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A Local Version Of The Two-Circles Theorem

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Abstract: A necessary and sufficient condition is given so that in a domain Ω there are no functions whose average over all balls contained in Ω of radii r_1, r_2 vanish except the zero function.

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A local version of the two circles theorem

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1. Introduction. -

One of the oldest questions in integral geometry has been that of recovering a function f in $\mathbf{R}^n$ from the knowledge of its average over balls. It is easy to see that unless f decays sufficiently fast at infinity the average over all balls of a fixed radius could vanish without f being identically zero. It is not always possible to assume such decay but a very elegant result of Zalcman [20] and, independently, Brown-Schreiber-Taylor [10], describes explicitly a countable set E_n such that averages over all balls of radii r_1, r_2 suffice as long as $r_1/r_2 \notin E_n$. This-"two circles" theorem can be described as saying that the map

$$C(\mathbf{R}^n) \to C(\mathbf{R}^n) \oplus C(\mathbf{R}^n)$$

$$f \to \left(\int_{B(x,r_1)} f(y)dy, \int_{B(x,r_2)} f(y)dy \right)$$

is injective if and only if $r_1/r_2 \notin E_n$. $\left(B(x,r)=\{y \; R^n : |x-y|<r\}\right)$. Under slightly stronger conditions on the quotient r_1/r_2 this map has also a continuous and explicit inverse [8]. This result and other variants of the so-called Pompeiu problem have been generalized to symmetric spaces (see the surveys [20], [1] for positive results and their limitations)

In practical situations of a tomographic nature one is limited to balls that fit into a fixed region Ω. One could take smaller and smaller balls when approaching the boundary $\partial\Omega$ of Ω, this is roughly the situation when we consider the case

Ω = unit ball of $\mathbf{R}^n$ as the hyperbolic space, but it is clear that it might be hard to accomplish if we are dealing with physical devices whose size cannot be made infinitesimally small or cannot even be changed at will. It is this kind of problem that we call a local version of the two-circles theorem. The main difference with the above mentioned results is that we do not have any longer the whole group of Euclidean motions at our disposal which was the crucial ingredient lying behind the two circles theorem and its generalizations. The inversion formula of [8] would allow us to reconstruct f away from $\partial\Omega$ but gives no indication of whether we could change the values of f in a collar-like region near $\partial\Omega$ without affecting its average. There is some recent work on systems of convolution equations in convex domains which deals with this type of question [4] but the hypotheses required are far too restrictive to be satisfied by our simple looking problems. Nevertheless, using a combination of ideas from classical harmonic analysis and results of Cormack-Quinto on the Radon transform on spheres [12] we are able to prove the following.

<u>Theorem</u>. - Let r_1, r_2 be positive numbers, $r_1/r_2 \notin E_n$, Ω an open subset of $\mathbf{R}^n$ such that every point lies in an open ball contained in Ω of radius strictly larger than $r_1 + r_2$. If $f \in C(\Omega)$ satisfies

$$\int_{B(x,r_j)} f(y)dy = 0 \quad \text{for every} \quad \overline{B}(x,r_j) \subseteq \Omega, \ j = 1,2$$

then $f \equiv 0$. Furthermore, this statment does not hold if Ω fails the above geometrical restriction.

The method of proof allows us to generalize this theorem greatly, providing in particular new local mean-value theorems for harmonic functions.

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2. Preliminaries. -

We will follow the standard notation for distributions found in [14]. We denote $B(x,r) = \{y \in \mathbf{R}^n : |x-y| < r\}$ $(r>0)$, $\overline{B}(x,r)$ its closure and χ_r the characteristic function of $B(0,r)$. Let Ω be an open set in $\mathbf{R}^n$, $\Omega_r = \{x \in \Omega : d(x,\Omega^c) < r\}$. For a locally-integrable function f in an open set Ω the average

$$(1) \qquad A_r(f,x) = \frac{1}{\omega_n r^n} \int_{B(x,r)} f(y)dy$$

is defined for $x \in \Omega_r$. Here ω_n is the volume of $B(0,1)$. If we let $\mu_r = \chi_r/\omega_n r^n$, we can interpret this average as a convolution and hence it makes sense to define it for $f \in \mathcal{D}'(\Omega)$ giving a distribution $A_r(f)$ in $\mathcal{D}'(\Omega_r)$, namely $A_r(f) = f * \mu_r$. Therefore, for uniqueness questions, if the averages of f are zero, by restriction ourselves to Ω_ε, $\varepsilon > 0$ small, we can assume $f \in C^\infty$. Henceforth, all distributions with vanishing averages will be assumed to be C^∞ functions in Ω.

For $r > 0$, we denote by σ_r the distributions defining the spherical average

$$(2) \qquad \lambda_r(f,x) = \int_{S^{n-1}} f(x+ry)d\sigma(y) = (\sigma_r * f)(x)$$

$d\sigma$ is the normalized Lebesgue measure on S^{n-1}.

For $T \in E'$ the Fourier transform

$$\hat{T}(\zeta) = \langle T_x, e^{-i(x|\zeta)}\rangle, \quad (x|\zeta) = \sum_j x_j \zeta_j,$$

is an entire function in $\mathbb{C}^n$ which satisfies, for some $A, N > 0$, the estimates

$$(3) \qquad |\hat{T}(\zeta)| < A(1+|\zeta|)^N \exp(H(\mathrm{Im}\zeta)).$$

where $\zeta = \xi + i\eta$, $\xi, \eta \in R^n$, $\mathrm{Im}\zeta = \eta$ and H is the supporting function of the support of T, i.e.:

$$H(\eta) = \mathrm{Max}\{(x|\eta) : x \in \mathrm{supp}T\}$$

Note that H is also the supporting function of cv(suppT), the convex hull of suppT. The Fourier transform is an isomorphism between the convolution algebra $E'(\mathbf{R}^n)$ and $\hat{E}'(\mathbf{R}^n)$, the algebra of entire functions of exponential type and polynomial growth on the real axis.

A distribution T will be called invertible (or $\hat{T}$ is slowing decreasing) if whenever $S \in E'(\mathbf{R}^n)$ and $\hat{S}/\hat{T}$ is an entire function, then there is a distribution $U \in \mathcal{E}'(\mathbf{R}^n)$ such that $\hat{U} = \hat{S}/\hat{T}$, that is

$$(4) \qquad S = T * U$$

and we have the identity

$$(5) \qquad H_S = H_T + H_U$$

or, what amounts to the same thing

$$(6) \qquad cv(suppU*T) = cv(suppU) + cv(suppT),$$

where, for two sets $A,B \subseteq \mathbf{R}^n$ we have $A \pm B = \{x \pm y; x \in A, y \in B\}$. We will need to use that μ_r is an invertible distribution. This will follow from the explicit formula for $\hat{\mu}_r$ given below and the characterization of invertible distributions: T is invertible if and only if there is a positive constant a such that for all $\xi \in \mathbf{R}^n$

$$(7) \qquad \text{Max}\{|T(\xi+\eta)|: \eta \in \mathbf{R}^n, |\eta| < a.\log(2+|\xi|)\} > (a+|\xi|)^{-a}$$

The Fourier transform of a radial distribution T is a radial function, i.e.: if:

$$\langle T, foA^{-1} \rangle = \langle T, f \rangle$$

for every $A \in O(n)$ then

$$\hat{T}(\zeta) = \hat{T}(A.\zeta)$$

for every $A \in O(n)$, $\zeta \in \mathbb{C}^n$, and depends, for $\xi \in \mathbf{R}^n$, only on $|\xi|$. Hence we consider the associated even entire function $\tilde{T}$ of one variable by

$$(8) \qquad \tilde{T}(|\xi|) = \hat{T}(\xi) \quad \text{and} \quad \hat{T}(\zeta) = \tilde{T}\left((\zeta_1^2+\dots+\zeta_n^2)^{1/2}\right)$$

Let us call $E_o'(\mathbf{R}^n)$, the space of radial distributions. This correspondence establishes an isomorphism between the algebras $\hat{E}_o'(\mathbf{R}^n)$ and $\hat{E}_o'(\mathbf{R})$. Using this notation we have

$$(9) \qquad \tilde{\mu}_r(t) = n2^{\frac{n-2}{2}}\,\Gamma(\tfrac{n}{2})\,J_{\frac{n}{2}}(rt)/(rt)^{n/2}$$

$$(10) \qquad \tilde{\sigma}_r(t) = 2^{\frac{n-2}{2}}\,\Gamma(\tfrac{n}{2})\,J_{\frac{n-2}{2}}(rt)/(rt)^{\frac{n-2}{2}}$$

and, more generally, if $f(x) = \phi(|x|)$ is a radial function of compact support

$$(11) \qquad \hat{f}(\zeta) = \tilde{f}(t) = \frac{(2\pi)^{n/2}}{t^{\frac{n-2}{2}}} \int_0^\infty \phi(\rho)\, J_{\frac{n-2}{2}}(\rho t)\rho^{n/2} d\rho \quad (|\zeta|=t)$$

To show that μ_r is invertible it is now sufficient to recall the asymptotic development of the Bessel functions [19] on the positive real axis

(12) $$J_\nu(t) = \sqrt{\frac{2}{\pi}}\, t^{-1/2} \cos(t-\frac{\pi}{4}-\frac{\nu\pi}{2}) + O(t^{-3/2})$$

It follows, for $|\xi| > 1$ and some $C > 0$

$$\operatorname{Max}\{|\hat{\mu}_r(\xi+\eta)| : \eta\in \mathbf{R}^n, |\eta|<\pi\} > C|\xi|^{-\frac{n+1}{2}}$$

which is the condition of invertibility.

From (12) we also obtain Mac-Mahon's asymptotic development of the positive zeros $\alpha_{k,\nu}$ of J_ν

$$0 < \alpha_{1,\nu} < \alpha_{2,\nu} < \dots$$

(13) $$\alpha_{k,\nu} = (2k+1)\frac{\pi}{2} + (2\nu+1)\frac{\pi}{4} + O(1/k)$$

which will be used further on.

3. <u>Series development of mean-periodic functions.</u>

Let Ω be an open convex set in $\mathbf{R}^n$ and $K = cv(\operatorname{supp}\mu)$, $\mu \in \mathcal{E}'(\mathbf{R}^n)$. We say that a function $f \in C^\infty(\Omega-K)$ is mean-periodic with respect to μ if

(14) $$\mu * f(x) = \langle\mu_y, f(x-y)\rangle = 0 \quad \text{for all} \quad x \in \Omega$$

If an exponential-polynomial, that is a finite linear combination of terms of the form $x^j \; e^{i(x|\zeta)}$

$(x^j = x_1^{j_1} \ldots x_n^{j_n}$, $j_k \in \mathbf{N}$, $1 \leq k \leq n)$, is mean-periodic with respect to μ then the frequencies ζ must satisfy $\hat{\mu}(\zeta) = 0$ since

$$(15) \qquad (\mu * e^{i(\cdot|\zeta)})(x) = \hat{\mu}(\zeta) e^{i(x|\zeta)}$$

When the zeros of $\hat{\mu}$ are simple no non constant monomials can appear. More generally if a monomial x^j appears with non zero coefficient then

$$\left(\frac{\partial^{|j|}}{\partial \zeta^j}\right)\hat{\mu}(\zeta) = 0$$

for the corresponding frequency ζ.

For $n = 1$ there is a well-known series development for such functions in terms of the exponential polynomial solution of the same convolution equation (14) due to L. Schwartz [18], [15], [13]. The case of interest for us is $n \geq 2$, μ invertible. In this case, a development in terms of integrals over the zero set of $\hat{\mu}$ has been proved when $\Omega = \mathbf{R}^n$ [6]. For Ω arbitrary convex set, a similar development has been proved in [4] but only for a very restrictive class of invertible distributions. In all these cases one obtains also some knowledge of the behavior of the terms involved in this development. Regretfully, the distributions μ_r, though invertible, do not satisfy the conditions required in [4], as it was shown (for a different reason) in [3], besides we are interested in $\Omega = B(0,R)$, therefore we cannot depend on any of the previously known results. We obtain here a series development but no precisions on the coefficients that appear in it, nevertheless the existence of this development is all we need later.

Proposition 1 - Let Ω be an open convex subset in $\mathbf{R}^n$ $(n \geq 2)$, $\mu \in \mathcal{E}'(\mathbf{R}^n)$ an invertible distribution, $K = cv(\operatorname{supp}\mu)$. Any function $f \in C^\infty(\Omega - K)$, mean periodic with respect to μ can be written as

$$(16) \qquad f(x) = \sum_{j \geq 1} P_j(x) \qquad (x \in \Omega - K)$$

with P_j exponential-polynomials also mean-periodic with respect to μ, and the series is convergent in the C^∞-topology of $\Omega - K$. Furthermore, given a sequence $(s_j)_{j \geq 1}$ of positive numbers, let $P_0 = 0$, we can chose the P_j so that the absolute value of all frequencies in P_{j+1} exceeds the largest absolute value of the frequencies in P_j by at least s_{j+1}.

Proof. Let us show first that, for any $s > 0$, the exponential polynomials which are mean-periodic with respect to μ and whose frequencies lie outside the ball of center 0 and radius s in $\mathbf{C}^n$ are dense in the space $N = \{f \in C^\infty(\Omega - K) : \mu * f = 0 \text{ in } \Omega\}$. N is a closed subspace of a Frechet space and we only need to show that if $\nu \in \mathcal{E}'(\Omega - K)$ is orthogonal to the above exponential-polynomials then ν is orthogonal to N. Hence $(\hat{\nu})^{\vee}$ is divisible by $\hat{\mu}$ at every point of $\mathbf{C}^n \setminus \overline{B}(0,s)$. Since $n \geq 2$, by Hartogs' theorem, $(\hat{\nu})^{\vee}/\hat{\mu}$ is an entire function. Since μ is invertible there is a distribution $T \in \mathcal{E}'(\mathbf{R}^n)$ such that

$$\check{\nu} = \mu * \check{T}$$

We need to know where is the support of T. By (6)

$$cv(supp\check{\nu}) = cv(supp\mu) + cv(supp\check{T})$$

or

$$cv(suppT) - K = cv(supp\nu) \subseteq \Omega - K$$

By the Hahn - Banach theorem one concludes that

$$cv(suppT) \subseteq \Omega$$

Hence $\langle\nu,f\rangle = (\check{\nu}*f)(0) = (\check{T}*\mu*f)(0) = \langle T,\mu*f\rangle = 0$ for $f \in N$.

To end the proof of the proposition, we pick an exhaustion of $\Omega - K$ by convex compacts sets K_j, hence we can find P_1, exponential-polynomial with frequencies lying in $\{\zeta\in\mathbb{C}^n : \hat{\mu}(\zeta)=0, |\zeta|>s_1\}$ such that

$$\sup_{K_1}|f-P_1| < 1.$$

Let σ_1 = maximum of the absolute values of frequencies in P_1. We can find P_2 with frequencies in $\{\zeta\in\mathbb{C}^n : \hat{\mu}(\zeta) = 0, |\zeta| > s_2+\sigma_1\}$ such that

$$\max_{|\alpha|\leq 1} \sup_{K_2}|D^{\alpha}(f-P_1-P_2)| < 1/2$$

Continuing in this fashion we obtain the desired expansion. □

Remark. One can eliminate the requirement of μ being invertible by using [14,16.4.1].

From (9) we know that the zero variety of $\hat{\mu}_r$ is the union of the hypersurfaces

$$(17) \qquad \zeta^2 = \zeta_1^2+\ldots+\zeta_n^2 = \lambda_k^2 \qquad k = 1,2,\ldots$$

where $\lambda_k = \alpha_{k,n/2}/r$. We disregard temporarily the dependence on r though it will play a role later on. Furthermore the function $\tilde{\mu}_r(t)$ vanishes at $t = \lambda_k$ with multiplicity one, in fact

$$(18) \qquad \frac{d}{dt}\tilde{\mu}_r(t) = -n2^{\frac{n-2}{2}}\Gamma(\tfrac{n}{2})r\, J_{(n/2)+1}(rt)/(rt)^{n/2}$$

and well known properties of Bessel functions show that this expression does not vanish for $t = \lambda_k$. Using the asymptotic expressions (12) and (13) we obtain

(19)

$$0 \neq \frac{d\tilde{\mu}_r}{dt}(\lambda_k) = r\, n\, 2^{(n-1)/2}\,\Gamma\big(n/2\big)(-1)^{k+1}/(\lambda_k r)^{\frac{n+1}{2}} + O\big(k^{-\frac{n+3}{2}}\big)$$

We introduce some auxiliary radial distributions $T_{r,k}$ by the formula

$$(20) \qquad \tilde{T}_{r,k}(t) = \frac{\tilde{\mu}_r(t)}{t^2-\lambda_k^2} .$$

They are even and entire since $\tilde{\mu}_r(\pm\lambda_k) = 0$. Hence they correspond to radial distributions (in fact $C^{1,1}$ functions) whose supports are contained in the support of μ_r, i.e. $\overline{B}(0,r)$.

Furthermore they satisfy

(21) $$(\Delta + \lambda_k^2) T_{r,k} = -\mu_r \quad \text{and}$$

(22) $$\tilde{T}_{r,k}(\lambda_k) = \frac{\tilde{\mu}_r'(\lambda_k)}{2\lambda_k} = \text{const.}(-1)^{k+1} \lambda_k^{-\frac{n+3}{2}} + O\left(k^{-\frac{n+5}{2}}\right).$$

We remark that these distributions have conspicuously appeared in previous work on the Pompeiu problem [2], [7].

Proposition 2 - Let $r > 0$ be fixed. For any ρ, $0 < \rho < \infty$, we can decompose σ_ρ in the following form

(23) $$\sigma_\rho = \nu_\rho + \mu_r * S_\rho,$$

where S_ρ is a radial distribution, whose support satisfies

(24) $$\text{supp } S_\rho \subseteq \overline{B}\left(0, \text{Max}(r,\rho) - r\right)$$

and ν_ρ is given explicitly by:

(25) $$\nu_\rho = - \sum_{k \geq 1} \frac{\tilde{\sigma}_\rho(\lambda_k)}{\lambda_k^2 \, \tilde{T}_{r,k}(\lambda_k)} \Delta T_{r,k}$$

hence $\text{supp } \nu_\rho \subseteq \overline{B}(o,r)$.

Proof. We consider the series

$$g(t) = \sum_{k\geq 1} \frac{\tilde{\sigma}_\rho(\lambda_k)}{\lambda_k^2 \tilde{T}_{r,k}(\lambda_k)} t^2 \tilde{T}_{r,k}(t) \tag{26}$$

The coefficients $\lambda_k^{-2} \tilde{\sigma}_\rho(\lambda_k)/\tilde{T}_{r,k}(\lambda_k)$ are uniformly bounded by a constant depending only on ρ as it can be seen from (10), (13) and (22), since $\lambda_k \sim$ const. k. Therefore, if $|t| < R$, $\eta_k > 2R$ we have $|t^2 \tilde{T}_{r,k}(t)| <$ const. k^{-2} which guarantees the convergence of the series, and shows g is an even entire function. We can obtain more precise estimates by picking a sequence of circles of center 0 and radii

$$R_j = (4j+n+5)^{\pi}/4r, \quad j = 1,2,..$$

Decomposing the sum into those terms where $\lambda_k < 2R_j$ and $\lambda_k > 2R_j$ one can estimate the second sum over $|t| = R_j$ by

$$\operatorname*{Max}_{|t|=R_j} |t^2 \tilde{\mu}_r(t)| . \quad C_o(\rho)$$

The first (finite) sum can be estimated by

$$C_1(\rho)\left(\operatorname*{Max}_{|t|=R_j}|t^2\tilde{\mu}_r(t)|\right) \left(\operatorname*{Max}_{\Omega_{j,\varepsilon}} \sum_{0<\lambda_k<2R_j} \frac{1}{|\lambda_k^2-t^2|}\right)$$

where $\Omega_{j,\varepsilon}$ is the region obtained from $|t| < R_j$ by removing disks of radius ε, $0 < \varepsilon$ very small, about $\pm\lambda_k$. One can then see, without difficulty, that the last sum is estimated by const. ε^{-1} In any case we obtain as a final estimate

$$\max_{|t|<R_j} |g(t)| < C(\rho) \max_{|t|<R_j} |t^2\tilde{\mu}_r(t)|.$$

Thus g defines a radial distribution of order 2, ν_ρ, by $\tilde{\nu}_\rho = g$, one can see ν_ρ is given explicitly by (25). We also have

$$\tilde{\sigma}_\rho - \tilde{\nu}_\rho = \tilde{\mu}_r\, h,$$

with h even entire function since $g(\pm\lambda_k) = \nu_\rho(\pm\lambda_k) = \tilde{\sigma}_\rho(\pm\lambda_k)$ by (26). Since μ_r is an invertible distribtion it follows $h = \tilde{S}_\rho$ for some $S_\rho \in E'_o(R^n)$. The identity (6) gives

$$(27) \qquad cv(supp(\sigma_\rho - \nu_\rho)) = cv(supp S_\rho) + cv(supp \mu_r).$$

There are two cases to consider. If $\rho < r$, then the support on the left hand side of (27) is contained in $\overline{B}(o,r)$ and $cv(supp\ S_\rho) = \{0\}$, which says S_ρ is a polynomial in the Laplace operator; if $\rho > r$ then the left hand side of (27) is contained in $\overline{B}(o,\rho)$, which says $cv(supp S_\rho) \subseteq B(\overline{o,\rho - r})$. □

<u>Remark</u> The decomposition we have just given in proposition 2 works also if we replace σ_ρ by any radial distribution. We need only to change $(t/\lambda_k)^2$ by $(t/\lambda_k)^{2q}$ with q convenient non negative integer. In particular there is such a decomposition with $\sigma_0 = \delta$, the Dirac mass at the origin (take $q > \frac{n+1}{4}$)

Corollary 3 - Let f be a μ_r -mean-periodic function in $C^{\infty}(B(0,R))$ $(R>r)$. Let $|x_0| < R - r$. Then, for any ρ, $0 < \rho < R - |x_0|$ we have

$$(28) \qquad \lambda_{\rho}(f,x_0) = (\nu_{\rho}*f)(0) = -\sum_{k\geq 1} \frac{\tilde{\sigma}_{\rho}(\lambda_k)}{\lambda_k^2\, \tilde{T}_{r,k}(\lambda_k)} \Delta(T_{r,k}*f)(x_0)$$

Proof. It suffices to use (2) and (23). □

4. Local two-circles theorem.

Let r_1, r_2 be two positive numbers and consider the distributions μ_{r_1}, μ_{r_2}. They will have no common, mean-periodic, exponential-polynomials if and only if $\hat{\mu}_{r_1}$ and $\hat{\mu}_{r_2}$ have no common zeros. By (17) this occurs if and only if

$$r_1/r_2 \neq \text{quotient of two zeros of } J_{n/2}$$

The set

$$E_n = \{\alpha_{k,n/2}/\alpha_{j,n/2} : 1\leq j,k<\infty\}$$

is the exceptional set described in the two-circles theorem.

Proposition 4 Let $R > r_1 + r_2$, $r_1/r_2 \notin E_n$. The only function in $C^{\infty}(B(0,R))$ which is mean-periodic with respect to both μ_{r_1} and μ_{r_2} is the zero function.

Proof. We assume $r_1 < r_2$. Let $f \in C^{\infty}(B(0,R))$ be μ_{r_1} -mean periodic. By proposition 1 we have

$$f(x) = \sum_{j\geq 1} P_j(x) \quad (|x|<R)$$

where the frequencies appearing in the exponential sums P_j lie in

$$\{\zeta\in C^n : \hat{\mu}_{r_1}(\zeta)=0\} = \bigcup_{k\geq 1} \{\zeta\in C^n : \zeta^2=(\alpha_{k,n/2}/r_1)^2\} = \bigcup_{k\geq 1} V_k .$$

We fix now $k \geq 1$, and consider $T_{r_1,k}*f$ which is in $C^\infty(B(0,R-r_1))$, furthermore

(29) $$T_{r_1,k}*f = \sum_{j\geq 1} T_{r_1,k}*P_j$$

If $P_j(x) = \sum_{\ell} C_{j,\ell}\, e^{i(x|\zeta_{j,\ell})}$ then

$$T_{r_1,k}*P_j = \sum_{\ell} C_{j,\ell}\, T_{r_1,k}(\zeta_{j,\ell})e^{i(x|\zeta_{j,\ell})}$$

but $\hat{T}_{r_1,k}(\zeta_{j,\ell}) \neq 0$ only if $\zeta_{j,\ell}\in V_k$ in which case we obtain the value $\tilde{T}_{r_1,k}(\lambda_k) = 0$ (where λ_k is computed with respect to r_1). Therefore

(30) $$T_{r_1,k}*f = \tilde{T}_{r_1,k}(\lambda_k) \sum_{j\geq 1} P_{j,k}$$

where $P_{j,k}$ is the sum of the terms in P_j whose frequencies lie in V_k. This series is convergent in $C^\infty(B(0,R-r_1))$. We convolve now with μ_{r_2}. We obtain

(31) $$\mu_{r_2}*(T_{r_1,k}*f) = \tilde{T}_{r_1,k}(\lambda_k)\, \tilde{\mu}_{r_2}(\lambda_k) \sum_{j\geq 1} P_{j,k}$$

since μ_{r_2} is also a radial distribution. The expansion (31) is valid in $C^\infty(B(0,R-r_1-r_2))$. Since f is also μ_{r_2} - mean - periodic we have

$$0 = (T_{r_1,k} * \mu_{r_2} * f)(x) = \tilde{\mu}_{r_2}(\lambda_k)(T_{r_1,k} * f)(x)$$

for $|x| < R - r_1 - r_2$. The hypothesis $r_1/r_2 \notin E_n$ now implies that $\tilde{\mu}_{r_2}(\lambda_k) \neq 0$. Hence

$$(32) \qquad (T_{r_1,k} * f)(x) = 0 \quad \text{for} \quad |x| < R - r_1 - r_2$$

On the other hand we have (by (22))

$$(\Delta + \lambda_k^2)(T_{r_1,k} * f) = -(f * \mu_{r_1}) = 0 \text{ in } |x| < R - r_1$$

hence $T_{r_1,k} * f$ is a real analytic function in $|x| < R - r_1$. We conclude that

$$(33) \qquad (T_{r_1,k} * f)(x) = 0 \quad \text{for} \quad |x| < R - r_1$$

Applying now corollary 3, formula (28), we have

$$(34) \qquad \lambda_\rho(f,x) = 0 \quad \text{whenever} \quad |x| < R - r_1,\ 0 \leq \rho < R - |x|$$

(We are allowed to take $\rho = 0$ by continuity). In particular

$$f(x) = 0 \quad \text{for } |x| < R - r_1$$

To finish the proof of the proposition we need to show f is zero in the remaining annulus, we do that using (34). It is at this point that we use Cormack-Quinto [12]. For any y $B(0,R)$, consider $R(f)(y) = \lambda_{|y|/2}(f, y/2)$. This is the Radon transform on spheres through the origin discussed in [12]. We want to show $Rf(y) = 0$. We only need to verify that the conditions stated in (34) are valid. Here $\rho = |y|/2$, $x = y/2$, hence

$$R - |x| = R - \frac{|y|}{2} = R - \rho > R/2 > \rho$$

The only condition left to see is that $|x| < R - r_1$. We have $2r_1 < r_1 + r_2 < R$ hence $r_1 < R/2$ and $R - r_1 > R/2$, therefore $|x| < R - r_1$ holds.

By [12, corollary 2] $f(y) = 0$. (We note that in [12], they require that f $C^\infty(\mathbf{R}^n)$ while we only have $f \in C^\infty(B(0,R))$ but the proof of corollary 2 depends on an explicit inversion formula for the Radon transform on spheres which uses, for each y, values of f in a neighborhood of $\overline{B}(0,|y|)$.) □

Remark. The crucial point of the proof above is (32). One does not really need the whole strength of Propositon 1 to obtain it. One can get by using the density of the exponential polynomial solutions in the sub-space N introduced in Propositon 1. Nevertheless, we feel that the proof is clearer using the expansion (16) as we have done.

We want to show that the condition $R > r_1 + r_2$ is sharp. It is easier to show this under the slight restriction that r_2/r_1 is not too well approximated by elements in E_n.

Definition. - For $N > 0$, we say that a positive number is N-well approximated by points in E_n if, for every $\ell > 1$, there are indices j,k such that

(35) $$|r - \alpha_k/\alpha_j| < \frac{1}{\ell j^N}$$

where $\alpha_k = \alpha_{k,n/2}$

Proposition 5 For any $N > 2$, the set of numbers N-well approximates by E_n has zero measure in $(0,\infty)$

Proof. Given p,q, $0 < p < r < q$ and $\nu > 0$, from (13) we have

$$\alpha_{k,\nu} = \alpha_k = (2k+1)\pi/2 + (2\nu+1)\pi/4 + O(1/k)$$

Therefore, if r satisfies (35), for $\ell > 1$, we have

(36) $$|r\cdot j - k + Ar + B| < C$$

for some constants A,B,C. Hence

$$pj - C_1 < k < qj + C_2$$

for some constant $C_1, C_2 > 0$. Hence the cardinal of the set of k satisfying (36) is bounded by $(q-p)j + L$, L constant > 0.

Now, the set of N-well approximated numbers in [p,q] is

(37) $$\bigcap_{\ell \geq 1} \bigcup_{j,k \geq 1} \{r:\ p<r<q,\ |r-\alpha_k/\alpha_j| < 1/\ell j^N\}$$

For ℓ fixed, the Lebesgue measure

$$\left|\bigcup_{j,k} \{r:p<r<q, |r-\alpha_k/\alpha_j| < 1/\ell j^N\}\right| \leq \frac{2}{\ell} \sum_{j\geq 1} \frac{(q-p)j+L}{j^N} \leq \frac{C_3}{\ell}$$

($C_3 > 0$) since $N > 2$. Therefore the set (37) has zero measure and by letting $q = p + 1$, $p \in \mathbf{N}$ we obtain the proposition.

It is interesting to compare proposition 5 with [8, Lemma 2.1] where examples of numbers which are not 2-well approximated by $E_n(n=2)$ are discussed. It might be that these include all rationals $\neq 1$, all quadratic irrationals $\neq 1$, but no such theorem seems to be known. Also, it is easy to see that, for $N < 1$, every positive number is N-well approximated by E_n.

<u>Proposition 6</u>. - Let r_1, r_2 be two positive numbers such that r_2/r_1 is not N-well approximated by E_n. Denote by λ_k the positive zeros of $\tilde{\mu}_{r_1}$. There is a positive constant C such that

(38) $$|\tilde{\mu}_{r_2}(\lambda_k)| \geq \frac{C}{k^{N+\frac{n-1}{2}}}$$

<u>Proof</u>. Let us denote $\alpha_k = \alpha_{k,n/2}$. Recall that $\lambda_k = \alpha_k/r_1$ and that

$$\tilde{\mu}_{r_2}(t) = \text{const.} \frac{J_{n/2}(r_2 t)}{(r_2 t)^{n/2}}$$

From the asymptotic development (13) we have

$$\alpha_{k+1} - \alpha_k = \pi + O(1/k)$$

Hence, if k is fixed and j_k is chosen such that $|r_2\lambda_k - \alpha_j|$ is minimal we have

(39) $$\varepsilon_k = |r_2\lambda_k - \alpha_{j_k}| < \pi/2 + O(1/k).$$

Let us distinguish two cases: $\varepsilon_k < \pi/4$ or not. In the second case we have

$$\left|\cos\left(r_2\lambda_k - \frac{(n+1)}{4}\pi\right)\right| = \left|\cos(\pm\varepsilon_k + (2j_k+1)\ \frac{\pi}{2} + O(\tfrac{1}{k}))\right|$$

$$= |\sin(\varepsilon_k + O(1/k))| > \frac{\sqrt{2}}{2} + O(\tfrac{1}{k}) > c_0 > 0$$

for large k. In this case the asymptotic development (12) gives the estimate

$$|\tilde{\mu}_{r_2}(\lambda_k)| > C_1\ k^{-\frac{n+1}{2}}$$

for some $C_1 > 0$ and all large k.

By hypothesis we have that for all j,k

$$\left|\frac{r_2}{r_1} - \frac{\alpha_1}{\alpha_k}\right| > \frac{C_2}{k^N}\ (C_2 > 0)$$

Therefore $\varepsilon_k > C_3/k^{N-1}$ $(C_3 > 0)$. Suppose also $\varepsilon_k < \pi/4$. By the mean-value theorem there is a ξ between α_{j_k} and $r_2\lambda_k$ such that

$$\frac{J_{n/2}(r_2\lambda_k)}{(r_2\lambda_k)^{n/2}} = -r_2\frac{J_{n/2+1}(\xi)}{\xi^{n/2}}\cdot\left(r_2\lambda_k-\alpha_{j_k}\right)$$

(Recall $J_{n/2}\left(\alpha_{j_k}\right) = 0$.) Note that $\delta_k = \left|\xi-\alpha_{j_k}\right| < \varepsilon_k < \pi/4$. Again by (12) we have to estimate

$$\begin{aligned}\cos\left(\xi-\frac{\pi}{4}-\frac{(n+2)\pi}{4}\right) &= \cos\left(\pm\delta_k+\left(2j_k+1\right)\frac{\pi}{2}+\frac{(n+1)\pi}{4}-\frac{(n+2)\pi}{4}+O\left(1/k\right)\right)\\ &= \cos\left(\pm\delta_k+j_k\pi+O(1/k)\right)\\ &= \pm\cos\delta_k + O(1/k).\end{aligned}$$

Then

$$|\tilde{\mu}_{r_2}(\lambda_k)| > \frac{C_4}{k^{N-1+\frac{n+1}{2}}} \quad (C_4 > 0).$$

Since $N > 1$ the estimate (38) holds in both cases. □

<u>Proposition 7</u> let f be a function in $L^1_{loc}(B(0,R))$, $g \in L^1_{loc}(B(0,R))$, $\operatorname{supp} g \subseteq B(0,r)$, g radial. For $|x_0| < R - r$ and $\rho < R - r - |x_0|$ we have

$$\lambda_\rho(f*g,x_0) = \left(\lambda_{|\cdot|}(f,x_0)*g(\cdot)\right)(y) \quad (|y|=\rho) \tag{40}$$

(The notation indicates that we are convolving in the variable denoted by a dot).

__Proof.__ Recall that the average $\lambda_\rho(f,x_0)$ can also be computed by

$$\lambda_\rho(f,x_0) = \int_{O(n)} f(x_0+Ay)dA$$

where y is any point with $|y| = \rho$, $O(n)$ is the orthogonal group and dA is the normalized Haar measure. Let

$$\phi(y) = \left(\lambda_{|\cdot|}(f,x_0)*g(\cdot)\right)(y)$$

we have

$$\phi(y) = \int_{\mathbf{R}^n} \left(\int_{O(n)} f(x_0+A(y-x)dA\right)g(x)\ dx$$

$$= \int_{O(n)} \left(\int_{\mathbf{R}^n} f\left(x_0+A(y-x)\right)g(x)dx\right)dA$$

Set $u = Ax$ then $g(x) = g(u)$ and $dx = du$ Hence

$$\phi(y) = \int_{O(n)} \left(\int_{\mathbf{R}^n} f(x_0+Ay-u)g(u)du\right)dA$$

$$= \lambda_\rho(f*g,x_0). \qquad \square$$

__Corollary__ 8. Let g be radial integrable function of compact support and α a positive number. Then

$$(41) \qquad \left(g(x) * \frac{J_{\frac{n-2}{2}}(\alpha|x|)}{(a|x|)^{\frac{n-2}{2}}} \right)(y) = \tilde{g}(\alpha) \frac{J_{\frac{n-2}{2}}(\alpha|y|)}{(\alpha|y|)^{\frac{n-2}{2}}}$$

Proof. Let $\xi \in R^n$ be any vector with $|\xi| = \alpha$, then

$$(42) \qquad \left(g(\cdot) * e^{i(\xi|\cdot)}\right)(y) = \hat{g}(\xi) e^{i(\xi|y)}$$

$$= \tilde{g}(\alpha)\ e^{i(\xi|y)}.$$

On the other hand

$$\lambda_\rho\left(e^{i(\xi|\cdot)}, 0\right) = \tilde{\sigma}_\rho(\alpha)$$

$$= 2^{\frac{n-2}{2}} \Gamma(n/2) \frac{J_{\frac{n-2}{2}}(\alpha\rho)}{(\alpha\rho)^{\frac{n-2}{2}}}.$$

Applying now to (42) Proposition 7 we obtain the desired formula (41). □

Proposition 9. Let r_1, r_2 be two positive numbers such that r_2/r_1 is not N-well approximated by E_n. Let R be any number, $\max(r_1, r_2) < R < r_1 + r_2$. Then there is a non zero radial function $f \in C^\infty(B(0,R))$ which is mean periodic with respect to μ_{r_1} and μ_{r_2}.

Proof. Let $\phi \in \mathcal{D}(]0, r_1[), \phi \not\equiv 0$ such that $\mathrm{supp}\phi \subseteq [R-r_2, r_1[$ It follows from [16, theorem 2.1 page 247] that ϕ admits a series development of the form

(43) $$\phi(t) = \sum_{k\geq 1} \alpha_k \frac{J_{\frac{n-2}{2}}(\lambda_k t)}{(\lambda_k t)^{\frac{n-2}{2}}}$$

where $\lambda_k = \alpha_{k,n/2}/r_1$. This is the Sturm-Liouville expansion for a boundary value problem singular at $t = 0$ and derivative equal to zero at $t = r_1$. It can be seen by successive integrations of parts that

(44) $$|a_k| = O(k^{-p}) \quad \text{for every} \quad p > 0.$$

Since r_2/r_1 is not N-well approximated by E_n we see that

$$b_k = a_k/\tilde{\mu}_{r_2}(\lambda_k)$$

satisfies the same estimates as a_k (Proposition 6) Hence the function

(45) $$f(x) = \sum_{k\geq 1} b_k \frac{J_{\frac{n-2}{2}}(\lambda_k |x|)}{(\lambda_k |x|)^{\frac{n-2}{2}}}$$

is a C^∞ radial function in $\mathbb{R}^n$, $f \not\equiv 0$, . And, from corollary 8, it follows that f is μ_{r_1} mean-periodic. Furthermore

$$\left(\mu_{r_2} * f\right)(x) = \sum_{k\geq 1} b_k \tilde{\mu}_{r_2}(\lambda_k) \frac{J_{\frac{n-2}{2}}(\lambda_k |x|)}{(\lambda_k |x|)^{\frac{n-2}{2}}}$$

$$= \phi(|x|)$$

which is zero in $B(0,R-r_2)$ and therefore the function f restricted to $B(0,R)$ is μ_{r_1} and μ_{r_2} mean-periodic. □

The above propositions can be summarized by the following:

Theorem 10. Let $r_1 > 0$ and $r_2 > 0$ be such that $r_2/r_1 \in E_n$. The necessary and sufficient condition on a open set Ω of $\mathbf{R}^n$ so that the only distribution $T \in \mathcal{D}'(\Omega)$ which can be mean-periodic with respect to both μ_{r_1} and μ_{r_2} is $T = 0$, is that Ω is the reunion of balls of radii strictly larger than $r_1 + r_2$.

An amusing corollary of theorem 10 is the following:

Corollary 11 If $r_2/r_1 \in E_2$, $r_1 + r_2 < R$ and $f \in C(B(0,R))$ then the conditions

$$\int_{\partial B(z,r_j)} f(\zeta)d\zeta = 0 \quad \text{for every } z,\ |z| < R - r_j \ (j=1,2),$$

imply that f is holomorphic in the disk $B(0,R)$.

5. Generalizations

After a first version of this paper was written, Professor Zalcman pointed out to us that J. D. Smith [17] had also proved the two-circles, starting precisely with a local version of it. It turns out it was not as sharp as Proposition 4 since he required $R > 2r_1 + r_2$. Furthermore, Dr. Smith had also indicated to Professor Zalcman that his method did not generalize to the other problems discussed in [20], e.g. the converse of the mean value property for harmonic functions. The aim of this section is to show that the methods used above do generalize.

Definition. We say that a radial distribution μ of compact support is hyperbolic if:

(i) μ is invertible, and

(ii) there is a constant C such that every zero λ of $\tilde{\mu}$ satisfies

$$|\mathrm{Im}\lambda| \leq C \log(2+|\lambda|).$$

Theorem 12 Let $\mu_1, \mu_2, \ldots$ be a (possibly infinite) family of radial distributions of compact support, $cv(\mathrm{supp}\mu_j) = \overline{B}(0,r_j)$. Suppose $\{z\in C^n : \mu_j(z)=0 \ \forall j\} = \emptyset$, μ_1 is hyperbolic, and $R - r_1 > \sup_j r_j$. Then $\{f\in \mathcal{D}'(B(0,R) : \mu_j * f=0 \ \forall j\} = \{0\}$.

Proof: Due to the condition on R we can assume $f \in C^\infty(B(0,R))$ as done before. The proof that leads to (32) can be repeated almost verbatim just using for each λ_k, zero of $\tilde{\mu}_1$, a convenient $\mu_j (j\geq 2)$ with $\tilde{\mu}_j(\lambda_k) \neq 0$. We obtain

$$(47) \qquad T_{k,s} * f(x) = 0 \quad \text{for} \quad |x| \quad R - r_1 - \sup_j r_j,$$

where $\tilde{T}_{k,s}(t) = \tilde{\mu}_1(t)\,(t^2-\lambda_k^2)^{-s}$, $1 \leq s \leq m_k$, m_k = multiplicity of λ_k as a root of $\tilde{\mu}_1$.

On the other hand

$$(48) \qquad (-1)^s(\Delta+\lambda_k^2)^s\,(T_{k,s} * f) = \mu_1 * f = 0 \quad \text{in} \quad B(0,R-r_1),$$

therefore, $T_{k,s} * f$ is real analytic, and hence

(49) $$T_{k,s}*f = 0 \text{ in } B(0,R-r_1),$$

as before. It is at this point we have to be more careful to prove the correct version of Proposition 2. It will be replaced by the following:

Lemma 13 Let $\Lambda = \{\lambda_k\}$ = set of distinct zeros of $\tilde{\mu}_1$, then $\Lambda = \bigcup_{j=0}^{\infty} \Lambda_j$, where the Λ_j are finite and mutually disjoint sets. There is also a positive integer q such that for any ρ, $0 < \rho < \infty$ we can write

(50) $$\sigma_\rho = \nu_\rho + \mu_1 * S_\rho,$$

where ν_p, S_ρ are radial distributions satisfying

(51) $$\text{supp } \nu_\rho \subseteq \overline{B}(0,r_1)$$ and

(52) $$\text{supp } S_\rho \subseteq \overline{B}(0, \text{Max}(r_1,\rho)-r_1).$$

Furthermore,

(53) $$\nu_\rho = \sum_{j=0}^{\infty} \Delta^q \nu_{\rho,j},$$

a convergent series in $E_o'(\mathbf{R}^n)$, each $\nu_{\rho,j}$ a finite linear combination of the distributions $T_{k,s}$, $\lambda_k \in \Lambda_j$, $1 \leq s \leq m_k$ (if $m_o > 1$ then one denotes by $\Delta^q \nu_{\rho,o}$ not only a finite linear combination of $\Delta^q T_{k,s}$, $\lambda_k \in \Lambda_o$ but also of $T_{o,m_o}, \Delta T_{o,m_o}, \ldots, \Delta^{q-1} T_{o,m_o}$.)

Once this lemma has been proved, the proof of Theorem 12 is achieved the same way as it was done in Proposition 4 and we note that the hypotheses imply $2r_1 < R$.

Proof of Lemma 13 The proof of this lemma proceeds as in Proposition 2 by interpolating the values of $\tilde{\sigma}_\rho$ on the variety of zeros of $\tilde{\mu}_1$ (counted with multiplicities). We have to repeat with due care the procedure used in [13], [15], [18] since we need the precise statement (51), (52), and (53).

First we note that as in [5, lemma 4] (cf. also [11, p. 50]), the condition of hyperbolicity and the minimum modulus theorem allow us to construct a family of a Jordan quadrilaterals Γ_k, $k \in \mathbb{Z}$ symmetric with respect to the real axis and enjoying the following properties:

(54) for some $d > 0$ the horizontal sides lie on the curves

$$\operatorname{Im} z = \pm \log(d+|\operatorname{Re} z|),$$

and the vertical sides are arcs of circles.

(55) $0 \in \operatorname{int} \Gamma_o$ which is symmetric with respect to the origin (i.e. if $z \in \Gamma_o$ then $-z \in \Gamma_o$ also).

(56) for $k \neq 0$, Γ_{-k} is the symmetric of Γ_k with respect to the origin.

(57) for $j \neq k$, $\overline{\text{int}\,\Gamma_g} \cap \overline{\text{int}\,\Gamma_k} = \emptyset$, furthermore, for some positive number a we have that if $z \in \Gamma_j$, $\text{dist}(z,\Gamma_k) > (a+|z|)^{-a}$ for any $k \neq j$.

(58) for some positive constant b we have:

$$\text{diam}\,\Gamma_j < b(1+|z|)^b$$

and

$$\text{length}\,\Gamma_j < b\,(1+|z|)^b,$$

for any $z \in \overline{\text{int}\,\Gamma_j}$, any j.

(59) there is a constant $c > 0$ such that for any j, and any $z \in \Gamma_j$ we have

$$|\tilde{\mu}_1(z)| > (c+|z|)^{-c},$$

and this inequality is valid even for those z such that $\text{dist}(z,\Gamma_j) < 1/2\;(a+|z|)^{-a}$ (the same a as in (57))

(60) $$\Lambda \subseteq \bigcup_{j=-\infty}^{\infty} (\text{int}\,\Gamma_j).$$

(61) for some $d > 0$: if $j > 1$, $z \in \Gamma_j$, then $|z| > j/d$.

(62) $\Lambda_o = \Lambda \quad \text{int}\,\Gamma_o$, $\Lambda_j = \Lambda \quad (\text{int}\,\Gamma_j \cap \text{int}\,\Gamma_{-j})$. $j > 1$.

For the sake of definiteness we will index the points in Λ so that $\lambda_o = 0$, and, for $k \geq 1$, either $\mathrm{Re}\,\lambda_k > 0$ or $\mathrm{Re}\lambda_k = 0$ and $\mathrm{Im}\,\lambda_k > 0$, and, finally, $\lambda_{-k} = -\lambda_k$.
Now consider the even entire function

$$(63)\qquad f(t) = t^{2q}\,\tilde{\mu}_1(t),$$

for q a positive integer to be chosen conveniently later on.
We note that if $t \notin \overline{\mathrm{int}\,\Gamma_j} \cup \overline{\mathrm{int}\,\Gamma_j}$ then

$$(64)\qquad \phi_j(t) = \frac{1}{2\pi i}\int_{\Gamma_j} \frac{\tilde{\sigma}_\rho(s)}{f(s)}\frac{ds}{s-t} + \frac{1}{2\pi i}\int_{\Gamma_{-j}} \frac{\tilde{\sigma}_\rho(s)}{f(s)}\frac{ds}{s-t}$$

(where we disregard the second term if $j = 0$) is an even function which is a linear combination of terms of the form $(t^2-\lambda_k^2)^{-s}$, for $\lambda_k \in \Lambda_j$ and $1 \leq s \leq m_k$ if $k \geq 1$, $1 \leq s \leq m_o + 2q$ if $k = 0$. Hence ϕ_j can be defined as a rational function throughout $\mathbb{C}$ and the function $f(t)\,\phi_j(t)$ is an even entire function. We want to show now that q can be chosen so that

$$(65)\qquad g(t) = \sum_{j=0}^{\infty} f(t)\,\phi_j(t)$$

is in $\tilde{E}_o'(R^n)$ and the series converges in the topology of $\tilde{E}_o'(R^n)$.

In fact, we have that for $|\mathrm{Im}\,t| \leq \log(d+|\mathrm{Re} t|)$ there is some $N > 0$ such that

(66) $$|\tilde{\sigma}_\rho(t)| < C(\rho)\ (1+|t|)^N$$

and also

(67) $$|\tilde{\mu}_1(t)| < C_o(1+|t|)^N.$$

Therefore, for some $N_1 > 0$ sufficiently large, if $\mathrm{dist}(t, \overline{\mathrm{int}\,\Gamma_j} \cup \overline{\mathrm{int}\,\Gamma_{-j}}) > 1$ we have by (66), (59) and (58), that with respect to an arbitrary point $z \in \overline{\mathrm{int}\,\Gamma_j}$, which we can take it to be the point in the positive real axis closest to the origin,

$$|\phi_j(t)| < (N_1+|z|)^{N_1}|z|^{-2q} < \text{const. } j^{-2}$$

by (61) (just take $2q > N_1 + 2$). Therefore, under the same condition on t we have

(68) $$|f(t)\ \phi_j(t)| < C_1\ j^{-2}(1+|t|)^M\ e^{r_1|\mathrm{Im}t|}$$

Using the condition (58) on the diameter of Γ_j and (67), this estimate remains valid throughout $\mathbb{C}$, after possibly increasing C_1, M. This shows that the $\nu_\rho \in E_o'(\mathbb{R}^n)$ defined by

$$\tilde{\nu}_\rho(t) = g(t)$$

satisfies (51). It is also clear that the distributions $\nu_{\rho,j}$ such that $t^{2q}\ \nu_{\rho,j} = f(t)\ \phi_j(t)$ have the properties required

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defined by

$$\tilde{\mu}_j(t) = t^{-2}(\widetilde{\sigma_{r_j}-\delta})(t) = (\tilde{\sigma}_{r_j}(t)-1)/t^2$$

are hyperbolic. The hypothesis on r_1/r_2 guarantees these two entire functions have no common zeros. Theorem 13 shows now that the distribution Δu is zero in $B(0,R)$.

Remark As mentioned in [20], Delsarte proved this theorem in R^n. He also showed that H_n is finite and $H_3 = \{1\}$. Hence, at least for dimension 3, any pair of distinct positive value r_1, r_2 would work in the above corollary.

The several other results in [20] can now be carried over to the local case without difficulty. It remains as an open question for the moment the elimination of the invertibility condition on μ_1, which could probably be done following the Euclidean summation method of [6]. More interesting, in our view, is to try to extend this theorem to non-compact symmetric spaces of rank 1 or even to the Euclidean group thus obtaining a local version of the Pompeiu problem considered in [9].

As an example of this let us mention the following corollary of Theorem 13.

Corollary 15 Let $R > \sqrt{n}\, a$, if $f \in L^1_{loc}(B(0,R))$ has zero integral over any n-cube of side a contained in $B(0,R)$, then $f = 0$ a.e.

Proof Following the ideas from [9] we see we can consider all radial distributions μ whose Fourier tranforms are of the form

(69) $$\int_{O(n)} \hat{\chi}_Q(k\zeta)\hat{T}(k\zeta)\,dk = \hat{\mu}(\zeta)$$

where Q is the cube $[-a/2,a/2]^n$ and T is a distribution of compact support in the ball $B(0,\varepsilon)$, $\varepsilon + \sqrt{n}\,a < R$. Then, for any such μ, $cv(\mathrm{Supp}\mu)$ $\bar{B}(0,r)$, and f will satisfy the equations:

$$\mu * f = 0 \quad \text{in} \quad B(0,R-r).$$

Since this set of distributions generates the same closed ideal in $E'(R^n)$ as those are considered in [9, p. 602], then their Fourier tranforms have no common zeros [9, section 9]. It only remains to find a distribution that plays the role of μ_1 in Theorem 13. The easiest one is obtained when

$$T = \frac{\partial^{2n} \chi_Q}{\partial x_1^2 \ldots \partial x_n^2}$$

An easy computation shows that in this case, for

μ_1 = average over $O(\mu)$ of $\dfrac{\partial^{2n}}{\partial x_1^2 \ldots \partial x_n^2} \chi_Q$, we have

(70) $$\tilde{\mu}_1(t) = \text{const. } t^{(n/2)+1} J_{\frac{3n-2}{2}}(\sqrt{n}\, at/2)$$

which is clearly hyperbolic. (For $n = 2$, this can be obtained from Sonine second finite integral [19, p.376].)