# ABSTRACT

Title of dissertation: NETWORK TOMOGRAPHY

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Doctor of Philosophy, 2006

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While conventional tomography is associated to the Radon transform in Euclidean spaces, electrical impedance tomography, or EIT, is associated to the Radon transform in the hyperbolic plane. In this dissertation, we discuss some recent work on network tomography that can be associated to a problem similar to EIT on graphs and indicate how in some sense it may be also associated to the Radon transform on trees. We develop a strategy to determine the weight $\omega$ for the case of general weighted graphs. We begin by considering relatively simple regions of interest in a graph and suitable choices for the data of the $\omega$-Neumann boundary value problem to produce a linear system of equations for the values of $\omega$.

# NETWORK TOMOGRAPHY

by

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Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2006

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To my parents, my siblings, my wife Heffy and my son Nicolás.

# ACKNOWLEDGMENTS

First and foremost I'd like to thank my advisor, Carlos Berenstein for being both a wonderful advisor and a generous person as well as a paternal figure in my life. Carlos is a mathematician who has given so much to me and to the community. His amazingly broad vision for mathematics has changed positively my own, and inspired not only to me, but to many mathematicians and students from different backgrounds and cultures. I will never be able to repay for everything he has done for me in every aspect. Thanks to Professor John Baras for his enormous support in every step of this journey. Thanks to Professor Benjamin Kedem for his support and for the wonderful conversations we had. He is one of the best people I ever met. I learned a great deal of things from him. I would also like to thank Professor PS Krishnaprasad who made important remarks and crucial suggestions to my research. Special thanks to Professor S-Y. Chung for his academic backup while he was visiting University of Maryland and to Professor E. B. Curtis for the information provided. Thanks to Professor Udaya Shankar for accepting the role as the dean representative for my committee. Thanks to Professors Ricardo Nochetto, Ankur Srivastava and Jeffery Cooper for being members of my advisory committee. I want to particularly thank Professor Ricardo Nochetto for his constructive comments, suggestions, and corrections. Thanks to Haydee Hidalgo for making me feel a member of her family. I owe my deepest thanks to my family - my mother, father, and siblings- who have always backed me up and guided me through my life. I'd also like to express that this journey would not have been possible at all without the support and love from my wife Heffy and my son Nicolás. Thanks to my friends Onur Oktay, Gunay Dogan, Khamron Mekchay, Andy Kebo, Samvit Prakash, Aram Tangboondouangjit, Gustavo Rhode, and Juan Romero for their unconditional support and help, particularly to Gunay Dogan for their suggestions and lots of help. Special thanks again to Prof. Berenstein who spent a great deal of his time correcting the manuscript. I would like to acknowledge financial support from the Institute

for Systems Research (ISR) at University of Maryland through Carlos Berenstein and John Baras. Finally, I would like to thank everyone who somehow helped me along this journey.

# CONTENTS

# 1. INTRODUCTION

As networks, in particular, communication networks like internet have became an essential part of everyday life, disruptions may have very serious consequences. Therefore, the need to prevent, or, at least, detect such disruptions early on. The problem becomes one of obtaining information of the inner structure of a network from the collection of end to end measurements. This is analogous to tomographic problems that have been studied in the continuous setting and we tried to use this analogy as a source of inspiration under the supervision of Prof. Berenstein. The similarity to conventional tomography becomes closer when one examines traffic packets sent from the boundary of the network to check whether they ever reach the boundary again. When the packets do arrive at other boundary points, we could also keep track of how long it took for the information packets to reach a given boundary point. Another way of monitoring a network, such as the internet network, is done by using probes such as sending an empty message to check whether a given destination exists allowing us to gather information about the network status. The answer to the above problem is already known to be true on particular types of networks. For instance, Curtis and Morrow [28] show that in the case of a square resistor network (lattice), electrical impedance tomography ideas can in fact be effectively used in this context to determine the conductivity $\omega$ (weight) in the network from

the knowledge of the Neumann to Dirichlet map associated to $\omega$. They show that the conductivity $\omega$ can be uniquely determined and give an algorithm to compute $\omega$. They also show the continuity of the inverse. In the case of planar finite weighted graphs, Berenstein and Chung [9] more recently proved the uniqueness result, that is, any two weights $\omega_1$ and $\omega_2$ must coincide if the Neumann to Dirichlet map associated to $\omega_1$ is equal to the Neumann to Dirichlet map associated to $\omega_2$. Considering a related problem about finite networks, Y-C de Verdiere [49] gave a proof that the boundary input determines uniquely the traffic flow without using the analogy to the Neumann to Dirichlet problem. In this dissertation I go beyond the work of Berenstein, Morrow, Curtis, and others and find computationally effective methods to monitor specific connected subsets of arbitrary planar weighted graphs (*regions of interest*) from the input output map corresponding to paths that have crossed such regions and from this, to determine, for instance, congested areas or better yet anticipate areas that will get congested. This would allow the system manager to take measures to avoid the stoppage of traffic. For related continuous inverse problems, one observes that the vanishing condition of a wavelet corresponds exactly to the fact that any solution of the continuous Neumann problem has average zero by Green's theorem. Before we return to the problem at hand, we would like to point out another feature of the main result in [14], which I think it may prove very useful when studying large networks. It is the localization principle for the Radon transform in the continuous case. By that we mean the following: suppose one is only interested in determining the values of a function $f$ in a subregion $S_1$ of the whole region $S_0$ that one could $x$-ray. Then, one would only need to use $x$-rays

passing through $S_1$ to determine the values of $f$ in $S_1$ . (To be more precise, one needs $x$-rays passing through a set $S_1'$ slightly larger than $S_1$.) The key ingredient to prove this is feasible was the use of wavelets, mainly the fact that wavelets are functions with average zero. Returning to the corresponding network problem I note that for large networks the localization feature I mentioned above may prove to be very important, as it would allow us to monitor "only" the "region of interest" which is clearly more efficient than dealing with the whole network. The mathematical framework to study the problem mentioned above is that of weighted graphs. The assumption we use to study the network is that the underlying graph is known, which is very reasonable in this context and standard in the literature. In this discrete context we model the problem in the following way. Consider a finite connected graph $G = G(E, V)$, where $E$ denotes the set of links (edges) of $G$ and $V$ the set of nodes (vertices). Two nodes $x$ and $y$ are adjacent, written $x \backsim y$, if they are the endpoints of a link in $G$. We assume there is also a distinguished non-empty subset $\partial G$ of $G$, called the boundary of $G$ which represents the nodes that are accessible to us for the purpose of monitoring the traffic in the network. Furthermore, we assume that to every edge in $E$ there is an associated non-negative number $\omega(x, y)$ which corresponds to the traffic between the endpoints $x$ and $y$ of the edge. The degree $d_\omega x$ of a node $x$ in the weighted graph $G$ with weight $\omega$ is defined by $d_\omega x = \sum_{y \in V} \omega(x, y)$. The Laplacian operator corresponding to this weight $\omega$ is defined by $\Delta_\omega f(x) = \sum_{y \in V} [f(y) - f(x)] \cdot \frac{\omega(x,y)}{d_\omega x},\ x \in V$. A graph $S = S(V', E')$ is said to be a *subgraph* of $G(E, V)$ if $V' \subset V$ and $E' \subset E$. In this case, $G$ is called a *host graph* of $S$. The integration of a function $f : G \to \mathbb{R}$ on a graph $G = G(V, E)$

is defined by $\int_G f d_\omega = \sum_{x\in V} f(x) d_\omega x$. For a subgraph $S$ of a graph $G = G(V, E)$ the (node) *boundary* $\partial S$ of $S$ is defined to be the set of all nodes $z \in V$ not in $S$ but adjacent to some node in $S$, i.e., $\partial S = \{z \in V \mid z \notin S \text{ and } z \sim y \text{ for some } y \in S\}$ and, correspondingly, the *inner boundary* $\overset{\circ}{\partial} S$ is defined by $\overset{\circ}{\partial} S = \{z \in S \mid y \sim z \text{ for some } y \in \partial S\}$. Additionally, $\overline{S}$ denotes a graph whose nodes and edges are in $S \cup \partial S$. The (outward) normal derivative $\frac{\partial f}{\partial n_\omega}(z)$ at $z \in \partial S$ is defined to be $\frac{\partial f}{\partial\, n_\omega}(z) = \sum_{y\in S}[f(z) - f(y)] \cdot \frac{\omega(z,y)}{d'_\omega\, z}$, where $d'_\omega\, z = \sum\limits_{y\in S} \omega(z, y)$. In this model, there are two kinds of disruptions of traffic data that could arise. In one of them, disruptions occurs when an edge "ceases" to exist, in this case the "topology" of the graph has changed. This kind of disruption is outside the scope of the research proposed here. In the other, the weights change because of "increase" of traffic, that is, the network configuration remains the same but the weights have either increased or remained the same. In this second situation, we can appeal to the Berenstein and Chung's uniqueness theorem [9]. Namely, let $\omega_1$ and $\omega_2$ be weights with $\omega_1 \leq \omega_2$ on $\overline{S} \times \overline{S}$, and $f_1,\ f_2 : \overline{S} \to \mathbb{R}$ be functions satisfying for $j = 1, 2$,

$$\begin{cases} \Delta_{\omega_j} f_j(x) = 0,\ x \in S \\ \frac{\partial f_j}{\partial n_{\omega_j}}(z) = \psi(z),\ z \in \partial S \\ \int_S f_j d_{\omega_j} = K \end{cases}$$

for any given function $\psi : \partial S \to \mathbb{R}$ with $\int_{\partial S} \psi = 0$, and a given constant $K$ with $K > m_0$, where $m_0 = \max\limits_{j=1,2} |m_j| \cdot vol(S, w_j)$, $m_j = \min_{z\in\partial S} f_j(z), j = 1, 2$ and

$vol(S, w_j) = \sum_{x \in S} d_{\omega_j} x$. If it is assumed that

$$i)\, \omega_1(z, y) = \omega_2(z, y) \text{ on } \partial S \times \overset{\circ}{\partial} S,$$

$$ii)\, f_1|_{\partial S} = f_2|_{\partial S},$$

then

$$f_1 \equiv f_2,$$

and

$$\omega_1(x, y) = \omega_2(x, y),$$

for all $x$ and $y$ in $\overline{S}$. The condition that $\Delta_\omega f(x) = 0$ corresponds to the fact that the value $f(x)$ is the weighted average of the values of $f$ at the adjacent nodes. One concludes that by choosing a basis for the data one can distinguish between two weights. That is, one can decide whether there is an increase of traffic somewhere in the network or not. While this is only a uniqueness theorem, nevertheless, this leads to the possibility of effectively computing the actual weight from the knowledge of the Dirichlet data (output) for convenient choices of the Neumann data (input) in a way similar to that done in [26, 28] for lattices, it is equally related to the work of de Verdiere et al [49, 50]. Similarly, the Green function of this Neumann boundary value problem can be represented by an explicit matrix. Thus, it leads to the natural question, to be discussed below, of the effective determination of the weight $\omega$ in such a graph. What we want to discuss now is how to relate Berenstein and Chung's uniqueness theorem and similar mathematical results to the problem of understanding and monitoring a large network, may be even the internet. Munzner's work on the visualization of the internet [40] indicates that the natural geometric

domain to use might be the real hyperbolic space of dimensions two or three, the choice of the dimension being related to the density of the network. This observation makes it clear that there may be at least a formal relation with electrical impedance tomography, sometimes called the Neumann to Dirichlet problem, which is related to the hyperbolic Radon transform [5, 6].

To rephrase the above considerations, what we mean by monitoring a network in this mathematical context is the following: we are actually monitoring the amount of traffic $\omega$ (loads or weights) along the links during a period when the network is working and we are trying to take preventive measures in case the traffic along some of the links is getting close to overloading those links and causing a major disruption. Note that in this model, if a link becomes saturated, it is practically the same as saying the link does not exist any longer and the graph has changed. Of course, the point is then to have at hand measures to bypass this problem, but this is also outside the scope of my dissertation. Before proceeding let us note that I use the term network tomography really to indicate that I think about this problem in terms of a close analogy to the use of the Radon transform in a number of applications in the continuous case, for instance, that of CT scans, where CT stands for computerized tomography.

After describing the relation between the monitoring of a large network and tomography, here I study the question of how to go from the knowledge of the boundary data to indicate that the network is experiencing overloads along some links to the possibility of determining where exactly this congestion is taking place or may be close to take place. Moreover, it may become possible to monitor just a

portion of the network by using a significantly smaller amount of data. The natural question that arises here is to find the discrete analogue of wavelets. I consider that the combinatorial aspects of the work of de Verdiere [49] may become useful to provide a computable algorithm to reconstruct the weight of finite planar weighted graphs from the knowledge of the Neumann to Dirichlet map for a portion of the graph. At the end there is an appendix that contains a computer program that creates the Neumann-to-Dirichlet matrix for a 20 by 20 square weighted network and from the knowledge of this matrix it reconstructs the weight of the network.

## *1.1 Background and preliminaries*

Networks such as telephone networks, internet, and the like, have become ubiquitous in our society and thus it has become very important to avoid or at least, detect disruptions as soon as possible. For example, it is very important to prevent malicious intruders from disrupting a network. For this purpose, it is essential to count on a mathematical model that can allow early detection of attacks to the network. The mathematical tool that we consider to accomplish the early detection of disruptions is based on the use of tomographic ideas. One of the questions we consider is how to find out whether an attack against the network by traffic overload is taking place while monitoring traffic only at the periphery of the network (input-output map), and hence, we are lead to use a tomographic approach. We present first a general idea on how tomography can be used to do similar monitoring and implemented in its better known context, that is, in medical radiology.

A sketch of what a CT scanner does is the following. Consider two subsets $A$ and $\Omega$ of $\mathbb{R}^2$ with $A \subset \Omega$, $\Omega$ open and such that $\Omega \subset\subset \mathbb{R}^2$ and let $f$ be and integrable function such that $suppf \subseteq A$. The function $f$, for instance, could represent the density of tissue at every point of a planar cross section of human body, so that it is clear that $f$ has compact support. (In general, one could let $\Omega = \mathbb{R}^2$ and ask for $f$ to be rapidly decaying for $z$ approaching $\infty$.)

Let $\Phi$ be the collection of all the straight lines in $\Omega$ connecting any pair of points $a$ and $b$ where $a, b \in \partial\Omega$, $a \neq b$, i.e., $\Phi = \{\zeta \ line \ / a, b \in \zeta \cap \partial\Omega, \ a \neq b\}$, and $f(x)$ could represent the distribution of mass or density at the point $x \in \Omega$, which implies that $f \geq 0$ and we could assume $f$ is in $L^1(A)$. The Radon transform is defined by

$$R(f)(\xi) = \left\{ \int_{\xi} f(x)dx \right\}, \qquad \xi \in \Phi \tag{1.1}$$

that is, the set of all line integrals of the function $f$, where $\Phi$ represents the collection of all lines through $\Omega$.

Tomography then can be understood as the reconstruction of the function $f$ from the set of values given by $R(f)$. To recover $f$ is thus the same as finding the inverse of the operator $R$ . Thus, we would be able to know the value of $f$ at any point $x$ in $\Omega$ without having direct access to the interior points of $\Omega$.

A well-known example of tomography is transmission CT in diagnostic radiology, [44]. Essentially the setup consists of a detector and an X-rays beam source. A cross-section of the human body is scanned by a thin X-rays beam. Because the density of the tissue of the human body changes from its surface to its interior, there

is a intensity loss which is recorded by the detector and processed by a computer to produce a two-dimensional image which in turn is displayed on a screen. Given that the X-rays go through the tissue, it is clear the X-rays absorption is related to the attenuation coefficient. Let $f(x)$ be the X-ray attenuation coefficient of the tissue at the point $x$. Taking a close look, X-rays traversing a small distance $\Delta x$ at $x$ suffer a relative intensity loss,

$$\frac{\Delta I}{I} = -f(x)\Delta x. \tag{1.2}$$

If the X-rays are considered as straight lines, as indeed they essentially are, we let $\zeta$ be the straight line representing the beam, $I_o$ , the initial intensity of the beam, and $I_1$ its intensity after having traversed the body. It follows from (1.2) that

$$\frac{I_1}{I_o} = \exp\{-\int_\zeta f(x)dx\} \tag{1.3}$$

thus the scanning process provides us with the line integral of the function $f$ along each of the lines $\zeta$. From the knowledge of all of these integrals the problem is to reconstruct $f$. Equally well known by now is MRI, magnetic resonance tomography, where the underlying space is $\mathbb{R}^3$ and the integrals take place over the family of all planes in $\mathbb{R}^3$.

See [36] for other examples of imaging equipment based on tomographic principles and [48] as a recent overview of the kind of problems discussed here. We refer to [4] for more details on MRI.

Chapter 1 provides the background information on the Radon transform. In this chapter we also consider tomographic examples like the geodesic Radon transform in the hyperbolic plane, which appears naturally in relation to the inverse

conductivity problem and also internet tomography. In chapter 2 we provide some new results on this last subject. The key ingredient is the attempt to understand what happens in a network from "boundary measurements", that is, to determine whether all the nodes and routers are working or not and also measure congestion in the links between nodes by means of introducing test packets (ICMP packets) in the "external" nodes, the routers. The question of finding out whether there are nodes that are in working order is a classical question in graph theory. For networks, it is also interesting to try to predict future problems due to congestion. (Note that nodes could fail to work for other reasons than congestion on the links starting at a given edge.) This requires to monitor also traffic intensity, also known as load, congestion, etc., in different contexts. There is another analogy to mathematical tomography that arose independently and maybe closer to the consideration of this question in the context of electrical networks. Curtis and Morrow have done very interesting work in this context, both theoretical and in simulations. For instance, consider a square resistor network $\Gamma(\Omega_0, \Omega_1, \omega)$ consisting of $4n$ exterior or boundary nodes, where $\Omega_0$ is the set of the $n^2$ interior nodes, $\Omega_1$ the set of the $2n(n+1)$ edges connecting the nodes such that each boundary node is connected to exactly one node, and $\omega : \Omega_1 \to \mathbb{R}^+$ a function called the conductivity. For each edge $pq$ in $\Omega_1$, the number $\omega(pq)$ is the conductance of $pq$ and $\frac{1}{\omega(pq)}$ the resistance of $pq$. Consider Kirchoff's law

$$\sum_{q \sim p} \omega(pq)(u(p) - u(q)) = 0 \tag{1.4}$$

and Ohm's law

$$I(pq) = \omega(pq)(u(p) - u(q)) \tag{1.5}$$

where $u(p)$ is the voltage at node $p$, $q \sim p$ means that $q$ is to vary around the neighbors of $p$, and $I(pq)$ the current passing through edge $pq$, then by setting up a Kirchoff's law equation for each *interior* node and an Ohm's law equation for each *exterior* node we have the matrix equation

$$Au = b \tag{1.6}$$

where the set $\{u(p)\}$ for $p$ an exterior or boundary node is used for $b$. Let $\partial\Omega$ be the set of boundary nodes, then for any given current $\Phi$ on $\partial\Omega$, there is a voltage $v$ on $\partial\Omega$. This defines a map $N_\Omega$ called the Neumann-to-Dirichlet map that takes currents on $\partial\Omega$ and gives voltages on $\partial\Omega$ and is represented by the so called Neumann matrix $N$. To create the $N$ matrix, the basis chosen is the set $\{\Phi_j\}$, $j = 1, ..., 4n$ where $\Phi_j$ is the current that represents 1 amp entering the boundary node $j$ and 1 amp coming out at boundary node $j + 1$ with no current in or out the other boundary nodes. For each data $\Phi_j$ the equation $Au = b$ is solved and this gives the $j^{th}$ column of $N$ as follows: The boundary nodes $i$ are labeled in a clockwise direction, starting in the left-most node of the north side. The matrix $N = \{N_{i,j}\}$ is a $4n$ by $4n$ matrix where $N_{i,j}$ is the voltage difference between node $i$ and node $i + 1$ due to $\Phi_j$.

The inverse problem consists of calculating the values of $\omega(pq)$ in the network from the knowledge of the matrix $N$, *i.e.*, from the knowledge of current to voltage boundary information. We begin by finding the values of $\omega(pq)$ at the corners of the network. A zero current is placed on $\partial\Omega$ except the $i^{th}$ and $(i + 1)^{th}$ nodes

which must be around the corner. A current of 1 is sent in at the $i^{th}$ node and $-1$ at the $(i+1)^{th}$ node. A voltage equal to 0 is set on the face opposite the $(i+1)^{th}$ node. This implies that the two values of the conductances at the corner are obtained by knowing the values of the voltages at the $i^{th}$ and $(i+1)^{th}$ boundary nodes and this is obtained directly from the matrix $N$. For instance if $i = 2$, the voltages for the $i^{th}$ and $(i+1)^{th}$ nodes are read from the second column of $N$. In fact, $v_2 = -N_{1,2}$ and $v_3 = -N_{3,2}$. where $v_i$ represents the voltage at node $i$ To calculate the four interior values of $\omega$ for the respective corner, the structure of the $N$ matrix is used. In conclusion, given the current to voltage boundary information, solving the inverse problem determines the $2n(n+1)$ values of $\omega$. For more details see [28] and [27]. Another analogy in the same direction arises when we consider very large networks, as the internet, which could be considered as the discretization of an underlying continuous model. In this way, we can see the analogy with the well-known inverse conductivity problem and we could try to profit from the large body of mathematical research in this area. The analogy with this particular inverse problem indicates that if one were to pursue this "abstract" approach the "correct" geometry is closer to be hyperbolic than to be Euclidean [6]. On the other hand, as of this moment, we have found that those tomographic analogies are more useful for providing directions of research and methods to consider these problems than providing an exact correspondence between the two phenomena. It is in this context that [9] modelled "internet tomography" as an inverse Neumann-to-Dirichlet problem for a graph with weights. In this situation, one can prove that characteristics of the graph, namely, its connectivity and the traffic along links can

be uniquely determined by boundary-value measurements as shown in [9] which is the natural analogue of the continuous inverse conductivity problem.

Among the questions that arise naturally using the inverse conductivity problem as a guiding model there are a number of questions that have been previously addressed using other points of view. Namely, the problems already addressed in [24] for internet tomography are:

1. Link-level inference, in other words, link-level parameter estimation based on end-to-end path-level traffic measurements. Examples of this are unicast inference of link loss rates, unicast inference of link delay distributions, topology identification, loss rates by using multicast probing and so on.

2. Path-level inference (origin-destination tomography OD) in other words sender-receiver path-level traffic intensity estimation based on link-level traffic measurements. One example of this is time-varying OD traffic matrix estimation.

## *1.2 The Radon transform in* $\mathbb{R}^2$

Let $\omega \in S^1$, then $\omega = (\cos\theta, \sin\theta)$, and take $p \in \mathbb{R}$. The locus of the equation $x \cdot \omega = p$ represents the line $l$ that is perpendicular to the line $r$ passing through the origin and forming an angle $\theta$ with the real line $\mathbb{R}$. If $B$ is the intersection of $l$ and $r$, the signed Euclidean distance $d$ from $B = p\omega$ to the origin is equal to $p$.

Consider a "nice" function $f$ defined on $\mathbb{R}^2$, for instance $f$ in $\mathcal{C}^\infty$ and compactly supported, then consider the line integral with respect to the Euclidean arc length

$ds$,

$$Rf(\omega, p) := \int\limits_{x\cdot\omega=p} f(x)ds = \int\limits_{-\infty}^{\infty} f(x_o + t\omega^{\perp})dt, \tag{1.7}$$

where $x_o$ is a fixed point in $l$, i.e. it satisfies $x_o \cdot \omega = p$, and $\omega^{\perp} = (\cos\theta, -\sin\theta)$, the rotation of $\omega$ by $\pi/2$. When $p$ and $\omega$ range over $\mathbb{R}$ and $S^1$ respectively, we get all of the lines in $\mathbb{R}^2$. Usually $x_o$ is taken as $p\omega$.

The map $f \rightarrow Rf$ is called the Radon transform in $\mathbb{R}^2$ and $Rf$ is called the *Radon transform of* $f$. We refer to [44] and references therein, for a detailed exposition of the Radon transform. Clearly $Rf$ is a function defined on $S^1 \times \mathbb{R}$, i.e., the family of all lines in $\mathbb{R}^2$ with the compatibility condition mentioned in [8]:

$$(Rf)(-\omega, -p) = Rf(\omega, p), \tag{1.8}$$

Given that $l$ doesn't change when $\omega$ and $p$ are changed to $\lambda\omega$ and $\lambda p$, $\lambda \neq 0$, $\lambda \in \mathbb{R}$, then the Radon transform can be extended from $S^1 \times \mathbb{R}$ to $\mathbb{R}^2 \times \mathbb{R}$. The pair ($\lambda\omega$,$\lambda p$) is identified with ($\omega$,$p$), and the extension of the Radon transform satisfies

$$R\mu(\lambda\omega, \lambda p) = R\mu(\omega, p), \tag{1.9}$$

therefore, the Radon transform can be extended as a homogeneous function of degree -1 on the spatial variable $\lambda$. Namely,

$$Rf(\lambda, p) = \frac{1}{|\lambda|} Rf(\frac{\lambda}{|\lambda|}, p)$$

, a very important property of this transform. $Rf$ can also be defined as

$$Rf(\omega, p) := \int_{\mathbb{R}^2} f(x)\delta(p - \omega \cdot x)dx, \tag{1.10}$$

with $\delta$ the 1-dimensional delta function, which allows us to obtain easily the properties below in the natural coordinates for the space of lines. In particular for $\omega = (\omega_1, \omega_2)$ with $|\omega| = 1$, then

$$R(\frac{\partial f(\omega, p)}{\partial x_i}) = \omega_i \ \frac{\partial Rf(\omega, p)}{\partial p}, \tag{1.11}$$

and

$$\frac{\partial Rf(\omega, p)}{\partial \omega_i} = -\frac{\partial}{\partial p} R(x_i \ f)(\omega, p) \tag{1.12}$$

hence, it follows that if $P_m(x)$ is a homogeneous polynomial with constant coefficients, degree $m$, and $|\omega| = 1$,

$$R(P_m(\partial x) \ f(\omega, p)) = P_m(\omega) \cdot \frac{\partial^m Rf(\omega, p)}{\partial p^m}, \tag{1.13}$$

and

$$P_m(\partial_\omega)(Rf(\omega, p)) = (-1)^m \frac{\partial^m}{\partial p^m} R(P_m(x) \ f(\omega, p)) \tag{1.14}$$

where

$$\partial_x = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}), \qquad \partial_\omega = (\frac{\partial}{\partial \omega_1}, \frac{\partial}{\partial \omega_2}) \tag{1.15}$$

and obviously $x = (x_1, x_2)$.

One can similarly define the Radon transform in $\mathbb{R}^n$ and verify that the properties (1.11) and (1.13) extend to this case. In particular for the Laplacian $\Delta$ in $\mathbb{R}^n$,

$$R(\Delta f)(\omega, p) = \frac{\partial^2 Rf(\omega, p)}{\partial p^2}, \tag{1.16}$$

where, for each direction $\omega \in S^{n-1}$ the right hand side is the Laplace operator in dimension 1. Note that in the general case of $\omega \in \mathbb{R}^n \setminus \{0\}$,

$$R(\Delta f)(\omega, p) = (\omega_1^2 + \cdots + \omega_n^2) \frac{\partial^2 Rf(\omega, p)}{\partial p^2} \tag{1.17}$$

As a consequence, if the function $f$ depends also on time, and $\Box_n$ represents the wave operator in $n$ dimensions we conclude that

$$R\,\Box_n f = \Box_1 Rf. \tag{1.18}$$

Therefore, the Radon transform in $n$ dimensions is localizable if and only if the wave equation is localizable. Fixing $\omega \in S^{n-1}$, one can express this identity by saying that the Radon transform interwines the wave operator $\Box_n = \Delta - \frac{\partial^2}{\partial t^2}$ in $n-$dimensions with the wave operator $\Box_1 = \frac{\partial^2}{\partial p^2} - \frac{\partial^2}{\partial t^2}$ in $1-$space dimension. It follows that the Radon transform cannot be localized in even dimensions [10]. In spite of this observation one can obtain an *almost* localization of the Radon transform in $\mathbb{R}^2$. The key elements is the use of wavelets as it will be described in the next section. Meanwhile, for the sake of completeness we remind the reader of the standard inversion formula for the Radon transform in $\mathbb{R}^2$. It depends on the following identity, usually called the Fourier slice theorem. Namely, writing the Fourier transform $F_2(f)$ of a nice function $f$ in $\mathbb{R}^2$ in polar coordinates $(s, \omega)$ we have

$$\int_{\mathbb{R}^2} f(x)\, e^{-is\omega\cdot x}\, dx = \int_{-\infty}^{\infty} Rf(\omega, p)\, e^{-isp}\, dp, \quad x \in \mathbb{R}^2 \tag{1.19}$$

or, in a more concise form,

$$F_2(f) = F_1(Rf) \tag{1.20}$$

where $F_2$ stands for the 2-dimensional Fourier transform and $F_1$ stands for the 1-dimensional Fourier transform in the variable $p$ which provides one of the standard inversion formulae for the Radon transform in $\mathbb{R}^2$

$$f = F_2^{-1} F_1(Rf) \tag{1.21}$$

There is another inversion formula that has a number of advantages for us, and we proceed to explain it now. To simplify, we work in $X = \mathbb{S}(\mathbb{R}^2)$, the Schwartz space of functions $f$ and $Y = \mathbb{S}(S^1 \times \mathbb{R})$ the Schwartz space of functions $g$. Let $f_1,\ f_2 \in X$ and $g_1, g_2 \in Y$, and $\langle f_1,\ f_2 \rangle_X$ , $\langle g_1,\ g_2 \rangle_Y$ the $L^2$-inner products in $X$ and $Y$ respectively, then because of the linearity of the operator $R$, we write the equation that defines $R^*$, the adjoint operator of $R$

$$\langle Rf, g \rangle_Y = \langle f, R^* g \rangle_X \tag{1.22}$$

The explicit expression for $R^* g$ is given by

$$\int_{S^1} g(\omega, \omega \cdot x) d\omega = R^* g \tag{1.23}$$

$R^*$ is called the *backprojection operator*. The function $R^* g$ is such that for $x$ fixed

$$R^* g(x) = \int_{S^1} g(\omega, \omega \cdot x) d\omega \tag{1.24}$$

is the integral of $g$ over all lines passing through $x$.

In order to get a formula for $f$ from the Radon transform values, one uses the following important property of the backprojection operator.

$$(R^* g) * f = R^*(g \circledast Rf) \tag{1.25}$$

where $\circledast$ stands for the convolution with respect to the second argument (1-dimensional), and the Radon inversion formula is then given by

$$\int_{\mathbb{R}^2} \frac{|\varsigma|}{2} F_2(R^* Rf)(\varsigma) e^{i2\pi x \cdot \varsigma} d\varsigma = f(x), \tag{1.26}$$

Introducing $\Lambda$, the square root of the Laplacian operator $\Delta$, we have

$$\Lambda(R^* Rf)(x) = f(x) \tag{1.27}$$

which is usually called *the backprojection inversion formula*.

## *1.3 Localization of the Radon transform*

As explained above, we cannot in general reconstruct the function $f$ in a disk $D(a,r)$ of $\mathbb{R}^2$ using only lines $l$ passing through $D(a,r)$. One can localize $f$ up to a baseline value of the function $f$, that is, one can recover $f$ on a disk $D(a,r)$ by using only the data $Rf(l)$ for passing through $D(a,r+\varepsilon)$, for arbitrary $\varepsilon > 0$, up to an additive constant [12] and [13] . The key element is the use of wavelets.

In what follows for $f \in L_1(\mathbb{R})$ (or $f \in S(\mathbb{R})$ ) we denote

$$\widetilde{f}(\varsigma) = \int_{-\infty}^{\infty} f(x)\ e^{-\ i2\pi\varsigma x} dx, \tag{1.28}$$

the usual Fourier transform of $f$ in $\mathbb{R}$. Let us recall the basic properties of the continuous wavelet transform (CWT) and the discrete wavelet transform (DWT). Let $b \in \mathbb{R}$ be and $f_b$ the translation of $f$ by $b$, i.e. $f_b(x) = f(x-b)$, then

$$\widetilde{f}_b(\varsigma) = e^{-i2\pi b\varsigma}\widetilde{f}(\varsigma) \tag{1.29}$$

Now let $D_a f$ be the dilation of $f$ by the scaling factor $a \in \mathbb{R}$, $a > 0$ where $D_a f$ is defined as $D_a f(x) = \frac{1}{\sqrt{a}} f(\frac{x}{a})$ where the term $\frac{1}{\sqrt{a}}$ is chosen such that $\|f\|_2 = \|D_a f\|_2$ , i.e., $f$ and $D_a f$ have the same energy, then one has

$$\widetilde{(D_a f)}\,(\varsigma) = D_{1/a}\widetilde{f}(\varsigma) = \sqrt{a}\widetilde{f}(a\varsigma), \tag{1.30}$$

As pointed out in [39], equation (1.30) tells us that the Fourier transform $\widetilde{(D_a f)}\,(\varsigma)$ is dilated by $1/a$ , then we lose in the $\varsigma-$domain (frequency) what we

gained in the $x-$domain (time). In other words, there is a trade-off between time and frequency localization if $\varsigma$ and $x$ stand for frequency and time respectively.

### *1.3.1 Wavelets as a tool*

Let us recall from [39] the definition of the continuous wavelet transform (CWT) associated to a "mother" wavelet $\Psi$. Namely, following [8], given a "mother" wavelet $\Psi \in L_2(\mathbb{R}) \cap L_1(\mathbb{R})$ and $f \in L_2(\mathbb{R})$, we define the wavelet transform of $f$ as

$$W_\Psi f(a,b) := \int_{-\infty}^{\infty} f(t)\ \overline{\Psi}(\frac{t-b}{a})\frac{dt}{\sqrt{a}} =< f, D_a\Psi_b(t) >_{L_2} \tag{1.31}$$

$b, a \in \mathbb{R}$, $a > 0$, where, for a function $g$ and $b \in \mathbb{R}$ we let $g_b(t) = g(t-b)$. One requires that the "mother" wavelet $\Psi$ be oscillatory, i.e. $\int_{-\infty}^{\infty} \Psi(x)dx = 0$. In fact, one assumes the stronger condition

$$C_\Psi = \int_{-\infty}^{\infty} \frac{\left|\widetilde{\Psi}(\varsigma)\right|^2}{|\varsigma|} d\varsigma < \infty, \tag{1.32}$$

called the *admissibility condition*. The admissibility condition is satisfied when $\Psi$ has several vanishing moments,i.e., for $0 \leq k < s$

$$\int_{-\infty}^{\infty} x^k\ \Psi(x)dx = 0 \tag{1.33}$$

the functions $D_a\Psi_b$ are called the wavelets

The function $f$ can be reconstructed from its wavelet transform by means of the "*resolution of the identity*" formula

$$f = C_\Psi^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} < f, D_a\Psi_b(t) >_{L_2}\ D_a\Psi_b(t)dt \tag{1.34}$$

where the constant $C_\Psi < \infty$ since $\Psi \in L_1(\mathbb{R})$. We refer to [39] for the general theory of wavelets.

The following proposition explains how to use wavelets to obtain (almost) localization of the Radon transform in $\mathbb{R}^2$.

**Proposition 1.1.** *[10] Let $n$ be an even integer, and $h \in L_2(\mathbb{R})$ a function with compact support such that for some integer $m \geq 0$ $\overset{\sim}{h}$ is n+m-1 times differentiable and satisfies*

1. $\gamma^{\,j}\overset{\sim}{h}^{(k)}(\gamma) \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ *for* $0 \leq j \leq m$, $0 \leq k \leq m+n-1$
2. $\int_{-\infty}^{\infty} t^{\,j} h(t)dt = 0$ *for* $0 \leq j < m+1$, *i.e., $h$ has $m+1$ vanishing moments*

*Then*

$$I^{\,1-n}h(t) = o(|t|^{-n-m+1}) \quad as \ |t| \longmapsto \infty$$

*and*

$$t^{\,n+m-1} I^{1-n} h \in L_2(\mathbb{R}) where$$

$$I is the identity function$$

where $I$ is the identity function. The fact that $I^{\,1-n}h(t) = o(|t|^{-n-m+1})$ as $|t| \longmapsto \infty$ tells us that $I^{\,1-n}h$ decays as $|t|^{-(n+m-1)}$, and therefore, it does a good localization job.

For practical purposes, the continuous wavelet transform, CWT, is discretized and the discrete wavelet transform, DWT, is obtained. In order to discretize it, consider $m, n \in Z$ and the values $a, b$ that appear in $W_\Psi f(a,b)$ are restricted to only discrete values $a = a_o^m$, $b = nb_o a_o^m$, $a_o > 1$, $b_o > 1$ fixed. (The fact that $a_o > 1$, $b_o > 1$ really does not matter because $m, n$ can be negative). The discrete wavelet transform DWT of $f$ is defined as

$$W_{m,n}^{\Psi}(f) = a_o^{-m/2} \int_{-\infty}^{\infty} f(t)\overline{\Psi}(a_o^{-m}t - nb_o), \tag{1.35}$$

where, as before, it holds that $\int_{-\infty}^{\infty} \Psi(t) = 0$, and the wavelets are given by

$$\Psi_{m,n}(x) = a_o^{-m/2}\Psi(a_o^{-m}x - nb_o) = a_o^{-m/2}\Psi(a_o^{-m}(x - nb_o a_o^m) \tag{1.36}$$

hence $\Psi_{m,n}$ is localized around $nb_o a_o^m$ in time, (1.35) can be also expressed as $\langle f, \Psi_{m,n}\rangle$ which are called the wavelet coefficients.

It is important to point out that in the discrete case, in general, there does not exist a *resolution of the identity* formula to recover $f$, so the recovering of $f$ must be done by using some other means, for instance numerical methods. The choice of the wavelet $\Psi$ is essentially only restricted by the requirement that the admissibility condition holds, i.e., $C_\Psi = \int_{-\infty}^{\infty} \frac{\left|\widetilde{\Psi}(\varsigma)\right|^2}{|\varsigma|} d\varsigma$ is finite. Following [29], the discretization is only restricted to positive values of $a$ then the admissibility condition becomes

$$C_\Psi = \int_0^{\infty} \frac{\left|\widetilde{\Psi}(\varsigma)\right|^2}{|\varsigma|} d\varsigma = \int_{-\infty}^{0} \frac{\left|\widetilde{\Psi}(\varsigma)\right|^2}{|\varsigma|} d\varsigma < \infty \tag{1.37}$$

and since $a, b$ will take discrete values only, then the dilation parameter is chosen as $a_o^m$, $m \in Z$ and $a_o \neq 1$ is fixed (usually $a_o > 1$). The value $b_o$ is also fixed and it is chosen such that the union of the supports of the functions $\Psi(x - n\ b_o)$ covers the whole line. Now, for reasonable $\Psi$ and suitable $a_o, b_o$, there exist $\Psi_{m,n}$ so that the discrete wavelet coefficients $\langle\ f, \Psi_{m,n}\rangle$ characterize completely $f$ which is given by

$$f = \sum_{m,n} \langle\ f, \Psi_{m,n}\rangle\ \Psi_{m,n} \tag{1.38}$$

then any function in $L_2(\mathbb{R})$ can be written as a superposition of the wavelets $\Psi_{m,n}$.

### *1.3.2 Wavelets and the Radon transform*

Now we want to state some results that relate wavelets and the Radon transform, which are of interest for tomography, [12], [13].

**Proposition 1.2.** *Let $\rho \in L_2(\mathbb{R})$ real valued, even, and satisfying*

$$\int_{-\infty}^{\infty} \frac{\left|\widetilde{\rho}(r)\right|^2}{r^3} dr < \infty \tag{1.39}$$

*where $\widetilde{\rho}$ stands for the 1-dimensional Fourier transform of $\rho$. Define the radial function $\Psi$ in $\mathbb{R}^2$ by*

$$F_2\Psi(\varsigma) = \frac{\widetilde{\rho}(|\varsigma|)}{|\varsigma|}$$

*where as before, $F_2$ is the 2-dimensional Fourier transform, then $\Psi$ is a wavelet for n=2 and the wavelet transform of $f$ is such that*

$$W_\Psi f(a,b) = a^{-1/2} \int_{S^1} (W_\rho R_\omega\ f)(a, b\omega) d\omega$$

*where $R_\omega\ f$ is such that $R_\omega\ f(p) = Rf(w,p)$.*

**Proposition 1.3.** *Let $\Psi$ be a separable 2-dimensional wavelet, i.e.,*

$$\Psi(x) = \Psi^1(x_1)\Psi^2(x_2), \quad x \equiv (x_1, x_2)$$

*where for i=1,2* $\left|\widetilde{\Psi^i}(\gamma)\right| \leq C_1(1+|\gamma|)^{-1}$ *for all $\gamma \in \mathbb{R}$. Defining the family of the one-dimensional functions $\{\rho_\omega\}_{\omega \in S^1}$ by*

$$\widetilde{\rho_\omega}(\gamma) = \frac{1}{2}|\gamma|\ \widetilde{\Psi^1}(\gamma\omega_1)\widetilde{\Psi^2}(\gamma\omega_2), \quad \omega = (\omega_1, \omega_2)$$

*i.e., $\rho_\omega = F_1^{-1}(\widetilde{\rho_\omega})$. Then for every $f \in L_1(R^2) \cap L_2(R^2)$,*

$$(W_\Psi f)(a,x) = a^{-1/2} \int_{S^1} \left(W_{\rho_\omega}\ R_\omega f\right)(a, x\cdot\omega) d\omega$$

The proposition shows that the wavelet transform of a function $f(x)$ for any mother wavelet and at any scale can be obtained by backprojecting the wavelet transform of the Radon transform of $f$ using wavelets that vary with each angle, the argument of $\omega$, but which are admissible for each angle, i.e., $C_\Psi < \infty$.

### *1.3.3 The Radon transform on homogeneous trees*

Let us now remind the reader what do we mean by a tree $T$. A tree $T$ is a finite or countable collection $V$ of vertices $\{v_j,\ j = 0, 1, ....\}$ and a collection $E$ of line segments connecting two vertices $v_j$ and $v_k$) called edges and denoted by $e_{jk} = (v_j, v_k)$ where the number of edges is equal to the number of vertices minus one. We orient the edge $e_{jk}$ by thinking that $v_j$ is the first node and $v_k$ the second node of the edge. We always include the edges $e_{kj}$ in this collection, which have the reverse orientation. Given two vertices $u$ and $v$, we say they are neighbors if $(u, v)$ is an edge and write $u \backsim v$ in this case. A geodesic $\gamma$ from $u_0$ to $u_l$ is a collection $u_0\ , u_1\ , ....., u_{l-1}\ , u_l$ of pairwise distinct vertices such that $u_0 \backsim u_1$ , $u_1 \backsim u_2\ , ...., u_{l-1} \backsim u_l$ . If it turns out that $u_0 \backsim u_l$ then we consider the closed geodesic path $\bar{\gamma}$ by adding the edge $(u_l\ , u_0\ )$ to $\gamma$. Closed geodesics are also known as cycles, hence one can say that a tree is a connected graph without cycles. To simplify the notation, for any geodesic $\gamma = u_0 \backsim u_1 \backsim u_1 \backsim u_2 \backsim .... \backsim u_{l-1} \backsim u_l$ , we denote by $-\gamma$ the geodesic with the opposite orientation , i.e., -$\gamma = u_l \backsim u_{l-1} \backsim .... \backsim u_0$ . The collection of all (open) geodesics is denoted by $\Gamma$. If $T$ is infinite, then a complex valued function is defined to be in $L^1(T)$ if

$$\sum_{v \in V} |f(v)| < \infty \tag{1.40}$$

The Radon transform $R$ of a function $f \in L^1(T)$ is simply the bounded function $Rf$ on $\Gamma$ defined by

$$Rf(\gamma) = \sum_{v \in \gamma} f(v) \tag{1.41}$$

Given a node $v$ we denote by $\nu(v)$ the number of edges that contain $v$ as an endpoint. This number is sometimes called the degree of the node. We will assume throughout that we always have $\nu(v) \geq 3$ to ensure that the Radon transform is injective. (In our applications this is only needed for nodes $v$ that lie in $\text{supp}(f)$. In the terminology of [7] we are assuming there are neither black holes nor flat points in $T$. Under these conditions, the Radon transform in a tree is invertible. In fact, the explicit inversion formula resembles that of the inversion for the Radon transform in the Euclidean plane [8, 10, 12, 32]. Unfortunately, even in this case, we need to introduce a significant amount of auxiliary notation. For the purpose of illustration we describe the inversion formula here only for the case of homogeneous trees and we refer to [7] for the general case.

### *1.3.4 Inversion of the Radon transform on homogeneous trees*

Consider a homogeneous tree $T$ in which each vertex touches $q+1$ edges with $q \geq 2$. If $n$ is a nonnegative integer, let $v_{(n)}$ be the number of vertices of $T$ at distance $n$ from a fixed vertex of $T$. It follows that

$$v(n) = \begin{cases} 1 & if \quad n = 0 \\ (q+1)q^{n-1} & if \quad n \geq 1 \end{cases} \tag{1.42}$$

We give the following definitions. Let $v, w$ be two vertices in $T$ that are connected by a path $(v = v_0, ...., v_m = w)$, then the *distance* between $v$ and $w$ is the nonnegative integer $|v, w| = m$. Also, for $f \in L^1(T)$, let $\mu_n$ be the average operator defined by

$$\mu_n f(v) = \frac{1}{v(n)} \sum_{|v,w|=n} f(w), \quad \text{for } v \in T \tag{1.43}$$

It can be seen that $\mu_n$ is basically a convolution with radial kernel

$$h_n(v,w) = \begin{cases} \frac{1}{v(n)} & if \ \ |v,w| = n \\ 0 & if \ \ |v,w| \neq n \end{cases} \tag{1.44}$$

Let $\beta = q/(2(q+1))$ and $R^*$ be the dual Radon transform defined for $\Phi \in L^\infty(\Gamma)$ by

$$R^*\Phi(v) = \int_{\Gamma_v} \Phi(\gamma) d\rho_v(\gamma) \tag{1.45}$$

for each vertex $v \in T$, with respect to a suitable family $\{\rho_v : v \in T\}$ of measures on $\Gamma_v$, where $\Gamma_v$ is the set of all of the geodesics containing the vertex $v$. In order to obtain the inversion of $R$ we observe that $R^*R$ acts as a convolution operator given by the radial kernel $h = \beta h_0 + \sum_{n=1}^{\infty} 2\beta h_v$. The identity

$$R^*R = \beta\mu_0 + \sum_{n=1}^{\infty} 2\beta\mu_n \ \text{ on } L^1(T),$$

holds in $L^1(T)$, where the series is absolutely convergent in the convolution operator norm on $L^2(T)$, thus providing a bounded extension of $R^*R$ to $L^2(T)$ The unique bounded extension to $L^2(T)$ of the operator $R^*R$ is invertible on $L^2(T)$, and its inverse is the operator

$$E = \frac{2(q+1)^3}{q(q-1)^2} \left[ \ \mu_0 + \sum_{n=1}^{\infty} (-1)^n 2\mu_n \ \right] \tag{1.46}$$

which acts as the convolution operator with the radial kernel

$$\frac{2(q+1)^3}{q(q-1)^2}[h_0 + \sum_{n=1}^{\infty}(-1)^n 2h_n].$$

As above, this series converges absolutely in the convolution operator norm on $L^2(T)$; in particular, $E$ is bounded.

**Corollary 1.4.** *The Radon transform* $R : \mathcal{L}^1(T) \to L^\infty(\Gamma)$ *is inverted by*

$$ER^*Rf = f. \tag{1.47}$$

## *1.4 The hyperbolic Radon transform and EIT*

In this section we discuss the Radon transform on the hyperbolic plane, state some formulae analogous to the ones that were given in section 1.2 to invert the Radon transform. The backprojection inversion formula is one of them, and later we will see how the hyperbolic Radon transform is related to electric impedance tomography (EIT).

In [5] and [6] it is shown that the hyperbolic Radon transform is involved in the problem of reconstructing the conductivity distribution of a plate by using electrical impedance tomography (EIT).

### *1.4.1 The hyperbolic Radon transform*

Let D be the unit disk of the complex plane, i.e., $D = \{z \in \mathbb{C} : |z| < 1\}$. In $D$, a Riemannian structure is defined through the hyperbolic metric of arc-length $ds$ given by

$$ds = \frac{2\,|dz|}{(1 - |z|^2)}, \tag{1.48}$$

with $dz$ the Euclidean distance in $\mathbb{R}^2$, and the hyperbolic distance between two points $z, \omega \in D$ is given by

$$d(z, \omega) = \operatorname{arcsin} h \left( \frac{|z - \omega|}{(1 - |z|^2)^{1/2}(1 - |\omega|^2)^{1/2}} \right) \tag{1.49}$$

The set of lines that are diameters of $D$, and the set of intersections between the Euclidean circles and $D$ such that the resultant lines (intersections) are perpendicular to the boundary $\partial D$ of $D$ are the geodesics or h-lines for the metric (1.48). If $z \in D$ is expressed in polar coordinates by $(\omega, r)$ where $\omega = z/|z|$, $r = d(z, 0)$, then the metric (4.1) becomes

$$ds^2 = dr^2 + \sinh^2 r \; d\omega^2 \tag{1.50}$$

where $\dot{\varphi}^2$ is the usual metric on $\partial D$, i.e., arc lenght. This indicates that the area in hyperbolic geometry is exponential on the radius $r$. Let us recall that if $E$ is a set contained in the hyperbolic disk $D$, then the hyperbolic area of $E$, $h - area(E)$, is given by

$$h - area(E) = \iint_E dxdy \frac{4}{[(1-|\ z|^2)]^2} dxdy, \quad z = (x, y)$$

and the hyperbolic length of any curve $\gamma$ in $D$, $h - length(\gamma)$, is given by

$$h - length(\gamma) = \int_\gamma \frac{2|dz|}{1-|\ z|^2}$$

In terms of the Euclidean Laplacian $\Delta$, the Laplace-Beltrami operator $\Delta_H$ in polar coordinates on $D$ can be expressed as

$$\Delta_H = \frac{\left(1 - |z|^2\right)^2}{4} \Delta = \frac{\partial^2}{\partial r^2} + \coth r \frac{\partial}{\partial r} + \sinh^{-1} r \frac{\partial^2}{\partial \omega^2}, \tag{1.51}$$

and in the Euclidean coordinates, $z = (x, y)$, (1.51) becomes

$$\Delta_H = \frac{(1 - x^2 - y^2)^2}{4} (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}).$$

Following [8], we denote the Radon transform of a function $f$ within the hyperbolic plane by $R_H f$ which is a function on the family of geodesics in $D$. It is defined as

follows,

$$Rf(\gamma) = R_H f(\gamma) = \int_\gamma f(z) ds(z), \quad \gamma \text{ geodesic in } D, \tag{1.52}$$

where $f$ is a function such that (1.52) makes sense. For instance, if $f$ is a function with compact support, i.e., $f \in C_o(D)$ or $f \in \mathbb{S}(D)$, the Schwartz space, which is the space of functions rapidly decaying as $|z| \to 1$. In fact, $f$ has to decay a bit faster than $e^{-r}$ because the element of length $ds(z)$ grows like $e^r$, since $ds^2 = dr^2 + \sinh^2 r \, d\omega^2$, $ds$ grows as $e^r$. If $\Gamma$ is the space of all the geodesics in $D$, then the dual (adjoint) transform $R^*$ (backprojection operator) is given by

$$R^*\Phi(z) = \int_{\Gamma_z} \Phi(\gamma) d\mu_z(\gamma)$$

where $\Gamma_z$ is the set of all the geodesics passing through $z$ and $d\mu_z$ is the normalized measure of $\Gamma_z$ .

Any geodesic passing through the point $z \in D$ depends only on one of the end points. (The other end point determines the same geodesic through $z$.) Therefore, $\Gamma_z$ is completely determined by $\partial D = S^1$ hence $\Gamma_z$ can be parametrized by $\omega = e^{i\theta} \in S^1$ and $d\mu_z$ is naturally associated to $\frac{1}{2\pi} d\theta$.

Having introduced this parametrization, the purpose now is to invert the operator $R_H$. In order to invert $R_H$ one can proceed in the spirit of the classical Radon's inversion formula, see [32]. Following [10], we try to find a filtered backprojection type formula similar to (1.26). Recall that if $k \in L^1_{loc}([0,\infty))$ we can associate a radial kernel such that for $f \in C_o(D)$, the convolution operator with respect to this radial kernel $k$ is defined as

$$k * f(z) = k *_H f(z) = \int_D f(w) k(d(z,w)) dm(w) \tag{1.53}$$

where $dm(w)$ is the measure for the hyperbolic area which in polar coordinates is given by

$$dm = (\sinh r) dr d\theta \tag{1.54}$$

Recall the formula for $R^*R$ in $\mathbb{R}^2$ can be written as

$$R^*Rf = \frac{2}{|x|} * f. \tag{1.55}$$

The analogous result for the hyperbolic Radon transform $R_H$ is given by

$$R_H^* R_H f = k * f, \tag{1.56}$$

where

$$k(t) = \frac{1}{\pi \sinh t},$$

In [6] and [8] it is shown that by letting f $S(t) = \coth t - 1$, one obtains

$$\frac{1}{4\pi} \Delta_H \; S *_H R_H^* R_H = I, \tag{1.57}$$

the analogue to the backprojection inversion formula given before.

The Fourier transform in the hyperbolic disk $D$ for a radial function $k$ is given by

$$\widetilde{k}(\lambda) = 2\pi \int_0^\infty k(t) P_{i\lambda - 1/2}(\cosh t) \sinh t dt, \quad for \; \lambda \in \mathbb{R} \tag{1.58}$$

where $P_\nu(r)$ is the *Legendre* function if index $\nu$. If $m$ is another radial function then

$$\widetilde{(k * m)}(\lambda) = \widetilde{k}(\lambda) \widetilde{m}(\lambda), \tag{1.59}$$

as shown in [8]. It follows that as $\widetilde{k}(\lambda) \neq 0 \; \forall \lambda \in \mathbb{R}$ then the operator $R_H$, which takes $f$ to $k *_H f$, is injective.

### *1.4.2 Electrical impedance tomography (EIT)*

EIT has a number of applications to medicine and non-destructive evaluation. For instance, to determine the existence and lengths of internal cracks in the wings of an airplane. These applications are related to the inverse problem which is formulated now.

Let $D$ the unit disk in $\mathbb{R}^2$ and $\beta$ an strictly positive function defined on $\overline{D}$ which is unknown and represents the conductivity distribution inside the disk. When currents are introduced at the boundary $\partial D$, let $\Psi$ be a given integrable function representing such currents and such that the average of the values of $\Psi$ on $\partial D$ is zero

$$\int_{\partial D} \Psi ds = 0$$

and consider the boundary value problem with Neumann conditions

$$\begin{cases} div(\beta grad\ u) = 0 \quad in \quad D \\ \beta \frac{\partial u}{\partial n} = \Psi \quad on \quad \partial D \end{cases} \tag{1.60}$$

where $\Psi$ is given and $n$ is the outer unit normal vector on $\partial D$. This problem has a unique solution $u$, where the uniqueness of $u$ is only up to an additive constant. The function $u$ is the potential distribution on $D$ so $grad\ u$ is the electrical field. The variation of $u$ on $\partial D$ has to correspond to the known values of $\Psi$ on $\partial D$, then, if $s$ represents the tangent vector to $\partial D$, it follows that the tangential derivative of $u$, $\frac{\partial u}{\partial s}$, depends linearly on $\Psi$. So, for $\Psi$ given and $\beta$, the unknown conductivity, there exists a solution $u$. This defines a mapping

$$\beta, \Psi \longrightarrow \frac{\partial u}{\partial s} \tag{1.61}$$

where $\beta$ is the only remaining function to be found.

Let $\Lambda_\beta$ be

$$\Lambda_\beta : \Psi \longrightarrow \frac{\partial u}{\partial s}$$

$\Lambda_\beta$ is a linear operator from the Sobolev space $H^\alpha(\partial D)$ into $H^\alpha(\partial D)$ , and $\beta$ determines $\Lambda_\beta$. Given that $\beta$ is to be found, then we consider the nonlinear mapping

$$\beta \longrightarrow \Lambda_\beta \tag{1.62}$$

Now the problem, sometimes called Calderón's problem, consists in determining $\beta$ once $\Lambda_\beta$ is given. In other words, the problem is to find the inverse of the mapping (1.62), and this problem is currently known as the *inverse conductivity problem.*

Several questions arise here. Is the mapping (1.62) injective?. If so, how can the inverse of $\Lambda_\beta$ be found?. The injectivity of it in two dimensions was proven by Nachman [42] and for dimensions higher than two by Sylvester and Uhlmann [48] and [47]. For the linearized problem, the injectivity was proven by Calderón [18]. What we explain next is how to try to find an approximate inverse.

### *1.4.3 The approximate solution to the EIT problem*

As mentioned above, $\beta$ is called the conductivity distribution and $1/\beta$ is the impedance, hence the name of EIT. The value of $\beta$ corresponding to different constituents like human lungs tissue, blood and so on are already known, then one only looks for a profile of the areas occupied by them. EIT can measure the rate of pumping of the heart. In fact, there is already a patented device based on EIT that measures that rate.

In the case of the determination of cracks, the conductivity function of the material in normal conditions, i.e., without the cracks is known. To simplify the notation we call this function $\beta_o$ which we assume is positive (this is the conductivity that corresponds to the conductivity of the material without the cracks). Let $\beta$ is the function that corresponds to the actual conductivity of the material, which is also positive. For the determination of cracks in a homogeneous material, we can assume that the function $\beta_o$ is a constant, which we can assume to be 1

We want to emphasize that $\beta_o$ is generally not a constant but in the case we consider now $\beta_o$ is close to being a constant positive value. Assuming that $\beta$ is initially known, what we want to know is how much it deviates from $\beta_o$. We set $\beta_o = 1$ so the deviation $\delta\beta$ of $\beta$ is governed by $\beta = 1 + \delta\beta$ where $|\delta\beta| << 1$, and $\delta\beta$ is a function depending on the position. If $\delta\beta = 0$ at some point $w$ in the object being studied $D$, then there is no "abnormal" situation at $w$. It is also assumed that there is no any crack on $\partial D$, i.e., $\delta\beta = 0$ on $\partial D$. Let $U$ be the solution of (1.60) for $\beta = 1$, *i.e.*,

$$\begin{cases} div(grad\ U) = 0 \ \ in \ \ D \\ \frac{\partial U}{\partial n} = \Psi \ \ on \ \ \partial D \end{cases} \tag{1.63}$$

and since $div(grad\ U) = \Delta U$ (the Laplacian of $U$), it follows that

$$\begin{cases} \Delta U = 0 \ \ in \ \ D \\ \frac{\partial U}{\partial n} = \Psi \ \ on \ \ \partial D \end{cases} \tag{1.64}$$

Now let $u$ be the corresponding solution of (1.60) for the perturbed conductivity $\beta = 1 + \delta\beta$ , then there is a corresponding perturbation $\delta U$, so that $u = U + \delta U$.

The perturbation $\delta U$ satisfies

$$\begin{cases} \Delta(\delta U) = -\langle grad\ \delta\beta, grad\ U\rangle \ \ in \ \ D \\ \frac{\partial U}{\partial n} = -(\delta\beta)\Psi \ \ on \ \ \partial D \end{cases} \tag{1.65}$$

and since $\Psi$ represents the input by boundary currents, it can be arbitrarily chosen with the only constraint

$$\int_{\partial D} \Psi ds = 0,$$

for instance, a linear combination of dipoles. Recall that a dipole at a point $w \in \partial D$ is given by $-\pi\frac{\partial}{\partial s}\delta_w$, $\delta_w$ the Dirac delta at $w$. It follows that the linearized version of the problem (1.64) for the dipole (input) $-\pi\frac{\partial}{\partial s}\delta_w$ at $w$ becomes

$$\begin{cases} \Delta U_w = 0 \ \ in \ \ D \\ \frac{\partial U_w}{\partial n} = -\pi\frac{\partial}{\partial s}\delta_w \ \ on \ \ \partial D \end{cases} \tag{1.66}$$

and since the solution $U_w$ of (1.66) has level curves which are arcs of circles that pass through $w$ and are perpendicular to $\partial D$. Therefore, the level curves of $U_w$ are exactly the geodesics given by the hyperbolic metric, therefore the hyperbolic Radon transform appears naturally in the search for a solution to this problem.

In [6] is shown that the linearized problem can in fact be described explicitly in the context of hyperbolic geometry using $R_H$ and a radial convolution operator with kernel $k$. Let $k$ be given by

$$k(t) = \frac{\cosh^{-2}(t) - 3\cosh^{-4}(t)}{8\pi} \tag{1.67}$$

then, as the boundary data function $\mu = \frac{\partial(\partial U)}{\partial s}$ defined on the space of the geodesics in $D$, the relation between $\delta\beta$ and $\mu$ can be shown to be

$$R_H(k *_H \delta\beta) = \mu \tag{1.68}$$

and because of the backprojection operator, one obtains

$$R_H^* R_H(k *_H \delta\beta) = R_H^* \mu \tag{1.69}$$

hence

$$\frac{1}{4\pi}\Delta_H(S *_H (R_H^* \mu)) = k *_H \delta\beta, \tag{1.70}$$

Computing the hyperbolic Fourier transform of $k$, $\widetilde{k}$, which can be done exactly, it can be seen that $\widetilde{k}(\lambda) \neq 0,\ \forall \lambda \in \mathbb{R}$, and consequently, the convolution operator with kernel or symbol $k$, $k*_H$ is invertible. Formula (1.70) requires to invert the convolution operator of symbol $k$ to compute $\delta\beta$. Barber and Brown [2] proposed an approximate inversion and Santosa and Vogelius [45] showed that the inversion formula suggested by [2] is a generalized Radon transform.

To numerically implement the reconstruction of $\delta\beta$ it is necessary to invert the geodesic Radon transform and perform a deconvolution. The difficulty of numerically implementing (1.70) lies in the fact that it is complicated to numerically implement a two-dimensional non-Euclidean convolution on the hyperbolic space. In [38], Lissianoi and Ponomarev focussed on the problem of numerically inverting the geodesic Radon transform by developing an algorithm, and the problem regarding the deconvolution is also considered there. For this purpose, they consider the inversion formula (1.60) and use it to derive an inversion formula for the geodesic Radon transform that it is more suitable for computations. The interesting open problem here is to be able to define a class of "discrete hyperbolic wavelets" that provides the localization described in section 3 for the Euclidean Radon transform and has computation properties similar to those of the Euclidean wavelets. For

examples of discrete hyperbolic wavelets, we refer to [30, 28].

# 2. NETWORK TOMOGRAPHY

As communication networks have become an essential part of everyday life, disruptions may have very serious consequences. Thus, the need to prevent or, at least, detect them as early as possible, has become very important. In order to do that we discuss two models of the problem, one based on weighted graphs and the second based on trees. The first one is the discrete equivalent of the inverse conductivity problem, that is, of Electrical Impedance Tomography. The second model was mentioned recently by E. Jonckheere and his collaborators [35]. The reason we can think about this problem as a tomographic problem is that in both cases, the data we collect are obtained by monitoring traffic only at distinguished subsets of the network. We think about this subset as being the periphery of the network.

## *2.1 The Problem in Networks*

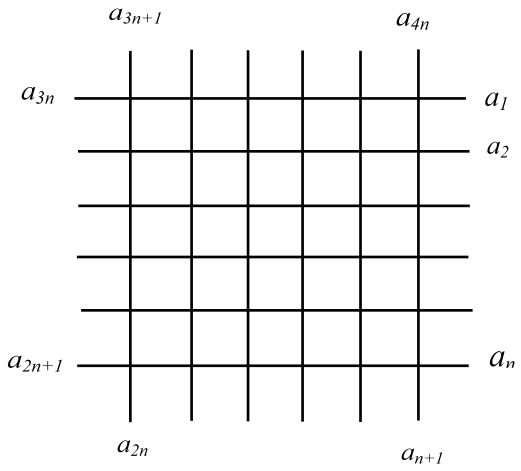
The problem we refer to is to be able to find out whether a network, usually a communications network, is suffering some sort of breakdown. By that we mean that traffic along the network either can not reach every node in the network, or when we add a measure of traffic around nodes, the traffic is so large in some parts of the network that it would take very long to go from one node to another. When

the network is large, the information is naturally gathered at the "periphery" of the network and hence the name of internet tomography. The similarity to usual tomography becomes closer when one uses it as a way to measure the traffic "packets" sent from the "boundary of the network" and measures whether they arrive to the other boundary points and, more often, how long it takes to get there.

Computer scientists have done "experimental" work on this subject and have suggested that the natural model of internet tomography is a graph situated in a portion of the 2-sphere or, what is essentially the same thing, in the hyperbolic plane [40]. Before we proceed further, let us note that we have alluded to two natural types of "disruptions" of the network. First, when thought as a planar graph, if a node or collection of nodes have ceased to exist due to an "intrusion" the "topology" of the network has changed. There has been very significant work on this direction by experts on graph theory. The important work of Fan Chung and her collaborators offers crucial insights into this question. (See, for instance [19], [20, 22].) Another situation, that resembles more what "conventional" tomography is supposed to help with, arises when traffic among certain nodes starts to increase to levels where the graph structure remains intact but there is significant slow down due to this large amount of traffic. Communication networks and, regretfully, road networks are a well known example of this second phenomenon. In either case, the desire is to be able to detect this problem when it is incipient and thus one may try to devise a solution to it. It is the latter problem that is of interest to us. One can see that Munzner's suggestion leads to a question closely resembling EIT, and it is natural to consider it as a problem in hyperbolic tomography of the kind described earlier. On

the other hand, recently has been obtained a significant result on the inversion of the Neumann-Dirichlet problem by studying it directly on "weighted" graphs [9]. A similar result was obtained by Colin de Verdiere by considering a different approach.

Let us explain now a bit more in detail what these recent results are and what new questions they open up. To understand these ideas better let us consider a very simple example of a planar network, the square network $G$ [28]. This network is constructed as follows. The nodes of $G$ are the integer lattice points $p = (i, j)$ with $0 \leq i \leq n+1$ and $0 \leq j \leq n+1$ and exclude the points $(0,0)$, $(n+1,0)$, $(0,n+1)$, and $(n+1,n+1)$.




Let $V$ be the set of nodes, and $intV$, the *interior* of $V$, consists of the nodes $p = (i, j)$ with $1 \leq i \leq n$ and $1 \leq j \leq n$. The boundary of $G$ is denoted $\partial G$ and it is equal to $V \setminus intV$. Any interior node $p$ has four neighboring nodes which are the nodes at unit distance from $p$. Denote the set of these neighboring nodes $N(p)$. If $p$ is an interior node then $N(p)$ is in $V$, and if $p$ is on $\partial G$ then it has only one neighboring node which is the interior node that has unit distance from $p$. If a line segment $l$

connects a pair of neighboring nodes $p$ and $q$ in $intV$ or if it connects a boundary node $p$ to its neighboring interior node $q$ is called *edge* or *conductor* and denoted $(p, q)$. In the case in which $p$ is on the boundary, the edge is called a boundary edge. The set of edges is denoted by $E$, and usually the graph $G$ is denoted by $G(V, E)$.

Let $\omega$ a non-negative real-valued function on $E$, the value $\omega(p, q)$ is called the *conductance* of $(p, q)$ and $1/\omega(p, q)$, the *resistance* of $(p, q)$, and $\omega$ is the *conductivity* ($\omega$ is sometimes called a *weight*). A function $u : V \to \mathbb{R}$ gives a current across each edge $(p, q)$ by *Ohm's law*, the current from $p$ to $q$, $I = \omega(p, q)(u(p) - u(q))$. The function $u$ is called $\omega$-*harmonic* if for each interior node $p$,

$$\sum_{q \in N(p)} \omega(p, q)(u(q) - u(p)) = 0 \tag{2.1}$$

In other words, the sum of the currents flowing out of each interior node is zero, which is the discrete equivalent of *Kirkhoff's* law. Let $\Phi$ a function defined at the boundary nodes, the network will acquire a unique $\omega$-harmonic function $u$ with $u(p) = \Phi(p)$ for each $p \in \partial G$ in other words, $\Phi$ induces $u$ and $u$ is called the *potential* induced by $\Phi$. Considering a conductor $(p, q)$ then the potential drop across this conductor is $\Delta u(p, q) = u(p) - u(q)$. The potential function $u$ determines a current $I_\Phi(p)$ through each boundary node $p$, by $I_\Phi(p) = \omega(p, q)(u(p) - u(q))$, $q$ being the interior neighbor of $p$. As in the continuous case, for each conductivity $\omega$ on $E$, the linear map $\Lambda_\omega$ from boundary functions to boundary functions is defined by $\Lambda_\omega \Phi = I_\Phi$, where the boundary function $\Phi$ is called the Dirichlet data, the boundary current $I_\Phi$ is called Neumann data, and the map $\Lambda_\omega$ is called the Dirichlet-to-Neumann map.

The problem to consider is to recover the conductivity $\omega$ from $\Lambda_\omega$, which is

analogous to the corresponding inverse problem in the continuous case. The two basic problems are the connectivity and conductivity of the network. Note that the connectivity of the network or the situation where the network remains connected but some edges disappear is a topological problem, the *configuration* of the graph has changed. For the detailed theory about electrical networks, planar graphs, identification of a graph, and harmonic functions, we refer to [25] and the work of Curtis and Morrow [27].

The discrete or finite nature of graphs makes working on graphs basically easier than investigating these problems in the continuous case, although it gives rise to several disadvantages. For example, solutions of the Laplace equation for graphs have neither the local uniqueness property nor is their uniqueness guaranteed by the Cauchy data, contrary to the continuous case where they are the most important mathematical tools used to study the inverse conductivity problem and related problems [9]. The inverse problem that we study is to identify the connectivity of the nodes and the conductivity on the edges between each adjacent pair of nodes.

## *2.2 The weighted graph model*

### *2.2.1 Calculus on weighted graphs*

We begin with some definitions of graph theoretic notions

Given a network with a pattern of traffic measured as the "usual" load between adjacent nodes (e.g., number of messages) one can associate to it a Laplace operator denoted $\Delta_\omega$, where the weight $\omega$ is a sequence of values representing the usual loads

between every pair of adjacent nodes in the network. We model our network in the context of graphs in the following way. We have a collection of nodes and edges between the nodes in a finite planar simple connected graph $G$. A graph $G$ is said to be simple if it has neither a multiple edge nor loops and G is said to be *connected* if for every pair of vertices $x$ and $y$ there exist a sequence (termed a *path*) of vertices $x = x_0, x_1, x_2, \cdots, x_{n-1}, x_n = y$ such that $x_{j-1}$ and $x_j$ are connected by an edge (termed *adjacent*) for $j = 1, 2, \cdots, n$. We denote by $V$ the set of nodes of $G$ and by $E$ the set of edges of $G$. Usually, the graph $G$ is denoted by $G(E, V)$. A particular subset of the vertices of this graph $G$ is denoted by $\partial G$ and called the boundary of $G$. In our context these are the nodes accessible to whoever is trying to monitor the traffic in $G$. The boundary edges are those edges whose two endpoints are in $\partial G$. We assume that $G$ remains connected even if we remove the boundary edges. For our present purposes, the boundary edges play no role, thus we may as well assume that there are none. We also assume that $\partial G$ is not empty. Furthermore, we assume that to every edge in $E$ we have an associated non-negative number $\omega(x, y)$ which corresponds to the traffic between the endpoints $x$ and $y$ of the edge. The weight $\omega(x, y)$ can be extended to all $V \times V$ by defining $\omega(x, y) = 0$ for the pairs $(x, y)$ of vertices which are not linked by an edge. The function $\omega$ satisfies the following:

*(i)* $\omega(x, x) = 0, \quad x \in V,$

*(ii)* $\omega(x, y) = \omega(y, x), \quad$ if $x \sim y,$

*(iii)* $\omega(x, y) = 0 \quad$ if and only if $\quad \{x, y\} \notin E.$

Here, $x \sim y$ means that two vertices $x$ and $y$ are connected (adjacent) by an edge in $E$. In that case, $\{x, y\}$ denotes the edge connecting the vertices $x$ and $y$.

In particular, a weight function $\omega$ satisfying

$$\omega(x, y) = 1, \text{ if } x \sim y \tag{2.2}$$

is called the *standard* weight on $G$. The physical meaning of the weight function will be discussed later. Note that this is a static model and we are really thinking that the graph is a planar graph, although this is not used anywhere in the reasoning. We define the degree $d_\omega x$ of a node $x$ in the weighted graph $G$ with weight $\omega$ by

$$d_\omega x = \sum_{y \in V} \omega(x, y) \tag{2.3}$$

A graph $S = S(V', E')$ is said to be a *subgraph* of $G(E, V)$ if $V' \subset V$ and $E' \subset E$. In this case, we call $G$ a *host graph* of $S$. If $E'$ consists of all the edges from $E$ which connect the vertices $V'$ in its host graph $G$, then $S$ is called an *induced* subgraph. Throughout this section, all the subgraphs are assumed to be induced subgraphs of a host graph, which is simple and connected, with a weight, and a function on a graph is understood as a function defined only on the set of vertices.

The integration of a function $f : G \to \mathbb{R}$ on a graph $G = G(V, E)$ is defined by

$$\int_G f d_\omega \text{ or simply } \int_G f \equiv \sum_{x \in V} f(x) d_\omega x \tag{2.4}$$

We now define the directional derivative of a function $f : G \to \mathbb{R}$. For each $x$ and $y \in V$ we define

$$D_{\omega,y} f(x) = [f(y) - f(x)] \sqrt{\frac{\omega(x, y)}{d_\omega x}} \ . \tag{2.5}$$

The gradient $\nabla_\omega$ of function $f$ is defined to be the vector

$$\nabla_\omega f(x) = \big(D_{\omega,y} f(x)\big)_{y \in V} \ , \tag{2.6}$$

which is indexed by the vertices $y \in V$. Then it is easy to see that

$$\begin{aligned}\int_G |\nabla_\omega f(x)|^2 &= \sum_{x\in V} |\nabla_\omega f(x)|^2 \, d_\omega x \\ &= \sum_{x\in V}\sum_{y\in V} |f(y)-f(x)|^2 \, \omega(x,y) \\ &= 2 \sum_{\{x,y\}\in E} |f(y)-f(x)|^2 \, \omega(x,y), \end{aligned} \tag{2.7}$$

which is called the energy of $f$ on $G$.

For a subgraph $S$ of a graph $G = G(V, E)$ the (node) *boundary* $\partial S$ of $S$ is defined to be the set of all nodes $z \in V$ not in $S$ but adjacent to some node in $S$, i.e.,

$$\partial S = \{z \in V \mid z \notin S \text{ and } z \sim y \text{ for some } y \in S\} \tag{2.8}$$

and the *inner boundary* $\mathring{\partial} S$ by

$$\mathring{\partial} S = \{z \in S \mid y \sim z \text{ for some } y \in \partial S\} \tag{2.9}$$

where $z \sim y$ means that the two nodes $z$ and $y$ are connected by an edge in $E$. Also, by $\overline{S}$ we denote a graph whose nodes and edges are in $S \cup \partial S$. The (outward) normal derivative $\frac{\partial f}{\partial n_\omega}(z)$ at $z \in \partial S$ is defined to be

$$\frac{\partial f}{\partial n_\omega}(z) = \sum_{y\in S} [f(z) - f(y)] \cdot \frac{\omega(z,y)}{d'_\omega \, z}, \tag{2.10}$$

where $d'_\omega \, z = \sum_{y\in S} \omega(z,y)$

The weighted $\omega$-Laplacian $\Delta_\omega f$, the Laplacian operator corresponding to this weight $\omega$, is defined by

$$\Delta_\omega f = \sum_{y\in V} [f(y) - f(x)] \cdot \frac{\omega(x,y)}{d_\omega x}, \ x \in V \tag{2.11}$$

Using arguments similar to those in [28] we can see that this Laplacian satisfies the maximum principle. On the other hand, as pointed out in [26], the elipticity in the usual sense does not hold. For instance, one can construct non-trivial $\omega-$harmonic functions that are constant in a proper subset of $G$, the analogue of an open set in the non discrete case, but the function is not identically constant in $G$. In fact, let's see some results related to the first derivative, gradient, and Laplacian previously defined. In what follows, a function $f$ defined on $\overline{S}$ may be understood as a function on its host graph $G$ such that $f = 0$ on $G \setminus \overline{S}$, if necessary.

**Theorem 2.1.** *Let $S$ be a subgraph of a host graph $G$. Then for any pair of functions $f : \overline{S} \to \mathbb{R}$ and $h : \overline{S} \to \mathbb{R}$, we have*

$$2\int_{\overline{S}} h(-\Delta_\omega f) = \int_{\overline{S}} \nabla_\omega h \cdot \nabla_\omega f. \tag{2.12}$$

*Proof.* A direct use of the definitions mentioned above gives

$$\begin{aligned}
2\int_{\overline{S}} h(-\Delta_\omega f) &= 2\sum_{x\in\overline{S}} h(x)[-\Delta_\omega f(x)]d_\omega x \\
&= -2\sum_{x\in\overline{S}} h(x)\Big\{\sum_{y\in V(G)} [f(y)-f(x)]\omega(x,y)\Big\} \\
&= 2\sum_{x\in\overline{S}}\sum_{y\in\overline{S}} h(x)[f(x)-f(y)]\omega(x,y) \\
&= \sum_{x\in\overline{S}}\sum_{y\in\overline{S}} h(x)[f(x)-f(y)]\omega(x,y) + \sum_{x\in\overline{S}}\sum_{y\in\overline{S}} h(y)[f(y)-f(x)]\omega(x,y) \\
&= \sum_{x\in\overline{S}}\sum_{y\in\overline{S}} \Big\{[f(y)-f(x)]\sqrt{\omega(x,y)}\Big\}\cdot\Big\{[h(y)-h(x)]\sqrt{\omega(x,y)}\Big\} \\
&= \sum_{x\in\overline{S}} \big\{\nabla_\omega f(x)\cdot\nabla_\omega h(x)\big\} d_\omega x \\
&= \int_{\overline{S}} \nabla_\omega f \cdot \nabla_\omega h.
\end{aligned}$$

□

The above theorem yields many useful formulae such as the Green theorem.

**Corollary 2.2.** *Under the same hypotheses as above we have the following identities:*

*(i)*

$$2\int_{\overline{S}} f(-\Delta_\omega f) = \int_{\overline{S}} |\nabla_\omega f|^2 .$$

*(ii)*

$$\int_{\overline{S}} h\Delta_\omega f = \int_{\overline{S}} f\Delta_\omega h.$$

*(iii) (Green's formula)*

$$\int_S (f\Delta_\omega h - h\Delta_\omega f) = \int_{\partial S} (f\frac{\partial h}{\partial_\omega n} - h\frac{\partial f}{\partial_\omega n}).$$

*Proof.* (i) is trivial and (ii) can be easily obtained by the symmetry in (2.12). We prove (iii). In view of (ii) we have

$$\begin{aligned} 0 &= \int_{\overline{S}} [f\Delta_\omega h - h\Delta_\omega f] \\ &= \int_S [f\Delta_\omega h - h\Delta_\omega f] + \int_{\partial S} [f\Delta_\omega h - h\Delta_\omega f] \end{aligned}$$

Then, since $S$ is the induced subgraph, it follows that $\omega(z, y) = 0$ for all $z$ and $y \in \partial S$ and

$$\begin{aligned} \int_S [f\Delta_\omega h - h\Delta_\omega f] &= \int_{\partial S} [h\Delta_\omega f - f\Delta_\omega h] \\ &= \sum_{z\in\partial S} [h(z)\Delta_\omega f(z) - f(z)\Delta_\omega h(z)]d_\omega z \\ &= \sum_{z\in\partial S}\sum_{y\in S} \Big\{ h(z)\big[f(y) - f(z)\big]\omega(z, y) - f(z)\big[h(y) - h(z)\big]\omega(z, y)\Big\} \end{aligned}$$

$$= \sum_{z \in \partial S} \Big[ h(z) \{ - \tfrac{\partial f}{\partial_\omega n} \} + f(z) \tfrac{\partial h}{\partial_\omega n}(z) \Big] d_\omega z$$

$$= \int_{\partial S} \left[ f \tfrac{\partial h}{\partial_\omega n} - h \tfrac{\partial f}{\partial_\omega n} \right] \quad \square$$

In the continuous case, the following are well-known formulae:

$$\Delta(fg) = f\Delta g + 2\nabla f \cdot \nabla g + g\Delta f$$

$$\int_\Omega \nabla f \cdot \nabla g + \int_\Omega f \Delta g = \int_{\partial\Omega} f \frac{\partial g}{\partial n}$$

Here, we introduce a discrete analogue of these formulae.

**Theorem 2.3.** *Under the same hypotheses as in Theorem (2.1), the following identities hold:*

*(i)*

$$\Delta_\omega(fh) = f\Delta_\omega h + \nabla_\omega f \cdot \nabla_\omega h + h\Delta_\omega f$$

*(ii)*

$$\int_S \nabla_\omega f \cdot \nabla_\omega h + \int_S [f\Delta_\omega h + h\Delta_\omega f] = \int_{\partial S} \frac{\partial(fh)}{\partial_\omega n}$$

*Proof.* (i) can be obtained by an elementary manipulation. Using now (i) and Theorem (2.1), (iii) with $h \equiv 1$ we obtain (ii). $\square$

### *2.2.2 Harmonic Functions*

In this section we will discuss the functional properties of functions which satisfy the equation

$$\Delta_\omega f(x) = \sum_{y \in \overline{S}} \big[ f(y) - f(x) \big] \frac{\omega(x,y)}{d_\omega x} = 0, \tag{2.13}$$

For a subgraph $S$ with boundary $\partial S \neq \phi$ of a host graph $G$ with a weight $\omega$ we say that a function $f : \overline{S} \to \mathbb{R}$ is $\omega$-*harmonic* on $S$ if it satisfies (2.13) for all $x \in S$, i.e.,

$$f(x) = \frac{1}{d_\omega x} \sum_{y \in \overline{S}} f(y)\omega(x, y), \quad x \in S. \tag{2.14}$$

this implies that the value of $f$ at $x$ is given by a weighted average of the values of $f$ at its neighboring vertices. From this point of view, we can clearly expect the following result to be true:

**Theorem 2.4.** *(Minimum and Maximum Principle) Let $S$ be a subgraph of a host graph $G$ with a weight $\omega$ and $f : \overline{S} \to \mathbb{R}$ be a function.*

*(i) If $\Delta_\omega f(x) \geq 0, x \in S$ and $f$ has a maximum at a vertex in $S$, then $f$ is constant.*

*(ii) If $\Delta_\omega f(x) \leq 0, x \in S$ and $f$ has a minimum at a vertex in $S$, then $f$ is constant.*

*(iii) If $\Delta_\omega f(x) = 0, x \in S$ and $f$ has either a minimum or maximum in $S$, then $f$ is constant.*

*(iv) If $\Delta_\omega f(x) = 0, x \in S$ and $f$ is constant on the boundary $\partial S$, then $f$ is constant.*

*Proof.* (ii) can be done in a similar way as in (i). (iii) and (iv) are easily obtained from (i) and (ii).

We prove (i). Assume that $f$ has a maximum at a vertex $x_0 \in S$. Then

$$f(x_0) \geq f(y), \quad y \in \overline{S} \tag{2.15}$$

and

$$f(x_0) \leq \sum_{y \in \overline{S}} f(y) \frac{\omega(x_0, y)}{d_\omega x_0} \tag{2.16}$$

Suppose that there exists $y_0 \in \overline{S}$ such that $x_0 \sim y_0$ and $f(x_0) \neq f(y_0)$, i.e., $f(x_0) > f(y_0)$ in view of (2.15). Then it follows from (2.16) that

$$\begin{aligned} f(x_0) &\leq \sum_{y \in \overline{S}} \sum_{y \neq y_0} \frac{f(y)\omega(x_0, y)}{d_\omega x_0} + \frac{f(y_0)\omega(x_0, y_0)}{d_\omega x_0} \\ &< \sum_{y \in \overline{S}} \sum_{y \neq y_0} \frac{f(x_0)\omega(x_0, y)}{d_\omega x_0} + \frac{f(x_0)\omega(x_0, y_0)}{d_\omega x_0} \\ &= f(x_0), \end{aligned}$$

which implies that $f(x_0) = f(y)$ for all $y \in \overline{S}$ such that $y \sim x_0$. Now for any $x \in \overline{S}$, there exists a path

$$x_0 \sim x_1 \sim x_2 \sim \cdots \sim x_{n-1} \sim x_n = x,$$

since $S$ is connected. By applying the same argument as above inductively we see that $f(x_0) = f(x)$. □

The following is an easy consequence of the above theorem:

**Corollary 2.5.** *Under the same hypotheses as in Theorem (2.4), the following statements are true:*

*(i) If $\Delta_\omega f \geq 0$ on $S$ and $f|_{\partial S} \leq 0$ $(< 0)$, then $f \leq 0$ $(< 0)$ on $S$.*

*(ii) If $\Delta_\omega f \leq 0$ on $S$ and $f|_{\partial S} \geq 0$ $(> 0)$, then $f \geq 0$ $(> 0)$ on $S$.*

**Corollary 2.6.** *The following statements are true:*

*(i) If two functions $f$ and $g$ on $\overline{S}$ satisfy*

$$\Delta_\omega f = 0 \text{ and } \Delta_\omega g \geq 0$$

*on $S$, then $g|_{\partial S} \leq f|_{\partial S}$ implies $g \leq f$ on $S$.*

*(ii) If a function $f : \overline{S} \to \mathbb{R}$ satisfies*

$$\Delta_\omega f(x) = 0, \quad x \in S$$

*and $|f|$ has a maximum in $S$, then $f$ is constant.*

In the continuous case, it is well known that a local maximum principle holds for a harmonic function in an open subset $\Omega \subset \mathbb{R}^n$. But it is not hard to see that the local maximum principle is no longer true in general in our case. Moreover, the local uniqueness principle does not hold in general. As a matter of fact, it is rather natural to expect that such discrepancies are caused by the discrete nature of graphs.

A nonempty subset $\Gamma$ of vertices of a subgraph $\overline{S}$ is said to be a *surface in* $\overline{S}$ if $\Gamma = \partial T$ for a subgraph $T$ whose vertices belong to $S$. In this case, we denote by $\overset{\circ}{\Gamma}$ the inner boundary $\overset{\circ}{\partial} T$. For each vertex $z \in \Gamma$ and $x \in \overset{\circ}{\Gamma}$ we define

$$d'_\omega z = \sum_{y \in \overset{\circ}{\Gamma}} \omega(y, z) \quad \text{(inward degree)}$$

and

$$d''_\omega x = \sum_{z \in \Gamma} \omega(x, z) \quad \text{(outward degree)}.$$

In addition, for a function $f$ on $\overline{S}$ we write

$$\int_\Gamma f(z) d'_\omega z = \sum_{z \in \Gamma} f(z) d'_\omega z \quad \text{(inward integral)}$$

and

$$\int_{\overset{\circ}{\Gamma}} f(x) d''_\omega x = \sum_{x \in \overset{\circ}{\Gamma}} f(x) d''_\omega x \quad \text{(outward integral)}.$$

We use these notions to obtain the following interesting properties of $\omega$-harmonic functions.

**Theorem 2.7.** *Let $S$ be a subgraph of a host graph with weight $\omega$ and let $f : \overline{S} \to \mathbb{R}$. Then $f$ is $\omega$-harmonic on $S$, i.e., for all $x \in S$,*

$$\Delta_\omega f(x) = 0, \tag{2.17}$$

*if and only if for every surface $\Gamma$ in $\overline{S}$*

$$\int_\Gamma f(z) d'_\omega z = \int_{\overset{\circ}{\Gamma}} f(y) d''_\omega y. \tag{2.18}$$

*Proof.* Let $x \in S$ and $\Gamma_x = \left\{ y \in \overline{S} | x \sim y \right\}$. Then $\Gamma_x$ is a surface in $\overline{S}$ and $\overset{\circ}{\Gamma}_x = \{x\}$. Since $d_\omega x = d''_\omega x$ on $\overset{\circ}{\Gamma}_x$ and $d'_\omega z = \omega(x, z)$, (2.18) implies

$$f(x) d_\omega x = \sum_{z \in \Gamma_x} f(z) \omega(x, z),$$

which implies (2.17) immediately.

Assume now that (2.17) holds and let $\Gamma$ be a surface in $\overline{S}$ such that $\Gamma = \partial T$ for a subgraph $T \subset S$. We use Green's formula (Corollary 2.2, (iii)) to obtain

$$\begin{aligned} 0 &= \int_T \Delta_\omega f \\ &= \int_\Gamma \frac{\partial f}{\partial_\omega n} \\ &= \sum_{z \in \Gamma} \frac{\partial f}{\partial_\omega n}(z) d'_\omega z \\ &= \sum_{z \in \Gamma} \sum_{y \in \overset{\circ}{\Gamma}} \big[ f(z) - f(y) \big] \omega(z, y). \end{aligned} \tag{2.19}$$

Then it follows that

$$\sum_{z\in\Gamma}\sum_{y\in\overset{\circ}{\Gamma}} f(z)\omega(z,y) = \sum_{z\in\Gamma}\sum_{y\in\overset{\circ}{\Gamma}} f(y)\omega(z,y)$$

or, equivalently

$$\sum_{z\in\Gamma} f(z)\Big[\sum_{y\in\overset{\circ}{\Gamma}}\omega(z,y)\Big] = \sum_{y\in\overset{\circ}{\Gamma}} f(y)\Big[\sum_{z\in\Gamma}\omega(z,y)\Big],$$

which yields (2.18). □

In view of (2.19) we obtain the edge version of Theorem 2.7, the so-called dual theorem as follows.

**Corollary 2.8.** *Under the same conditions as in Theorem 2.7 , the formula (2.18) is equivalent to*

$$\sum_{\{x,y\}\in E(\Gamma,\overset{\circ}{\Gamma})} \big[f(z) - f(y)\big]\omega(z,y) = 0$$

*where* $E(\Gamma,\overset{\circ}{\Gamma})$ *denotes the set of all edges joining a vertex in* $\Gamma$ *and a vertex in* $\overset{\circ}{\Gamma}$.

For two vertices $x$ and $y$ in a connected graph, the distance $d(x,y)$ between $x$ and $y$ is the number of edges in a shortest path joining $x$ and $y$.

For a vertex $x_0$ in a subgraph $S$ we write

$$\Gamma_j(x_0) := \left\{y \in \overline{S} | \quad d(x_0,y) = j\right\}, j = 0,1,2,\cdots$$

which is called a neighborhood of $x_0$ with radius $j$.

Then the following is a variant of Theorem 2.7:

**Corollary 2.9.** *Let $S$ and $f$ be the same as in Theorem 2.7. Then $f$ is $\omega$-harmonic on $S$ if and only if for every $x_0 \in S$*

$$\int_{\Gamma_j(x_0)} f(x) d''_\omega x = \int_{\Gamma_{j+1}(x_0)} f(x) d'_\omega x \tag{2.20}$$

*for each $j$ with $\Gamma_j(x_0) \subset S$.*

*Proof.* Letting $j = 0$ in (2.20) we have the sufficiency. To prove the necessity, consider an induced subgraph $T$ whose vertices are exactly those of $\bigcup_{k=0}^{j} \Gamma_k(x_0)$. Then it is easy to see that

$$\partial T = \Gamma_{j+1}(x_0) \text{ and } \overset{\circ}{\partial} T \subset \Gamma_j(x_0).$$

But a vertex $x$ in $\Gamma_j(x_0)$, which does not belong to $\overset{\circ}{\partial} T$, does not make any contribution to the outer integral $\int_{\Gamma_j(x_0)} f(x) d''_\omega x$, since $d''_\omega x = 0$. Hence, condition (2.18) in Theorem 2.7 shows the condition is necessary. □

The following is the dual version of the above corollary:

**Corollary 2.10.** *Under the same conditions as in Corollary 2.9 the formula (2.20) is equivalent to*

$$\sum_{\{x,y\} \in E\left(\Gamma_j(x_0), \Gamma_{j+1}(x_0)\right)} \big[f(x) - f(y)\big]\omega(x,y) = 0$$

*where $E\big(\Gamma_j(x_0), \Gamma_{j+1}(x_0)\big)$ denotes the set of all edges joining a vertex in $\Gamma_j(x_0)$ and a vertex in $\Gamma_{j+1}(x_0)$*

### *2.2.3 The Dirichlet and Neumann Boundary Value Problems: Direct Problems*

In this section, we discuss the direct problems such as the Dirichlet BVP and Neumann BVP [15, 16, 20, 22, 21]. We start this section with a physical interpretation

of the $\omega$-Laplace and $\omega$-Poisson equations. Consider a host graph $G$ with a weight $\omega$ and an (induced) subgraph $S$. For a surface $\Gamma$ in $\overline{S}$ with $\Gamma = \partial T$ for some $T \subset S$ and $z \in \Gamma$, the flux of energy passing through $z$ to its adjacent nodes in $T$ is given by

$$-\sum_{y\sim z}\big[f(z)-f(y)\big]\cdot\frac{\omega(z,y)}{d'z} \tag{2.21}$$

where $d'z = \sum_{y\sim z, y\in T}\omega(z,y)$ and $f$ is a potential function in a diffusion field on a network. (For example, an electrostatic field, a thermal field, or an elastic membrane.) Here, the weight $\omega(z,y)$ plays the role of the conductivity of the diffusion along the edge $\{z,y\}$. In fact, (2.21) is exactly $-\frac{\partial f}{\partial_\omega n}(z)$ on $\Gamma$ by definition and thus, by Green's formula we have

$$\int_T(-\Delta_\omega f) = \int_\Gamma\big(-\frac{\partial f}{\partial_\omega n}\big),$$

which is the flow across $\Gamma$. On the other hand, assume that $T$ gains (or loses) an amount of energy $\int_T g$ where $g$ is the energy density. Then we have

$$\int_T(-\Delta_\omega f) = \int_T g$$

Therefore, since $T$ is arbitrary, by taking $T$ to be any single vertex $x \in S$ we obtain the vertex equation

$$-\Delta_\omega f(x) = g(x), \quad x \in S. \tag{2.22}$$

Thus, it is reasonable to say that the conductivity equation on a graph can be represented as in (2.22), where $\omega(x,y)$ corresponds to the edge conductivity on the edge $\{x,y\}$.

Following the work of Fan Chung and her collaborators [20, 22, 21], we will discuss first the equation (2.22) on a graph $G = G(V, E)$ with a weight $\omega$ and no boundary. We consider the matrix

$$\Delta_\omega(x,y) = \begin{cases} -1 & \text{if } x = y \\ \frac{\omega(x,y)}{d_\omega x}, & \text{if } x \sim y \\ 0, & \text{otherwise .} \end{cases}$$

Considering the function $f$ as a $|V|$-dimensional vector, the equation (2.22) can be understood as a matrix linear equation. Let $D$ denote the diagonal matrix with $(x,x)$-th entry having the value $d_\omega x$ for each $x$ and $L_\omega = D^{1/2}\Delta_\omega D^{-1/2}$. Then $(-L_\omega)$ is a nonnegative definite symmetric matrix, so that it has the eigenvalues

$$\lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{N-1}$$

and the corresponding eigenfunctions

$$\Phi_0, \Phi_1, \Phi_2, \cdots \Phi_{N-1}, \tag{2.23}$$

which are orthonormal in the sense that for each pair of distinct $i$ and $j$

$$\sum_{x \in V} \Phi_i(x) \cdot \Phi_j(x) = 0,$$

while, for all j,

$$\sum_{x \in V} |\Phi_j(x)|^2 = 1.$$

Here, $N$ denotes $|V|$, the number of vertices in $G$. Then it is easy (see [20]) to show that $\lambda_0 = 0$, $\lambda_1 > 0$ and $\Phi_0(x) = \frac{\sqrt{d_\omega x}}{\sqrt{\text{vol}(G)}}$, $\quad x \in V$, and $\text{vol}(G) := \sum_{x \in V} d_\omega x$.

In what follows, we occasionally use the notation $\langle\ ,\ \rangle_X$, defined by

$$\langle f, g \rangle_X = \sum_{x \in X} f(x) g(x)$$

for simplicity. Now we have the following solvability result for the Poisson equation:

**Theorem 2.11.** *Let $G = G(V, E)$ be a graph with a weight $\omega$ and $f : G \to \mathbb{R}$ be a function. Then the equation*

$$\Delta_\omega f(x) = g(x), \quad x \in V \tag{2.24}$$

*has a solution if and only if $\int_G g = 0$. In this case, the solution is given by*

$$f(x) = a_0 + \langle \Gamma_\omega(x, \cdot), g \rangle_V, \quad x \in V \tag{2.25}$$

*where $a_0$ is an arbitrary constant and*

$$\Gamma_\omega(x, y) = \sum_{j=1}^{N-1} \big( -\frac{1}{\lambda_j} \big) \Phi_j(x) \Phi_j(y) \sqrt{\frac{d_\omega y}{d_\omega x}}, \quad x, y \in V. \tag{2.26}$$

*Proof.* Assume that $\int_G g = 0$. Then

$$\begin{aligned} \langle D^{1/2} g, \Phi_0 \rangle &= \sum \sqrt{d_\omega x} g(x) \cdot \frac{\sqrt{d_\omega x}}{\sqrt{\mathrm{vol} G}} \\ &= \frac{1}{\sqrt{\mathrm{vol} G}} \int_G g \\ &= 0, \end{aligned}$$

where $D$ is the diagonal matrix whose $x$-th diagonal entry is $d_\omega x$. Consider the orthogonal expansion

$$\big( D^{1/2} f \big)(x) = \sum_{j=0}^{N-1} a_j \Phi_j(x), \quad x \in V$$

where $a_j = \langle D^{1/2} f, \Phi_j \rangle, j = 0, 1, 2, \cdots, N-1$. Then, since $L_\omega D^{1/2} = D^{1/2} \Delta_\omega$ and

$$\begin{aligned} -\lambda_j a_j &= \langle D^{1/2} f, L_\omega \Phi_j \rangle \\ &= \langle L_\omega D^{1/2} f, \Phi_j \rangle \\ &= \langle D^{1/2} g, \Phi_j \rangle, \end{aligned}$$

we have

$$a_j = \Big( -\frac{1}{\lambda_j}\Big)\langle D^{1/2}g, \Phi_j\rangle, \quad j = 1, 2, \cdots, N-1$$

and $a_0$ is an arbitrary constant. Hence

$$\sqrt{d_\omega x} f(x) = a_0 \frac{\sqrt{d_\omega x}}{\sqrt{\mathrm{vol}G}} + \sum_{j=1}^{N-1} \Big( -\frac{1}{\lambda_j}\Big) \Big[ \sum_{y\in V} g(y)\Phi_j(y)\sqrt{d_\omega y}\Big] \Phi_j(x),$$

or, equivalently,

$$f(x) = \frac{a_0}{\sqrt{\mathrm{vol}G}} + \sum_{j=1}^{N-1} \Big( -\frac{1}{\lambda_j}\Big) \sum_{y\in V} g(y)\Phi_j(y) \frac{\sqrt{d_\omega y}}{\sqrt{d_\omega x}} \Phi_j(x)$$

which gives (2.25) with a different constant $a_0$. With a simple calculation we see that

$$\Delta_\omega \cdot \Gamma_\omega\, g(x) = g(x) + \frac{1}{\mathrm{vol}G} \int_G g \quad , \quad x \in V,$$

which implies that every function of the form (2.25) gives a solution to the equation (2.24). The proof of the converse is easy. □

The matrix $\Gamma_\omega$ in (2.26) is called the Green function of $\Delta_\omega$. The following corollary is a Liouville type theorem for $\omega$-harmonic functions.

**Corollary 2.12.** *Under the same conditions as in Theorem 2.11, every solution $f$ of*

$$\Delta_\omega f(x) = 0, \quad x \in V$$

*is constant.*

The following corollary describes all functions which are $\omega$-harmonic except possibly on a given (singularity) set $T$.

**Corollary 2.13.** *Under the same conditions as in Theorem 2.11, let $T \subset V$. Then every solution to*

$$\Delta_\omega f(x) = 0, \quad x \in V \setminus T$$

*can be represented as*

$$f(x) = a_0 + \sum_{y \in T} \Gamma_\omega(x, y)\alpha(y), \quad x \in V \tag{2.27}$$

*where $a_0$ is an arbitrary constant and*

$$\alpha(y) = \Delta_\omega f(y), \quad y \in T.$$

In particular, if $T = \{x_0\},\ x_0 \in V$, then (2.27) can be written as

$$f(x) = a_0 + \alpha_0 \Gamma_\omega(x, x_0), \quad x \in V$$

where $\alpha_0 = \Delta_\omega f(x_0)$.

Let us now turn to boundary value problems and their eigenvalues. For a subgraph $S$ of a host graph $G$ with a weight $\omega$, the *Dirichlet eigenvalues* of $-L_\omega = -D^{1/2}\Delta_\omega D^{-1/2}$ are defined to be the eigenvalues

$$\nu_1 \leq \nu_2 \leq \cdots \leq \nu_n$$

of the matrix $-L_{\omega,S}$ where $L_{\omega,S}$ is a submatrix of $L_\omega$ with rows and columns restricted to those indexed by vertices in $S$ and $n = |S|$. Let $\phi_1, \phi_2, \cdots, \phi_n$ be the functions on $\overline{S}$ such that for each $j = 1, 2, \cdots, n$,

$$L_{\omega,S}\phi_j(x) = (-\nu_j)\phi_j(x), \quad x \in S \quad \text{and} \quad \phi_j|_{\partial S} = 0.$$

In fact, $\phi_1, \phi_2, \cdots, \phi_n$ are the eigenfunctions corresponding to $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_n$ and are orthonormal in the sense that for each pair of distinct $i$ and $j$

$$\sum_{x \in S} \phi_i(x) \cdot \phi_j(x) = 0,$$

while, for all j,

$$\sum_{x \in S} |\phi_j(x)|^2 = 1.$$

Then it is easy to verify that the first eigenvalue $\nu_1 > 0$, (see for instance, [20]).

One can follow now the standard procedure to define Green's functions $\gamma_{\omega,S}$ as follows :

$$\gamma_{\omega,S}(x,y) = \sum_{j=1}^{|S|} \big( - \frac{1}{\nu_j} \big) \phi_j(x) \phi_j(y) \frac{\sqrt{d_\omega y}}{\sqrt{d_\omega x}}, \quad x, y \in S \tag{2.28}$$

Letting $D_S$ stand for the diagonal matrix whose $x$-th entry is $d_\omega x$ for each $x \in S$ and setting $\Delta_{\omega,S} = D_S^{-1/2} L_{\omega,S} D_S^{1/2}$, one can easily verify that

$$\gamma_{\omega,S} \Delta_{\omega,S} = \Delta_{\omega,S} \gamma_{\omega,S} = I \tag{2.29}$$

and

$$\Delta_{\omega,S}(x,y) = \sum_{j=1}^{|S|} (-\nu_j) \phi_j(x) \phi_j(y) \frac{\sqrt{d_\omega y}}{\sqrt{d_\omega x}}, \quad x, y \in S. \tag{2.30}$$

where $I$ denotes the $|S|$-dimensional identity matrix.

The Dirichlet boundary value problem was solved by F.R.K.Chung in [21], when the graph has the standard weight. (For the interested reader, despite some minor errata, the proof given there is correct.) The following is the proof for the solvability of the Dirichlet boundary value problem for graphs with arbitrary weights using a different method.

**Theorem 2.14.** *Let $S$ be a subgraph of a host graph with a weight $\omega$ and $\sigma : \partial S \to \mathbb{R}$ be a given function. Then the unique solution $f$ to the Dirichlet boundary value problem (DBVP)*

$$\begin{cases} \Delta_\omega f(x) = 0, \quad x \in S, \\ f|_{\partial S} = \sigma \end{cases}$$

*can be represented as*

$$f(x) = -\langle \gamma_\omega(x, \cdot), B_\sigma \rangle_{y \in S}, \quad x \in S, \tag{2.31}$$

*where*

$$B_\sigma(y) = \sum_{z \in \partial S} \frac{\sigma(z)\omega(y,z)}{d_\omega y}, \quad y \in S. \tag{2.32}$$

*Proof.* Let $f$ be a solution of DBVP. Then

$$\begin{aligned} 0 &= \sum_{y \in S} \gamma_{\omega,S}(x,y) \Delta_\omega f(y) \qquad (2.33) \\ &= \sum_{j=1}^{|S|} \big( -\frac{1}{\nu_j} \big) \frac{\phi_j(x)}{\sqrt{d_\omega x}} \Big[ \sum_{y \in S} \phi_j(y) \sqrt{d_\omega y} \Delta_\omega f(y) \Big] \\ &= \sum_{j=1}^{|S|} \big( -\frac{1}{\nu_j} \big) \frac{\phi_j(x)}{\sqrt{d_\omega x}} \Big[ \int_S \big( D_S^{-1/2} \phi_j \big) \cdot \Delta_\omega f \Big] \\ &= \sum_{j=1}^{|S|} \big( -\frac{1}{\nu_j} \big) \frac{\phi_j(x)}{\sqrt{d_\omega x}} \Big[ \int_S f \cdot \Delta_\omega \big( D_S^{-1/2} \phi_j \big) \\ &\quad + \int_{\partial S} \Big\{ \big( D_S^{-1/2} \phi_j \big) \cdot \frac{\partial f}{\partial_\omega n} - f \cdot \frac{\partial}{\partial_\omega n} \big( D_S^{-1/2} \phi_j \big) \Big\} \Big]. \end{aligned}$$

Here, we have used Green's formula from the Corollary 2.2. On the other hand, one can show that

$$\Delta_\omega \big( D_S^{-1/2} \phi_j \big)(x) = (-\nu_j) \big( D_S^{-1/2} \phi_j \big)(x), \quad x \in S,$$

since $\phi_j = 0$ on $\partial S$. From this identity and orthonormality of $\phi_j$ we can conclude that

$$\begin{aligned}
&\sum_{j=1}^{|S|} \big( -\frac{1}{\nu_j}\big) \frac{\phi_j(x)}{\sqrt{d_\omega x}} \Big[ \int_S f \cdot \Delta_\omega \big( D_S^{-1/2} \phi_j \big) \Big] \\
&= \sum_{j=1}^{|S|} \frac{\phi_j(x)}{\sqrt{d_\omega x}} \cdot \Big[ \sum_{y \in S} f(y) \cdot \frac{\phi_j(y)}{\sqrt{d_\omega y}} \cdot d_\omega y \Big] \\
&= \sum_{y \in S} f(y) \Big[ \sum_{j=1}^{|S|} \phi_j(x) \phi_j(y) \sqrt{\frac{d_\omega y}{d_\omega x}} \ \Big] \\
&= f(x).
\end{aligned}$$

Hence, from the equality (2.33) and the fact that $\phi_j = 0$ on $\partial S$, we have

$$\begin{aligned}
f(x) &= \sum_{j=1}^{|S|} \big( -\frac{1}{\nu_j}\big) \frac{\phi_j(x)}{\sqrt{d_\omega x}} \int_{\partial S} \Big[ f \cdot \frac{\partial}{\partial_\omega n} \big( D_S^{-1/2} \phi_j \big) \Big] \\
&= \sum_{j=1}^{|S|} \big( -\frac{1}{\nu_j}\big) \frac{\phi_j(x)}{\sqrt{d_\omega x}} \Big[ \sum_{z \in \partial S} f(z) \cdot \frac{\partial}{\partial_\omega n} \big( D_S^{-1/2} \phi_j \big)(z) \cdot dz \Big] \\
&= \sum_{j=1}^{|S|} \big( -\frac{1}{\nu_j}\big) \frac{\phi_j(x)}{\sqrt{d_\omega x}} \sum_{z \in \partial S} \sigma(z) dz \Big[ \sum_{y \in S} \big\{ \frac{\phi_j(z)}{\sqrt{d_\omega z}} - \frac{\phi_j(y)}{\sqrt{d_\omega y}} \big\} \frac{\omega(z,y)}{d_\omega z} \Big] \\
&= -\sum_{j=1}^{|S|} \big( -\frac{1}{\nu_j}\big) \frac{\phi_j(x)}{\sqrt{d_\omega x}} \sum_{y \in S} \phi_j(y) \sqrt{d_\omega y} \big( \sum_{z \in \partial S} \frac{\sigma(z)\omega(z,y)}{d_\omega y} \big) \\
&= -\sum_{y \in S} \gamma_{\omega,S}(x,y) B_\sigma(y) \\
&= -\langle \gamma_{\omega,S}(x,\cdot), B_\sigma \rangle_S
\end{aligned}$$

for each $x \in S$. Moreover, with a simple calculation we see that every function of the form (2.31) gives a solution. The desired uniqueness result now follows easily from Theorem (2.4). $\square$

**Remark 2.15.** *(i) The identity (2.31) can be rewritten as*

$$f(x) = \sum_{j=1}^{|S|} \frac{1}{\nu_j} \sum_{y \in S} \Big[ \sum_{z \in \partial S} \frac{\sigma(z)\omega(y,z)}{d_\omega y} \Big] \phi_j(y)\phi_j(x) \sqrt{\frac{d_\omega y}{d_\omega x}}, \quad x \in S.$$

*In fact, $B_\sigma$ is a function on $S$ depending only on the value of $\sigma$ on $\partial S$ and $B_\sigma(y) = 0$ for $y \in S \backslash \overset{\circ}{\partial} S$. On the other hand, two different boundary conditions $\sigma_1$ and $\sigma_2$ may give rise to the same solution whenever $B_{\sigma_1} = B_{\sigma_2}$.*

*(ii) (2.31) can be understood as a matrix multiplication by*

$$f = -\gamma_{\omega,S} \cdot B_\sigma \text{ on } S \tag{2.34}$$

*or, equivalently,*

$$\Delta_{\omega,S} f = -B_\sigma \text{ on } S \tag{2.35}$$

*in view of (2.28). The relation(2.35) enables us to identify uniquely the boundary values from a $\omega$-harmonic function $f$ with $\Delta_\omega f = 0$ on $S$.*

Now we characterize the $\omega$-harmonic functions with a set of singularities in a subgraph with nonempty boundary.

**Theorem 2.16.** *Let $S$ be a subgraph of a graph with weight $\omega$ and $T \subset S$. Then every $f : \overline{S} \to \mathbb{C}$ satisfying*

$$\Delta_\omega f(x) = 0, \quad x \in S \setminus T$$

*can be uniquely represented as*

$$f(x) = h(x) + \sum_{y \in T} \gamma_{\omega,S}(x,y)\beta(y), \quad x \in \overline{S}, \tag{2.36}$$

*where $h$ is a $\omega$-harmonic function on $S$ satisfying $h|_{\partial S} = f|_{\partial S}$ and $\beta(y) = \Delta_\omega f(y)$, $y \in T$.*

*Proof.* The uniqueness is easy, by Theorem 2.4. Now let $\beta(y) = \Delta_\omega f(y)$, $y \in T$. Then we have

$$\Delta_\omega f(x) = \begin{cases} 0, & x \in S \setminus T, \\ \beta(x), & x \in T. \end{cases}$$

Define, for $x \in \overline{S}$,

$$f_1(x) := \sum_{y \in T} \gamma_{\omega,S}(x,y)\beta(y),$$

and

$$h(x) := f(x) - f_1(x).$$

Then $h|_{\partial S} = f|_{\partial S}$ and for each $x \in S$,

$$\begin{aligned} \Delta_\omega h(x) &= \Delta_\omega f(x) - \Delta_\omega \Big[ \sum_{y \in T} \sum_{j=1}^{|S|} -\frac{1}{\nu_j} \cdot \frac{\phi_j(x)}{\sqrt{d_\omega x}} \cdot \phi_j(y) \sqrt{d_\omega y} \beta(y) \Big] \\ &= \Delta_\omega f(x) - \sum_{y \in T} \sum_{j=1}^{|S|} \frac{\phi_j(x)}{\sqrt{d_\omega x}} \cdot \Big[ \phi_j(y) \sqrt{d_\omega y} \beta(y) \Big] \\ &= \Delta_\omega f(x) - \sum_{y \in T} \delta(x,y)\beta(y) \\ &= 0, \end{aligned}$$

which completes the proof. □

**Remark 2.17.** *(i) In particular, if* $T = \{x_0\}$, $x_0 \in S$, *then (2.36) can be written simply as*

$$f(x) = h(x) + \gamma_{\omega,S}(x, x_0)\beta(x_0),$$

*where* $\beta(x_0) = \Delta_\omega f(x_0)$.

*(ii) In fact, in view of (2.36) and Theorem 2.14, the solution to the nonhomoge-*

*neous DBVP*

$$\begin{cases} \Delta_\omega f(x)=g(x), \quad x \in S, \\ f|_{\partial S}=\sigma \end{cases}$$

*can be represented by*

$$f(x) = -\langle \gamma_{\omega,S}(x,\cdot), B_\sigma \rangle_S + \langle \gamma_{\omega,S}(x,\cdot), g \rangle_S.$$

*Now we will discuss the Neumann boundary value problem (NVBP). First, we recall Green's formula*

$$\int_S \Delta_\omega f = \int_{\partial S} \frac{\partial f}{\partial_\omega n}.$$

*Hence, if there exists a solution to*

$$\begin{cases} \Delta_\omega f = g \quad on \quad S, \\ \frac{\partial f}{\partial_\omega n} = \psi \quad on \quad \partial S, \end{cases}$$

*then by Green's formula it is necessary that*

$$\int_S g = \int_{\partial S} \psi$$

.

**Theorem 2.18.** *Let $S$ be a subgraph of a host graph $G$ with a weight $\omega$ and let $f : \overline{S} \to \mathbb{R}$, $g : S \to \mathbb{R}$, and $\psi : \partial S \to \mathbb{R}$ be functions with $\int_{\partial S} \psi = \int_S g$. Then the solution to the NBVP*

$$\begin{cases} \Delta_\omega f(x) = g(x), \quad x \in S, \\ \frac{\partial f}{\partial_\omega n}(z) = \psi(z), \quad z \in \partial S \end{cases}$$

*is given by*

$$f(x) = a_0 + \langle \Gamma_\omega(x,\cdot), g \rangle_S - \langle \Gamma_\omega(x,\cdot), \psi \rangle_{\partial S},$$

*where $\Gamma_\omega$ is the Green's function of $\Delta_\omega$ on the graph $\overline{S}$ as a new host graph of $S$ and $a_0$ is an arbitrary constant.*

*Proof.* We rewrite (NBVP) as

$$\begin{cases} \sum_{y\in\overline{S}} \big[f(y)-f(x)\big]\frac{\omega(x,y)}{d_\omega x} = g(x), x\in S, \\ \sum_{y\in S} \big[f(y)-f(z)\big]\frac{\omega(y,z)}{d'_\omega z} = -\psi(z), z\in \partial S. \end{cases} \tag{2.37}$$

To solve the system (2.37), consider $\overline{S}$ as a new host graph with the weight $\omega$ and with no boundary. Then $S$ is still a subgraph of $\overline{S}$. (In fact, we should note here that if we regard $\overline{S}$ as a subgraph of $G$, then its boundary $\partial\overline{S}$ may not be empty.) Then, for each $z\in\partial S$, the inner degree $d'_\omega z$ is equal to $d_\omega z$ in this new graph $\overline{S}$, since the induced subgraph has no edges between the vertices on $\partial S$. Hence the equation (2.37) can be written as

$$\begin{cases} \sum_{y\in\in V_0} \big[f(y)-f(x)\big]\frac{\omega(x,y)}{d_\omega x} = g(x), x\in S, \\ \sum_{y\in V_0} \big[f(y)-f(z)\big]\frac{\omega(y,z)}{d'_\omega z} = -\psi(z), z\in \partial S. \end{cases} \tag{2.38}$$

where $V_0$ is the set of vertices in $\overline{S}$. Hence (2.38) is equivalent to

$$\sum_{y\in V_0} \big[f(y)-f(x)\big]\frac{\omega(x,y)}{d_\omega x} = \Psi(x), \quad x\in\overline{S}, \tag{2.39}$$

where

$$\Psi(x) = \begin{cases} g(x), x\in S, \\ -\psi(x), x\in\partial S. \end{cases}$$

Therefore, (NBVP) is equivalent to

$$\Delta_\omega f(x) = \Psi(x), \quad x\in\overline{S}.$$

Thus, it follows from Theorem 2.11 that

$$\begin{aligned} f(x) &= a_0 + \langle \Gamma_\omega(x,\cdot), \Psi \rangle_{x\in V_0} \\ &= a_0 + \sum_{y\in V_0} \Gamma_\omega(x,y)\Psi(y) \\ &= a_0 + \sum_{y\in S} \Gamma_\omega(x,y)g(y) - \sum_{z\in \partial S} \Gamma_\omega(x,z)\psi(z) \\ &= a_0 + \langle \Gamma_\omega(x,\cdot), g\rangle_S - \langle \Gamma_\omega(x,\cdot), \psi\rangle_{\partial S}, \end{aligned}$$

where $a_0$ is an arbitrary constant. This completes the proof.

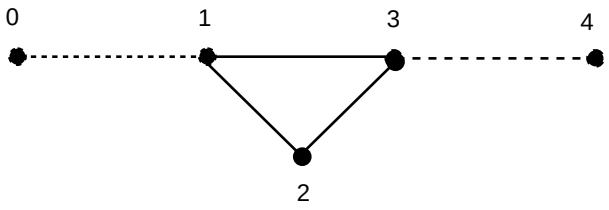
### *2.2.4 Inverse Problems*

In the previous section, we have seen that for a function $\psi : \partial S \to \mathbb{R}$ with $\int_{\partial S} \psi = 0$ the Neumann boundary value problem

$$(NBVP) \begin{cases} \Delta_\omega f(x) = 0, \quad x \in S, \\ \frac{\partial f}{\partial_\omega n}(z) = \psi(z), \quad z \in \partial S \end{cases}$$

has a unique solution up to an additive constant. Therefore, the Dirichlet data $f|_{\partial S}$ is well-defined up to an additive constant. In this section, we will discuss the inverse conductivity problem on the network (graph) $S$ with nonempty boundary, which consists in recovering the conductivity (connectivity or weight) $\omega$ of the graph by using, the so called input-output map, for example by using the Dirichlet data induced by the Neumann data (Neumann-to-Dirichlet map), with one boundary measurement. In order to deal with this inverse problem, we need at least to know or be given the boundary data such as $f(z)$, $\frac{\partial f}{\partial_\omega n}(z)$ for $z \in \partial S$ and $\omega$ near the boundary. So it is natural to assume that $f|_{\partial S}$, $\frac{\partial f}{\partial_\omega n}|_{\partial S}$ and $\omega|_{\partial S \times \mathring{\partial} S}$ are known

(given or measured). But even though we are given all these data on the boundary, we are not guaranteed, in general, to be able to identify the conductivity $\omega$ uniquely. To illustrate this we consider a graph $S$ whose vertices are $\{1, 2, 3\}$ and $\partial S = \{0, 4\}$ as follows:




Let the weight $\omega$ given by

$$\omega(0,1) = 1,\ \omega(0,k) = 0\ \ (k = 2,\ 3,\ 4),$$

and

$$\omega(3,4) = 1,\ \omega(k,4) = 0\ \ (k = 0,\ 1,\ 2).$$

Let $f : \overline{S} \to \mathbb{R}$ be a function satisfying $\Delta_\omega f(k) = 0,\ k = 1, 2, 3$. Assume that

$$f(0) = 0,\ f(1) = 1,\ f(3) = 3,\ f(4) = 4,\ f(2) = (\text{unknown}).$$

Thus, since $\overset{\circ}{\partial} S = \{1, 3\}$, the boundary data $f|_{\partial S}$, $\frac{\partial f}{\partial_\omega n}|_{\partial S}$ and $\omega|_{\partial S \times \overset{\circ}{\partial} S}$ are known. In fact,

$$\frac{\partial f}{\partial_\omega n}(0) = f(0) - f(1) = -1,$$
$$\frac{\partial f}{\partial_\omega n}(4) = f(4) - f(3) = 1.$$

The problem is to determine

$$\omega(1,2) = x,\ \ \omega(2,3) = y,\ \ \omega(1,3) = z,\quad \text{and}\quad f(2).$$

From $\Delta_\omega f(k) = 0,\ k = 1, 2, 3$, we have

$$f(1) = \frac{f(0) + xf(2) + 3z}{1 + x + z} = 1,$$

$$f(2) = \frac{xf(1) + yf(3)}{x + y},$$

$$f(3) = \frac{zf(1) + yf(2) + f(4)}{z + y + 1} = 3.$$

This system is equivalent to

$$\begin{cases} x(y-1) + y(x-1) + 2z(x+y) = 0, \\ f(2) = \frac{x+3y}{x+y}. \end{cases} \tag{2.40}$$

*The above system has infinitely many solutions. For instance, assume $z = 0$, that is, the two vertices 1 and 3 are not adjacent. Then (2.40) is reduced to*

$$\begin{cases} \frac{1}{x} + \frac{1}{y} = 2, \\ f(2) = \frac{x+3y}{x+y}. \end{cases} \tag{2.41}$$

For example, $(x, y, z) = (1, 1, 0)$ or $(2, 2/3, 0)$ satisfy the first equation. In fact, it is easy to see that there are infinitely many pairs $(x, y)$ of nonnegative numbers satisfying the first equation in (2.41), so that $f(2)$ is undetermined as a result.

In view of the above example, in order to determine the weight $\omega$ uniquely we need some more information than just $f|_{\partial S}$, $\frac{\partial f}{\partial_\omega n}|_{\partial S}$ and $\omega|_{\partial S \times \mathring{\partial} S}$. To motivate the main theorem we impose in this example the additional constraints that

$$x \geq 1,\ y \geq 1 \text{ and } z \geq 0 \tag{2.42}$$

in (2.40). Then the equation (2.40) yields a unique triple of solution $x = 1,\ y = 1,\ z = 0$ and $f(2) = 2$.

As a matter of fact, even the inverse conductivity problem of the following diffusion equation in a bounded open subset $\Omega \subset \mathbb{R}^n$

$$P[a;u] = \begin{cases} div[a(x)\nabla u(x)] = 0, \quad x \in \Omega, \\ u|_{\partial\Omega} = \sigma \end{cases} \tag{2.43}$$

has been studied under some additional constraints besides Dirichlet and Neumann data (see [17, 18, 33, 34, 47]). In particular, in [33] it is shown that there is a global uniqueness result under the condition that

(i) $\mathrm{a}_1 = a_2$ near $\partial\Omega$, $a_1 \le a_2$ in $\Omega$,

(ii) $\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n}$ on $\partial\Omega$,

(iii) $\int_\Omega u_1 = \int_\Omega u_2 = 0$, where $\mathrm{P}[\mathrm{a}_j; u_j] = 0,\ j = 1, 2$ in (2.43).

Now we are in position to state the first main theorem of this section.

***Theorem 2.19.*** *Let $\omega_1$ and $\omega_2$ be weights with $\omega_1 \le \omega_2$ on $\overline{S} \times \overline{S}$, and $f_1$, $f_2 : \overline{S} \to \mathbb{R}$ be functions satisfying for $j = 1, 2$,*

$$\begin{cases} \Delta_{\omega_j} f_j(x) = 0,\ x \in S \\ \frac{\partial f_j}{\partial n_{\omega_j}}(z) = \Psi(z),\ z \in \partial S \end{cases}$$

*for any given function $\Psi : \partial S \to \mathbb{R}$ with $\int_{\partial S} \Psi = 0$, $j = 1, 2$. If it is assumed that*

$$i)\, \omega_1(z, y) = \omega_2(z, y) \text{ on } \partial S \times \overset{\circ}{\partial} S,$$

$$ii)\, f_1|_{\partial S} = f_2|_{\partial S},$$

*then*

$$f_1 \equiv f_2, \quad \text{on } \overline{S},$$

*and*

$$\omega_1(x,y) = \omega_2(x,y), \quad \textit{whenever} \quad f_1(x) \neq f_1(y), \quad \textit{or} \quad f_2(x) \neq f_2(y)$$

To prove this result we adapt the method of energy functionals, extensively used for theory of nonlinear partial differential equations. For functions $\sigma : \partial S \to \mathbb{R}$ and $g : \overline{S} \to \mathbb{R}$ we define a functional by

$$I_\omega[h] := \int_{\overline{S}} \quad [\frac{1}{4}|\nabla_\omega h|^2 - hg] \tag{2.44}$$

for every function $h$ in the set

$$A := \left\{ h : \overline{S} \to \mathbb{R} | \quad h|_{\partial S} = \sigma \right\}, \tag{2.45}$$

which is called the admissible set. In the continuous case, the well known Dirichlet's principle states that the energy minimizer in the admissible set is a solution of the Dirichlet boundary value problem. We derive here the discrete version of Dirichlet's principle as follows :

**Theorem 2.20.** *(Dirichlet's principle) Assume that $f : \overline{S} \to \mathbb{R}$ is a solution to*

$$\begin{cases} -\Delta_\omega f = g \ \textit{on} \ S, \\ f|_{\partial S} = \sigma. \end{cases} \tag{2.46}$$

*Then*

$$I_\omega[f] = \min_{h \in A} I_\omega[h]. \tag{2.47}$$

*Conversely, if $f \in A$ satisfies (2.45), then $f$ is the solution of (2.44), and the only one.*

*Proof.* Let $h$ be a function in $A$. Then making use of (2.12) in Theorem 2.1 we have

$$\begin{aligned}
0 &= \int_{\overline{S}} (-\Delta_\omega f - g)(f-h) \\
&= \int_{\overline{S}} \big[(-\Delta_\omega f)(f-h) - g(f-h)\big] \\
&= \int_{\overline{S}} \big[\frac{1}{2}\nabla_\omega f \cdot \nabla_\omega (f-h) - g(f-h)\big] \\
&= \frac{1}{2}\int_{\overline{S}} |\nabla_\omega f|^2 - \frac{1}{2}\int_{\overline{S}} \nabla_\omega f \cdot \nabla_\omega h - \int_{\overline{S}} g(f-h).
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{\overline{S}} \big[\frac{1}{2}|\nabla_\omega f|^2 - gf\big] &= \int_{\overline{S}} \big[\frac{1}{2}\nabla_\omega f \cdot \nabla_\omega h - gh\big] \\
&\le \frac{1}{2}\sum_{x\in\overline{S}}\sum_{y\in\overline{S}} \big|\big[f(y)-f(x)\big]\cdot\big[h(y)-h(x)\big]\big| \cdot \omega(x,y) - \int_{\overline{S}} gh \\
&\le \frac{1}{2}\sum_{x\in\overline{S}}\sum_{y\in\overline{S}} \frac{\big[f(y)-f(x)\big]^2 + \big[h(y)-h(x)\big]^2}{2} \cdot \omega(x,y) - \int_{\overline{S}} gh \\
&= \frac{1}{4}\int_{\overline{S}} |\nabla_\omega f|^2 + \frac{1}{4}\int_{\overline{S}} |\nabla_\omega h|^2 - \int_{\overline{S}} gh,
\end{aligned}$$

where we used the triangular inequality

$$|ab| \le \frac{a^2+b^2}{2}, \quad a, b \in \mathbb{R}.$$

Thus, it follows that

$$\int_{\overline{S}} \big[\frac{1}{4}|\nabla_\omega f|^2 - gf\big] \le \int_{\overline{S}} \big[\frac{1}{4}|\nabla_\omega h|^2 - gh\big],$$

which implies

$$I_\omega[f] \le I_\omega[h], \quad h \in A.$$

Since $f \in A$, we have

$$\min_{h\in A} I_\omega[h] = I_\omega[f].$$

Now we prove the converse. Let $T$ be a subset of vertices in $S$ and

$$\chi_T(x) = \begin{cases} 1, \, x \in T \\ \\ 0, \text{ otherwise.} \end{cases}$$

Then $f + \tau\chi_T \in A$ for each real number $\tau$, since $\chi_T = 0$ on $\partial S$. Define

$$i(\tau) := I_\omega[f + \tau\chi_T], \quad \tau \in \mathbb{R}.$$

Then

$$\begin{aligned} i(\tau) &= \int_{\overline{S}} \big[\frac{1}{4}|\nabla_\omega f + \tau\nabla_\omega\chi_T|^2 - (f + \tau\chi_T)g\big] \\ &= \frac{1}{4}\int_{\overline{S}} |\nabla_\omega f|^2 + 2\tau\nabla_\omega f \cdot \nabla_\omega\chi_T + \tau^2|\nabla_\omega\chi_T|^2 - \int_{\overline{S}}(f + \tau\chi_T)g. \end{aligned}$$

Note that the scalar function $i(\tau)$ has a minimum at $\tau = 0$ and thus $\frac{di}{d\tau}(0) = 0$. That is,

$$\begin{aligned} 0 &= \frac{1}{2}\int_{\overline{S}} \nabla_\omega f \cdot \nabla_\omega\chi_T - \int_{\overline{S}} \chi_T \cdot g \\ &= \int_{\overline{S}} \big[\chi_T(-\Delta_\omega f - g)\big] \\ &= \sum_{x\in T} \big[ - \Delta_\omega f(x) - g(x)\big] d_\omega x. \end{aligned}$$

In particular, taking $T = \{x\}, x \in S$, we obtain

$$-\Delta_\omega f(x) - g(x) = 0,$$

which is the required result. The uniqueness follows from Theorem 2.14. □

Now we are ready to prove Theorem 2.19.

*Proof.* (Proof of Theorem 2.19.)

(i) Let $\sigma : \partial S \to \mathbb{R}$ be the function defined by

$$\sigma(z) = f_1(z) = f_2(z), \quad z \in \partial S,$$

using the hypothesis (ii). Define

$$I_{\omega_1}[h] := \frac{1}{4}\int_{\overline{S}} |\nabla_{\omega_1} h|^2 d_{\omega_1}$$

for every $h$ in the admissible set

$$A = \left\{ h : \overline{S} \to \mathbb{R} | \quad h|_{\partial S} = \sigma \right\}.$$

Then, by virtue of Theorem 2.1 we have

$$\begin{aligned} I_{\omega_1}[h] &= \frac{1}{2}\int_{\overline{S}} h(-\Delta_{\omega_1} h) d_{\omega_1} \\ &= \frac{1}{2}\int_{S} h(-\Delta_{\omega_1} h) d_{\omega_1} + \frac{1}{2}\int_{\partial S} h(-\Delta_{\omega_1} h) d_{\omega_1}. \end{aligned}$$

Moreover, by the coincidence of the Dirichlet and Neumann data we can see that the boundary $\partial S$ and the inner boundary $\overset{\circ}{\partial} S$ are well-defined independently of the values of the weights $\omega_1,\ \omega_2$ and, moreover, for $z \in \partial S$

$$d_{\omega_1} z = \sum_{y \in \overset{\circ}{\partial} S} \omega_1(z,y) = \sum_{y \in \overset{\circ}{\partial} S} \omega_2(z,y) = d_{\omega_2} z, \tag{2.48}$$

$$\begin{aligned} \Delta_{\omega_1} f_1(z) &= \sum_{y \in \overset{\circ}{\partial} S} \big[ f_1(y) - f_1(z) \big] \frac{\omega_1(z,y)}{d_{\omega_1} z} \\ &= \sum_{y \in \overset{\circ}{\partial} S} \big[ f_2(y) - f_2(z) \big] \frac{\omega_2(z,y)}{d_{\omega_2} z} \\ &= \Delta_{\omega_2} f_2(z). \end{aligned} \tag{2.49}$$

Then, it follows from the condition $\omega_1 \leq \omega_2$ that

$$\begin{aligned}
I_{\omega_1}[f_1] &= \frac{1}{2}\int_{\partial S} f_1(-\Delta_{\omega_1} f_1) d_{\omega_1} \\
&= \frac{1}{2}\int_{\partial S} f_2(-\Delta_{\omega_2} f_2) d_{\omega_1} \\
&= \frac{1}{2}\int_{S} f_2(-\Delta_{\omega_2} f_2) d_{\omega_2} + \frac{1}{2}\int_{\partial S} f_2(-\Delta_{\omega_2} f_2) d_{\omega_2} \\
&= \frac{1}{2}\int_{\overline{S}} f_2(-\Delta_{\omega_2} f_2) d_{\omega_2} \\
&= \frac{1}{4}\int_{\overline{S}} |\nabla_{\omega_2} f_2|^2 d_{\omega_2} \\
&= \frac{1}{4}\sum_{x\in\overline{S}}\sum_{y\in\overline{S}} \big[f_2(x) - f_2(y)\big]^2 \omega_2(x,y) \\
&\geq \frac{1}{4}\sum_{x\in\overline{S}}\sum_{y\in\overline{S}} \big[f_2(x) - f_2(y)\big]^2 \omega_1(x,y) \\
&= \frac{1}{4}\int_{\overline{S}} |\nabla_{\omega_1} f_2|^2 d_{\omega_1} \\
&= I_{\omega_1}[f_2]
\end{aligned}$$

Using Dirichlet's principle (Theorem 2.20) one sees that $f_1 = f_2$ on $\overline{S}$.

(ii) In the proof of (i) it is actually proven that $I_{\omega_1}[f_1] = I_{\omega_1}[f_2]$. In other words, taking $f := f_1 = f_2$ on $\overline{S}$

$$\sum_{x\in\overline{S}}\sum_{y\in\overline{S}} \big[f(x) - f(y)\big]^2 \omega_2(x,y) = \sum_{x\in\overline{S}}\sum_{y\in\overline{S}} \big[f(x) - f(y)\big]^2 \omega_1(x,y),$$

or, equivalently

$$\sum_{x\in\overline{S}}\sum_{y\in\overline{S}} \big[f(x) - f(y)\big]^2 \cdot \big[\omega_2(x,y) - \omega_1(x,y)\big] = 0.$$

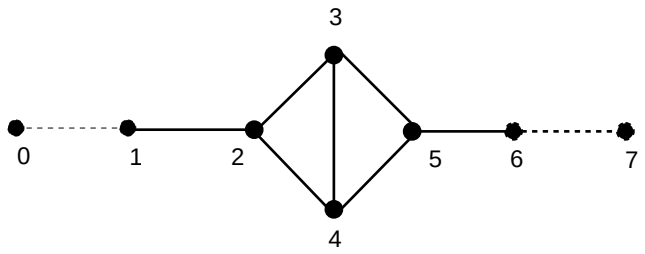
Therefore, we have

$$\big[f(x) - f(y)\big]^2 \cdot \big[\omega_2(x,y) - \omega_1(x,y)\big] = 0,$$

for all $x \in \overline{S}$ and $y \in \overline{S}$. This gives (ii). □

**Remark 2.21.** *In Theorem 2.19 above, if $f := f_1 = f_2$ is injective on $S$ then we are able to get $\omega_1 = \omega_2$ on $\overline{S} \times \overline{S}$. For example, if $S$ is the path $P_n$ on $n$ vertices with arbitrary weight $\omega$, then it is not hard to see that every nonconstant $\omega$-harmonic function $f$ on $P_n$ is strictly monotonic and hence all the weights are identified. But, in general, most graphs, even with the standard weight do not admit an injective solution to the DBVP or NBVP. Therefore, it will be quite interesting to figure out a pair of graphs and weights which admits an injective solution to the DBVP or NBVP.*

To get an idea on how to improve Theorem 2.19 we consider a graph $S = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$ with $\partial S = \{0,\ 7\}$ as follows:




Suppose that $\omega_1$ is the standard weight and $\omega_2$ is the weight given by $\omega_1 = \omega_2$ except only $\omega_2(3,\ 4) = k,\ k \geq 1$. Then $\omega_1 \leq \omega_2$ throughout the graph $\overline{S}$ and $\omega_1 = \omega_2$ except on the edge $\{3, 4\}$. Now define a function $f : \overline{S} \to \mathbb{R}$ as

$$f(0) = a,\ f(1) = a - \alpha,\ f(2) = a - 2\alpha,\ f(3) = f(4) = \frac{2a - 5\alpha}{2},$$

$$f(5) = a - 3\alpha,\ f(6) = a - 4\alpha,\ f(7) = a - 5\alpha,$$

where $a$ and $\alpha$ are arbitrary real numbers. Then it is easy to verify that $f$ satisfies both the equations

$$\Delta_{\omega_1} f(x) = 0 = \Delta_{\omega_2} f(x), \quad x \in S.$$

Here, we note that $f$ is uniquely determined by the Dirichlet data

$$f(0) = a, \quad f(7) = a - 5\alpha$$

and the Neumann data

$$\frac{\partial f}{\partial_\omega n}(0) = f(0) - f(1) = \alpha,$$

$$\frac{\partial f}{\partial_\omega n}(7) = f(7) - f(6) = -\alpha$$

and each value $f(x)$ is determined regardless of the weight $\omega_2(3,4) = k$. This implies that we cannot identify the weight $\omega_2(3,\ 4) = k$ even with all possible boundary data. To figure out how to overcome this difficulty, we take $a > 0$ and $\alpha$ so that $f(0) > 0$ and $f(7) > 0$. By a direct calculation (or, using Corollary 2.5) we see that

$$f(m) > 0,\ m = 0,\ 1,\ 2,\ \cdots,\ 7.$$

Suppose that $f$ satisfies the relation

$$\int_S f d_{\omega_1} = \int_S f d_{\omega_2}\ . \tag{2.50}$$

Then, since

$$\int_S f d_{\omega_1} = 2f(1) + 3f(2) + 3f(3) + 3f(4) + 3f(5) + 2f(6)$$

and

$$\int_S f d_{\omega_2} = 2f(1) + 3f(2) + (2+k)f(3) + (2+k)f(4) + 3f(5) + 2f(6),$$

it follows that

$$k\big[f(3) + f(4)\big] = f(3) + f(4),$$

which gives $k = 1$. Therefore, in order to identify the weight over all edges we need to impose an additional condition such as (2.50).

Now we return to the general situation. We know that for a function $\psi : \partial S \to \mathbb{R}$ with $\int_{\partial S} \psi = 0$ and $j = 1,\ 2$, the equation

$$\begin{cases} \Delta_{\omega_j} h_j(x) = 0,\ x \in S \\ \frac{\partial h_j}{\partial n_{\omega_j}}(z) = \psi(z),\ z \in \partial S \\ \int_S h_j d_{\omega_j} = 0 \end{cases} \tag{2.51}$$

has a unique pair of solution $(h_1, h_2)$. Let

$$m_j = \min_{z \in \partial S}\ h_j(z),\ j = 1, 2 \tag{2.52}$$

and

$$m_0 = \max_{j=1,2}\ |m_j| \cdot \mathrm{vol}(S, \omega_j), \tag{2.53}$$

where $\mathrm{vol}(S, \omega_j) = \sum_{x \in S} d_{\omega_j} x$. Motivated by the above example we refine Theorem 2.19 as follows:

**Theorem 2.22.** *Let $\omega_1$ and $\omega_2$ be weights with $\omega_1 \leq \omega_2$ on $\overline{S} \times \overline{S}$, and $f_1,\ f_2 : \overline{S} \to \mathbb{R}$ be functions satisfying for $j = 1, 2$,*

$$\begin{cases} \Delta_{\omega_j} f_j(x) = 0,\ x \in S \\ \frac{\partial f_j}{\partial n_{\omega_j}}(z) = \psi(z),\ z \in \partial S \\ \int_S f_j d_{\omega_j} = K \end{cases} \tag{2.54}$$

*for any given function $\psi : \partial S \to \mathbb{R}$ with $\int_{\partial S} \psi = 0$, and a given constant $K$ with the condition $K > m_0$, where $m_0 = \max_{j=1,2} |m_j| \cdot vol(S, w_j)$, $m_j = \min_{z \in \partial S} f_j(z), j = 1, 2$*

*and* $vol(S, w_j) = \sum_{x \in S} d_{\omega_j} x$. *If it is assumed that*

$$i)\, \omega_1(z, y) = \omega_2(z, y) \text{ on } \partial S \times \overset{\circ}{\partial} S,$$

$$ii)\, f_1|_{\partial S} = f_2|_{\partial S},$$

*then*

$$f_1 \equiv f_2,$$

*and*

$$\omega_1(x, y) = \omega_2(x, y),$$

*for all* $x$ *and* $y$ *in* $\overline{S}$.

*Proof.* We have already shown in Theorem 2.19 that $f_1 \equiv f_2$ . Now, for each $j = 1,\ 2$, we choose a constant $C_j$ so that $C_j \cdot \text{vol}(S, \omega_j) = K$. Then, it follows that $C_j > |m_j|$ and, hence, $h_j(x) + C_j > 0,\ x \in S$ by the maximum principle (or, Corollary 2.5). Moreover, the function $\tilde{h}(x) := h_j(x) + C_j$ satisfies the equation (2.54). By the uniqueness of the solution we have

$$f_j(x) = \tilde{h}(x) = h_j(x) + C_j > 0,\ x \in S.$$

Let $f := f_1 = f_2$ on $\overline{S}$. Then it follows from the condition

$$\int_S f_1\ d_{\omega_1} = K = \int_S f_2\ d_{\omega_2}$$

that

$$\sum_{x \in S} f(x) d_{\omega_1}(x) = \sum_{x \in S} f(x) d_{\omega_2}(x),$$

or, equivalently

$$\sum_{x \in S} f(x) \big[ d_{\omega_2}(x) - d_{\omega_1}(x) \big] = 0.$$

Since $f(x) > 0$ and $d_{\omega_1}(x) \geq d_{\omega_1}(x)$ for all $x \in S$, we have

$$\begin{aligned} 0 &= d_{\omega_2}(x) - d_{\omega_1}(x) \\ &= \sum_{\in \overline{S}} \big[\omega_2(x,y) - \omega_2(x,y)\big]. \end{aligned}$$

Since $\omega_1(x,y) \leq \omega_2(x,y)$, we obtain

$$\omega_1(x,y) = \omega_2(x,y)$$

for all $x$ and $y$ in $\overline{S}$, which is the required. □

The second conclusion of Theorem 2.22 shows not only whether or not each pair of nodes is connected by a link, but also how nice the link is. Moreover, the proof gives an algorithm to detect if the weights change on the edges.

**Remark 2.23.** *In the above proof, the condition $K > m_0$ was used only to guarantee that $f_j(x) > 0,\ x \in S$. Hence, if we replace this condition by $f|_{\partial S} > 0,\ j = 1,\ 2$ in Theorem 2.22, we arrive at the same conclusion. Practically, all the positive solutions are easily available by adjusting the boundary values.*

## *2.3 Results*

We have discussed the inverse problem of recovering the conductivity of the links between adjacent pair of nodes in a network based on the Neumann-to-Dirichlet map. In order to do this we consider the weighted Laplacian $\Delta_\omega$ and an $\omega$-harmonic function on the graph with its physical interpretation as a diffusion equation on the graph, which models an electric network. Basic properties of $\omega$-harmonic functions

are derived and then the solvability of problems such as the Neumann and Dirichlet boundary value problems is proven. The main result is the global uniqueness of the inverse conductivity problem for a network where the conditions $\omega_1 \leq \omega_2$ (monotonicity condition) and $\int_G f_j d_{\omega_j} = K$ (the normalization condition) are essential for the uniqueness of the result.

## *2.4 Conclusions and Future Research*

### *2.4.1 Conclusions*

In this thesis, some theoretical aspects of Electrical Impedance Tomography have been adapted to the weighted graph model. In this model, there are two kinds of disruptions of traffic data that could arise. In one of them, disruptions occurs when an edge "ceases" to exist, in this case the "topology" of the graph has changed, and we refer to the important work of Fan Chung and her collaborators which offers crucial insights into this question. (See, for instance [19, 20, 22]. In the other, the weights change because of "increase" of traffic, that is, the network configuration remains the same but the weights have either increased or remained the same (substantially in some parts of the network). Associated to the weight $\omega$ there is a Laplace operator in the network we consider the response to diagnostic "proves" applied to the outside boundary. The boundary observations (outputs) correspond to the Neumann-to-Dirichlet map for the Laplacian $\Delta_\omega$. In this second situation, we appeal to Theorem 2.22 which shows that the Neumann-to-Dirichlet map for $\Delta_{\omega'}$ is different to that for $\Delta_\omega$. The condition that $\Delta_\omega f(x) = 0$ corresponds to the fact

that the value $f(x)$ is the weighted average of the values of $f$ at the adjacent nodes.

Note that the Neumann boundary value problem $NBVP$

$$\begin{cases} \Delta_\omega f(x) = 0, \ x \ \ in \ \ S \\ \frac{\partial f}{\partial_\omega n}(z) = \Phi(z), \ z \in \ \ \partial S \end{cases} \tag{2.55}$$

has a unique solution up to an additive constant; therefore, the Dirichlet data $f\,|_{\partial S}$, $z \in \partial S$ is well-defined up to an additive constant. In this thesis we have discussed the inverse conductivity problem on the network (graph) $S$ with nonempty boundary. In order to deal with this problem, we need at least to know or be given the boundary data such as $f(x)$, $\frac{\partial f}{\partial_\omega n}(z)$ for $z \in \partial S$ and $\omega$ near the boundary. So it is natural to assume that $f\,|_{\partial S}$, $\frac{\partial f}{\partial_\omega n}|_{\partial S}$ and $\omega|_{\partial S \times \overset{\circ}{\partial S}}$ are known (given or measured). But even though we are given all these data on the boundary, we are not guaranteed, in general, to be able to identify the conductivity $\omega$ uniquely.

While Theorem 2.22 is only a uniqueness theorem, nevertheless, we can effectively compute the actual weights from the knowledge of the Dirichlet data for convenient choices of the input Neumann data in a way similar to that done in [26] and [28] for lattices. Similarly, the Green function of this Neumann boundary value problem can be represented by an explicit matrix. What we have discussed is the relationship between the above results to the problem of understanding a large network like the internet. One way to make more concrete this problem was discussed by T. Munzner in [40, 41] on visualizing the internet. It implies that the natural domain might be a hyperbolic space of dimension higher than 2. As we stated before, Munzner's suggestion leads to a question closely resembling EIT, and it is natural to consider it a problem in hyperbolic tomography [5, 6]. On the other hand, we have

just obtained a significant result on the inversion of the Neumann-Dirichlet problem by studying it directly on "weighted graphs "[9]. Similarly, the Radon transform in the hyperbolic plane has been studied in [5, 6, 32]. Also, experimental evidence indicates that at least locally, the network could be modelled as being part of a tree and therefore it can be visualized using 2-dimensional hyperbolic geometry. As a consequence, a different way to study locally this kind of networks would be by the use of the Radon transform on trees. As it turns out, an inversion formula for the Radon transform on trees is already known and was shown in section 1.3 and it can also be found in [7]. For the sake of completeness, we have described in sections 1.3.3 and 1.3.4 a simplified version of the Radon transform on trees and its inversion formula. As explained there, this would seem enough to deal with the network problems we are interested in.

### *2.4.2 Future Research*

There are many questions still to be answered, for instance what happens if the number of nodes is not finite? What is the hyperbolic version of the discrete case for arbitrary weighted graphs?. What is the analogue of the Radon transform in arbitrary graphs?. If we allow to consider also $\omega = 0$ then the presence of zero weights tells us that the conductivity on the edge (a particular one) is either down or the nodes connected to that edge "disappear" in the sense that the edge length becomes infinite and this is because uniqueness is not true.

It is natural to try to develop a wavelet-like theory to implement explicitly the determination of the traffic loads along the edges of a network. As seen in the study

of the inverse conductivity problem in the continuous case (see, for instance, [31] [18], [33], [34], [47]) it would be worthwhile to prove the uniqueness under a condition weaker than the monotonicity condition $\omega_1 \leq \omega_2$ imposed above. Moreover, it would also be interesting to consider a stability theorem for the same conductivity equation.

We have seen that the monotonicity condition $\omega_1 \leq \omega_2$ is essential for the uniqueness result in Theorem 2.22 but if the weight decreased then one could think that the topology of the graph has changed. Therefore, if we were to solve a traffic congestion problem in a network (graph) it is my guess that we could consider adding suitable links to the network.

We also wondered if it is possible to simultaneously recover the weight and determine the configuration for a sparse weighted graph.

I have made some minor modifications to a computer code written by S. H. Shepard IV and based on the algorithm of Curtis and Morrow just to show that for a square resistor network the knowledge of the boundary data allows to solve the inverse conductivity problem. I plan to implement a code to solve the problem for more general networks. It is expected to consider also realistic scale networks and this will require to analize very effective linear algebra methods underlying the solution of these problems since the systems of equations are very large.

# APPENDIX

Solution of the inverse conductivity problem for a square network

This program creates the Neumann-to Dirichlet matrix $\Lambda$ for a 20x20 square resistor network where the weight (conductivity) is the standard weight.

```
with(linalg);
Digits:=30:
N:=20:
edges:=:
TN:=N*(N+4):
for i from 1 by 1 to N do
edges:=edges union [i, (i+4)*N]:
edges:=edges union  [N+i, TN+1-i]:
edges:=edges union [2*N +i, TN - i*N +1]
od:
for j from 4*N+1 by 1 to TN do
edges:=edges union [j-N, j]
od:
Nd:=N*(N-1):
for i from 1 to N do
for j from 2 to N do
edges:=edges union [(3+i)*N+j-1, (3+i)*N+j]
od:
od:
k:=matrix(TN, TN, 0):
```

```
conduct:=1.0:

edge[1.0]:=edges:

for i in conduct do

for p in edge[i] do

k[p[1], p[1]]:=k[p[1], p[1]] + i:

k[p[2], p[2]]:=k[p[2], p[2]] + i:

k[p[1], p[2]]:= -i:

k[p[2], p[1]]:= -i

od;

od;

c:=submatrix(k, 4*N+1..TN, 4*N+1..TN):

b:=submatrix(k, 1..4*N, 4*N+1..TN):

a:=submatrix(k, 1..4*N, 1..4*N):

x:=linsolve(c, transpose(b)):

Lambda:=matadd(a,-multiply(b,x));save(N,conduct,Lambda,‘Lambda.m‘);
```

This program takes the Dirichlet-to-Neumann map stored in the file Lambda.m and uses it to compute the conductivity in the network.

```
with(linalg):
read(‘Lambda.m‘):
Digits:=40:
bN:=4*N:
gammav:=matrix(N+1, N, 0):
gammah:=matrix(N, N+1, 0):
mu:=matrix(N, N, 0):
j:=4*N: k:=1: i:=3*N:
gammav[1, 1]:= Lambda[j, j] - Lambda[i, j]*Lambda[j,k]/Lambda[i, k]:
gammah[1, 1]:= Lambda[k, k] - Lambda[i,k]*Lambda[k,j]/Lambda[i,j]:
for i from 2 to N do
c:=4*N+1-i:
A1:=submatrix(Lambda, 2*N+1..2*N+i, 1..i):
d1:=-subvector(Lambda, 2*N+1..2*N+i, c):
alpha1:=vector(bN, 0):
alpha:=linsolve(A1, d1):
for j from 1 to i do
alpha1[j]:= alpha[j]
od:
alpha1[c]:=1:
gammav[1, i]:= dotprod(row(Lambda, c), alpha1):
```

```
gammah[i, 1]:= dotprod(row(Lambda, i), alpha1)/alpha1[i]:
for j from 1 to i-1 do
mu[j, 1]:= alpha1[j] - dotprod(row(Lambda, j), alpha1)/gammah[j, 1]
od:
if i > 2 then
for h from 2 to i-1 do
mu[1, h]:= -dotprod(row(Lambda, 4*N+1-h), alpha1)/gammav[1, h]
od
fi:
if i > 3 then
for k from 2 to i-2 do
mu[2, k]:= ((gammav[1, k]+gammav[2,k]+gammah[1, k]+gammah[1, k+1])*
mu[1, k]-gammah[1,k]*mu[1,k-1] -gammah[1, k+1]*mu[1, k+1])/gammav[2,k]
od
fi:
if i > 4 then
for p from 3 to i-2 do
for q from 2 to i-p do
mu[p,q]:=((gammav[p,q]+gammav[p-1,q]+gammah[p-1,q]+
+gammah[p-1 ,q+1])*mu[p-1,q]
-gammav[p-1,q]*mu[p-2,q]-gammah[p-1,q]*mu[p-1,q-1]
-gammah[p-1,q+1]*mu[p-1,q+1])/gammav[p,q]
od
```

```
od
fi:
gammah[1,i]:=-gammav[1,i]/mu[1,i-1]:
gammav[i,1]:=-alpha1[i]*gammah[i,1]/mu[i-1,1]:
if i > 2 then
a:= 2:
b:= i-1:
gammav[a, b]:=-(gammah[a-1,b+1]+gammav[a-1,b]+gammah[a-1,b])
+gammah[a-1,b]*mu[a-1,b-1]/mu[a-1,b]:
gammah[a, b]:=-gammav[a,b]*mu[a-1,b]/mu[a,b-1]:
if i > 3 then
for count from 1 to i-3 do
a:=a+1:
b:=b-1:
gammav[a, b]:=-(gammah[a-1,b+1]+gammav[a-1,b]+gammah[a-1,b])+
+(gammah[a-1,b]*mu[a-1,b-1]+ gammav[a-1,b]*mu[a-2,b])/mu[a-1,b]:
gammah[a,b]:=-gammav[a,b]*mu[a-1,b]/mu[a,b-1]
od
fi
fi
od:
j:=2*N: k:=2*N+1: i:=N:
gammav[N+1, N]:= Lambda[j, j] - Lambda[i, j]*Lambda[j,k]/Lambda[i, k]:
```

```
gammah[N, N+1]:= Lambda[k, k] - Lambda[i,k]*Lambda[k,j]/Lambda[i,j]:
for i from 2 to N do
c:=2*N+1-i:
mu:=matrix(N, N, 0):
A1:=submatrix(Lambda, 1..i, 2*N+1..2*N+i):
d1:=-subvector(Lambda, 1..i, c):
alpha1:=vector(bN, 0):
alpha:=linsolve(A1, d1):
for j from 1 to i do
alpha1[2*N+j]:= alpha[j]
od:
alpha1[c]:=1:
gammav[N+1, N+1-i]:= dotprod(row(Lambda, c), alpha1):
gammah[N+1-i, N+1]:= dotprod(row(Lambda, 2*N+i), alpha1)/alpha1[2*N+i]:
for j from 1 to i-1 do
mu[N+1-j, N]:= alpha1[2*N+j]-
- dotprod(row(Lambda, 2*N+j), alpha1)/gammah[N+1-j, N+1]
od:
if i > 2 then
for h from 2 to i-1 do
mu[N, N+1-h]:=
-dotprod(row(Lambda, 2*N+1-h), alpha1)/gammav[N+1,N+1-h]
od
```

```
fi:
if i > 3 then
for k from 2 to i-2 do
mu[N-1, N+1-k]:= ((gammav[N+1, N+1-k]+
+gammav[N, N+1-k]+gammah[N , N+1-k]+
+gammah[N, N+2-k])*mu[N,N+1-k]-
-gammah[N, N+2-k]*mu[N,N+2- k] -
-gammah[N, N+1-k]*mu[N, N-k])/gammav[N,N+1-k]
od
fi:
if i > 4 then
for p from 3 to i-2 do
for q from 2 to i-p do
mu[N+1-p,N+1-q]:=((gammav[N+3-p,N+1-q]+gammav[N+2-p,N+1-q]+
+gammah[N+2-p, N+1-q]+
+gammah[N+2-p,N+2-q])*mu[N+2-p,N+1-q]-
-gammav[N+3-p,N+1-q]*mu[N+3-p,N+1-q]-
-gammah[N+2-p, N+2-q]*mu[N+2-p,N+ 2-q]
-gammah[N+2-p, N+1-q]*mu[N+2-p, N-q])/gammav[N+2-p, N+1-q]
od
od
fi:
gammah[N,N+2-i]:=-gammav[N+1,N+1-i]/mu[N,N+2-i]:
```

```
gammav[N+2-i,N]:=-alpha1[2*N+i]*gammah[N+1-i,N+1]/mu[N+2-i,N]:
if i > 2 then
a:= N:
b:= N+2-i:
gammav[a, b]:=-(gammah[a,b]+gammav[a+1,b]+gammah[a,b+1])+
+gammah[a,b+1]*mu[a,b+1]/mu[a,b]:
gammah[a-1, b+1]:=-gammav[a,b]*mu[a,b]/mu[a-1,b+1]:
if i > 3 then
for count from 1 to i-3 do
a:=a-1:
b:=b+1:
gammav[a, b]:=-(gammah[a,b]+gammav[a+1,b]+gammah[a,b+1])+
+(gammah[a,b+1]*mu[a,b+1]+ gammav[a+1,b]*mu[a+1,b])/mu[a,b]:
gammah[a-1,b+1]:=-gammav[a,b]*mu[a,b]/mu[a-1,b+1]
od
fi
fi
od:
print(gammah, gammav);
Digits:=4:
gammahorizontaledges:=evalm(gammah*1.000);
gammaverticaledges:=evalm(gammav*1.000);
```

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