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# On Stabilization with a Prescribed Region of Asymptotic Stability

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# ON STABILIZATION WITH A PRESCRIBED REGION OF ASYMPTOTIC STABILITY

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## ABSTRACT

An important unsolved problem in nonlinear control is that of stabilization with a prescribed region of stability. In this paper, sufficient conditions are obtained for the existence of a *linear* feedback stabilizing an equilibrium point of a given nonlinear system with the resulting region of asymptotic stability (RAS) containing a ball of given radius. Conditions for global stabilization are also given. Feedback stabilization is achieved while satisfying a certain robustness property. The technique is applied to planar systems, resulting in a complete design methodology for this case. Examples and simulations illustrating the method are presented.

## 1. INTRODUCTION

Given a nonlinear control system $\dot{x} = f(x,u)$ where $f : I\!R^n \times I\!R^m \to I\!R^n$ is a smooth mapping satisfying $f(0,0) = 0$, it is well known that if the associated linearized system is stabilizable, then there exists a linear feedback control which also stabilizes the (null solution of) the original nonlinear system. The null solution of the closed-loop nonlinear system of course possesses a nonempty region of asymptotic stability (RAS). The *size* of the RAS is usually not stated as an *explicit* control objective. The reason for this is the lack of systematic analytical tools for the synthesis of feedback control laws achieving specifications on the RAS. The importance of obtaining such tools is clear, and has been emphasized in [1].

The traditional approach based solely on linearization at an operating point is often considered unreliable from a stability point of view. An alternative design method consists in repeated testing of the performance of the closed-loop system for each of a set of possible stabilizing control laws. Since approximation of the obtained RAS is often very difficult, each of these tests typically involves many simulations of the closed-loop dynamics, and the method is hence very costly [2].

In this paper, sufficient conditions are obtained for the existence of a *linear* feedback stabilizing an equilibrium point of a given nonlinear system with the resulting region of asymptotic stability (RAS) containing a ball of given radius. Conditions for global stabilization are also given. Feedback stabilization is achieved while satisfying a certain robustness property. Synthesis of the desired feedback control laws rests on the solution of certain nonstandard questions in linear systems. These questions are addressed successfully for the case of planar systems, for which a complete design methodology is achieved. Examples and simulations illustrating the method are presented.

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## 2. SUFFICIENT CONDITIONS FOR STABILIZATION WITH PRESCRIBED RAS

Before presenting the main results of this section, we establish notation. With $S$ a subset of $\mathbb{C}$, $\Re e(S)$ denotes the set $\{\Re e(s) : s \in S\}$. For a real matrix $M$, $[M]_s$ and $[M]_{ss}$ denote its symmetric and skew-symmetric parts, respectively:

$$[M]_s := \frac{1}{2}(M + M^T),$$
$$[M]_{ss} := \frac{1}{2}(M - M^T).$$

The spectrum of $M$ is denoted by $\sigma(M)$. For $x$ a vector in $\mathbb{R}^n$, $|x|$ denotes its Euclidean norm. Denote by $B(R)$ the open ball in $\mathbb{R}^n$ of radius $R$ centered at the origin. By $\mathbb{R}_-$ and $\mathring{\mathbb{C}}_-$, we intend $(-\infty, 0)$ and the open left-half of the complex plane, respectively. For $S$ a given set, $S^n$ denotes the Cartesian product $S \times S \times \cdots \times S$ ($n$ times).

Two definitions relating to stabilizability of linear systems are now introduced. Let $A$ and $B$ be real matrices of dimensions $n \times n$ and $n \times m$, respectively.

Definition 1: Say that the pair $(A,B)$ is *symmetrically stabilizable* [1] if $\exists K \in \mathbb{R}^{m\times n}$ such that $\sigma([A + BK]_s) \subset \mathbb{R}_-^n$. For $\Delta$ a nonempty subset of $\mathbb{R}_-^n$, the pair $(A,B)$ is said to be *symmetrically stabilizable within* $\Delta$ if $\forall \Lambda \in \Delta$, $\exists K \in \mathbb{R}^{m\times n}$ such that $\sigma([A + BK]_s) = \Lambda$.

Definition 2: Say that the pair $(A,B)$ is *normally stabilizable* if $\exists K \in \mathbb{R}^{m\times n}$ such that $\sigma(A+BK) \subset \mathring{\mathbb{C}}_-^n$ with $A + BK$ a normal matrix. Let $\Delta \subset \mathring{\mathbb{C}}_-^n$ be nonempty. Say that $(A,B)$ *is normally stabilizable within* $\Delta$ if $\forall \Lambda \in \Delta$, $\exists K \in \mathbb{R}^{m\times n}$ such that $\sigma(A + BK) = \Lambda$ with $A + BK$ normal.

It is a simple exercise to show that if $(A,B)$ is normally stabilizable within $\Delta$ then it is symmetrically stabilizable within $\Re e(\Delta)$.

We consider nonlinear multi-input control systems $\dot{x} = F(x) + Bu$ where $F : \mathbb{R}^n \to \mathbb{R}^n$ is analytic over $\mathbb{R}^n$ and satisfies $F(0) = 0$, and where $B$ is an $n \times m$ matrix.[2] It is convenient to rewrite the model in the equivalent form

$$\dot{x} = Ax + Bu + h(x), \tag{1}$$

where $A := \frac{\partial F}{\partial x}(0)$ and $h(x)$ represents higher order terms.

Let

$$\mathcal{H} := \{\ h : \mathbb{R}^n \to \mathbb{R}^n,\ \text{analytic over } \mathbb{R}^n : h(0) = 0,\ \frac{\partial h}{\partial x}(0) = 0\ \}$$

and define for a fixed $R > 0$ the following two functions on $\mathcal{H}$

(i) $\|.\|_R : \mathcal{H} \to \mathbb{R}_+$ such that

$$\|h\|_R = \sup_{\substack{x\in B(R)\\ x\neq 0}} \frac{|h(x)|}{|x|}$$

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[1] Recall that Hurwitz stability of $[M]_s$ implies that of $M$.

[2] Notice that this model is not restrictive since a more general model $\dot{z} = f(z, v)$ may always be put in the form above by letting $x$ be the augmented state $(z,v)^T$ and taking $u = \dot{v}$.

(ii) $\|.\| : \mathcal{H} \to I\!R_+ \cup \{\infty\}$ such that

$$\|h\| = \sup_{\substack{x \in I\!R^n \\ x \neq 0}} \frac{|h(x)|}{|x|}$$

Notice that $\lim_{|x|\to 0} \frac{|h(x)|}{|x|} = 0$ is well defined and that we allow the possibility $\|h\| = \infty$ with the understanding that inequalities such as $\|h_1 + h_2\| \leq \|h_1\| + \|h_2\|$ are to be interpreted in the obvious way when $\|h_1\|$ or $\|h_2\|$ is infinite. Then $(\mathcal{H}, +, .)$ with either $\|.\|_R$ or $\|.\|$ is a normed vector space. This can be seen easily, noting that analyticity of the elements of $\mathcal{H}$ implies that a function which vanishes within $B(R)$ must also vanish everywhere. We naturally define two balls in $\mathcal{H}$:

$$\mathcal{B}_R(\rho) := \{h \in \mathcal{H} : \ \|h\|_R < \rho\},$$

$$\mathcal{B}(\rho) := \{h \in \mathcal{H} : \ \|h\| < \rho\}.$$

Theorem 1: Fix $R > 0$ and let $(A,B)$ be symmetrically stabilizable within $\Delta \subset (-\infty, -\|h\|_R)^n$ (resp. $(-\infty, -\|h\|)^n$). Then the nonlinear control system (1) is stabilizable within $B(R)$ (resp. globally stabilizable) using linear state feedback.

The proof of this theorem relies on Proposition 1, given next. Let

$$\dot{x} = F(x), \quad F(0) = 0 \tag{2}$$

where $F$ is analytic over $I\!R^n$. Let the null solution of (2) be asymptotically stable in the sense of Lyapunov. Denote the associated RAS by $\mathcal{D}^x$. Consider a change of coordinates $z = Q^T x$ where $Q$ is an *orthogonal* matrix (i.e., $Q$ satisfies $Q^T Q = Q Q^T = I$). In the new coordinates,

$$\dot{z} = \tilde{F}(z) \tag{3}$$

where

$$\tilde{F}(z) := Q^T F(Qz).$$

Clearly, the origin is also asymptotically stable for Eq. (3). The sets $\mathcal{D}^x$ and $\mathcal{D}^z$ are in general different. However, we can use the fact that orthogonal transformations preserve norms and angles to obtain the following proposition.

Proposition 1: The largest Euclidean balls in $\mathcal{D}^x$ and $\mathcal{D}^z$ are identical.

Consequently, for each $R > 0$

$$B(R) \subset \mathcal{D}^x \quad \Longleftrightarrow \quad B(R) \subset \mathcal{D}^z.$$

Proof of Theorem 1: Let $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Delta$. Since $(A, B)$ is symmetrically stabilizable within $\Delta$, there is a feedback gain matrix $K$ such that

$$\sigma([A + BK]_s) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}.$$

By setting $u(x) = Kx$ in (1), we obtain the closed loop system

$$\begin{aligned} \dot{x} &= (A + BK)x + h(x) \\ &= [A + BK]_{ss} x + [A + BK]_s x + h(x). \end{aligned} \tag{4}$$

Since $[A + BK]_s$ is symmetric, it can be diagonalized using an orthogonal transformation. Let $Q$ be such a transformation and define new coordinates $z = Q^T x$. Then $z$ satisfies

$$\dot{z} = Gz + Dz + \tilde{h}(z) \tag{5}$$

where

$$D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$
$$G = Q^T [A + BK]_{ss} Q$$

and

$$\tilde{h}(z) = Q^T h(Qz).$$

Now consider the Lyapunov function candidate $V(z) = \frac{1}{2}|z|^2$ and note that $B(R)$ is a level set for $V$. The derivative of $V$ along trajectories of (5) is

$$\dot{V}(z) = z^T Gz + z^T Dz + z^T \tilde{h}(z).$$

The first term vanishes since $G$ is skew-symmetric. Thus,

$$\begin{aligned}\dot{V}(z) &\leq \sum_{i=1}^{n} \lambda_i z_i^2 + |z| \left|\tilde{h}(z)\right| \\ &\leq \max_{1 \leq i \leq n} (\lambda_i) |z|^2 + \|h\|_R |z|^2 \\ &= \left( \max_{1 \leq i \leq n} (\lambda_i) + \|h\|_R \right) |z|^2 \end{aligned} \tag{6}$$

for all $z \in B(R)$. Noting that $\Delta \subset (-\infty, -\|h\|_R)^n$, we have that $\dot{V}(z) < 0$ for all nonzero $z \in B(R)$. A standard Lyapunov stability result now implies $B(R) \subset \mathcal{D}^z$. In view of Proposition 1, an analogous statement also holds for Eq. (4). This proves the first assertion of Theorem 1. The second assertion similarly follows from the observation

$$\begin{aligned}\dot{V}(z) &\leq \left( \max_{1 \leq i \leq n} (\lambda_i) + \|h\| \right) |z|^2 \\ &< 0\end{aligned}$$

for all $z \in I\!R^n$, $z \neq 0$. This proves global asymptotic stability.

□

Robustness of the stabilization property of Theorem 1 with respect to perturbations in the nonlinear terms $h(x)$ is now considered. The higher order terms $h(x)$ do not affect asymptotic stability of the null solution of a hyperbolic system (linearization with no imaginary eigenvalues). In our framework, we note that for $u(x) = Kx$ a linearly stabilizing feedback, the null solution of $\dot{x} = (A + BK)x + h(x)$ is asymptotically stable for all $h \in \mathcal{H}$. However, the RAS does indeed depend on variations in $h$. The next result states that the linear feedback $u(x) = Kx$ in Theorem 1 is robust to variations in $h$. Specifically, the assertion is that $B(R)$ is guaranteed to be within the RAS for each member of a family of systems each of whose linear parts is $\dot{x} = (A + BK)x$.

Theorem 1 (Robustness Form): Let $R > 0$ be fixed. Suppose that $(A, B)$ is symmetrically stabilizable within $\Delta \subset I\!R^n_-$, and let

$$\alpha := \sup_{\Lambda \in \Delta} \max_{1 \leq i \leq n} (\Lambda_i).$$

If $h \in \mathcal{B}_R(|\alpha|)$ (resp. $\mathcal{B}(|\alpha|)$ ), then the nonlinear control system $\dot{x} = Ax + Bu + h(x)$ is stabilizable within $B(R)$ (resp. globally stabilizable) using linear state feedback.

Under the foregoing assumption, this asserts the existence of a feedback gain matrix $K \in I\!R^{m \times n}$ for which the associated RAS contains $B(R)$, for each $h \in \mathcal{B}_R(|\alpha|)$. No $h \in \mathcal{B}_R(|\alpha|)$ results in an RAS not entirely containing $B(R)$.

## 3. PLANAR SYSTEMS

In this section, we apply the results of Section 2 to planar systems, i.e., systems as in (1) with $A \in I\!R^{2 \times 2}$ and $b \in I\!R^2 \setminus \{0\}$.[1] Let

$$\Delta := \{ (\lambda_1, \lambda_2) \in I\!R_- \times I\!R_- : \lambda_1 \neq \lambda_2 \text{ and } \lambda_1, \lambda_2 \notin \sigma(A) \},$$

$$\nu(A,b) := \frac{b^T \mathrm{Adj}(A) b}{|b|^2} \quad \text{and} \quad \mu(A,b) := \frac{|\mathrm{Adj}(A) b|^2}{|b|^2}.$$

Also, define the set $\Delta(A,b)$ by

$$\Delta(A,b) = \{(\lambda_1, \lambda_2) \in \Delta : \lambda_1 \lambda_2 - \nu(A,b)(\lambda_1 + \lambda_2) + \mu(A,b) = 0\}, \tag{7}$$

where $\mathrm{Adj}(A)$ denotes the adjugate of $A$. The defining equation in (7) is that of a hyperbola. This hyperbola is equivalently characterized by

$$\lambda_2 = \frac{\nu(A,b)\lambda_1 - \mu(A,b)}{\lambda_1 - \nu(A,b)}. \tag{8}$$

For the next result, we assume that the hyperbola (8) is nondegenerate, i.e. it is not a horizontal line. It will be seen shortly that this assumption amounts to $(A, b)$ being controllable.

**Theorem 2.** Assume that $(A, b)$ is controllable. Then the pair $(A, b)$ is normally stabilizable within $\Delta(A, b)$ if and only if $\nu(A, b) < 0$. Furthermore, given any set of desired closed-loop eigenvalues $(\lambda_1, \lambda_2) \in \Delta(A, b)$, the corresponding normally stabilizing feedback gain is given by

$$k = [1 \quad 1] \, [ \, (\lambda_1 I - A)^{-1} b \quad (\lambda_2 I - A)^{-1} b \, ]^{-1}. \tag{9}$$

**Proof.** First, we show that $\Delta(A, b) \neq \emptyset$. From Eq. (8), we obtain

$$\frac{\partial \lambda_2}{\partial \lambda_1} = \frac{\mu(A,b) - \nu^2(A,b)}{(\lambda_1 - \nu(A,b))^2}. \tag{10}$$

[1] If we let $B$ be a nonzero $2 \times 2$ matrix, then it is either nonsingular, in which case the stabilization problem becomes trivial, or of rank one. The latter case is equivalent to considering $B$ to be a vector $b$ in $I\!R^2$.

Define the controllability matrices $C$, $C_{\mathrm{a}}$ by $C := [b\ Ab]$, $C_{\mathrm{a}} := [b\ \mathrm{Adj}(A)b]$ and note that $\det(C_{\mathrm{a}}) = -\det(C) \neq 0$. Then it easily follows that

$$\det(C_{\mathrm{a}}^T C_{\mathrm{a}}) = (\det(C_{\mathrm{a}}))^2 = |b|^4 \left(\mu(A,b) - \nu^2(A,b)\right) > 0. \tag{11}$$

and hence that $\lambda_2$ is a monotonically strictly increasing function of $\lambda_1$. A quick sketch of the plot of $\lambda_2$ as a function of $\lambda_1$ convinces us that $\Delta(A,b) \cap I\!R^2_- \neq \emptyset$ precisely when $\nu(A,b) < 0$. The sketch just referred to is also useful in finding pairs $(\lambda_1, \lambda_2) \in \Delta(A,b)$. To find such a pair, we may simply pick a value $\lambda_1$ in $(\nu(A,b), 0) \setminus \sigma(A)$, and then use Eq. (8) to compute the corresponding value of $\lambda_2$.

Next, we show that for $(\lambda_1, \lambda_2) \in \Delta(A,b)$, the vectors

$$v_i = (\lambda_i I - A)^{-1} b\ , \quad i = 1,2$$

are orthogonal. It is equivalent to show that the vectors $w_1$ and $w_2$ are orthogonal, where

$$w_i = \chi_A(\lambda_i)\ v_i\ , \quad i = 1,2$$

and $\chi_A(s)$ denotes the characteristic polynomial of $A$. (Recall that $\lambda_1,\ \lambda_2 \notin \sigma(A)$.) We have

$$\begin{aligned} w_1^T w_2 &= b^T \left(\mathrm{Adj}(\lambda_1 I - A)\right)^T \mathrm{Adj}(\lambda_2 I - A) b \\ &= b^T \left(\lambda_1 \lambda_2 I - \lambda_1 \mathrm{Adj}(A) - \lambda_2 \mathrm{Adj}^T(A) + \mathrm{Adj}^T(A)\mathrm{Adj}(A)\right) b \\ &= |b|^2 \lambda_1 \lambda_2 - (\lambda_1 + \lambda_2) b^T \mathrm{Adj}(A) b + |\mathrm{Adj}(A) b|^2 . \end{aligned}$$

Since $(\lambda_1, \lambda_2) \in \Delta(A,b)$, it follows that $w_1^T w_2 = 0$.

We now show that $v_1$ and $v_2$ are eigenvectors of $A + bk$ corresponding to $\lambda_1$ and $\lambda_2$, respectively. Let $V = [v_1\ \ v_2]$. Then $k = [1\ \ 1]V^{-1}$ and

$$\begin{aligned} [\ (A+bk)v_1\ \ (A+bk)v_2\ ] &= (A+bk)V = AV + bkV \\ &= [Av_1 + b\ \ Av_2 + b]. \end{aligned}$$

On the other hand, $Av_i + b = \left(A(\lambda_i I - A)^{-1} + I\right) b = \lambda_i v_i$ for $i = 1,2$. Therefore

$$(A+bk)v_i = \lambda_i v_i, \quad i = 1,2,$$

i.e., $\sigma(A+bk) = \{\lambda_1, \lambda_2\}$. Since the eigenvectors $v_1$ and $v_2$ are orthogonal, we obtain that $A + bk$ is a normal matrix (in fact symmetric since $\lambda_1$ and $\lambda_2$ are real).

$\square$

Theorem 2 states that if $(A,b)$ is controllable and $\nu(A,b) < 0$, then every pair $(\lambda_1, \lambda_2) \in \Delta(A,b)$ may be assigned via linear linear feedback while achieving the normality requirement. It is not necessary however that a pair $(A,b)$ be controllable for it to be normally stabilizable.

Remarks.

1. It is easily shown that by allowing complex eigenvalues in $\Delta$ and $\Delta(A,b)$, one obtains one additional pair of assignable eigenvalues; namely $\nu(A,b) \pm i\,|det(C)|$. Thus the set of all distinct assignable eigenvalues not in $\sigma(A)$ is essentially real.

2. The condition $\nu(A,b) < 0$ implies that $\det[b \;\; \mathrm{Adj}(A)b] \neq 0$, hence $(A,b)$ controllable since $\det[b \; Ab] = -\det[A \; \mathrm{Adj}(A)b]$, in all but the case when $b$ and $\mathrm{Adj}(A)b$ are of opposite directions.
3. Note that it is not possible to force *both* eigenvalues to be arbitrarily large. Clearly, this is a consequence of the normality requirement. It can be shown that in order for a pair $(A,B)$ to be "arbitrarily" normally stabilizable (normally stabilizable with arbitrarily negative assignable closed-loop eigenvalues) it is necessary and sufficient that rank $(B) = n$, $n$ being the size of $A$.

The next theorem is a direct consequence of Theorems 1 and 2. Let $R > 0$ be a fixed number.

Theorem 3: Assume that $\nu(A,b) < 0$. If $h \in \mathcal{B}_R(|\nu(A,b)|)$ (resp. $\mathcal{B}(|\nu(A,b)|)$ ) then there exists a linear feedback $u(x) = kx$ such that (the origin of) the closed-loop system $\dot{x} = (A + bk)x + h(x)$ is asymptotically stable within $B(R)$ (resp. globally asymptotically stable). Furthermore, for any desired closed-loop eigenvalues $\lambda_1 \in (\, \nu(A,b), -\|h\|_R \,) \setminus \sigma(A)$ (resp. $(\nu(A,b), -\|h\| \,) \setminus \sigma(A)$ ) and $\lambda_2$ given by (8), the feedback gain $k$ is given by (9).

The analogue of Theorem 1 (Robustness Form) in the two-dimensional case follows by taking $\alpha$ to be any number strictly between $\nu(A,b)$ and 0. The desired feedback $k$ is obtained in a manner identical to that outlined in Theorem 3, with $-\|h\|_R$ (resp. $-\|h\|$) replaced by $\alpha$.

## 4. EXAMPLES

Example 1: Let $R = 1$ and consider the system

$$\dot{x}_1 = -\frac{3}{2}x_1 + x_2 + x_1^2$$
$$\dot{x}_2 = x_2 + u - x_2^2.$$

The origin of the unforced system is unstable since $\sigma(A) = \{-\frac{3}{2}, 1\}$. It is easily checked that $(A,b)$ is controllable, $\nu(A,b) = -\frac{3}{2}$, $\|h\|_{R=1} = 1 < |\nu(A,b)|$ and $\mu(A,b) = \frac{13}{4}$. By picking $\lambda_1$ in $(-\frac{3}{2}, -\|h\|_R\,) \setminus \sigma(A) = (-\frac{3}{2}, -1)$, say $\lambda_1 = -\frac{5}{4}$, we get from (8) that $\lambda_2 = -\frac{11}{2}$ and from (9) that $k = [1 \quad -\frac{25}{4}]$. The closed-loop system is

$$\dot{x}_1 = -\frac{3}{2}x_1 + x_2 + x_1^2$$
$$\dot{x}_2 = x_1 - \frac{21}{4}x_2 - x_2^2.$$

Simulations of the closed-loop system for various initial conditions, shown in Fig. 1, corroborate the fact that $B(1)$ is contained in the actual RAS. Note that some initial conditions in the immediate vicinity of $B(1)$ lead to instability (e.g., $x_0 = (1.4, 0)$ and $(1.2, 1.2)$).

Example 2: We consider globally stabilizing the system

$$\dot{x}_1 = -\frac{3}{2}x_1 + x_2$$
$$\dot{x}_2 = -x_1 + x_2 + u + \sin(x_1).$$

Here, $A$ and $b$ are as in Example 1, $h(x) = [0, \sin(x_1) - x_1]^T$ and $\|h\| = 1.217$. Proceeding as in Example 1, we obtain by choosing $\lambda_1 = -\frac{5}{4}$ that the closed-loop system is

$$\dot{x}_1 = -\frac{3}{2}x_1 + x_2$$
$$\dot{x}_2 = -\frac{21}{4}x_2 + \sin(x_1).$$

With the Lyapunov function $V(x) = x_1^2 + x_2^2$, we find that $\dot{V}(x) < -x_1^2 - \frac{17}{2}x_2^2 < 0$ for all $x \neq 0$. Therefore, the null solution of the closed-loop system is globally asymptotically stable, as predicted by Theorem 3.

## 5. CONCLUSION

Sufficient conditions for stabilizability of nonlinear systems with a region of asymptotic stability containing a prescribed ball in $I\!R^n$ have been presented. Under a symmetric stabilizability condition on the system linearization, it was shown that there is a linear stabilizing controller, and that the closed-loop system stability is robust to certain model perturbations. Necessary and sufficient conditions for normal stabilizability of a two-dimensional linear time-invariant system were obtained. These facilitated identification of a closed-form formula for a stabilizing feedback gain $k$ which guarantees stabilization within a given ball. Necessary and sufficient conditions for normal stabilizability of a general pair $(A,B)$ is an issue that is currently being considered.

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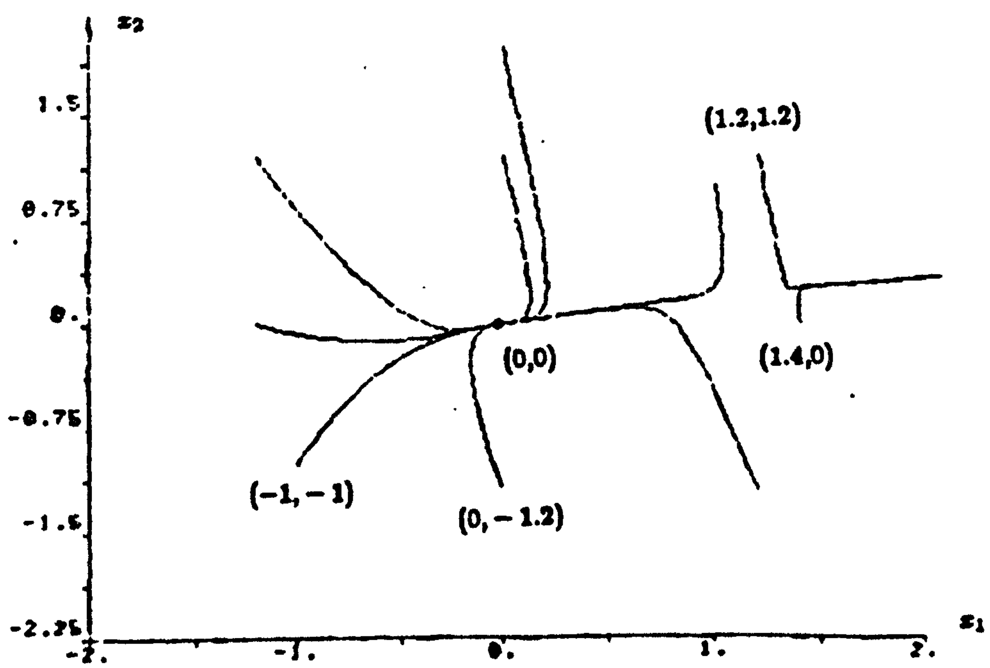


**Fig. 1 Closed loop trajectories for Example 1**