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How Violent Are Fast Controls?

by

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ABSTRACT:

Consider a controllable system: $\dot{x} = \mathbf{A}x + \mathbf{B}u$, with $x(0) = 0$. Given any time $T > 0$ there is then a control operator $\mathbf{C}_T : \xi \mapsto u(\cdot)$ giving the (unique) minimum norm control such that $x(T) = \xi$. We show that $\|\mathbf{C}_T\| \sim \gamma T^{-(K+1/2)}$ where γ is computable from \mathbf{A}, \mathbf{B} and K is the minimal exponent giving the rank condition.

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1. Introduction

Consider a linear control system

$$(1.1) \quad \dot{x} = \mathbf{A}x + \mathbf{B}u \quad , \quad x(0) = 0$$

with \mathbf{A}, \mathbf{B} constant matrices ($n \times n$ and $n \times m$, respectively, so $x(\cdot)$ is \mathbb{R}^n -valued and the control $u(\cdot)$ is \mathbb{R}^m -valued). Assuming this is controllable, we know that for each terminal time $T > 0$ and each target $\xi \in \mathbb{R}^n$ there exist controls $u(\cdot)$ giving $x(T) = \xi$ and, indeed, that there is unique such control

$$(1.2) \quad u_{opt} = u_{opt}(\cdot; T, \xi) \in L^2([0, T] \rightarrow \mathbb{R}^m) =: \mathcal{U} = \mathcal{U}_T$$

minimizing $\|u\|$. Of course the norm to be minimized is that of $\mathcal{U} = L^2([0, T] \rightarrow \mathbb{R}^m)$ so $\|u_{opt}\|^2$ gives the least control energy needed to reach the target ξ at time T .

It is to be expected that more violent control would be needed as the time T available becomes shorter³. Our object in this paper is to give a precise (asymptotic) answer to the question of the title. Since the optimal control u_{opt} is given by a linear operator

$$(1.3) \quad \mathbf{C}_T : \xi \mapsto u_{opt}(\cdot; T, \xi) : \mathbb{R}^n \longrightarrow \mathcal{U} = \mathcal{U}_T ,$$

the principal result can be stated as

$$(1.4) \quad \|\mathbf{C}_T\| \sim \gamma T^{-(K+1/2)} \quad \text{as } T \rightarrow 0$$

where K is the minimal exponent giving the well known *rank condition* for controllability:

$$(1.5) \quad \text{rank } [\mathbf{B}, \mathbf{A}\mathbf{B}, \dots, \mathbf{A}^K \mathbf{B}] = n$$

and $\gamma \neq 0$ is also computable from \mathbf{A}, \mathbf{B} .

It is worth noting that this also estimates sensitivity for observation of the adjoint problem. If one can observe $y := \mathbf{B}^* z$ for z satisfying $\dot{z} = -\mathbf{A}^* z$, then one easily sees that one recovers the state through $z(T) = \mathbf{C}_T^* y(\cdot)$. This means that the uncertainty in the recovered state due to noise or measurement error $e(\cdot)$ in the observation is estimated by $\|\mathbf{C}_T^* e(\cdot)\| \leq \|\mathbf{C}_T\| \|e(\cdot)\|$. The same result (1.4) shows how sensitivity to error increases as the observation time shrinks. Plausibly, we might anticipate that the expected noise energy is proportional to times i.e.,

$$\text{Exp}[\int_0^T |e|^2 dt] \sim \sigma^2 T, \text{ so } \text{Exp}[\|e(\cdot)\|] \sim \sigma T^{1/2}.$$

³The uniqueness of u_{opt} and the linearity of the map: $\xi \mapsto u_{opt}$ follow from the Hilbert space projection theorem under more general conditions than here. From uniqueness it follows that $\|u_{opt}\|$ is strictly decreasing in T for each $\xi \neq 0$.

The sensitivity estimate then becomes

$$(1.6) \quad \|\text{expected uncertainty in } z(T)\| \sim \gamma \sigma T^{-K} \text{ as } T \rightarrow 0.$$

The formula (2.3) is classical but it is interesting to observe historically that the question of the title seems to have been considered first for distributed parameter systems⁴ although it was posed for the present finite dimensional case at least as far back as 1975 [3].

2. Formulation

Treatment of (1.1) is expressible in terms of the matrix exponential, given by the convergent series

$$(2.1) \quad e^{s\mathbf{A}} := \sum_0^\infty (s^k/k!) \mathbf{A}^k.$$

The solution x of (1) is then given by

$$x(t) = \int_0^t e^{(t-s)\mathbf{A}} \mathbf{B} u(s) ds$$

so, in particular, one has $x(T) = \mathbf{V} u(\cdot)$ where

$$(2.2) \quad \mathbf{V} = \mathbf{V}_T : \mathcal{U} \rightarrow \mathbb{R}^n : u(\cdot) \mapsto \int_0^T e^{(T-s)\mathbf{A}} \mathbf{B} u(s) ds.$$

A standard argument shows $\|u(\cdot)\|_{\mathcal{U}}$ is minimized, subject to the condition $\mathbf{V} u = \xi$, by taking $u \in \mathcal{R}(\mathbf{V}^*)$ whence

$$u_{opt}(\cdot; T, \xi) = \mathbf{V}^* \omega \text{ with, necessarily, } \mathbf{V} \mathbf{V}^* \omega = \xi;$$

so one has $\mathbf{C}_T : \xi \mapsto u_{opt}$ given by

$$(2.3) \quad \mathbf{C}_T = \mathbf{V}_T^* (\mathbf{V}_T \mathbf{V}_T^*)^{-1}$$

where controllability gives, as we see, invertibility of $\mathbf{V}_T \mathbf{V}_T^* =: \mathbf{W}_T = \mathbf{W}$. We easily see that $\mathbf{V}^* : \mathbb{R}^n \rightarrow \mathcal{U}$ is given by

$$(2.4) \quad [\mathbf{V}^* \omega](t) = [e^{(T-t)\mathbf{A}} \mathbf{B}]^* \omega \quad \text{for } t \in [0, T]$$

so the $n \times n$ matrix $\mathbf{W} := \mathbf{V} \mathbf{V}^*$ is given by

$$(2.5) \quad \mathbf{W} = \mathbf{W}_T = \int_0^T e^{s\mathbf{A}} \mathbf{B} [e^{s\mathbf{A}} \mathbf{B}]^* ds.$$

⁴One has $\log \|\mathbf{C}_T\| = \mathcal{O}(1/T)$ (sharply) for the known infinite dimensional cases[4], [1], [2].

Clearly, \mathbf{W} is self-adjoint and (at least) semidefinite from its form. We have the identity

$$(2.6) \quad \begin{aligned} \|u_{opt}\|^2 &= \|\mathbf{C}_T \xi\|^2 = \langle \mathbf{V}^* \mathbf{W}^{-1} \xi, \mathbf{V}^* \mathbf{W}^{-1} \xi \rangle \\ &= (\mathbf{W}^{-1} \xi) \cdot (\mathbf{V} \mathbf{V}^* \mathbf{W}^{-1} \xi) = (\mathbf{W}_T^{-1} \xi) \cdot \xi \end{aligned}$$

which makes it clear that our object must be to compute \mathbf{W}_T^{-1} asymptotically.

The key to our approach is the invertibility of

$$(2.7) \quad \mathbf{Q} := \lim_{T \rightarrow 0} T^{-(2K+1)} \mathbf{\Gamma}_T \mathbf{W}_T \mathbf{\Gamma}_T,$$

using a suitable family of operators $\mathbf{\Gamma} = \mathbf{\Gamma}_T$ such that

$$(2.8) \quad \mathbf{\Gamma}_T \text{ invertible for } T \neq 0, \quad \mathbf{\Gamma}_T = \mathbf{\Gamma}_0 + \mathcal{O}(T);$$

see (2.12), below.

Given the matrices \mathbf{A}, \mathbf{B} we consider the nested sequence (S_0, S_1, \dots) of subspaces of \mathbb{R}^n given recursively by

$$(2.9) \quad \begin{aligned} S_k &= S_{k-1} + \mathcal{R}(\mathbf{A}^k \mathbf{B}) \quad \text{with } S_{-1} := \{0\}, \\ S_0 &= \mathcal{R}(\mathbf{B}), \quad S_1 = \mathcal{R}(\mathbf{B}) + \mathcal{R}(\mathbf{A} \mathbf{B}), \dots \end{aligned}$$

so each S_k is the column space (range) of the composite matrix $[\mathbf{B}, \mathbf{A} \mathbf{B}, \dots, \mathbf{A}^k \mathbf{B}]$. The assumption of controllability means that $S_K = \mathbb{R}^n$ for large enough K (i.e., (1.5)) and we fix K as the *minimal* exponent/index giving this.

For each k ($0 \leq k \leq K$) we can find the orthogonal complement of S_{k-1} in S_k and let \mathbf{E}_k be the orthogonal projection on this subspace. This gives the important fact that

$$(2.10) \quad \mathbf{E}_k \mathbf{A}^j \mathbf{B} = \mathbf{0} \text{ for } j < k \leq K$$

since $j < k$ gives $\mathcal{R}(\mathbf{A}^j \mathbf{B}) \subset S_{k-1} \subset \mathcal{N}(\mathbf{E}_k)$. We observe, although we do not need the fact, that

$$m \geq \dim \mathcal{R}(\mathbf{B}) = \dim \mathcal{R}(\mathbf{E}_0) \geq \dim \mathcal{R}(\mathbf{E}_1) \geq \dots \geq \dim \mathcal{R}(\mathbf{E}_K);$$

we *will* need the fact that $S_{K-1} \neq S_K = \mathbb{R}^n$ by the definition of K so $\dim \mathcal{R}(\mathbf{E}_K) \neq 0$ and $\mathbf{E}_K \neq \mathbf{0}$. The construction of $\{\mathbf{E}_k\}$ gives a direct sum decomposition

$$(2.11) \quad \begin{aligned} \mathbf{1} &= \mathbf{E}_0 + \dots + \mathbf{E}_K, & \mathbb{R}^n &= \oplus_0^K \mathcal{R}(\mathbf{E}_k), \\ S_k &= \mathcal{R}(\mathbf{E}_0) \oplus \dots \oplus \mathcal{R}(\mathbf{E}_k) \text{ for } k = 0, \dots, K. \end{aligned}$$

Thus, introducing

$$(2.12) \quad \mathbf{\Gamma} = \mathbf{\Gamma}_T := \sum_0^K k! T^{K-k} \mathbf{E}_k$$

we see that (2.8) holds with $\mathbf{\Gamma}_0 = K! \mathbf{E}_K \neq \mathbf{0}$.

3. Principal Computation

Our object in this section is to obtain (2.7), with $\mathbf{\Gamma}_T$ as in (2.12), computing \mathbf{Q} and showing it is invertible.

The integral expression (2.5) gives, on substituting $s = T\sigma$,

$$T^{-(2K+1)}\mathbf{\Gamma}\mathbf{W}\mathbf{\Gamma} = \int_0^1 [T^{-K}\mathbf{\Gamma}e^{T\sigma\mathbf{A}}\mathbf{B}][T^{-K}\mathbf{\Gamma}e^{T\sigma\mathbf{A}}\mathbf{B}]^* d\sigma.$$

Using (2.12) and (2.1), we have (for $T > 0$)

$$\begin{aligned} T^{-K}\mathbf{\Gamma}e^{T\sigma\mathbf{A}}\mathbf{B} &= \sum_{k=0}^K k!T^{-k}\mathbf{E}_k \sum_{j=0}^{\infty} \frac{\sigma^j}{j!} T^j \mathbf{A}^j \mathbf{B} \\ &= \sum_{k=0}^K \sum_{j=0}^{\infty} \frac{k!\sigma^j}{j!} T^{j-k} \mathbf{E}_k \mathbf{A}^j \mathbf{B}. \end{aligned}$$

By (2.10), the terms with $j < k$ vanish so no negative powers of T actually appear on the right; we then split the sum into the terms with $j = k$ and those with $j \geq k+1$ for which we set $i = j - (k+1) = 0, 1, \dots$. Thus,

$$\begin{aligned} T^{-K}\mathbf{\Gamma}e^{T\sigma\mathbf{A}}\mathbf{B} &= \sum_{k=0}^K \sigma^k \mathbf{E}_k \mathbf{A}^k \mathbf{B} \\ (3.1) \quad &+ T \left[\sum_{k=0}^K \sum_{i=0}^{\infty} \frac{k!\sigma^{i+k+1} T^i}{(i+k+1)!} \mathbf{E}_k \mathbf{A}^j \mathbf{B} \right] \\ &= \mathbf{P}(\sigma) + T\mathbf{R}_1(T, \sigma). \end{aligned}$$

Restricting our attention to $T \leq 1$, which is certainly permissible as we are only interested in the limit $T \rightarrow 0$, an easy estimation gives the uniform bound

$$\begin{aligned} \|\mathbf{R}_1(T, \sigma)\| &\leq \sum_{k=0}^K \sum_{i=0}^{\infty} \|\mathbf{A}\|^{i+k+1} \|\mathbf{B}\| / (i+1)! \\ &= (1 + \dots + \|\mathbf{A}\|^K) \|\mathbf{B}\| (e^{\|\mathbf{A}\|} - 1) \end{aligned}$$

since $(i+k+1)! \geq (i+1)!k!$. Hence, (3.1) gives⁵

$$T^{-K}\mathbf{\Gamma}e^{T\sigma\mathbf{A}}\mathbf{B} = \mathbf{P}(\sigma) + \mathcal{O}(T)$$

and

$$\begin{aligned} (3.2) \quad &(T^{-K}\mathbf{\Gamma}e^{T\sigma\mathbf{A}}\mathbf{B})(T^{-K}\mathbf{\Gamma}e^{T\sigma\mathbf{A}}\mathbf{B})^* \\ &= [\mathbf{P}(\sigma) + T\mathbf{R}_1(T, \sigma)][\mathbf{P}(\sigma) + T\mathbf{R}_1(T, \sigma)]^* \\ &= \mathbf{P}(\sigma)\mathbf{P}^*(\sigma) + T\mathbf{R}_2(T, \sigma), \end{aligned}$$

⁵Note that our estimation of \mathbf{R}_1 precisely legitimates the use of the $\mathcal{O}(T)$ notation.

with $\mathbf{R}_2(T, \sigma) := (\mathbf{R}_1 \mathbf{P}^* + \mathbf{P} \mathbf{R}_1^* + T \mathbf{R}_1 \mathbf{R}_1^*)$ uniformly bounded. Thus, integrating,

$$(3.3) \quad T^{-(2K+1)} \mathbf{T} \mathbf{W} \mathbf{T} = \mathbf{Q} + \mathcal{O}(T)$$

with $\mathcal{O}(T) = T \int \mathbf{R}_2 d\sigma =: T \mathbf{R}_3(T)$ and

$$(3.4) \quad \begin{aligned} \mathbf{Q} &:= \int_0^1 \mathbf{P}(\sigma) \mathbf{P}^*(\sigma) d\sigma \\ &= \sum_{j,k=-\infty}^K (j+k+1)^{-1} \mathbf{E}_j \mathbf{A}^j \mathbf{B} \mathbf{B}^* \mathbf{A}^{*k} \mathbf{E}_k. \end{aligned}$$

We must show that \mathbf{Q} is invertible.

Lemma : $\mathbf{B}^* \mathbf{A}^{*k} \mathbf{E}_k \xi = 0 \implies \mathbf{E}_k \xi = 0$.

PROOF : For any $\xi \in \mathbb{R}^n$ we have $\mathbf{E}_k \xi \in \mathcal{S}_k := \mathcal{S}_{k-1} + \mathcal{R}(\mathbf{A}^k \mathbf{B})$ by definition so we may write

$$\mathbf{E}_k \xi = \mathbf{A}^k \mathbf{B} \eta + \xi' \quad (\xi' \in \mathcal{S}_{k-1})$$

for some $\eta \in \mathbb{R}^m$. Then, assuming $\mathbf{B}^* \mathbf{A}^{*k} \mathbf{E}_k \xi = 0$, we would have

$$\begin{aligned} \|\mathbf{E}_k \xi\|^2 &= (\mathbf{A}^k \mathbf{B} \eta + \xi') \cdot (\mathbf{E}_k \xi) \\ &= \eta \cdot (\mathbf{B}^* \mathbf{A}^{*k} \mathbf{E}_k \xi) + (\mathbf{E}_k \xi') \cdot \xi = 0 \end{aligned}$$

since $\mathbf{E}_k \xi' = 0$ for $\xi' \in \mathcal{S}_{k-1}$. \square

From (3.4) we see that

$$\xi \cdot \mathbf{Q} \xi = \int_0^1 \xi \cdot [\mathbf{P}(\sigma) \mathbf{P}^*(\sigma) \xi] d\sigma = \int_0^1 \|\mathbf{P}^*(\sigma) \xi\|^2 d\sigma$$

so $\mathbf{Q} \xi = 0$ only if $\mathbf{P}^*(\sigma) \xi \equiv 0$.

From the definition of $\mathbf{P}(\cdot)$, this would mean that each term $\sigma^k \mathbf{B}^* \mathbf{A}^{*k} \mathbf{E}_k \xi$ would have to vanish and, by the Lemma, this would imply $\mathbf{E}_k \xi = 0$ for each k . Hence, from (2.11), $\mathbf{Q} \xi = 0$ would give $\xi = 0$. We have thus shown that $\mathbf{Q} \xi = 0$ only for $\xi = 0$; for an $n \times n$ matrix \mathbf{Q} , this ensures invertibility.

4. Results

We must draw the desired conclusions from (3.3).

It is clear from the bound on $\mathbf{R}_1(T, \sigma)$ and the obvious fact that $\mathbf{P}(\sigma)$ is bounded uniformly on $[0, 1]$ that $\mathbf{R}_2(T, \sigma)$ is uniformly bounded so $\mathbf{R}_3(T)$ is uniformly bounded – say, $\|\mathbf{R}_3(T)\| \leq M_3$ for $0 \leq T \leq 1$. Restricting attention to $T \leq 1/2M_3 \|\mathbf{Q}^{-1}\| =: \tau$, we have

$$\begin{aligned} (\mathbf{Q} + T \mathbf{R}_3)^{-1} &= \mathbf{Q}^{-1} (\mathbf{1} + T \mathbf{Q}^{-1} \mathbf{R}_3)^{-1} \\ &= \mathbf{Q}^{-1} + T \mathbf{R}_4(T) \end{aligned}$$

with

$$\begin{aligned}
\|\mathbf{R}_4\| &\leq \|\mathbf{Q}^{-1}\| \|(\mathbf{1} + T\mathbf{Q}^{-1}\mathbf{R}_3)^{-1} - \mathbf{1}\|/T \\
&= \|\mathbf{Q}^{-1}\| \|\mathbf{Q}^{-1}\mathbf{R}_3(\mathbf{1} + T\mathbf{Q}^{-1}\mathbf{R}_3)^{-1}\| \\
&\leq \|\mathbf{Q}^{-1}\| \|\mathbf{Q}^{-1}\mathbf{R}_3\|/(1 - T\|\mathbf{Q}^{-1}\mathbf{R}_3\|) \\
&\leq 2M_3\|\mathbf{Q}^{-1}\|^2 \quad \text{for } 0 \leq T \leq \tau,
\end{aligned}$$

Now, inverting each side of (3.3) is legitimate for $T \neq 0$ and gives

$$\begin{aligned}
(4.1) \quad T^{2K+1}\mathbf{\Gamma}_T^{-1}\mathbf{W}_T\mathbf{\Gamma}_T^{-1} &= \mathbf{Q}^{-1} + T\mathbf{R}_4, \\
T^{2K+1}\mathbf{W}_T^{-1} &= \mathbf{\Gamma}_T(\mathbf{Q}^{-1} + T\mathbf{R}_4)\mathbf{\Gamma}_T \\
&= \mathbf{\Gamma}_0\mathbf{Q}^{-1}\mathbf{\Gamma}_0 + T\mathbf{R}_5(T)
\end{aligned}$$

with, obviously, $\|\mathbf{R}_5(T)\|$ uniformly bounded on $0 \leq T \leq \tau$. In particular, this proves (independently of the standard controllability arguments) the invertibility of \mathbf{W}_T — at least for small $T > 0$ and so *a fortiori* for all $T > 0$ by the non-negativity of the integrand in (2.5).

From (2.6) and the positivity of $\mathbf{W}, \mathbf{W}^{-1}$ we then have

$$\begin{aligned}
(4.2) \quad \|\mathbf{C}_T\|^2 &:= \max\{\|\mathbf{C}_T\xi\|^2 : \|\xi\| = 1\} \\
&= \max\{\xi \cdot \mathbf{W}_T^{-1}\xi : \|\xi\| = 1\} = \|\mathbf{W}_T^{-1}\| \\
&= T^{-(2K+1)}[\|\mathbf{\Gamma}_0\mathbf{Q}^{-1}\mathbf{\Gamma}_0\| + \mathcal{O}(T)].
\end{aligned}$$

This, of course, is just (1.4) with, from (2.12),

$$(4.3) \quad \gamma := \|\mathbf{\Gamma}_0\mathbf{Q}^{-1}\mathbf{\Gamma}_0\|^{1/2} = K!\|\mathbf{E}_K\mathbf{Q}^{-1}\mathbf{E}_K\|^{1/2}$$

once one show $\mathbf{E}_K\mathbf{Q}^{-1}\mathbf{E}_K \neq \mathbf{0}$ so $\gamma \neq 0$. Note that the positivity of \mathbf{Q} , hence of \mathbf{Q}^{-1} , gives

$$\begin{aligned}
\|\mathbf{E}_K\mathbf{Q}^{-1}\mathbf{E}_K\| &= \max\{\xi \cdot (\mathbf{E}_K\mathbf{Q}^{-1}\mathbf{E}_K\xi) : \|\xi\| = 1\} \\
&= \max\{\xi \cdot \mathbf{Q}^{-1}\xi : \|\xi\| = 1, \xi = \mathbf{E}_K\xi\} \\
&= \|\mathbf{Q}^{-1}|_{\mathcal{R}(\mathbf{E}_K)}\|.
\end{aligned}$$

Since $\mathcal{R}(\mathbf{E}_K) \neq \{0\}$ by the minimality of K , this is clearly non-zero.

At this point we work out in somewhat greater detail the case of scalar control ($m = 1$). The $n \times 1$ matrix \mathbf{B} is now just a vector and we set

$$(4.4) \quad \beta_0 = \mathbf{B}, \quad \beta_k = \mathbf{A}^k\beta_0 \quad \text{for } k = 0, \dots, n-1.$$

Note that controllability gives $K = n-1$ in this case so, for scalar control, our result (4.2) becomes

$$(4.5) \quad \|\mathbf{C}_T\| \sim \gamma T^{-(n+1/2)} + \mathcal{O}(T^{-(n-1/2)}).$$

To compute γ here, note first that $(\beta_0, \dots, \beta_{n-1})$ is a basis for \mathbb{R}^n and let $(\varepsilon_0, \dots, \varepsilon_{n-1})$ be the orthonormal basis obtained from that by the Gram-Schmidt procedure so $\mathbf{E}_k : \xi \mapsto (\xi \cdot \varepsilon_k) \varepsilon_k$ and $\mathbf{B}^* \mathbf{A}^{*k} \mathbf{E}_k \xi$ becomes $(\beta_k \cdot \varepsilon_k) (\xi \cdot \varepsilon_k)$. Then (3.4) becomes

$$(4.6) \quad \mathbf{Q} \xi = \sum_{j=0}^{K-1} \left[\sum_{k=0}^{K-1} \frac{(\beta_j \cdot \varepsilon_j)(\beta_k \cdot \varepsilon_k)}{j+k+1} (\xi \cdot \varepsilon_k) \right] \varepsilon_j.$$

This shows that, *re-written in terms of the orthonormal basis* $(\varepsilon_0, \dots, \varepsilon_{n-1})$, the new matrix for \mathbf{Q} is just $\mathbf{D}\mathbf{H}\mathbf{D}$ where $\mathbf{D} := \text{diag} [\beta_j \cdot \varepsilon_j]$ and \mathbf{H} is the $n \times n$ Hilbert matrix. Then

$$\begin{aligned} \|\mathbf{E}_K \mathbf{Q}^{-1} \mathbf{E}_K\| &= \varepsilon_K \cdot \mathbf{Q}^{-1} \varepsilon_K \\ &= [\text{lower right corner element of } \mathbf{D}^{-1} \mathbf{H}^{-1} \mathbf{D}^{-1}] \\ &= (\beta_{n-1} \cdot \varepsilon_{n-1})^2 [\text{lower right corner element of } \mathbf{H}^{-1}] \end{aligned}$$

whence

$$(4.7) \quad \gamma = C_n (\beta_{n-1} \cdot \varepsilon_{n-1})$$

with $C_n := (n-1)! [\text{lower right corner element of the inverse of the } \mathbf{H}^{-1}]^{1/2}$. The coefficient C_n grows extremely rapidly with n but, of course, is fixed for any given dimensionality. Thus, $\beta_{n-1} \cdot \varepsilon_{n-1}$ provides the only dependence on the *particular* system (1.1); it is just the norm of the component of $\mathbf{A}^{n-1} \beta_0$ orthogonal to $\text{span}\{\beta_0, \dots, \mathbf{A}^{n-2} \beta_0\}$.

Returning to the general case, we now consider the asymptotics for a particular target ξ (rather than the ‘worst case’ treatment above). We have, from (2.6),

$$x(T; u(\cdot)) = \xi \implies \|u(\cdot)\| \geq \|\mathbf{C}_T \xi\| = (\xi \cdot \mathbf{W}_T^{-1} \xi)^{1/2}.$$

From (4.1) we have

$$\|\mathbf{C}_T \xi\| = T^{-(K+1/2)} K! (\xi_K \cdot \mathbf{Q}^{-1} \xi_K)^{1/2} + \mathcal{O}(T^{-(K-1/2)})$$

where we have abbreviated $\xi_K := \mathbf{E}_K \xi$. We may write this, assuming⁶ $\xi_K \neq 0$, as

$$(4.8) \quad \|\mathbf{C}_T \xi\| \sim (K! \|\mathbf{Q}^{-1/2} \xi_K\|) T^{-(K+1/2)},$$

which gives the same asymptotic growth rate for (almost all) targets.

⁶This is ‘almost always’ true — it fails only when ξ happens to lie exactly in the (proper) subspace $\mathcal{N}(\mathbf{E}_K)$ in which case one has slower blowup. Even in that case slight perturbations would, almost inevitably, give *some* component in $\mathcal{R}(\mathbf{E}_K)$ so this analysis would dominate.

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