

**Feedback Control for an Abstract  
Parabolic Equation**

**by**

**R. Rostamian and T. Seidman**

*FEEDBACK CONTROL FOR AN ABSTRACT  
PARABOLIC EQUATION*

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## 1. Introduction

In our previous paper [6] we considered the problem of the exponential stabilization of the heat equation with Neumann boundary conditions

$$\begin{aligned}\frac{\partial u}{\partial t} &= \Delta u && \text{in } \Omega \times (0, \infty) \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \partial\Omega \times (0, \infty) \\ u(x, 0) &= u_0(x) && x \text{ in } \Omega\end{aligned}\tag{1.1}$$

in the smooth and bounded domain  $\Omega$  in  $\mathbf{R}^n$ , where  $\partial/\partial\nu$  denotes the differentiation in the direction normal to the boundary  $\partial\Omega$ . The solutions of this problem, although stable (in the sense of Liapunov, considered for any  $L^p$  space, say) are not asymptotically stable, because spatially homogeneous functions do not decay. A practical way to induce asymptotic stability is to introduce a feedback mechanism which observes the temperature at a patch of the boundary  $\partial\Omega$  and causes an appropriate

exchange of heat through the boundary or through an interior subset of  $\Omega$ . A feedback mechanism of the latter type may be modeled by introducing a term of the form

$$\Phi(x, t) := -\epsilon \left( \int_{\partial\Omega} u(y, t) \varphi(y) dy \right) \sigma(x) \quad (1.2)$$

on the right hand side of the equation in (1.1). The functions  $\varphi$  and  $\sigma$ , prescribed on  $\partial\Omega$  and  $\Omega$ , respectively, are indicators of the sites where the observation and the feedback take place. The coefficient  $\epsilon$  is the ‘gain factor’ of the feedback. Feedback mechanisms of the former type may also be treated by combining the method of section 6 of [6] and the estimates developed in the present paper.

More generally, we may consider a more complex diffusion process governed by a general (autonomous) linear elliptic operator  $A$ , and several feedback mechanisms of the type above that act simultaneously and independently, so that the governing equation becomes

$$\frac{\partial u}{\partial t} = Au - \epsilon \sum_{k=1}^p \sum_{j=1}^q \gamma_{jk} \left( \int_{\partial\Omega} u(y, t) \varphi_k(y) \right) \sigma_j(x). \quad (1.3)$$

For instance,

$$Au = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i}$$

may correspond to a heat conduction problem in an anisotropic and inhomogeneous medium.

The method introduced in [6] for the stability analysis extends to this and more general situations. In this paper we study an abstract evolution equation which includes the problem in [6] and the problem corresponding to equation (1.3) as special cases. Our main result, in the context of the latter problem is, roughly, that if certain hypotheses are satisfied and if  $\epsilon$  is sufficiently small then the system is exponentially asymptotically stable. Studies of similar kind, albeit with quite more complicated sufficient stability conditions, appear in various places in the literature; we refer the reader to the articles in [1], [5], [7], and [8].

The smallness of the feedback gain factor  $\epsilon$  is essential for stability. In [6] we give an example where for a large  $\epsilon$  the solution grows exponentially as  $t \rightarrow \infty$ . For a

comprehensive study of the spectral properties and the spectral resolution of elliptic operators of the type occurring in (1.3) see the recent article of van Harten[3].

## 2. The Abstract Problem

Let  $A$  be a sectorial operator (cf. [4]) on a Hilbert space  $H$ , and denote by  $e^{tA}$  the analytic semigroup generated by  $A$ . Suppose that the null-space  $H_0$  of  $A$  is finite-dimensional and non-trivial, and that the following hypotheses hold:

*Hypotheses on  $A$ :*

- (i) There exists a continuous projection operator  $Q$ , not necessarily orthogonal, of  $H$  onto  $H_0$ , which commutes with  $A$ ; thus  $AQ = QA = 0$ .
- (ii) If  $P = I - Q$ , where  $I$  is the identity mapping in  $H$ , and  $H_1 = \text{range}(P) = \text{null space}(Q)$ , then there exist positive constants  $a$  and  $\alpha$  such that

$$|e^{tA}v| \leq ae^{-\alpha t}|v|, \quad v \in H_1. \quad (2.1)$$

Thus, in the dynamical system corresponding to  $e^{tA}$ , all orbits in  $H_1$  are asymptotically exponentially stable. The points in  $H_0$ , on the other hand, are equilibria and remain stationary. We now introduce a feedback process, motivated by the concrete application in [6], which makes *all* orbits in  $H$  tend exponentially to zero as  $t \rightarrow \infty$ .

We suppose we are given  $q$  elements  $\{\sigma_1, \dots, \sigma_q\}$  of  $H$  and  $p$  linear (possibly unbounded) functionals  $\{l_1, \dots, l_p\}$  on  $H$ , for which there exists a  $p \times q$  real matrix  $\gamma_{jk}$  such that the feedback operator  $B$ , formally defined by

$$B(u) = - \sum_{k=1}^p \sum_{j=1}^q \gamma_{jk} l_k(u) \sigma_j, \quad (2.2)$$

satisfies the following hypotheses:

*Hypotheses on  $B$ :*

- (i) For each  $t > 0$  and  $k = 1, 2, \dots, p$ , the expression  $l_k(e^{tA})$  defines a *bounded* linear functional on  $H$ .

(ii) There exists a  $\delta \in (0, 1)$  such that

$$|l_k(e^{tA}u)| \leq at^{-\delta}e^{-\alpha t}|u|, \quad u \in H_1, \quad (2.3)$$

and

$$|l_k(u)| \leq a|u|, \quad u \in H_0. \quad (2.4)$$

(iii) If we define the operator  $B_0 : H_0 \rightarrow H_0$  by

$$B_0 w = - \sum_{k=1}^p \sum_{j=1}^q \gamma_{jk} l_k(w) Q \sigma_j, \quad w \in H_0, \quad (2.5)$$

then there exist positive constants  $b$  and  $\beta$  such that

$$|e^{tB_0}w| \leq be^{-\beta t}|w|, \quad w \in H_0. \quad (2.6)$$

*Remark:* The inequalities (2.4) and (2.6) imply the estimate

$$|l_k(e^{tB_0}w)| \leq abe^{-\beta t}|w|, \quad w \in H_0. \quad (2.7)$$

Note that, formally,  $B_0 = QBQ$ , and that the operator  $Be^{tA}$  is bounded on  $H$  for each  $t > 0$ . In view of this we are able to show that the equation

$$\begin{aligned} \dot{u} &= Au + \epsilon Bu \\ u(0) &= u_0 \end{aligned} \quad (2.8)$$

makes sense in  $H$ . We prove the

**Stabilization Theorem:** *There exists a number  $\epsilon_0 > 0$ , depending on the constants  $a, \alpha, b, \beta, \delta$  entering in estimates in (2.1), (2.3), and (2.6) as well as on  $p, q, \gamma_{jk}$ , and the norms of the  $\sigma_j$ 's, such that, for any  $\epsilon \in (0, \epsilon_0)$ , the solution of (2.8) decays exponentially as  $t \rightarrow \infty$ .*

We defer the proof to the next section but first make some remarks:

(i) It is possible to obtain an *explicit* upper bound  $\epsilon_0$  in terms of the quantities mentioned above. The computation proceeds along the lines described in section 4 of [6].

(ii) It is worthwhile to observe that in our computation we use the estimates in (2.1), (2.3), (2.4), (2.6) very sparingly. In fact, the bulk of the computation in the following section, up to the derivation of the system of integral equations (3.7), is in terms of equalities rather than estimations. This has the advantage of keeping the results as sharp as possible; moreover, the general theorem on integral equations developed in [6] applies here directly.

(iii) The inequality in (2.1) applies in situations where the initial condition is in the space  $H$ . This assumption is of course unnecessarily strong; in concrete cases the initial conditions are often allowed to be outside  $H$ . For instance,  $H$  may be  $L^2(\Omega)$  whereas  $u_0$  is a distribution. The regularizing effect of the analytic semigroup puts the orbit in  $H$  in positive times. The estimate (2.1) can be modified to apply to such irregular data; see [6] for details.

(iv) The estimate in (2.6) is an assumption of the positivity of the operator  $-B_0$  on the space  $H_0$ . One might also consider a possible order-preserving property of  $-B_0$ , in the form

$$w \geq 0 \Rightarrow -B_0 w \geq 0, \quad \forall w \in H_0$$

where the inequalities are understood in the sense of a partial order on  $H$  for which one has a maximum principle for  $A$ . It seems likely, but we have not verified this, that under this stronger condition the equation (2.8) is asymptotically stable with no restrictions on the magnitude of the coefficient  $\epsilon$ . See [6] for further discussion and practical implications.

(v) It is possible to show, although we do not do it here, that the operator  $A + \epsilon B$  entering in (2.8) is sectorial for arbitrary  $\epsilon$ . Hence it generates an analytic semigroup on  $H$ .

### 3. The Proof of the Stabilization Result

We apply the projections  $P$  and  $Q$  to the evolution equation (2.8) and use the notation  $v = Pu$ ,  $w = Qu$ , to arrive at the following coupled system of differential equations:

$$\dot{v} = Av - \epsilon \sum_{k=1}^p \sum_{j=1}^q \gamma_{jk} l_k(u) P \sigma_j \quad (3.1)$$

$$\dot{w} = -\epsilon \sum_{k=1}^p \sum_{j=1}^q \gamma_{jk} l_k(u) Q \sigma_j \quad (3.2)$$

We also set  $v_0 = v(0) = Pu(0)$  and  $w_0 = w(0) = Qu(0)$ . Note that the equation (3.2) may be re-written, in view of the definition of the operator  $B_0$  in (2.5) and the identity  $u = v + w$ , as:

$$\dot{w} + \epsilon B_0 w = -\epsilon \sum_{k=1}^p \sum_{j=1}^q l_k(v) Q \sigma_j.$$

Applying the variation of constants formula, we get

$$v(t) = e^{tA} v_0 - \epsilon \sum_{k=1}^p \sum_{j=1}^q \int_0^t \gamma_{jk} \left[ e^{(t-s)A} P \sigma_j \right] l_k(v(s) + w(s)) ds, \quad (3.3)$$

$$w(t) = e^{\epsilon t B_0} w_0 - \epsilon \sum_{k=1}^p \sum_{j=1}^q \int_0^t \gamma_{jk} \left[ e^{\epsilon(t-s)B_0} Q \sigma_j \right] l_k(v(s)) ds. \quad (3.4)$$

Now define

$$\begin{aligned} \rho_i(t) &= l_i(v(t)), \\ \lambda_{ik}(t) &= \sum_{j=1}^q \gamma_{jk} l_i \left( e^{tA} P \sigma_j \right), \\ \eta_{ik}(t) &= \sum_{j=1}^q \gamma_{jk} l_i \left( e^{\epsilon t B_0} Q \sigma_j \right), \end{aligned}$$

and apply  $l_i$  to both of equations (3.3) and (3.4) to obtain

$$\begin{aligned} \rho_i(t) &= l_i \left( e^{tA} v_0 \right) - \epsilon \sum_{k=1}^p \int_0^t \lambda_{ik}(t-s) \left( \rho_k(s) + l_k(w(s)) \right) ds, \\ l_i(w(t)) &= l_i \left( e^{\epsilon t B_0} w_0 \right) - \epsilon \sum_{k=1}^p \int_0^t \eta_{ik}(t-s) \rho_k(s) ds. \end{aligned}$$



Substitute the second of these equations in the first, interchange the order of integration in the resulting double-integral (justified by the estimates in (2.1) and (2.3),) and simplify, to arrive at

$$\begin{aligned}\rho_i(t) = & l_i(e^{tA}v_0) - \epsilon \sum_{k=1}^p \int_0^t \lambda_{ik}(t-s) l_k(e^{\epsilon s B} w_0) ds \\ & - \epsilon \sum_{k=1}^p \int_0^t \lambda_{ik}(t-s) \rho_k(s) ds \\ & + \epsilon^2 \sum_{k,k'=1}^p \int_0^t \int_0^s \lambda_{ik}(t-s) \eta_{kk'}(s-\tau) \rho_{k'}(\tau) d\tau ds.\end{aligned}\quad (3.5)$$

Further simplification is achieved by letting  $\rho_i^*(t)$  denote the sum of the first two terms on the right hand side of (3.5) (i.e., the terms which contain the initial data) and setting

$$\mu_{ij}(t) = \lambda_{ij}(t) - \epsilon \sum_{k=1}^p \int_0^t \lambda_{ik}(\tau) \eta_{kj}(t-\tau) d\tau. \quad (3.6)$$

After a bit of manipulation, (3.5) then yields the following system of integral equations:

$$\rho_i(t) = \rho_i^*(t) - \epsilon \sum_{j=1}^p \int_0^t \mu_{ij}(t-\tau) \rho_j(\tau) d\tau \quad (3.7)$$

for  $i = 1, 2, \dots, p$ .

It is easy to see that by the assumption  $0 < \delta < 1$  in (2.3), the functions  $\rho_i^*$  and  $\mu_{ik}$  have *integrable* singularities, hence the system of Volterra equations (3.7) has a unique solution. By a computation similar to the one in [6], it is possible to show that this solution decays exponentially as  $t \rightarrow \infty$ , if  $\epsilon > 0$  is sufficiently small. Going back to (3.4), recalling that  $l_k(v(t)) = \rho_k(t)$ , and using the estimations in (2.1) and (2.3), we conclude that the norm of  $w(t)$  decays. Now since

$$l_k(u(t)) = l_k(v(t)) + l_k(w(t)) = \rho_k(t) + l_k(w(t))$$

we get exponential decay for  $l_k(u(t))$ . Finally, using this in (3.3) we obtain exponential decay for  $v(t)$  and, consequently, for  $u(t) = v(t) + w(t)$ . This completes the proof of the stabilization result.

*Remark:* A variant of the control problem treated in this paper is that of the boundary observation and *boundary control*. In this situation the feedback mechanism

involves interaction with  $\Omega$  entirely through the boundary of  $\Omega$ . The corresponding mathematical model is:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \sum_{i,j=1}^n \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) && \text{in } \Omega \times (0, \infty) \\ \frac{\partial u}{\partial \nu} &= -\epsilon \sum_{j,k}^p \gamma_{jk} \left( \int_{\partial\Omega} u(y, t) \varphi_k(y) dy \right) \sigma_j && \text{on } \partial\Omega \times (0, \infty) \\ u(x, 0) &= u_0(x) && x \in \Omega.\end{aligned}$$

Abstractly, this corresponds to having  $e^{tA}\sigma_j \in H$  for  $t > 0$  rather than  $\sigma_j \in H$ ; we must then assume an estimate like (2.3) for  $|e^{tA}\sigma_j|$ , corresponding to having  $\sigma_j \in \mathcal{D}(A^\delta)$ . It is possible to show that this system is also asymptotically exponentially stable provided that the coefficient  $\epsilon$  is sufficiently small. The proof is along the lines of the argument in section 6 of [6], combined with the generalized framework that we have developed in the resent paper.

*Acknowledgments:* The research of R. Rostamian was partially supported by grants DMS-8602006 from the National Science Foundation and N00014-86-K-0498 from the Office of Naval Research. The research of T. I. Seidman was partially supported by grants AFOSR-82-0271 from the Air Force Office of Scientific Research and CDR-85-00108 from the National Science Foundation. T. Nambu, while visiting UMBC, was on leave from Kumamoto University and was also partially supported by grant AFOSR-82-0271.

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in the smooth and bounded domain  $\Omega$  in  $\mathbb{R}^n$ , where  $\partial/\partial\nu$  denotes the differentiation in the direction normal to the boundary  $\partial\Omega$ . The solutions of this problem, although stable (in the sense of Liapunov, considered for any  $L^p$  space, say) are not asymptotically stable, because spatially homogeneous functions do not decay. A practical way to induce asymptotic stability is to introduce a feedback mechanism which observes the temperature at a patch of the boundary  $\partial\Omega$  and causes an appropriate

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For instance,

$$Au = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i}$$

may correspond to a heat conduction problem in an anisotropic and inhomogeneous medium.

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## 2. The Abstract Problem

Let  $A$  be a sectorial operator (cf. [4]) on a Hilbert space  $H$ , and denote by  $e^{tA}$  the analytic semigroup generated by  $A$ . Suppose that the null-space  $H_0$  of  $A$  is finite-dimensional and non-trivial, and that the following hypotheses hold:

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and

$$|l_k(u)| \leq a|u|, \quad u \in H_0. \quad (2.4)$$

(iii) If we define the operator  $B_0 : H_0 \rightarrow H_0$  by

$$B_0 w = - \sum_{k=1}^p \sum_{j=1}^q \gamma_{jk} l_k(w) Q \sigma_j, \quad w \in H_0, \quad (2.5)$$

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*Remark:* The inequalities (2.4) and (2.6) imply the estimate

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Note that, formally,  $B_0 = QBQ$ , and that the operator  $Be^{tA}$  is bounded on  $H$  for each  $t > 0$ . In view of this we are able to show that the equation

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makes sense in  $H$ . We prove the

**Stabilization Theorem:** *There exists a number  $\epsilon_0 > 0$ , depending on the constants  $a, \alpha, b, \beta, \delta$  entering in estimates in (2.1), (2.3), and (2.6) as well as on  $p, q, \gamma_{jk}$ , and the norms of the  $\sigma_j$ 's, such that, for any  $\epsilon \in (0, \epsilon_0)$ , the solution of (2.8) decays exponentially as  $t \rightarrow \infty$ .*

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$$\dot{w} + \epsilon B_0 w = -\epsilon \sum_{k=1}^p \sum_{j=1}^q l_k(v) Q \sigma_j.$$

Applying the variation of constants formula, we get

$$v(t) = e^{tA} v_0 - \epsilon \sum_{k=1}^p \sum_{j=1}^q \int_0^t \gamma_{jk} \left[ e^{(t-s)A} P \sigma_j \right] l_k(v(s) + w(s)) ds, \quad (3.3)$$

$$w(t) = e^{\epsilon t B_0} w_0 - \epsilon \sum_{k=1}^p \sum_{j=1}^q \int_0^t \gamma_{jk} \left[ e^{\epsilon(t-s)B_0} Q \sigma_j \right] l_k(v(s)) ds. \quad (3.4)$$

Now define

$$\begin{aligned} \rho_i(t) &= l_i(v(t)), \\ \lambda_{ik}(t) &= \sum_{j=1}^q \gamma_{jk} l_i \left( e^{tA} P \sigma_j \right), \\ \eta_{ik}(t) &= \sum_{j=1}^q \gamma_{jk} l_i \left( e^{\epsilon t B_0} Q \sigma_j \right), \end{aligned}$$

and apply  $l_i$  to both of equations (3.3) and (3.4) to obtain

$$\begin{aligned} \rho_i(t) &= l_i \left( e^{tA} v_0 \right) - \epsilon \sum_{k=1}^p \int_0^t \lambda_{ik}(t-s) \left( \rho_k(s) + l_k(w(s)) \right) ds, \\ l_i(w(t)) &= l_i \left( e^{\epsilon t B_0} w_0 \right) - \epsilon \sum_{k=1}^p \int_0^t \eta_{ik}(t-s) \rho_k(s) ds. \end{aligned}$$

Substitute the second of these equations in the first, interchange the order of integration in the resulting double-integral (justified by the estimates in (2.1) and (2.3),) and simplify, to arrive at

$$\begin{aligned}\rho_i(t) &= l_i \left( e^{tA} v_0 \right) - \epsilon \sum_{k=1}^p \int_0^t \lambda_{ik}(t-s) l_k \left( e^{sB} w_0 \right) ds \\ &\quad - \epsilon \sum_{k=1}^p \int_0^t \lambda_{ik}(t-s) \rho_k(s) ds \\ &\quad + \epsilon^2 \sum_{k,k'=1}^p \int_0^t \int_0^s \lambda_{ik}(t-s) \eta_{kk'}(s-\tau) \rho_{k'}(\tau) d\tau ds.\end{aligned}\quad (3.5)$$

Further simplification is achieved by letting  $\rho_i^*(t)$  denote the sum of the first two terms on the right hand side of (3.5) (i.e., the terms which contain the initial data) and setting

$$\mu_{ij}(t) = \lambda_{ij}(t) - \epsilon \sum_{k=1}^p \int_0^t \lambda_{ik}(r) \eta_{kj}(t-r) dr. \quad (3.6)$$

After a bit of manipulation, (3.5) then yields the following system of integral equations:

$$\rho_i(t) = \rho_i^*(t) - \epsilon \sum_{j=1}^p \int_0^t \mu_{ij}(t-\tau) \rho_j(\tau) d\tau \quad (3.7)$$

for  $i = 1, 2, \dots, p$ .

It is easy to see that by the assumption  $0 < \delta < 1$  in (2.3), the functions  $\rho_i^*$  and  $\mu_{ik}$  have *integrable* singularities, hence the system of Volterra equations (3.7) has a unique solution. By a computation similar to the one in [6], it is possible to show that this solution decays exponentially as  $t \rightarrow \infty$ , if  $\epsilon > 0$  is sufficiently small. Going back to (3.4), recalling that  $l_k(v(t)) = \rho_k(t)$ , and using the estimations in (2.1) and (2.3), we conclude that the norm of  $w(t)$  decays. Now since

$$l_k(u(t)) = l_k(v(t)) + l_k(w(t)) = \rho_k(t) + l_k(w(t))$$

we get exponential decay for  $l_k(u(t))$ . Finally, using this in (3.3) we obtain exponential decay for  $v(t)$  and, consequently, for  $u(t) = v(t) + w(t)$ . This completes the proof of the stabilization result.

*Remark:* A variant of the control problem treated in this paper is that of the boundary observation and *boundary control*. In this situation the feedback mechanism

involves interaction with  $\Omega$  entirely through the boundary of  $\Omega$ . The corresponding mathematical model is:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \sum_{i,j=1}^n \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) && \text{in } \Omega \times (0, \infty) \\ \frac{\partial u}{\partial \nu} &= -\epsilon \sum_{j,k}^p \gamma_{jk} \left( \int_{\partial\Omega} u(y, t) \varphi_k(y) dy \right) \sigma_j && \text{on } \partial\Omega \times (0, \infty) \\ u(x, 0) &= u_0(x) && x \in \Omega.\end{aligned}$$

Abstractly, this corresponds to having  $e^{tA}\sigma_j \in H$  for  $t > 0$  rather than  $\sigma_j \in H$ ; we must then assume an estimate like (2.3) for  $|e^{tA}\sigma_j|$ , corresponding to having  $\sigma_j \in \mathcal{D}(A^\delta)$ . It is possible to show that this system is also asymptotically exponentially stable provided that the coefficient  $\epsilon$  is sufficiently small. The proof is along the lines of the argument in section 6 of [6], combined with the generalized framework that we have developed in the recent paper.

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