SRC TR 87-116

# Controlled Diffusions in a Random Medium

by

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# CONTROLLED DIFFUSIONS IN A RANDOM MEDIUM

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**Abstract**

Using homogenization theory we treat the problem of controlled diffusions in a random medium with rapidly varying composition. This involves homogenization of a nonlinear Bellman dynamic programming equation with rapidly varying random coefficients. The appropriate "averaged form" of this equation is derived to define the limiting control problem; and a precise convergence result is given.

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A. BENSOUSSAN[2] and G. BLANKENSHIP[3]

## Introduction

Classical homogenization theory deals with the limit of solutions of elliptic problems of the form

$$-\frac{\partial}{\partial x_i}\left(a_{ij}(\frac{x}{\epsilon})\frac{\partial u^\epsilon}{\partial x_j}\right) = f \text{ in } \mathcal{O}, \quad u^\epsilon|_\Gamma = 0 \tag{1}$$

as $\epsilon \to 0$, where $\mathcal{O}$ is a bounded (smooth) domain of $\Re^n$, whose boundary is denoted by $\Gamma$. In (1) $a_{ij}(y)$ is a periodic function of all components, with period 1. For details, see among other works, A. BENSOUSSAN, J.L. LIONS, and G. PAPANICOLAOU [2].

In treating the control of diffusion processes with highly oscillatory coefficients, one is led to Bellman equations which are of the form

$$-\frac{\partial}{\partial x_i}\left(a_{ij}(\frac{x}{\epsilon})\frac{\partial u^\epsilon}{\partial x_j}\right) = H(x, \frac{x}{\epsilon}, Du^\epsilon, u^\epsilon) \text{ in } \mathcal{O} \tag{2}$$
$$u^\epsilon|_\Gamma = 0.$$

On the right hand side of (2) $H$ represents the Hamiltonian function. It is a nonlinear operator which has in general quadratic growth in $Du^\epsilon$. In

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A. BENSOUSSAN, L. BOCCARDO, F. MURAT [1] the limit of the solution of equations of the type (2) was studied for quite general $H$ having quadratic growth. The limiting solution of (2) as $\epsilon \to 0$, satisfies a limit equation which is also a Bellman equation that "averages" the original control problem.

Periodic functions are a particular case of *stationary processes* $a_{ij}(y, \omega)$. It makes sense to consider such a generalization to model materials with characteristics which oscillate rapidly but randomly. On a probability space $(\Omega, \mathcal{A}, P)$ the analogue of (1) in this case is

$$-\frac{\partial}{\partial x_i}\left(a_{ij}(\frac{x}{\epsilon}, \omega)\frac{\partial u^\epsilon}{\partial x_j}\right) = f \text{ in } \mathcal{O}, \quad u^\epsilon|_\Gamma = 0 \tag{3}$$

Note that $u^\epsilon(x, \omega)$ is a random field. The homogenization theory for equations of the type (3) was first treated by JURINSKII [4] and KOZLOV [5]. The most complete treatment is that of G.C. PAPANICOLAOU and S.R.S. VARADHAN [7]. In this paper we consider the Bellman equation related to (3) namely

$$-\frac{\partial}{\partial x_i}\left(a_{ij}(\frac{x}{\epsilon}, \omega)\frac{\partial u^\epsilon}{\partial x_j}\right) = H(x, \frac{x}{\epsilon}, Du^\epsilon, u^\epsilon, \omega) \text{ in } \mathcal{O} \tag{4}$$

$$u^\epsilon|_\Gamma = 0.$$

where $H$ is a random Hamiltonian having quadratic growth in $Du^\epsilon$.

Such systems describe the control of diffusions in a random environment. They are also related to the control of beams and related structures with random infrastructure. We shall treat this case in further work.

One important mathematical difficulty related to (4), which does not appear in the periodic case is worth describing at the outset. Suppose one proves an estimate of the form

$$E \int_{O} (u^{\epsilon})^2 \, dx \; + \; E \int_{O} |Du^{\epsilon}|^2 dx \leq C. \tag{5}$$

Extracting a subsequence which converges weakly, one cannot assert that $u^{\epsilon}$ converges in $L^2$ strongly and pointwise. Usually this compactness property is used in homogenization theory, especially in nonlinear problems. We show here (since we do not have compactness) that the estimate (5) is sufficient by itself. This complicates the proof slightly, with respect to that of classical homogenization; but it is a useful fact to know.

# 1 Preliminary Set Up: Definition of the Cell Problem

## 1.1 Notation - Assumptions

We shall consider a framework for stationary processes as presented in G. PAPANICOLAOU and S.R.S. VARADHAN [7]. Let $(\Omega, \mathcal{A}, P)$ be a probability space and define $\mathcal{H} = L^2(\Omega, \mathcal{A}, P)$. We assume:

$$\mathcal{H} \text{ is separable;} \tag{6}$$

$$\text{there exists a strongly continuous unitary group } T_y \text{ on } \mathcal{H}, y \in \Re^n \tag{7}$$

$$T_y \text{ is ergodic, which means if } \tilde{f} \in \mathcal{H} \text{ satisfies} \tag{8}$$

$$T_y\tilde{f} = \tilde{f}, \quad \forall y, \text{ then } \tilde{f} \text{ is a constant.}$$

$$\text{if } \tilde{f} \geq 0, \text{ then } T_y\tilde{f} \geq 0 \text{ and } T_y 1 = 1 \tag{9}$$

The group $T_y$ has a spectral resolution defined by

$$T_y = \int_{\Re^n} e^{i\lambda y} U(d\lambda)$$

where $U(d\lambda)$ is a projection valued measure. We consider the complex extension of $\mathcal{H}$, provided with the scalar product $E\tilde{f}\bar{\tilde{g}}$, whenever $\tilde{f}$, $\tilde{g}$ are two elements of $\mathcal{H}$. The measure $U(\Delta)$, $\Delta$ Borel subset of $\Re^n$, satisfies

$$EU(\Delta)\tilde{f}\overline{U(\Delta')\tilde{g}} = 0 \quad \forall \tilde{f}, \tilde{g} \in \mathcal{H}, \Delta \cap \Delta' = \phi$$

$$EU(\Delta)\tilde{f}\overline{U(\Delta)\tilde{g}} = EU(\Delta)\tilde{f}\bar{\tilde{g}}$$

and by ergodicity

$$U(\{0\})\tilde{f} = E\tilde{f} \tag{10}$$

We next define

$$D_i\tilde{f}(\omega) = \frac{\partial}{\partial y_i}\left(T_y\tilde{f}\right)(\omega)|_{y=0} \tag{11}$$

which are closed, densely defined linear operators with domains $\mathcal{D}(D_i)$ in $\mathcal{H}$. Note that

$$E\{\tilde{g}D_i\tilde{f}\} = -E\{D_i\tilde{g}\tilde{f}\} \quad \forall \tilde{f}, \tilde{g} \in \mathcal{D}(D_i), \tag{12}$$

and

$$D_j\tilde{f} = i\int_{\Re^n} \lambda_j U(d\lambda)\tilde{f}. \tag{13}$$

If $D_j\tilde{f} = 0 \quad \forall j$, then since

$$E|D_j\tilde{f}|^2 = \int_{\Re^n} \lambda_j^2 E|U(d\lambda)\tilde{f}|^2$$

it follows that $U(d\lambda)\tilde{f} = 0 \quad \forall\lambda \neq 0$; hence

$$\tilde{f} = U(\{0\})\tilde{f} = E\tilde{f}.$$

Let $\mathcal{H}^1 = \bigcap_{j=1}^{n} \mathcal{D}(D_j)$ which is dense in $\mathcal{H}$. We equip $\mathcal{H}^1$ with the Hilbert scalar product

$$((\tilde{f},\tilde{g}))_{\mathcal{H}^1} = E\tilde{f}\tilde{g} + \sum_{j=1}^{d} ED_j\tilde{f}D_j\tilde{g} \tag{14}$$

We identify $\mathcal{H}$ with its dual and call $\mathcal{H}^{-1}$ the dual of $\mathcal{H}^1$. We have the inclusions

$$\mathcal{H}^1 \subset \mathcal{H} \subset \mathcal{H}^{-1}$$

each space being dense in the next one with continuous injection.

The family $T_y$ is also a strongly continuous unitary group on $\mathcal{H}^1$, since

$$T_y D_i \tilde{f} = D_i T_y \tilde{f}.$$

It can be extended as a strongly continuous unitary group on $\mathcal{H}^{-1}$ by the formula

$$< T_y\tilde{f}_*, \tilde{f} > = < \tilde{f}_*, T_{-y}\tilde{f} > \quad \forall\tilde{f} \in \mathcal{H}^1, \tilde{f}_* \in \mathcal{H}^{-1}$$

and $<,>$ refers to the duality between $\mathcal{H}^1$ and $\mathcal{H}^{-1}$.

REMARK 1.1 *The periodic case.*

Let $\Omega$ be the unit $n$ dimensional torus, $\mathcal{A}$ the $\sigma$-algebra of Lebesgue measurable sets and $P$ Lebesgue measure on $\Omega$. Then $\mathcal{H}$ is the space of measurable periodic functions (period 1 in each component) such that

$$\int_{\Omega} (\tilde{f}(\omega))^2 d\omega < \infty.$$

We define

$$T_y \tilde{f}(\omega) = \tilde{f}(\omega + y)$$

hence

$$D_i \tilde{f} = \frac{\partial}{\partial \omega_i} \tilde{f}.$$

An important fact in the periodic case which does not carry over to the stochastic case, is that there is no analogue of Poincare's inequality. The consequence is following. Consider the quotient space $\mathcal{H}^1/\Re$ of elements of $\mathcal{H}^1$ which differ by a constant. Denote by $[\tilde{f}]$ the equivalence class related to an element $\tilde{f}$, then the quotient norm is given by

$$\| [\tilde{f}] \| = \{E|\tilde{f} - E\tilde{f}|^2 + \sum_j E|D_j \tilde{f}|^2\}^{1/2}$$

This is not equivalent to $\left(\sum_j E|D_j \tilde{f}|^2\right)^{1/2}$. In the periodic case one has

$$\| [\tilde{f}] \|^2 = \int_\Omega |\tilde{f} - m(\tilde{f})|^2 d\omega + \sum_j \int_\Omega |\frac{\partial \tilde{f}}{\partial \omega_i}|^2 d\omega$$

where $m(\tilde{f}) = \int_\Omega \tilde{f}(\omega) d\omega$. Poincare's inequality implies that $\| \; [\tilde{f}] \; \|$ is equivalent to $\left(\sum_j \int_\Omega |\frac{\partial}{\partial \omega_j} \tilde{f}|^2 d\omega\right)^{1/2}$.

$\square$

Consider now random variables not necessarily in $\mathcal{H}$. We assume that

$T_y$ is a linear group on the set of complex random variables such that $\forall \tilde{\eta}_1 \ldots \tilde{\eta}_k$ complex random variables, $\phi$ Borel bounded function on $C^k$, then

$$E\phi(T_y \tilde{\eta}_1, \ldots, T_y \tilde{\eta}_k) = E\phi(\tilde{\eta}_1, \ldots, \tilde{\eta}_k) \tag{15}$$

$$y, \omega \to T_y \tilde{\eta} \text{ is measurable,}$$

$$T_y \tilde{\eta} \geq 0 \text{ if } \tilde{\eta} \geq 0.$$

A *stationary process* is a stochastic process $\eta(y;\omega)$ which can be represented in the form

$$\eta(y;\omega) = T_y \tilde{\eta}(\omega). \tag{16}$$

The space of square integrable stationary processes can be identified with $\mathcal{H}$. Moreover,

$$\frac{\partial \eta}{\partial y_i}(y;\omega) = D_i T_y \tilde{\eta}(\omega) = T_y D_i \tilde{\eta} \tag{17}$$

$$\text{if } \tilde{\eta} \in \mathcal{H}^1.$$

Note that the continuity assumption on $T_y$ implies that the square integrable stationary processes are necessarily continuous functions of $y$ with values in $\mathcal{H}$. Hence, if $\tilde{\eta} \in H$, $\eta(y;\omega) \in C^o(\Re^n;\mathcal{H})$, the space of uniformly continuous functions on $\Re^n$ with values in $\mathcal{H}$. If $\tilde{\eta} \in \mathcal{H}^1$, then $\eta(y;\omega) \in C^1(\Re^n;\mathcal{H})$.

Note that $D_i \in \mathcal{L}(\mathcal{H}^1;\mathcal{H})$. If $\tilde{\eta} \in \mathcal{H}$, we can consider the distribution derivative $\frac{\partial \eta}{\partial y_i}$ with values in $\mathcal{H}$, defined by

$$\int_{\Re^n} \frac{\partial \eta}{\partial y_i} \theta(y) dy = -\int_{\Re^n} \eta(y;\omega) \frac{\partial \theta}{\partial y_i} dy, \quad \forall \theta \in C_0^\infty(\Re^n).$$

Let us check that

$$-\int_{\Re^n} \eta(y,\omega) \frac{\partial \theta}{\partial y_i} dy = \int_{\Re^n} D_i T_y \tilde{\eta} \theta(y) dy \tag{18}$$

which is an equality in $\mathcal{H}^{-1}$. This proves that $\eta(y,\omega) \in C^1(\Re^n; \mathcal{H}^{-1})$ and the distribution derivative $\partial\eta/\partial y_i$ with values in $\mathcal{H}$ can be considered as a continuous function with values in $\mathcal{H}^{-1}$.

To prove (18) pick $\tilde{\phi} \in \mathcal{H}^1$, then

$$\begin{aligned} E\tilde{\phi}\int_{\Re^n} D_i T_y \tilde{\eta}\theta(y)dy &= -E\int_{\Re^n} D_i\tilde{\phi} T_y\tilde{\eta}\theta(y)dy \\ &= -E\int_{\Re^n} D_i T_{-y}\tilde{\phi}\tilde{\eta}\theta(y)dy \\ &= -E\int_{\Re^n} \frac{\partial\phi}{\partial y_i}(-y,\omega)\tilde{\eta}\theta(y)dy \\ &= -E\int_{\Re^n} \phi(-y,\omega)\tilde{\eta}\frac{\partial\theta}{\partial y_i}(y)dy \\ &= -E\int_{\Re^n} \tilde{\phi}\eta(y,\omega)\frac{\partial\theta}{\partial y_i}(y)dy \end{aligned}$$

which completes the proof of (18).

## 1.2 The Cell Problem

We consider stationary processes $a_{ij}(y;\omega)$ such that

$$a_0|\xi|^2 \le \sum_{i,j} a_{ij}(y,\omega)\xi_i\xi_j \le \frac{1}{a_0}|\xi|^2 \qquad (19)$$
$$\forall \xi \in \Re^n, \text{ some } a_0 > 0$$

Let $g_j(y,\omega) = T_y\tilde{g}_j$ be square integrable stationary processes, $j = 1, \ldots n$. We shall solve the problem

$$\chi(y;\omega) \in C^1(\Re^n; \mathcal{H}), \quad \chi(0;\omega) = 0, \quad E\chi(y) = 0 \quad \forall y \qquad (20)$$
$$\frac{\partial\chi}{\partial y_j} \text{ is a square integrable stationary process} \quad \forall j$$

$$-\frac{\partial}{\partial y_i}\left(a_{ij}(y,\omega)\frac{\partial \chi}{\partial y_j}\right) = \frac{\partial g_j}{\partial y_j}(y,\omega)$$

in the sense of distributions with values in $\mathcal{H}$

(or as continuous functions with values in $\mathcal{H}^{-1}$).

G. PAPANICOLAOU and S.R.S. VARADHAN [7] have shown the existence and uniqueness of the solution of (20). For the sake of completeness we shall reproduce their proof with minor changes. Note that $\chi(y;\omega)$ itself is *not* a stationary process. This is a major difference relative to the periodic case and relates to Remark 1.1. Note also that $\chi(y;\omega) \in C^2(\Re^n;\mathcal{H}^{-1})$.[4]

Let $\tilde{\chi}_j \in H$ such that

$$\frac{\partial \chi}{\partial y_j}(y;\omega) = T_y\tilde{\chi}_j$$

we can assert that

$$E\tilde{\chi}_j D_k\tilde{\phi} = E\tilde{\chi}_k D_j\tilde{\phi}, \quad \forall\tilde{\phi} \in \mathcal{H}^1. \tag{21}$$

Indeed we have to show that

$$D_k\tilde{\chi}_j = D_j\tilde{\chi}_k \tag{22}$$

as an equality in $\mathcal{H}^{-1}$. But

$$\begin{aligned} T_y D_k\tilde{\chi}_j &= \frac{\partial}{\partial y_k}\chi_j(y;\omega) \\ &= \frac{\partial^2 \chi}{\partial y_j \partial y_k}(y;\omega) \end{aligned}$$

[4]By virtue of (9) it is sufficient to have (19) for $y = 0$.

hence

$$T_y D_k \tilde{\chi}_j = T_y D_j \tilde{\chi}_k$$

which implies (22).

We have also

$$E\tilde{\chi}_j = 0. \tag{23}$$

This follows from

$$\chi(y;\omega) = \sum_j \int_0^1 T_{\theta y} \tilde{\chi}_j y_j d\theta$$

hence

$$E\chi(y) = \sum_j E\tilde{\chi}_j y_j = 0$$

by the assumption. Therefore (23) follows.

We can then state

THEOREM 1.1 *There exists one and only one solution of* (20).

PROOF.

*Uniqueness.*

Assume that $\partial g_j / \partial y_j = 0$. Define

$$\tilde{\phi}^\beta(\omega) = \int_{\Re^n} \sum_j \frac{(-i\lambda_j - \beta)}{|i\lambda - \beta|^2} U(d\lambda) \tilde{\chi}_j(\omega). \tag{24}$$

Note that (20) can be written as

$$-D_i T_y (\tilde{a}_{iy} \tilde{\chi}_j) = 0 \text{ in } \mathcal{H}^{-1}$$

Hence, as is easily seen,

$$E\tilde{a}_{ij} \tilde{\chi}_j D_i \tilde{\phi}^\beta = 0. \tag{25}$$

Because of (22) we have

$$\int_{\Re^n} \frac{\lambda_k U(d\lambda)\tilde{\chi}_j}{|i\lambda - \beta|^2} = \int_{\Re^n} \frac{\lambda_j U(d\lambda)\tilde{\chi}_k}{|i\lambda - \beta|^2}$$

which implies

$$D_k\tilde{\phi}^\beta - \beta\tilde{\phi}^\beta = \int_{\Re^n} \sum_j \frac{(-i\lambda_j - \beta)(i\lambda_k - \beta)}{|i\lambda - \beta|^2} U(d\lambda)\tilde{\chi}_j(\omega)$$

$$= \int_{\Re^n} \sum_j \frac{(-i\lambda_j - \beta)(i\lambda_j - \beta)}{|i\lambda - \beta|^2} U(d\lambda)\tilde{\chi}_k(\omega) = \tilde{\chi}_k.$$

Therefore (25) reads

$$E\tilde{a}_{ij}\tilde{\chi}_j\tilde{\chi}_i + \beta E\tilde{a}_{ij}\tilde{\chi}_j\tilde{\phi}^\beta = 0. \tag{26}$$

However,

$$E|\beta\tilde{\phi}^\beta|^2 = \int_{\Re^n} \beta^2 \sum_{j,k} \frac{(-i\lambda_j - \beta)(-i\lambda_k - \beta)}{|i\lambda - \beta|^4} EU(d\lambda)\tilde{\chi}_j\tilde{\chi}_k$$

$$\rightarrow \sum_{j,k} EU\{0\}\tilde{\chi}_j\tilde{\chi}_k \text{ as } \beta \rightarrow 0$$

and by ergodicity and property (23), we get

$$E|\beta\tilde{\phi}^\beta|^2 \rightarrow 0 \text{ as } \beta \rightarrow 0.$$

Therefore, (26) implies

$$E\tilde{a}_{ij}\tilde{\chi}_j\tilde{\chi}_i = 0$$

Hence, $\tilde{\chi}_i = 0$, which implies also $\chi = 0$.

*Existence:*

Let $\beta > 0$, we solve the problem

$$-\frac{\partial}{\partial y}(a_{ij}\frac{\partial \chi^\beta}{\partial y_j}) + \beta\chi^\beta = \frac{\partial g_j}{\partial y_j}.$$

$$\chi^\beta(y;\omega) = T_y\tilde{\chi}^\beta, \quad \tilde{\chi}^\beta \in \mathcal{H}^1 \tag{27}$$

This problem is equivalent to

$$E\tilde{a}_{ij}D_j\tilde{\chi}^\beta D_i\tilde{\phi} + \beta E\tilde{\chi}^\beta\tilde{\phi} = -E\tilde{g}_jD_j\tilde{\phi} \quad \forall\tilde{\phi} \in \mathcal{H}^1 \tag{28}$$

We deduce easily the estimates

$$E|D_j\tilde{\chi}^\beta|^2 \leq C, \quad \beta E(\tilde{\chi}^\beta)^2 \leq C.$$

Let us extract a subsequence such that

$$D_j\tilde{\chi}^\beta \to \tilde{\chi}_j \text{ in } \mathcal{H} \text{ weakly.}$$

$$ED_j\tilde{\chi}^\beta D_k\tilde{\phi} = ED_k\tilde{\chi}^\beta D_j\tilde{\phi}$$

$$ED_j\tilde{\chi}^\beta = 0$$

we deduce (22) and (23). Going to the limit in (28) we have

$$E\tilde{a}_{ij}\tilde{\chi}_jD_j\tilde{\phi} = -E\tilde{g}_jD_j\tilde{\phi} \quad \forall\tilde{\phi} \in \mathcal{H}^1. \tag{29}$$

Define then

$$\chi(y;\omega) = \int_{\mathbb{R}^n}(e^{i\lambda y} - 1)\frac{1}{|\lambda|^2}\sum_j(-i\lambda_j)U(d\lambda)\tilde{\chi}_j(\omega) \tag{30}$$

then

$$\frac{\partial\chi}{\partial y_k}(y;\omega) = \int_{\mathbb{R}^n} e^{i\lambda y}\frac{1}{|\lambda|^2}\sum_j \lambda_j\lambda_k U(d\lambda)\tilde{\chi}_j(\omega)$$

$$= T_y \tilde{\chi}_k$$

and $\chi(0;\omega) = 0$, $E\chi(y) = 0$. Then (29) can be written as

$$-D_i \tilde{a}_{ij} \tilde{\chi}_j = D_i \tilde{g}_i \text{ equality in } \mathcal{H}^{-1}$$

which is indeed (20).

$\square$

## 1.3 Some Technical Results

We shall prove some useful technical results in this paragraph. Let $\mathcal{O}$ be a smooth bounded domain of $\Re^n$. We define $C_0^\infty(\mathcal{O};\mathcal{H})$ to be the space of infinitely differentiable $\mathcal{H}$ valued functions that vanish outside a compact subset of $\mathcal{O}$. We define $H^1(\mathcal{O};\mathcal{H})$ and $H_0^1(\mathcal{O};\mathcal{H})$ as done for Sobolev spaces. Note that since $\mathcal{H}$ is separable, a function $\phi \in H^1(\mathcal{O};\mathcal{H})$ can be written as the expansion

$$\phi(x;\omega) = \sum_k \phi_k(x) \tilde{h}_k(\omega)$$

where $\tilde{h}_k$ is an orthonormal basis in $\mathcal{H}$ and $\phi_k(x) \in H^1(\mathcal{O})$ with

$$\phi_k(x) = E\tilde{\phi}(x)\tilde{h}_k.$$

Setting

$$\tilde{\phi}(x)(\omega) = \phi(x;\omega) \in \mathcal{H}.$$

Furthermore,

$$\| \phi \|^2_{H^1(\mathcal{O};\mathcal{H})} = \sum_k \| \phi_k \|^2_{H^1(\mathcal{O})} .$$

We begin with

LEMMA 1.1 *Let $\tilde{\phi} \in \mathcal{H}$ and $\phi(y;\omega) = T_y\tilde{\phi}(\omega)$. Let $z^\epsilon \in H_0^1(\mathcal{O};\mathcal{H})$. Assume that*

$$\| z^\epsilon \|_{H^1(\mathcal{O};\mathcal{H})} \leq C \tag{31}$$

$$E\tilde{\phi} = 0 \tag{32}$$

*then*

$$E\int_{\mathcal{O}} \phi(\frac{x}{\epsilon},\omega) z^\epsilon(x;\omega)dx \to 0 \text{ as } \epsilon \to 0 \tag{33}$$

PROOF.

Define

$$\psi_\ell(y;\omega) = \int_{\Re^n} (e^{i\lambda y} - 1)\frac{(-i\lambda_\ell)}{|\lambda|^2} U(d\lambda)\tilde{\phi}(\omega) \in C^1(\Re^n;\mathcal{H})$$

then

$$\frac{\partial \psi_\ell}{\partial y_\ell}(y;\omega) = \phi(y;\omega)$$

hence

$$\phi(\frac{x}{\epsilon};\omega) = \epsilon\frac{\partial}{\partial x_\ell}\psi_\ell(\frac{x}{\epsilon};\omega).$$

Using this representation in (33) yields

$$X_\epsilon = E\int_{\mathcal{O}} \phi(\frac{x}{\epsilon},\omega) z^\epsilon(x;\omega)dx$$

$$= -\epsilon E\int_{\mathcal{O}} \psi_\ell(\frac{x}{\epsilon};\omega)\frac{\partial z^\epsilon}{\partial x_\ell}(x;\omega)dx$$

Hence, using (31)

$$|X_\epsilon|^2 \leq C\epsilon^2 E\int_{\mathcal{O}} \sum_\ell \psi_\ell(\frac{x}{\epsilon};\omega)^2 dx.$$

But

$$\epsilon^2 E \sum_\ell \psi_\ell(\frac{x}{\epsilon})^2 = \int_{\Re^n} \frac{|e^{i\lambda\frac{x}{\epsilon}} - 1|^2}{|\frac{\lambda}{\epsilon}|^2} E(U(d\lambda)\tilde{\phi}, \tilde{\phi})$$
$$\rightarrow E(U\{0\}\tilde{\phi}, \tilde{\phi}) = |E\tilde{\phi}|^2 = 0$$

which implies the desired result (33).

□

REMARK 1.2 By the mean ergodic theorem it is easy to check that $\phi^\epsilon(x;\omega) = \phi(x/\epsilon, \omega) \rightarrow 0$ in $L^2(\mathcal{O};\mathcal{H})$ weakly. Indeed, we have to prove that

$$X_\epsilon = E\int_{\mathcal{O}} \phi(\frac{x}{\epsilon}, \omega) z(x;\omega) dx \rightarrow 0 \quad \forall z \in L^2(\mathcal{O};\mathcal{H}).$$

Clearly, we can take $z$ to be a step function, then it is sufficient to prove that $\quad \forall M \in \mathcal{O}$

$$E\int_{B_M} \phi(\frac{x}{\epsilon}, \omega) dx = 0 \tag{34}$$

where $B_M$ is a cube with center $M$, such that $\bar{B}_M \subset \mathcal{O}$. But the left hand side of (34) is equal to

$$E\tilde{\phi}\ \mathrm{Mes}\, B_M = 0 \quad \text{since } E\tilde{\phi} = 0$$

This property, however, does not immediately imply (33) since, despite (31), we cannot extract a subsequence of $z^\epsilon$ which converges strongly in $L^2(\mathcal{O};\mathcal{H})$. Indeed, there is no compactness of the injection of $\mathcal{H}^1(\mathcal{O};\mathcal{H})$ in $L^2(\mathcal{O};\mathcal{H})$, unlike for the usual Sobolev spaces where $\mathcal{H} = \Re$.

$\square$

LEMMA 1.2 *Let $\tilde{\phi}(x) = \tilde{\phi}(x;\omega) \in L^2(\mathcal{O};\mathcal{H})$ and $\phi(x,y,\omega) = T_y\tilde{\phi}(x;\omega)$. Let $z^\epsilon \in H_0^1(\mathcal{O};\mathcal{H})$ satisfying (31). Assume also that*

$$E\tilde{\phi}(x) = 0 \tag{35}$$

*then one has*

$$E\int_{\mathcal{O}} \phi(x,\frac{x}{\epsilon},\omega) z^\epsilon(x,\omega)dx \to 0 \text{ as } \epsilon \to 0 \tag{36}$$

PROOF.

Let us consider a triangulation of $\mathcal{O}$

$$\mathcal{O} = \bigcup_{k=1}^{N(h)} \mathcal{O}_h^k$$

$$\mathcal{O}_h^k \cup \mathcal{O}_h^{k'} = \phi \text{ if } k \neq k'$$

$$\text{diameter of } \mathcal{O}_h^k \leq Ch$$

We define

$$\tilde{\phi}_h(x) = \sum_{k=1}^{N(h)} \mathbf{1}_{\mathcal{O}_h^k} \frac{1}{|\mathcal{O}_h^k|}\int_{\mathcal{O}_h^k} \tilde{\phi}(x)dx$$

then

$$\tilde{\phi}_h \to \tilde{\phi} \text{ in } L^2(\mathcal{O};\mathcal{H}).$$

Let

$$\phi_h(x,y,\omega) = T_y\tilde{\phi}_h(x;\omega)$$

then

$$\left|E\int_{\mathcal{O}} \left(\phi(x,\frac{x}{\epsilon},\omega) - \phi_h(x,\frac{x}{\epsilon},\omega)\right) z^\epsilon(x,\omega)dx\right|^2 \tag{37}$$

$$\leq CE\int_{O}\left(\phi(x,\frac{x}{\epsilon},\omega)-\phi_h(x,\frac{x}{\epsilon},\omega)\right)^2 dx$$
$$= C\int_{O}\parallel \tilde{\phi}_h(x)-\tilde{\phi}(x)\parallel_{\mathcal{X}}^2 dx = \delta(h).$$

On the other hand

$$E\int_{O}\phi_h(x,\frac{x}{\epsilon},\omega)z^{\epsilon}(x;\omega)dx = \sum_{k=1}^{N(h)} E\int_{O_h^k}\phi_h^k(\frac{x}{\epsilon},\omega)z^{\epsilon}(x,\omega)dx$$

where

$$\tilde{\phi}_h^k(\omega) = \frac{1}{|O_h^k|}\int_{O_h^k}\tilde{\phi}(x,\omega)dx$$

and

$$\phi_h^k(y,\omega) = T_y\tilde{\phi}_h^k(\omega).$$

Since $E\tilde{\phi}_h^k = 0$, we deduce from Lemma 1.1 that for fixed $h$,

$$E\int_{O}\phi_h(x,\frac{x}{\epsilon},\omega)z^{\epsilon}(x,\omega)dx \to 0$$

which with the estimate (37) implies (36).

□

LEMMA 1.3 *Let* $\tilde{\phi} \in L^1(\Omega,\mathcal{A},P)$, *such that*

$$E\tilde{\phi} = 0. \tag{38}$$

*Let*

$$\phi(y;\omega) = T_y\tilde{\phi}(\omega)$$

*and* $z^{\epsilon}(x,\omega)$ *such that*

$$z^\epsilon \in L^\infty(\mathcal{O} \times \Omega) \cap H_0^1(\mathcal{O};\mathcal{H}), \ \textit{satisfies (31)} \tag{39}$$

*then one has*

$$E\int_{\mathcal{O}} \phi(\frac{x}{\epsilon},\omega) z^\epsilon(x,\omega)dx \to 0 \text{ as } \epsilon \to 0. \tag{40}$$

PROOF:

Let

$$\tilde{\phi}_N(\omega) = \left| \begin{array}{lll} \tilde{\phi}(\omega) & \text{if} & |\tilde{\phi}| < N \\ N & \text{if} & \tilde{\phi} \geq N \\ -N & \text{if} & \tilde{\phi} \leq -N \end{array} \right. .$$

and

$$\phi_N(y;\omega) = T_y \tilde{\phi}_N(\omega).$$

From Lemma 1.1 we have

$$E\int_{\mathcal{O}} (\phi_N(\frac{x}{\epsilon},\omega) - E\tilde{\phi}_N) z^\epsilon(x,\omega)dx \to 0 \text{ as } \epsilon \to 0, \tag{41}$$

for fixed $N$. But also

$$|E\int_{\mathcal{O}} (\phi(\frac{x}{\epsilon},\omega) - \phi_N(\frac{x}{\epsilon},\omega)) z^\epsilon(x,\omega)dx| \leq C|\mathcal{O}|E|\tilde{\phi} - \tilde{\phi}_N| \tag{42}$$

$$= C|\mathcal{O}|E|\tilde{\phi}|\mathbf{1}_{|\tilde{\phi}|\leq N} \to 0 \text{ as } N \to \infty.$$

and

$$|E\tilde{\phi}_N E\int_{\mathcal{O}} z^\epsilon(x,\omega)dx| \leq C|E\tilde{\phi}_N|$$

$$= C|E(\tilde{\phi}_N - \tilde{\phi})| \to 0 \text{ as } N \to \infty. \tag{43}$$

From the uniform estimates (42),(43), together with (41), the desired result follows.

□

LEMMA 1.4 *Let* $\tilde{\phi}(x) = \tilde{\phi}(x;\omega) \in C^\circ(\overline{\mathcal{O}}; L^1(\Omega, \mathcal{A}, P))$, *and* $\phi(x,y,\omega) = T_y\tilde{\phi}(x;\omega)$. *Let* $z^\epsilon$ *verifying (39). Assume also (35) then one has*

$$E\int_{\mathcal{O}} \phi(x, \frac{x}{\epsilon}, \omega) z^\epsilon(x,\omega)dx \to 0, \text{ as } \epsilon \to 0.$$

PROOF.

Consider the sequence $\tilde{\phi}_h$ as in Lemma 1.2. We have

$$|E\int_{\mathcal{O}} \left(\phi(x, \frac{x}{\epsilon}, \omega) - \phi_h(x, \frac{x}{\epsilon}, \omega)\right) z^\epsilon(x,\omega)dx| \leq C\sum_{k=1}^{N(h)} E\int_{\mathcal{O}_h^k} |\tilde{\phi}(x) - \tilde{\phi}_h^k|dx$$
$$= \delta(h) \to 0 \text{ as } h \to \infty.$$

We also have

$$E\tilde{\phi}_h^k = 0.$$

From Lemma 1.3, we deduce that for fixed $h$, one has

$$E\int_{\mathcal{O}} \phi_h(x, \frac{x}{\epsilon}, \omega) z^\epsilon(x,\omega)dx \to 0$$

and the desired result (1.3) follows.

□

# 2 Setting of the Problem

## 2.1 Notation

We shall consider the operator

$$A^\epsilon = -\frac{\partial}{\partial x_i}\left(a_{ij}(\frac{x}{\epsilon}, \omega)\frac{\partial}{\partial x_j}\right)$$

where $a_{ij}(y,\omega)$ is the stationary process introduced in (19). Define also

$$H(x,y,p,z,\omega) = T_y\ \tilde{H}(x,p,z,\omega) \tag{44}$$

to be a family of stationary processes, depending on $x,p,z$. The functional $H$ depends measurably on all parameters. Moreover, assume

$$H(x,y,p,z+b,\omega) \leq -\beta b + H(x,y,p,z,\omega) \tag{45}$$

$$\forall b \text{ positive constant, } \beta > 0$$

$$H(x,y,p,z+b,\omega) \geq -\beta b + H(x,y,p,z,\omega) \tag{46}$$

$$\forall b \text{ negative constant,}$$

$$|H(x,y,0,0,\omega)| \leq M \quad \text{constant}$$

$$|H(x,y,p,z_1,\omega) - H(x,y,p,z_2,\omega)| \leq \bar{H}|z_1 - z_2| \tag{47}$$

$$|H(x,y,p,z,\omega) - H(x,y,q,p,\omega)| \leq \bar{H}|p-q|(1+|p|+|q|+|z|^{1/2}), \tag{48}$$

where $\bar{H}$ is a constant[5]

$$|H(x,y,p,z,\omega)-H(x',y,p,z,\omega)| \leq \bar{H}|x-x'|^{\theta}(1+|p|^2+|z|),\ 0<\theta\leq 1 \tag{49}$$

We introduce the problem: Find $u^\epsilon(x,\omega)$ the solution of

$$-\frac{\partial}{\partial x_i}\left(a_{ij}(\frac{x}{\epsilon},\omega)\frac{\partial u^\epsilon}{\partial x_j}\right) = H(x,\frac{x}{\epsilon},Du^\epsilon,u^\epsilon,\omega) \tag{50}$$

$$u^\epsilon \Big|_{\partial O} = 0$$

[5] As for (19) it is sufficient to assume (45)(47)(48) for $y = 0$.

where $\mathcal{O}$ is a smooth bounded subset of $R^n$, as considered in §1.3. The precise function space in which the solution of (50) is sought is

$$u^\epsilon \in L^\infty(\mathcal{O} \times \Omega); \;\; u^\epsilon \in H_0^1(\mathcal{O}; \mathcal{H}). \tag{51}$$

Of course, for each $\omega$, we can solve (50) relying on standard results on elliptic equations (LADYZHENSKAYA – URALT'SEVA [6]). But measurability questions remain. Therefore, it is preferable to consider functions of $x$ which are $\mathcal{H}$ valued. One can first approximate $H$ by

$$H_\delta = \frac{H}{1+\delta|H|}.$$

This will permit solution of (50) by standard variational techniques (i.e., the Lax–Milgram theorem). The study of convergence, as $\delta \to 0$, of these solutions will result in *a priori* estimates which we shall establish on solutions of (50),(51). Therefore, to simplify the writing, we shall operate on the solution of (50),(51), instead of its approximate $u^{\epsilon\delta}$.

## 2.2 Statement of the Results

We shall introduce "correctors," which are processes $\chi^\ell(y;\omega)$ solving

$$-\frac{\partial}{\partial y_i}\left(a_{ij}(y,\omega)\frac{\partial \chi^\ell}{\partial y_j}\right) = \frac{\partial a_{\ell j}}{\partial y_j}(y,\omega). \tag{52}$$

This problem is similar to (20) with $g_j(y,\omega) = a_{\ell j}(y,\omega)$, for each $\ell$. We shall denote by $\tilde{\chi}_j^\ell(\omega)$ the element of $\mathcal{H}$ such that

$$\frac{\partial \chi^\ell}{\partial y_j}(y,\omega) = T_y \; \tilde{\chi}_j^\ell(w) \tag{53}$$

Let next

$$q_{ij} = E(\tilde{a}_{ij} + \tilde{a}_{ik}\tilde{\chi}_k^j). \tag{54}$$

We set $(\widetilde{D\chi})_{ij} = \tilde{\chi}_i^j$. Consider then the function

$$\mathbf{H}(x,p,z) = E\tilde{H}(x,(I - \widetilde{D\chi})p, z). \tag{55}$$

Let us begin with a few remarks. First, the matrix $q_{ij}$ is symmetric and positive definite. Indeed we have

$$q_{ij} = E\left(\tilde{a}_{ij} + \tilde{a}_{ik}\tilde{\chi}_k^j + \tilde{a}_{jk}\tilde{\chi}_k^i + \tilde{\chi}_\ell^i \tilde{a}_{\ell k}\tilde{\chi}_k^j\right) \tag{56}$$

since

$$\begin{aligned} E\left[\tilde{a}_{jk}\tilde{\chi}_k^i + \tilde{\chi}_\ell^i \tilde{a}_{\ell k}\tilde{\chi}_k^j\right] &= E\left(\tilde{a}_{\ell j} + \tilde{a}_{\ell k}\tilde{\chi}_k^j\right)\tilde{\chi}_\ell^i \\ &= 0. \end{aligned} \tag{57}$$

This relation can be established by considering the function $\tilde{\chi}^{i,\beta}$ used in the existence proof of the correctors. We have from (29)

$$E\tilde{a}_{\ell k}\tilde{\chi}_k^j D_\ell \tilde{\chi}^{i,\beta} = -E\tilde{a}_{\ell j} D_\ell \tilde{\chi}^{i,\beta}.$$

Letting $\beta \to 0$ and using the weak convergence of $D_\ell\tilde{\chi}^{i,\beta}$ to $\tilde{\chi}_\ell^i$ in $\mathcal{H}$, we deduce (57). The function $\mathbf{H}$ is measurable and satisfies

$$\mathbf{H}(x,p,z+b) \le -\beta b + \mathbf{H}(x,p,z) \quad \forall\, b > 0 \tag{58}$$

$$\mathbf{H}(x,p,z+b) \ge -\beta b + \mathbf{H}(x,p,z) \quad \forall\, b < 0$$

$$|\mathbf{H}(x,0,0)| \le M \tag{59}$$

$$|\mathbf{H}(x,p,z_1) - \mathbf{H}(x,p,z_2)| \leq \bar{H}|z_1 - z_2| \tag{60}$$

$$|\mathbf{H}(x,p,z) - \mathbf{H}(x,q,z)| \leq \bar{\mathcal{H}}|p-q|(1+|p|+|q|+|z|^{1/2}). \tag{61}$$

where $\bar{H}$ and $\bar{\mathcal{H}}$ are constants. We can thus solve the Dirichlet problem

$$\begin{aligned} - q_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} &= \mathbf{H}(x, Du, u) \\ u \in H_0^1(\mathcal{O}) &\cap L^\infty(\mathcal{O}) \end{aligned} \tag{62}$$

THEOREM 2.1 *We assume (19),(44),(45),(47). Then one has*

$$u^\epsilon \to u \quad \text{in} \quad H_0^1(\mathcal{O};\mathcal{H}) \quad \text{weakly} \tag{63}$$

$$\begin{cases} Du^\epsilon - (I + D_y\chi(\frac{x}{\epsilon}))Du \to 0 & \text{in} \quad L^2(\mathcal{O};\mathcal{H}) \\ u^\epsilon \to u \quad \text{in} \quad L^2(\mathcal{O};\mathcal{H}) \end{cases} \tag{64}$$

We shall write

$$\mathcal{A}\, u = -q_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}$$

# 3 *A Priori* Estimates

In this section, we shall derive some *a priori* estimates on the solution $u^\epsilon$ of (50). This will permit us to study the limit in the next section.

## 3.1 An $L^\infty$ Estimate

LEMMA 3.1 *One has the estimate*

$$|u^\epsilon(x,\omega)| \quad \leq \frac{M}{\beta} \tag{65}$$

PROOF.

Let us proceed formally, considering that a.s. (50) holds $\forall x$. At a point of maximum $x_0$, which is not on the boundary, one has from (50)

$$H(x_0, \frac{x_0}{\epsilon}, 0, u^\epsilon(x_0), \omega) \geq 0$$

and from (45) it follows that if $u^\epsilon(x_0) > 0$

$$0 \leq -\beta u^\epsilon(x_0) + M$$

hence $u^\epsilon(x_0) \leq M/\beta$. Similar reasoning proves that if $x_0$ is a point of minimum, then

$$u^\epsilon(x_0) \geq -\frac{M}{\beta}$$

hence (65).

□

## 3.2 An $H_0^1$ Estimate

LEMMA 3.2 *One has the estimate*

$$\|u^\epsilon\|_{H_0^1(\mathcal{O};\mathcal{H})} \leq C \tag{66}$$

PROOF.

It is clear that, since $\mathcal{O}$ is bounded and the estimate (65) holds, one has

$$\|u^\epsilon\|_{L^2(\mathcal{O};\mathcal{H})} \leq C$$

Set

$$\mathbf{H}^\epsilon = \mathbf{H}(x, \frac{x}{\epsilon}, Du^\epsilon, u^\epsilon, \omega)$$

and for $s > 0$, note that $\exp\{s(u^\epsilon)^2\}u^\epsilon \in H_0^1(\mathcal{O}; \mathcal{H}) \cap L^\infty(\mathcal{O}; \mathcal{H})$ (the estimate (65) is used here). Multiply (2.5) by $\exp\{s(u^\epsilon)^2\}u^\epsilon$ and integrate over $\mathcal{O}$ and take the mathematical expectation. Setting $a^\epsilon_{ij} = a_{ij}(\frac{x}{\epsilon}, \omega)$, we deduce

$$E\int_{\mathcal{O}} a^\epsilon_{ij} \frac{\partial u^\epsilon}{\partial x_j}\frac{\partial u^\epsilon}{\partial x_i}\left(1 + 2s(u^\epsilon)^2\right) e^{s(u^\epsilon)^2} dx = E\int_{\mathcal{O}} H^\epsilon u^\epsilon e^{s(u^\epsilon)^2} dx.$$

From (19), the assumptions (45)(47)(48) and the estimate (65), it follows

$$a_0 E\int_{\mathcal{O}} \left(2s(u^\epsilon)^2 + 1\right) e^{s(u^\epsilon)^2} |Du^\epsilon|^2 dx \le CE\int_{\mathcal{O}} e^{s(u^\epsilon)^2} |u^\epsilon|(1 + |Du^\epsilon|^2 + |u^\epsilon|) dx$$

and picking $s$ sufficiently large, we deduce

$$E\int_{\mathcal{O}} e^{s(u^\epsilon)^2} |Du^\epsilon|^2 dx \le C$$

which implies the desired result (66).

□

# 4 Convergence

## 4.1 The Limit Problem

The solution $u$ of (62) satisfies the same estimate as (65)

$$\|u\|_{L^\infty} \le \frac{M}{\beta}.$$

It follows that

$$|\mathbf{H}(x, Du, u)| \le \bar{\mathbf{H}}_1(1 + |Du|^2) \tag{67}$$

where $\bar{\mathbf{H}}_1$ is a constant. This estimate and the equation (62) imply $u \in W^{2,p}(\mathcal{O}), \quad \forall\ 2 \le p < \infty$ (see for instance J. FREHSE [3]). In particular $u \in W^{1,\infty}(\mathcal{O})$. We shall need in the sequel to consider the following approximation scheme

$$\begin{gathered} -q_{ij}\frac{\partial^2 u^k}{\partial x_i \partial x_j} + Nu^k = \mathbf{H}(x, Du^k, u^k) + Nu^{k-1} \\ u^k \in H_0^1(\mathcal{O}) \cap L^\infty(\mathcal{O}), \quad k \ge 1, \end{gathered} \tag{68}$$

which we initialize with $u^0 = 0$. In (68) $N$ is a constant, which will be chosen later.

We have the

LEMMA 4.1 *The sequence $u^k$ satisfies*

$$\|u^{k+1} - u^k\|_{L^\infty} \le \frac{N}{N+\beta}\|u^k - u^{k-1}\|_{L^\infty} \tag{69}$$

PROOF.

This is derived from a maximum principle type argument. We proceed formally, assuming that $u^{k+1} - u^k$ is $C^2$. Let $x_0$ be a point of maximum of $u^{k+1} - u^k$ and suppose that $(u^{k+1} - u^k)(x_0) > 0$, hence $x_0 \in \mathcal{O}$. We have

$$\mathcal{A}(u^{k+1} - u^k)(x_0) + N(u^{k+1} - u^k)(x_0) = \mathbf{H}(x_0, Du^{k+1}(x_0), u^{k+1}(x_0))$$

$$-\mathbf{H}(x_0, Du^k(x_0), u^k(x_0)) \ + \ N(u^k - u^{k-1})(x_0)$$

$$\leq -\beta(u^{k+1} - u^k)(x_0) \ + \ \mathbf{H}(x_0, Du^{k+1}(x_0), u^k(x_0))$$

$$-\mathbf{H}(x_0, Du^k(x_0), u^k(x_0)) \ + \ N \ \|u^k - u^{k-1}\|.$$

But from (61)

$$|\mathbf{H}(x_0, Du^{k+1}(x_0), u^k(x_0)) - \mathbf{H}(x_0, Du^k(x_0), u^k(x_0))|$$

$$\leq \bar{\mathbf{H}}|D(u^{k+1} - u^k)(x_0)|(1 + |Du^k(x_0)| + |Du^{k+1}(x_0)|$$

$$+ \ |u^k(x_0)|^{1/2}) = 0.$$

Since also

$$\mathcal{A}(u^{k+1} - u^k)(x_0) > 0$$

we deduce

$$(u^{k+1} - u^k)(x_0) \leq \frac{N\|u^k - u^{k-1}\|}{N + \beta}$$

which proves (69).

□

It is easy to deduce also the estimate

$$\|u^k\|_{L^\infty} \leq \frac{M}{\beta} \tag{70}$$

which does not depend on $N$. Note also that from (69)

$$\|u^{k+1} - u^k\|_{L^\infty} \leq \left(\frac{N}{N + \beta}\right)^k \|u^1\|$$

$$\leq \left(\frac{N}{N + \beta}\right)^k \frac{M}{N + \beta} \leq \frac{M}{N + \beta}$$

Hence,

$$|\mathcal{A}u^k| \leq M \; + \; |\mathbf{H}(x, Du^k, u^k)|$$
$$\leq M \; + \; \bar{\mathbf{H}}|Du^k|(1+|Du^k|) + \bar{H}\frac{M}{\beta} \tag{71}$$
$$\leq C_0(1+|Du^k|^2)$$

and we note that the constant $C_0$ does not depend on $k$. This permits us to deduce (see J. FREHSE [3] for instance)

LEMMA 4.2

$$\|u^k\|_{W^{2,p}} \leq C_p, \qquad 2 \leq p < \infty \tag{72}$$

*where the constant $C_p$ does not depend on $k$.*

It is of course clear that

$$u^k \to u \text{ in } W^{2,p} \text{ weakly}$$

## 4.2 A Fundamental Relation

We also introduce the sequence $u^{\epsilon k}$ defined by

$$-\frac{\partial}{\partial x_i}\left(a_{ij}(\frac{x}{\epsilon},\omega)\frac{\partial u^{\epsilon k}}{\partial x_j}\right) + N\; u^{\epsilon k} = H(x, \frac{x}{\epsilon}, Du^{\epsilon k}, u^{\epsilon k}, \omega) + Nu^{\epsilon k-1} \tag{73}$$
$$u^{\epsilon}|_{\partial O} = 0.$$

Using the assumptions (45)(47)(48), we deduce in a manner similar to that of Lemma 4.1

$$\|u^{\epsilon,k+1} - u^{\epsilon k}\|_{L^\infty} \leq \frac{N}{N+\beta}\|u^{\epsilon,k} - u^{\epsilon,k-1}\|_{L^\infty} \tag{74}$$

$$\|u^{\epsilon,k}\|_{L^\infty} \leq \frac{M}{\beta} \tag{75}$$

$$\|u^{\epsilon,k+1} - u^{\epsilon,k}\|_{L^\infty} \leq \frac{M}{N+\beta}. \tag{76}$$

Of course the $L^\infty$ norm refers to $L^\infty(\mathcal{O} \times \Omega)$. We now consider approximation $u_\delta^k$ to $u^k$ such that

$$u_\delta^k \quad \text{is smooth,} \quad u_\delta^k \to u^k \quad \text{in} \quad H_0^1(\mathcal{O}), \quad \text{as} \quad \delta \to 0 \tag{77}$$

$$\|u_\delta^k\|_{W^{1,\infty}} \leq \bar{u}$$

where the constant $\bar{u}$ does not depend on $k$ nor $\delta$. This last uniform estimate is possible, by virtue of the estimate (72).

Define next

$$u_\delta^{\epsilon,k} = u_\delta^k \; + \; \epsilon \frac{\partial u_\delta^k}{\partial x_\ell} \frac{\chi^\ell}{[1+\epsilon^2(\chi^\ell)^2]^{1/2}}(\frac{x}{\epsilon}, \omega). \tag{78}$$

We shall also write

$$(\chi^\ell)^\epsilon(x,\omega) = \chi^\ell(\frac{x}{\epsilon}, \omega)$$

and

$$\chi^\epsilon = ((\chi^1)^\epsilon, ..., (\chi^n)^\epsilon).$$

We have

$$\|u_\delta^{\epsilon,k}\|_{L^\infty} \leq C_0 \bar{u}. \tag{79}$$

We then compute

$$\frac{\partial u_\delta^{\epsilon,k}}{\partial x_j} = \frac{\partial u_\delta^k}{\partial x_j} \; + \; \epsilon \frac{\partial^2 u_\delta^k}{\partial x_\ell \partial x_j} \frac{\chi^\ell}{[1+\epsilon^2(\chi^\ell)^2]^{1/2}} \tag{80}$$

$$+\frac{\partial u_\delta^k}{\partial x_\ell}\frac{\partial \chi^\ell}{\partial y_j}\frac{1}{[1+\epsilon^2(\chi^\ell)^2]^{1/2}}$$

which yields the estimate

$$|Du_\delta^{\epsilon,k}| \leq C_1\bar{u}\,(1+\|D_y\chi^\epsilon\|)+\epsilon h_\delta^k|\chi^\epsilon| \tag{81}$$

where

$$h_\delta^k = \|D^2u_\delta^k\|_{L^\infty},$$

and $\|D_y\chi^\epsilon\|$, $|\chi^\epsilon|$ represent the norms of the matrix $D_y\chi$, and of the vector $\chi$ evaluated at $x/\epsilon$, and $|Du_\delta^{\epsilon,k}|$ is the norm of the vector $Du_\delta^{\epsilon,k}$ at point $x$. The matrix $D_y\chi$ is precisely

$$(D_y\chi)_{j\ell} = \frac{\partial \chi^\ell}{\partial y_j}.$$

Let $F_\delta^{\epsilon,k} = \exp[\ s(u^{\epsilon,k}-u_\delta^{\epsilon,k})^2]$, where $s$ will be chosen later (as will $N$). We have

LEMMA 4.3 *The following relation holds*

$$E\int_O a_{ij}^\epsilon \frac{\partial}{\partial x_j}(u^{\epsilon,k}-u_\delta^{\epsilon,k})\frac{\partial}{\partial x_i}(u^{\epsilon,k}-u_\delta^{\epsilon,k})[2s(u^{\epsilon,k}-u_\delta^{\epsilon,k})^2+1]F_\delta^{\epsilon,k}dx \tag{82}$$

$$-E\int_O \frac{\partial^2 u_\delta^k}{\partial x_i\partial x_j}(a_{ij}+a_{i\ell}\frac{\partial \chi^j}{\partial y_\ell})^\epsilon(u^{\epsilon,k}-u_\delta^{\epsilon,k})F_\delta^{\epsilon,k}dx$$

$$+E\int_O\left[\epsilon a_{ij}^\epsilon\frac{\partial^2 u_\delta^k}{\partial x_\ell\partial x_j}\frac{\chi^\ell}{[1+\epsilon^2(\chi^\ell)^2]^{1/2}}\right.$$

$$\left.+\frac{\partial u_\delta^k}{\partial x_\ell}a_{ij}^\epsilon\frac{\partial \chi^\ell}{\partial y_j}\left(\frac{1}{(1+\epsilon^2(\chi^\ell)^2)^{3/2}}-1\right)\right]\frac{\partial}{\partial x_j}\left(u^{\epsilon,k}-u_\delta^{\epsilon,k}\right)\cdot$$

$$F_\delta^{\epsilon,k}(1+2s(u^{\epsilon,k}-u_\delta^{\epsilon,k})^2)dx$$

$$+NE\int_{\mathcal{O}}(u^{\epsilon,k}-u_\delta^{\epsilon,k})^2F_\delta^{\epsilon,k}dx+NE\int_{\mathcal{O}}u_\delta^k(u^{\epsilon,k}-u_\delta^{\epsilon,k})F_\delta^{\epsilon,k}dx$$
$$+\epsilon NE\int_{\mathcal{O}}\frac{\partial u_\delta^k}{\partial x_\ell}\frac{\chi^\ell}{[1+\epsilon^2(\chi^\ell)^2]^{1/2}}(u^{\epsilon,k}-u_\delta^{\epsilon,k})F_\delta^{\epsilon,k}dx=$$
$$E\int_{\mathcal{O}}H^\epsilon(u^{\epsilon,k}-u_\delta^{\epsilon,k})F_\delta^{\epsilon,k}dx+NE\int_{\mathcal{O}}(u^{\epsilon,k-1}-u_\delta^{\epsilon,k-1})(u^{\epsilon,k}-u_\delta^{\epsilon,k})F_\delta^{\epsilon,k}dx$$
$$+NE\int_{\mathcal{O}}u_\delta^{k-1}(u^{\epsilon,k}-u_\delta^{\epsilon,k})F_\delta^{\epsilon,k}dx$$
$$+\epsilon NE\int_{\mathcal{O}}\frac{\partial u_\delta^{k-1}}{\partial x_\ell}\frac{\chi^\ell}{[1+\epsilon^2(\chi^\ell)^2]^{1/2}}(u^{\epsilon,k}-u_\delta^{\epsilon,k})F_\delta^{\epsilon,k}dx$$

PROOF.

Note that $(u^{\epsilon,k}-u_\delta^{\epsilon,k})F_\delta^{\epsilon,k}$ belongs to $H_0^1(\mathcal{O};\mathcal{H})\cap L^\infty$. We deduce easily (82) from (73) and some integrations by parts.

□

## 4.3 Estimating the Hamiltonian

We first have

$$E\int_{\mathcal{O}}(H(x,\frac{x}{\epsilon},Du^{\epsilon,k},u^{\epsilon,k})-H(x,\frac{x}{\epsilon},Du^{\epsilon,k},u_\delta^{\epsilon,k}))(u^{\epsilon,k}-u_\delta^{\epsilon,k})F_\delta^{\epsilon,k}dx \quad (83)$$
$$\leq -\beta E\int_{\mathcal{O}}(u^{\epsilon,k}-u_\delta^{\epsilon,k})^2F_\delta^{\epsilon,k}dx$$
$$E\int_{\mathcal{O}}(H(x,\frac{x}{\epsilon},Du^{\epsilon,k},u_\delta^{\epsilon,k})-H(x,\frac{x}{\epsilon},Du^{\epsilon,k},u_\delta^k))(u^{\epsilon,k}-u_\delta^{\epsilon,k})F_\delta^{\epsilon,k}dx \quad (84)$$
$$\leq \bar{H}E\int_{\mathcal{O}}\frac{\epsilon|\chi^\ell|}{[1+\epsilon^2(\chi^\ell)^2]^{1/2}}\left|\frac{\partial u_\delta^k}{\partial x_\ell}\right|\,|u^{\epsilon,k}-u_\delta^{\epsilon,k}|F_\delta^{\epsilon,k}dx$$

Then from (48)

$$E\int_O (H(x,\frac{x}{\epsilon},Du^{\epsilon,k},u_\delta^k) - H(x,\frac{x}{\epsilon},Du_\delta^{\epsilon,k},u_\delta^{\epsilon,k}))(u^{\epsilon,k}-u_\delta^{\epsilon,k})F_\delta^{\epsilon,k}dx \tag{85}$$

$$\leq \bar{H}E\int_O |D(u^{\epsilon,k}-u_\delta^{\epsilon,k})|(1+|Du^{\epsilon,k}|$$

$$+|Du_\delta^{\epsilon,k}|+|u_\delta^k|^{1/2})|u^{\epsilon,k}-u_\delta^{\epsilon,k}|F_\delta^{\epsilon,k}dx$$

$$\leq \bar{H}E\int_O |D(u^{\epsilon,k}-u_\delta^{\epsilon,k})|^2|u^{\epsilon,k}-u_\delta^{\epsilon,k}|F_\delta^{\epsilon,k}dx$$

$$+\bar{H}E\int_O |D(u^{\epsilon,k}-u_\delta^{\epsilon,k})|\left(1+2|Du_\delta^{\epsilon,k}|+|u_\delta^k|^{1/2}\right)|u^{\epsilon,k}-u_\delta^{\epsilon,k}|F_\delta^{\epsilon,k}dx$$

$$\leq \bar{H}\frac{\rho}{2}E\int_O |D(u^{\epsilon,k}-u_\delta^{\epsilon,k})|^2F_\delta^{\epsilon,k}dx$$

$$+\frac{\bar{H}}{2\rho}E\int_O |D(u^{\epsilon,k}-u_\delta^{\epsilon,k})|^2(u^{\epsilon,k}-u_\delta^{\epsilon,k})^2F_\delta^{\epsilon,k}dx$$

$$+\bar{H}\frac{\sigma}{2}E\int_O |D(u^{\epsilon,k}-u_\delta^{\epsilon,k})|^2F_\delta^{\epsilon,k}dx$$

$$+\frac{\bar{H}}{\sigma}E\int_O (1+4|Du_\delta^{\epsilon,k}|^2+|u_\delta^k|)(u^{\epsilon,k}-u_\delta^{\epsilon,k})^2F_\delta^{\epsilon,k}dx.$$

We also notice that

$$E\int_O |Du_\delta^{\epsilon,k}|^2(u^{\epsilon,k}-u_\delta^{\epsilon,k})^2F_\delta^{\epsilon,k}dx \leq 2C_1^2\bar{u}^2(1+E\|D_y\chi\|^2)\cdot \tag{86}$$

$$E\int_O (u^{\epsilon,k}-u_\delta^{\epsilon,k})^2F_\delta^{\epsilon,k}dx$$

$$+E\int_O 2C_1^2\bar{u}^2(\|D_y\chi^\epsilon\|^2-E\|D_y\chi\|^2)(u^{\epsilon,k}-u_\delta^{\epsilon,k})^2F_\delta^{\epsilon,k}dx$$

$$+2\epsilon^2(h_\delta^k)^2E\int_O |\chi^\epsilon|^2(u^{\epsilon,k}-u_\delta^{\epsilon,k})^2F_\delta^{\epsilon,k}dx.$$

Next we have

$$E\int_O (H(x,\frac{x}{\epsilon},Du_\delta^{\epsilon,k},u_\delta^k) \tag{87}$$

$$-H(x,\frac{x}{\epsilon},(I+D_y\chi)Du_\delta^k,u_\delta^k))(u^{\epsilon,k}-u_\delta^{\epsilon,k})F_\delta^{\epsilon,k}dx$$

$$\leq \bar{H}\left[E\int_O\sum_j\left(\epsilon\frac{\partial^2 u_\delta^k}{\partial x_\ell\partial x_j}\frac{\chi^\ell}{[1+\epsilon^2(\chi^\ell)^2]^{1/2}}\right.\right.$$

$$\left.\left.+\frac{\partial u_\delta^k}{\partial x_\ell}\frac{\partial\chi^\ell}{\partial y_j}(\frac{1}{[1+\epsilon^2(\chi^\ell)^2]^{3/2}}-1)\right)^2dx\right]^{1/2}$$

$$\cdot C_2(1+\bar{u}+\epsilon h_\delta^k)$$

## 4.4 Convergence Proof

We now fix the value of various constants. We pick

$$\sigma=\rho=\frac{a_0}{2\bar{H}}$$

$$s=\frac{\bar{H}^2}{2a_0^2} \tag{88}$$

$$N=\frac{2\bar{H}^2}{a_0}\left[1+\bar{u}+8C_1^2\bar{u}^2(1+E\|D_y\chi\|^2)\right].$$

With this choice, we deduce from (82) the estimate

$$\frac{a_0}{2}E\int_O|D(u^{\epsilon,k}-u_\delta^{\epsilon,k})|^2F_\delta^{\epsilon,k}dx+\beta E\int_O(u^{\epsilon,k}-u_\delta^{\epsilon,k})^2F_\delta^{\epsilon,k}dx$$

$$+E\int_O\left[-\frac{\partial^2 u_\delta^k}{\partial x_i\partial x_j}(a_{ij}+a_{i\ell}\frac{\partial\chi^j}{\partial y_\ell})^\epsilon+Nu_\delta^k\right.$$

$$-H(x,\frac{x}{\epsilon},(I+D_y\chi)Du_\delta^k,u_\delta^k)$$

$$\left.-Nu_\delta^{k-1}\right](u^{\epsilon,k}-u_\delta^{\epsilon,k})F_\delta^{\epsilon,k}dx$$

$$+E\int_O\left[\epsilon a_{ij}^\epsilon\frac{\partial^2 u_\delta^k}{\partial x_\ell\partial x_j}\frac{\chi^\ell}{[1+\epsilon(\chi^\ell)^2]^{1/2}}\right.$$

$$+\frac{\partial u_\delta^k}{\partial x_\ell}a_{ij}^\epsilon\frac{\partial \chi^\ell}{\partial y_j}\left(\frac{1}{[1+\epsilon^2(\chi^\ell)^2]^{3/2}}-1\right)\bigg]\frac{\partial}{\partial x_i}(u^{\epsilon,k}-u_\delta^{\epsilon,k})$$

$$F_\delta^{\epsilon,k}(1+2s(u^{\epsilon,k}-u_\delta^{\epsilon,k})^2)dx$$

$$+E\int_{\mathcal{O}}\epsilon N\frac{\partial(u_\delta^k-u_\delta^{k-1})}{\partial x_\ell}\frac{\chi^\ell}{[1+\epsilon^2(\chi^\ell)^2]^{1/2}}(u^{\epsilon,k}-u^{\epsilon,k-1})F_\delta^{\epsilon,k}dx \tag{89}$$

$$\leq NE\int_{\mathcal{O}}(u^{\epsilon,k-1}-u_\delta^{\epsilon,k-1})(u^{\epsilon,k}-u_\delta^{\epsilon,k})F_\delta^{\epsilon,k}dx+$$

$$+\bar{H}E\int_{\mathcal{O}}\frac{\epsilon\chi^\ell}{[1+\epsilon^2(\chi^\ell)^2]^{1/2}}\left|\frac{\partial u_\delta^k}{\partial x_\ell}\right|\,|u^{\epsilon,k}-u_\delta^{\epsilon,k}|F_\delta^{\epsilon,k}dx$$

$$+2C_1^2\bar{u}^2E\int_{\mathcal{O}}(\|D_y\chi^\epsilon\|^2-E\|D_y\chi\|^2)(u^{\epsilon,k}-u_\delta^{\epsilon,k})^2F_\delta^{\epsilon,k}dx$$

$$+2\epsilon^2(h_\delta^k)^2E\int_{\mathcal{O}}|\chi^\epsilon|^2(u^{\epsilon,k}-u_\delta^{\epsilon,k})^2F_\delta^{\epsilon,k}dx$$

$$+\bar{H}C_2(1+\bar{u}+\epsilon h_\delta^k)\left[E\int_{\mathcal{O}}\sum_j\left(\epsilon\frac{\partial^2u_\delta^k}{\partial x_\ell\partial x_j}\frac{\chi^\ell}{[1+\epsilon^2(\chi^\ell)^2]^{1/2}}\right.\right.$$

$$\left.\left.+\frac{\partial u_\delta^k}{\partial x_\ell}\frac{\partial\chi^\ell}{\partial y_j}\left(\frac{1}{[1+\epsilon^2(\chi^\ell)^2]^{1/2}}-1\right)\right)^2dx\right]^{1/2}.$$

We also note from (73) and (50) that

$$\|u^\epsilon-u^{\epsilon,k}\|_{L^\infty}\leq\frac{N}{N+\beta}\|u^\epsilon-u^{\epsilon,k-1}\|_{L^\infty} \tag{90}$$

hence

$$\|u^\epsilon-u^{\epsilon,k}\|_{L^\infty}\leq(\frac{N}{N+\beta})^k\frac{M}{\beta}. \tag{91}$$

Similarly,

$$\|u-u^k\|_{L^\infty}\leq(\frac{N}{N+\beta})^k\frac{M}{\beta}. \tag{92}$$

We also have the estimate

$$\|u^{\epsilon,k}\|_{H_0^1(\mathcal{O};\mathcal{N})}\leq C \tag{93}$$

where the constant does not depend on $\epsilon, k$. This is obtained as in Lemma 3.2, multiplying (73) by

$$u^{\epsilon,k} \exp\left[s(u^{\epsilon,k})^2\right]$$

and using the $L^\infty$ estimates (74)(75)(76). Now from (81) we also deduce

$$\|u_\delta^{\epsilon,k}\|_{H_0^1(\mathcal{O};\mathcal{Y})} \leq C_1(1+\epsilon h_\delta^k). \tag{94}$$

By Lemma 1.4 and the assumptions on $H$, we conclude that

$$\begin{aligned} E\int_{\mathcal{O}} \Big[ -\frac{\partial^2 u_\delta^k}{\partial x_i \partial x_j}(a_{ij} + a_{i\ell}\frac{\partial \chi^j}{\partial y_\ell})^\epsilon + Nu_\delta^k - H(x, \frac{x}{\epsilon}, (I + D_y\chi)Du_\delta^k, u_\delta^k) \quad (95) \\ -Nu_\delta^{k-1}\Big] (u^{\epsilon,k} - u_\delta^{\epsilon,k})F_\delta^{\epsilon,k}dx \\ -E\int_{\mathcal{O}} \Big[ -\frac{\partial^2 u_\delta^k}{\partial x_i \partial x_j} q_{ij} + Nu_\delta^k - \mathbf{H}(x, Du_\delta^k, u_\delta^k) \\ -Nu_\delta^{k-1}\Big] (u^{\epsilon,k} - u_\delta^{\epsilon,k})F_\delta^{\epsilon,k}dx \\ \to 0 \quad \text{as} \quad \epsilon \to 0, \quad \text{for} \quad \delta, k \quad \text{fixed.} \end{aligned}$$

Moreover, we have

$$\begin{aligned} \left| E\int_{\mathcal{O}} \left[ -\frac{\partial^2 u_\delta^k}{\partial x_i \partial x_j} q_{ij} + Nu_\delta^k - \mathbf{H}(x, Du_\delta^k, u_\delta^k) - Nu_\delta^{k-1} \right] (u^{\epsilon,k} - u_\delta^{\epsilon,k})F_\delta^{\epsilon,k}dx \right| \\ (96) \\ = \left| E\int_{\mathcal{O}} q_{ij}\frac{\partial u_\delta^k}{\partial x_j}\frac{\partial}{\partial x_i}(u^{\epsilon,k} - u_\delta^{\epsilon,k})F_\delta^{\epsilon,k}dx(1 + 2s(u^{\epsilon,k} - u_\delta^{\epsilon,k})^2) \right| \\ + E\int_{\mathcal{O}}(Nu_\delta^k - \mathbf{H}(x, Du_\delta^k, u_\delta^k) - Nu_\delta^{k-1})(u^{\epsilon,k} - u_\delta^{\epsilon,k})F_\delta^{\epsilon,k}dx \Big| \\ = \left| E\int_{\mathcal{O}} q_{ij}\frac{\partial(u_\delta^k - u^k)}{\partial x_j}\frac{\partial}{\partial x_i}(u^{\epsilon,k} - u_\delta^{\epsilon,k})F_\delta^{\epsilon,k}(1 + 2s(u^{\epsilon,k} - u_\delta^{\epsilon,k})^2)dx \right. \quad (97) \end{aligned}$$

$$+ \; E\int_O \Big[N(u_\delta^k - u^k) - \mathbf{H}(x, Du_\delta^k, u_\delta^k) \tag{98}$$
$$+\mathbf{H}(x, Du^k, u^k) - N(u_\delta^{k-1} - u^{k-1})\Big]\,(u^{\epsilon,k} - u_\delta^{\epsilon,k})F_\delta^{\epsilon,k}dx$$
$$\leq \phi_k(\delta)(1 + \epsilon h_\delta^k)$$

where $\phi_k(\delta) \to 0$ as $\delta \to 0$. Let us check next that

$$\epsilon^2 E\sum_\ell \left(\chi^\ell(\frac{x}{\epsilon})\right)^2 \to 0 \qquad \forall x. \tag{99}$$

This follows from the formula (see (30))

$$\chi^\ell(y,\omega) \;=\; \int_{\Re^n}(e^{i\lambda y} - 1)\frac{1}{|\lambda|^2}\sum_j(-i\lambda_j)U(d\lambda)\tilde{\chi}^\ell(\omega) \tag{100}$$

and the proof is the same as that used in Lemma 1.1. Also, noting that

$$\epsilon^2\sum_\ell \left(\chi^\ell(\frac{x}{\epsilon})\right)^2 \leq |x|^2 E|D\chi^\ell|^2 \tag{101}$$

we have by Lebesgue's theorem

$$\epsilon^2\int_O\sum_\ell \left(\chi^\ell(\frac{x}{\epsilon})\right)^2 dx \to 0. \tag{102}$$

Let us also check that

$$X_\epsilon = E\int_O \left(\frac{\partial\chi^\ell}{\partial y_j}\right)^2 \left(\frac{1}{[1+\epsilon^2(\chi^\ell)^2]^{3/2}} - 1\right)^2 (\frac{x}{\epsilon})dx \to 0. \tag{103}$$

Indeed,

$$X_\epsilon = E\int_O (\frac{\partial\chi^\ell}{\partial y_j})^2 \left(\frac{1}{[1+\epsilon^2(\chi^\ell)^2]^{3/2}} - 1\right)^2 (\mathbf{1}_{|\frac{\partial\chi^\ell}{\partial y_j}|>k} + \mathbf{1}_{|\frac{\partial\chi^\ell}{\partial y_j}|<k})(\frac{x}{\epsilon})dx \tag{104}$$

$$\leq |\mathcal{O}| E(\tilde{\chi}_j^\ell)^2 \mathbf{1}_{|\tilde{\chi}_j^\ell|>k} \; + \; 9k^2 \int_{\mathcal{O}} E\epsilon^2 \chi^\ell(\frac{x}{\epsilon})^2 dx$$

and since $k$ is arbitrary, one obtains (103). Also from Lemma 1.3

$$E\int_{\mathcal{O}} (\|D_j\chi^\epsilon\|^2 - E\|D_j\chi\|^2)(u^{\epsilon,k} - u_\delta^{\epsilon,k})^2 F_\delta^{\epsilon,k} dx \to 0 \tag{105}$$

as $\epsilon \to 0$, $\quad \forall k, \delta$ fixed. Finally, we note that

$$\begin{aligned}
u^{\epsilon,k} - u^k &= u^{\epsilon,k} - u_\delta^{\epsilon,k} \; + \; u_\delta^k - u^k \; + \; \epsilon \frac{\partial u_\delta^k}{\partial x_\ell} \frac{\chi^\ell}{[1+\epsilon^2(\chi^\ell)^2]^{1/2}}(\frac{x}{\epsilon}) \qquad (106) \\
&\quad - \frac{\partial u^{\epsilon,k}}{\partial x_j} - \frac{\partial u^k}{\partial x_j} - \frac{\partial u^k}{\partial x_\ell}\frac{\partial \chi^\ell}{\partial y_j}(\frac{x}{\epsilon}) \\
&= \frac{\partial}{\partial x_j}(u^{\epsilon,k} - u_\delta^{\epsilon,k}) \; + \; \epsilon \frac{\partial^2 u_\delta^k}{\partial x_\ell \partial x_j} \frac{\chi^\ell}{[1+\epsilon^2(\chi^\ell)^2]^{1/2}} \\
&\quad + \frac{\partial (u_\delta^k - u^k)}{\partial x_j} + \frac{\partial (u_\delta^k - u_\delta^k)}{\partial x_\ell}\frac{\partial \chi^\ell}{\partial y_j}(\frac{x}{\epsilon}) \\
&\quad + \frac{\partial u_\delta^k}{\partial x_\ell}\frac{\partial \chi^\ell}{\partial y_j}\left(\frac{1}{[1+\epsilon^2(\chi^\ell)^2]^{3/2}} - 1\right)
\end{aligned}$$

Hence,

$$\begin{aligned}
E\int_{\mathcal{O}} (u^{\epsilon,k} - u^k)^2 dx &\leq E\int_{\mathcal{O}} (u^{\epsilon,k} - u^k)^2 F_\delta^{\epsilon,k} dx \qquad (107) \\
&\leq 2E\int_{\mathcal{O}} (u^{\epsilon,k} - u_\delta^{\epsilon,k})^2 F_\delta^{\epsilon,k} dx + \phi_k(\delta) + 2\bar{u}^2 E\int_{\mathcal{O}} \epsilon^2 \left(\chi(\frac{x}{\epsilon})\right)^2 dx
\end{aligned}$$

and

$$\begin{aligned}
&E\int_{\mathcal{O}} |Du^{\epsilon,k} - (I + D_y\chi)^\epsilon Du^k|^2 dx \\
&\leq E\int_{\mathcal{O}} |Du^{\epsilon,k} - (I + D_y\chi)^\epsilon Du^k|^2 F_\delta^{\epsilon,k} dx \\
&\leq 2E\int_{\mathcal{O}} |D(u^{\epsilon,k} - u_\delta^{\epsilon,k})|^2 F_\delta^{\epsilon,k} dx + \phi_k(\delta) \qquad (108) \\
&\quad + 2(h_\delta^k)^2 E\int_{\mathcal{O}} \epsilon^2 \left(\chi(\frac{x}{\epsilon})\right)^2 dx
\end{aligned}$$

$$+2\bar{u}^2 \sum_{\ell,j} E \int_O (\frac{\partial \chi^\ell}{\partial y_j})^2 \left( \frac{1}{[1+\epsilon^2(\chi^\ell)^2]^{3/2}} - 1 \right)^2 (\frac{x}{\epsilon}) dx$$

Using (95)(96)(99) (103)(105), we deduce from (89) and (107) that (letting first $\epsilon$ tend to 0, then $\delta$)

$$\overline{\lim}_{\epsilon \to 0} \left\{ E \int_O |Du^{\epsilon,k} - (I + D_y\chi)^\epsilon Du^k|^2 dx + E \int_O (u^{\epsilon,k} - u^k)^2 dx \right\} \quad (109)$$

$$\leq \overline{\lim}_{\epsilon \to 0} E \int_O (u^{\epsilon,k-1} - u^{k-1})^2 dx, \qquad \forall k$$

However, $u^{\epsilon,0} = u^0 = 0$; hence, by induction

$$E \int_O |Du^{\epsilon,k} - (I + D_y\chi)^\epsilon Du^k|^2 dx + E \int_O (u^{\epsilon,k} - u^k)^2 dx \to 0 \qquad (110)$$

as $\epsilon \to 0$, $\forall k$ fixed. This result and the estimates (91)(92) show at least that

$$E \int_O (u^\epsilon - u)^2 dx \to 0 \quad \text{as} \quad \epsilon \to 0. \qquad (111)$$

Once this is proved it is possible to consider (50) and (62) and to perform operations similar to those made for (73) and (68). In particular, one can prove a fundamental relation similar to (82) and on inequality similar to (89). Operating as for (110) we deduce

$$E \int_O |Du^\epsilon - (I + D_y\chi)^\epsilon Du|^2 dx \to 0 \qquad (112)$$

and thus the proof of Theorem 2.1 has been completed.

□

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