

# ABSTRACT

Title of Dissertation: OPTIMAL PREVENTIVE MAINTENANCE  
POLICIES FOR UNRELIABLE QUEUEING  
AND PRODUCTION SYSTEMS

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Preventive Maintenance (PM) models have traditionally concentrated on utilizing machine “technical” state information such as the degree of deterioration. However, in real manufacturing systems, additional system operational information such as work-in-process (WIP) inventory levels critically impact actual PM decisions. Surprisingly, the literature on models incorporating this important aspect is relatively sparse. This thesis attempts to fill some of the research gaps in this area by considering problems of optimal preventive maintenance explicitly under the context of unreliable queueing and production-inventory systems.

We propose a two-level hierarchical modeling framework for PM planning and scheduling problems. In the higher level, our objective is to characterize structure of optimal PM policies. We start with a simple case in which queueing is not

taken into account in the model. We show that a randomized PM policy, like the widely used “time-window” policy in industry, is in general not optimal. We then consider the problem of optimal PM policies for an M/G/1 queueing system with an unreliable server. The decision problem is formulated as a semi-Markov decision process. We establish some structural properties, e.g., “control-limit” type structure, that optimal policies will satisfy.

We then take the optimal PM problem a step further by considering optimal joint PM and production control policies for unreliable production-inventory systems with time-dependent or operation-dependent failures. We show the optimal joint policies retain the “control-limit” type structure in terms of the PM portion of the policy. For the production portion of the policy, some properties are also derived, but numerical studies show that in general optimal policies have more complicated structure than the simple control-limit form.

The last part of the thesis is devoted to the lower level of the framework where the objective is to optimally schedule multiple PM tasks across a group of tools. We take into account information such as interdependence of PM tasks, WIP data and resource constraints, and formulate the problem as a mixed-integer program. Results of a simulation study comparing the performance of the model-based PM schedule with that of a baseline reference schedule are presented to illustrate the fitness of our solutions.

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AND PRODUCTION SYSTEMS**

by

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2003

# Dedication

to my wife Winnie  
for her endless love and support

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# Table of Contents

<b>List of Tables</b>	<b>viii</b>
<b>List of Figures</b>	<b>ix</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Hierarchical PM Planning and Scheduling in Semiconductor Manufacturing . . . . .	4
1.2 Contributions of the Thesis . . . . .	6
1.3 Organizations of Chapters . . . . .	7
<b>2 Preliminaries</b>	<b>10</b>
2.1 MDP . . . . .	11
2.1.1 Bellman’s Equation and Value Iteration . . . . .	12
2.2 SMDP . . . . .	14
2.3 Monotonic Policies . . . . .	15
2.3.1 Stochastic Order Relations . . . . .	18
2.4 Literature Review . . . . .	20



2.4.1	Classic PM Models . . . . .	21
2.4.2	Recent Work on Production-Inventory Systems . . . . .	25
<b>3</b>	<b>Optimal PM Policies</b>	<b>29</b>
3.1	Related Work . . . . .	30
3.2	The Classic Case: Single Machine without Queue . . . . .	33
3.2.1	Problem Setting . . . . .	33
3.2.2	Optimality of Deterministic PM Policy . . . . .	36
3.2.3	Numerical Examples . . . . .	40
3.3	M/G/1 with Infinite Queue Capacity . . . . .	42
3.3.1	Discounted SMDP Formulation . . . . .	43
3.3.2	Structural Results . . . . .	47
3.4	A Simplified Discrete-Time Model . . . . .	52
3.4.1	MDP Formulation . . . . .	54
3.4.2	Structural Properties . . . . .	55
<b>4</b>	<b>Optimal Joint PM and Production Control Policies</b>	<b>59</b>
4.1	Related Work . . . . .	60
4.2	The Model with Time-Dependent Failures . . . . .	63
4.2.1	MDP Formulation . . . . .	64
4.2.2	Structural Properties of Optimal Policies . . . . .	67
4.3	The Model with Operation-Dependent Failures . . . . .	80
4.3.1	MDP Formulation . . . . .	81
4.3.2	Structural Properties of Optimal Policies . . . . .	82
4.4	Conclusions . . . . .	97

<b>5</b>	<b>Optimal PM Scheduling in Semiconductor Fabs</b>	<b>99</b>
5.1	Background and Related Work . . . . .	101
5.2	Mixed Integer Programming . . . . .	104
5.2.1	Problem Definition . . . . .	106
5.2.2	MIP Formulation . . . . .	107
5.2.3	Implementation Issues . . . . .	114
5.3	Solving the MIP Model . . . . .	116
5.4	Simulation Case Study . . . . .	119
5.5	Conclusions . . . . .	124
<b>6</b>	<b>Conclusions and Future Work</b>	<b>125</b>
6.1	Future Work . . . . .	127
	<b>Bibliography</b>	<b>129</b>

# List of Tables

5.1	Configuration matrix for a cluster tool (legend: 0: down; 1: up; X: up/down) . . . . .	108
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# List of Figures

1.1 Two-level hierarchical modeling framework for PM planing and scheduling . . . . .	5
3.1 Problem setting. . . . .	34
3.2 The function $S(t)$ when $F(t)$ is exponential distribution. . . . .	40
3.3 The function $S(t)$ when $F(t)$ is Weibull distribution. . . . .	42
3.4 Model 1 – M/G/1 with infinite queue capacity . . . . .	42
3.5 The discrete-time model . . . . .	53
4.1 An unreliable production system with time-dependent failures . .	63
4.2 Joint optimal policy. (a) The optimal actions on the whole state space. (b) The optimal actions on the space $[-5,5] \times [0, \text{MAXLIFE}]$ .	78
4.3 The comparative difference of cost functions under the joint optimal policy and independently optimized policy. . . . .	79
4.4 An unreliable production system with operation-dependent failures	80
4.5 A specific policy – postpone optimal control until the time period $t_0 + \tau_r^*$ . . . . .	85

4.6	$\tau(s)$ : minimum time for inventory reaching 0 for a sample path starting with initial level $s$ ; $\Phi(s)$ : the sample-path inventory cost under the policy $\mu_0$ until the time $\tau(s)$ . . . . .	92
4.7	Inventory sample path versus boundary path . . . . .	93
4.8	Computing $E(\Phi(s + 1)) - E(\Phi(s))$ . . . . .	95
4.9	Optimal policy for unreliable production system with operation-dependent failures. (The blank area in the state space is where the optimal action is to stay idle.) . . . . .	96
5.1	PM tasks with associated time windows, where asterisk points are the optimal times computed by the model to perform PM tasks. .	121
5.2	Simulation result for throughput increases (in percentages) under the model-based schedule over those under the reference schedule.	122
5.3	Simulation result for WIP changes (in percentages) under the model-based schedule over those under the reference schedule. . .	123

# Chapter 1

## Introduction

The main problems we study in this research concern optimal Preventive Maintenance (PM) under the context of unreliable queueing or production systems. This research has been motivated by the challenging problem of optimal PM planning and scheduling in semiconductor manufacturing. The reliability of equipment in semiconductor manufacturing fabs has become an important issue in yield improvement, cost reduction, and cycle time reduction. The fabrication equipment is extremely sophisticated. It requires extensive PM and calibration, but is still subject to unpredictable failures. The unpredictable equipment downtime has been identified as the main cause of uncertainty in semiconductor manufacturing [55]. A “good” PM schedule can increase tools’ availability by trading off between the “planned” unproductive downtime (due to PM) versus the risk of much costlier “unplanned” downtime (due to tools’ failure). Thus, in order to maximize the profits from fabs operation, PM tasks have to be scheduled carefully and comprehensively.

Two distinguishing features of fabs make PM scheduling a challenging task. First, a fab is a highly integrated system in the sense that it is a large-scale

re-entrant system, involving about ten to twenty stations (modules) and a hundred or more stages of processing for each wafer [4]. These stations are highly coupled by re-entrant wafers and processing steps. Second, many new advanced techniques have been deployed on the shop floor, such as the wide use of cluster tools. A cluster tool is a highly integrated tool that is composed of several chambers and robots, where different PM tasks on different chambers have to be coordinated carefully in order to maximize the availability (and hence throughput) of the entire tool. The increasing complexity and integration of fab systems and tools call for new operational models to be applied to PM planning and scheduling in semiconductor manufacturing.

Interestingly, our literature survey reveals that optimal PM problems have not been addressed sufficiently for modern manufacturing systems, such as semiconductor manufacturing fabs. In the effort to obtain optimal PM policies, conventional maintenance theories have concentrated on utilizing data solely on the reliability of machines. This approach has been very successful in dealing with single-unit systems [56], where the status of unreliable devices (machines) is of most interest. Although there are also many PM models developed for multi-unit (multi-component) systems, most of them focus on group/block or opportunistic maintenance that make use of economies of scale to perform preventive replacement at the failure of one unit, or on the effect of repairman/spare parts inventory on maintenance policies; see the survey paper [14] and the references therein. Conventional methods have ignored the fact that each tool is only a part of the whole production system, and so the entire state of the system, such as the operating status of up-stream or down-stream tools, as well as buffer levels, has significant impact on PM for that tool and should be incorporated

into the PM policy in order to achieve maximal overall equipment effectiveness.

Queueing systems, on the other hand, have long been used in modeling complex manufacturing systems and computer systems [32, 13]. Most results have been derived either under the assumption of reliable servers, or of unreliable servers but no PM considered. A very similar class of problems is that of queueing systems with server vacations, where server vacations can be viewed as server breakdowns [19, 2, 20, 21]. However, there is very little literature on PM policies for unreliable queueing systems. As a matter of fact, optimal PM policies and their performance evaluation have not been investigated under the context of queueing systems until very recently [24, 23].

Our approach to optimal PM problems in the context of queueing and production systems differs from other work in that that we believe the system’s “operating” information, specifically, Work-In-Process (WIP) level or queue length, has an important effect on the PM decision-making process, and thus should be reflected in the PM policies. In our models developed in this thesis, we include this information explicitly into the system’s state. Essentially, we expand the state space of PM decision problems to two dimensions: machine deterioration degree (“technical” state) and system buffer level (“operating” state). This formulation increases the difficulty of finding an optimal policy. It is thus our main research objective to investigate structural properties of optimal PM policies.

The research reported in this thesis is part of a comprehensive effort at developing models, algorithms and software tools for PM scheduling in semiconductor manufacturing. Our research was funded by the joint Semiconductor Research Corporation (SRC) & International SEMATECH (ISMT) “Factory Operations Research Center” (FORCe) program.



## 1.1 Hierarchical PM Planning and Scheduling in Semiconductor Manufacturing

In semiconductor manufacturing, hierarchical planning and scheduling for maintenance activities is common practice. The vast majority of PM tasks follow a “generalized age replacement” structure, in which a PM is scheduled for a time after a tool’s “age” exceeds some threshold, but there is flexibility on the actual start time within some associated interval. Here, “age” means calendar-time or operation history, according to the type of PM. In semiconductor manufacturing practice, this is often called a “time window” policy, where such a window is associated with each PM task. Usually, for each PM task, its frequency, or interval between two consecutive PM windows, is planned first. The exact time to do a PM is then scheduled within the time window by taking into account other factors, such as WIP levels and interdependence among different PM tasks. It turns out that this hierarchical structure is necessary for dealing with the complexity of complicated PM scheduling in semiconductor fabs, for example, the challenging task of PM scheduling for cluster tools.

The semiconductor industry practice motivates us to propose a two-level hierarchical modeling framework to model the structure. The framework is illustrated in Figure 1.1 and its components are explained below.

The goal of the higher level planning model is to obtain optimal policies for PM tasks. Depending on different operations objectives, the policy could be *optimized* within a preferred policy family, for example, a generalized age-replacement policy, or the so-called time-window policy. Differing from conventional PM models [43, 56, 14], which include only the system’s “technical” state

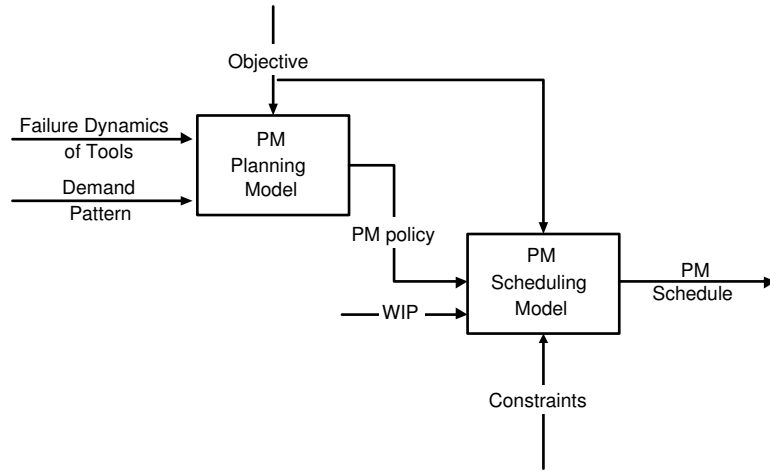


Figure 1.1: Two-level hierarchical modeling framework for PM planing and scheduling

information, the PM modeling should consider the system’s “operating” states and “technical” states simultaneously. The higher level model takes tool failure processes and incoming demand processes (possibly stochastic) together with appropriate system objective functions as model inputs, and generates an optimized PM policy. In order to obtain computationally tractable models, under the hierarchical framework, the information about interdependence among various PM tasks and other constraints is ignored intentionally in the higher level, and be handled by the lower level scheduling model.

It is up to the lower level scheduling model to determine the exact time to do a PM by considering other factors that have been ignored in the upper level model. For example, in the context of cluster tools, the scheduling model will consider the interdependence of different PM tasks in terms of their joint impact on the entire tool’s throughput, as well as the match between the tool’s

availability and projected incoming WIP. The scheduling model obtains optimal PM schedules, under some objective function and resource constraints. Mathematical programming can be used to formulate such models, and we develop a mixed integer program to solve the scheduling problem for a group of cluster tools. Our solution will be presented in Chapter 6.

It is worth pointing out that the hierarchical framework can be interpreted from other perspectives as well. For example, from the viewpoint of model computation, the higher level is operating at a slower time scale with time unit possibly in weeks, whereas the lower level is operating at a faster time scale with time unit likely in shifts/days.

## 1.2 Contributions of the Thesis

The main contributions of the thesis are as follows:

- We develop a Semi-Markov Decision Process (SMDP) model for the problem of finding optimal PM policies for an M/G/1 queue with an unreliable server. We prove in Theorem 3.2 that under some conditions, the optimal PM policy has a “control-limit” structure in the dimension of system “technical” state, such that for fixed queue length, if and only if the machine (server)’s age is above the control-limit (threshold), then it is optimal to do PM.
- We study joint PM and production control policies for production-inventory systems with an unreliable machine. We develop two Markov Decision Process (MDP) models for systems with *time-dependent* failures and *operation-dependent* failures, respectively. We obtain several results on the structure

of optimal joint PM and production control policies. We prove again the “control-limit” structure of the optimal joint policies as stated in Theorem 4.1 and 4.4. Other structural results regarding optimal policies are shown as in Proposition 4.1 and Theorems 4.3 and 4.5.

- For a single machine case, we prove in Theorem 3.1 that the widely used “time-window” policy is not optimal. This result extends Barlow and Proschan’s theorem to a more general problem structure.
- We develop a mixed integer program (Model MIP1) in Chapter 5 for optimal PM scheduling in fabs, especially for a group of cluster tools. Our prototype implementation and preliminary simulation study suggest our solutions can be a significant aid for (human) decision makers to rule out errors and oversights, and have an impact on practical use.
- We have also proposed a decomposition approach to the practical problem of PM planning and scheduling in semiconductor manufacturing, i.e., a two-level hierarchical modeling framework. At the higher level is a *planning* model used to optimize PM policies, whereas at the lower level is a *scheduling* model used to project the best time to do specific PM tasks.

### 1.3 Organizations of Chapters

The remainder of the thesis is organized as follows.

Chapter 2 provides some technical background. It starts with a brief introduction of Markov decision processes (MDP) and Semi-Markov decision processes (SMDP), and some well-known results about conditions on monotonic

policies, followed by a literature survey on PM models.

Chapter 3 contains the results we achieved on optimal PM policies for unreliable queueing systems in different problem settings. We first study the simplest case in which queueing has not been taken into account, and we prove that there exists a deterministic optimal PM policy. We then study optimal PM policies for an M/G/1 queueing system with an unreliable server (machine). A discounted SMDP model is developed for the optimization problem. Under some conditions, we show that for fixed queue length, there exists a *control-limit* such that if and only if the machine's age is greater than the control-limit, then it is optimal to do PM. A simplified discrete-time queueing model is also discussed. With a simplified cost structure and system stochastic processes, more properties of the optimal cost function and corresponding optimal policies are established.

In chapter 4, we add a new dimension to the optimal PM problem. Previously, we assume the system's production control policy is fixed. In this chapter, we start to look at joint PM and production control policies, and we derive some structural properties that an optimal joint PM and production control policy will have. We first study an unreliable production-inventory system that experiences *time-dependent* failures, and some properties of an optimal joint policy are derived. Later on, we consider another unreliable production-inventory system that experiences *operation-dependent* failures. Numerical examples are provided to illustrate the structure of optimal policies.

In chapter 5, we solve an optimal PM scheduling problem for cluster tools in semiconductor manufacturing fabs. A Mixed-Integer Programming (MIP) model is developed in detail. Interdependence among different PM tasks, production planning data such as projected WIP, manpower constraints, and associated PM

costs, are incorporated in the MIP model. After an appropriate transformation, the model can be solved using any commercial LP/IP software. Results of a simulation study comparing the performance of the model-based PM schedule with that of a baseline reference schedule are presented to illustrate the usefulness of our solutions.

Chapter 6 concludes this thesis with a summary of this research work. Some future work relevant to this research is also discussed briefly.

## Chapter 2

# Preliminaries

Fundamentally, finding optimal PM policies is a sequential decision process in the presence of uncertainty. Examples of uncertainties are the stochastic processes of machine failures and the randomness of projected demand or work-in-process (WIP). Sequential decision problems can be well formulated as MDPs or SMDPs [44, 10]. MDP and SMDP models enable us to evaluate rigorously a trade-off between immediate and future benefits and costs, and they provide analytic and computational tools to investigate structures of optimal cost functions and policies.

In this chapter, we will first provide a brief introduction to MDP and SMDP, as well as a discussion of monotonic policies. A literature review on various PM models and policies is then presented.

## 2.1 MDP

The optimization problem for system maintenance is to a large extent a sequential decision problem, as the decision to do PM or not to do PM has to be made sequentially as time goes on. Many sequential decision problems can be formulated as MDP problems. A general MDP can be characterized as a five-tuple  $\langle \mathcal{X}, \mathcal{U}, g, \mathcal{Q}, \mathcal{N} \rangle$ , where  $\mathcal{X}$  is the state space,  $\mathcal{U}$  a set of actions (controls),  $g$  a one-stage cost function,  $\mathcal{Q}$  the state transition function, and  $\mathcal{N}$  the problem horizon.

Let us consider a discrete-time dynamic system given by:

$$x_{k+1} = f(x_k, u_k, w_k), k = 0, 1, \dots, \quad (2.1)$$

where for all  $k$ ,  $x_k \in \mathcal{X}$  is the system state,  $u_k \in U(x_k) \subseteq \mathcal{U}$  the system control, and  $w_k \in \mathcal{D}$  the random disturbance. For the sake of simplicity, we assume  $\mathcal{D}$  is a countable set, and  $w_k, k = 0, 1, \dots$ , are i.i.d. The state transition function  $\mathcal{Q}$  can be characterized by the distribution of  $w_k$  and the function  $f$ .

We are particularly interested in MDP problems with infinite horizon, because their optimal policies are typically stationary, and thus the implementation of optimal policies is often simple. There are basically two classes of infinite horizon problems.

The first class is the discounted cost problem given by:

$$J_\pi(x_0) = \lim_{N \rightarrow \infty} E \left[ \sum_{k=0}^{N-1} (\beta^k g(x_k, \mu_k(x_k))) \mid x_0 \right], \quad (2.2)$$

where  $J_\pi(x_0)$  denotes the cost associated with an initial state  $x_0$  and a policy  $\pi = (\mu_0, \mu_1, \dots)$  and  $0 < \beta < 1$  is discount factor. A stationary policy has the form  $\pi = \{\mu, \mu, \dots\}$  and its corresponding cost function is denoted by  $J_\mu$ . Let



$J^*(x)$  be the optimal cost functions, defined by:

$$J^*(x) = \inf_{\pi} J_{\pi}(x). \quad (2.3)$$

We say  $\mu$  is optimal if  $J_{\mu}(x) = J^*(x)$  for all states  $x$ .

The second type of problem is the average cost problem whose cost function under a policy  $\pi$  is given by:

$$J_{\pi}(x_0) = \lim_{N \rightarrow \infty} \frac{1}{N} E \left[ \sum_{k=0}^{N-1} (g(x_k, \mu_k(x_k))) \mid x_0 \right]. \quad (2.4)$$

The objective is to find a policy  $\pi$  to achieve the optimal average cost  $J^*(x)$  for all states  $x$ . Again,  $\pi$  is usually a stationary policy denoted by  $\mu$ .

### 2.1.1 Bellman's Equation and Value Iteration

In the following discussion, we concentrate on the discounted MDP problem only. Furthermore, we assume the cost per stage  $g$  is non-negative, but possibly unbounded, as stated in the following assumption.

**Assumption P:(Positive)** The cost per stage  $g$  satisfies

$$g(x, u) \geq 0, \quad \text{for all } (x, u) \in \mathcal{X} \times \mathcal{U}. \quad (2.5)$$

It is a well-known result that the optimal cost function (2.3) satisfies Bellman's equation.

**Theorem 2.1.** *Under the Assumption P, the optimal cost function  $J^*$  satisfies*

$$J^*(x) = \inf_{u \in \mathcal{U}(x)} E \{g(x, u) + \beta \cdot J^*(f(x, u, w))\}, x \in \mathcal{X}. \quad (2.6)$$

*Proof.* see Proposition 1.1, Chapter 3, Vol. II of [10]. □

Note: Bellman's equation is often written as

$$J^* = TJ^*,$$

where  $T$  is a mapping defined by

$$(TJ)(x) = \inf_{u \in \mathcal{U}(x)} E \{g(x, u) + \beta \cdot J(f(x, u, \omega))\}, x \in \mathcal{X} \quad (2.7)$$

### Value Iteration Algorithm

The value iteration method is a computational algorithm for solving Bellman's equation. It proceeds as follows. Let  $J_0$  be the zero function on the state space  $\mathcal{X}$ ,

$$J_0(x) = 0, \quad x \in \mathcal{X}. \quad (2.8)$$

Then under Assumption P, we have

$$J_0 \leq TJ_0 \leq T^2J_0 \leq \cdots \leq T^k J_0 \leq \cdots,$$

and denote the limit function by

$$J_\infty(x) = \lim_{k \rightarrow \infty} (T^k J_0)(x), \quad x \in \mathcal{X}. \quad (2.9)$$

The value iteration method converges to the optimal cost function  $J^*$  under some sufficient conditions as stated in the following theorem; see Proposition 1.6 in Chapter 3, Vol. II of [10].

**Theorem 2.2.** *Under Assumption P, if the control set  $\mathcal{U}$  is finite for every  $x \in \mathcal{X}$ , then*

$$J_\infty = TJ_\infty = J^*.$$

If the control set is not finite, then it can also be shown that the value iteration converges to the optimal cost function  $J^*$  under other conditions; please refer to the Sec. 3.1 in Chapter 3, Vol. II of [10] for more details.

## 2.2 SMDP

An important feature of MDP problems is that the underlying probability structures can be modeled as Markov chains. However, many sequential decision making problems have underlying probability structures that cannot be characterized by Markov chains, such as systems with sojourn times that are drawn from general probability distributions other than exponential distributions, e.g., the service time for an  $M/G/1$  queue. Such problems can often be modeled as SMDP. SMDPs can be regarded as a generalization of MDP models, where the times between transitions are general random variables (if the random variables are exponentially distributed, then the problem can be formulated easily as a MDP problem). The random variables may depend on the current state, the action taken, and even the next state.

Considering a system with a discrete state space. The transition distribution functions  $Q_{ij}(\tau, u)$  specify the joint distribution of the transition interval and the next state for a given state-action pair  $(i, u)$ , that is:

$$Q_{ij}(\tau, u) = \Pr(t_{k+1} - t_k \leq \tau, x_{k+1} = j \mid x_k = i, u_k = u). \quad (2.10)$$

One form of discounted cost problem for SMDP is:

$$J_\pi(x_0) = \lim_{N \rightarrow \infty} E \left[ \int_0^{t_N} e^{-\beta t} g(x(t), u(t)) dt \mid x_0 \right]. \quad (2.11)$$

where  $t_N$  is the completion time of  $N$ th transition, and  $\beta > 0$  the discounting parameter.

Let

$$m_{ij}(u) := \int_0^\infty e^{-\beta \tau} Q_{ij}(d\tau, u), \quad (2.12)$$

and denote by  $G(i, u)$  the expected single stage cost corresponding to  $(i, u)$ , which is given by:

$$\begin{aligned} G(i, u) &:= g(i, u) \cdot \sum_j \int_0^\infty \left( \int_0^\tau e^{-\beta t} dt \right) Q_{ij}(d\tau, u), \\ &= g(i, u) \cdot \sum_j \int_0^\infty \frac{1 - e^{-\beta\tau}}{\beta} Q_{ij}(d\tau, u). \end{aligned} \quad (2.13)$$

Under some mild conditions (see Chapter 5 in [10]), the discounted cost problem is equivalent to an ordinary MDP problem, i.e.,  $J^*$  is the unique solution of following Bellman equation:

$$J(i) = \inf_u \left[ G(i, u) + \sum_j m_{ij}(u) J(j) \right].$$

For the average cost problem, we have

$$J_\pi(x_0) = \lim_{N \rightarrow \infty} \frac{1}{E[t_N]} E \left[ \int_0^{t_N} g(x(t), u(t)) dt \mid x_0 \right]. \quad (2.14)$$

If the state space and action space are finite, under the assumption of a unichain policy, i.e., a policy whose associated Markov chains have a single recurrent class, the average cost problem can be transformed to an equivalent MDP problem via the embedded Markov chain; see [10] for more details.

## 2.3 Monotonic Policies

The literature on maintenance theory suggests that monotonicity properties are very common among maintenance policies. For one-dimensional problems, a monotone policy is often in the form of a control-limit; for two-dimensional problems, it is often in the form of a switching curve [28]. The formulations of MDP and SMDP and the corresponding computational algorithms (specifically,

value iteration) are an important vehicle for us to exploit the structure of optimal policies.

It has been observed by many researchers that *submodularity* of cost (value) functions is the key in proving the existence of a monotonic optimal policy [28, 62, 3, 25].

A *submodular* function is defined on a *lattice* set.

**Definition (Lattice):** A partially ordered set  $\Omega$  is a *lattice* if  $x \wedge y \in \Omega$  and  $x \vee y \in \Omega$  for all  $x$  and  $y$  in  $\Omega$ , where  $x \wedge y = \sup\{z | z \leq x, z \leq y, z \in \Omega\}$ ,  $x \vee y = \inf\{z | z > x, z > y, z \in \Omega\}$ . If  $\Omega \subseteq \mathcal{R}$ , then  $x \wedge y = \min(x, y)$ ,  $x \vee y = \max(x, y)$ .

The following definition of submodular function is due to [54]:

**Definition (Submodular function):** Let  $g$  be a real-valued function whose domain is a lattice  $\Omega$ . Then  $g$  is *submodular* if

$$g(x \wedge y) + g(x \vee y) \leq g(x) + g(y),$$

for all  $x$  and  $y$  in  $\Omega$ . If  $-g$  is submodular, then  $g$  is *supermodular*.

Using submodularity of value functions, sufficient conditions for a monotonic optimal policy can be established under a general dynamic programming setting as follows; see also [30].

Consider the dynamic programming recursion:

$$J_n^*(x) = \inf (J_n(x, u) : u \in U(x)) \quad x \in \mathcal{X} \subset \mathcal{R}, \quad (2.15)$$

$$J_n(x, u) = g(x, u) + \beta E [J_{n-1}^*(f(x, u, w))] \quad (x, u) \in \mathcal{C} \subset \mathcal{R}^2, \quad (2.16)$$

where  $U(x)$  is the set of all actions for state  $x$ , and  $\mathcal{C} = \{(x, u) : x \in \mathcal{X}, u \in U(x)\}$  the state-action space. The system state at the next stage when the system's current state is  $x$  and action  $u$  is applied,  $f(x, u, w)$ , is a r.v., taking values in

$\mathcal{X}$ . Denote the tail distribution of  $f(x, u, w)$  by:

$$\gamma_y(x, u) := \Pr(f(x, u, w) > y). \quad (2.17)$$

**Definition (Lower semi-continuous):** A real-valued function  $h(\cdot)$  is *lower semi-continuous* at  $x \in \mathcal{R}$ , if for all  $\epsilon > 0$ , there is  $\delta > 0$  such that  $h(x) \leq h(y) + \epsilon$  for all  $y$  with  $|x - y| < \delta$ .

**Definition (Contracting set):** The collection of sets  $\{U(x) : x \in \mathcal{X}\}$  is *contracting* if  $x \leq x'$  implies  $U(x) \supseteq U(x')$ .

**Definition (Ascending set):** The collection of sets  $\{U(x) : x \in \mathcal{X}\}$  is *ascending* if  $x \leq x'$ ,  $b \in U(x)$ , and  $b' \in U(x') \implies b \wedge b' \in U(x)$  and  $b \vee b' \in U(x')$ .

The following theorem can be found in [30] (see Theorem 8-5, pp. 381).

**Theorem 2.3.** *Suppose for each  $x \in \mathcal{X}$  that  $U(x)$  is compact (i.e., closed and bounded),  $J_n(x, \cdot)$  is lower semi-continuous for each  $n \in \mathbb{Z}^+$ , the state-action space  $\mathcal{C}$  is a lattice,  $J_0^*(\cdot)$  is non-decreasing and bounded below on  $\mathcal{X}$ , the inf in the above DP recursion is attained at all  $x \in \mathcal{X}$ , and*

- $g(\cdot, u)$  is non-decreasing for each  $u$ ,
- $g(\cdot, \cdot)$  is submodular and bounded below,
- $\gamma_y(\cdot, \cdot)$  is submodular on  $\mathcal{C}$  for each  $y$ ,
- $\gamma_y(\cdot, u)$  is non-decreasing for each  $y$  and  $u$ ,
- $\{U(x) : x \in \mathcal{X}\}$  is contracting and ascending.

Then for each  $n$  there exists  $\mu_n^*(\cdot)$  non-decreasing on  $\mathcal{X}$  such that

$$J_n^*(x) = J_n(x, \mu_n^*(x)).$$

Submodular functions play an important role in the work of Altman and Stidham [3] on sufficient conditions for monotonic optimal policies for two-action Markov decision processes. They consider a discrete time MDP with action space  $\mathcal{U} = \{0, 1\}$ . They point out that, under some appropriate conditions, the submodularity of  $J_n(x, u)$  can be propagated to the next stage, and as a result, a monotonic optimal policy can be derived. Roughly, for the MDP with binary actions, if transition probabilities are stochastically monotone, and  $g(x, u)$  is submodular in  $(x, u)$ , it is reduced to checking whether or not

$$\xi(x) := g(x, 1) + \beta E [J_n(f(x, 1), 0)] - g(x, 0) - \beta E [J_n(f(x, 0), 1)]$$

is monotone in  $x \in \mathcal{X}$ .  $\xi(x)$  represents the difference in expected total discounted cost between taking action 0 now and action 1 in the next stage, and taking action 1 now and action 0 in the next stage, and then following an optimal policy thereafter. In general, proving this condition is a very complicated task.

It is worth noting that these established conditions are basically for problems with one dimensional state space. For models with two-dimensional state space, we need to define the binary relation of  $\preceq$  first on the space, for example, we can define  $(i_1, n_1) \preceq (i_2, n_2)$  if and only if  $i_1 \leq i_2$  and  $n_1 \leq n_2$ .

### 2.3.1 Stochastic Order Relations

In proving monotonic structures of optimal value functions or optimal policies, we often need to make a comparison between two random variables. There are three types stochastic order relations that are commonly used: stochastically larger, hazard-rate ordering, and likelihood-ratio ordering.

The following definitions are based on the materials by Sheldon Ross; see the

Appendix in [46].

**Definition (Stochastically Larger):** The random variable  $X$  is stochastically larger than the random variable  $Y$ , written as  $X \geq_{st} Y$ , if

$$\Pr(X > a) \geq \Pr(Y > a), \quad \text{for all } a.$$

The next theorem provides an equivalent definition for stochastically larger.

**Theorem 2.4.**  $X \geq_{st} Y \Leftrightarrow E(f(X)) \geq E(f(Y))$  for all increasing functions  $f$ .

The hazard-rate ordering relation is defined among non-negative random variables only. Let  $X$  be such a random variable with distribution  $F$  and density  $f$ . The hazard (or failure) rate function of  $X$  is defined by

$$\lambda(t) = \frac{f(t)}{1 - F(t)}.$$

**Definition (Hazard-Rate Ordering):** Let  $\lambda_X(t)$  and  $\lambda_Y(t)$  be the hazard-rate functions of  $X$  and  $Y$ . We say  $X$  is larger than  $Y$  in the sense of hazard-rate ordering, denoted by  $X \geq_{HR} Y$ , if

$$\lambda_X(t) \geq \lambda_Y(t), \quad \text{for all } t \geq 0.$$

Another stochastic order relation is called Likelihood-Ratio ordering, also defined for non-negative random variables.

**Definition (Likelihood-Ratio Ordering):** Let  $f$  and  $g$  be the density functions of  $X$  and  $Y$ , respectively. We say that  $X$  is larger than  $Y$  in the sense of



likelihood ratio, denoted by  $X \geq_{LR} Y$ , if

$$\frac{f(x)}{g(x)} \leq \frac{f(y)}{g(y)}, \quad \text{for all } x \leq y.$$

It can be easily shown that the following relations exist among the above three orderings:

$$X \geq_{LR} Y \Rightarrow X \leq_{HR} Y \Rightarrow X \geq_{st} Y. \quad (2.18)$$

## 2.4 Literature Review

In recent years, there has been increasing interest in investigating optimal maintenance policies for production-inventory systems. As we have pointed out previously, in conventional models, system states often include only the “technical” information of the system, i.e., deterioration degree or age of the machine. In the context of production systems, it is natural to include in the system state both “technical” information and “operating” information about the system, e.g. inventory (buffer) levels. This section is divided into two parts, with the first part devoted to the classic PM models, and the second one focused on the more recent work that studies maintenance policies for production systems.

Before we proceed, there are a couple of related terms that need to be defined. Two types of maintenance are categorized, i.e., PM and corrective maintenance (CM). A repair activity performed at system failures falls into the class of CM. Other repair activities performed before system failures are called PM. Unless specifically defined, the term “repair” does not necessarily mean a CM. It can be also a PM, for example, a deteriorated machine can be “repaired” to an upgraded status before it fails.

### 2.4.1 Classic PM Models

Maintenance theory has a well-established body of literature. Many maintenance models have been developed and applied in manufacturing and logistic systems, since the pioneering work by Barlow and Proschan [6] in the area of reliability. Four excellent survey papers since the 1960s [37, 43, 56, 14] provide a wide range of models describing the degrading process of equipment, cost structure and admissible maintenance actions.

A large class of models falls into the so-called inspection model [56]. In an inspection model, usually two decisions have to be made, i.e., when to do inspection, and when to do preventive maintenance. Many inspection models are in the form of Markov Decision Processes, which is a natural way to formulate maintenance problems, and these models trace their origin to the basic model introduced by Derman [18]. The basic model is a discrete-time Markov chain. At each time point, a decision is made, i.e., to replace or continue to let system run. If the device is replaced, then the system moves into the “new” state; if “continue to run”, then it deteriorates from state  $i$  to  $j$  with some probability  $p_{ij}$  in one period. Costs considered are only preventive replacement cost and failure replacement cost, which is greater than preventive replacement. Under the objective of minimizing expected long run average cost, and fairly realistic conditions, such as increasing failure rate (IFR), the optimal policy is proved to be of “control-limit” form. A “control limit” policy is defined as follows:

**Control Limit Policy:** There exists a state  $k^*$  (control limit) such that if the observed state is  $k \leq k^*$ , then it is optimal to replace; otherwise it is optimal to “continue”.

This basic model has been extended to include the case where the state of a device is not always known at each time period. Since inspection is a costly activity, a decision should be made when to inspect the device. Now at each decision epoch, there are three admissible actions: repair (replacement), inspection or doing nothing. The system state becomes a pair  $(i, k)$ , where  $i$  denotes the status of the system,  $k$  the number of time periods since the last inspection. Under the action of repair, state  $(i, k)$  moves to  $(0, 0)$  with probability 1; under the action of inspect, state  $(i, k)$  moves to  $(j, 0)$  with probability  $p_{ij}(k+1)$ ; lastly, under the action of doing nothing, the system moves from state  $(i, k)$  to  $(i, k+1)$  with probability 1. The cost structure includes repair cost and operating cost (both dependent only on system status), as well as inspection cost, which is a constant. It has been shown the optimal policy is often a monotonic policy.

Recently, Stadjje and Zuckerman(1996) [52] proposed a generalized maintenance model with a largely extended action space by permitting at any stage the system to be restored to any state that is better than its current one, so that the degree of repair becomes a decision variable. They assume the state of system is always known with certainty, and thus no inspection activity is considered. Maintenance models with imperfect observation of states are also studied. These models have been formulated as partially observed Markov decision processes, cf. [51, 40]. By introducing a new information vector, partially observed Markov decision processes can be transformed to standard MDP problems.

Shock models are another type of models used commonly in the literature, cf. [53, 56]. Instead of simply assuming the system state undergoes deterioration according to a Markov chain, shock models try to model system deterioration due to randomly occurring shocks. Each shock causes a random amount of damage to

the system, and when damage is accumulated to some degree, the system fails. The arrival process of shocks, which can be Poisson or non-Poisson, plays an important role in model development. Accordingly, the model can be formulated as a Markov decision process or a semi-Markov decision process problem. Again, it has been shown the optimal policies are a control-limit policy.

Ozekici (1995) [41] studies optimal maintenance policies in random environments. The so-called intrinsic age of a device, which ticks differently in different environments to measure the intrinsic age of the device, is introduced to reflect the deterioration and failure processes dependency on random environments. The intrinsic age is represented by the cumulative hazard accumulated in time, and thus the lifetime cumulative distribution function. Moreover, the random environment is modeled by a semi-Markov process. Environment states take values in a discrete set, and with a given failure rate function. Decision epochs are restricted only when the environment state changes. A Markov renewal process has been formulated, and for the discounted cost problem with infinite horizon, the control limit type optimal replacement and repair policies are shown to be valid under usual assumptions, e.g., IFR life distribution, and reasonable cost structure.

Benyamini and Yechiali (1999) [7] investigate the optimal PM policy under a non-stationary deterioration process. In a non-stationary deterioration process, system parameters, e.g., transition probabilities and maintenance/operation costs, depend explicitly on both the system's state and its total cumulative age. Thus, the system model is a two-dimensional problem with system state  $(i, t)$ , where  $i$  denotes system status, and  $t$  system age. Under some mild conditions, for discounted cost and average cost problems, they show the familiar results for

stationary models (of one dimension), i.e., optimal PM policies in control limit form, can be extended to the non-stationary model (of two dimensions), in the following sense: for any fixed age  $t$ , the system is replaced if and only if the state is above  $i^*(t)$ ; also for any fixed state  $i$ , the system is replaced when its age is greater than  $t^*(i)$ . A computationally efficient algorithm is proposed for finding the optimal policy by using the control-limit form structure.

Depending on problem context, these models have different structures. In general, the underlying stochastic processes can be modeled as a Markov chain or semi-Markov process. At each decision epoch, the admissible action could be replacement, repair, inspection or just no action. Normally, the cost structure could include inspection cost, preventive replacement cost, failure replacement cost, repair cost, and operation cost. One important result about optimal policies derived from these models is that they are often of the control-limit form. This policy structure has been studied by So (1992) [50] under the context of replacement/maintenance models for systems subject to deterioration, and sufficient conditions for the optimality of control limit policy has been established. This structure provides both insight and practical implications in deriving optimal policies to preventive maintenance problems in actual industrial application.

Along with the development of maintenance models for single-unit systems, some efforts have been directed to the so-called multi-unit maintenance models, where several machines are stochastically or economically dependent on each other, cf. the survey paper [14]. Most efforts along this line have been focused on group/block or opportunistic maintenance models that make use of economies of scale to perform preventive replacement at the failure of one unit [8, 9, 14], or on the investigation of the effect of repairmen/spare parts inventory on maintenance

policies.

## 2.4.2 Recent Work on Production-Inventory Systems

In the conventional maintenance models surveyed above, the system state includes only the “aging” or deterioration information; other system information, like buffer levels or queue lengths, has not been taken into account. Until recently, optimal preventive maintenance has not been investigated explicitly under the context of production systems, specifically under the context of queueing systems.

Van der Duyn Schouten and Vanneste (1995) [17] investigate an integrated maintenance-production problem, in which the preventive maintenance policy is based not only on the information about the age of the device, but also on the level of the subsequent buffer. Specifically, they consider a production system, where a buffer with finite capacity  $K$  is between an input generating installation and a production machine. The production machine never fails, whereas the installation is subject to failure, and preventive maintenance should be scheduled on it. By assuming the times for PM/CM are geometrically distributed, they formulate the problem as a Markov Decision Process, with system state  $(i, x)$ , where  $i$  is the age of the installation, and  $x$  the content of the buffer. Under some reasonable conditions, they show that for the average cost problem, the optimal maintenance policy has the following control-limit form for fixed buffer level  $x, 0 \leq x \leq K$ , that is, there exists a threshold level  $i^*(x)$  such that a PM is not performed whenever  $i < i^*$  and a PM is performed for  $i \geq i^*$ . By identifying this structure of the optimal policy, they propose the so-called  $(n, N, k)$  PM policy, which is very close to the optimal policy structure but is much easier to

implement. The  $(n, N, k)$  policy prescribes to do PM if and only if the age  $i$  and buffer level  $x$  satisfy  $i \geq N$  and  $k \leq x \leq K$ , or  $N > i \geq n$  and  $x = K$ . Numerical results suggest the  $(n, N, k)$  policy yields a near-optimal policy.

Meller and Kim (1996) [38] consider a similar production-inventory system with two machines connected by a finite buffer between them, but under a different problem setting. The first machine is required to do PM from time to time, and a PM policy is predefined such that a PM will be performed whenever the buffer level reaches a predefined level  $b$ . The effect of such a PM policy has been investigated with respect to the cost structure including costs of PM, unscheduled repairs, starving of the downstream machine, and holding inventory. The objective is to determine the optimal buffer level  $b^*$  that triggers PM by using numerical comparisons.

Hsu and Tapiero (1987) [24] and Hsu (1992) [23] consider a problem of age-dependent maintenance in an  $M/G/1$  job shop. Specifically, they study the  $(n, T)$  maintenance policy, that is, to perform PM after the  $n$ th job has been processed since the last maintenance/repair, or do (corrective) maintenance when the process breaks down (at random time  $T$ ). Some operating characteristics of this maintenance-queue, e.g., effective service time distribution, are analyzed, and a long run average cost problem for the optimization of the maintenance policy has been formulated by using renewal theory. Analytical results for the  $M/M/1$  queue with exponential breakdown rate are presented in detail. However, it is worth noting that the analysis is intractable for general distributions.

Das et al. (1999,2000) [15, 16] consider a production-inventory system, where inventory is maintained according to an  $(s, S)$  policy, and the production process is subject to failure. An a priori preventive maintenance policy has been studied,

which can be stated as: if the current inventory is  $i$ , and the number of products made since the last maintenance/repair is at least  $N_i$ , then perform a PM on the machine. A SMDP model has been formulated with system state  $(w, i, c)$ , where  $w$  is system status (working, maintenance or repair),  $i$  inventory level and  $c$  product count. They derive performance measures, such as service level (the average percentage of the demands that are satisfied), average inventory, system productivity (the percentage of time the system is working) and cost benefit due to maintenance (from increased service level, savings in repair/maintenance cost). A simulation-based optimization algorithm, i.e., reinforcement learning, was employed to obtain optimal policy parameters.

Liu et al. (1996) [35] consider a different maintenance problem in an  $M/G/1$  type production system. Preventive maintenance tasks are performed whenever there are no jobs in the queue, and the machine breakdown rate is assumed to be a non-increasing, convex function with respect to the mean maintenance time. Their objective is to optimize the mean maintenance time in order to minimize the average time spent by a job in the system. The problem is formulated as a  $M/G/1$  vacation queue model.

Federgruen and So (1990) [22] consider an  $M/G/1$  queueing system with an unreliable server. Although there is no PM available, there are alternate corrective maintenance. Basically, when the server breaks down, there are two repair operations available, which are characterized by their repair time distributions and the associated costs. Roughly, one is faster than the other. The system operating costs include customer holding costs, repair costs and running costs. The objective is to find a repair policy that minimizes the long run average costs of the system. The optimal stationary policy has been shown to be monotone,



i.e., the faster repair operation is used if and only if the number of customers in the system exceeds a threshold value.

## Chapter 3

# Optimal PM Policies

We will study optimal PM policies under different problem settings in this chapter. It is our primary objective to characterize structural properties of optimal policies. Many models have been proposed and developed in the research literature, with a wide variety of underlying assumptions [7, 21, 22, 33, 14, 16, 23, 50, 41, 52]. However, our models differ from most of them in that we consider explicitly not only machine deterioration degree but also system operational states, e.g., queue length or buffer level.

We start by considering the (simple) classic PM problem, in which a single machine has random lifetime. The problem is to find the optimal time to do PM so as to minimize the average total cost. The analysis is based on renewal theory, and we extend Barlow and Proschan's result [6] by showing there exists a deterministic PM policy under a more general problem setting.

We then study optimal PM under the context of an M/G/1 queueing system with unreliable server. The server has to be preventively maintained in order to avoid failures. Upon failures, the server has to be repaired, obviously at higher cost than as for PM. In order to investigate the effect of WIP level on PM, we

include the system’s queue length, corresponding to WIP level, explicitly in the system state, along with the server deterioration degree. The main components in the cost structure are PM cost, repair cost, and jobs holding (queueing) cost. The objective is to minimize the discounted total cost over the infinite planning horizon. Under some conditions, we show that for fixed queue length, the optimal PM policy has the so-called “control-limit” form with respect to the server’s age. However, sufficient conditions have not been established under which the optimal policy has the same property with respect to the queue length.

Under a simplified problem setting, a discrete-time model is developed and analyzed in the last section with the aim of uncovering more structural results related to the optimal policy and optimal cost function. Technically, the model is very close to [17], in which the unreliable server produces items to the buffer which are then consumed by constant demand. The model has simplified cost structure: only jobs holding cost is considered, whereas PM and repair cost are indirectly included.

### **3.1 Related Work**

The problem of optimal PM policy for a single machine, that is, only the machine’s deterioration state is considered without taking queueing into account, is a very classical problem, and has been studied extensively as early as the 1960’s; see the survey paper [37] and the references therein. The basic problem is the following. Assuming a machine experiences stochastic failures, we want to find an optimal PM policy to minimize machine’s operating costs, given there are different costs associated with machine failures, maintenances, and operations.

This kind of problems is also technically equivalent to the classical problems of replacement policies that balances the cost of failures against the cost of planned replacement.

In their influential work on reliability [6], Barlow and Proschan study optimal replacement policies for both a finite horizon and an infinite horizon. They assume a cost  $c_1$  incurred for each failure and a cost  $c_2 < c_1$  incurred for each planned replacement. They also assume times for failure replacement and planned replacement are *negligible*. For the case with infinite horizon, one main result is that the optimal age replacement policy is deterministic.

The optimization problem of PM policies under the context of unreliable queueing systems is a much more difficult problem, and has not been studied until recently by some researchers.

Hsu and Tapiero [24] and Hsu [23] considered an optimization problem of age-dependent maintenance in an  $M/G/1$  job shop. Some operating characteristics of a pre-determined maintenance policy, which prescribes to perform PM after  $n$ th job has been processed since the last maintenance/repair, or do (corrective) maintenance when the process breaks down, are analyzed, and a long run average cost problem for the optimization of the maintenance policy has been formulated by using renewal theory. It is worth noting that their objective is to optimize the policy within the class of pre-determined policies. They present analytical results for the  $M/M/1$  queue with exponential breakdown rate. However, the analysis is intractable for general distributions.

One of the work related to our model is by Koyanagi and Kawai (1997) [33], in which the unreliable server of a queueing system has multiple states, and the transition of states are governed by a continuous-time Markov chain. When the

server reaches the highest state, which corresponds to the server's failure, a CM must be conducted. A PM can be performed at the end of service, or at server state transition points. One important assumption is that at the beginning of either type of maintenance, the jobs in the system are rejected and any arriving customers during maintenance are also rejected, with each incurring a unit cost. No other costs are included. The objective is to find an optimal maintenance policy that minimizes the total expected discounted cost over an infinite time horizon. Formulating the problem as a semi-Markov decision process with  $(i, l)$  as system state, where  $i$  is queue length and  $l$  server state, they show that the optimal policy has an intuitive monotone property in the following sense: if  $(i, l) \preceq (j, k)$  [a partial ordering,  $(i, l) \preceq (j, k)$  iff  $i \leq j$  and  $l \geq k$ ], then if it is optimal to do PM on  $(i, l)$ , then it is also optimal on  $(j, k)$ .

The work by Van der Duyn Schouten et. al (1995) [17] motivates us for the development of the simplified discrete-time PM model. Though their problem setting is quite different, the technical tool employed is similar. Specifically, they consider a production system, where a buffer is placed between an input generating installation and a production machine. The production machine never fails, whereas the installation is subject to failure, and preventive maintenance should be scheduled on it. By assuming the times for PM/CM are geometrically distributed, they formulate the problem as a Markov Decision Process. Under some mild conditions, they show that for the average cost problem, the optimal maintenance policy has the control-limit structure with respect to machine's age.

## 3.2 The Classic Case: Single Machine without Queue

In this section, we consider the optimal PM policies under the classic problem setting: a single machine with random lifetime. We extend Barlow and Proschan's result, i.e., non-randomized optimal PM policy, (Theorem 2.1, Chapter 4 in [6]) to a more general case in which times for PM and repair are non-negligible. Moreover, we consider a general cost structure which includes not only setup costs but also running costs for PM/repair. It is also shown that there exists a unique optimal time to do PM under some appropriate conditions.

Our interest in the investigation of optimality of non-randomized PM policy is motivated by the observation that in semiconductor manufacturing, a randomized policy called "*time-window*" policy is commonly employed in machine PM planning. A time-window policy defines a *time window* within which the PM can be performed anytime. The finding shows that the so-called time-window policy is not optimal for the problem setting under our consideration. Rather, a deterministic policy is optimal.

In the following sections, we will first define the problem, and then show the optimality of a deterministic PM policy. Two numerical examples are presented in the last section.

### 3.2.1 Problem Setting

We consider a failure-prone machine. Let machine's lifetime be  $T$ , a r.v., with distribution  $F(\cdot)$ . Upon failures, the machine has to be repaired with setup cost  $c_f$  and running cost  $k_f \cdot T_f$ , where  $k_f$  is the running cost rate for machine down

time, and  $T_f$  the time for repair, a r.v. with expected value  $a$ . In order to avoid failures, the machine can be preventively maintained before failures with setup cost  $c_p$  and running cost  $k_p \cdot T_p$ , where  $k_p$  is the running cost rate for machine maintenance, and  $T_p$  the time for PM, also a r.v. with expected value  $b$ . After either a repair or a PM, it is assumed that the machine is renewed.

We study a randomized policy in which the time to do PM  $\hat{T}$  is a r.v. with general distribution  $G(\cdot)$ . For example, the time-window policy commonly practiced in industry, in which PM tasks can be started uniformly within the window, is a randomized policy. Also note that a deterministic policy, for example, the familiar age replacement policy, is the special case of a randomized policy. The objective is to find the optimal  $G(\cdot)$  to minimize the average cost.

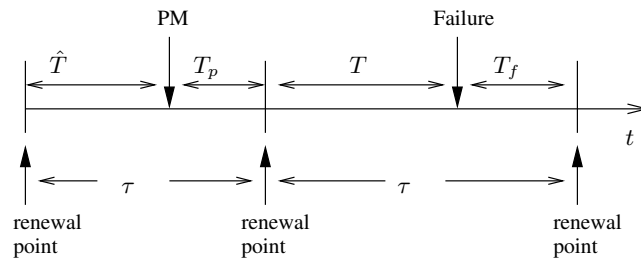


Figure 3.1: Problem setting.

The notations used hereafter are summarized as follows:

$T$ : machine's life time, a r.v. with c.d.f.  $F(t)$ ;

$\hat{T}$ : time to do PM, a r.v., with c.d.f.  $G(t)$ ;

$T_f$ : time for repair, r.v.,  $E(T_f) = a$ ;

$T_p$ : time for PM, r.v.,  $E(T_p) = b$ ;

$c_p$ : setup cost for PM;

$c_f$ : setup cost for repair;

$k_f$ : running cost rate for machine in failure;

$k_p$ : running cost rate for machine in PM;

$\tau$ : the length of renewal cycle (time between two consecutive renewals of the machine);

$C$ : the total costs incurred during a cycle  $\tau$ .

We assume  $T_f$  and  $T_p$  are independent of  $T$  and  $\hat{T}$ . It is obvious that

$$\tau = \min(T, \hat{T}) + T_f \cdot 1(T < \hat{T}) + T_p \cdot 1(T > \hat{T}), \quad (3.1)$$

where  $1(A)$  is the indicator function of event  $A$ , with value being 1 if the event  $A$  is true. So,

$$E(\tau) = E\left(\min(T, \hat{T})\right) + E(T_f) \cdot \Pr(T < \hat{T}) + E(T_p) \cdot \Pr(T > \hat{T}). \quad (3.2)$$

but,

$$\begin{aligned} E\left(\min(T, \hat{T})\right) &= E\left(E\left(\min(T, \hat{T}) \mid \hat{T}\right)\right) \\ &= \int_0^\infty \left(\int_0^t \bar{F}(y) dy\right) dG(t). \end{aligned} \quad (3.3)$$

$$\begin{aligned} \Pr(T < \hat{T}) &= E\left(E\left(1(T < \hat{T}) \mid \hat{T}\right)\right) \\ &= \int_0^\infty E(1(T < t)) dG(t) \\ &= \int_0^\infty F(t) dG(t) \end{aligned} \quad (3.4)$$

$$\begin{aligned} \Pr(T > \hat{T}) &= 1 - \Pr(T < \hat{T}) \\ &= \int_0^\infty \bar{F}(t) dG(t). \end{aligned} \quad (3.5)$$



Therefore,

$$E(\tau) = \int_0^\infty \left( \int_0^t \bar{F}(y) dy + a \cdot F(t) + b \cdot \bar{F}(t) \right) dG(t). \quad (3.6)$$

The total cost  $C$  during a cycle  $\tau$  is:

$$C = (c_f + k_f \cdot T_f) \cdot 1(T < \hat{T}) + (c_p + k_p \cdot T_p) \cdot 1(T > \hat{T}). \quad (3.7)$$

Therefore, the total expected cost in a cycle  $\tau$  is:

$$\begin{aligned} E(C) &= (c_f + k_f \cdot E(T_f)) \cdot \int_0^\infty F(t) dG(t) + (c_p + k_p \cdot E(T_p)) \int_0^\infty \bar{F}(t) dG(t) \\ &= \int_0^\infty (a' \cdot F(t) + b' \cdot \bar{F}(t)) dG(t), \end{aligned} \quad (3.8)$$

where  $a'$  and  $b'$  are expected total cost of repair and PM, respectively, and given by:

$$a' := c_f + k_f \cdot a, \quad (3.9)$$

$$b' := c_p + k_p \cdot b. \quad (3.10)$$

### 3.2.2 Optimality of Deterministic PM Policy

Under the randomized PM policy  $G$ , by renewal theory, the average cost in a cycle is  $E(C)/E(\tau)$ , denoted by  $H(G)$ . Our objective is to find an optimal policy  $G^*$  to minimize the average cost  $H$ , i.e.,

$$\begin{aligned} H(G^*) &= \min_G H(G) \\ &= \min_G \frac{\int_0^\infty (a' \cdot F(t) + b' \cdot \bar{F}(t)) dG(t)}{\int_0^\infty \left( \int_0^t \bar{F}(y) dy + a \cdot F(t) + b \cdot \bar{F}(t) \right) dG(t)}. \end{aligned} \quad (3.11)$$

**Theorem 3.1.** *There exists a deterministic PM policy  $G^*$  such that it is optimal to do PM at some time  $t^*$ , where  $G^*$  has a mass of probability, i.e.,  $\Pr(\hat{T} = t^*) = 1$ .*

*Proof.* Our proof basically follows the method used by Barlow and Proschan [6].

$$\text{Let } P(t) := a' \cdot F(t) + b' \cdot \bar{F}(t), \quad (3.12)$$

$$Q(t) := \int_0^t \bar{F}(y) dy + a \cdot F(t) + b \cdot \bar{F}(t), \quad (3.13)$$

$$\text{Thus, } H(G) = \frac{\int_0^\infty P(t) dG(t)}{\int_0^\infty Q(t) dG(t)}. \quad (3.14)$$

Now denote

$$S(t) := \frac{P(t)}{Q(t)}. \quad (3.15)$$

Observe that  $S(t)$  is non-negative and right-continuous, so there exists infimum(s) of  $S(t)$ . Denote  $t^-$  be such that  $\forall \epsilon > 0, t - \epsilon < t^- < t$ . An infimum of  $S(t)$  might be achieved at some point  $t_0$  or  $t_0^-$ , dependent on the continuity of  $S(t)$  at the point  $t_0$ . If  $S(t)$  is continuous at the point  $t_0$ , let  $t^* = t_0$ ; if  $S(t)$  is right-continuous at  $t_0$ , then let  $t^* = t_0^-$ . It follows that

$$\begin{aligned} S(t) &\geq S(t^*), \text{ for } t \neq t^* \\ \Rightarrow \frac{P(t)}{Q(t)} &\geq \frac{P(t^*)}{Q(t^*)} \\ \Rightarrow P(t) &\geq \frac{P(t^*)}{Q(t^*)} \cdot Q(t). \end{aligned}$$

Since  $G(t)$  is a non-negative, increasing function, so

$$\begin{aligned} &\int_0^\infty \left( P(t) - \frac{P(t^*)}{Q(t^*)} \cdot Q(t) \right) dG(t) \geq 0 \\ \Rightarrow \frac{\int_0^\infty P(t) \cdot dG(t)}{\int_0^\infty Q(t) \cdot dG(t)} &\geq \frac{P(t^*)}{Q(t^*)} = S(t^*) \\ \Rightarrow H(G) &\geq H(G^*), \end{aligned}$$

where  $G^*$  is the degenerate distribution with all its mass of probability on point  $t^*$ , i.e.,  $\Pr(\hat{T} = t^*) = 1$ . □

*Remark:* We have relaxed in the theorem Barlow and Proschan's condition on the continuity of  $F(t)$ , which is a sufficient condition for  $S(t)$  to be continuous, and by doing so, we can cover the case when machine's lifetime is deterministic. It is obvious that for the deterministic case, the optimal PM policy is either to do PM just right before failures or never to do PM, dependent on the real costs data. The optimal policy can be derived from the theorem. If the lifetime is deterministic, say is  $t_0$ , it follows from (3.15) that

$$S(t) = \begin{cases} \frac{b'}{t+b} & \text{if } t < t_0, \\ \frac{a'}{t_0+a} & \text{if } t \geq t_0. \end{cases}$$

It is obvious that if  $\frac{b'}{t_0+b} < \frac{a'}{t_0+a}$ , then  $S(t)$  achieves infimum at point  $t^* = t_0^-$ , which implies the optimal time to do PM be right before failures; otherwise,  $t^*$  could be any value in  $[t_0, \infty)$ , which implies never do PM.

**Corollary 3.1.** *If there exists a unique  $t^*$ , then there is a unique optimal policy, and it is deterministic.*

To examine if there exists a unique  $t^*$ , take the derivative of  $S(t)$ , i.e.,

$$\begin{aligned} S'(t) &= \frac{P'Q - PQ'}{Q^2}, \\ &= \frac{\left[ (a' - b')r(t) \cdot \int_0^t \bar{F}(y)dy + r(t) \cdot (a'b - ab') - (a' - b')F(t) - b' \right] \bar{F}(t)}{Q^2}, \\ &:= M(t) \cdot \frac{\bar{F}(t)}{Q^2(t)}, \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} M(t) &:= (a' - b')r(t) \cdot \int_0^t \bar{F}(y)dy + r(t) \cdot (a'b - ab') - (a' - b')F(t) - b' \\ r(t) &:= \frac{f(t)}{\bar{F}(t)}, \text{ i.e., hazard (failure) rate of } F(t). \end{aligned}$$

Observe

$$S'(t) = 0 \Leftrightarrow M(t) = 0.$$

We further have

$$M'(t) = (a' - b')r'(t) \cdot \int_0^t \bar{F}(y)dy + r'(t) \cdot (a'b - ab') \quad (3.17)$$

In the following discussion, we assume two conditions are satisfied:

- (1)  $r'(t) > 0, \forall t > 0$ , i.e., the distribution of machine's lifetime is IFR;
- (2)  $a' > b'$ , i.e., the expected cost for failure is greater than for PM.

Let  $t^*$  be a point where  $S(t)$  achieves the minimum. In the following analysis, we consider the normal case where  $a'b - ab' \geq 0$ .

*Remark:* The condition  $a'b - ab' \geq 0$  can be rewritten as

$$\frac{a'}{a} \geq \frac{b'}{b},$$

or

$$\frac{c_f}{a} + k_f \geq \frac{c_p}{b} + k_p.$$

This means that the average cost rate of failure is greater than or equal to the average cost rate of PM, which is the case for most situations.

By the condition  $a'b - ab' \geq 0$ , it follows that  $M'(t) > 0$ .

(i) If  $M(0) \geq 0$ , i.e.,  $r(0) \geq \frac{b'}{a'b-ab'}$ , then  $M(t) \geq 0, \forall t > 0$ . It then follows that  $S(t)$  is an increasing function. Therefore,  $t^* = 0$ .

(ii) If  $M(\infty) \leq 0$ , i.e.,  $r(\infty) \leq \frac{a'}{(a'-b')\mu+(a'b-ab')}$ , then  $M(t) \leq 0, \forall t > 0$ . It then follows that  $S(t)$  is a decreasing function. Therefore,  $t^* = \infty$ .

(iii) If  $M(0) < 0$ , and  $M(\infty) > 0$ , i.e.,  $r(0) < \frac{b'}{a'b-ab'}$ , and  $r(\infty) \geq \frac{a'}{(a'-b')\mu+(a'b-ab')}$ , then there is a unique point  $t^*$ , such that  $M(t^*) = 0$ , i.e.,  $S'(t) |_{t=t^*} = 0$ . It then

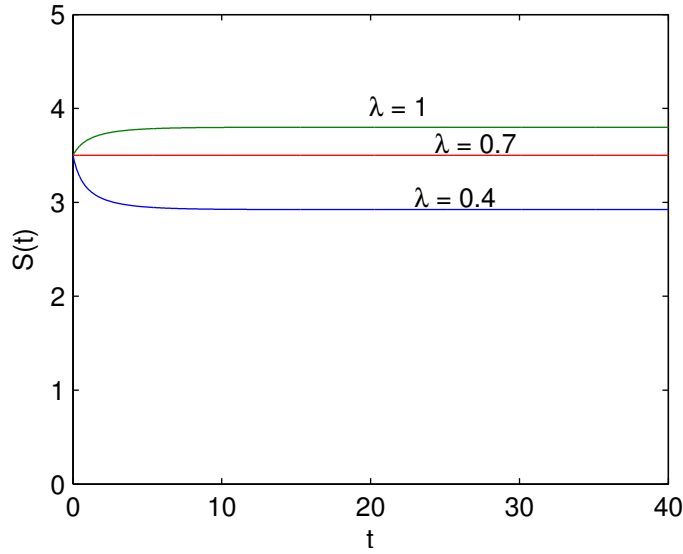


Figure 3.2: The function  $S(t)$  when  $F(t)$  is exponential distribution.

follows that  $S''(t^*) > 0$ . Therefore,  $S(t^*)$  is a minimum. To see the uniqueness, assume there are two points  $t_0^*, t_1^*$ , and  $t_0^* \neq t_1^*$ , such that,  $S'(t_0^*) = S'(t_1^*) = 0$ , i.e.,  $M(t_0^*) = M(t_1^*) = 0$ . Without loss of generality, assume  $t_0^* < t_1^*$ , because  $M' > 0$ ,  $M(t)$  is an increasing function, so it follows that  $M(t_1^*) > M(t_0^*) = 0$ . Contradiction.

### 3.2.3 Numerical Examples

**Example 1:** (Exponentially distributed lifetime)

$$\begin{aligned} k_p &= k_f = 1, \\ c_p &= 5, c_f = 15, \\ b &= 2, a = 4, \\ F(t) &= 1 - e^{-\lambda t}. \end{aligned}$$

Since  $F(t)$  is an exponential distribution, it has a constant failure rate, i.e.,  $r(t) = \lambda$ . It then follows that  $M'(t) = 0$ , by (3.17). Therefore,  $M(t)$  is a constant,

and it follows from (3.16) that  $S(t)$  is a monotonic function. To determine the optimal PM time  $t^*$ , it suffices to compare the value of  $S(t)$  at  $t = 0$  and  $t = \infty$ . If  $S(0) < S(\infty)$ , then  $t^* = 0$ ; if  $S(0) > S(\infty)$ , then  $t^* = \infty$ ; if  $S(0) = S(\infty)$ , then  $t^*$  is any value between  $[0, \infty]$ . In terms of  $\lambda$ , this can be written as:

$$\begin{cases} \lambda < \frac{c_p+k_p \cdot b}{(c_f+k_f \cdot a)b-(c_p+k_p \cdot ba)} & t^* = \infty, \\ \lambda > \frac{c_p+k_p \cdot b}{(c_f+k_f \cdot a)b-(c_p+k_p \cdot ba)} & t^* = 0, \\ \lambda = \frac{c_p+k_p \cdot b}{(c_f+k_f \cdot a)b-(c_p+k_p \cdot ba)} & t^* \text{ is any value in } [0, \infty]. \end{cases} \quad (3.18)$$

Figure 3.2 shows the function of  $S(t)$  with different values of  $\lambda$ .

**Example 2:** (Weibull distributed lifetime)

$$\begin{aligned} k_p &= k_f = 1, \\ c_p &= 1, c_f = 3, \\ b &= 0.5, a = 1, \\ F(t) &= 1 - e^{-\lambda t^\alpha}. \end{aligned}$$

The failure rate of Weibull distribution is  $\lambda\alpha t^{\alpha-1}$ . It has increasing failure rate for  $\alpha > 1$ , and in this case, the failure rate at  $t = 0$  is 0, but is unbounded as  $t$  goes to  $\infty$ . Therefore, according to our above analysis, if  $\frac{a'}{a} \geq \frac{b'}{b}$ , there is a unique point  $t^*$  at which  $S(t)$  achieves minimum. Figure 3.3 shows the function  $S(t)$  under different values of  $\alpha$ .

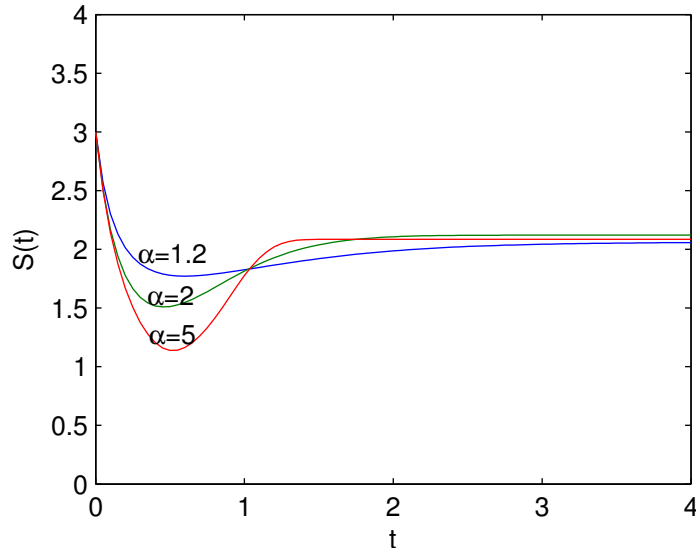


Figure 3.3: The function  $S(t)$  when  $F(t)$  is Weibull distribution.

### 3.3 M/G/1 with Infinite Queue Capacity

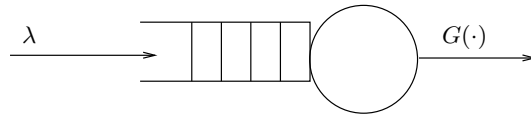


Figure 3.4: Model 1 – M/G/1 with infinite queue capacity

In this section, we study the optimal PM problem under the context of queueing systems. We consider an M/G/1 queueing system with an unreliable server. The server experiences random failures during operation. For simplicity, we assume the server failures are only at the completion of a job, and the server's lifetime can be specified by the number of jobs it has processed since the last repair. Once the server fails, a repair/CM must be carried out at some cost. During the repair, the server is inoperative, and so coming jobs will build up in the system. We assume the queueing system has infinite queue capacity. In

order to avoid costly server failures, PM can be scheduled at some lower cost. Furthermore, we assume the server never stays idle if the job queue is not empty. If the queue is indeed empty, the server is then forced idle; however, it will start to serve as soon as a new job arrives.

We consider the jobs waiting/queueing cost explicitly in the optimization problem. The objective is to find an optimal PM policy to minimize the total discounted job waiting cost and PM/CM costs over the infinite horizon. In the following, we formulate the optimization problem as a Semi-Markov Decision Process (SMDP), and then derive some structural properties regarding the optimal cost function and optimal policy. The main analytic result is the “control-limit” structure of the optimal policy with respect to the age or the deterioration degree of the server.

### 3.3.1 Discounted SMDP Formulation

We assume the decision epochs are at the completion of a job, repair or PM. When the server is up but idle (the job queue is empty), we assume the next decision epoch is at the arrival of next incoming job, and the server can choose to process the job or to start PM.

The system state will be denoted by  $(s_t, n_t)$  at the decision epoch  $t$ , where  $s_t$  is called the system’s *operational* state, i.e., the number of jobs in the system, and  $n_t$  the system’s *technical* state, i.e., the number of jobs finished since last maintenance (CM or PM). When the server is failed, we denote the system’s *technical* state by  $F$ . Therefore, the state space is  $\mathcal{Z}^+ \times \{\mathcal{Z}^+ \cup \{F\}\}$ .

When the technical state is not  $F$ , then at each decision epoch, two actions (controls)  $u$  are available: to do PM ( $u = PM$ ), or to process ( $u = \overline{PM}$ , not to



do PM). When the technical state is  $F$ , it is mandatory to do CM ( $u = CM$ ).

The following notations about the system will be employed hereafter:

$q_n$ : server's conditional probability of failure at the completion of the  $n$ th job,  
where the server's time to failure is a discrete r.v.

$\tau_r$ : time for repair, which is a continuous r.v.

$G_r(\cdot)$ : c.d.f. of  $\tau_r$

$c_r$ : CM cost when the server is failed.

$\tau_p$ : time for PM, a continuous r.v.

$G_p(\cdot)$ : c.d.f. of  $\tau_p$ .

$c_p$ : PM cost.

$\lambda$ : arrival rate of incoming jobs.

$G_w(\cdot)$ : c.d.f of the service time for a job. We assume the service time is independent of other r.v.s.

$h$ : holding cost rate for jobs in the system (whether in waiting status or in processing).

The cost function we consider has the form

$$\lim_{N \rightarrow \infty} E \left\{ \int_0^{t_N} e^{-\beta t} g(s_t, n_t; u_t) dt \right\}, \quad (3.19)$$

where  $t_N$  is the time of the  $N$ th decision epoch,  $\beta$  is the discount factor, and  $g$  is given by

$$g(s_t, n_t; u_t) = h \cdot s_t + c_p \cdot \delta(u_t = PM) + c_r \cdot \delta(u_t = CM), \quad (3.20)$$

Note that the maintenance costs  $c_r$  and  $c_p$  are incurred only at the time the control  $u$  is chosen, and are independent of the length of the transition interval.

Let  $J$  be the optimal cost function, which satisfies the following Bellman's equation:

$$J(s, n) = \min \left\{ Q^{\overline{PM}}(s, n), Q^{PM}(s, n) \right\},$$

where  $Q^{\overline{PM}}(s, n)$  and  $Q^{PM}(s, n)$  are the so-called Q-functions of taking action  $\overline{PM}$  and  $PM$ , respectively, at the current decision epoch and optimal actions thereafter.

Throughout this chapter, we use the notation  $p_k(\lambda t) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$ . Following the standard SMDP formulation (for example, see [30, 10]), we have the following equations:

1. For  $s > 0, n \geq 0, n \neq F$ ,

$$\begin{aligned} Q^{\overline{PM}}(s, n) &= h \cdot s \cdot \int_0^\infty \frac{1 - e^{-\beta t}}{\beta} dG_w(t) + TC_w + (1 - q_{n+1}) \cdot \\ &\quad + \sum_{k=0}^\infty \int_0^\infty e^{-\beta t} p_k(\lambda t) dG_w(t) \cdot J(s - 1 + k, n + 1) \\ &\quad + q_{n+1} \cdot \sum_{k=0}^\infty \int_0^\infty e^{-\beta t} p_k(\lambda t) dG_w(t) \cdot J(s - 1 + k, F). \end{aligned} \tag{3.21}$$

$$\begin{aligned} Q^{PM}(s, n) &= c_p + h \cdot s \cdot \int_0^\infty \frac{1 - e^{-\beta t}}{\beta} dG_p(t) + TC_p \\ &\quad + \sum_{k=0}^\infty \int_0^\infty e^{-\beta t} p_k(\lambda t) dG_p(t) \cdot J(s + k, 0). \end{aligned} \tag{3.22}$$

Note: the first term in the RHS of  $Q^{\overline{PM}}(s, n)$  is the one-stage expected inventory cost of the  $s$  jobs already in the system, and  $TC_w$  is the expected holding cost of new jobs arriving until the next decision epoch. The third and fourth terms account for the possibilities of the next transition state

depending on whether the production succeeds or fails, respectively. Similarly, the first term in the RHS of  $Q^{PM}(s, n)$  is the PM cost, the second and third terms  $TC_p$  are the expected total holding costs, and the last term accounts for the next transition states.

We now compute the  $TC_w$  and  $TC_p$ , the expected holding cost of new jobs arriving in the transition interval, in the following. Assume the transition interval is  $[0, t]$ . Given there is  $k$  jobs arriving during the interval, for a sample path  $0 < t_1 < t_2 < \dots < t_k < t$ , the holding cost is:

$$\begin{aligned} \sum_{i=1}^k \int_{t_i}^t h e^{-\beta\tau} d\tau &= \frac{h}{\beta} \sum_{i=1}^k (e^{-\beta t_i} - e^{-\beta t}) \\ &= \frac{h}{\beta} \sum_{i=1}^k e^{-\beta t_i} - \frac{kh}{\beta} e^{-\beta t}. \end{aligned}$$

As we know for Poisson arrivals in a given period  $[0, t]$ , the unordered arrival epoches  $t_i$  are distributed uniformly in  $[0, t]$ ; see, for example [47]. So by unconditioning on the sample path, we have:

$$\begin{aligned} E \left[ \sum_{i=1}^k \int_{t_i}^t h e^{-\beta\tau} d\tau \right] &= \frac{h}{\beta} \sum_{i=1}^k E [e^{-\beta t_i}] - \frac{kh}{\beta} e^{-\beta t} \\ &= \frac{kh}{\beta} \int_0^t \frac{1}{t} e^{-\beta\tau} d\tau - \frac{kh}{\beta} e^{-\beta t} \\ &= \frac{kh}{t\beta^2} (1 - e^{-\beta t}) - \frac{kh}{\beta} e^{-\beta t} \\ &= \frac{kh}{\beta^2 t} [1 - (1 + \beta t)e^{-\beta t}]. \end{aligned}$$

Now, unconditioning on  $k$  and  $t$  yields

$$\begin{aligned} TC_w &= \sum_{k=0}^{\infty} \int_0^{\infty} p_k(t) \frac{kh}{\beta^2 t} [1 - (1 + \beta t)e^{-\beta t}] dG_w(t) \\ &= \int_0^{\infty} \frac{\lambda h}{\beta^2} [1 - (1 + \beta t)e^{-\beta t}] dG_w(t). \end{aligned} \tag{3.23}$$

Similarly, for  $TC_p$ , we have the following equality,

$$TC_p = \int_0^\infty \frac{\lambda h}{\beta^2} [1 - (1 + \beta t)e^{-\beta t}] dG_p(t). \quad (3.24)$$

2. For  $s \geq 0, n = F$ ,

$$\begin{aligned} J(s, F) &= c_r + h \cdot s \cdot \int_0^\infty \frac{1 - e^{-\beta t}}{\beta} dG_r(t) + TC_r \\ &+ \sum_{k=0}^\infty \int_0^\infty e^{-\beta t} p_k(\lambda t) dG_r(t) \cdot J(s + k, 0). \end{aligned} \quad (3.25)$$

Note: When the server fails, a CM must be performed at the instant cost  $c_r$ . The corresponding cost  $TC_r$  is given by

$$TC_r = \int_0^\infty \frac{\lambda h}{\beta^2} [1 - (1 + \beta t)e^{-\beta t}] dG_r(t). \quad (3.26)$$

3. For  $s = 0, n \neq F$ ,

$$\begin{aligned} Q^{\overline{PM}}(0, n) &= \int_0^\infty e^{-\beta t} \lambda e^{-\lambda t} dt \cdot J(1, n) \\ &= \frac{\lambda}{\lambda + \beta} \cdot J(1, n); \end{aligned} \quad (3.27)$$

$$Q^{PM}(0, n) = c_p + TC_p + \sum_{k=0}^\infty \int_0^\infty e^{-\beta t} p_k dG_p(t) \cdot J(k, 0). \quad (3.28)$$

Note: When the queue is empty at a decision epoch, i.e.,  $s = 0$ , the next decision epoch is either the arrival of next job if the action chosen is not doing PM, or the completion of the PM if the action is doing PM. The holding cost for  $Q^{\overline{PM}}(0, n)$  is zero, since there is no jobs in the system during the transition interval.

### 3.3.2 Structural Results

**Lemma 3.1.** *The optimal cost function  $J(s, n)$  is non-decreasing with respect to  $s$ , i.e.,  $J(s, n) \leq J(s + 1, n)$ , for all  $s, n$ .*

*Proof.* We proceed by induction using value iteration. Assuming  $J_m(s, n) \leq J_m(s + 1, n)$ . Then we consider  $J_{m+1}(s, n)$  at the next step. We first prove it holds for the case  $s > 0, n \neq F$ .

If  $s > 0, n \neq F$ ,

$$\begin{aligned}
Q_{m+1}^{\overline{PM}}(s + 1, n) &= h \cdot (s + 1) \cdot \int_0^\infty \frac{1 - e^{-\beta t}}{\beta} dG_w(t) + TC_w + (1 - q_{n+1}) \cdot \\
&\quad + \sum_{k=0}^\infty \int_0^\infty e^{-\beta t} p_k(\lambda t) dG_w(t) \cdot J_m(s + k, n + 1) \\
&\quad + q_{n+1} \cdot \sum_{k=0}^\infty \int_0^\infty e^{-\beta t} p_k(\lambda t) dG_w(t) \cdot J_m(s + k, F) \\
&\geq h \cdot s \cdot \int_0^\infty \frac{1 - e^{-\beta t}}{\beta} dG_w(t) + TC_w + (1 - q_{n+1}) \cdot \\
&\quad + \sum_{k=0}^\infty \int_0^\infty e^{-\beta t} p_k(\lambda t) dG_w(t) \cdot J_m(s - 1 + k, n + 1) \\
&\quad + q_{n+1} \cdot \sum_{k=0}^\infty \int_0^\infty e^{-\beta t} p_k(\lambda t) dG_w(t) \cdot J_m(s - 1 + k, F) \\
&= Q_{m+1}^{\overline{PM}}(s, n); \tag{3.29}
\end{aligned}$$

Similarly,  $Q_{m+1}^{PM}(s + 1, n) \geq Q_{m+1}^{PM}(s, n)$ . It follows that

$$\begin{aligned}
J_{m+1}(s + 1, n) &= \min \left( Q_{m+1}^{\overline{PM}}(s + 1, n), Q_{m+1}^{PM}(s + 1, n) \right) \\
&\geq \min \left( Q_{m+1}^{\overline{PM}}(s, n), Q_{m+1}^{PM}(s, n) \right) \\
&= J_{m+1}(s, n). \tag{3.30}
\end{aligned}$$

We can use the same arguments to prove it holds for the optimal cost function  $J(s, F)$  for all  $s$ . But we need some extra steps for the case when  $s = 0, n \neq F$ .

We first note that  $J_{m+1}(s, n) \geq J_m(s, n)$  for all  $(s, n)$ , by the monotonicity of the

dynamic programming operator (see [10], Lemma 1.1 on page 7, Vol II). Thus

$$\begin{aligned}
Q_{m+1}^{\overline{PM}}(1, n) &\geq J_{m+1}(1, n), \text{ by definition,} \\
&\geq J_m(1, n), \text{ by monotonicity of DP} \\
&= \left(1 + \frac{\beta}{\lambda}\right) Q_{m+1}^{\overline{PM}}(0, n), \\
&> Q_{m+1}^{\overline{PM}}(0, n).
\end{aligned}$$

It is obvious that  $Q_{m+1}^{PM}(1, n) > Q_{m+1}^{PM}(0, n)$ . Therefore  $J_{m+1}(1, n) > J_{m+1}(0, n)$ .

Because  $J(s, n)$  is the limit function of  $J_m(s, n)$ , i.e.,

$$J(s, n) = \lim_{m \rightarrow \infty} J_m(s, n),$$

so  $J(s, n)$  is increasing in  $s$ . This ends the proof.  $\square$

**Condition 3.1.**  $q_n$  is non-decreasing (i.e., time to failure is IFR).

**Condition 3.2.**  $c_r \geq c_p$  (i.e., repair is costlier than PM).

**Condition 3.3.**  $\tau_r = \tau_p$  in stochastic sense, i.e., equal in distribution.

**Lemma 3.2.** Under Conditions 3.1, 3.2 and 3.3, for all  $s$ ,  $Q^{\overline{PM}}(s, n)$  and  $J(s, n)$  are non-decreasing with respect to  $n$ . Moreover  $J(s, F) \geq J(s, n), \forall n$ .

*Proof.* We proceed by induction using value iteration.

(1) Step 1:  $J_0(\cdot, \cdot) = 0$ .

(2) Step 2: Assume  $Q_m^{\overline{PM}}(s, n), J_m(s, n)$  non-decreasing,  $J_m(s, F) \geq J_m(s, n)$ ,

then

$$\begin{aligned}
Q_{m+1}^{\overline{PM}}(s, n) &= h \cdot s \cdot \int_0^\infty \frac{1 - e^{-\beta t}}{\beta} dG_w(t) + TC_w + (1 - q_{n+1}) \cdot \\
&\quad \sum_{k=0}^\infty \int_0^\infty e^{-\beta t} p_k(\lambda t) dG_w(t) \cdot J_m(s - 1 + k, n + 1) \\
&\quad + q_{n+1} \cdot \sum_{k=0}^\infty \int_0^\infty e^{-\beta t} p_k(\lambda t) dG_w(t) \cdot J_m(s - 1 + k, F). \\
Q_{m+1}^{\overline{PM}}(s, n + 1) &= h \cdot s \cdot \int_0^\infty \frac{1 - e^{-\beta t}}{\beta} dG_w(t) + TC_w + (1 - q_{n+2}) \cdot \\
&\quad \sum_{k=0}^\infty \int_0^\infty e^{-\beta t} p_k(\lambda t) dG_w(t) \cdot J_m(s - 1 + k, n + 2) \\
&\quad + q_{n+2} \cdot \sum_{k=0}^\infty \int_0^\infty e^{-\beta t} p_k(\lambda t) dG_w(t) \cdot J_m(s - 1 + k, F).
\end{aligned}$$

So,

$$\begin{aligned}
&Q_{m+1}^{\overline{PM}}(s, n + 1) - Q_{m+1}^{\overline{PM}}(s, n) \\
&= \sum_{k=0}^\infty \int_0^\infty e^{-\beta t} p_k dG_w \cdot (J_m(s - 1 + k, n + 2) - J_m(s - 1 + k, n + 1)) \\
&\quad + q_{n+2} \cdot \sum_{k=0}^\infty \int_0^\infty e^{-\beta t} p_k dG_w \cdot (J_m(s - 1 + k, F) - J_m(s - 1 + k, n + 2)) \\
&\quad - q_{n+1} \cdot \sum_{k=0}^\infty \int_0^\infty e^{-\beta t} p_k dG_w \cdot (J_m(s - 1 + k, F) - J_m(s - 1 + k, n + 1)) \\
&\geq \sum_{k=0}^\infty \int_0^\infty e^{-\beta t} p_k dG_w \cdot (J_m(s - 1 + k, n + 2) - J_m(s - 1 + k, n + 1)) \\
&\quad + q_{n+2} \sum_{k=0}^\infty \int_0^\infty e^{-\beta t} p_k dG_w \cdot (J_m(s - 1 + k, n + 1) - J_m(s - 1 + k, n + 2)) \\
&= (1 - q_{n+2}) \sum_{k=0}^\infty \int_0^\infty e^{-\beta t} p_k dG_w \cdot (J_m(s - 1 + k, n + 2) - J_m(s - 1 + k, n + 1)) \\
&\geq 0.
\end{aligned}$$

So,  $Q_{m+1}^{\overline{PM}}(s, n)$  is non-decreasing with respect to  $n$ .

$$J_{m+1}(s, n) = \min \left( Q_{m+1}^{\overline{PM}}(s, n), Q_{m+1}^{PM}(s, n) \right) \quad (3.31)$$

$Q_{m+1}^{PM}(s, n)$  is constant with respect to  $n$  by (3.22), so  $J_{m+1}(s, n)$  is non-decreasing in  $n$ . Moreover,

$$\begin{aligned} J_{m+1}(s, F) &= c_r + h \cdot s \cdot \int_0^\infty \frac{1 - e^{-\beta t}}{\beta} dG_r + TC_r \\ &\quad + \sum_{k=0}^\infty \int_0^\infty e^{-\beta t} p_k(\lambda t) dG_r \cdot J_m(s + k, 0). \\ Q_{m+1}^{PM}(s, n) &= c_p + h \cdot s \cdot \int_0^\infty \frac{1 - e^{-\beta t}}{\beta} dG_p + TC_p \\ &\quad + \sum_{k=0}^\infty \int_0^\infty e^{-\beta t} p_k(\lambda t) dG_p \cdot J_m(s + k, 0). \end{aligned}$$

Under the condition 3.1, 3.2 and 3.3,  $Q_{m+1}^{PM}(s, n) \leq J_{m+1}(s, F)$ . Since

$$J_{m+1}(s, n) = \min \left( Q_{m+1}^{\overline{PM}}(s, n), Q_{m+1}^{PM}(s, n) \right) \quad (3.32)$$

$$\leq Q_{m+1}^{PM}(s, n). \quad (3.33)$$

so,  $J_{m+1}(s, n) \leq J_{m+1}(s, F), \forall n$ .

(3) Step 3: Let  $m \rightarrow \infty$ , we have

$$Q^{\overline{PM}}(s, n) = \lim_{m \rightarrow \infty} Q_m^{\overline{PM}}(s, n), \quad (3.34)$$

$$J(s, n) = \lim_{m \rightarrow \infty} J_m(s, n). \quad (3.35)$$

So,  $Q^{\overline{PM}}(s, n)$  and  $J(s, n)$  are non-decreasing in  $n$ , and  $J(s, n) \leq J(s, F), \forall n$ .

This concludes the proof.  $\square$

We are now ready to prove the following theorem regarding the optimal policy.



**Theorem 3.2.** For any fixed  $s$ , under Conditions 3.1, 3.2 and 3.3, the optimal PM policy is of control-limit type, i.e.,  $\exists n^*(s)$  s.t. for system state  $(s, n)$ , if  $n \geq n^*(s)$ , then the optimal action  $\mu^*(s, n) = PM$ ; else,  $\mu^*(s, n) = \overline{PM}$ .

*Proof.* Observe that  $Q^{PM}(s, n)$  is constant with respect to  $n$ . By the monotonicity of  $Q^{\overline{PM}}(s, \cdot)$ , we know

$$Q^{\overline{PM}}(s, n) - Q^{PM}(s, n) \leq Q^{\overline{PM}}(s, n+1) - Q^{PM}(s, n+1), \forall n. \quad (3.36)$$

Thus, at some point  $n^*$ ,

$$Q^{\overline{PM}}(s, n^*) - Q^{PM}(s, n^*) \geq 0, \quad (3.37)$$

and for  $n \geq n^*$ ,

$$\begin{aligned} Q^{\overline{PM}}(s, n) - Q^{PM}(s, n) &\geq Q^{\overline{PM}}(s, n^*) - Q^{PM}(s, n^*) \\ &\geq 0. \\ \text{i.e., } Q^{\overline{PM}}(s, n) &\geq Q^{PM}(s, n). \end{aligned}$$

Therefore,  $J(s, n) = Q^{PM}(s, n)$ , i.e.,  $\mu^*(s, n) = PM$ . □

### 3.4 A Simplified Discrete-Time Model

In this section, we develop a simplified queueing model to study the PM optimization problem. As in [17], we assume the system is monitored at discrete, equidistant time epochs, and the job arrival process and machine's service process are assumed deterministic. A simplified cost structure is also adopted.

Specifically, we assume jobs arrive to the system at the rate of  $d$  per time unit. If the machine is in working state, then it can produce at the maximal

production capacity  $P$ , given there is enough jobs in the system waiting for processing. Otherwise, if there is not enough waiting jobs, then it will produce exactly the same jobs as in the system. From this regard, the machine production policy is fixed.

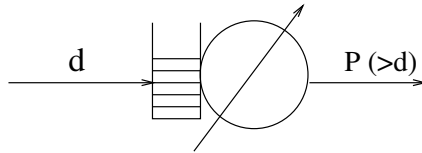


Figure 3.5: The discrete-time model

Because the machine is not reliable, if the machine fails, it has to be repaired. During the repair, the jobs keep arriving and build up in the queue. The machine can have PM so as to avoid failures. Although the PM makes the machine unavailable, the time for PM is relatively shorter than the repair time. Both the time for PM and for repair are assumed to be random variables.

We consider a simplified cost structure for this model – only jobs holding cost is considered, that is, a unit cost for each job waiting in the system will be incurred per each time unit. It is worth noting that although the costs for PM and CM are not included directly, they are implicitly considered. Since when the machine is down due to either PM or CM, jobs build up and holding costs are incurred. The objective is to minimize the discounted cost over the infinite horizon.

For simplicity, we further assume that both repair time and PM time are geometrically distributed, with parameters  $r$  and  $p$ , respectively. As a result, we are able to reduce the problem's state space. The machine's lifetime  $L$  is a discrete random variable, and denote its failure rate for age  $n + 1$  by  $f_n$ , for

$n \in \mathcal{Z}^+$  (we assume the machine can't fail at age 0, i.e., right after it is just renewed).

We summarize the model's notation as follows:

$d$ : number of jobs arriving in each time unit. Without loss of generality, it is assumed integer;

$P$ : server's maximal production rate, assumed integer;

$L$ : machine's lifetime, a discrete random variable;

$f_n$ : machine's failure rate at age  $n + 1$ , i.e.,  $f_n = \Pr(L = n + 1 \mid L \geq n + 1)$ ;

$p$ : parameter of geometric distribution for PM time;

$r$ : parameter of geometric distribution for repair time;

$\beta$ : discounting factor,  $0 < \beta < 1$ .

### 3.4.1 MDP Formulation

The problem can be easily formulated as an MDP problem, with system state denoted by  $(s_t, n_t)$  at the beginning of time period  $t$ , where  $s_t \in \mathcal{Z}^+$  is the number of jobs in the system, and  $n_t$  is the machine state. Because of the assumption of geometric distributions of repair/PM time, it is sufficient to denote the server state by  $PM$  when it is in the state of PM, and by  $CM$  when in the state of repair. Therefore,  $n_t \in \{\mathcal{Z}^+ \cup \{CM\} \cup \{PM\}\}$ .

The equidistant discrete time epochs are decision points to select to start maintenance or continue production.

Let  $J$  be the optimal cost function, and then we can write the following dynamic programming equations.

- When the server is in repair:

$$J(s, CM) = s + \beta \cdot r \cdot J(s + d, 0) + \beta \cdot (1 - r) \cdot J(s + d, CM). \quad (3.38)$$

- When the server is in PM:

$$J(s, PM) = s + \beta \cdot p \cdot J(s + d, 0) + \beta \cdot (1 - p) \cdot J(s + d, PM). \quad (3.39)$$

- When the server is in working state, let  $\Delta = P - d$ ,

$$J(s, n) = \min \left\{ J(s, PM), J^{\overline{PM}}(s, n) \right\} \quad (3.40)$$

$$\begin{aligned} J^{\overline{PM}}(s, n) &= s + \beta \cdot f_n \cdot J((s - \Delta)^+, CM) \\ &\quad + \beta(1 - f_n) \cdot J((s - \Delta)^+, n + 1) \end{aligned} \quad (3.41)$$

Equations (3.38) and (3.39) represent the dynamics of the system when the machine is in repair and in PM, respectively. The first terms in both equations are the holding cost for jobs in the system. The second terms state that with probability  $r$  (or  $p$ ), the repair (or PM) will be finished by next time and thus the machine age is renewed, whereas the third terms state that with probability  $1 - r$  (or  $1 - p$ ), the repair (or PM) will be continued by next time and the machine stays the  $CM$  (or  $PM$ )state. The first term in equation (3.40) corresponds to the action taking PM, and the second term corresponds to the action continuing production.

### 3.4.2 Structural Properties

In this section, we will first derive some properties of the optimal cost function  $J$ , and then obtain important structural properties of the optimal PM policy. The proofs are based on value iteration of optimal cost function, as used in [17].

Let  $J_k(\cdot, \cdot)$  be the cost after  $k$  steps of value iteration.

$$J_0(\cdot, \cdot) = 0; \quad (3.42)$$

$$J_{k+1}(s, CM) = s + \beta \cdot r \cdot J_k(s + d, 0) + \beta \cdot (1 - r) \cdot J_k(s + d, CM); \quad (3.43)$$

$$J_{k+1}(s, PM) = s + \beta \cdot p \cdot J_k(s + d, 0) + \beta \cdot (1 - p) \cdot J_k(s + d, PM); \quad (3.44)$$

$$J_{k+1}(s, n) = \min\{J_{k+1}(s, PM), J_{k+1}^{\overline{PM}}(s, n)\}; \quad (3.45)$$

$$J_{k+1}^{\overline{PM}}(s, n) = s + \beta \cdot f_n \cdot J_k((s - \Delta)^+, CM) \quad (3.46)$$

$$+ \beta(1 - f_n) \cdot J_k((s - \Delta)^+, n + 1). \quad (3.47)$$

Theorem 2.2 states that the value iteration will converge to the optimal cost function, i.e.,

$$J = \lim_{k \rightarrow \infty} J_k.$$

It is obvious that if we can show a property holds for function  $J_k$ , for all  $k$ , then it also holds for the optimal cost function  $J$ .

**Condition 3.4.**  $f_n$  is increasing in  $n$ .

**Condition 3.5.**  $p \geq r$ .

**Lemma 3.3.**  $J(s, \cdot) \leq J(s + 1, \cdot)$ , for all  $s \geq 0$ .

*Proof.* We will show the property holds for  $J_k$ , for all  $k$ , by using induction.

Obviously, it holds for  $J_0$ . Assuming it holds for  $J_k$ , then at  $k + 1$ ,

$$\begin{aligned} J_{k+1}(s, CM) &= s + \beta \cdot r \cdot J_k(s + d, 0) + \beta \cdot (1 - r) \cdot J_k(s + d, CM) \\ &< s + 1 + \beta \cdot r \cdot J_k(s + d, 0) + \beta \cdot (1 - r) \cdot J_k(s + d, CM) \\ &\leq (s + 1) + \beta \cdot r \cdot J_k(s + d + 1, 0) + \beta(1 - r)J_k(s + d + 1, CM) \\ &= J_{k+1}(s + 1, CM). \end{aligned}$$

The proof is similar for  $J_{k+1}(s, PM)$ .

We then examine  $J(s, n)$ . Observe that  $(s - \Delta)^+ \leq (s + 1 - \Delta)^+$ , and thus  $J_k((s - \Delta)^+, \cdot) \leq J_k((s + 1 - \Delta)^+, \cdot)$ , by induction hypothesis. It follows easily that  $\overline{J_{k+1}^{PM}}(s, n)$  is increasing in  $s$ . So is  $J_{k+1}(s, n)$ .

Let  $k \rightarrow \infty$ , it follows that  $J(s, \cdot) \leq J(s + 1, \cdot)$ . □

**Lemma 3.4.** *Under Condition (3.5), for fixed  $s$ , we have*

$$J(s, n) \leq J(s, PM) \leq J(s, CM), \text{ for all } n.$$

*Proof.* The first part inequality is obvious by (3.40). Assuming  $J_k(s, PM) \leq J_k(s, CM)$ , then

$$\begin{aligned} J_{k+1}(s, PM) &= s + \beta \cdot p \cdot J_k(s + d, 0) + \beta \cdot (1 - p) \cdot J_k(s + d, PM) \\ &\leq s + \beta \cdot p \cdot J_k(s + d, 0) + \beta \cdot (1 - p) \cdot J_k(s + d, CM) \\ &= s + \beta \cdot J_k(s + d, CM) - \beta \cdot p \cdot (J_k(s + d, CM) - J_k(s + d, 0)) \\ &\leq s + \beta \cdot J_k(s + d, CM) - \beta \cdot r \cdot (J_k(s + d, CM) - J_k(s + d, 0)) \\ &= J_{k+1}(s, CM). \end{aligned}$$

The first inequality is due to the induction hypothesis. The second inequality follows from the condition (3.5) and the induction hypothesis that  $J_k(s + d, 0) \leq J_k(s + d, CM)$ . Let  $k \rightarrow \infty$ , it follows that  $J(s, n) \leq J(s, PM) \leq J(s, CM)$ . □

**Lemma 3.5.** *Under Conditions (3.4) and (3.5), we have  $J(s, n) \leq J(s, n + 1)$ .*

*Proof.* We will first show by induction that  $\overline{J_k^{PM}}(s, n) \leq \overline{J_k^{PM}}(s, n + 1)$ , for all

$k$ . Then it follows from (3.45) that  $J_k(s, n) \leq J_k(s, n + 1)$  for all  $k$ .

$$\begin{aligned}
J_{k+1}^{\overline{PM}}(s, n) &= s + \beta \cdot f_n \cdot J_k((s - \Delta)^+, CM) + \beta(1 - f_n) \cdot J_k((s - \Delta)^+, n + 1) \\
&\leq s + \beta \cdot f_n \cdot J_k((s - \Delta)^+, CM) + \beta(1 - f_n) \cdot J_k((s - \Delta)^+, n + 2) \\
&= s + \beta \cdot J_k((s - \Delta)^+, n + 2) \\
&\quad + \beta \cdot f_n \cdot (J_k((s - \Delta)^+, CM) - J_k((s - \Delta)^+, n + 2)) \\
&\leq s + \beta \cdot J_k((s - \Delta)^+, n + 2) \\
&\quad + \beta \cdot f_{n+1} \cdot (J_k((s - \Delta)^+, CM) - J_k((s - \Delta)^+, n + 2)) \\
&= J_{k+1}(s, n + 1).
\end{aligned}$$

The first inequality is due to the induction hypothesis. The second inequality follows from the condition (3.4) and Lemma 3.4 that  $J_k((s - \Delta)^+, CM) \geq J_k((s - \Delta)^+, n + 2)$ .  $\square$

**Theorem 3.3.** *Under Conditions (3.4) and (3.5), the optimal PM policy is of control-limit type, i.e., for system state  $(s, n)$ ,  $\exists n^*(s)$  such that the optimal action is to do PM if and only if  $n \geq n^*(s)$ .*

*Proof.* The proof is similar to that of Theorem 3.2. It follows from the monotonicity of  $Q^{\overline{PM}}(s, n)$  in  $n$  and the fact that  $Q^{PM}(s, n)$  is constant with respect to  $n$ .  $\square$

## Chapter 4

# Optimal Joint PM and Production Control Policies

In previous chapter, we have assume implicitly that system production policies are fixed. In this chapter, we will take the optimal PM problem a step further by considering joint PM and production control policies for failure-prone production systems. A joint PM and production control policy determines when PM should be performed, and if PM is not being performed, then how much should be produced. The basic problem setting is the following. Consider an unreliable make-to-stock production system with the stock of completed goods consumed by external demand (possibly random). The production unit (machine) can produce at maximal rate if it is in working state. However, the unit experiences random failures, and if it fails, a costly repair/CM (Corrective Maintenance) has to be performed. In order to avoid failures, PM can be scheduled for the unit. Either CM or PM will take some random time to be finished with some costs. When it is in CM or PM, it produces nothing; when it is not in CM or PM, then it can produce at any rate up to its maximal production rate. Backlogged demands are allowed, but they will incur larger cost than holding completed



goods in stock. The objective is to find an optimal joint policy to minimize the expected discounted costs over the infinite horizon.

Two types of failures are considered. Type I is called *time-dependent* failure, with which a machine deteriorates whether or not it is producing, and can fail when idle. Type II is called *operation-dependent* failure, with which a machine deteriorates only when it is producing, and cannot fail when idle. From the viewpoint of semiconductor manufacturing, the time-dependent failures can be prevented by performing *calendar-based* PMs, while the operation-dependent failures can be prevented by performing *wafer-count based* or *operation-time based* PMs.

The main focus of this chapter will be characterizing the structures of optimal joint policies. In the remainder of the chapter, we start with a brief discussion of related work, followed by the development of the model with time-dependent failures. An MDP formulation is presented in detail, which is the basis of our analysis for optimal policy structures. The development and analysis for the model with operation-dependent failures then follows. Some concluding remarks are provided in the last section.

## 4.1 Related Work

The problem of optimal production control in failure prone production systems has been studied extensively since the pioneering work by Akella and Kumar [1]; see also [27] for a discrete-time model. Interestingly, the problem of joint production control and maintenance policy has not been studied until very recently. One reason for the neglect is possibly due to the modeling of machine failure

processes as two-state (on-off) continuous-time Markov chains, which essentially means that the machine's lifetime is exponentially distributed, and thus has constant failure rate, which as a result precludes PM from being included in these traditional models.

Recently, Boukas and Liu study the production and maintenance control problem of a failure prone manufacturing system [11]. They consider the continuous flow model in which the machine has three working states: good, average and bad, and one failure state. The state transition is governed by a continuous-time Markov chain. The jump rates from average and bad states to the good state are PM rates and the one from failure rate to good state is the CM rate. The objective is to optimize the production rate and maintenance (PM and CM) rates in order to minimize discounted total costs including inventory holding, backlog, and maintenance costs. Finding the optimal policy involves solving the corresponding Hamilton-Jacobi-Bellman equations, which often lack closed-form solution.

Iravani and Duenyas also consider an integrated maintenance and production control policy for a single-machine make-to-stock manufacturing system [31]. But instead of a continuous flow model, a semi-Markov decision process model is developed, which in addition allows the incorporation of stochastic demand and production processes. Again, the machine is assumed to have multiple operational states and one failure state, and to be deteriorating as random shocks take the system to a worse state. An optimal policy will determine at each decision epoch, whether to produce one more item, to stay idle, or perform maintenance in order to minimize the total average inventory and maintenance costs. They investigate the structure of the optimal policy through some numer-

ical examples, and it turns out the structure is extremely complex. Therefore, they propose and analyze a heuristic policy with simple structure, the so-called double-threshold, which performs very close to the optimal policy, according to their numerical study.

Other related work includes the papers by Sloan and Shanthikumar [48, 49], in which they address the problems of equipment maintenance scheduling and production scheduling for multiple-product, single-machine production systems and multiple-stage production systems, where they formulate the problems as average reward MDP problems.

Our work presented in the following sections is somewhat close to the paper by Iravani and Duenyas [31]. However, our effort has been placed on the investigation of structural properties of optimal policies other than on the study of heuristic sub-optimal policies. Some of results was presented in TECHCON 2003 [65].

## 4.2 The Model with Time-Dependent Failures

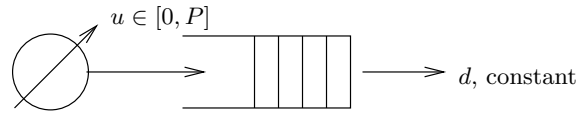


Figure 4.1: An unreliable production system with time-dependent failures

Consider an unreliable make-to-stock production system. The stock of completed goods is consumed by a constant demand  $d$ . The production unit (machine) can produce at maximal rate  $P$  ( $P > d$ ). However, the machine experiences time-dependent random failures, and if the machine is failed, a CM has to be performed. In order to avoid machine failures, PM can be scheduled for the machine. Either CM or PM will take some random time to be finished. When machine is in PM or CM, it produces nothing. When machine is not in PM or CM, i.e., in working state, then it can produce amount up to  $P$ . Our problem is to find out an optimal policy to decide when to do PM, and if decide not to do PM, how much should be produced, under an appropriate objective function. In the following discussion, we will consider a discrete-time system model. One of the reasons for using a discrete-time model is that it allows us to study more general distributions other than exponential one, which is most often seen in continuous-time models.

The following cost structure will be imposed. Completed goods in stock will incur inventory cost, while backlogged demands are allowed but with higher cost. If the machine fails, then a CM has to be conducted at the cost of  $c_r$ . If a PM is decided to be conducted, then the cost of  $c_p$  will be incurred. The objective is to minimize the expected discounted (with discount factor  $\beta < 1$ ) cost over the

infinite horizon.

### 4.2.1 MDP Formulation

The corresponding optimal control problem can be formulated as a discounted MDP model. We start by giving the notations that will be used hereafter.

$\mathcal{S}_t$ : inventory level at the beginning of period  $t$ .

$\mathcal{X}_t$ : machine state at the beginning of period  $t$ .  $\mathcal{X}_t \in \{0, 1, 2\}$ , where  $\mathcal{X}_t = 1$  if the machine is up;  $\mathcal{X}_t = 0$ , if the machine is down for CM;  $\mathcal{X}_t = 2$ , if machine is down for PM.

$a_t$ : number of time periods that the machine has been in the current state  $\mathcal{X}_t$  at the beginning of period  $t$ . If  $\mathcal{X}_t = 1$ ,  $a_t$  is the age of the machine since the last PM or CM. If  $\mathcal{X}_t = 0$  or 2,  $a_t$  is the time that the machine has been in CM or PM, respectively.

$f_n$ : conditional failure probability at age  $n + 1$ , given the machine is up at age  $n$ , i.e.,  $f_n = \Pr(\mathcal{X}_{t+1} = 0 \mid \mathcal{X}_t = 1, a_t = n)$ . In other words,  $f_n$  is the failure rate of a machine of age  $n + 1$ .

$p_n$ : conditional PM completion probability at the beginning of period  $t + 1$ , given the machine has been in PM for  $n$  periods at the beginning of period  $t$ , i.e.,  $p_n = \Pr(\mathcal{X}_{t+1} = 1 \mid \mathcal{X}_t = 2, a_t = n)$ .

$r_n$ : conditional CM completion probability at the beginning of period  $t + 1$ , given the machine has been in CM for  $n$  periods at the beginning of period  $t$ , i.e.,  $r_n = \Pr(\mathcal{X}_{t+1} = 1 \mid \mathcal{X}_t = 0, a_t = n)$ .

$d$ : incoming demand in each period, assumed constant.

$c_r$ : cost for performing a CM on the machine.

$c_p$ : cost for performing a PM on the machine.

$c^+$ : unit inventory cost/per period when inventory level is positive.

$c^-$ : unit inventory cost/per period when inventory level is negative.

$\tau_r$ : time for CM, which is a random variable.

$\tau_p$ : time for PM, which is also a random variable.

The system state at the beginning of period  $t$  is denoted by  $(\mathcal{S}_t, \mathcal{X}_t, a_t)$ . When the machine is in state  $\mathcal{X}_t = 1$ , the available control  $u_t$  is either to do PM or to produce amount  $u \in \{0, 1, \dots, P\}$ . There is no admissible control when the machine is in state  $\mathcal{X}_t = 0$  or 2.

Let  $g(\mathcal{S}_t)$  be the one-period inventory cost function given by

$$g(\mathcal{S}_t) = c^+ \cdot |\mathcal{S}_t| \cdot 1(\mathcal{S}_t \geq 0) + c^- \cdot |\mathcal{S}_t| \cdot 1(\mathcal{S}_t < 0), \quad (4.1)$$

and let  $h(\mathcal{X}_{t-}, \mathcal{X}_t, u_t)$  be the one-period maintenance/repair cost function given by

$$h(\mathcal{X}_{t-}, \mathcal{X}_t, u_t) = \begin{cases} c_r & \text{if } \mathcal{X}_{t-} = 1, \mathcal{X}_t = 0; \\ c_p & \text{if } \mathcal{X}_t = 1, u_t = PM; \\ 0 & \text{otherwise.} \end{cases} \quad (4.2)$$

Then the total discounted inventory/maintenance/repair costs of a stationary policy  $\mu$  over the infinite horizon when the system starts in state  $(s, \alpha, n)$  is given

by

$$J_\mu(s, \alpha, n) = E \left[ \sum_{t=0}^{\infty} \beta^t (g(\mathcal{S}_t) + h(\mathcal{X}_{t-}, \mathcal{X}_t, u_t)) \mid \mathcal{S}_0 = s, \mathcal{X}_0 = \alpha, a_0 = n \right], \quad (4.3)$$

(assuming  $\mathcal{X}_{0-} = \mathcal{X}_0$ ).

The objective is to find an optimal joint PM and production policy  $\mu^*$  to minimize the cost  $J_\mu$ . We denote the corresponding optimal value function by

$$J(s, \alpha, n) = \min_{\mu} J_\mu(s, \alpha, n). \quad (4.4)$$

Based on the system dynamics, we have the following dynamic programming optimality equations:

$$\begin{aligned} J(s, 0, n) &= g(s) + \beta \cdot r_n \cdot J(s - d, 1, 0) \\ &\quad + \beta \cdot (1 - r_n) \cdot J(s - d, 0, n + 1); \end{aligned} \quad (4.5)$$

$$\begin{aligned} J(s, 2, n) &= g(s) + \beta \cdot p_n \cdot J(s - d, 1, 0) \\ &\quad + \beta \cdot (1 - p_n) \cdot J(s - d, 2, n + 1); \end{aligned} \quad (4.6)$$

$$J(s, 1, n) = \min\{Q^{PM}(s, 1, n), \min_{0 \leq u \leq P} Q^u(s, 1, n)\}; \quad (4.7)$$

where

$$\begin{aligned} Q^{PM}(s, 1, n) &= c_p + g(s) + \beta \cdot p_0 \cdot J(s - d, 1, 0) \\ &\quad + \beta \cdot (1 - p_0) \cdot J(s - d, 2, 1); \end{aligned} \quad (4.8)$$

$$\begin{aligned} Q^u(s, 1, n) &= g(s) + \beta \cdot f_n \cdot (c_r + J(s + u - d, 0, 0)) \\ &\quad + \beta \cdot (1 - f_n) \cdot J(s + u - d, 1, n + 1). \end{aligned} \quad (4.9)$$

Equation (4.5) states that when the machine has been under CM for  $n$  periods, it will finish by the next period with probability  $r_n$ , and the inventory level will decrease by  $d$  in each period. Similarly, equation (4.6) states that when

the machine has been under PM for  $n$  periods, it will finish by the next period with probability  $p_n$ , and the inventory will be depleted by  $d$  in each period. In equation (4.7),  $Q^{PM}(s, 1, n)$  is the Q-function corresponding to the policy that chooses to do PM at the state  $(s, 1, n)$  and then follows an optimal policy.  $Q^u(s, 1, n)$  is the Q-function corresponding to the policy that chooses not to do PM but to produce at the rate of  $u$  at the state of  $(s, 1, n)$ , and then follows an optimal policy thereafter.

## 4.2.2 Structural Properties of Optimal Policies

In this section, we show some structural properties that the optimal policy satisfies.

**Lemma 4.1.**  $J(s, \cdot, \cdot)$  is decreasing function in  $s$ , for  $s \leq 0$ .

*Proof.* We proceed by value iteration. Let  $J_k(s, \cdot, \cdot)$  be the approximated cost function at the step  $k$ . Assume  $J_k(s, \cdot, \cdot)$  is decreasing function in  $s$ , for  $s \leq 0$ . If we can show at step  $k + 1$ ,  $J_{k+1}(s, \cdot, \cdot)$  is also decreasing in  $s$ , for  $s \leq 0$ . Then  $J(s, \cdot, \cdot)$  is decreasing in  $s$ , for  $s \leq 0$ , since  $J(s, \cdot, \cdot) = \lim_{k \rightarrow \infty} J_k(s, \cdot, \cdot)$ .

(1). At step 0, let  $J_0(s, \cdot, \cdot) = 0$ .

(2). Assume  $J_k(s, \cdot, \cdot)$  is decreasing in  $s$ , for  $s \leq 0$ . Since  $J_{k+1}(s, 0, n) = g(s) + \beta \cdot r_n \cdot J_k(s - d, 1, 0) + \beta \cdot (1 - r_n) \cdot J_k(s - d, 0, n + 1)$ , and  $g(s)$  is decreasing in  $s \leq 0$ , it is obvious that  $J_{k+1}(s, 0, n)$  is decreasing in  $s$ , for  $s \leq 0$ . Similarly,  $J_{k+1}(s, 2, n)$  is decreasing in  $s$ , for  $s \leq 0$ .

Now we show  $J_{k+1}(s, 1, n)$  is also decreasing in  $s \leq 0$ .

$$J_{k+1}(s, 1, n) = \min \{ Q_{k+1}^{PM}(s, 1, n), \min Q_{k+1}^u(s, 1, n) \},$$



where  $Q_{k+1}^{PM}(s, 1, n) = c_p + J_{k+1}(s, 2, 0)$ , so  $Q_{k+1}^{PM}(s, 1, n)$  is decreasing in  $s$ , for  $s \leq 0$ . Let

$$\begin{aligned} \overline{Q}_{k+1}^{PM}(s, 1, n) &= \min_{0 \leq u \leq P} Q_{k+1}^u(s, 1, n) \\ &= g(s) + \beta \cdot f_n \cdot c_r + \min_u (\beta \cdot f_n \cdot J_k(s + u - d, 0, 0) + \\ &\quad \beta \cdot (1 - f_n) \cdot J_k(s + u - d, 1, n + 1)), \end{aligned}$$

denote  $W_k(s) := \beta \cdot f_n \cdot J_k(s, 0, 0) + \beta \cdot (1 - f_n) \cdot J_k(s, 1, n + 1)$ , so  $W_k(s)$  is decreasing in  $s$ , for  $s \leq 0$ . Thus,

$$\begin{aligned} \overline{Q}_{k+1}^{PM}(s, 1, n) &= g(s) + \beta \cdot f_n \cdot c_r + \min_{u=0, \dots, P} W_k(s + u - d) \\ &= g(s) + \beta \cdot f_n \cdot c_r + \begin{cases} W_k(s + P - d) & \text{if } s \leq d - P, \\ \min_{x=0, \dots, s+P-d} W_k(x) & \text{otherwise.} \end{cases} \end{aligned}$$

Similarly, for  $(s - 1, 1, n)$ , we have

$$\overline{Q}_{k+1}^{PM}(s - 1, 1, n) = g(s - 1) + \beta \cdot f_n \cdot c_r + \begin{cases} W_k(s + P - d - 1) & \text{if } s \leq d - P, \\ \min_{x=0, \dots, s+P-d-1} W_k(x) & \text{otherwise.} \end{cases}$$

Therefore,  $\overline{Q}_{k+1}^{PM}(s - 1, 1, n) > \overline{Q}_{k+1}^{PM}(s, 1, n)$ , for  $s \leq 0$ . It follows that  $J_{k+1}(s, 1, n)$  is decreasing in  $s$ , for  $s \leq 0, \forall n$ .  $\square$

The lemma yields the following proposition immediately.

**Proposition 4.1.** *If  $s < 0$ , and the optimal action is not to do PM in state  $(s, 1, n)$ , then it is optimal to make the inventory non-negative as quickly as possible.*

We next show there is a control-limit policy with respect to machine's age to do PM. Observe

$$Q^{PM}(s, 1, n) = c_p + J(s, 2, 0),$$

is a constant with respect to the machine's age  $n$ . Recall

$$J(s, 1, a) = \min \left\{ Q^{PM}(s, 1, n), \min_u Q^u(s, 1, n) \right\}.$$

If we can show  $\min_u Q^u(s, 1, n)$  is increasing in  $n$ , then it follows that the optimal policy is a “control-limit” policy, as stated in the following lemma.

**Lemma 4.2.** *If  $Q^u(s, 1, n)$  is increasing in  $n$ , then the optimal policy is of “control-limit” type, such that if in state  $(s, 1, n)$ , it is optimal to do PM, then it is also optimal to do PM in state  $(s, 1, n + 1)$ .*

*Proof.* From the Bellman equation, we have

$$J(s, 1, a) = \min \left\{ Q^{PM}(s, 1, n), \min_u Q^u(s, 1, n) \right\},$$

where  $Q^{PM}(s, 1, n) = c_p + J(s, 2, 0)$ , is a constant with respect to  $n$ . If  $Q^u(s, 1, n)$  is increasing in  $n$ , so is  $\min_u Q^u(s, 1, n)$ . Therefore, at some age  $n^*$ , for  $n \geq n^*$ ,  $Q^{PM}(s, 1, n) \leq \min_u Q^u(s, 1, n)$ , and for  $n < n^*$ ,  $Q^{PM}(s, 1, n) > \min_u Q^u(s, 1, n)$ , i.e., for  $n \geq n^*$ ,  $\mu^*(s, 1, n) = PM$ , and for  $n < n^*$ ,  $\mu^*(s, 1, n) \neq PM$ .  $\square$

Let  $\tau_r$  be time for CM, with distribution  $\{l_n\}, n = 1, 2, \dots$ , where  $l_n = \Pr(\tau_r = n)$ . Recall  $\{r_n\}, n = 0, 1, \dots$ , the conditional CM completion probability, is actually the failure rate of  $\tau_r$ , i.e.,  $r_n = \frac{l_{n+1}}{\bar{L}_{n+1}}$ , where  $\bar{L}_{n+1} = 1 - \sum_{i=1}^n l_i$ , the tail distribution of  $\tau_r$ . Thus, we have following relations between  $\{r_n\}$  and  $\{l_n\}$ :

$$\begin{aligned} r_0 &= l_1/1, & \Leftrightarrow & \quad l_1 = r_0, \\ r_1 &= l_2/(1 - l_1), & \Leftrightarrow & \quad l_2 = r_1 \cdot (1 - r_0), \\ r_2 &= l_3/(1 - l_1 - l_2), & \Leftrightarrow & \quad l_3 = r_2 \cdot (1 - r_0) \cdot (1 - r_1), \\ & & \vdots & \Leftrightarrow \quad \vdots \end{aligned}$$

So

$$l_k = r_{k-1} \cdot \prod_{i=0}^{k-2} (1 - r_i), \text{ for } k \geq 1,$$

$$\bar{L}_n = 1 - \sum_{i=1}^{n-1} l_i = \prod_{i=0}^{n-1} (1 - r_i).$$

We can rewrite  $J(s, 0, 0)$  as follows:

$$\begin{aligned} J(s, 0, 0) &= g(s) + \beta \cdot r_0 \cdot J(s - d, 1, 0) + \beta \cdot (1 - r_0) \cdot J(s - d, 0, 1), \\ J(s - d, 0, 1) &= g(s - d) + \beta \cdot r_1 \cdot J(s - 2d, 1, 0) + \beta \cdot (1 - r_1) \cdot J(s - 2d, 0, 2), \\ &\vdots \end{aligned}$$

It follows that

$$\begin{aligned} J(s, 0, 0) &= g(s) + \sum_{k=1}^{\infty} \beta^k \prod_{i=0}^{k-1} (1 - r_i) g(s - kd) \\ &\quad + \sum_{k=1}^{\infty} \beta^k r_{k-1} \prod_{i=0}^{k-2} (1 - r_i) J(s - kd, 1, 0) \text{ (let } r_{-1} = 0) \\ &= \sum_{k=0}^{\infty} \beta^k \bar{L}_k g(s - kd) + \sum_{k=1}^{\infty} \beta^k l_k J(s - kd, 1, 0) \\ &= \sum_{k=0}^{\infty} \beta^k \bar{L}_k g(s - kd) + E[\beta^{\tau_r} J(s - \tau_r d, 1, 0)]. \end{aligned} \quad (4.10)$$

Likewise, let  $\tau_p$  be time for PM, with c.m.f.  $G$ , and we can rewrite  $J(s, 2, 0)$  as follows:

$$J(s, 2, 0) = \sum_{k=0}^{\infty} \beta^k \bar{G}_k g(s - kd) + E[\beta^{\tau_p} J(s - \tau_p d, 1, 0)]. \quad (4.11)$$

**Theorem 4.1.** *The optimal policy is of control-limit type with respect to the machine's age, if the following conditions are satisfied:*

- (1) machine's failure rate is IFR, i.e.,  $f_n$  is increasing in  $n$ ;
- (2)  $c_p \leq c_r$ ;
- (3)  $r_n = p_n$ , for  $n = 0, 1, \dots$ .

*Proof.* By condition (3), we know the time for CM  $\tau_r$  and the time for PM  $\tau_p$  have the same distribution, and thus  $J(s, 0, 0) = J(s, 2, 0)$  for all  $s$ , by equations (4.10) and (4.11).

Denote  $J(s, 1, F) = c_r + J(s, 0, 0)$ , so  $Q^u(s, 1, n)$  can be rewritten as

$$Q^u(s, 1, n) = g(s) + \beta f_n J(s + u - d, 1, F) + \beta(1 - f_n)J(s + u - d, 1, n + 1).$$

Moreover, we have

$$\begin{aligned} J(s, 1, n) &= \min \left\{ Q^{PM}(s, 1, n), \min_u Q^u(s, 1, n) \right\} \\ &\leq Q^{PM}(s, 1, n) \\ &= c_p + J(s, 2, 0) \\ &\leq c_r + J(s, 0, 0), \text{ by condition (2)} \\ &= J(s, 1, F). \end{aligned}$$

Now, we claim  $J(s, 1, n)$  is increasing in  $n$ . To prove the monotonicity of  $J(s, 1, n)$ , we proceed by value iteration.

(1) Obviously, it holds for  $J_0(s, 1, n)$  at the step 0;

(2) Assume at the step  $k$ ,  $J_k(s, 1, n)$  is increasing in  $n$ . At the step  $k + 1$ , we have

$$\begin{aligned} &Q_{k+1}^u(s, 1, n + 1) - Q_{k+1}^u(s, 1, n) \\ &= \beta[(f_{n+1} - f_n)J_k(s + u - d, 1, F) + (1 - f_{n+1})J_k(s + u - d, 1, n + 2) \\ &\quad - (1 - f_n)J_k(s + u - d, 1, n + 1)] \\ &\geq \beta[(f_{n+1} - f_n)J_k(s + u - d, 1, F) + (1 - f_{n+1})J_k(s + u - d, 1, n + 1) \\ &\quad - (1 - f_n)J_k(s + u - d, 1, n + 1)] \\ &= \beta[(f_{n+1} - f_n)(J_k(s + u - d, 1, F) - J_k(s + u - d, 1, n + 1))] \\ &\geq 0, \end{aligned}$$

i.e.,

$$Q_{k+1}^u(s, 1, n+1) \geq Q_{k+1}^u(s, 1, n). \quad (4.12)$$

By value iteration, let  $k \rightarrow \infty$ , we have

$$Q^u(s, 1, n+1) \geq Q^u(s, 1, n).$$

By Lemma 4.2, the optimal policy is of control-limit type.  $\square$

The condition of  $\tau_r$  and  $\tau_p$  having the same distribution may not be satisfied in practice. If we consider a simpler case where there is no PM/repair costs, i.e.,  $c_p = c_r = 0$ , then we can show the optimal cost function still retains desirable structural properties under more relaxed conditions, as stated in the following theorem.

**Theorem 4.2.** *If the following conditions are satisfied:*

- (1)  $f_n$  is increasing in  $n$ ;
- (2)  $c_p = c_r = 0$ ;
- (3)  $p_n \geq r_n$ , for all  $n$ ;
- (4)  $p_n$  is non-increasing in  $n$ ;
- (5)  $r_n$  is non-increasing in  $n$ ;

*then the following relations hold:*

- (i)  $J(s, 2, n) \leq J(s, 2, n+1)$ ;
- (ii)  $J(s, 0, n) \leq J(s, 0, n+1)$ ;
- (iii)  $J(s, 2, n) \leq J(s, 0, n)$ ;
- (iv)  $J(s, 1, n) \leq J(s, 1, n+1)$ .

*Proof.* The proof uses value iteration, and proceeds by induction. We prove (i) first. (ii) and (iii) can be proved similarly using the same procedure. Assume  $J_k(s, 2, n)$  satisfies the relationship (i). Then at  $k + 1$ ,

$$\begin{aligned}
J_{k+1}(s, 2, n) &= g(s) + \beta p_n J_k(s - d, 1, 0) + \beta(1 - p_n) J_k(s - d, 2, n + 1) \\
&\leq g(s) + \beta p_n J_k(s - d, 1, 0) + \beta(1 - p_n) J_k(s - d, 2, n + 2) \\
&= g(s) + \beta J_k(s - d, 2, n + 2) \\
&\quad - \beta p_n (J_k(s - d, 2, n + 2) - J_k(s - d, 1, 0)) \\
&\leq g(s) + \beta J_k(s - d, 2, n + 2) \\
&\quad - \beta p_{n+1} (J_k(s - d, 2, n + 2) - J_k(s - d, 1, 0)) \\
&= J_{k+1}(s, 2, n + 1).
\end{aligned}$$

The first inequality is by the induction assumption at the step  $k$ , and the second inequality follows from the condition  $p_n \geq p_{n+1}$ , and the fact that

$$J_k(s - d, 1, 0) \leq J_k(s - d, 2, 0) \leq J_k(s - d, 2, n), \text{ for all } n.$$

Recall under the condition (2),

$$J_{k+1}(s, 1, n) = \min \left\{ J_{k+1}(s, 2, 0); \min_u \{ Q_{k+1}^u(s, 1, n) \} \right\},$$

where

$$\begin{aligned}
Q_{k+1}^u(s, 1, n) &= g(s) + \beta f_n J_k(s + u - d, 0, 0) + \beta(1 - f_n) J_k(s + u - d, 1, n + 1) \\
&\leq g(s) + \beta f_n J_k(s + u - d, 0, 0) + \beta(1 - f_n) J_k(s + u - d, 1, n + 2) \\
&= g(s) + \beta J_k(s + u - d, 1, n + 2) \\
&\quad + \beta f_n (J_k(s + u - d, 0, 0) - J_k(s + u - d, 1, n + 2)) \\
&\leq g(s) + \beta J_k(s + u - d, 1, n + 2) \\
&\quad + \beta f_{n+1} (J_k(s + u - d, 0, 0) - J_k(s + u - d, 1, n + 2)) \\
&= Q_{k+1}^u(s, 1, n + 1).
\end{aligned}$$

It follows immediately that  $J^{k+1}(s, 1, n) \leq J^{k+1}(s, 1, n + 1)$ . By taking the appropriate limits, we have the relations of (i)-(iv).  $\square$

**Corollary 4.1.** *The optimal policy is of control-limit type with respect to the machine's age.*

*Proof.* It follows straightforward from the monotonic structure of the optimal cost function  $J(s, 1, n)$ .  $\square$

The following theorem states an intuitive property of the optimal policy  $\mu^*$  when the system has a high inventory level.

**Theorem 4.3.** *There exists  $s^*$  such that  $\forall s > s^*$ ,  $\mu^*(s, 1, n) = 0$  or  $PM$ , for all  $n$ .*

*Proof.* The optimal cost function  $J(s, 1, n)$  can be broken down into two parts

$$J(s, 1, n) = G(s, 1, n) + H(s, 1, n) \tag{4.13}$$

where  $G$  is the total discounted inventory holding / backlogging cost and  $H$  is the total discounted PM/CM cost. Now consider a starting inventory level  $s = k \cdot d + \delta$  with  $k = 0, 1, 2, \dots$ , and  $0 \leq \delta < d$ . Obviously, the total discounted inventory holding cost satisfies

$$G(s, 1, n) \geq \sum_{t=0}^k c^+(s - t \cdot d) \beta^t \equiv A(s). \quad (4.14)$$

Let  $\mu_1$  be any policy such that  $\mu_1(s, 1, n) \geq 1$ . Thus the total discounted inventory holding cost starting at  $(s, 1, n)$  under  $\mu_1$  satisfies

$$G^{\mu_1}(s, 1, n) \geq c^+ s + \sum_{t=1}^k c^+(s + 1 - t \cdot d) \beta^t \quad (4.15)$$

$$= A(s) + c^+ \frac{\beta - \beta^{k+1}}{1 - \beta}. \quad (4.16)$$

Next, consider the cost related to PM/CM only. Let  $h(s, 1, n)$  be the minimal total discounted PM/CM cost. This will be achieved by the optimal PM policy. (Actually, because the machine deterioration process is not affected by the inventory process, the problem can be reduced to the classic problem of finding the optimal PM policy for a Markov Chain; see, for example [18].) Denote the corresponding optimal PM policy by a function  $v(s, 1, n)$ .

Obviously, for all policies  $\mu$ , the total discounted PM/CM cost  $H^\mu$  satisfies:

$$H^\mu(s, 1, n) \geq h(s, 1, n). \quad (4.17)$$

Thus, from (4.14) and (4.17),

$$J(s, 1, n) \geq A(s) + h(s, 1, n). \quad (4.18)$$

From (4.16) and (4.17)

$$J^{\mu_1}(s, 1, n) \geq A(s) + c^+ \frac{\beta - \beta^{k+1}}{1 - \beta} + h(s, 1, n). \quad (4.19)$$



We now construct a simple joint PM/production policy  $\mu_0$  from the optimal PM policy  $v$ , such that

$$\mu_0(s, 1, n) = \begin{cases} 0, & \text{if } v(s, 1, n) \neq PM, \\ PM, & \text{if } v(s, 1, n) = PM. \end{cases}$$

It is obvious that  $H^{\mu_0}(s, 1, n) = h(s, 1, n)$ , and

$$\begin{aligned} G^{\mu_0}(s, 1, n) &= \sum_{t=0}^{\infty} g(s - td)\beta^t \\ &= \sum_{t=0}^k c^+(s - td)\beta^t + \sum_{t=k+1}^{\infty} c^-(td - s)\beta^t \\ &= A(s) + \beta^k \sum_{t=1}^{\infty} c^-(td - \delta)\beta^t \\ &\leq A(s) + \beta^k \sum_{t=1}^{\infty} c^- \cdot td \cdot \beta^t \\ &= A(s) + \frac{\beta^{k+1}c^-d}{(1-\beta)^2}. \end{aligned} \tag{4.20}$$

Therefore,

$$J^{\mu_0}(s, 1, n) \leq A(s) + h(s, 1, n) + \frac{\beta^{k+1}c^-d}{(1-\beta)^2}. \tag{4.21}$$

From (4.18) and (4.21),

$$A(s) + h(s, 1, n) \leq J(s, 1, n) \leq A(s) + h(s, 1, n) + \frac{\beta^{k+1}c^-d}{(1-\beta)^2}. \tag{4.22}$$

It then follows from (4.19) that  $\mu(s, 1, n) \in \{0, PM\}$  if

$$c^+ \frac{\beta - \beta^{k+1}}{1 - \beta} > c^- \frac{\beta^{k+1}d}{(1 - \beta)^2}, \tag{4.23}$$

or equivalently if

$$k > \left\lceil \frac{\ln \frac{c^+}{c^+ + c^-d/(1-\beta)}}{\ln \beta} \right\rceil = k^*, \tag{4.24}$$

where  $\lceil x \rceil$  is the minimal integer that is greater or equal to  $x$ . So,  $s^* = k^*d$ .  $\square$

## Example: Optimal joint PM and production control policy for unreliable production system with time-dependent failures

As the Weibull distribution is perhaps the most popular parametric family of failure distributions, we assume the machine's lifetime is Weibull distributed, with pdf given by

$$f(t) = \frac{\alpha}{\eta} \left(\frac{t}{\eta}\right)^{\alpha-1} e^{-\left(\frac{t}{\eta}\right)^\alpha}.$$

Let MAXLIFE be the maximal lifetime of the machine; then the sequence of failure rates  $\{f_n, n = 0, 1, \dots\}$  is obtained by discretizing the Weibull distribution. The time for PM is uniformly distributed in  $[0, \text{MAXTM}]$ ; the time for CM is also uniformly distributed, in  $[0, \text{MAXTR}]$ . Let

---

$\alpha = 4,$	$\eta = 5,$
$\beta = 0.95,$	$\text{MAXLIFE} = 100,$
$\text{MAXTM} = 3,$	$\text{MAXTR} = 6,$
$d = 1,$	$P = 3,$
$c_p = 50,$	$c_r = 2 * c_p,$
$c^+ = 1,$	$c^- = 10.$

---

The optimal policy can be obtained numerically by solving the corresponding DP equations. Figure 4.2 shows the optimal policy.

It is interesting to compare the performances of the jointly optimized policy and the conventional independently optimized policies. Traditionally, the PM and production policies are optimized independently. First, the production policy is optimized assuming the machine is reliable, and then the PM policy is

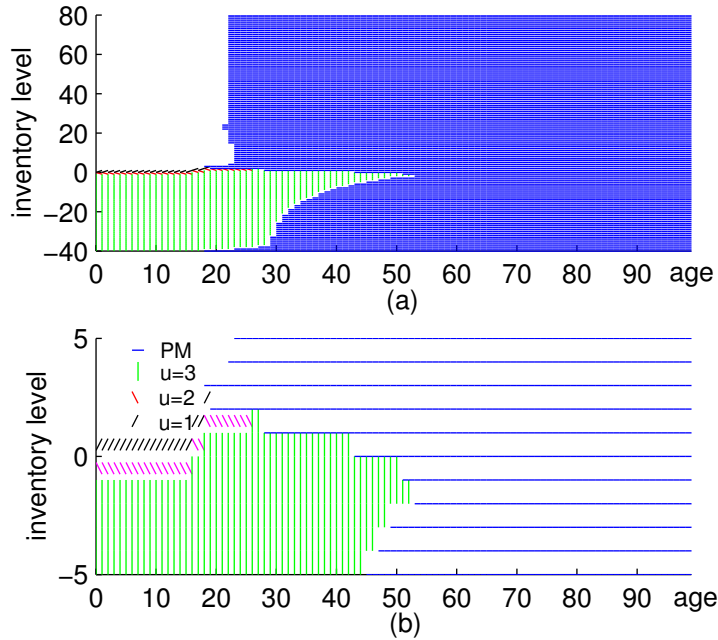


Figure 4.2: Joint optimal policy. (a) The optimal actions on the whole state space. (b) The optimal actions on the space  $[-5,5] \times [0, \text{MAXLIFE}]$ .

optimized without consideration of systems inventory level. For this example, due to the constant demand, the independently optimized production policy  $\mu_{ind}$  is very simple, and is given by

$$\mu_{ind}(s) = \begin{cases} 0 & \text{if } s \geq d, \\ P & \text{if } s \leq (d - P), \\ d - s & \text{otherwise.} \end{cases}$$

The independently optimized PM policy also has a simple structure. It is well known it is a control-limit policy under some appropriate conditions, usually, IFR; see [18]. The control limit can be obtained by solving the DP equations. For this example, the control limit is 22.

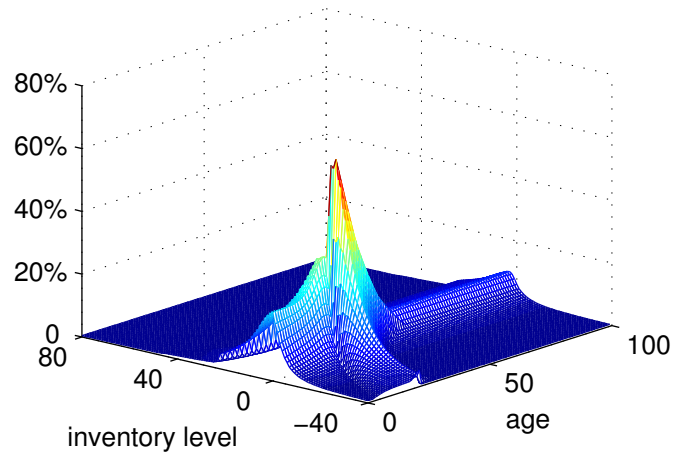


Figure 4.3: The comparative difference of cost functions under the joint optimal policy and independently optimized policy.

Let  $J^*$  be the cost function of the joint optimal policy, and  $J_{ind}$  the cost function of the independently optimized policy. Figure 4.3 shows the comparative difference of cost functions,  $(J_{ind} - J^*)/J^*$ . The maximal difference is about 65%, which shows the classic independently optimized PM and production policy has very poor performance for some system states.

### 4.3 The Model with Operation-Dependent Failures

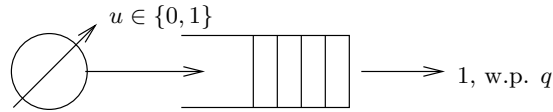


Figure 4.4: An unreliable production system with operation-dependent failures

In this section, we consider a machine experiencing operation-dependent failures, i.e., the machine deteriorates only when it is in production. It does not deteriorate when it stays idle. This is different from time-dependent failures, where the machine deteriorates even if it is idle. In the context of semiconductor manufacturing, the operation-dependent failures are related to deteriorations due to wafer processing and can be corrected preventively by wafer-count-based or operation-time based PMs.

It is assumed that the machine failures can occur only at the completion of a job. We further assume the machine can produce at rate either 1 or 0. The reason behind this assumption is for the fact of operation-dependent failures. If the machine can produce more than one unit in one period, then failures could be possibly occurred in the middle of a time-unit (when a job is finished). We also assume the incoming demand is a random variable. For simplicity, we assume that a unit demand arrives in each period with probability  $q$ .

### 4.3.1 MDP Formulation

We use the same notation as in the previous model, with some modifications in the following.

$u_t$ : control applied at the beginning of period  $t$ . It is admissible only when  $\mathcal{X}_t = 1$ , i.e., machine is up. The admissible values of  $u_t$  are  $PM$ , 0, or 1, i.e., doing PM, idle, or producing 1 item, respectively.

$f_n$ : conditional failure probability at age  $n + 1$  given it is up at age  $n$ , i.e.,  $f_n = \Pr(\mathcal{X}_{t+1} = 0 \mid \mathcal{X}_t = 1, a_t = n, u_t = 1)$ . Note:  $\Pr(\mathcal{X}_{t+1} = 0 \mid \mathcal{X}_t = 1, a_t = n, u_t = 0) = 0$ , i.e., the state of machine at the next period will not change if the machine is up and  $u_t = 0$ , due to the assumption of operation dependent failure.

$q$ : probability of unit demand arriving in each period.

Our objective is to find an optimal joint production and PM policy to minimize the total discounted inventory/PM/repair costs over the infinite horizon, as stated below:

$$J(s, \alpha, n) = \min E \left[ \sum_{t=0}^{\infty} \beta^t (g(\mathcal{S}_t) + h(\mathcal{X}_{t-}, \mathcal{X}_t, u_t)) \mid \mathcal{S}_0 = s, \mathcal{X}_0 = \alpha, a_0 = n \right], \quad (4.25)$$

where  $(\mathcal{X}_{0-} = \mathcal{X}_0)$  and the one-period cost functions  $g$  and  $h$  are given by (4.1) and (4.2), respectively.

Based on the system dynamics, we have the following dynamic programming optimality equations:

$$J(s, 0, n) = g(s) + \beta \{q \cdot r_n \cdot J(s-1, 1, 0) + q(1-r_n) \cdot J(s-1, 0, n+1) + (1-q)r_n \cdot J(s, 1, 0) + (1-q)(1-r_n) \cdot J(s, 0, n+1)\}, \quad (4.26)$$

$$J(s, 2, n) = g(s) + \beta \{q \cdot p_n \cdot J(s-1, 1, 0) + q(1-p_n) \cdot J(s-1, 2, n+1) + (1-q)p_n \cdot J(s, 1, 0) + (1-q)(1-p_n) \cdot J(s, 2, n+1)\}, \quad (4.27)$$

$$J(s, 1, n) = \min \{Q^{PM}(s, 1, n), Q^0(s, 1, n), Q^1(s, 1, n), \} \quad (4.28)$$

where  $Q^{PM}(s, 1, n)$  is the Q-function of applying PM control,  $Q^0(s, 1, n)$  the Q-function of producing 0 unit, and  $Q^1(s, 1, n)$  the Q-function of producing 1 unit, i.e.,

$$Q^{PM}(s, 1, n) = c_p + J(s, 2, 0), \quad (4.29)$$

$$Q^0(s, 1, n) = g(s) + \beta \{q \cdot J(s-1, 1, n) + (1-q) \cdot J(s, 1, n)\}, \quad (4.30)$$

$$\begin{aligned} Q^1(s, 1, n) = & g(s) + \beta \{q \cdot f_n \cdot (c_r + J(s, 0, 0)) \\ & + q \cdot (1-f_n) \cdot J(s, 1, n+1) \\ & + (1-q) \cdot f_n \cdot (c_r + J(s+1, 0, 0)) \\ & + (1-q) \cdot (1-f_n) \cdot J(s+1, 1, n+1)\}. \end{aligned} \quad (4.31)$$

### 4.3.2 Structural Properties of Optimal Policies

The following lemma states a connection between the value iteration of DP and corresponding finite horizon MDP.

**Lemma 4.3.** *For a discounted infinite horizon MDP problem, if  $J_1(\cdot), J_2(\cdot), \dots$ , is the sequence generated by the following value iteration starting from  $J_0(\cdot) = 0$ ,*

$$J_k(x) = \min_{u \in U(i)} E [g(x, u) + \beta J_{k-1}(f(x, u, w))], k = 1, 2, \dots, \quad (4.32)$$

then,

$$J_k(x) = \min_{\pi^k} E \left( \sum_{t=0}^{k-1} \beta^t g(x_t, u_t) \mid x_0 = x \right), \quad (4.33)$$

where  $\pi^k = \{\mu_0, \mu_1, \dots, \mu_{k-1}\}$ , i.e.,  $J_k(x)$  is the optimal  $k$ -stage discounted cost when system is starting from the state  $x$ .

We now write the value iteration for our MDP problem as follows:

$$\begin{aligned} J_0(\cdot, \cdot, \cdot) &= 0, \\ J_{k+1}(s, 0, n) &= g(s) + \beta \{q \cdot r_n \cdot J_k(s-1, 1, 0) + q(1-r_n) \cdot J_k(s-1, 0, n+1) \\ &\quad + (1-q)r_n \cdot J_k(s, 1, 0) + (1-q)(1-r_n) \cdot J_k(s, 0, n+1)\}, \\ J_{k+1}(s, 2, n) &= g(s) + \beta \{q \cdot p_n \cdot J_k(s-1, 1, 0) + q(1-p_n) \cdot J_k(s-1, 2, n+1) \\ &\quad + (1-q)p_n \cdot J_k(s, 1, 0) + (1-q)(1-p_n) \cdot J_k(s, 2, n+1)\}, \\ J_{k+1}(s, 1, n) &= \min \{Q_{k+1}^{PM}(s, 1, n), Q_{k+1}^0(s, 1, n), Q_{k+1}^1(s, 1, n)\}, \\ Q_{k+1}^{PM}(s, 1, n) &= c_p + J_{k+1}(s, 2, 0), \\ Q_{k+1}^0(s, 1, n) &= g(s) + \beta \{q \cdot J_k(s-1, 1, n) + (1-q) \cdot J_k(s, 1, n)\}, \\ Q_{k+1}^1(s, 1, n) &= g(s) + \beta \{q \cdot f_n \cdot (c_r + J_k(s, 0, 0)) \\ &\quad + q \cdot (1-f_n) \cdot J_k(s, 1, n+1) \\ &\quad + (1-q) \cdot f_n \cdot (c_r + J_k(s+1, 0, 0)) \\ &\quad + (1-q) \cdot (1-f_n) \cdot J_k(s+1, 1, n+1)\}. \end{aligned}$$

The following conditions will be needed in our later analysis.

**Condition 4.1.**  $f_n$  is IFR, i.e., increasing with respect to  $n$ .

**Condition 4.2.**  $c_r \geq c_p$ .

**Condition 4.3.**  $\tau_r \geq_{st} \tau_p$ .



**Lemma 4.4.** *Under conditions 4.1,4.2 and 4.3,  $J_{k+1}(s, 0, 0) \geq J_{k+1}(s, 2, 0)$  for  $k=0, 1, \dots$*

*Proof.* By Lemma 4.3,

$$\begin{aligned} J_{k+1}(s, 0, 0) &= \min_{\pi^{k+1}} E \left( \sum_{t=0}^k \beta^t (g(\mathcal{S}_t) + h(\mathcal{X}_{t-}, \mathcal{X}_t, u_t)) \mid \mathcal{S}_0 = s, \mathcal{X}_0 = 0, a_0 = 0 \right), \\ J_{k+1}(s, 2, 0) &= \min_{\pi^{k+1}} E \left( \sum_{t=0}^k \beta^t (g(\mathcal{S}_t) + h(\mathcal{X}_{t-}, \mathcal{X}_t, u_t)) \mid \mathcal{S}_0 = s, \mathcal{X}_0 = 2, a_0 = 0 \right). \end{aligned}$$

Because  $\tau_r \geq_{st} \tau_p$ , by coupling, there exists a r.v.  $\tau_r^*$  that has the same distribution as  $\tau_r$  such that

$$\tau_r^* = \tau_p + \Delta\tau, \quad (4.34)$$

where  $\Delta\tau \geq 0$ , w.p.1. Moreover,  $\tau_r^*$  can be generated as follows,

$$\tau_r^* = F_{\tau_r}^{-1} (F_{\tau_p}(\tau_p)), \quad (4.35)$$

where  $F_{\tau_r}(\cdot)$  and  $F_{\tau_p}(\cdot)$  are the c.d.f of  $\tau_r$  and  $\tau_p$ , respectively, and  $F^{-1}$  is the inverse function of  $F$ . Note: In order for the inverse function  $F^{-1}$  to be well defined for discrete random variables, we define  $F^{-1}$  as follows:

$$F^{-1}(p) = \min \{x : F(x) \geq p\}. \quad (4.36)$$

We decompose the cost of  $J_{k+1}(s, 0, 0)$  and  $J_{k+1}(s, 2, 0)$ , respectively, into two parts: the partial cost incurred from the beginning to the point when the CM (or PM) is finished at time  $\tau_r$  (or  $\tau_p$ ), and the cost incurred thereafter. The main idea in the following steps is that assuming the machine starts from state  $(s, 2, 0)$  at time  $t_0$ , it will finish the PM at a random time  $t_0 + \tau_p$ . After the PM is finished, the machine is up and an admissible control could be applied. We consider a policy such that a control  $u_t = 0$  is applied first for a time duration of

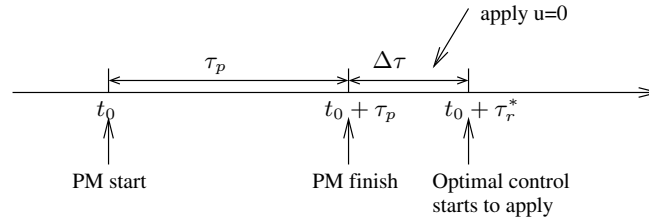


Figure 4.5: A specific policy – postpone optimal control until the time period  $t_0 + \tau_r^*$ .

$\Delta\tau$ , i.e., forcing the machine into idle between  $t_0 + \tau_p$  and  $t_0 + \tau_r^*$ , and then an optimal control is followed thereafter; see the illustration of Figure 4.5. It can be seen that this special policy will incur the same cost as the machine starting from state  $(s, 0, 0)$ . Therefore, the optimal cost-to-go from state  $(s, 2, 0)$  should be less than the optimal cost-to-go from state  $(s, 0, 0)$ .

Note also that from time  $t_0$  to  $t_0 + \tau_r - 1$  and  $t_0 + \tau_p - 1$ , the system is in CM and PM, respectively, so during those periods, the cost is only from inventory, and there is no control at all.

$$\begin{aligned}
J_{k+1}(s, 0, 0) &= \min_{\pi^{k+1}} E \left( E \left( \sum_{t=0}^{\tau_r-1} \beta^t g(\mathcal{S}_t) 1(\tau_r \leq k) \right. \right. \\
&\quad \left. \left. + \sum_{t=\tau_r}^k \beta^t (g(\cdot) + h(\cdot, \cdot)) 1(\tau_r \leq k) \right. \right. \\
&\quad \left. \left. + \sum_{t=0}^k \beta^t g(\mathcal{S}_t) 1(\tau_r > k) \mid (s, 0, 0), \tau_r \right) \mid (s, 0, 0) \right) \\
&= E \left( E \left( \sum_{t=0}^{\tau_r-1} \beta^t g(\mathcal{S}_t) 1(\tau_r \leq k) \mid (s, 0, 0), \tau_r \right) \mid (s, 0, 0) \right) \\
&\quad + E \left( E \left( \sum_{t=0}^k \beta^t g(\mathcal{S}_t) 1(\tau_r > k) \mid (s, 0, 0), \tau_r \right) \mid (s, 0, 0) \right) \\
&\quad + \min_{\pi^{k+1}} E \left( E \left( \sum_{t=\tau_r}^k \beta^t (g + h) 1(\tau_r \leq k) \mid (s, 0, 0), \tau_r \right) \mid (s, 0, 0) \right).
\end{aligned} \tag{4.37}$$

Similarly, for  $J_{k+1}(s, 2, 0)$ , we have

$$\begin{aligned}
J_{k+1}(s, 2, 0) &= E \left( E \left( \sum_{t=0}^{\tau_p-1} \beta^t g(\mathcal{S}_t) 1(\tau_p \leq k) \mid (s, 2, 0), \tau_p \right) \mid (s, 2, 0) \right) \\
&\quad + E \left( E \left( \sum_{t=0}^k \beta^t g(\mathcal{S}_t) 1(\tau_p > k) \mid (s, 2, 0), \tau_p \right) \mid (s, 2, 0) \right) \\
&\quad + \min_{\pi^{k+1}} E \left( E \left( \sum_{t=\tau_p}^k \beta^t (g + h) 1(\tau_p \leq k) \mid (s, 2, 0), \tau_p \right) \mid (s, 2, 0) \right).
\end{aligned} \tag{4.38}$$

By conditioning on  $\tau_r^*$ , the last term on the RHS of (4.38) can be written as:

$$\begin{aligned}
& \min_{\pi^{k+1}} E(E(\sum_{t=\tau_p}^k \beta^t (g+h) 1(\tau_p \leq k) \mid (s, 2, 0), \tau_p) \mid (s, 2, 0)) \\
= & \min_{\pi^{k+1}} E(E(E(\sum_{t=\tau_p}^{\tau_r^*-1} \beta^t (g+h) 1(\tau_p \leq k) 1(\tau_r^* \leq k) + \sum_{t=\tau_r^*}^k \beta^t (g+h) \\
& 1(\tau_p \leq k) 1(\tau_r^* \leq k) + \sum_{t=\tau_p}^k \beta^t (g+h) 1(\tau_p \leq k) 1(\tau_r^* > k) \\
& \mid (s, 2, 0), \tau_p, \tau_r^*) \mid (s, 2, 0), \tau_p) \mid (s, 2, 0)) \tag{4.39}
\end{aligned}$$

Note that at the time of  $\tau_p$ , the PM is finished and the system state changes to  $(s', 1, 0)$ . One policy after the time of  $\tau_p$  is that we can choose the action of producing 0 units, until the time of  $\tau_r^*$ . During these time periods, the machine's state  $\mathcal{X}$  will remain at 1, and the system cost is incurred only by the inventory. The cost of (4.39) should be less than the cost of applying this specific policy. Thus we have the following inequality:

$$\begin{aligned}
& \min_{\pi^{k+1}} E(E(\sum_{t=\tau_p}^k \beta^t (g+h) 1(\tau_p \leq k) \mid (s, 2, 0), \tau_p) \mid (s, 2, 0)) \\
\leq & E(E(E(\sum_{t=\tau_p}^{\tau_r^*-1} \beta^t g(\mathcal{S}_t) 1(\tau_r^* \leq k) \mid s, \tau_p, \tau_r^*) \mid s, \tau_p) \mid s) \\
& + E(E(E(\sum_{t=\tau_p}^k \beta^t g(\mathcal{S}_t) 1(\tau_p \leq k < \tau_r^*) \mid s, \tau_p, \tau_r^*) \mid s, \tau_p) \mid s) \\
& + \min_{\pi^{k+1}} E(E(E(\sum_{t=\tau_r^*}^k \beta^t (g+h) 1(\tau_r^* \leq k) \mid s, \tau_p, \tau_r^*) \mid s, \tau_p) \mid s). \tag{4.40}
\end{aligned}$$

For the first two terms on the RHS, the condition on the initial state  $(s, 2, 0)$  can be written compactly as  $s$ , because only the initial inventory level is relevant.

Also the conditioning on  $\tau_p$  in the third term can be dropped. So it becomes

$$\begin{aligned}
& \min_{\pi^{k+1}} E \left( E \left( \sum_{t=\tau_p}^k \beta^t (g+h) 1(\tau_p \leq k) \mid (s, 2, 0), \tau_p \mid (s, 2, 0) \right) \right) \\
& \leq E \left( E \left( E \left( \sum_{t=\tau_p}^{\tau_r^*-1} \beta^t g(\mathcal{S}_t) 1(\tau_r^* \leq k) \mid s, \tau_p, \tau_r^* \mid s, \tau_p \mid s \right) \right. \right. \\
& \quad \left. \left. + E \left( E \left( \sum_{t=\tau_p}^k \beta^t g(\mathcal{S}_t) 1(\tau_p \leq k < \tau_r^*) \mid s, \tau_p, \tau_r^* \mid s, \tau_p \mid s \right) \right) \right. \right. \\
& \quad \left. \left. + \min_{\pi^{k+1}} E \left( E \left( \sum_{t=\tau_r^*}^k \beta^t (g+h) 1(\tau_r^* \leq k) \mid (s, 2, 0), \tau_r^* \mid (s, 2, 0) \right) \right) \right). \tag{4.41}
\end{aligned}$$

Substituting this into the equation (4.38):

$$\begin{aligned}
& J_{k+1}(s, 2, 0) \\
& \leq E \left( E \left( \sum_{t=0}^{\tau_p-1} \beta^t g(\mathcal{S}_t) 1(\tau_p \leq k) \mid s, \tau_p \right) \mid s \right) \\
& \quad + E \left( E \left( \sum_{t=0}^k \beta^t g(\mathcal{S}_t) 1(\tau_p > k) \mid s, \tau_p \right) \mid s \right) \\
& \quad + E \left( E \left( E \left( \sum_{t=\tau_p}^{\tau_r^*-1} \beta^t g(\mathcal{S}_t) 1(\tau_r^* \leq k) \mid s, \tau_p, \tau_r^* \right) \mid s, \tau_p \right) \mid s \right) \\
& \quad + E \left( E \left( E \left( \sum_{t=\tau_p}^k \beta^t g(\mathcal{S}_t) 1(\tau_p \leq k < \tau_r^*) \mid s, \tau_p, \tau_r^* \right) \mid s, \tau_p \right) \mid s \right) \\
& \quad + \min_{\pi^{k+1}} E \left( E \left( \sum_{t=\tau_r^*}^k \beta^t (g+h) 1(\tau_r^* \leq k) \mid s, \tau_r^* \right) \mid s \right)
\end{aligned}$$

$$\begin{aligned}
&= E \left( E \left( E \left( \sum_{t=0}^{\tau_p-1} \beta^t g(\mathcal{S}_t) 1(\tau_p \leq k) 1(\tau_r^* \leq k) \mid s, \tau_p, \tau_r^* \right) \mid s, \tau_p \right) \mid s \right) \\
&+ E \left( E \left( E \left( \sum_{t=0}^{\tau_p-1} \beta^t g(\mathcal{S}_t) 1(\tau_p \leq k) 1(\tau_r^* > k) \mid s, \tau_p, \tau_r^* \right) \mid s, \tau_p \right) \mid s \right) \\
&+ E \left( E \left( \sum_{t=0}^k \beta^t g(\mathcal{S}_t) 1(\tau_p > k) \mid s, \tau_p \right) \mid s \right) \\
&+ E \left( E \left( E \left( \sum_{t=\tau_p}^{\tau_r^*-1} \beta^t g(\mathcal{S}_t) 1(\tau_r^* \leq k) \mid s, \tau_p, \tau_r^* \right) \mid s, \tau_p \right) \mid s \right) \\
&+ E \left( E \left( E \left( \sum_{t=\tau_p}^k \beta^t g(\mathcal{S}_t) 1(\tau_p \leq k < \tau_r^*) \mid s, \tau_p, \tau_r^* \right) \mid s, \tau_p \right) \mid s \right) \\
&+ \min_{\pi^{k+1}} E \left( E \left( \sum_{t=\tau_r^*}^k \beta^t (g+h) 1(\tau_r^* \leq k) \mid s, \tau_r^* \right) \mid s \right).
\end{aligned}$$

Combining the 1st and 4th terms, as well as the 2nd and 5th terms on the RHS,

i.e.,

$$\begin{aligned}
&J_{k+1}(s, 2, 0) \\
&\leq E \left( E \left( \sum_{t=0}^{\tau_r^*-1} \beta^t g(\mathcal{S}_t) 1(\tau_r^* \leq k) \mid s, \tau_r^* \right) \mid s \right) \\
&+ E \left( E \left( \sum_{t=0}^k \beta^t g(\mathcal{S}_t) 1(\tau_r^* > k) \mid s, \tau_r^* \right) \mid s \right) \\
&+ \min_{\pi^{k+1}} E \left( E \left( \sum_{t=\tau_r^*}^k \beta^t (g(\cdot) + h(\cdot, \cdot)) 1(\tau_r^* \leq k) \mid s, \tau_r^* \right) \mid s \right) \\
&= J_{k+1}(s, 0, 0). \tag{4.42}
\end{aligned}$$

□

Now we are ready to prove the following main theorem.

**Theorem 4.4.** *Under Conditions 4.1, 4.2, and 4.3,  $J(s, 1, n)$  is increasing with respect to  $n$ .*

*Proof.* We proceed by value iteration.

(1).  $J_0(\cdot, \cdot, \cdot) = 0$ ;

(2). Assume  $J_k(s, 1, n)$  is increasing in  $n$ ,

$$J_{k+1}(s, 1, n) = \min \{Q_{k+1}^{PM}(s, 1, n), Q_{k+1}^0(s, 1, n), Q_{k+1}^1(s, 1, n)\},$$

where  $Q_{k+1}^{PM}(s, 1, n) = c_p + J_{k+1}(s, 2, 0)$ , is constant in  $n$ , and

$$Q_{k+1}^0(s, 1, n) = g(s) + \beta \{q \cdot J_k(s-1, 1, n) + (1-q) \cdot J_k(s, 1, n)\},$$

so  $Q_{k+1}^0(s, 1, n)$  is increasing in  $n$  as is  $J_k$ .

Next we want to show  $Q_{k+1}^1(s, 1, n)$  is increasing.

$$\begin{aligned} & Q_{k+1}^1(s, 1, n+1) - Q_{k+1}^1(s, 1, n) \\ = & \beta \{q \cdot f_{n+1}(c_r + J_k(s, 0, 0)) - q \cdot f_n(c_r + J_k(s, 0, 0)) \\ & + q \cdot (1 - f_{n+1}) \cdot J_k(s, 1, n+2) - q \cdot (1 - f_n) \cdot J_k(s, 1, n+1) \\ & + (1 - q) \cdot f_{n+1} \cdot (c_r + J_k(s+1, 0, 0)) - (1 - q) \cdot f_n \cdot (c_r + J_k(s+1, 0, 0)) \\ & + (1 - q) \cdot (1 - f_{n+1}) \cdot J_k(s+1, 1, n+2) \\ & - (1 - q) \cdot (1 - f_n) \cdot J_k(s+1, 1, n+1)\} \\ \geq & \beta \{q(f_{n+1} - f_n)(c_r + J_k(s, 0, 0)) + (1 - q)(f_{n+1} - f_n)(c_r + J_k(s+1, 0, 0)) \\ & - q(f_{n+1} - f_n)J_k(s, 1, n+1) - (1 - q)(f_{n+1} - f_n)J_k(s+1, 1, n+1)\} \\ = & \beta \{q(f_{n+1} - f_n)(c_r + J_k(s, 0, 0) - J_k(s, 1, n+1)) \\ & + (1 - q)(f_{n+1} - f_n)(c_r + J_k(s+1, 0, 0) - J_k(s+1, 1, n+1))\} \\ \geq & 0. \end{aligned}$$

The last inequality is due to Condition 4.1 and

$$J_k(s, 1, n+1) \leq c_p + J_k(s, 2, 0) \leq c_r + J_k(s, 0, 0).$$

Therefore,  $Q_{k+1}^1(s, 1, n)$  is increasing in  $n$ . So does  $J_{k+1}(s, 1, n)$ .

(3). Since  $J(s, 1, n) = \lim_{k \rightarrow \infty} J_k(s, 1, n)$ , therefore,  $J(s, 1, n)$  is increasing in  $n$ . □

**Corollary 4.2.** *Under Conditions 4.1, 4.2 and 4.3, the joint policy is of control-limit type with respect to  $n$ .*

*Proof.* It follows immediately from the monotonicity of the optimal cost function  $J(s, 1, n)$  in  $n$ , and the fact that  $Q^{PM}(s, 1, n)$  is constant in  $n$ . □

The following theorem states that when the inventory level is sufficient high, no matter what the deterioration degree of the machine is, it is optimal neither to do PM nor to produce.

**Theorem 4.5.**  $\forall n, \exists s^*$  such that for all  $s > s^*$ , the optimal action for state  $(s, 1, n)$  is to stay idle, i.e.,  $\mu^*(s, 1, n) = 0$ .

*Proof.* Without loss of generality, assume  $s > 0$ . Consider a policy  $\mu_0$  that keeps the machine idle until the inventory level  $s = 0$ . Denote by  $\tau(s)$  the random time when the inventory reaches 0 under  $\mu_0$ . Let  $\{d_i, i = 1, 2, \dots\}$  be the sequence of incoming i.i.d. demands, i.e.,  $d_i = 1$ , w.p.  $q$ ; 0, w.p.  $1 - q$ . Then  $\tau(s)$  can be equivalently defined as

$$\tau(s) = \min\left\{t : \sum_{i=1}^t d_i \geq s\right\}. \quad (4.43)$$

Obviously,  $\tau(s)$  is a stopping time of the demand sequence, and  $E(\tau(s)) = s/q$ . Moreover,

$$\tau(s) \geq s, \text{ w.p.1.} \quad (4.44)$$



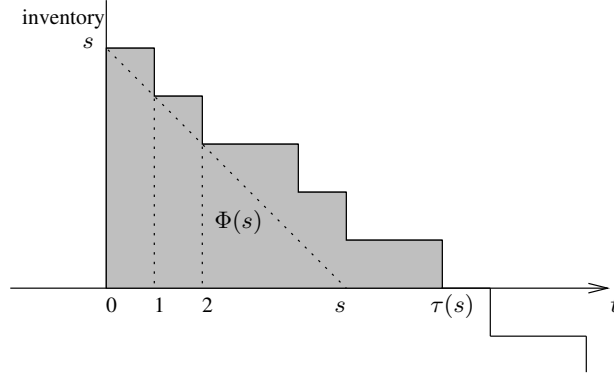


Figure 4.6:  $\tau(s)$ : minimum time for inventory reaching 0 for a sample path starting with initial level  $s$ ;  $\Phi(s)$ : the sample-path inventory cost under the policy  $\mu_0$  until the time  $\tau(s)$ .

We now denote by  $\Phi(s)$  the sample-path total *inventory* cost under the policy  $\mu_0$ , for the system starting at the inventory level  $s$  until the time  $\tau(s)$ , i.e.,

$$\Phi(s) = \sum_{t=0}^{\tau(s)} \beta^t \cdot c^+ \cdot \mathcal{S}^s(t), \quad (4.45)$$

where  $\mathcal{S}^s(t)$  is the inventory level at the time  $t$  if the system starts with the initial level  $s$ . It is obvious that  $E[\Phi(s)]$  is actually the minimum inventory cost for the system with the initial inventory level  $s$ , under any policy, i.e.,

$$J(s, \cdot, \cdot) \geq E[\Phi(s)]. \quad (4.46)$$

Under the policy  $\mu_0$ , the corresponding cost function  $J^{\mu_0}(s, 1, n)$  satisfies

$$J^{\mu_0}(s, 1, n) = E[\Phi(s)] + E[\beta^{\tau(s)} J(0, 1, n)]. \quad (4.47)$$

Combining (4.46) and (4.47), we have the following inequalities for the optimal cost function  $J(s, 1, n)$ :

$$E[\Phi(s)] \leq J(s, 1, n) \leq E[\Phi(s)] + E[\beta^{\tau(s)} J(0, 1, n)]. \quad (4.48)$$

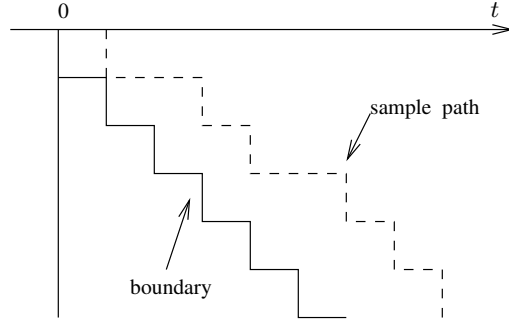


Figure 4.7: Inventory sample path versus boundary path

Suppose now the machine starts from the state  $(0, 1, n)$ , and we apply a policy  $\mu_1$  that puts the machine in idle forever. The cost function  $J^{\mu_1}(0, 1, n)$  satisfies

$$\begin{aligned} J^{\mu_1}(0, 1, n) &\leq \sum_{t=0}^{\infty} \beta^t \cdot c^- \cdot t \\ &= c^- \frac{\beta}{(1-\beta)^2}. \end{aligned} \quad (4.49)$$

Therefore, the optimal cost function  $J(0, 1, n)$  is bounded by:

$$\begin{aligned} J(0, 1, n) &\leq J^{\mu_1}(0, 1, n) \\ &\leq c^- \frac{\beta}{(1-\beta)^2}. \end{aligned} \quad (4.50)$$

It follows that

$$\begin{aligned} J(s, 1, n) &\leq E[\Phi(s)] + E[\beta^{\tau(s)}] c^- \frac{\beta}{(1-\beta)^2} \\ &\leq E[\Phi(s)] + \beta^s c^- \frac{\beta}{(1-\beta)^2}. \end{aligned} \quad (4.51)$$

But we also have

$$\begin{aligned} Q^{PM}(s, 1, n) &= c_p + J(s, 2, 0) \\ &\geq c_p + E[\Phi(s)], \end{aligned} \quad (4.52)$$

Therefore,  $Q^{PM}(s, 1, n) > J(s, 1, n)$ , if

$$c_p > \beta^s c^- \frac{\beta}{(1-\beta)^2},$$

or equivalently if

$$s > \frac{\ln \frac{c_p(1-\beta)^2}{c^- \cdot \beta}}{\ln \beta}. \quad (4.53)$$

Next, we consider the action of producing one more item at state  $(s, 1, n)$ . By the system dynamics (4.31),

$$\begin{aligned} Q^1(s, 1, n) &= g(s) + \beta \{q \cdot f_n \cdot (c_r + J(s, 0, 0)) + q \cdot (1 - f_n) \cdot J(s, 1, n + 1) \\ &\quad + (1 - q) \cdot f_n \cdot (c_r + J(s + 1, 0, 0)) \\ &\quad + (1 - q) \cdot (1 - f_n) \cdot J(s + 1, 1, n + 1)\} \\ &\geq g(s) + \beta \{q \cdot f_n \cdot J(s, 1, n + 1) + q \cdot (1 - f_n) \cdot J(s, 1, n + 1) \\ &\quad + (1 - q) \cdot f_n \cdot J(s + 1, 1, n + 1) \\ &\quad + (1 - q) \cdot (1 - f_n) \cdot J(s + 1, 1, n + 1)\} \\ &= g(s) + \beta \{q \cdot J(s, 1, n + 1) + (1 - q) \cdot J(s + 1, 1, n + 1)\} \\ &\geq g(s) + \beta \{qJ(s, 1, n) + (1 - q)J(s + 1, 1, n)\} \\ &= Q^0(s + 1, 1, n) - g(s + 1) + g(s) \\ &\geq E[\Phi(s + 1)] - c^+. \end{aligned}$$

The first and second inequalities are due to Theorem 4.4, and the last inequality is due to (4.46).

Since  $\{d_i\}$  is i.i.d, it can be seen that the sample pathes with initial levels  $s + 1$  and  $s$  satisfy

$$\mathcal{S}^{s+1}(t) =_{st} \mathcal{S}^s(t) + 1.$$

Therefore,

$$\begin{aligned} E[\Phi(s + 1)] - E[\Phi(s)] &= E \left[ \sum_{t=0}^{\tau(s+1)} \beta^t \cdot c^+ \right] \\ &\geq \frac{c^+(1 - \beta^{s+2})}{1 - \beta}, \end{aligned}$$

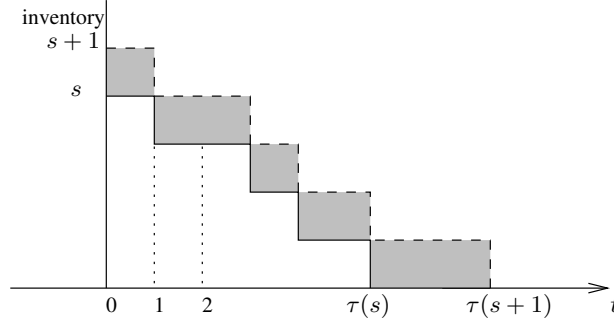


Figure 4.8: Computing  $E(\Phi(s+1)) - E(\Phi(s))$

Thus,

$$\begin{aligned}
 Q^1(s, 1, n) &\geq E[\Phi(s)] + \frac{c^+(1 - \beta^{s+2})}{1 - \beta} - c^+ \\
 &= E[\Phi(s)] + \frac{c^+\beta(1 - \beta^{s+1})}{1 - \beta}.
 \end{aligned} \tag{4.54}$$

Comparing (4.51) and (4.54), we see  $Q^1(s, 1, n) > J(s, 1, n)$  if

$$\frac{c^+\beta(1 - \beta^{s+1})}{1 - \beta} > \frac{c^-\beta^{s+1}}{(1 - \beta)^2}, \tag{4.55}$$

or equivalently if

$$\beta^s < \frac{c^+(1 - \beta)}{c^- + c^+\beta(1 - \beta)}, \tag{4.56}$$

$$\text{i.e., } s > \frac{\ln \frac{c^+(1 - \beta)}{c^- + c^+\beta(1 - \beta)}}{\ln \beta}. \tag{4.57}$$

Combining (4.53) and (4.57), it follows that  $\mu^*(s, 1, n) = 0$  if

$$s > \max \left\{ \frac{\ln \frac{c_p(1 - \beta)^2}{c^-\beta}}{\ln \beta}, \frac{\ln \frac{c^+(1 - \beta)}{c^- + c^+\beta(1 - \beta)}}{\ln \beta} \right\}. \tag{4.58}$$

□

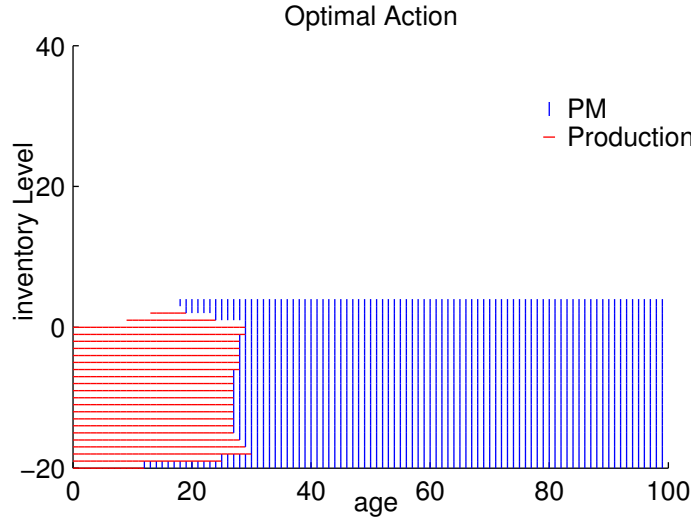


Figure 4.9: Optimal policy for unreliable production system with operation-dependent failures. (The blank area in the state space is where the optimal action is to stay idle.)

## Example

Again assume the machine's lifetime is Weibull distributed. The time for PM and time for CM are uniformly distributed in  $[0, \text{MAXTM}]$  and  $[0, \text{MAXTR}]$ , respectively. The following are the model parameters.

---

$\alpha = 4,$	$\eta = 5,$
$\beta = 0.95,$	$\text{MAXLIFE} = 100,$
$\text{MAXTM} = 3,$	$\text{MAXTR} = 6,$
$q = 0.8,$	$\beta = 0.95,$
$c_p = 50,$	$c_r = 2 * c_p,$
$c^+ = 1,$	$c^- = 10.$

---

The optimal policy is shown in Figure 4.9.

## 4.4 Conclusions

In this chapter, we have studied the problems of optimal joint PM and production policies for unreliable systems with time-dependent and operation-dependent failures. Two different models have been developed and analyzed. Both models have applications in semiconductor manufacturing systems: the first model is appropriate for tools with calendar-based PMs, whereas the second model is useful for tools with wafer-based PMs.

Using MDP modeling, we have analyzed the optimal joint PM and production policies. Some structural properties of the optimal value function and corresponding optimal policies have been derived. Particularly, we show, under some reasonable conditions, the optimal policies are control-limit with respect to machine's age (deterioration degree).

It is worth noting that the problems studied in this chapter have some connections with the so-called *restless bandit problems* [59]. A restless bandit problem differs from the classical *bandit processes* [26] in that the restless bandits ('projects') continue to change state even when they are not being operated. Specifically, the model with operation-dependent failures is much like a problem with two-armed restless bandit processes, with one process corresponding to production and the other corresponding to preventive maintenance. The state for the production process is the inventory level, which changes even when the system is in maintenance. On the other hand, the state for the preventive maintenance process is the age of the machine, which changes when the system is in production.

While the classical multi-armed bandit problems are well-known to be solved by Gittins index policy [26], the restless bandit problems are much harder to solve

[59, 58], and actually have been proven to be *PSPACE-hard* [42]. The additional constraints in our problems make them even harder. For instance, one constraint is that once the system is in maintenance, it cannot switch to production until the maintenance is finished. In addition, for the model with time-dependent failures, the multiple-valued production control ( $u \in \{0, 1, \dots, P\}$ ) adds even more complexity to the restless bandit problem formulation.

## Chapter 5

# Optimal PM Scheduling in Semiconductor Fabs

In this chapter, we address the problem of optimal PM *scheduling* for semiconductor manufacturing systems. Our emphasis here will be placed other than on the investigation of optimal PM policies structures but on the *scheduling* of multiple PM tasks within a specified time horizon. As the difficulty in PM scheduling for cluster tools is representative of the complexity found in most semiconductor fabs, we will consider the problem under the setting of a group of cluster tools with multiple PM tasks on each tool.

The optimal PM scheduling problem considered in this chapter corresponds to the lower level of the proposed hierarchical framework. It coordinates interdependent multiple PM tasks by taking into account individual tool PM policies, WIP levels, and resource constraints, so as to obtain an optimal schedule under a predefined objective function. The basic problem is stated as follows. Consider a cluster tool with multiple processing chambers. During a certain time period, there are different PM tasks that need to be scheduled on different chambers. Since the throughput of the entire tool is dependent on the status of each



chamber, the question is how to schedule the PM tasks so as to maximize the throughput while satisfying each PM's requirement. Following common practice in the semiconductor industry, we assume each individual PM follows the so-called "time window" policy, where each PM is associated with a time interval specifying an earliest and latest start time for the PM.

We address the PM scheduling problem using mixed integer programming (MIP) models, and the formulation and solution of the models will be discussed in detail in this chapter. Although our proposed solutions are applicable to all tool groups in a fab, those groups with highly complex and interdependence PM tasks, and high utilization rates, would clearly be most positively impacted. Commonly, groups with cluster tools fall in the latter category, and we focus on these to illustrate our solutions.

In order to test the models and algorithms developed, a simulation study was conducted for a tool group in a large fab, using that fab's actual simulation model and real data. Performance obtained by scheduling PM tasks using our solutions was compared to a baseline PM schedule over a one-week period during medium to high workload level. The baseline schedule used was the one implementing actual historical PM scheduling decisions by tool group managers. The simulation results indicate that our solutions perform quite efficiently, in the sense that they outperform the baseline reference schedule, and they exhibit a preference for consolidated PM tasks when searching for an optimal schedule. The proposed MIP model is now being implemented and integrated into a real fab operational environment at a major semiconductor manufacturer. The work presented in this chapter has been submitted for publication [64] where more implementation details are provided; a preliminary version of this work was once

presented at the IEEE Conference on Control Applications in 2001 [63].

The remainder of the chapter is organized as follows. Section 5.1 provides some background and contains a brief literature review on related work. The MIP model for the PM scheduling problem is developed and discussed in section 5.2, with more technical details provided in the section 5.3. The simulation study is contained in Section 5.4. Finally we provide some concluding remarks.

## 5.1 Background and Related Work

PM scheduling in semiconductor fabs has long been seen as a very hard problem; for example, see [34, 55]. The scheduling of PM tasks for cluster tools is a good representative example. Cluster tools are highly integrated machines that can perform a sequence of semiconductor manufacturing processes. A general configuration of a cluster tool includes load/unload locks, orientor/degas, transfer robots and several processing chambers.

The difficulty of PM scheduling for cluster tools is largely due to their complex behavior, as they usually have several chambers, and each chamber has several different PM tasks that have to be performed. To improve the availability of the entire tool requires coordination of PM tasks in different chambers, because the entire tool's availability is dependent on the status of each chamber. In addition, fab production data such as Work-In-Process (WIP) should be considered in PM scheduling. For instance, PM tasks should be avoided if possible during periods when a significant amount of work is expected to arrive soon. It would be wise to “pull” or “push” a planned PM task beyond a certain period under such circumstances. Hence, PM tasks should be scheduled by looking ahead

at both the effect from WIP and the impact on WIP. In addition, it may be advantageous sometimes to consolidate PM tasks, e.g., doing one task “early” when a tool or chamber is brought down for another task. Costs for supplies and lost production, as well as technician availability constraints, should also be accounted for.

It is obvious that the uncertain (stochastic) nature of WIP and tool failures, and the interdependence of PM tasks in fabs, require new models to be developed to deal with these complicated situations. Unfortunately, there does not appear to be such models readily applicable to PM scheduling problems. On the other hand, there are enormous amounts of data in the fab databases readily available to modelers and planners; yet most of this potential goes unutilized.

Performance evaluation for cluster tools operation has been studied extensively in the literature [5, 61, 36, 29]. However, there is little study related to PM for cluster tools. On the other hand, many PM models on multi-component systems have been developed for systems where several machines are stochastically or economically dependent on each other; for example, see the survey paper [14] and the references therein. Most of these efforts have been focused on group/block or opportunistic maintenance models that make use of economies of scale to perform preventive replacement upon the failure of one unit, e.g. [9], or on the investigation of the effect of repairmen/spare parts inventory on maintenance policies. However, there are very few papers on PM scheduling under the specific context of semiconductor manufacturing.

Recently, Mosley et al. [39] study maintenance dispatching and staffing policies for a group of fabs sharing maintenance resources. Yet their objective is to study various policies for scheduling maintenance personnel, by using a discrete-

event simulation model.

In semiconductor manufacturing, hierarchical planning and scheduling for maintenance activities is widely adopted. Although this is common in the industry as well as in many other applications, this type of hierarchical PM scheduling structure has not been addressed formally until very recently.

The recent work by van Dijkhuizen and van Harten [57] appears to be the first effort to address the issue. They study a two-stage maintenance policy, where the first stage is to determine a time window  $[t, t + \Delta t]$ , and the second stage is to determine the actual start time of a PM within the time interval. Specifically, they assume, in the first stage, a generalized age maintenance policy is used, i.e., a PM must be carried out somewhere between  $t$  and  $t + \Delta t$  since the last PM or repair. In the second stage, the initiation of a PM task is driven by the operating state of the system, which is assumed to be deterministic over the interval  $[t, t + \Delta t]$ . The PM is carried out at the optimal time  $\hat{t} \in [t, t + \Delta t]$ . However, their problem setting is for a single PM on a single tool, and the model is not well suited for scheduling multiple PM tasks in the context of multiple tools. In addition, they assume the time for a PM is negligible, which is clearly not the case for most PMs in semiconductor manufacturing.

The value of consolidation of different PM tasks, e.g., for cluster tools, is commonly recognized in semiconductor manufacturing; yet it has not been addressed in a rigorous way in the literature. One study of the problem of grouping maintenance activities, which doesn't consider production costs, is conducted by Wildeman et al. [60], in a generic problem setting. They consider a multi-component system where preventive maintenance activities can be carried out on each component with a system-dependent cost (i.e., setup cost, which is the

same for all activities) and a component-dependent cost. It is desirable to group maintenance activities, since execution of a group of activities requires only one setup. They develop a rolling horizon dynamic policy for grouping PMs.

## 5.2 Mixed Integer Programming

We will study the problem of optimal PM scheduling for a group of cluster tools. There are a few important issues we need to address.

To begin with, there are various PM activities on each component of the cluster tool. Roughly, they can be categorized into two types of PMs: calendar-based and operation-based. A calendar-based PM must be performed at some interval of calendar time, e.g., every 7, 14, 30, 90, 180 or 360 days. For an operation-based PM, the interval between two consecutive tasks is determined by the tool's operation history, which can be characterized by either wafer count or cumulative operating time since the last PM. For example, for each processing chamber, a kit change is supposed to be undertaken at every specified number of wafers produced since the last PM. The vast majority of PM policies follow a "generalized age replacement" structure, in which a PM is scheduled for a time after a tool's "age" exceeds some threshold, but there is flexibility on the actual start time within some associated interval. Here, "age" means calendar-time or operation history, according to the type of PM. In semiconductor manufacturing practice, this is often called a "PM window" policy, where such a window is associated with each PM task. Even if a PM window is operation-based, e.g., "2000 wafers  $\pm$  10%" since last PM, tasks must be scheduled on a calendar basis, e.g., work shift and day. Furthermore, if an optimization model were

to track wafer count, this would lead to a very high level of computational complexity, and scheduling decisions of the form “schedule PM task A at 1,860 wafer count”, which would need to be converted to an equivalent calendar date. For these reasons, our models and algorithms operate on a calendar base, and PM window specifications are assumed to be given on this base. Converting operation-based data to equivalent calendar dates is commonly handled in ad hoc ways in practice. Efficient methods and algorithms have also been developed recently in [45].

Another issue in PM scheduling is that several key factors affecting the decision-making process have to be taken into account. First of all, careful coordination of PM tasks in different chambers is required to improve the entire tool’s throughput, because it is dependent on the status of each chamber. Usually, it is advantageous to consolidate PM tasks when another PM is planned on the near horizon, or tools are shut down due to unexpected “out of control” events. Second, WIP has to be considered in the PM schedule. For example, it would be ideal to do PM in a period when WIP is low, and not to do PM in a period of high WIP or when many lots of wafers are scheduled to arrive. Finally, resource constraints such as the headcount of maintenance technicians for the entire tool group of interest have to be taken into consideration, since manpower is usually the most critical constraint in PM scheduling.

In the following, after the definition of the problem, a mixed integer program is presented. Specific issues on solving the MIP model and its software implementation will then be discussed.

### 5.2.1 Problem Definition

We consider PM scheduling for a group of generic tools. We assume all PM tasks are calendar-based. For non-calendar-based PMs, such as wafer-count-based and operation-time-based PMs, we assume that these are first converted from their wafer-count or operation time into calendar time for the purpose of scheduling. For the sake of generality, we give our presentation below in terms of cluster tools, keeping in mind that non-clustered tools can be viewed as single chamber tools for purpose of our model. Moreover, tools with coupled operations, e.g., litho steppers and trackers, should be modeled as a single tool with two chambers (in series).

Now consider a group of  $M$  cluster tools. The indexing of PM tasks for tool  $i$  is from 1 to  $\rho_i$ , where  $\rho_i$  is the total number of PM tasks applicable to tool  $i$ . (A type of PM that is to be scheduled multiple times over the planning horizon, whether on the same chamber or different chambers of a same tool, must be given a separate distinct index for each possible occurrence of the PM task.)

For each cluster tool, the joint impact of PM tasks on its relative throughput, defined with respect to a fully operational tool, is characterized through a so-called “configuration matrix”. Table 5.1 illustrates such a matrix for a cluster tool that has five chambers indexed by Ch1, ..., Ch5. The first row represents the scenario when all chambers 1 to 5 are up, indicated with “1”, and so its availability is by definition 100%. The second and third rows represent the scenarios when either Ch1 or Ch2 is shut down, indicated with “0”, for PM, respectively, with relative availability of only 60%. However, the 4th row shows that relative availability is 0 when both Ch1 and Ch2 are down at the same time, regardless of the status of all other chambers (indicated with “X”). This

suggests that each wafer likely has to go through either Ch1 or Ch2, so when both chambers are down, no wafers can be processed. This also indicates that it is unwise to consolidate PM tasks for Ch1 and Ch2. Similarly, the last row suggests that each wafer has to go through Ch3, and so when it is down, there is no throughput, and its availability is therefore 0.

Similar to the “configuration matrix”, a table describing resource requirements will list the resources required and duration for each PM and any consolidated PMs.

Now, given a set of PM tasks that need to be scheduled on these tools in a scheduling horizon, with each PM task associated with a time window in which the PM has to be started, the problem is to determine the best time for doing each PM, with the objective of maximizing overall tool availability and minimizing WIP, under some resource or operation constraints.

We formulate below the problem as an MIP model.

### 5.2.2 MIP Formulation

Let  $t$  denote a generic time period, or PM decision epoch, and  $T$  the planning horizon; hence  $t = 1, \dots, T$ . For example, time could be divided in periods of one work shift or one day, and the planning period could be two weeks, i.e.,  $T = 42$  shifts (assuming 3 shifts per day) or  $T = 14$  days, respectively. The following notation will be used hereafter.

$a_i^l(t)$ : binary decision variables for PM task  $l$  on tool  $i$  in period  $t$ , (1: do PM; 0: do not do PM). Define  $\underline{a}_i(t) = [a_i^1(t) \ a_i^2(t) \ \dots \ a_i^{\rho_i}(t)]^T$ , the control vector for all PM tasks on tool  $i$ .



Table 5.1: Configuration matrix for a cluster tool (legend: 0: down; 1: up; X: up/down)

Ch1	Ch2	Ch3	Ch4	Ch5	Availability
1	1	1	1	1	100%
0	1	1	1	1	60%
1	0	1	1	1	60%
0	0	X	X	X	0%
1	1	1	0	1	80%
1	1	1	1	0	80%
X	X	X	0	0	0%
1	0	1	1	0	60%
1	0	1	0	1	60%
0	1	1	1	0	60%
0	1	1	0	1	60%
X	X	0	X	X	0%

$w_i^l, u_i^l$ : time window [min, max] associated with PM task  $l$  on tool  $i$ .

$k_i$ : number of periods for the PM task with the longest duration on tool  $i$ .

$V_i(t)$ : availability of tool  $i$  in period  $t$ .

$I_i(t)$ : workload level (total in buffer and in process) for tool  $i$  in period  $t$ .

$d_i(t)$ : projected incoming WIP for tool  $i$  in period  $t$ .

$b_i$ : profit coefficient for availability of tool  $i$ .

$c_i^I$ : cost coefficient for inventory on tool  $i$ .

$c_i^l$ : PM cost for performing PM task  $l$  on tool  $i$ .

$L_i$ : WIP buffer size for tool  $i$ .

$K_i$ : coefficient of wafer throughput for tool  $i$ 's availability.

$f_i(\cdot)$ : availability function for tool  $i$ ; constructed from the “configuration matrix”, e.g., Table 5.1.

$N$ : number of resource types considered.

$r_i^j(\cdot)$ : resource function calculating the requirement of resource type  $j$  for tool  $i$ ,  
 $j = 1, \dots, N$ ; constructed from a resource requirement matrix.

$R^j(t)$ : amount of resource type  $j$  available in period  $t$ ,  $j = 1, \dots, N$ .

Our model is then given as follows.

Model **MIP1**:

$$\max \sum_{t=1}^T \sum_{i=1}^M \left( b_i \cdot V_i(t) - c_i^I \cdot I_i(t) - \sum_{l=1}^{\rho_i} c_i^l \cdot a_i^l(t) \right) \quad (5.1)$$

subject to:

$$\sum_{t=w_i^l}^{u_i^l} a_i^l(t) = 1, \text{ for those PM tasks that have to be finished} \\ \text{in the time window } [w_i^l, u_i^l] \subseteq [1, T]. \quad (5.2)$$

$$V_i(t) = f_i(\underline{a}_i(t), \underline{a}_i(t-1), \dots, \underline{a}_i(t - (k_i - 1))), \\ \text{for } i = 1, \dots, M; t = 1, \dots, T; \underline{a}_i(t) = 0, \\ \text{for } t \leq 0. \quad (5.3)$$

$$R^j(t) \geq \sum_{i=1}^M r_i^j(\underline{a}_i(t), \underline{a}_i(t-1), \dots, \underline{a}_i(t - (k_i - 1))), \\ \text{for } t = 1, \dots, T; j = 1, \dots, N; \underline{a}_i(t) = 0, \\ \text{for } t \leq 0. \quad (5.4)$$

$$I_i(t+1) = (I_i(t) - K_i \cdot V_i(t) + d_i(t))^+, \\ \text{for } i = 1, \dots, M; t = 1, \dots, T-1. \quad (5.5)$$

$$I_i(t) \leq L_i, \text{ for } i = 1, \dots, M; t = 1, \dots, T. \quad (5.6)$$

In (5.5), the operation  $(\cdot)^+$  is defined as  $(x)^+ = \max(0, x)$ . The objective is to maximize profits from tool availability, minus costs from inventory build-up and performing the PM tasks. Equation (5.2) states that the scheduled PM tasks have to be performed within their individual time windows. Equation (5.3) computes the availability for each tool for each time period. A particular sequence of  $\underline{a}_i(t), \underline{a}_i(t-1), \dots, \underline{a}_i(t - (k_i - 1))$ , determines a particular row of the “configuration matrix” and the value of  $f_i(\cdot)$  would be the corresponding value of availability for that row. Equation (5.4) states that for each type of resources the

sum of resource requirement over all tools must be less than available resource in each period. Equation (5.5) describes the WIP dynamics for each tool, and implies that for each tool  $i$ , it would produce as many wafers as possible, using all availability at hand, if there is enough in-buffer WIP; otherwise it would produce wafers matching up with the in-buffer WIP. Equation (5.6) states that the WIP level of tool  $i$  should not exceed its buffer size at any time. Model parameters such as  $b_i, c_i^I, c_i^L, L_i, R^j$ , are fab specific data, and can be obtained from fab operation.

In the above formulation, without loss of generality, we have assumed that during the scheduling horizon, each PM is performed at most one time on each tool, as reflected in Equation (5.2). This assumption does not affect PM tasks of the same type performed on different chambers, because they should have been indexed differently due to their association with different chambers. In the case when the same type of PM needs to be scheduled more than once for the same tool during the horizon, different indices should have been assigned to them, so that they will be treated as different PMs.

Equations (5.3) and (5.4) contain the respective availability and resource functions  $f_i$  and  $r_i^j$ . Albeit nonlinear in general, these can be easily implemented as look-up tables for computational purposes. Moreover, they can be also transformed into an equivalent set of linear equations, exploiting the fact that all its arguments are binary.

Note that the constraint (5.5) is non-linear, due to the operator  $(\cdot)^+$ . However, since it is piecewise linear, we define the following related problem:

**Model MIP1'**: Same as **MIP1**, but with (5.5) replaced by the following two linear constraints:

$$I_i(t+1) \geq I_i(t) - K_i \cdot V_i(t) + d_i(t), \quad (5.7)$$

for  $i = 1, \dots, M; t = 1, \dots, T-1$ .

$$I_i(t) \geq 0, \text{ for } i = 1, \dots, M; t = 1, \dots, T. \quad (5.8)$$

We have the following proposition that states model **MIP1** and **MIP1'** are equivalent.

**Proposition 5.1.** *MIP1' is equivalent to MIP1.*

*Proof.* Let  $\mathcal{D}(\mathbf{MIP1})$  and  $\mathcal{D}(\mathbf{MIP1}')$  be all feasible solutions of **MIP1** and **MIP1'** respectively. We first show  $\mathcal{D}(\mathbf{MIP1}') \supseteq \mathcal{D}(\mathbf{MIP1})$ . Note that the constraints (5.7) and (5.8) of **MIP1'** can be written combinatorially as

$$I_i(t+1) \geq (I_i(t) - K_i \cdot V_i(t) + d_i(t))^+, \text{ for } i = 1, \dots, M; t = 1, \dots, T-1. \quad (5.9)$$

Indeed, it is a relaxed constraint (5.5) of **MIP1**. So  $\mathcal{D}(\mathbf{MIP1}') \supseteq \mathcal{D}(\mathbf{MIP1})$ .

It can be easily verified that any optimal solution to **MIP1'** will achieve  $I_i(t+1) = (I_i(t) - K_i \cdot V_i(t) + d_i(t))^+$ , which implies this is also an optimal solution to **MIP1**. To see this, assume there is an optimal solution such that  $I_{i_1}^*(t_1+1) > (I_{i_1}(t_1) - K_{i_1} \cdot V_{i_1}(t_1) + d_{i_1}(t_1))^+$ . Obviously, if we choose  $I'_{i_1}(t_1+1) = (I_{i_1}(t_1) - K_{i_1} \cdot V_{i_1}(t_1) + d_{i_1}(t_1))^+ < I_{i_1}^*(t_1+1)$ , it will achieve a larger objective value. Contradiction.

Thus **MIP1** and **MIP1'** are equivalent. □

It is worth noting here that in the objective function, minimizing the cost of a tool's WIP level implies maximizing its wafer throughput, due to the following

relationship:

$$I_i(t + 1) = I_i(t) - X_i(t) + d_i(t), \quad (5.10)$$

where  $X_i(t)$  is the wafer throughput of tool  $i$  in the time period  $t$ , and is given by

$$X_i(t) = \min\{K_i \cdot V_i(t), I_i(t) + d_i(t)\}. \quad (5.11)$$

It can be easily seen that (5.5) is just a compact expression of (5.10) and (5.11).

There is an operational difference, though, between maximizing tool availability and maximizing wafer throughput. The former is relevant only to the tool technical state, i.e., keeping the tool operational as long as possible, whereas the latter is not only relevant to the tool technical state, but also strongly affected by projected incoming WIP. However, for bottleneck tools, maximizing availability is equal to maximizing wafer throughput, and vice versa, because there is always enough in-buffer WIP to be processed.

On the other hand, in situations where one specifically wants to maximize wafer throughput, the MIP formulation would become a little different, as defined by the following problem:

$$\text{Model MIP2: } \max \sum_{t=1}^T \sum_{i=1}^M \left( b'_i \cdot X_i(t) - \sum_{l=1}^{\rho_i} c_i^l \cdot a_i^l(t) \right) \quad (5.12)$$

subject to:

$$X_i(t) \leq K_i \cdot f_i(\underline{a}_i(t), \underline{a}_i(t-1), \dots, \underline{a}_i(t - (k_i - 1))), \quad (5.13)$$

for  $i = 1, \dots, M; t = 1, \dots, T; \underline{a}_i(t) = 0$ , for  $t \leq 0$ .

$$I_i(t+1) = I_i(t) - X_i(t) + d_i(t),$$

for  $i = 1, \dots, M; t = 1, \dots, T - 1$  (5.14)

$$I_i(t) \geq 0, \text{ for } i = 1, \dots, M; t = 1, \dots, T. \quad (5.15)$$

and constraints (5.2), (5.4) and (5.6), where  $b'_i$  in the objective function is the profit coefficient for wafer throughput of tool  $i$ .

In general, **MIP1** and **MIP2** are not equivalent.

### 5.2.3 Implementation Issues

Given our discussion above, we choose to maximize availability, and thus we follow models **MIP1** and **MIP1'**, the latter of which is more amenable for practical implementation purposes. A model implemented in practice will usually have a simpler structure than the general formulation above. For example, it is not uncommon to consider a group of homogeneous (identical) cluster tools. In that case, all tools have the same physical structures and PM tasks. Hence, their availability functions are the same, as well as the resource functions. Thus  $f_i(\cdot)$  and  $r_i^j(\cdot)$  will reduce to  $f(\cdot)$  and  $r^j(\cdot)$ , respectively. In addition, if manpower, i.e., the number of available maintenance technicians, is the only resource constraint of interest, which seems often to be the case based on our experience, then the

resource vector  $\underline{R}$  becomes a scalar.

In order to deal with the non-linearity of functions  $f_i(\cdot)$  and  $r_i^j(\cdot)$ , we introduce a new set of decision variables – PM task vectors. A task vector contains a set of PM tasks, which could be consolidated and performed on a tool. At each time there is only one PM task vector active on each tool. Each task vector corresponds to one scenario of PM consolidation; Section 5.3 provides more details about the definition of PM task vector and the model transformation. The set of task vector can be generated dynamically according to which PM tasks have to be scheduled in a given scheduling scenario. For a predefined planning horizon, only those PM tasks whose time windows fall into the horizon will be taken into consideration.

Based on the MIP model described above, an optimal preventive maintenance scheduling system has been implemented within a real fab setting with capability of optimizing PM tasks for any module consisting of a group of tools. With different data interfaces, the system is integrated with other information systems in the fab in such a way that the specific MIP model to apply can be generated automatically by extracting PM data from a tool maintenance database, and WIP information from a real-time dispatch system, or a fab simulation model. A summary report is presented to users, e.g., tool managers, after an optimal scheduling solution is found, with information of projected availability and WIP of each tool in each period along the scheduling horizon. A comparison list with initial PM schedules and model-optimized schedules is also generated, and users can decide whether or not the model-optimized schedule will be put into effect.



### 5.3 Solving the MIP Model

There are two technical problems that must be addressed in order to solve the MIP model. To begin with, there may be PM tasks with a duration exceeding a single period. As seen in availability function  $f_i$  and resource function  $r_i^j$ , this results in the difficulty that chambers statuses (thus, the tool's state) will depend not only on PM tasks initiated in current time period  $t$ , but also on those unfinished PM tasks that were initiated in  $t - 1, t - 2, \dots$ , etc. One method to work around the complexity is to introduce some “artificial” PM tasks as follows.

Assume PM  $l$  lasts for 3 periods. We introduce two “artificial” PMs  $l'$  and  $l''$  such that  $l'$  must be performed in the next period following  $l$ , and  $l''$  is following  $l'$ , and we now treat  $l$  as a PM task with a duration of only one period instead of three. This relationship can be formulated as “precedence” constraints as follows:

$$a_i^{l'}(t+1) = a_i^l(t), \quad (5.16)$$

$$a_i^{l''}(t+1) = a_i^{l'}(t). \quad (5.17)$$

Thus any PM task with a duration exceeding one period can be transformed into a sequence of PMs of one-period duration. Hence, in the following analysis, we will assume without loss of generality that no PMs have a duration exceeding one period.

The second difficulty is that the availability function  $f_i$  and resource function  $r_i$  are non-linear functions of chamber status. To deal with the non-linearity, the main idea is to transform these non-linear functions into linear form by changing control variables. Observe that availability and resource functions can be expressed in “look-up” table form, as the “configuration matrix”. Explicitly,

if we denote the state of tool  $i$  (i.e., all chambers statuses, up or down) by  $s_i$ , then the availability function will become  $f_i(s_i)$ , and we denote its value by  $f_i^{s_i}$ . Now the function can be expressed as a data set  $\{f_i^{s_i}\}$ .

The decision variable in the MIP model is  $a_i^l(t)$ , i.e., to determine whether PM task  $l$  is conducted on tool  $i$  in period  $t$ , for every feasible  $l$ . This is equivalent to determining a group of PM tasks (task vector) conducted on tool  $i$  for every period  $t$ . Because there is a finite number of PM tasks, it is easy to obtain all combinations, i.e., vectors of these tasks. For example, if there are  $n$  tasks on tool  $i$ , then there are  $2^n - 1$  task vectors, which include all possible combinations of these  $n$  tasks. We denote the task vector by  $v$ , and for the sake of simplicity, we assume every vector  $v$  is associated with only one tool, i.e., it can be only applied to a specific tool. We denote by  $\mathcal{V}(i)$ , the set of all feasible task vectors for tool  $i$ . The information of element tasks included in a vector  $v$  is contained in data  $e(l, v)$ , where  $e(l, v) = 1$  if  $v$  contains  $l$ ,  $e(l, v) = 0$  otherwise.

Now, define new binary decision variables  $z(i, v, t)$  for  $v \in \mathcal{V}(i)$ , where  $z(i, v, t) = 1$  if task vector  $v$  is performed on tool  $i$  in the period  $t$ ,  $z(i, v, t) = 0$  otherwise. Obviously, for  $v \notin \mathcal{V}(i)$ ,  $z(i, v, t) = 0$ . It is also obvious that on each tool in any period, there is only one vector that can be active. So, the following new constraints on  $z(i, v, t)$  will be enforced:

$$\sum_v z(i, v, t) \leq 1, \text{ for } i = 1, \dots, M; t = 1, \dots, T. \quad (5.18)$$

The original decision variable  $a_i^l(t)$  can be expressed as follows:

$$a_i^l(t) = \sum_{v \in \mathcal{V}(i)} z(i, v, t) \cdot e(l, v), \text{ for } i = 1, \dots, M; l = 1, \dots, \rho_i; t = 1, \dots, T. \quad (5.19)$$

Since tool state  $s_i$  is completely dependent on task vector  $v$ , their relationship can be characterized by  $\delta(v, s_i)$ , where  $\delta(v, s_i) = 1$  if  $v$  changes the tool state to

$s_i$ , otherwise  $\delta(v, s_i) = 0$ . The availability function now can be expressed as a linear function of the control variable  $z(i, v, t)$  as follows:

$$V_i(t) = \sum_{v \in \mathcal{V}(i)} \sum_{s_i} f_i^{s_i} \cdot \delta(v, s_i) \cdot z(i, v, t), \text{ for } i = 1, \dots, M, t = 1, \dots, T. \quad (5.20)$$

Similarly the resource requirement of the tool is dependent only on task vector, the corresponding resource function can be expressed as a data set  $\{r_i^{j,v}\}$ . Hence, equation (5.4) can be written as:

$$R^j(t) \geq \sum_i \sum_{v \in \mathcal{V}(i)} r_i^{j,v} \cdot z(i, v, t), \text{ for } j = 1, \dots, N; t = 1, \dots, T. \quad (5.21)$$

Thus, we are able to transform non-linear functions into linear functions of the new decision variables, and the transformed MIP model can be solved by a commercial IP/LP package. Equations (5.18) and (5.19) are the new constraints added in the transformed MIP model. The number of new constraints due to equations (5.18) and (5.19) is  $M \cdot T + \sum_{i=1}^M \rho_i \cdot T$ .

The drawback of introducing the new task vectors is that it will lead to a set of decision variables with much larger size. There are many ways generating the set of task vectors. The basic requirement is that the set of  $v$  must cover all possibilities of PM consolidations. One of the easiest ways is to list all possible combinations of PM tasks for each tool, and then put these combinations together and give them different index numbers one by one. This method ensures all possibilities of PM consolidation would be covered by the set, and its size is  $\sum_{i=1}^M 2^{\rho_i} - M$ . The actual number of new decision variables due to  $z(i, v, t)$  is  $\left(\sum_{i=1}^M 2^{\rho_i} - M\right) \cdot T$ .

## 5.4 Simulation Case Study

In order to evaluate the performance of the MIP scheduling model, we conducted a simulation case study employing a simulation model of a real fab. We ran the simulation with historical PM schedule data, and with a PM schedule that was optimized through our MIP model, and then compared their performance in terms of throughput and WIP level.

The simulation model used had been developed in Brooks Automation’s AutoSched AP software [12] (ASAP for short) by a large semiconductor manufacturer. ASAP has been used widely in semiconductor manufacturing for capacity analysis, planning and scheduling, and it is capable of modeling cluster tools in a fab. Some features of cluster tool modeling in ASAP are: (a) modeling robots, single or dual load locks with serial or parallel cassette processing, pump and vent delays, tool failure and chamber failure; (b) modeling a PM task as PM calendar or PM order.

We present only summary information for this initial case study, with sensitive company-specific data removed. The module employed has 11 cluster tools, i.e.,  $M = 11$ . These tools are homogeneous in the sense that they can perform the same processing steps and have the same configuration, i.e., same processing chambers and robots. Coincidentally, there are 11 PM tasks of interest in the study on *each* tool, and they are indexed from 1 to 11. The “configuration matrix” of these tools is the one given in Table 5.1. The longest duration of any PM task is 2 days, and the only resource we considered was the manpower (headcount) of available maintenance technicians. The resource function of PM task vectors, i.e., resource requirement for any joint or single PM tasks, is listed as a table with each row corresponding to a PM scenario, its duration and its

resource requirement.

The time unit is one day, and the scheduling horizon is one week, i.e.,  $T = 7$  and  $t = 1, 2, 3, 4, 5, 6, 7$ . Those PM tasks that are to be scheduled in the horizon per tool are shown in Figure 5.1 together with their time windows defined as a pair of (earliest\_start\_date, late\_date). So, for example, PM task 1 on tool 1 should be performed between Monday and Wednesday. PM task 5 on tool 1 is performed between Wednesday and Friday, while task 10 is between Friday and Sunday. For tool 8, the PM task 6 has to be performed on Monday as its time window is shrunk to a point in this specific case.

A specific MIP model instance was then generated from these PM tasks, and their individual time windows, along with all other relevant data, such as availability and resource requirement data sets, were fed into the scheduling system. The model instance was then solved. (For this simulation case study, the implemented MIP model has a total of 686 decision variables and 698 constraints.) The corresponding model outputs, i.e., optimal PM schedules for these tools along the scheduling horizon, are also shown in Figure 5.1 as the asterisk points. One main feature that can be seen in the figure is that the optimal PM schedule tends to consolidate PM tasks, as on tools 1, 6, 9, and 11.

We simulated one week of fab operations with the two different PM schedules: the schedule that was actually performed in operations, which is referred to as the “reference” schedule, and the optimized, model-based schedule. Ten replications of such a simulation were made, averaging output results. Each PM task was modeled as a “PM order” in ASAP in both simulations. Two statistics, the average number of wafers completed on each tool and average number of WIP wafers on each tool, were selected as performance measures. Figure 5.2 shows the

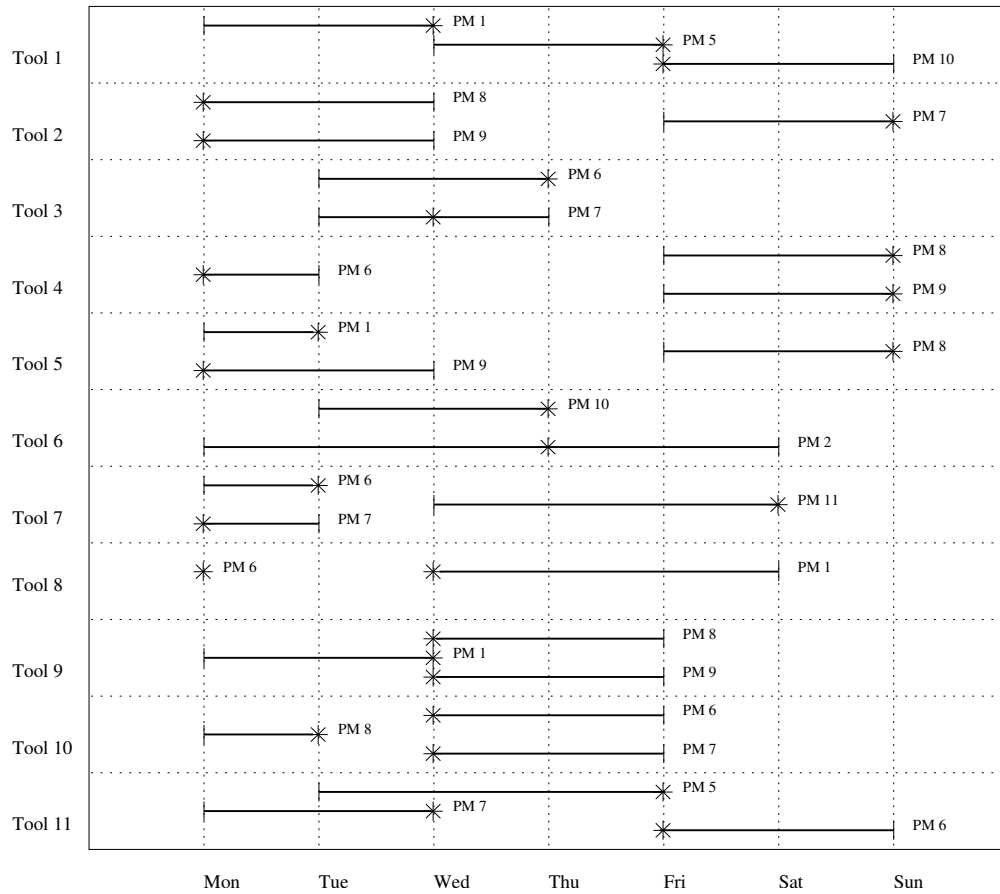


Figure 5.1: PM tasks with associated time windows, where asterisk points are the optimal times computed by the model to perform PM tasks.

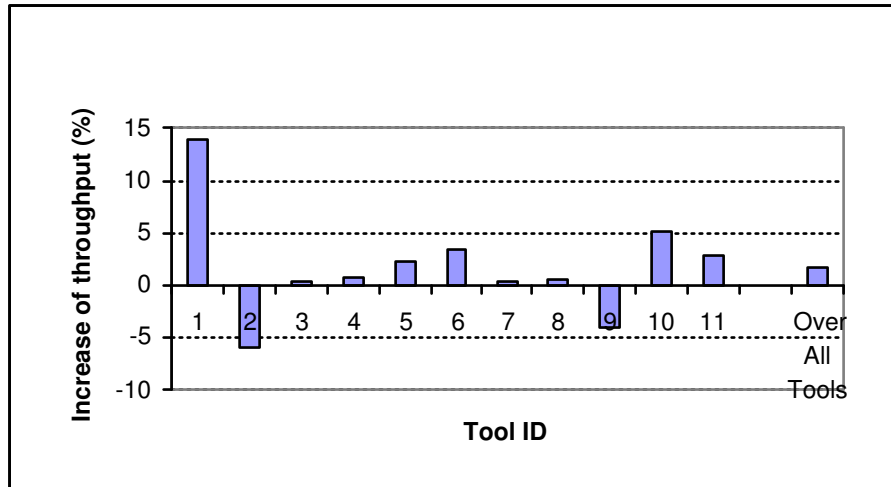


Figure 5.2: Simulation result for throughput increases (in percentages) under the model-based schedule over those under the reference schedule.

percentage throughput increase on each individual tool as well as the average over all tools for the model-based schedule versus the reference schedule, i.e.,  $\text{increase}\% = \frac{\text{TP}_{\text{model}} - \text{TP}_{\text{ref}}}{\text{TP}_{\text{ref}}} \times 100\%$ , where  $\text{TP}_{\text{model}}$  and  $\text{TP}_{\text{ref}}$  are the throughput under the model-based schedule and reference schedule, respectively. Figure 5.3 shows the same for WIP levels.

The simulation results show that the performance of the model-based schedule and reference schedule is relatively close, but overall the model-based schedule outperforms the reference schedule in both performance metrics. Although the average increase of wafer throughput over all tools is only slightly more than 1.6%, economically it is still a significant improvement, because these tools under investigation are critical tools in the fab. For example, assuming a fab's throughput is 5,000 wafers per week and the average price for finished wafers is \$15,000 per wafer, then the 1.6% improvement in throughput would result in a revenue increase of up to \$1.2 million per week, or over \$60 million per year.

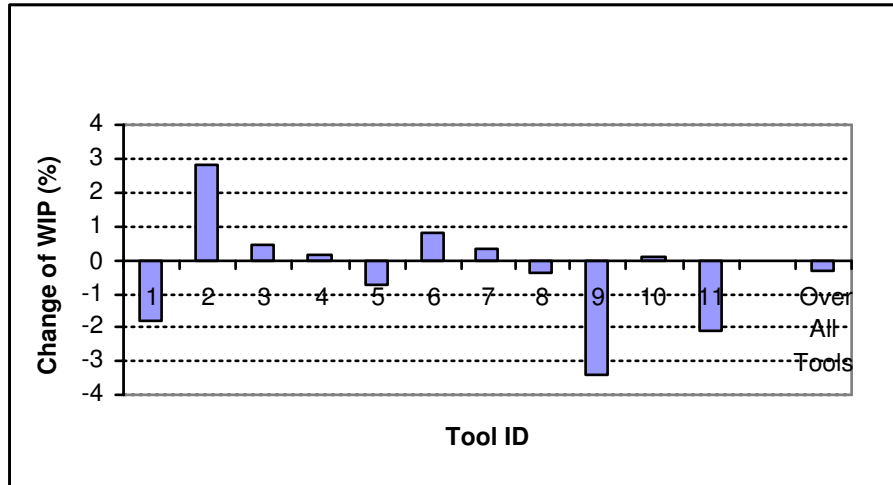


Figure 5.3: Simulation result for WIP changes (in percentages) under the model-based schedule over those under the reference schedule.

We surmise that the close performance between the model-based performance and the reference schedule can probably be attributed to the following factors. First, fab engineers have done a good job in PM scheduling on the basis of their rich experience in considering critical factors such as PM consolidation, and so they come up with a near-optimal reference schedule, especially for the most critical tools. Second, the benefits of model-based schedule have not been revealed fully through the ASAP simulation, because of the simplified modeling structure of cluster tools as well as the fab model in ASAP. For instance, the relation between entire tool throughput and chamber statuses cannot be modeled in ASAP as precisely as in a “configuration matrix”.



## 5.5 Conclusions

We develop a mixed integer programming model for optimal PM scheduling in fabs. These solutions are applicable for all tool groups in a fab, but higher impact is to be expected when purely ad hoc scheduling may be too complex to handle, e.g., for cluster tools. In general, given their optimization base, our solutions can be a significant aid for (human) decision makers to rule out errors and oversights.

The chapter has focused on optimal scheduling model, and a mixed integer programming model has been developed. Our efforts have concentrated on the problem of PM scheduling for a module composed of a group of highly integrated cluster tools. The most recognized and implemented PM time window policies are assumed in the model. The objective is to determine the exact time to start each PM task within its associated time window. Interdependence among PM tasks, in terms of tool availability (or throughput) changes, is characterized by availability functions, which can be expressed as a “configuration matrix”. The MIP model considers resource requirements over the tool group, as well as projected WIP data for each tool, with the objective of maximizing overall tool group availability while reducing WIP costs. By introducing new decision variables, non-linear functions appearing in the general MIP model can be transformed into linear functions, resulting in a model easily solvable using any commercial LP/IP software package. A simulation case study using real fab data shows that our model-based PM scheduling solution is very promising.

## Chapter 6

# Conclusions and Future Work

This thesis studies the problem of optimal preventive maintenance for unreliable queueing and production-inventory systems. Our research was motivated by the challenging problems of preventive maintenance planning and scheduling in semiconductor manufacturing.

The main contents of the thesis are roughly composed of two parts. The first part studies structural properties of optimal PM and joint PM/production control policies under different problems settings. The principal technique employed are (Semi-)Markov decision processes and dynamic programming. The second part studies the optimal scheduling problem of multiple PM tasks on a group of tools, and mathematical programming is employed.

In Chapter 3, our first result is Theorem 3.1 on the optimality of a deterministic PM policy for a simple case in which no queueing is considered. Our second result is the study of optimal PM policies for an M/G/1 queueing system with an unreliable server. Under some conditions, we show in Theorem 3.2 that for fixed queue length, the optimal PM policy exhibits a control-limit structure such that if and only if the machine's age is greater than the control limit, then

it is optimal to do PM. More properties of the optimal cost function and corresponding optimal policies are established for a simplified discrete-time queueing model with simple cost structure and stochastic processes.

In Chapter 4, we have obtained several results. We first study an unreliable production-inventory system that experiences *time-dependent* failures. We prove again the control-limit type structure of the optimal joint policy under some conditions as stated in Theorem 4.1. We also prove the intuitive result in Proposition 4.1 that when the production system has backlogged demand, if the optimal action is not to do PM, then the production should make the inventory non-negative as soon as possible. Another intuitive result proved in Theorem 4.3 states that when the production-inventory system is at a high inventory level, then it is optimal to not produce at all. For systems experiencing *operation-dependent* failures, we show in Theorem 4.4 that the control-limit type structure holds for the optimal joint policy under relaxed conditions. Another interesting result (Theorem 4.5) regarding the optimal joint policy, is that when the system has high enough inventory, the optimal action is to just stay idle, no matter how deteriorated the system is.

In Chapter 5, we solve an optimal PM scheduling problem for cluster tools in semiconductor manufacturing fabs. A mixed-integer programming model is developed. After an appropriate transformation, the model can be solved using any commercial LP/IP software. Results of a simulation study comparing the performance of the model-based PM schedule with that of a baseline reference schedule are presented to illustrate the usefulness of our solutions.

## 6.1 Future Work

In characterizing optimal PM policies, we have shown that the control-limit type structure always holds with respect to system deterioration degree, under some reasonable conditions, e.g., IFR, and/or PM costs lower than CM costs. However, we have not been able to show analytically the same structure holds in the other dimension, i.e., system buffer/inventory level, although numerical studies indicate such structure exists. Thus one obvious possible direction of future work is to establish some sufficient conditions under which the optimal PM policies have such monotonic structure with respect to system buffer level.

The MDP/SMDP formulation of the optimization problems allows us to derive optimal policies and analyze structures they might have, and provides us with insights. However, for practical problems with large state space, the technique is not computationally efficient, due to the well-known “curse of dimensionality”. It is thus of great importance to come up with some heuristics policies that have simpler structures. Like the  $(n, N, k)$  policy proposed in [17] and the “double-threshold” policy proposed in [31], such sub-optimal heuristics policies can provide fast performance evaluation and will have significant use in real situations.

Many opportunities exist for incorporation of PM models into other areas. One interesting problem that has not been addressed in depth is the consideration of PM policies in capacity planning. In current capacity planning, the issue of PM is not considered at all, or the projected capacity spent on maintenance is often too conservative. With the incorporation of the PM planning models and derived optimal policies, more accurate estimation of the capacity used for maintenance can be achieved. As a result, the capital cost on capacity expansion

could be possibly reduced.

Regarding future work in the scheduling model, there are several directions in which the model can be extended. One direction is to incorporate statistical process control (SPC) data into the model. The idea is that the PM schedule would be able to respond to possible “out of control” events, by triggering a “pull” or “push” of the planned time window of the corresponding PM. Another direction is to consider PM policies without time windows, with a penalty imposed if its starting time differs from a planned time. It would be easy to extend our developed MIP model to this case simply by removing time window constraints and adding a penalty function into the model objective.

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