

SRC TR 88-48



**TECHNICAL
RESEARCH
REPORT**

**A Canonical Form For Controllable
Singular Systems**

by

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A CANONICAL FORM FOR CONTROLLABLE SINGULAR SYSTEMS

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Abstract

A new canonical form for the action of restricted system equivalence on controllable singular systems is given. The construction of this form is based on the Weierstrass decomposition of the singular system into a slow and a fast subsystem. Both subsystems are transformed into Hermite canonical form. The resulting Hermite canonical form for singular systems has a particularly simple structure and is expected to be useful for e.g. identification purposes. Continuity properties of the Hermite form are investigated and the nonexistence of a globally defined continuous canonical form for controllable singular systems is shown.

1. Introduction

In this paper we consider canonical forms for controllable singular systems

$$E\dot{x} = Ax + Bu \quad (1.1)$$

with E, A $n \times n$ and B $n \times m$ matrices, which satisfy the admissibility condition

$$\det(\lambda E + \mu A) \not\equiv 0. \quad (1.2)$$

Two such systems (E, A, B) , (E', A', B') are called *restricted system equivalent* if they are related by suitable invertible coordinate transformations M, N with

$$(E', A', B') = (MEN^{-1}, MAN^{-1}, MB). \quad (1.3)$$

For regular systems (E, A, B) with $\det E \neq 0$, canonical forms for the group action of restricted system equivalence (1.3) are well known and form now a cornerstone of the structure theory of controllable regular systems. They play an important role in diverse topics such as parametrization theory, model reduction theory and the identification problem, to name a few. See e.g. [5], [9], [12], [17].

¹Research partially supported by the National Science Foundation under grants ECS-8696108, CDR-8500108, and by a grant from the Monsanto Company.

In the past few years there has been a growing interest in the analysis and classification of general singular systems, see e.g. [19], [20], [22], and the references cited there. However, only very recently there has been some progress on the general canonical form problem for arbitrary controllable singular systems (1.1), (1.2). *Pseudo-canonical form* for restricted system equivalence were derived by Christodoulou and Mertzios [3], Tan and Vandewelle [21] and applied to the observer design for singular systems in [22]. However, these pseudo-canonical forms are *not* canonical forms in the usual sense and there exist in general several inequivalent singular systems having the same pseudo-canonical form. Glüsing-Lüerßen and Hinrichsen [8] were the first who described a canonical form for controllable singular systems in the terminological sense; see also [6]. They use the well-known Weierstrass decomposition of a singular pencil $\lambda E - A$ to decompose the system into a controllable slow subsystem (A_1, B_1) and a controllable fast subsystem (A_2, B_2) . The slow subsystem is then transformed into standard control canonical form while the fast subsystem (A_2, B_2) , with A_2 nilpotent, is transformed into Jordan canonical form. They also analyse the continuity properties of this resulting *Jordan control canonical form* for singular systems. Due to the inherent structural complexity of both Popov's control canonical form [17] and the Jordan canonical form [15] the Jordan control canonical form for singular systems is quite complicated to describe.

In this paper we propose a different canonical form which is based on the *Hermite canonical form for controllable pairs* (A, B) described in [16], see also [14]. Our construction is similar to that in [8] but transforms *both* subsystems (A_1, B_1) and (A_2, B_2) into Hermite canonical form. The resulting canonical form for singular systems has a considerably simpler structure than the Jordan control canonical form [8] and should be of value for structural considerations. Our analysis runs parallel to the basic treatment of Glüsing-Lüerßen and Hinrichsen [8] and we refer to [8], [13] for additional information.

The paper is organized as follows: Section 2 contains the preliminaries. In section 3 we recall the construction of the Hermite canonical form for controllable pairs. The piecewise continuity of the Hermite form is shown. Section 4 gives the Hermite canonical form for controllable singular systems. The discrete invariants of the form are given by pairs of combinations $(K_1, \dots, K_m; L_1, \dots, L_m)$. Continuity properties of the form are investigated in section 5. Based on a topological version of the Weierstrass decomposition due to [4] we prove the continuity of the Hermite canonical form on the subsets, obtained by fixing the discrete invariants $(K_1, \dots, K_m; L_1, \dots, L_m)$. Finally we prove the nonexistence of continuous canonical forms for singular systems. This depends on certain results on the topology of singular systems, given in [15].

2. Preliminaries

Let \mathbf{F} denote either the field \mathbb{R} or real numbers of the field \mathbb{C} of complex numbers.

A singular system

$$E\dot{x} = Ax + Bu \quad (2.1)$$

with $(E, A, B) \in \mathbf{F}^{n \times n} \times \mathbf{F}^{n \times n} \times \mathbf{F}^{n \times m}$ arbitrary is called *admissible* if the genericity condition for the homogenous polynomial

$$\det(\lambda E + \mu A) \neq 0 \quad (2.2)$$

in (λ, μ) holds. In the sequel we will only consider singular systems (2.1) which satisfy the admissibility condition (2.2). Let $\tilde{\mathfrak{A}}_{n,m}(\mathbf{F})$ denote the subset of $\mathbf{F}^{n \times (2n+m)}$ of all admissible singular systems:

$$\tilde{\mathfrak{A}}_{n,m}(\mathbf{F}) := \{(E, A, B) \in \mathbf{F}^{n \times (2n+m)}; \det(\lambda E + \mu A) \neq 0\}.$$

$\tilde{\mathfrak{A}}_{n,m}(\mathbf{F})$ is a Zariski-open subset of the affine space $\mathbf{F}^{n \times (2n+m)}$ and is thus open and dense.

Two linear state space systems $(A, B), (A', B') \in \mathbf{F}^{n \times n} \times \mathbf{F}^{n \times m}$ are called *similar*, $(A, B) \sim_{\sigma} (A', B')$ if

$$(A', B') = (SAS^{-1}, SB) \quad (2.3)$$

for some invertible transformation $S \in \text{GL}_n(\mathbf{F})$. \sim_{σ} defines an equivalence relation on $\mathbf{F}^{n \times (n+m)}$ and the equivalence classes

$$[A, B]_{\sigma} := \{(SAS^{-1}, SB); S \in \text{GL}_n(\mathbf{F})\} \quad (2.4)$$

are by definition the orbits of the group action

$$\begin{aligned} \sigma: \text{GL}_n(\mathbf{F}) \times \mathbf{F}^{n \times (n+m)} &\longrightarrow \mathbf{F}^{n \times (n+m)} \\ (S, (A, B)) &\longmapsto (SAS^{-1}, SB). \end{aligned} \quad (2.5)$$

σ is called the *similarity action* of $\text{GL}_n(\mathbf{F})$.

Two singular systems $(E, A, B), (E', A', B') \in \tilde{\mathfrak{A}}_{n,m}(\mathbf{F})$ are called *equivalent* (in symbols $(E, A, B) \sim_{\eta} (E', A', B')$) if they belong to the same orbit of the *group action of (restricted) system equivalence*

$$\begin{aligned} \eta: (\text{GL}_n(\mathbf{F}) \times \text{GL}_n(\mathbf{F})) \times \tilde{\mathfrak{A}}_{n,m}(\mathbf{F}) &\longrightarrow \tilde{\mathfrak{A}}_{n,m}(\mathbf{F}) \\ ((M, N), (E, A, B)) &\longmapsto (MEN^{-1}, MAN^{-1}, MB). \end{aligned} \quad (2.6)$$

The orbits of η are denoted by

$$[E, A, B]_\eta := \{(MEN^{-1}, MAN^{-1}, MB); M, N \in GL_n(\mathbf{F})\} \quad (2.7)$$

Thus

$$(E, A, B) \sim_\eta (E', A', B') \iff [E, A, B]_\eta = [E', A', B']_\eta. \quad (2.8)$$

By the well-known Weierstrass decomposition for singular systems [7], every system $(E, A, B) \in \tilde{\mathcal{U}}_{n,m}(\mathbf{F})$, $r := \deg \det(\lambda E - A)$, is equivalent to a singular system in the *standard form*

$$\dot{x}_1 = A_1 x_1 + B_1 u \quad (2.9a)$$

$$A_2 \dot{x}_2 = x_2 + B_2 u \quad (2.9b)$$

with A_2 nilpotent and $B_1 \in \mathbf{F}^{r \times m}$, $B_2 \in \mathbf{F}^{(n-r) \times m}$. (2.9a), (2.9b) are called the *slow* resp. *fast subsystems* of (1.1). They are defined uniquely up to a similarity which leaves the slow and fast subspaces invariant.

Lemma 1.2. Let

$$E = \begin{bmatrix} I_r & 0 \\ 0 & A_2 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}$$

$$E' = \begin{bmatrix} I_r & 0 \\ 0 & A'_2 \end{bmatrix}, \quad A' = \begin{bmatrix} A'_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}$$

be in standard form (2.9) with A_2, A'_2 nilpotent. Assume there exist $M, N \in GL_n(\mathbf{F})$ with $(E', A') = (MEN^{-1}, MAN^{-1})$. Then

$$M = N = \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix}$$

with $M_{11} \in GL_r(\mathbf{F})$, $M_{22} \in GL_{n-r}(\mathbf{F})$.

Proof.

We partition

$$M = \begin{bmatrix} M_{11} & M_{22} \\ M_{21} & M_{22} \end{bmatrix}, \quad N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$$

with $M_{11}, N_{11} r \times r$, $M_{22}, N_{22} (n-r) \times (n-r)$, etc. Then $(E', A') = (MEN^{-1}, MAN^{-1})$ is equivalent to the following set of equations:

$$M_{12}A_2 = N_{12}, \quad M_{12} = A'_1 N_{12} \quad (2.10a)$$

$$A'_2 N_{21} = M_{21}, \quad N_{21} = M_{21} A_1 \quad (2.10b)$$

$$M_{ii}A_i = A'_i N_{ii}, \quad M_{ii} = N_{ii}, \quad i = 1, 2. \quad (2.10c)$$

(2.10a) implies that $A_1' N_{12} A_2 = N_{12}$ and thus (upon iteration) that $N_{12} = 0$, since A_2 is nilpotent. Similarly for N_{12} , M_{12} and M_{21} . \square

Recall that a singular system (E, A, B) is called *controllable* if and only if the pencil condition

$$\text{rank}(\lambda E - \mu A, B) = n \quad \forall (\lambda, \mu) \in \mathbb{C}^2 - \{(0, 0)\} \quad (2.11)$$

is satisfied [18]. The following characterization is due to Yip and Sincovec [23].

Lemma 1.3 ([23]). A system $(E, A, B) \in \tilde{\mathcal{A}}_{n,m}(\mathbb{F})$ in standard form (2.9) is controllable if and only if the subsystems $(A_1, B_1), (A_2, B_2)$ are controllable. \square

Let

$$\tilde{C}(n, m) := \{(E, A, B) \in \tilde{\mathcal{A}}_{n,m}(\mathbb{F}); (E, A, B) \text{ is controllable}\}$$

denote the set of all controllable systems $(E, A, B) \in \tilde{\mathcal{A}}_{n,m}(\mathbb{F})$. $\tilde{C}(n, m)$ is a Zariski-open subset of $\mathbb{F}^{n(2n+m)}$, [13], and thus open and dense in $\mathbb{F}^{n(2n+m)}$.

Similarly let

$$\tilde{\Sigma}_{n,m} := \{(A, B) \in \mathbb{F}^{n \times n} \times \mathbb{F}^{n \times m}; \text{rk}(B, AB, \dots, A^{n-1}B) = n\}$$

denote the Zariski-open subset of $\mathbb{F}^{n(n+m)}$ of all controllable pairs (A, B) .

The group actions σ, η restrict to actions on the spaces $\tilde{\Sigma}_{n,m}$ resp. $\tilde{C}(n, m)$ of controllable systems.

$$\begin{aligned} \sigma: \text{GL}_n(\mathbb{F}) \times \tilde{\Sigma}_{n,m} &\longrightarrow \tilde{\Sigma}_{n,m} \\ (S, (A, B)) &\longmapsto (SAS^{-1}, SB) \end{aligned} \quad (2.12)$$

resp.

$$\begin{aligned} \eta: (\text{GL}_n(\mathbb{F}) \times \text{GL}_n(\mathbb{F})) \times \tilde{C}(n, m) &\longrightarrow \tilde{C}(n, m) \\ ((M, N), (E, A, B)) &\longmapsto (MEN^{-1}, MAN^{-1}, MB). \end{aligned} \quad (2.13)$$

We refer to (2.12) resp. (2.13) as the similarity action resp. equivalence action on $\tilde{\Sigma}_{n,m}$ resp. $\tilde{C}(n, m)$.

Let

$$C(n, m) := \tilde{C}(n, m) / (\text{GL}_n(\mathbb{F}) \times \text{GL}_n(\mathbb{F}))$$

denote the set of all orbits $[E, A, B]_\eta$ of controllable systems. We endow $C(n, m)$ with the quotient topology, i.e. with the finest topology on $\tilde{C}(n, m)$ for which the quotient map

$$\begin{aligned} \pi: \tilde{C}(n, m) &\longrightarrow C(n, m) \\ (E, A, B) &\longmapsto [E, A, B]_\eta \end{aligned}$$

is continuous. Since equivalent systems $(E, A, B), (E', A', B')$ have the same system theoretic properties, we may consider $C(n, m)$ as the proper space of all (abstract) controllable singular systems.

3. Hermite indices

In this section we recall the construction of the Hermite canonical form for the similarity action on controllable pairs (A, B) . This canonical form due to Mayne [16] and Hinrichsen and Prätzel-Wolters [14], will be the basic tool for our construction of a canonical form for singular systems, described in section 4.

A *combination* of $n \in \mathbb{N}$ into m parts is any m -tuple of nonnegative integers $K = (K_1, \dots, K_m)$ with $K_1 + \dots + K_m = n$. Combinations K of fixed length m and weight n form a finite set $K_{n,m} := \{K \in \mathbb{Z}^m; K_i \geq 0, \sum_{i=1}^m K_i = n\}$ of cardinality

$$\binom{n+m-1}{n} = \frac{(n+m-1)!}{n!(m-1)!}.$$

Let $(A, B) \in \tilde{\Sigma}_{n,m}$ be controllable and let b_1, \dots, b_m denote the first, \dots , m -th column of the matrix B . We consider the following deletion procedure on the columns of the controllability matrix of (A, B) :

Delete in the list

$$(b_1, Ab_1, \dots, A^{n-1}b_1, \dots, b_m, Ab_m, \dots, A^{n-1}b_m),$$

while going from left to right, all vectors $A^k b_l$ which are linearly dependent on the set of all previous vectors.

By controllability, the remaining vectors

$$T_{AB} := (b_1, \dots, A^{K_1-1}b_1, \dots, b_m, \dots, A^{K_m-1}b_m) \quad (3.1)$$

form a basis of the state space \mathbb{F}^n with nonnegative integers K_1, \dots, K_m satisfying $K_1 + \dots + K_m = n$. The combination $K = (K_1, \dots, K_m) \in K_{n,m}$ is called the list of *Hermite indices* of (A, B) .

In the sequel we use the following terminology which is standard in control theory, see [1], [5], [12].

Definition 3.1. Let X be a set with equivalence relation \sim . A *canonical form* for \sim is a map $\Gamma: X \longrightarrow X$ which satisfies:

- (i) $\Gamma(x) \sim x$ for all $x \in X$
- (ii) $\Gamma(x) = \Gamma(y) \iff x \sim y$ for all $y, x \in X$.

It is now easy to describe a canonical form for the similarity action σ on the set $\tilde{\Sigma}_{n,m}$ of controllable pairs (A, B) . For any $(A, B) \in \tilde{\Sigma}_{n,m}$ let

$$(A_H, B_H) := (T_{AB}^{-1} A T_{AB}, T_{A,B}^{-1} B) \quad (3.2)$$

with $T_{AB} \in \text{GL}_n(\mathbb{F})$ defined by (3.1). Then

$$\begin{aligned} \Gamma_H: \tilde{\Sigma}_{n,m} &\longrightarrow \tilde{\Sigma}_{n,m} \\ (A, B) &\longmapsto (A_H, B_H) \end{aligned} \quad (3.3)$$

is called the *Hermite canonical form* for the similarity action $\sigma: \text{GL}_n(\mathbb{F}) \times \tilde{\Sigma}_{n,m} \longmapsto \tilde{\Sigma}_{n,m}$.

Theorem 3.2 ([14],[16]). Γ_H is a canonical form for the similarity action on controllable systems. Every $(A, B) \in \tilde{\Sigma}_{n,m}$ with Hermite indices $K = (K_1, \dots, K_m)$ is similar to a unique system $(A_H, B_H) \in \tilde{\Sigma}_{n,m}$ of the form

$$A_H = (A_{ij})_{i,j \in \underline{m}} \text{ with } A_{ij} = \begin{cases} \begin{bmatrix} 0 & \cdots & 0 & \alpha_{ij1} \\ 1 & \cdots & 0 & \\ 0 & & & \\ \vdots & & & \vdots \end{bmatrix}_{K_i \times K_i} & \text{if } i = j \\ \begin{bmatrix} 0 & \cdots & 1 & \alpha_{ijK_i} \\ 0 & \cdots & 0 & \alpha_{ij1} \\ \vdots & & & \vdots \end{bmatrix}_{K_i \times K_j} & \text{if } i < j \\ 0_{K_i \times K_j} & \text{if } i > j \end{cases} \quad (3.4)$$

$$B_H = [b_1, \dots, b_m] \text{ with } b_j = \begin{cases} e_{K_1 + \dots + K_{j-1} + 1} & \text{for } K_j > 0 \\ \sum_{\substack{l=1 \\ i < j}}^{K_i} \alpha_{ijl} e_{K_1 + \dots + K_{i-1} + l} & \text{for } K_j = 0 \end{cases}.$$

Here e_1, \dots, e_n are the standard basis vectors of \mathbb{F}^n . (If $K_j = 0$ for some $i \in \underline{m} = \{1, \dots, m\}$ the corresponding block in A_H is not present.) \square

The Hermite canonical form induces a canonical form

$$\begin{aligned} \Gamma_H^0: \tilde{\mathcal{N}}_{n,m} &\longrightarrow \tilde{\mathcal{N}}_{n,m} \\ (A, B) &\longmapsto (A_H, B_H) \end{aligned} \quad (3.5)$$

on the $\text{GL}_n(\mathbb{F})$ -invariant subvariety $\tilde{\mathcal{N}}_{n,m} \subset \tilde{\Sigma}_{n,m}$ of all controllable pairs $(A, B) \in \tilde{\Sigma}_{n,m}$ with A nilpotent. Since a block matrix $A_H = (A_{ij})_{i,j \in \underline{m}}$, A_{ij} defined by (3.4), is nilpotent if and only if the diagonal blocks A_{11}, \dots, A_{mm} are nilpotent, Thm. 3.2 implies:

Theorem 3.3. $\Gamma_H^0: \tilde{\mathcal{N}}_{n,m} \longrightarrow \tilde{\mathcal{N}}_{n,m}$, $(A, B) \longmapsto (A_H, B_H)$, is a canonical form for the similarity action on $\tilde{\mathcal{N}}_{n,m}$. Every $(A, B) \in \tilde{\mathcal{N}}_{n,m}$, A nilpotent, with Hermite indices $K = (K_1, \dots, K_m)$ is similar to a unique system $(A_H, B_H) \in \tilde{\mathcal{N}}_{n,m}$ of the form

$$A_H = (A_{ij})_{i,j \in \underline{m}} \text{ with } A_{ij} = \begin{cases} \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 \\ 0 & & & \\ \vdots & & & \end{bmatrix}_{K_i \times K_i} & \text{for } i = j \\ \begin{bmatrix} 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & \alpha_{ij1} \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & \alpha_{ijK_i} \end{bmatrix}_{K_i \times K_j} & \text{for } i < j \\ 0_{K_i \times K_j} & \text{for } i > j \end{cases} \quad (3.6)$$

$$B_H = [b_1, \dots, b_m] \text{ with } b_j = \begin{cases} e_{K_1 + \dots + K_{j-1} + 1} & \text{for } K_j > 0 \\ \sum_{\substack{l=1 \\ i < j}}^{K_i} \alpha_{ijl} e_{K_1 + \dots + K_{i-1} + l} & \text{for } K_j = 0 \end{cases}.$$

(For $K_j = 0$ the corresponding block in A_H is not present.) □

In order to analyze the continuity properties of the Hermite canonical forms Γ_H, Γ_H^0 we consider the subsets of $\tilde{\Sigma}_{n,m}$ (resp. $\tilde{\mathcal{N}}_{n,m}$) consisting of all controllable pairs (A, B) with fixed Hermite indices $K = K(A, B)$.

For any combination $K \in K_{n,m}$ a *Hermite stratum* of $\tilde{\Sigma}_{n,m}$ (resp. of $\tilde{\mathcal{N}}_{n,m}$) is defined as

$$\text{Her}(K) := \{(A, B) \in \tilde{\Sigma}_{n,m}; K(A, B) = K\}$$

resp.

$$\text{Her}^0(K) := \{(A, B) \in \tilde{\mathcal{N}}_{n,m}; K(A, B) = K\}.$$

The next result follows immediately from the definition (3.2), (3.3) of the Hermite canonical form.

Theorem 3.4. The Hermite canonical forms Γ_H, Γ_H^0 are continuous (even analytic) on each Hermite stratum. □

4. Canonical forms

The Weierstrass decomposition (2.9) allows us to construct canonical forms for singular systems from arbitrary canonical forms for the similarity action σ . For example, the Jordan control canonical form of [8] is obtained in this way, by transforming the slow subsystem (A_1, B_1) into Popov's control canonical form [17] while the fast subsystem is transformed into Jordan canonical form [15].

For $0 \leq r \leq n$ let

$$\begin{aligned} \Gamma: \tilde{\Sigma}_{r,m} &\longrightarrow \tilde{\Sigma}_{r,m} \\ (A_1, B_1) &\longmapsto (A_1^\Gamma, B_1^\Gamma) \end{aligned} \quad (4.1)$$

resp.

$$\begin{aligned} \Gamma^0: \tilde{\mathcal{N}}_{n-r,m} &\longrightarrow \tilde{\mathcal{N}}_{n-r,m} \\ (A_2, B_2) &\longmapsto (A_2^{\Gamma^0}, B_2^{\Gamma^0}) \end{aligned} \quad (4.2)$$

be arbitrary canonical forms for the similarity action (2.12) on $\tilde{\Sigma}_{r,m}$ resp. $\tilde{\mathcal{N}}_{n-r,m}$. We define a mapping $\hat{\Gamma}$ on the space of controllable singular systems by

$$\begin{aligned} \hat{\Gamma}: \tilde{C}(n, m) &\longrightarrow \tilde{C}(n, m) \\ (E, A, B) &\longmapsto (E^{\hat{\Gamma}}, A^{\hat{\Gamma}}, B^{\hat{\Gamma}}) \end{aligned} \quad (4.3)$$

Here $(E^{\hat{\Gamma}}, A^{\hat{\Gamma}}, B^{\hat{\Gamma}})$ are defined by

$$E^{\hat{\Gamma}} := \begin{bmatrix} I_r & 0 \\ 0 & A_2^{\Gamma^0} \end{bmatrix} \quad A^{\hat{\Gamma}} := \begin{bmatrix} A_1^\Gamma & 0 \\ 0 & I_{n-r} \end{bmatrix} \quad (4.4a)$$

$$B^{\hat{\Gamma}} := \begin{bmatrix} B_1^\Gamma \\ B_2^{\Gamma^0} \end{bmatrix} \quad (4.4b)$$

where $(A_1, B_1) \in \tilde{\Sigma}_{r,m}$ resp. $(A_2, B_2) \in \tilde{\mathcal{N}}_{n-r,m}$ denotes the slow resp. fast subsystem of (E, A, B) .

By Lemmas (2.1), (2.2) the Weierstrass decomposition (2.9) of (E, A, B) into a slow and a fast subsystem (A_1, B_1) resp. (A_2, B_2) is unique up to similarity

$$(A_i, B_i) \longmapsto (M_i A_i M_i^{-1}, M_i B_i), \quad i = 1, 2. \quad (4.5)$$

Since Γ, Γ^0 are invariant under similarity transformations (4.5), the map $\hat{\Gamma}$ in (4.3) is well defined. Obviously $\hat{\Gamma}$ is a canonical form for (restricted) system equivalence η on $\tilde{C}(n, m)$, called the (Γ, Γ^0) -canonical form on $\tilde{C}(n, m)$; see Lemma 1.1, 1.2. This shows

Proposition 4.1. Let Γ resp. Γ^0 be canonical forms for the similarity actions on $\tilde{\Sigma}_{r,m}$ resp. $\tilde{\mathcal{N}}_{n-r,m}$, $r = 0, \dots, n$. Then $\hat{\Gamma}$, defined by (4.3), (4.4) is a canonical form for the group action (2.14) of restricted system equivalence on $\tilde{C}(n, m)$. Each singular system

$(E, A, B) \in \tilde{C}(n, m)$ is equivalent to a unique system $(\tilde{E}, \tilde{A}, \tilde{B}) \in \tilde{C}(n, m)$ in Weierstrass form

$$\left(\begin{bmatrix} I_r & 0 \\ 0 & \tilde{A}_2 \end{bmatrix}, \begin{bmatrix} \tilde{A}_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}, \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} \right)$$

with

- (a) the slow subsystem $(\tilde{A}_1, \tilde{B}_1)$ is in Γ -canonical form
- (b) the fast subsystem $(\tilde{A}_2, \tilde{B}_2)$ is in Γ^0 -canonical form. □

By choosing Γ as the Kronecker-Popov form [17] on $\tilde{\Sigma}_{r,m}$ and Γ^0 as the Jordan canonical form on $\tilde{\mathcal{N}}_{n-r,m}$, [15], [8], the Jordan control canonical form [8] is obtained. This canonical form on $\tilde{C}(n, m)$ is quite complicated to describe, due to the rather complicated structure of both the Popov canonical form as well as the Jordan canonical form; cf. [8]. By choosing $\Gamma := \Gamma_H$ and $\Gamma^0 := \Gamma_H^0$ as the Hermite canonical form on $\tilde{\Sigma}_{r,m}$ resp. $\tilde{\mathcal{N}}_{n-r,m}$, the canonical form $\hat{\Gamma}$ has a particularly simple structure; compare with Thm. 3.3 in [8].

Theorem 4.2. To every $(E, A, B) \in \tilde{C}(n, m)$ there belongs a unique pair of combinations $K = (K_1, \dots, K_m)$, $L = (L_1, \dots, L_m)$ of r resp. $n - r$ and a unique system (E_H, A_H, B_H) in Weierstrass form

$$(E_H, A_H, B_H) = \left(\begin{bmatrix} I_r & 0 \\ 0 & A_2^H \end{bmatrix}, \begin{bmatrix} A_1^H & 0 \\ 0 & I_{n-r} \end{bmatrix}, \begin{bmatrix} B_1^H \\ B_2^H \end{bmatrix} \right) \quad (4.5)$$

which is system equivalent to (E, A, B) , such that

- (i) $(A_1^H, B_1^H) \in \tilde{\Sigma}_{r,m}$ is in Hermite canonical form (3.5) with Hermite indices (K_1, \dots, K_m) .
- (ii) $(A_2^H, B_2^H) \in \tilde{\mathcal{N}}_{n-r,m}$ is in Hermite canonical form (3.7) with Hermite indices (L_1, \dots, L_m) . □

We refer to

$$\begin{aligned} \hat{\Gamma}_H: \tilde{C}(n, m) &\longrightarrow \tilde{C}(n, m) \\ (E, A, B) &\longmapsto (E_H, A_H, B_H) \end{aligned} \quad (4.6)$$

defined by (4.5) as the *generalized Hermite canonical form* on $\tilde{C}(n, m)$. By Prop. 4.1, $\hat{\Gamma}_H$ is a canonical form for restricted system equivalence.

For $0 \leq r \leq n$ let $(E, A, B) \in \tilde{C}(n, m)$ with $\deg \det(\lambda E - A) = r$. The pair of combinations $(K, L) = ((K_1, \dots, K_m), (L_1, \dots, L_m)) \in K_{r,m} \times K_{n-r,m} \subset K_{n,2m}$ appearing in Theorem 4.2 is called the list of *generalized Hermite indices* of (E, A, B) . $(K(E, A, B), L(E, A, B)) = (K, L)$ are *discrete invariants* for system equivalence, i.e. for all $M, N \in \text{GL}_n(\mathbb{F})$

$$K(MEN^{-1}, MAN^{-1}, MB) = K(E, A, B) \quad (4.7a)$$

$$L(MEN^{-1}, MAN^{-1}, MB) = L(E, A, B) \quad (4.7b)$$

Let

$$I(n, m) := \{(r, K, L); 0 \leq r \leq n, K \in K_{r, m}, L \in K_{n-r, m}\}. \quad (4.8)$$

$I(n, m) \simeq K_{n, 2m}$ is a finite set of cardinality

$$\text{card } I(n, m) = \sum_{r=0}^n \binom{m+r-1}{r} \binom{m+n-r-1}{n-r} = \binom{n+2m-1}{n}. \quad (4.9)$$

For any $(r, K, L) \in I(n, m)$ there exists $(E, A, B) \in \tilde{C}(n, m)$ with $\deg \det(\lambda E - A) = r$, $K(E, A, B) = K$, $L(E, A, B) = L$. Thus $I(n, m) \simeq K_{n, 2m}$ is the set of all discrete invariants for the Hermite canonical form.

Let (E_H, A_H, B_H) be in Hermite canonical form (4.5) with Hermite indices (K, L) . Then for the slow and fast subsystems (A_1^H, B_1^H) , resp. (A_2^H, B_2^H) , A_1^H, A_2^H are upper block triangular matrices

$$A_1^H = \begin{bmatrix} A_{11} & \cdot & \cdot & \cdot & A_{1m} \\ & \cdot & & & \cdot \\ & & \cdot & & \cdot \\ & & & \cdot & \cdot \\ 0 & & & & A_{mm} \end{bmatrix} \quad (4.10)$$

where

$$A_{ii} = \begin{bmatrix} 0 & \cdot & \cdot & \cdot & 0 & * \\ 1 & \cdot & \cdot & \cdot & 0 & * \\ \cdot & \cdot & & & \cdot & \cdot \\ \cdot & & \cdot & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ 0 & & & & 1 & * \end{bmatrix}_{K_i \times K_i}, \quad A_{ij} = \begin{bmatrix} 0 & \cdots & 0 & * \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & * \end{bmatrix}_{K_i \times K_j}, \quad i < j, \quad (4.11)$$

and

$$A_2^H = \begin{bmatrix} \hat{A}_{11} & \cdot & \cdot & \cdot & \hat{A}_{1m} \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \hat{A}_{mm} \end{bmatrix} \quad (4.12)$$

with

$$\hat{A}_{ii} = \begin{bmatrix} 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 1 & \cdot & \cdot & \cdot & 0 & 0 \\ & \cdot & & & \cdot & \\ & & \cdot & & \cdot & \\ & & & \cdot & \cdot & \\ 0 & & & & 1 & 0 \end{bmatrix}_{L_i \times L_i}, \quad \hat{A}_{ij} = \begin{bmatrix} 0 & \cdots & 0 & * \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & * \end{bmatrix}_{L_i \times L_j}, \quad i < j. \quad (4.13)$$

Example 1. For $m = 1$, the Hermite canonical form on $\tilde{C}(n, 1)$ has the structure ($r = \deg \det(\lambda E - A)$):

$$A_1^H = \begin{bmatrix} 0 & \cdot & \cdot & \cdot & 0 & * \\ 1 & \cdot & \cdot & \cdot & 0 & * \\ & \cdot & & \cdot & & \\ & & \cdot & \cdot & & \\ & & & \cdot & & \\ 0 & \cdot & \cdot & \cdot & 1 & * \end{bmatrix}_{r \times r}, \quad B_1^H = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{r \times 1} \quad (4.14a)$$

$$A_2^H = \begin{bmatrix} 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 1 & \cdot & \cdot & \cdot & 0 & 0 \\ & \cdot & & \cdot & & \\ & & \cdot & \cdot & & \\ & & & \cdot & & \\ 0 & \cdot & \cdot & \cdot & 1 & 0 \end{bmatrix}_{(n-r) \times (n-r)}, \quad B_2^H = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(n-r) \times 1} \quad (4.14b)$$

similar to the Jordan control canonical form on $\tilde{C}(n, 1)$ [8].

Example 2. Let $n = 5, r = 2, m = 2, (K_1, K_2; L_1, L_2) = (2, 0; 1, 2)$. Each $(E, A, B) \in \tilde{C}(n, m)$ with generalized Hermite indices $(K_1, K_2; L_1, L_2)$ has the Hermite canonical form

$$A_1^H = \begin{bmatrix} 0 & * \\ 1 & * \end{bmatrix}, \quad B_1^H = \begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix} \quad (4.15a)$$

$$A_2^H = \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_2^H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (4.15b)$$

□

5. Continuity aspects

For any $(r, K, L) \in I(n, m)$ a *Hermite stratum* of $\tilde{C}(n, m)$ is defined by

$$\text{Her}(K, L) := \{(E, A, B) \in \tilde{C}(n, m); (K(E, A, B), L(E, A, B)) = (K, L)\}.$$

Thus

$$\tilde{C}(n, m) = \bigcup_{(r, K, L) \in I(n, m)} \text{Her}(K, L)$$

is a finite decomposition into finitely many disjoint η -invariant subsets. For $0 \leq r \leq n$ let

$$\tilde{C}^r(n, m) := \{(E, A, B) \in \tilde{C}(n, m); \deg \det(\lambda E - A) = r\}.$$

$\tilde{C}^r(n, m)$ is a quasi-affine subvariety of $\tilde{C}(n, m)$ and

$$\tilde{C}(n, m) = \bigcup_{r=0}^n \tilde{C}^r(n, m).$$

The following lemma is a special case of a result of Cobb [4].

Lemma 5.1 ([4]). Let $(\bar{E}, \bar{A}, \bar{B}) \in \tilde{C}^r(n, m)$. There exists an open neighborhood $U \subset \tilde{C}(n, m)$ of $(\bar{E}, \bar{A}, \bar{B})$ and a continuous map

$$\begin{aligned} F: \tilde{C}^r(n, m) \cap U &\longrightarrow \tilde{C}^r(n, m) \\ (E, A, B) &\longmapsto \left(\begin{bmatrix} I_r & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \right) \end{aligned} \quad (5.1)$$

which associates with every (E, A, B) an equivalent system in standard form.

Using Lemma 5.1 we have defined a continuous map

$$\begin{aligned} \varphi: \tilde{C}^r(n, m) \cap U &\longrightarrow \tilde{\Sigma}_{r,m} \times \tilde{\mathcal{N}}_{n-r,m} \\ (E, A, B) &\longmapsto ((A_1, B_1), (A_2, B_2)) \end{aligned}$$

with (A_i, B_i) defined by (5.1).

Theorem 5.2. The Hermite canonical form $\hat{\Gamma}_H: \tilde{C}(n, m) \longrightarrow \tilde{C}(n, m)$ is continuous on each Hermite stratum $\text{Her}(K, L)$.

Proof.

Let $\Gamma_H: \tilde{\Sigma}_{r,m} \longrightarrow \tilde{\Sigma}_{r,m}$ and $\Gamma_H^0: \tilde{\mathcal{N}}_{n-r,m} \longrightarrow \tilde{\mathcal{N}}_{n-r,m}$ be the Hermite canonical forms defined in section 3. By Thm. 3.5, Γ_H resp. Γ_H^0 are continuous on each Hermite stratum $\text{Her}(K)$ resp. $\text{Her}^0(L)$ of $\tilde{\Sigma}_{r,m}$ resp. $\tilde{\mathcal{N}}_{n-r,m}$. Thus the composed map

$$\begin{aligned} (\Gamma_H, \Gamma_H^0) \circ \varphi: \tilde{C}^r(n, m) \cap U &\longrightarrow \tilde{\Sigma}_{r,m} \times \tilde{\mathcal{N}}_{n-r,m} \\ (E, A, B) &\longmapsto (\Gamma_H(A_1, B_1), \Gamma_H^0(A_2, B_2)) \end{aligned}$$

is continuous on $\text{Her}(K, L) \cap U = \varphi^{-1}(\text{Her}(K) \times \text{Her}^0(L))$. The result follows. \square

However the Hermite canonical form is discontinuous as a map $\Gamma_H: \tilde{C}(n, m) \longrightarrow \tilde{C}(n, m)$. More generally we have

Theorem 5.3. Let $n, m \geq 1$ be arbitrary. There does not exist a global continuous canonical form $\Gamma: \tilde{C}(n, m) \longrightarrow \tilde{C}(n, m)$ for the system equivalence action η .

Proof.

By [8], [13] the quotient map

$$\begin{aligned} \pi: \tilde{C}(n, m) &\longrightarrow C(n, m) \\ (E, A, B) &\longmapsto [E, A, B]_\eta \end{aligned}$$

is a principal fiber bundle with structure group $\text{GL}_n(\mathbf{F}) \times \text{GL}_n(\mathbf{F})$. Thus the existence of a continuous canonical form $\Gamma: \tilde{C}(n, m) \longrightarrow \tilde{C}(n, m)$ for η is equivalent to the triviality of the bundle π [12]. While for $m \geq 2$ this would follow immediately from results of [2], [9], [10], we prefer to give the following independent argument.

Let $\text{PGL}_2(\mathbf{F}) = \text{GL}_2(\mathbf{F})/\mathbf{F}^* \cdot I_2$ denote the projective general linear group, i.e. the quotient

of $GL_2(\mathbf{F})$ by the subgroup of nonzero scalar multiples of the identity. Choose n pairwise linearly independent vectors $(e_i, a_i) \in \mathbf{F}^2$, $i = 1, \dots, n$. Let $b^1, \dots, b^n \in \mathbf{F}^m - \{0\}$ denote n arbitrary nonzero row vectors of \mathbf{F}^m . Then (E, A, B) defined by

$$E := \text{diag}(e_1, \dots, e_n), \quad A := \text{diag}(a_1, \dots, a_n), \quad B = \begin{bmatrix} b^1 \\ \vdots \\ b^n \end{bmatrix} \quad (5.2)$$

is admissible and controllable. For such an (E, A, B) fixed, consider the map

$$\begin{aligned} \sigma: GL_2(\mathbf{F}) &\longrightarrow \tilde{C}(n, m) \\ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} &\longmapsto (\alpha E + \beta A, \gamma E + \delta A, B). \end{aligned} \quad (5.3)$$

σ descends to a continuous map

$$\begin{aligned} \bar{\sigma}: PGL_2(\mathbf{F}) &\longrightarrow C(n, m) \\ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} &\longmapsto [\alpha E + \beta A, \gamma E + \delta A, B]_\eta. \end{aligned} \quad (5.4)$$

We have the commutative diagram

$$\begin{array}{ccc} GL_2(\mathbf{F}) & \xrightarrow{\sigma} & \tilde{C}(n, m) \\ p \downarrow & & \downarrow \pi \\ PGL_2(\mathbf{F}) & \xrightarrow{\bar{\sigma}} & C(n, m) \end{array} \quad (5.5)$$

where $p: GL_2(\mathbf{F}) \longrightarrow PGL_2(\mathbf{F})$ denotes the canonical quotient map. $p: GL_2(\mathbf{F}) \longrightarrow PGL_2(\mathbf{F})$ is a nontrivial $GL_1(\mathbf{F})$ -principal fibre bundle. Suppose π were trivial. Then by (5.5) also $p: GL_2(\mathbf{F}) \longrightarrow PGL_2(\mathbf{F})$ were trivial. Contradiction. \square

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