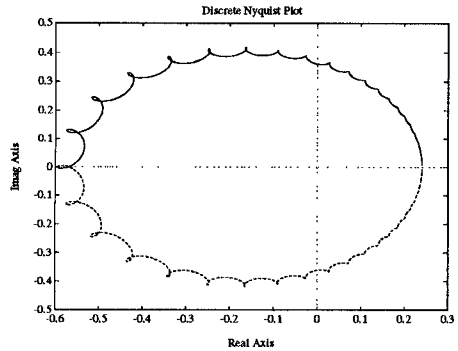

Figure 7: The Nyquist plot for the case with $P = 8$

4. Conclusions

This paper provides a method for obtaining the weights used in softening output constraints of MPC algorithms. The technique results in the largest weight (hardest constraint) that will not cause any stability problems. The task can be carried out as a second design step following the design of the unconstrained MPC algorithm. Thus, although the control system is nonlinear because of the constraint, the methods that have been developed in the literature for designing unconstrained controllers can still be used in the first step.

The method is based on the idea of handling the weights as "uncertain" parameters. Finding their largest value can be thought of as a robust stability problem. As a result the technique can be extended to MIMO systems, as well as to the case where model-plant mismatch exists.

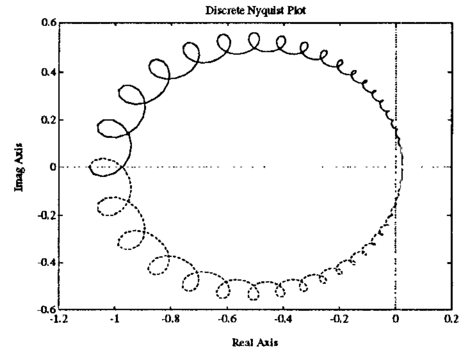

Figure 8: The Nyquist plot for the case with $P = 60$

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