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**Pole Placement by Dynamic
Compensation for Descriptor
Systems**

by

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**POLE PLACEMENT BY DYNAMIC COMPENSATION
FOR DESCRIPTOR SYSTEMS***

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ABSTRACT

Using the recently introduced concept of homogeneous indices, we obtain a generalization to descriptor systems of the Brasch-Pearson Theorem for pole placement by dynamic compensation.

I. INTRODUCTION

There has been a continuing interest in descriptor systems, i.e. generalized linear models of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

where E, A, B, C are constant matrices of dimensions $n \times n$, $n \times n$, $n \times m$, $p \times n$, and E is possibly singular. (See e.g. [1] [2] [3].) Recently, Shayman and Zhou [4] have presented a unified theory of control synthesis for generalized linear systems using constant-ratio proportional and derivative (CRPD) feedback – i.e., feedback of the form

$$u = F(\cos \theta x - \sin \theta \dot{x}) + v \tag{2}$$

where θ is a real parameter and F is a gain matrix. Analogous CRPD output feedback is obtained by replacing x and \dot{x} with y and \dot{y} in (2).

The purpose of the present paper is to show how the approach in [4] can be extended to include dynamic compensation. Recently, we introduced the concept of homogeneous indices [5], a generalization of the concept of Kronecker indices. Using the homogeneous indices, we will derive a generalization to descriptor systems of the well-known result of Brasch and Pearson [6] concerning pole placement by dynamic compensation for regular state-space systems:

Theorem (Brasch-Pearson [6]): Let (A, B, C) be controllable and observable, and let ν denote the observability index of (C, A) . Let Λ be a self-conjugate set of $n + \nu - 1$ complex numbers. Then there exists a $(\nu - 1)$ -dimensional dynamic compensator

$$\begin{aligned} \dot{z} &= Fz + v \\ u &= Hz + Ky \\ v &= Gy \end{aligned} \tag{3}$$

such that when coupled with the system

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}\tag{4}$$

the resulting closed-loop plant matrix

$$\begin{bmatrix} A + BKC & BH \\ GC & F \end{bmatrix}$$

has Λ as its set of eigenvalues.

II. BACKGROUND

We briefly review terminology and results from [4] and [5] which will be needed in the sequel. However, some of the notation to be used differs from that in these references. We refer to (1) as a regular system if E is nonsingular, a singular system if E is singular, and an admissible system if $|sE - A| \neq 0$. A regular system is always admissible, but a singular system may or may not be admissible.

Let $\hat{\Sigma}(n) := \{(E, A) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}\}$, $\Sigma(n) := \{(E, A) \in \hat{\Sigma}(n) : (E, A) \text{ is admissible}\}$, and for each $\theta \in \mathbb{R}$, let $\Sigma_\theta(n) := \{(E, A) \in \Sigma(n) : |\cos \theta E - \sin \theta A| \neq 0\}$. For each $\theta \in \mathbb{R}$, $\Sigma_\theta(n)$ is an open and dense subset of $\Sigma(n)$; $\Sigma_{\theta+\pi}(n) = \Sigma_\theta(n)$; $\Sigma(n) = \bigcup_{\theta \in [0, \pi)} \Sigma_\theta(n)$. Thus, $\{\Sigma_\theta(n) : \theta \in [0, \pi)\}$ is a covering of $\Sigma(n)$ by a family of open and dense subsets.

We denote by $\hat{\Sigma}(n, m)$, $\Sigma(n, m)$, $\Sigma_\theta(n, m)$ the spaces of triples (E, A, B) in $\hat{\Sigma}(n) \times \mathbb{R}^{n \times m}$, $\Sigma(n) \times \mathbb{R}^{n \times m}$, and $\Sigma_\theta(n) \times \mathbb{R}^{n \times m}$ respectively. We denote by $\hat{\Sigma}(n, m, p)$, $\Sigma(n, m, p)$, $\Sigma_\theta(n, m, p)$ the spaces of quadruples (E, A, B, C) in $\hat{\Sigma}(n) \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n}$, $\Sigma(n) \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n}$, and $\Sigma_\theta(n) \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n}$ respectively. Note that $\Sigma_0(n, m, p)$ consists of those systems (1) which are regular.

Next, we define a group of symmetries of the covering $\{\Sigma_\theta(n)\}$, transformations which map these subsets into each other. For each $\phi \in \mathbb{R}$, define a mapping $R_\phi : \hat{\Sigma}(n) \rightarrow \hat{\Sigma}(n)$ by

$$R_\phi(E, A) := (\cos \phi E + \sin \phi A, -\sin \phi E + \cos \phi A).$$

It is easy to check that $(R_\phi)^{-1} = R_{-\phi}$. The set $\Sigma(n)$ of admissible systems is invariant under R_ϕ , and R_ϕ maps $\Sigma_\theta(n)$ isomorphically onto $\Sigma_{\theta+\phi}(n)$.

For our purposes, it is useful to have a definition of system eigenvalue which treats all the points in the extended complex plane in a symmetric manner. This leads us to a definition given in [4] for the multiplicity of ∞ as a system eigenvalue which is different from that which is commonly used. It is usual to say that α is a finite eigenvalue of multiplicity q of $(E, A) \in \Sigma(n)$ if 0 is an eigenvalue of algebraic multiplicity q of the matrix $\alpha E - A$. If E is singular, then ∞ is an eigenvalue of (E, A) , and its multiplicity is defined to be $\text{rank } E - \deg \det(sE - A)$ - i.e. the number of independent impulsive modes. Thus, according to these usual definitions, (E, A) has $\text{rank } E$ eigenvalues, counting multiplicities.

To understand the different definition which we employ, it is useful to recall the identification of the extended complex plane, $\mathbb{C} \cup \{\infty\}$, with the complex projective space $\mathbb{CP}(1)$. Define an equivalence relation \sim on $\mathbb{C}^2 - \{(0, 0)\}$ whereby $(\lambda, \mu) \sim (\hat{\lambda}, \hat{\mu})$ if and only if there exists a nonzero complex number β such that $(\hat{\lambda}, \hat{\mu}) = \beta(\lambda, \mu)$. (Equivalently, $\mathbb{CP}(1)$ can be regarded as the set of all lines through the origin in \mathbb{C}^2 .) If $(\lambda, \mu) \in \mathbb{C}^2 - \{(0, 0)\}$, we denote by $[(\lambda, \mu)]$ the corresponding element of $\mathbb{CP}(1)$ - i.e., the equivalence class containing (λ, μ) . We refer to (λ, μ) as the homogeneous coordinates of $[(\lambda, \mu)]$. If $(\lambda, \mu) \sim (\hat{\lambda}, \hat{\mu})$, then $\lambda/\mu = \hat{\lambda}/\hat{\mu}$. Consequently, we can identify $\mathbb{CP}(1)$ with the extended complex plane via the map $[(\lambda, \mu)] \rightarrow \lambda/\mu$. If $\mu = 0$, then $[(\lambda, \mu)]$ is identified with the point at infinity in the complex plane.

Let $(E, A) \in \Sigma(n)$. We will say that $[(\lambda, \mu)] \in \mathbb{CP}(1)$ is a system eigenvalue of (E, A) if and only if 0 is an eigenvalue of the matrix $\lambda E - \mu A$. In this case, we define the multiplicity of $[(\lambda, \mu)]$ to be the algebraic multiplicity of 0 as an eigenvalue of $\lambda E - \mu A$. If $(\lambda, \mu) \sim (\hat{\lambda}, \hat{\mu})$, then 0 is an eigenvalue of $\lambda E - \mu A$ of multiplicity q if and only if 0 is an eigenvalue of $\hat{\lambda} E - \hat{\mu} A$ of multiplicity q . Thus, the system eigenvalues and their multiplicities are well-defined.

Since $[(\lambda, \mu)]$ is identified with the extended complex number $\alpha = \lambda/\mu$, we will also refer to α as a system eigenvalue of multiplicity q when $[(\lambda, \mu)]$ is such. If $\mu \neq 0$, then α is a (finite) complex number and is a system eigenvalue of multiplicity q if and only if 0 is an eigenvalue of $\alpha E - A$ of multiplicity q . This coincides with the usual definition of a finite eigenvalue of a generalized linear system. If $\mu = 0$, then $\alpha = \infty$ and is a system eigenvalue of multiplicity q if and only if 0 is an eigenvalue of E of multiplicity q . It follows easily (for example, using the Weierstrass decomposition) that the total number of system eigenvalues

(according to our definition) is n regardless of the value of rank E . If E is singular, this is greater than the number of independent dynamic modes. However, this alternative definition of system eigenvalue is very useful for control synthesis [4].

Next, we consider the question of how the system eigenvalues transform under system rotation. Define $r_\phi : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ by

$$r_\phi(\lambda, \mu) := ((\cos \phi)\lambda + (\sin \phi)\mu, -(\sin \phi)\lambda + (\cos \phi)\mu).$$

Proposition 1 [4]: Let $(E, A) \in \Sigma(n)$, and let $(\hat{E}, \hat{A}) := R_\phi(E, A)$. Then $[(\lambda, \mu)]$ is an eigenvalue of (E, A) of multiplicity q if and only if $[r_{-\phi}(\lambda, \mu)]$ is an eigenvalue of (\hat{E}, \hat{A}) of multiplicity q . Equivalently, the extended complex number α is an eigenvalue of (E, A) of multiplicity q if and only if the extended complex number $\hat{\alpha}$ is an eigenvalue of (\hat{E}, \hat{A}) of multiplicity q , where

$$\hat{\alpha} := \frac{(\cos \phi)\alpha - \sin \phi}{(\sin \phi)\alpha + \cos \phi}.$$

$(E, A, B, C) \in \Sigma(n, m, p)$ is controllable if every state in \mathbb{R}^n is reachable in positive time from the initial state $x(0-) = 0$. (E, A, B, C) is observable if only the initial state $x(0-) = 0$ gives $y(0-) = 0$ and the free response identically zero on $[0, \infty)$. (For a detailed discussion of controllability and observability for descriptor systems, see [7]).

Proposition 2 [4]: Let $(E, A, B, C) \in \Sigma(n, m, p)$, and let $(\hat{E}, \hat{A}) := R_\phi(E, A)$. Then (E, A, B, C) is controllable (observable) if and only if (\hat{E}, \hat{A}, B, C) is controllable (observable).

Recently, we have introduced the concept of homogeneous indices [5] as a generalization to descriptor systems of the Kronecker indices of regular systems. They are defined as follows: Associate to $(E, A, B) \in \hat{\Sigma}(n, m)$ the degree one matrix polynomial in two variables given by $[\lambda E - \mu A, B]$. Let z_1 be a column vector with entries in the ring $\mathbb{R}[\lambda, \mu]$ of polynomials in two variables which is a minimal degree nonzero solution to the equation

$$[\lambda E - \mu A, B]z = 0. \tag{5}$$

Let z_2 be a minimal degree solution which is linearly independent over $\mathbb{R}[\lambda, \mu]$ of z_1 . Let z_3 be a minimal degree solution which is linearly independent of $\{z_1, z_2\}$. Proceeding in this way, we obtain a sequence z_1, \dots, z_q ($q \leq n + m$) of solutions, which we refer to as a

fundamental series of solutions of (5). The degrees $\delta_1 \leq \dots \leq \delta_q$ of z_1, \dots, z_q are independent of the choice of fundamental series. We refer to $(\delta_1, \dots, \delta_q)$ as the homogeneous indices of the system (E, A, B) .

Proposition 3 [5]: Let $(E, A, B) \in \hat{\Sigma}(n, m)$ and let $(\hat{E}, \hat{A}) := R_\phi(E, A)$. Then (E, A, B) and (\hat{E}, \hat{A}, B) have the same set of homogeneous indices.

Proposition 4 [5]: Let (E, A, B) be a controllable regular system. Then the homogeneous indices of (E, A, B) are equal to the Kronecker indices of $(E^{-1}A, E^{-1}B)$.

Let (C, A) be an observable pair. The observability index of (C, A) is defined [8] to be

$$\nu := \min \left\{ j : \bigcap_{i=1}^j \ker CA^{i-1} = 0 \right\}.$$

Obviously, $1 \leq \nu \leq n$.

Remark 1: It follows that ν is the largest Kronecker index of the controllable dual pair (A', C') . By Proposition 4, ν is equal to the largest homogeneous index of the controllable regular system (I, A', C') .

III. MAIN RESULTS

To motivate the generalization which will be derived, we make the following three observations on the nature of the Brasch-Pearson Theorem: (1) It concerns systems belonging to the set $\Sigma_0(n, m, p)$ of regular systems, an open and dense subset of the space $\Sigma(n, m, p)$ of admissible systems. (2) It establishes pole-assignability on the extended complex plane with one point deleted, namely the point at infinity. (3) The upper bound on the order of the dynamic compensator is one less than the observability index of (C, A) — i.e., one less than the largest homogeneous index of the system (I, A', C') .

Our first result establishes pole placement using a generalized dynamic compensator and constant-ratio proportional and derivative connections.

Theorem 1: Let $(E, A, B, C) \in \Sigma_\theta(n, m, p)$ be controllable and observable, and let ν be the largest homogeneous index of (E', A', C') . Let Λ be a self-conjugate set of $n + \nu - 1$ numbers from $\mathbb{C} \cup \{\infty\} - \{\text{ctn}\theta\}$. Then there exists a $(\nu - 1)$ -dimensional generalized dynamic compensator

$$M\dot{z} = Fz + v$$

$$\begin{aligned}
u &= K(\cos \theta y - \sin \theta \dot{y}) + H(\cos \theta z - \sin \theta \dot{z}) \\
v &= G(\cos \theta y - \sin \theta \dot{y})
\end{aligned} \tag{6}$$

with $(M, F) \in \sum_{\theta}(\nu - 1)$ such that when coupled with the generalized system

$$\begin{aligned}
E\dot{x} &= Ax + Bu \\
y &= Cx
\end{aligned} \tag{7}$$

the resulting closed-loop composite system has Λ as its set of eigenvalues.

Remark 2: For each fixed choice of θ , the result in Theorem 1 is analogous to the Brasch-Pearson Theorem: It establishes pole assignment for the controllable and observable systems belonging to the open and dense subset $\sum_{\theta}(n, m, p)$, and the poles can be placed anywhere in the extended complex plane with one point deleted (namely $\text{ctn}\theta$). The upper-bound on the dimension of the compensator is one less than the largest homogeneous index of the dual system. In the special case when $\theta = 0$, Theorem 1 reduces to the Brasch-Pearson Theorem. Since $\{\sum_{\theta}(n, m, p) : \theta \in [0, \pi)\}$ is a covering of $\sum(n, m, p)$, Theorem 1 applies to every admissible general linear system which is controllable and observable. In fact, since an admissible system (E, A, B, C) belongs to $\sum_{\theta}(n, m, p)$ for all but at most finitely-many values of θ in $[0, \pi)$, we can always choose θ to be arbitrarily small, in which case the derivative terms in the CRPD connections in (6) can be regarded as small perturbations.

Proof of Theorem 1: Let $(\hat{E}, \hat{A}) := R_{-\theta}(E, A)$. Since $(E, A, B, C) \in \sum_{\theta}(n, m, p)$, $(\hat{E}, \hat{A}, B, C) \in \sum_0(n, m, p)$ — i.e. (\hat{E}, \hat{A}, B, C) is a regular system. Since controllability and observability are preserved by system rotation (Proposition 2), (\hat{E}, \hat{A}, B, C) is both controllable and observable. Since the homogeneous indices are invariant under system rotation (Proposition 3), ν is the largest homogeneous index of $R_{-\theta}(E', A', C') = (\hat{E}', \hat{A}', C')$. Since (\hat{E}', \hat{A}', C') is a controllable regular system, it follows from Proposition 4 that ν is the largest Kronecker index of the pair $((\hat{E}')^{-1}\hat{A}', (\hat{E}')^{-1}C')$, and hence the observability index of the observable pair $(C\hat{E}^{-1}, \hat{A}\hat{E}^{-1})$ (Remark 1). It is trivial to verify that $(C\hat{E}^{-1}, \hat{A}\hat{E}^{-1})$ and $(C, \hat{E}^{-1}\hat{A})$ have the same observability index. Thus, ν is the observability index of the observable pair $(C, \hat{E}^{-1}\hat{A})$.

Define a mapping $\bar{r}_{\theta} : \mathbb{C} \cup \{\infty\} \rightarrow \mathbb{C} \cup \{\infty\}$ by

$$\bar{r}_{\theta}(\alpha) := \frac{(\cos \theta)\alpha + \sin \theta}{-(\sin \theta)\alpha + \cos \theta}. \tag{8}$$

Let $\hat{\Lambda} := \bar{r}_\theta(\Lambda)$. Since $\text{ctn}\theta \notin \Lambda$, $\hat{\Lambda}$ does not contain the point at infinity. Thus, $\hat{\Lambda}$ is a self-conjugate set of $n + \nu - 1$ complex numbers. We apply the Brasch-Pearson Theorem to the controllable and observable state-space system

$$\begin{aligned}\dot{x} &= \hat{E}^{-1}\hat{A}x + \hat{E}^{-1}Bu \\ y &= Cx\end{aligned}\tag{9}$$

There exists a $(\nu - 1)$ -dimensional dynamic compensator

$$\begin{aligned}\dot{z} &= \hat{F}z + v \\ u &= Hz + Ky \\ v &= Gy\end{aligned}\tag{10}$$

such that when coupled with (9), the closed-loop plant matrix

$$Q := \begin{bmatrix} \hat{E}^{-1}\hat{A} + \hat{E}^{-1}BKC & \hat{E}^{-1}BH \\ GC & \hat{F} \end{bmatrix}\tag{11}$$

has $\hat{\Lambda}$ as its set of eigenvalues.

Let $(M, F) := R_\theta(I, \hat{F}) \in \sum_\theta(\nu - 1)$. Consider the interconnection of the compensator (6) with the system (7). The resulting closed-loop system is of the form

$$N \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = P \begin{bmatrix} x \\ z \end{bmatrix}$$

with

$$\begin{aligned}N &:= \begin{bmatrix} E + \sin\theta BKC & \sin\theta BH \\ \sin\theta GC & M \end{bmatrix} \\ P &:= \begin{bmatrix} A + \cos\theta BKC & \cos\theta BH \\ \cos\theta GC & F \end{bmatrix}.\end{aligned}$$

Using the fact that the system eigenvalues of a generalized system are invariant under left-multiplication by a nonsingular matrix, we have

$$\begin{aligned}\hat{\Lambda} &= \text{Spec } Q = \sigma(I, Q) \\ &= \sigma\left(\begin{bmatrix} \hat{E} & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} \hat{A} + BKC & BH \\ GC & \hat{F} \end{bmatrix}\right) \\ &= \sigma(R_{-\theta}(N, P)),\end{aligned}$$

where “Spec” denotes the spectrum of a square matrix, and “ σ ” denotes the set of system eigenvalues (counting multiplicities) of a generalized system. Thus, it follows from Proposition 1 that

$$\sigma(N, P) = \bar{r}_{-\theta}(\hat{\Lambda}) = \Lambda.$$

□

Remark 3: Since $R_{-\theta}(N, P)$ is a regular system — i.e., an element of $\sum_0(n + \nu - 1)$ — it follows that the closed-loop composite system (N, P) belongs to $\sum_\theta(n + \nu - 1)$.

For a given θ (such that $(E, A) \in \sum_\theta(n)$), the closed-loop system eigenvalues can be placed arbitrarily in the extended complex plane except at the point $\text{ctn } \theta$. In particular, we can require that all the closed-loop eigenvalues be finite. Then the closed-loop composite system is a regular system.

However, it is conceivable that the compensator

$$M\dot{z} = Fz + v \tag{12}$$

could turn out to be a singular system, even when the closed-loop composite system is regular. We will show that it is possible to achieve approximate eigenvalue assignment with arbitrary precision subject to the additional condition that the compensator dynamics (12) be regular. In order to prove this, we need a preliminary result which establishes that the system eigenvalues, as we have defined them, depend continuously on the system matrices.

We begin by describing the natural topology on the space of all possible sets of system eigenvalues. For each $(E, A) \in \sum(n)$, $\sigma(E, A)$ is an unordered set of n extended complex numbers, with repetitions allowed. Consequently, it is the space of all such sets which we must topologize.

Let (X, ρ_X) be a metric space, and let \tilde{X}_n denote the set consisting of all sets $\{x_1, \dots, x_n\}$ of n possibly repeated elements from X . Define a metric $\tilde{\rho}$ on \tilde{X}_n by

$$\tilde{\rho}(\{x_1, \dots, x_n\}, \{z_1, \dots, z_n\}) := \min_{\pi \in S_n} \max_{1 \leq j \leq n} \rho_X(x_j, z_{\pi(j)}),$$

where S_n denotes the group of permutations on n letters. It is easily verified that $\tilde{\rho}$ satisfies the axioms for a metric. Let (Y, ρ_Y) be a metric space, and let $f : X \rightarrow Y$. Then f induces a

map $\tilde{f} : \tilde{X}_n \rightarrow \tilde{Y}_n$ with $\tilde{f}(\{x_1, \dots, x_n\}) := \{f(x_1), \dots, f(x_n)\}$. It is easy to show that if f is continuous, then \tilde{f} is continuous (with respect to the induced metrics on \tilde{X}_n and \tilde{Y}_n).

Let X now denote the extended complex plane $\mathbb{C} \cup \{\infty\}$. Let $i : \mathbb{CP}(1) \rightarrow X$ be the map which identifies the extended complex plane with the complex projective space $\mathbb{CP}(1)$ — i.e.,

$$i([\lambda, \mu]) := \lambda/\mu.$$

The standard gap metric on $\mathbb{CP}(1)$ [8] induces a metric on X by the requirement that the map i be an isometry.

By restriction, we can view the map r_θ as a diffeomorphism of $\mathbb{C}^2 - \{(0, 0)\}$. r_θ induces a diffeomorphism of $\mathbb{CP}(1)$ which we denote by \hat{r}_θ . We have

$$\hat{r}_\theta([\lambda, \mu]) = [((\cos \theta)\lambda + (\sin \theta)\mu, -(\sin \theta)\lambda + (\cos \theta)\mu)].$$

The map $\bar{r}_\theta : X \rightarrow X$ defined earlier can be expressed as the composition

$$\bar{r}_\theta = i \circ \hat{r}_\theta \circ i^{-1}.$$

Consequently, \bar{r}_θ is a homeomorphism of the extended complex plane X .

We are now prepared to prove that the set $\sigma(E, A)$ of system eigenvalues depends continuously on the admissible generalized linear system (E, A) .

Lemma 1: The map $\sigma : \Sigma(n) \rightarrow \tilde{X}_n$ is continuous.

Proof: Let $\{(E_j, A_j)\}_{j=1}^\infty$ be a sequence in $\Sigma(n)$ which converges to $(E, A) \in \Sigma(n)$. We will show that $\{\sigma(E_j, A_j)\}_{j=1}^\infty$ converges to $\sigma(E, A)$ in \tilde{X}_n . Choose θ such that $(E, A) \in \Sigma_\theta(n)$. Since $\Sigma_\theta(n)$ is open, the sequence $\{(E_j, A_j)\}$ is contained in $\Sigma_\theta(n)$ once j becomes sufficiently large.

Let $(\hat{E}_j, \hat{A}_j) := R_{-\theta}(E_j, A_j)$, and let $(\hat{E}, \hat{A}) := R_{-\theta}(E, A) \in \Sigma_0(n)$. Since $R_{-\theta}$ is continuous, the sequence $\{(\hat{E}_j, \hat{A}_j)\}$ converges to (\hat{E}, \hat{A}) . For sufficiently large j , $(\hat{E}_j, \hat{A}_j) \in \Sigma_0(n)$. From now on, we assume j is large enough so that this is the case.

Since (\hat{E}_j, \hat{A}_j) and (\hat{E}, \hat{A}) are regular systems, $\sigma(\hat{E}_j, \hat{A}_j)$ and $\sigma(\hat{E}, \hat{A})$ are finite — i.e., belong to $\tilde{\mathcal{C}}_n \subset \tilde{X}_n$ — and

$$\sigma(\hat{E}_j, \hat{A}_j) = \text{Spec } \hat{E}_j^{-1} \hat{A}_j$$

$$\sigma(\hat{E}, \hat{A}) = \text{Spec } \hat{E}^{-1} \hat{A}.$$

Since $\{\hat{E}_j^{-1}\hat{A}_j\}$ converges to $\hat{E}^{-1}\hat{A}$, the standard fact that the spectrum of a square matrix is a continuous function of the matrix implies that $\{\sigma(\hat{E}_j, \hat{A}_j)\}$ converges to $\sigma(\hat{E}, \hat{A})$.

Let $\tilde{\tau}_{-\theta}$ denote the map on \tilde{X}_n induced by $\bar{r}_{-\theta}$. Since $\bar{r}_{-\theta}$ is a homeomorphism of X , $\tilde{\tau}_{-\theta}$ is a homeomorphism of \tilde{X}_n . By Proposition 1, we have

$$\begin{aligned}\sigma(E_j, A_j) &= \sigma(R_\theta(\hat{E}_j, \hat{A}_j)) = \tilde{\tau}_{-\theta}(\sigma(\hat{E}_j, \hat{A}_j)) \\ \sigma(E, A) &= \sigma(R_\theta(\hat{E}, \hat{A})) = \tilde{\tau}_{-\theta}(\sigma(\hat{E}, \hat{A})).\end{aligned}$$

Since $\tilde{\tau}_{-\theta}$ is continuous, this implies that $\{\sigma(E_j, A_j)\}$ converges to $\sigma(E, A)$. \square

Remark 4: The preceding result illustrates one of the motivations for our unconventional definition of the multiplicity of ∞ as a system eigenvalue. (See the discussion in Section 2). With the conventional definition, even the cardinality of the set of system eigenvalues (which would be rank E) is discontinuous at every singular system (E, A) .

We will now show that approximate eigenvalue assignment with arbitrary precision is possible while satisfying the additional requirement that the compensator dynamics be regular. Given any positive integer k , let $\tilde{\rho}$ denote the induced metric on \tilde{X}_k .

Theorem 2: Let $(E, A, B, C) \in \sum_\theta(n, m, p)$ be controllable and observable, and let ν be the largest homogeneous index of (E', A', C') . Let Λ be a self-conjugate set of $n + \nu - 1$ numbers from $\mathbb{C} \cup \{\infty\} - \{\text{ctn } \theta\}$, and let $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that for every ϕ satisfying $0 < |\phi - \theta| < \delta$, there exists a $(\nu - 1)$ -dimensional dynamic compensator

$$\begin{aligned}\overline{M}\dot{z} &= \overline{F}z + v \\ u &= \overline{K}(\cos \phi y - \sin \phi \dot{y}) + \overline{H}(\cos \phi z - \sin \phi \dot{z}) \\ v &= \overline{G}(\cos \phi y - \sin \phi \dot{y})\end{aligned}\tag{13}$$

with $(\overline{M}, \overline{F}) \in \sum_\phi(\nu - 1) \cap \sum_0(\nu - 1)$ such that when coupled with the generalized system

$$\begin{aligned}E\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}\tag{14}$$

the set $\overline{\Lambda}$ of eigenvalues of the resulting closed-loop composite system is such that $\tilde{\rho}(\overline{\Lambda}, \Lambda) < \epsilon$.

Remark 5: Since $(\overline{M}, \overline{F}) \in \sum_0(\nu - 1)$, the compensator is a regular system.

Proof of Theorem 2: Let \hat{F}, G, H, K, N, P be as in the proof of Theorem 2. Set $\overline{G} := G, \overline{H} := H, \overline{K} := K$. For each $\phi \in \mathbb{R}$, define $(\overline{M}(\phi), \overline{F}(\phi)) := R_\phi(I, \hat{F})$. Let $(\overline{N}(\phi), \overline{P}(\phi))$ denote the $(n + \nu - 1)$ -dimensional generalized linear system obtained by interconnecting the compensator

$$\begin{aligned}\overline{M}(\phi)\dot{z} &= \overline{F}(\phi)z + v \\ u &= \overline{K}(\cos \phi y - \sin \phi \dot{y}) + \overline{H}(\cos \phi z - \sin \phi \dot{z}) \\ v &= \overline{G}(\cos \phi y - \sin \phi \dot{y})\end{aligned}\tag{15}$$

with the system (14). Then

$$\begin{aligned}\overline{N}(\phi) &= \begin{bmatrix} E + \sin \phi BKC & \sin \phi BH \\ \sin \phi GC & \overline{M}(\phi) \end{bmatrix} \\ \overline{P}(\phi) &= \begin{bmatrix} A + \cos \phi BKC & \cos \phi BH \\ \cos \phi GC & \overline{F}(\phi) \end{bmatrix}.\end{aligned}$$

Since $(E, A) \in \sum_\theta(n)$, there exists $\delta > 0$ such that $(E, A) \in \sum_\phi(n)$ whenever $|\phi - \theta| < \delta$. Since

$$\cos \phi \overline{N}(\phi) - \sin \phi \overline{P}(\phi) = \begin{bmatrix} \cos \phi E - \sin \phi A & 0 \\ 0 & I \end{bmatrix},$$

it follows that $(\overline{N}(\phi), \overline{P}(\phi)) \in \sum_\phi(n + \nu - 1)$ for all such ϕ . In particular, $(\overline{N}(\phi), \overline{P}(\phi))$ is admissible. Since $\overline{N}(\phi)$ and $\overline{P}(\phi)$ are continuous functions of ϕ , it follows from Lemma 1 that the mapping $\phi \rightarrow \sigma(\overline{N}(\phi), \overline{P}(\phi))$ is a continuous mapping of the open interval $(\theta - \delta, \theta + \delta)$ into $\tilde{X}_{n+\nu-1}$. Since $(\overline{N}(\theta), \overline{P}(\theta)) = (N, P)$ and $\sigma(N, P) = \Lambda$, we can choose $\delta > 0$ smaller if necessary so that if $|\phi - \theta| < \delta$, then $\tilde{\rho}(\sigma(\overline{N}(\phi), \overline{P}(\phi)), \Lambda) < \epsilon$.

Since $(\overline{M}(\phi), \overline{F}(\phi)) = R_\phi(I, \hat{F})$ and $(I, \hat{F}) \in \sum_0(\nu - 1)$, $(\overline{M}(\phi), \overline{F}(\phi)) \in \sum_\phi(\nu - 1)$. $(\overline{M}(\phi), \overline{F}(\phi)) \in \sum_0(\nu - 1)$ if and only if $(I, \hat{F}) \in \sum_{-\phi}(\nu - 1)$, which is true for all but at most a discrete set of ϕ . Consequently, we can choose $\delta > 0$ smaller if necessary so that $(\overline{M}(\phi), \overline{F}(\phi)) \in \sum_0(\nu - 1)$ whenever $0 < |\phi - \theta| < \delta$. \square

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