ABSTRACT

Title of Dissertation: A CHEMOTAXIS FLUID MODEL
FOR MULTIPHASE TUMOR GROWTH

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Mathematical models for tumor growth can aid researchers in studying the evolution of tumors within the body and the effects of various drug treatments. Such models incorporate a variety of factors, including different cell populations, the presence of a drug and/or nutrient, and advection due to flow within the tumor.

We consider a chemotaxis fluid model for multiphase tumor growth. Our model assumes that the tumorous cells undergo chemotaxis in response to the presence of a nutrient, a consideration which is neglected in many other models. Additionally, we assume that the flow of cells through the extracellular matrix is modeled as flow through a porous medium, using either Darcy's Law or Brinkman's equation. Furthermore, we consider the model on a moving (time-dependent) domain in order to allow the shape of the tumor to evolve in time. These assumptions present several challenges for the analysis of the model.

We prove that there exist weak solutions to this model. The proof of existence relies on constructing an approximating system by means of time-discretization and an Arbitrary Lagrangian Eulerian (ALE) mapping. We then prove that the solutions to these approximating schemes converge to a solution to the original problem.

We also construct a convergent finite element scheme for this model. In the case of Darcy's Law, such a scheme can be constructed on either a fixed domain or a moving, polygonal domain, while for Brinkman's equation we focus only on the case of a fixed, polygonal domain. This numerical scheme can be used to simulate the evolution of a tumor under various assumptions on parameters, and the results of various numerical experiments are included here. These results illustrate the impact of the chemotactic effects, the moving domain, and different parameter choices.

A CHEMOTAXIS FLUID MODEL FOR MULTIPHASE TUMOR GROWTH

by

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Chapter 1: Introduction

Mathematical models for the evolution and growth of tumors describe the dynamics and interplay between tumor cells, drugs, nutrients, and the tumor medium. In conjunction with biological experiments these models provide crucial insights into the progression of cancerous tumors within the body. Models can help researchers to develop more effective drug treatments and to detect cancers at earlier stages.

Tumors change in both size and shape as they evolve. Malignant tumors are characterized by aggressive growth, in which the size of the tumor grows over time, while benign tumors either remain the same size or shrink. In either case, the domain of the tumor will not remain fixed in time. The time-dependence of the domain provides additional challenges in models which are spatially dependent.

The evolution of cells within the tumor is dependent on many factors, including the presence of drugs and nutrients. Typically, the presence of a drug, such as chemotherapy drugs, will inhibit the growth of active cells while nutrients, such as oxygen, will spur the growth and proliferation of active cells. Furthermore, tumor cells are known to undergo chemotaxis in response to the presence of certain chemicals (either drugs or nutrients).

Mathematical models are particularly useful when they can be used to simulate the growth of a tumor. Since many of the models do not have analytical solutions, it is necessary to develop numerical schemes which approximate the original model. For models based on differential equations the system is often discretized by means of finite difference or finite element schemes. Many of the aspects of the models which are mathematically interesting (in particular the nonlinearity of the models and the time-dependence of the domain) also make deriving a numerical scheme challenging. Furthermore, in order for numerical schemes to be useful in a practical sense the solutions to the numerical schemes must converge to a solution to the original model.

1.1 Background

There is a large body of work related to mathematical models for tumor growth [1]. Many different types of models have been proposed, but we will focus on models which consist of systems of differential equations. In their most basic form these models consist of ODEs (ordinary differential equations) which model the time-evolution of the tumor, ignoring any spatial dependence of the tumor [43]. Such models avoid the problem of solving the system on a moving domain but are less sophisticated since they cannot uncover any spatial inhomogeneities of the tumor.

We focus on PDE (partial differential equation) models which incorporate both temporal and spatial dependence. PDE models are more physically accurate, but the analysis of these models is also more complicated due to the time-dependence of the domain of the tumor, $\Omega(t)$, and the inherent difficulties of solving a (potentially nonlinear) system of PDEs. In some models the boundary of the tumor is taken to be a free boundary, in which case the motion of the domain is solved as part of the system ([8], [7], [24], [54]). The motion of the domain can also be prescribed a priori ([14], [15], [17]), or the problem can be solved on a larger, fixed reference domain which contains the moving domain ([51], [52]).

Many models also consider several different populations of cells within the tumor. Typically, a tumor could contain proliferating (or active) cells, quiescent (or dormant) cells, and dead cells. Some models assume that the tumor consists of a single cell population [19], while others consider the case of multiple phases of cells ([10], [20], [24], [26]). Furthermore, models differ in how the populations of cells interact. In some models, often called mixed models, the populations mix throughout the tumor ([10], [20], [26]), while in segregated models each population belongs to a distinct region, separated by surfaces which are typically allowed to move freely throughout the tumor (i.e. a free surface) [24].

In mixed models for tumor growth, phase transitions occur between populations in response to the presence of a nutrient or drug. The birth and death of new cells generates a flow field within the tumor, which is often modeled as a flow through a porous medium [24]. The transport effect from this flow field causes nonlinearities in the governing system.

Consideration of the effects of nutrients or drugs within the tumor are also important in many models. The presence of a nutrient aids in the proliferation of cancerous cells, while many drugs (such as chemotherapy drugs) inhibit the growth of the cells or decrease the concentration of nutrients. However, the body of research examining the effects of chemotaxis within a tumor is relatively small. Experimentation shows that chemotaxis is a real phenomenon in cancerous tumors [42], so it is important to be able to incorporate this effect in models.

1.2 A Chemotaxis-Fluid Model for Tumor Evolution

We assume that the tumor domain $\Omega(t) \subset \mathbb{R}^n$, $n = 2, 3$ is time-dependent. There are three types of cells present in the tumor region: *proliferating cells* with density n_p, *quiescent cells* with density n_q and *dead cells* with density n_l. The evolution of these cells densities is governed by a general transport equation with nonlinear diffusion:

$$\begin{cases} \partial_t n_p + \operatorname{div}(n_p \boldsymbol{v}) = d\Delta n_p - \operatorname{div}(n_p \psi_p(c)\nabla c) + G_p(c,u,n), \\ \partial_t n_q + \operatorname{div}(n_q \boldsymbol{v}) = d\Delta n_q - \operatorname{div}(n_q \psi_q(c)\nabla c) + G_q(c,u,n), \\ \partial_t n_l + \operatorname{div}(n_l \boldsymbol{v}) = d\Delta n_l + G_l(c,u,n). \end{cases} \tag{1.1}$$

Here $(n_p, n_q, n_l) = (n_p(t,x), n_q(x,t), n_l(x,t)) : \mathbb{R}^+ \times \Omega(t) \to \mathbb{R}^+$ denote the density of cells, $\boldsymbol{v} = \boldsymbol{v}(t,x) : \mathbb{R}^+ \times \Omega(t) \to \mathbb{R}^n$ the velocity field of the tumor, $c = c(t,x) : \mathbb{R}^+ \times \Omega(t) \to \mathbb{R}^+$ the concentration of the chemical attractant or nutrient, and $u = u(t,x) : \mathbb{R}^+ \times \Omega(t) \to \mathbb{R}^+$ the concentration of the drug. The functions $\psi_p(c), \psi_q(c)$ denote the chemotactic sensitivity of the proliferating and quiescent cells, respectively, while $d > 0$ is a diffusion parameter.

We note that the dead cells do not undergo chemotaxis, and thus there is no $\nabla \cdot (n_l \psi_l(c) \nabla c)$ term. The terms G_p, G_q, G_l represent the growth/loss of cells in each state. Following [25], we consider the growth terms to be of the form

$$G_p = (K_B(c) - K_Q(c) - K_A(c) - K_R(c,n) - K_1(u))n_p + K_P(c)n_q,$$

$$G_q = K_Q(c)n_p - (K_P(c) + K_D(c) + K_R(c,n) + K_2(u))n_q,$$

$$G_l = (K_A(c) + K_1(u))n_p + (K_D(c) + K_2(u))n_q - K_R(c,n)n_l.$$

Here, K_B represents the rate of birth of new proliferating cells, K_Q, K_P the rate at which proliferating cells transition to quiescent cells and vice versa, K_A, K_D the rate at which proliferating and quiescent cells die off due to apoptosis and lack of nutrients, respectively, and K_R the rate of removal of cells. The terms K_1, K_2 represent the rate at which proliferating and quiescent cells, respectively, die off due to the presence of the drug.

We assume that, for some critical nutrient density $\overline{c}$, the following hold:

$$\begin{cases} K_B(c), K_P(c) \geq 0 \text{ and increasing} & c \geq 0, \\ K_B(c), K_P(c) = 0 & c \leq 0, \\ K_Q(c), K_A(c), K_D(c) \geq 0 \text{ and decreasing} & c \leq \overline{c}, \\ K_Q(c), K_A(c), K_D(c) = 0 & c \geq \overline{c}. \end{cases}$$

For the sake of simplicity, we also assume that, for each $y \neq R$, K_y is linear in c when $K_y \geq 0$. Additionally, we assume that $K_i(u) = 0$ for $u \leq 0$ and $K_i(u) \geq 0$

is linear in u for $u \geq 0$. We note that

$$\sum_k G_k(c,u,n) = K_B(c)n_p - K_R(c,n)\sum_k n_k.$$

We assume that the total density of the cells, denoted by n_T, is constant and given by

$$n_T = n_p + n_q + n_l = constant.$$

Additionally, we define

$$K_R(c,n) = n_T^{-1}\int_{\Omega(t)} K_B(c)n_p. \tag{1.2}$$

This assumption guarantees that the total cell population will remain constant within the tumor, as the rate of removal of cells is proportional to the rate of birth of new cells. Then, we have

$$\sum_k G_k(c,u,n) = K_B(c)n_p - n_T K_R(c,n),$$

and we note that

$$\int_{\Omega(t)}\sum_k G_k(c,u,n) = \int_{\Omega(t)}(K_B(c)n_p - n_T K_R(c,n)) = 0$$

due to the definition of K_R. The constant density constraint can also be written in the form of a divergence constraint,

$$n_T \operatorname{div} \boldsymbol{v} = -\operatorname{div}((\psi_p(c)n_p + \psi_q(c)n_q)\nabla c) + K_B(c)n_p - n_T K_R(c,n), \tag{1.3}$$

obtained by summing the equations in (1.1).

The concentration of the chemical reactant is governed by a nonlinear equation that incorporates the diffusion mechanism as well as the consumption of the chemical by the cells and the destruction of the chemical by the drug:

$$\partial_t c = \mu \Delta c - n_p f_p(c) - n_q f_q(c) - K(u)c. \tag{1.4}$$

Here $f_p(c), f_q(c)$ denotes the consumption rate of the nutrient by the proliferating and quiescent cells, respectively, while $K(u)$ denotes the rate at which the nutrient is destroyed by the drug and $\mu > 0$ is a diffusion parameter. Note that the dead cells do not consume the nutrient.

The concentration of the drug obeys a similar governing equation:

$$\partial_t u = \alpha \Delta u - n_p g_p(u) - n_q g_q(u),$$

where α is a diffusion parameter and $g_p(u), g_q(u)$ are the consumption rates of the drug by the proliferating and quiescent cells, respectively.

Due to proliferation and removal of cells there is continuous motion of cells within the tumor; this movement is represented by the velocity field $\boldsymbol{v}$ given by Darcy's Law for flow through a porous medium,

$$\nabla p = -\frac{\lambda}{K}\boldsymbol{v} \tag{1.5}$$

with $p = p(t,x) : \mathbb{R}^+ \times \Omega(t) \to \mathbb{R}$ denoting the pressure of the tumor, λ, K positive constants describing the viscous like properties of tumor cells and the permeability, respectively. We also separately consider Brinkman's equation for the velocity field. Brinkman's equation is a model for flow through a porous medium, which incorporates diffusion and gravitational effects, given by

$$\nabla p = -\frac{\lambda}{K}\boldsymbol{v} + \mu\Delta\boldsymbol{v} - n_T\nabla\phi. \tag{1.6}$$

The coefficient μ represents the diffusion effects, while $\nabla\phi$ is a gravitational force.

Combining (1.6) with (1.3) yields the inhomogeneous Stokes' equations

$$\begin{cases} \nabla p = -\frac{\lambda}{K}\boldsymbol{v} + \mu\Delta\boldsymbol{v} - n_T\nabla\phi, \\ \operatorname{div}\boldsymbol{v} = -n_T^{-1}\operatorname{div}((n_p+n_q)\psi(c)\nabla c) + n_T^{-1}K_B(c)n_p - K_R(c,n). \end{cases} \tag{1.7}$$

In the case of Darcy's Law, (1.5) and (1.3) reduce to the elliptic equation

$$\Delta p = \frac{\lambda}{Kn_T}\left(\operatorname{div}((n_p+n_q)\psi(c)\nabla c) - K_B(c)n_p + n_TK_R(c,n)\right). \tag{1.8}$$

The model is closed by giving boundary conditions on the tumor boundary $\Gamma(t) = \partial\Omega(t)$ and initial conditions on the initial domain $\Omega(0)$. In the case of Darcy's Law, we assume that the boundary $\Gamma(t)$ is impermeable, meaning

$$\boldsymbol{v}(t,\cdot)\cdot\boldsymbol{n}|_{\Gamma(t)} = \boldsymbol{v}_s(t,\cdot)\cdot\boldsymbol{n}|_{\Gamma(t)},\ t\geq 0, \tag{1.9}$$

where $\boldsymbol{v}_s$ is the velocity of the boundary of the tumor $\Gamma(t)$. This translates to inhomogeneous Neumann boundary conditions for p, given by

$$\partial_{\boldsymbol{n}}p(t,\cdot)|_{\Gamma(t)} = -\frac{\lambda}{K}\boldsymbol{v}_s(t,\cdot)\cdot\boldsymbol{n},\ t>0. \tag{1.10}$$

In the case of Brinkman's equation, we impose no-slip boundary conditions of the form

$$\boldsymbol{v}(t,\cdot)|_{\Gamma(t)} = \boldsymbol{v}_s(t,\cdot)|_{\Gamma(t)},\ t\geq 0, \tag{1.11}$$

In addition, zero penetration boundary conditions for the density of cells and the density of the nutrient are imposed:

$$\partial_{\boldsymbol{n}} n_k(t,\cdot)|_{\Gamma(t)} = 0,\ \partial_{\boldsymbol{n}} c(t,\cdot)|_{\Gamma(t)} = 0,\ t > 0. \tag{1.12}$$

The drug can satisfy either Dirichlet or homogeneous Neumann boundary conditions:

$$u(t,\cdot)|_{\Gamma(t)} = u_B \text{ or } \partial_{\boldsymbol{n}} u(t,\cdot)|_{\Gamma(t)} = 0,\ t > 0.$$

Dirichlet boundary conditions would be used when the drug has a fixed supply through the boundary of the tumor, while homogeneous Neumann boundary conditions would be used when the drug cannot penetrate the boundary of the tumor.

Remark 1.2.1. We can also consider a model with four distinct cell populations: the three populations mentioned previously and a population of healthy cells with density n_h. This allows us to consider a problem where the tumor does occupy the entire domain. In this case the domain would consist of a tumorous region surrounded by healthy cells. The governing equation for n_h is similar to that for n_l:

$$\partial_t n_h + \operatorname{div}(n_h \boldsymbol{v}) = d\Delta n_h + G_h(c,n), \quad G_h(c,n) = -K_R(c,n)n_h.$$

We keep the assumption of constant total cell density, which now reads $n_p+n_q+n_l+n_h = n_T$. Thus, it follows that $\sum_k G_k(c,n) = K_B(c)n_p - n_T K_R(c,n)$ as before, so this does not alter the divergence constraint (1.3). Furthermore, if we take $n_l = n_l + n_h$, we see that n_l still satisfies $(1.1)_3$. Thus, for mathematical purposes, it is possible to consider a single population of both dead and healthy cells as neither type of cell undergoes chemotaxis.

1.3 Structure of Paper

In Chapter 2 we prove the existence of weak solutions to the models described in Section 1.2. We first introduce the ALE (Arbitrary Lagrangian Eulerian) method used to discretize the problem in time and set up a sequence of approximating schemes. We then prove that solutions to these approximating schemes converge to a weak solution to the original problem. Finally, we demonstrate the existence of solutions to the approximating schemes.

We construct a convergent finite element scheme in Chapter 3. First, we consider the case of the model with Darcy's Law on a fixed (time-independent) polygonal domain. We construct the finite element scheme and demonstrate that the scheme is both positivity-preserving and convergent. Then, we prove that a similar convergent scheme can be constructed for the same problem on a smooth domain and on a moving domain. Finally, we demonstrate that in the case of a fixed, polygonal domain a convergent finite element scheme can also be constructed for the model with Brinkman's equation.

In Chapter 4, we show the results of various numerical experiments for the finite element scheme with Darcy's Law. We consider both the fixed and moving domain problems in order to illuminate the effects of a moving domain. Additionally, we perform experiments that demonstrate the effects of various parameters, including chemotactic sensitivity, diffusivity, and drug application.

Conclusions and future work on a related free boundary problem are presented in Chapter 5.

Chapter 2: Analysis of a chemotaxis-fluid model on a moving domain

2.1 Introduction and Main Result

We examine the chemotaxis-fluid model with Darcy's Law. For the sake of simplicity, we examine the problem without drug application. However, the results described in this chapter will still hold in the case of drug application, provided $0 \leq u_0, u_B \leq u_M$ for some $u_M > 0$, and $K(u), g(u)$ are sufficiently smooth. Furthermore, we will assume that $f_p(c) = f_q(c) = f(c)$ and $\psi_p(c) = \psi_q(c) = \psi(c)$. Again, the same results will still hold in the case that f_p, f_q and/or ψ_p, ψ_q are distinct.

Without loss of generality, we assume that $\frac{K}{\lambda} = 1$. Furthermore, we define $G_{tot}(c,n) = K_B(c)n_p - n_T K_R(c,n)$, and let $Q_T = \{(x,t) : 0 \leq t \leq T,\ x \in \Omega(t)\}$ and $\Gamma_T = \{(x,t) : 0 \leq t \leq T,\ x \in \Gamma(t)\}$. This simplifies notation and allows us to write

$$
\begin{aligned}
&\partial_t n_k = d\Delta n_k - \operatorname{div}(n_k(\psi(c)\nabla c - \nabla p)) + G_k(c,n) && \text{in } Q_T,\\
&\partial_t c = \mu\Delta c - (n_p + n_q)f(c) + r_c(c_v - c) && \text{in } Q_T,\\
&\Delta p = n_T^{-1}\left(\operatorname{div}((n_p+n_q)\psi(c)\nabla c) - G_{tot}(c,n)\right) && \text{in } Q_T,\\
&(n_k, c)|_{t=0} = (n_{k,0}(x), c_0(x)) && \text{in } \Omega(0),\\
&\partial_{\boldsymbol{n}} p = -\boldsymbol{v}_s \cdot \boldsymbol{n}, \quad \partial_{\boldsymbol{n}} n_k = 0, \quad \partial_{\boldsymbol{n}} c = 0 && \text{on } \Gamma_T.
\end{aligned}
\tag{2.1}
$$

We require

$$\begin{cases} f : [0, +\infty) \to [0, +\infty) \text{ is a Lipschitz continuous function with } f(0) = 0, \\ \psi : [0, +\infty) \to [0, +\infty) \text{ is a Lipschitz continuous function.} \end{cases} \tag{2.2}$$

These assumptions are necessary to ensure that the scheme is positivity-preserving and to prove the weak convergence of approximating solutions.

Much of the work in this chapter will appear in the research article [50].

2.1.0.1 Estimates for the motion of the domain

We assume that there exists a mapping $\eta(t) : \Omega(0) \to \Omega(t)$ which defines the motion of the domain, such that $\boldsymbol{v}_s = \partial_t \eta$. Furthermore, we assume that:

1. $\eta \in W^{1,\infty}(0, T; H^1(\Omega(0))) \cap L^\infty(0, T; W^{2,\infty}(\Omega(0)))$,
2. η is bi-Lipschitz in the sense that there exist $c_L, C_L > 0$ such that

   $$c_L|\boldsymbol{x} - \boldsymbol{y}| \leq |\eta(t, \boldsymbol{x}) - \eta(t, \boldsymbol{y})| \leq C_L|\boldsymbol{x} - \boldsymbol{y}|$$

   uniformly for any $t \in [0, T]$,
3. η is a volume-preserving mapping, i.e. $\det \nabla \eta = 1$.

Several properties of the mapping η follow from the above assumptions.

1. For each $t \in [0, T]$, the function $\eta(t) : \Omega(0) \to \Omega(t)$ is invertible. We define $\eta^{-1}(t) = (\eta(t))^{-1}$ for each t.

2. The inverse mapping η^{-1} is bi-Lipschitz in the sense that

$$C_L^{-1}|\boldsymbol{x}-\boldsymbol{y}| \leq |\eta^{-1}(t,\boldsymbol{x}) - \eta^{-1}(t,\boldsymbol{y})| \leq c_L^{-1}|\boldsymbol{x}-\boldsymbol{y}|$$

   for $t \in [0,T]$ and $\boldsymbol{x},\boldsymbol{y} \in \Omega(t)$.

3. $\eta^{-1} \in W^{1,\infty}(0,T;H^1(\Omega(t))) \cap L^\infty(0,T;W^{2,\infty}(\Omega(t)))$.

4. If $\boldsymbol{x} \in \partial\Omega(0)$ then $\eta(t,\boldsymbol{x}) \in \partial\Omega(t)$.

5. The velocity of the domain satisfies $\boldsymbol{v}_s \in L^\infty(0,T;H^1(\Omega(0)))$.

6. Since η is volume preserving, the Leibniz integral rule yields

$$0 = \frac{d}{dt}\int_{\Omega(t)} dx = \int_{\partial\Omega(t)} \boldsymbol{v}_s \; ds = \int_{\Omega(t)} \operatorname{div} \boldsymbol{v}_s \; dx.$$

It is also possible to show that the constants in the Sobolev embedding theorem, trace theorem, and Poincare's inequality are independent of t for $0 \leq t \leq T$.

2.1.0.2 Existence of weak solutions

We seek weak solutions to the system (2.1).

Definition 2.1.1. We say that (n_p, n_q, n_l, c, p) is a weak solution of problem (2.1) provided that the following hold:

For $k = (p, q, l)$ and a.e. $t \in (0, T]$,

$$\begin{aligned}
&\int_0^t\int_{\Omega(s)} \Big(- n_k\Big(- \nabla p + \psi(c)\nabla c\Big) \cdot \nabla\varphi_k + d\nabla n_k \cdot \nabla\varphi_k - G_k(c, n)\varphi_k\Big) dxds \\
&\quad = \int_0^t\int_{\Omega(s)} n_k\partial_s\varphi_k dxds + \int_{\Omega(0)} n_{k0}\varphi_k(0)dx - \int_{\Omega(t)} n_k(t)\varphi_k(t)dx, \\
&\int_0^t\int_{\Omega(s)} \Big(\mu\nabla c \cdot \nabla\varphi_c + (n_p + n_q)f(c)\varphi_c + r_c(c - c_v)\varphi_c\Big) dxds \\
&\quad = \int_0^t\int_{\Omega(s)} c\partial_s\varphi_c dxds + \int_{\Omega(0)} c_0\varphi_c(0)dx - \int_{\Omega(t)} c(t)\varphi_c(t)dx, \\
&\int_0^t \int_{\Omega(s)} \nabla p \cdot \nabla\eta + \operatorname{div}(\boldsymbol{v}_s\eta) dxds \\
&\quad = n_T^{-1} \int_0^t \int_{\Omega(s)} ((n_p + n_q)\psi(c)\nabla c \cdot \nabla\eta + G_{tot}(c, n)\eta)\, dxds,
\end{aligned}$$

for any $\varphi_k, \varphi_c, \eta \in C^\infty(\overline{Q_T})$, $k \in (p, q, l)$.

The main result of this chapter follows.

Theorem 2.1.2. *Let $\Omega(t) \subset \mathbb{R}^n$, $n = 2, 3$ be a bounded domain of class $C^{1,1}$ for each $t \in [0, T]$ and assume that the initial data $(n_{k,0}, c_0)$ satisfy*

$$0 \le n_{k,0}, \quad n_{p,0} + n_{q,0} + n_{l,0} = n_T, \quad 0 \le c_0, c_v \le c_M,$$

and the assumptions in (2.2) and Section 2.1.0.1 hold. Then the problem (2.1) admits a weak solution in the sense of Definition 2.1.1.

For the sake of simplicity, we will examine the case that $n = 3$. However, the proof in the case that $n = 2$ follows in the same manner. In order to prove this theorem, we will construct a sequence of approximating problems and prove that the solutions to these approximating problems convergence to a solution to the system (2.1).

Similar work on the existence of weak solutions to a free boundary problem for polymeric fluids via convergence of approximate solutions has been conducted in [13].

2.2 Time-discrete, Regularized Approximating Schemes

2.2.1 Introduction to ALE method

We use an ALE (Arbitrary Lagrangian Eulerian) method to address the motion of the tumor domain. The general idea of the ALE method is to use a framework that is somewhere between the Lagrangian and Eulerian perspectives. In the Eulerian perspective the reference frame is fixed for all times, while the Lagrangian perspective the reference frame moves with the fluid. In an ALE approach, the reference frame is not fixed, but also is not required to move exactly with the fluid. This combines some of the advantages of both the Eulerian and Lagrangian methods while eliminating some of the disadvantages of each method.

2.2.2 Time-discrete problem

We will first discretize our problem in time. We consider the case of a uniform time-discretization, in which we split the interval $[0, T]$ into N subintervals $[t^{m-1}, t^m]$ of length Δt so that $t^m = m\Delta t$ for $0 \leq m \leq N$. Then, on each subinterval, we consider the problem on the fixed reference domain $\Omega^m = \Omega(t^m)$.

We define the mapping $A^m(t) : \Omega^m \to \Omega(t)$ by $A^m(t) = \eta(t) \circ \eta^{-1}(t^m)$. Then, given a function a defined on $\{(\tau, x) : \tau \in (t^{m-1}, t^m], x \in \Omega(\tau)\}$, we define $\tilde{a}(\tau, x) :=$

$$\begin{tikzcd}
\Omega^0 \arrow[r, bend left, "\eta^{m-1}"] \arrow[rr, bend left=60, "\eta^m"] & \Omega^{m-1} \arrow[l, bend left, "(\eta^{m-1})^{-1}"] \arrow[r, bend left, "(A^m)^{-1}"] & \Omega^m \arrow[l, bend left, "A^m"] \arrow[ll, bend left=60, "(\eta^m)^{-1}"]
\end{tikzcd}$$

Figure 2.1: Relationships between ALE mappings A^m and η^m.

$a(\tau, A^m(\tau, x))$. The function $\tilde{a}$ is then defined on $[t^{m-1}, t^m] \times \Omega(t^m)$. We see that

$$\begin{aligned}
\partial_{x_i}\tilde{a} &= (\nabla a) \circ A^m(\tau) \cdot \partial_{x_i} A^m(\tau) \\
\partial_\tau \tilde{a} &= (\partial_\tau a) \circ A^m(\tau) + (\nabla a) \circ A^m(\tau) \cdot \partial_\tau A^m(\tau) \\
&= (\partial_\tau a) \circ A^m(\tau) + (\nabla a) \circ A^m(\tau) \cdot \boldsymbol{v}_s(\tau) \circ \eta^{-1}(t^m), \\
\partial_{x_i}^2 \tilde{a} &= \partial_{x_i}\left((\nabla a) \circ A^m(\tau) \cdot \partial_{x_i} A^m(\tau)\right) \\
&= ((D^2 a) \circ A^m(\tau)) \cdot \partial_{x_i} A^m(\tau) \cdot \partial_{x_i} A^m(\tau) + (\nabla a) \circ A^m(\tau) \cdot \partial_{x_i}^2 A^m(\tau).
\end{aligned}$$

Noting that $A^m(t^m) : \Omega^m \to \Omega^m$ is the identity mapping, we see that when $\tau = t^m$,

$$\nabla \tilde{a} = \nabla a, \quad \Delta \tilde{a} = \Delta a, \quad \partial_\tau \tilde{a} = \partial_\tau a + \nabla a \cdot \boldsymbol{v}_s(t^m) \circ \eta^{-1}(t^m).$$

We use the notation $\eta^m = \eta(t^m)$ and $A^m := A^m(t^{m-1})$, $\boldsymbol{v}_s^m = \boldsymbol{v}_s(t^m) \circ (\eta^m)^{-1}$, so that $\boldsymbol{v}_s^m$ is defined on Ω^m for each m. The relationships between the mappings η^m, A^m and their inverses are visualized in Figure 2.1 and the ALE domains are visualized in Figure 2.2.

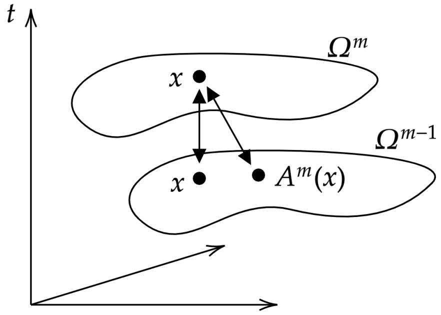

Figure 2.2: Visualization of the relationship between the ALE domains Ω^m and Ω^{m-1}.

We can then define a time-discrete approximate system by

$$\begin{aligned}
&D_t^- n_k^m + \operatorname{div}(n_k^m \boldsymbol{w}^m) = d\Delta n_k^m - n_k^m \operatorname{div} \boldsymbol{v}_s^m + G_k(c^m, n^m), && x \in \Omega^m,\\
&D_t^- c^m - \nabla c^m \cdot \boldsymbol{v}_s^m = \mu \Delta c^m - f^{m-1} + r_c(c_v^m - c^m), && x \in \Omega^m,\\
&\Delta p^m = n_T^{-1}\left(\operatorname{div}((n_p^m + n_q^m)\psi(c^m)\nabla c^m) - G_{tot}(c^m, n^m)\right), && x \in \Omega^m,\\
&\nabla p^m \cdot \boldsymbol{n} = -\boldsymbol{v}_s^m,\ \nabla n_k^m \cdot \boldsymbol{n} = 0,\ \nabla c^m \cdot \boldsymbol{n} = 0, && x \in \Gamma^m.
\end{aligned} \tag{2.3}$$

Here, we use the notation

$$\begin{aligned}
\boldsymbol{w}^m &= -\nabla p^m - \boldsymbol{v}_s^m + \psi(c^m)\nabla c^m,\\
f^{m-1} &= (f(c^{m-1})(n_p^{m-1} + n_q^{m-1})) \circ A^m,\\
D_t^- a^m &= \frac{a^m - a^{m-1} \circ A^m}{\Delta t},\\
c_v^m &= (\Delta t)^{-1} \int_{t^{m-1}}^{t^m} (c_v(t) \circ A^m(t)).
\end{aligned}$$

We will also regularize the initial cell and nutrient densities. We recall that we only assume $L^\infty(\Omega(0))$ regularity for $n_{k,0}, c_0$, so we define $\Omega_\kappa = \{x \in \Omega(0) : \operatorname{dist}(x, \Gamma(0)) > 2\kappa\}$, and let $\bar{c}_{\kappa,0}, \bar{n}_{k,\kappa,0} = c_0, n_{k,0}$ on Ω_κ. We extend $\bar{c}_{\kappa,0}, \bar{n}_{q,\kappa,0}, \bar{n}_{l,\kappa,0}$

to 0 on $\mathbb{R}^n$ and $\bar{n}_{p,\kappa,0}$ to n_T on $\mathbb{R}^n$. It follows that

$$\lim_{\kappa\to 0} \|\bar{c}_{\kappa,0} - c_0\|_{L^p(\Omega(0))}, \|\bar{n}_{k,\kappa,0} - n_{k,0}\|_{L^p(\Omega(0))} = 0 \text{ for any } 1 \leq p < \infty,$$

with

$$0 \leq \bar{n}_{k,\kappa,0}, \quad \sum_k \bar{n}_{k,\kappa,0} = n_T, \quad 0 \leq \bar{c}_{\kappa,0} \leq c_M.$$

Then, we take

$$n_{k,\kappa,0} = \bar{n}_{k,\kappa,0} * \sigma^\kappa, \;\; c_{\kappa,0} = \bar{c}_{\kappa,0} * \sigma^\kappa, \tag{2.4}$$

where $\{\sigma^\kappa\}$ is a family of standard mollifiers. Since $n_{k,\kappa,0}, c_{\kappa,0}$ are constant on $\Omega(0)\backslash\Omega_{\kappa/2}$, it follows that the regularized initial data satisfies homogeneous Neumann boundary conditions.

Remark 2.2.1. It is important to note that in order to guarantee the existence of a solution p^m to the elliptic equation with inhomogeneous Neumann boundary conditions, we must have

$$\int_{\Gamma^m} \nabla p^m \cdot \boldsymbol{n} = \int_{\Omega^m} n_T^{-1} \left(\operatorname{div}((n_p^m + n_q^m)\psi(c^m)\nabla c^m) - G_{tot}(c^m, n^m)\right). \tag{2.5}$$

The right hand side vanishes due to the homogeneous Neumann boundary conditions on c and the assumption (1.2). The boundary conditions on ∇p^m in $(2.3)_4$, along with the assumption that η is a volume-preserving mapping, imply that

$$\int_{\Gamma^m} \nabla p^m \cdot \boldsymbol{n} = \int_{\Gamma^m} -\boldsymbol{v}_s^m \cdot \boldsymbol{n} = \int_{\Omega^m} -\operatorname{div} \boldsymbol{v}_s^m = 0. \tag{2.6}$$

We are now ready to present the main result for the approximating scheme.

Theorem 2.2.2. *Under the assumptions of Theorem 2.1.2 for sufficiently small* $\kappa, \Delta t > 0$ *the system* (2.3),(2.4) *has a global weak solution* $(n_{k,\kappa,\Delta t}, c_{\kappa,\Delta t}, p_{\kappa,\Delta t})$ *satisfying*

$$0 \leq c_{\kappa,\Delta t} \leq c_M, \ 0 \leq n_{k,\kappa,\Delta t} \leq n_T, \ \sum_k n_{k,\kappa,\Delta t} = n_T,$$

where n_T, c_M *are independent of* κ*. Furthermore, the solution satisfies the bounds*

$$\|D^-_{\Delta t} c_{\kappa,\Delta t}\|_{L^2(0,T;H^{-1}(\Omega_{\Delta t}))} + \|c_{\kappa,\Delta t}\|_{L^2(0,T;H^1(\Omega_{\Delta t}))} \leq C,$$

$$\|D^-_{\Delta t} n_{k,\kappa,\Delta t}\|_{L^2(0,T;H^{-1}(\Omega_{\Delta t}))} + \|n_{k,\kappa,\Delta t}\|_{L^2(0,T;H^1(\Omega_{\Delta t}))} \leq C,$$

$$\|p_{\kappa,\Delta t}\|_{L^2(0,T;H^1(\Omega_{\Delta t}))} \leq C.$$

Here, C *is dependent on fixed quantities and is independent of* $\kappa, \Delta t$*.*

Remark 2.2.3. We will use the notation

$$\|a\|_{L^p(0,T;L^q(\Omega_t))} = \left(\int_0^T \|a(t)\|^p_{L^q(\Omega(t))}\, dt\right)^{1/p},$$

$$\|a\|_{L^p(0,T;L^q(\Omega_{\Delta t}))} = \left(\sum_{m=1}^n \Delta t \|a^m\|^p_{L^q(\Omega^m)}\right)^{1/p},$$

where $a^m = a(t^m)$, $n = T/\Delta t$. Similar definitions hold for more general Sobolev spaces. We also define

$$Q_{\Delta t} := \bigcup_{m=1}^n t^m \times \Omega^m, \text{ and } \|a\|_{L^p(Q_{\Delta t})} = \|a\|_{L^p(0,T;L^p(\Omega_{\Delta t}))}.$$

2.3 Proof of Theorem 2.1.2

The main theorem, Theorem 2.1.2, is a consequence of Theorem 2.2.2. We want to show that as $\kappa, \Delta t \to 0$, the approximate solution defined in Theorem 2.2.2 converges to a solution of the system (2.1).

We demonstrate convergence by mapping the functions $n_{k,\kappa,\Delta t}, c_{\kappa,\Delta t}, p_{\kappa,\Delta t}$ back to the initial domain $\Omega(0)$, and proving that the uniform bounds in Theorem 2.2.2 still hold. After proving convergence in the initial domain, we can then map back to the time-dependent domain.

Since our system is nonlinear, we must also prove strong convergence estimates for $n_{k,\kappa,\Delta t}, c_{\kappa,\Delta t}$, which we do by means of a semidiscrete analog of the Aubin-Lions Lemma. We replicate Theorem 2 of [9] below.

Lemma 2.3.1 (Time-Discrete Aubin-Lions Lemma [9])**.** *Let X, Y be Banach spaces and $M_+ \subset X$ be a nonnegative seminormed cone. Let either $1 \leq p < \infty$ and $r = 1$ or $p = \infty, r > 1$. Let $(u_{\Delta t}) \subset L^p(0,T; M_+ \cap Y)$ be a sequence of functions, which are constant on each interval $((m-1)\Delta t, m\Delta t] = (t^{m-1}, t^m]$, for $0 < t^m \leq T$. Assume that*

- *(i)* *M_+ is compactly embedded in X.*
- *(ii)* *If $\{w_n\} \subset X$ with $w_n \to w$ in X and $w_n \to 0$ in Y, then $w = 0$.*
- *(iii)* *The sequence $\{u_{\Delta t}\}$ is bounded in $L^p(0,T;M_+)$.*
- *(iv)* *There exists a constant $C > 0$ such that for all $\Delta t > 0$, $\|D_t^- u_{\Delta t}\|_{L^r(0,T;Y)} \leq C$.*

Then, if $p < \infty$*,* $\{u_\Delta t\}$ *is relatively compact in* $L^p(0,T;X)$ *and if* $p = \infty$*, there exists a subsequence of* $\{u_{\Delta t}\}$ *converging in* $L^q(0,T;X)$ *for all* $1 \leq q < \infty$ *to a limit function* $u \in C([0,T];X)$*.*

Remark 2.3.2. In the standard Aubin-Lions Lemma, if $p = \infty$ then $u_k \to u$ in $C([0,T];X)$, which we do not have here. Obviously, we do not have $u_{\Delta t} \in C([0,T];X)$ since $u_{\Delta t}$ is piecewise constant in time. However, this theorem also does not prove convergence in the space $L^\infty(0,T;X)$. To do this, we turn to a time-discrete version of the Arzela Ascoli Theorem (Lemma A.2.1) which guarantees that, under certain conditions on the spaces M_+, X, then we do have $u_{\Delta t} \to u$, up to a subsequence, in $L^\infty(0,T;X)$ with $u \in C(0,T;X)$. Furthermore, combining this estimate with Lemma 6.4 in [16], it follows that, if $u_{\Delta t} \to u$ in $L^\infty(0,T;X)$ and $t^m \to t$, then $u_{\Delta t}(t^m) \to u(t)$ in X. This will be useful when proving convergence of the scheme.

2.3.0.1 Convergence as $\Delta t \to 0$

We argue that, for fixed $\kappa > 0$, taking $\Delta t \to 0$ yields a weak solution to the system (2.1) with smooth initial data (2.4).

We first map the approximate solution $(n_{k,\kappa,\Delta t}, c_{\kappa,\Delta t}, p_{\kappa,\Delta t})$ from Ω^m onto Ω^0 via the inverse mapping $(\eta^m)^{-1}$ by setting

$$\tilde{a}_{\kappa,\Delta t} = a_{\kappa,\Delta t} \circ \eta^{-1}(t^m)$$

for $a = n_k, c, p$. This modified solution satisfies the weak formulation

$$\begin{aligned}
&\sum_{j=1}^{m} \Delta t \int_{\Omega(0)} \varphi_k^j \Big(D_{\Delta t}^- \tilde{n}_{k,\kappa}^j - \tilde{n}_{k,\kappa}^j (B^j \nabla) \cdot \tilde{\boldsymbol{v}}_s^j - G_k(\tilde{c}_\kappa^j, \tilde{n}_\kappa^j) \Big) \\
&\quad = \sum_{j=1}^{m} \Delta t \int_{\Omega(0)} \Big((-dB^j \nabla \tilde{n}_{k,\kappa}^j + \tilde{n}_{k,\kappa}^j (\tilde{\boldsymbol{v}}_\kappa^j + \psi(\tilde{c}_\kappa^j) B^j \nabla \tilde{c}_\kappa^j)) \cdot B^j \nabla \varphi_k^j \Big) \\
&\sum_{j=1}^{m} \Delta t \int_{\Omega(0)} \Big(\varphi_c^j \varphi_c D_{\Delta t}^- \tilde{c}_\kappa^j - \varphi_c^j B^j \nabla \tilde{c}_\kappa^j \cdot \tilde{\boldsymbol{v}}_s^j + \mu B^j \nabla \tilde{c}_\kappa^j \cdot B^j \nabla \varphi_c^j \Big) \\
&\quad = \sum_{j=1}^{m} \int_{\Omega(0)} \Big(- (\tilde{n}_{p,\kappa}^{j-1} + \tilde{n}_{q,\kappa}^{j-1}) f(\tilde{c}_\kappa^{j-1}) + r_c(\tilde{c}_v^j - \tilde{c}_\kappa^j) \Big) \varphi_c^j \qquad (2.7) \\
&\sum_{j=1}^{m} \Delta t \int_{\Omega(0)} B^j \nabla \tilde{p}_\kappa^j \cdot B^j \nabla \eta^j - (B^j \nabla) \cdot (\eta^j \tilde{\boldsymbol{v}}_s^j) \\
&\quad = n_T^{-1} \sum_{j=1}^{m} \Delta t \int_{\Omega(0)} (\tilde{n}_{p,\kappa}^j + \tilde{n}_{q,\kappa}^j) \psi(\tilde{c}_\kappa^j) B^j \nabla \tilde{c}_\kappa^j \cdot B^j \nabla \eta^j \\
&\qquad + n_T^{-1} \sum_{j=1}^{m} \Delta t \int_{\Omega(0)} (K_B(\tilde{c}_\kappa^j) \tilde{n}_{p,\kappa}^j - n_T K_R(\tilde{c}_\kappa^j, \tilde{n}_\kappa^j)) \eta^j,
\end{aligned}$$

for all $\varphi_k, \varphi_c, \eta \in C^\infty([0,T] \times \Omega(0))$ with $\varphi_k^j, \varphi_c^j, \eta^j = \varphi_k(t^j), \varphi_c(t^j), \eta(t^j)$. Here,

$$b_{ij}(t) = (\partial_{x_i} \eta_j^{-1}(t)) \circ \eta(t), \quad B(t) = (b_{ij}(t)), \quad B^m = B(t^m),$$

$$\tilde{\boldsymbol{v}}_\kappa^m = -B^m \nabla \tilde{p}_\kappa^m - \tilde{\boldsymbol{v}}_s^m.$$

We note that while we defined

$$D_{\Delta t}^- a^m = \frac{a^m - a^{m-1} \circ A^m}{\Delta t}$$

for a^m defined on Ω^m, we define

$$D_{\Delta t}^- \tilde{a}^m = \frac{\tilde{a}^m - \tilde{a}^{m-1}}{\Delta t}$$

for $\tilde{a}$ defined on Ω^0, so that this is the 'true' finite difference operator in time.

Using estimates for η^{-1}, we know that $(\tilde{n}_{k,\kappa,\Delta t}, \tilde{c}_{\kappa,\Delta t}, \tilde{p}_{\kappa,\Delta t})$ also satisfy the bounds in Theorem 2.2.2. Thus, we have

$$\tilde{c}_{\kappa,\Delta t} \rightharpoonup \tilde{c_\kappa} \text{ in } L^2(0,T;H^1(\Omega^0)) \cap L^r((0,T)\times\Omega^0),$$

$$\tilde{n}_{k,\kappa,\Delta t} \rightharpoonup \tilde{n}_{k,\kappa} \text{ in } L^2(0,T;H^1(\Omega^0)) \cap L^r((0,T)\times\Omega^0),$$

$$\tilde{p}_{\kappa,\Delta t} \rightharpoonup \tilde{p}_\kappa \text{ in } L^2(0,T;H^1(\Omega^0)),$$

for any $1 < r < \infty$. Furthermore, since $\partial_t \eta \in L^\infty(0,T;H^1(\Omega^0))$, it follows that $\tilde{\boldsymbol{v}}_{s,\Delta t}$ is uniformly bounded in $L^\infty(0,T;H^1(\Omega^0)$, and thus $\tilde{\boldsymbol{v}}_{s,\Delta t} \rightharpoonup \boldsymbol{v}_{s,\Delta t}$ in $L^r(0,T;H^1(\Omega^0)$ for all $r < \infty$.

Since the system is nonlinear, we also need to demonstrate strong convergence results for $\tilde{n}_k, \tilde{c}$.

Claim 2.3.3. *For fixed $\kappa > 0$, as $\Delta t \to 0$, $\tilde{c}_{\kappa,\Delta t} \to \tilde{c}_\kappa$ and $\tilde{n}_{k,\kappa,\Delta t} \to \tilde{n}_{k,\kappa}$ strongly in $L^r((0,T)\times\Omega^0)$ for any $1 < r < \infty$.*

Proof. We employ the semidiscrete version of the Aubin Lions Lemma given by Lemma 2.3.1. We have the uniform bounds

$$\|D^-_{\Delta t}\tilde{c}_{\kappa,\Delta t}\|_{L^2(0,T;H^{-1}(\Omega^0))} + \|\tilde{c}_{\kappa,\Delta t}\|_{L^2(0,T;H^1(\Omega^0))} \leq C,$$

$$\|D^-_{\Delta t}\tilde{n}_{k,\kappa,\Delta t}\|_{L^2(0,T;H^{-1}(\Omega^0))} + \|\tilde{n}_{k,\kappa\Delta t}\|_{L^2(0,T;H^1(\Omega^0))} \leq C.$$

Following the notation in Lemma 2.3.1, we take

$$X = L^p(\Omega^0),\ Y = H^{-1}(\Omega^0),\ M_+ = H^1(\Omega^0),$$

where $1 \leq p < 6$ so that $M_+ = H^1(\Omega^0)$ is compactly embedded in $X = L^p(\Omega^0)$. Thus, we have the estimate

$$\|D^-_{\Delta t}\tilde{c}_{\kappa,\Delta t}\|_{L^2(0,T;Y)} + \|\tilde{c}_{\kappa,\Delta t}\|_{L^2(0,T;M_+)} \leq C,$$

$$\|D^-_{\Delta t}\tilde{n}_{k,\kappa,\Delta t}\|_{L^2(0,T;Y)} + \|\tilde{n}_{k,\kappa,\Delta t}\|_{L^2(0,T;M_+)} \leq C.$$

We also note that if we have a sequence w_n which converges strongly to w in $X = L^p(\Omega^0)$ such that $w_n \to 0$ in $H^{-1}(\Omega^0)$, then $w = 0$, because $w_n \to w$ in $H^{-1}(\Omega^0)$. Therefore, Lemma 2.3.1 gives the strong convergence result

$$\tilde{c}_{\kappa,\Delta t} \to \tilde{c}_\kappa,\ \ \tilde{n}_{k,\kappa,\Delta t} \to \tilde{n}_{k,\kappa} \text{ strongly in } L^2(0,T;X) = L^2(0,T;L^p(\Omega^0)).$$

The proof of Claim 2.3.3 follows by noting that $0 \leq \tilde{c}_{\kappa,\Delta t} \leq c_M$ and $0 \leq n_{k,\kappa,\Delta t} \leq n_T$. ☐

Claim 2.3.4. *The limit solution $(n_{k,\kappa}, c_\kappa, p_\kappa)$ is a solution to the system (2.1) with initial data (2.4).*

Proof. We note that, by the properties of the mapping η and the inverse mapping η^{-1}, it follows that $B_{\Delta t} \to B$ in $C([0,T] \times \Omega^0)$. Furthermore, summation by parts implies that, for piecewise constant in time ϕ,

$$\sum_{j=1}^{m} \Delta t \int_{\Omega(0)} \phi^j D^-_{\Delta t}\tilde{a}^j = \int_{\Omega(0)} \phi(t^j)\tilde{a}(t^j) - \int_{\Omega(0)} \phi(0)\tilde{a}(0) - \sum_{j=0}^{m-1} \int_{\Omega(0)} D^+_{\Delta t}\phi^j\ \tilde{a}^j,$$

and, for any $\phi \in C^\infty([0,T] \times \Omega)$, if $\phi_{\Delta t}$ is the piecewise constant approximation of ϕ, then $D^+_{\Delta t}\phi_{\Delta t} \to \partial_t \phi$ in $C([0,T] \times \Omega(0))$. Additionally, we can use Lemma A.2.1

to note that

$$\tilde{c}_{\kappa,\Delta t}(t) \rightharpoonup \tilde{n}_\kappa(t),\ \tilde{n}_{k,\kappa,\Delta t}(t) \rightharpoonup \tilde{n}_{k,\kappa}(t) \text{ in } L^2(\Omega(0)) \text{ for a.e. } t \in [0,T].$$

Furthermore, for any $t \in [0,T]$ and any sequence $(\Delta t)_k \to 0$, there exists a sequence m_k such that $t^{m_k}_{(\Delta t)_k} \to t$, where $t^{m_k}_{(\Delta t)_k} = m_k(\Delta t)_k$. We also note that

$$\sum_{j=1}^{m} \Delta t \int_{\Omega(0)} \tilde{c}_v^j \varphi_c^j = \sum_{j=1}^{m} \int_{\Omega(0)} \int_{t^{j-1}}^{t^j} \tilde{c}_v(t)\varphi_c^j = \int_0^T \int_{\Omega(0)} \tilde{c}_v(t)\varphi_{c,\Delta t} \to \int_0^T \int_{\Omega(0)} \tilde{c}_v(t)\varphi_c.$$

Thus, it must be that the limit $(\tilde{n}_{k,\kappa}, \tilde{c}_\kappa, \tilde{p}_\kappa)$ of the modified solutions satisfies the weak formulation

$$\begin{aligned}
&\int_0^t \int_{\Omega(0)} \Big(-\partial_t\varphi_k \tilde{n}_{k,\kappa} - \varphi_k \tilde{n}_{k,\kappa}(B\nabla)\cdot \tilde{\boldsymbol{v}}_s - G_k(\tilde{c}_\kappa, \tilde{n}_\kappa)\varphi_k\Big) + \int_{\Omega(0)} \varphi_k(t)\tilde{n}_{k,\kappa}(t) \\
&\quad = -\int_0^t \int_{\Omega(0)} \Big((dB\nabla \tilde{n}_{k,\kappa} - \tilde{n}_{k,\kappa}(\tilde{\boldsymbol{v}}_\kappa + \psi(\tilde{c}_\kappa)B\nabla\tilde{c}_\kappa))\cdot B\nabla\varphi_k\Big) + \int_{\Omega(0)} \varphi_k(0)\tilde{n}_{k,\kappa}(0) \\
&\int_0^t \int_{\Omega(0)} \Big(-\partial_t\varphi_c\tilde{c}_\kappa - \varphi_c B\nabla\tilde{c}_\kappa \cdot \tilde{\boldsymbol{v}}_s + \mu B\nabla\tilde{c}_\kappa\cdot B\nabla\varphi_c\Big) + \int_{\Omega(0)} \varphi_c(t)\tilde{c}_\kappa(t) \\
&\quad = -\int_0^t \int_{\Omega(0)} \Big((\tilde{n}_{p,\kappa} + \tilde{n}_{q,\kappa})f(\tilde{c}_\kappa) - r_c(c_v - c_\kappa)\Big)\varphi_c + \int_{\Omega(0)} \varphi_c(0)\tilde{c}_\kappa(0) \\
&\int_0^t \int_{\Omega(0)} B\nabla\tilde{p}_\kappa \cdot B\nabla\eta dx - B : \nabla(\eta\tilde{\boldsymbol{v}}_s)\ dx \\
&\quad = n_T^{-1} \int_0^t \int_{\Omega(0)} (\tilde{n}_{p,\kappa}\tilde{n}_{q,\kappa})\psi(\tilde{c}_\kappa)B\nabla\tilde{c}_\kappa\cdot B\nabla\eta\ dx \\
&\qquad + n_T^{-1} \int_0^t \int_{\Omega(0)} (K_B(\tilde{c}_\kappa)\tilde{n}_{p,\kappa} - n_T K_R(\tilde{c}_\kappa, \tilde{n}_\kappa))\eta dx,
\end{aligned} \tag{2.8}$$

Here, $\tilde{\boldsymbol{v}}_\kappa = -B\nabla\tilde{p}_\kappa - \tilde{\boldsymbol{v}}_s$ By mapping these quantities back into $\Omega(t)$ from $\Omega(0)$, so that

$$(n_{k,\kappa}, n_\kappa, p_\kappa) = (\tilde{n}_{k,\kappa}, \tilde{c}_\kappa, \tilde{p}_\kappa)\circ\eta^{-1}(t)$$

we see that $(n_{k,\kappa}, n_\kappa, p_\kappa)$ satisfies the system (2.1) with initial data (2.4). One important note here is that

$$\partial_t \tilde{a} - (B(t)\nabla \tilde{a}) \cdot \boldsymbol{v}_s = (\partial_t a) \circ \eta.$$

□

2.3.0.2 Convergence as $\kappa \to 0$

Finally, we want to take the mollification parameter $\kappa \to 0$ and demonstrate that the limit solution (n_k, c, p) is a weak solution to the system (2.1) with initial data $n_{k,0}, c_0$, proving Theorem 2.1.2. We again consider the modified solution $(\tilde{n}_{k,\kappa}, \tilde{c}_\kappa, \tilde{p}_\kappa)$ on the reference domain $\Omega(0)$. The bounds in Theorem 2.2.2 are independent of κ, so $(\tilde{n}_{k,\kappa}, \tilde{c}_\kappa, \tilde{p}_\kappa)$ satisfy those bounds as well. However, in this case we have bounds on the actual time derivatives of $\tilde{c}_\kappa, \tilde{n}_{k,\kappa}$ instead of bounds on the finite difference approximations:

$$\|\partial_t \tilde{c}_\kappa\|_{L^2(0,T;H^{-1}(\Omega^0))} + \|\partial_t \tilde{n}_{k,\kappa}\|_{L^2(0,T;H^{-1}(\Omega^0))} \leq C.$$

Thus, using the same estimates as in the previous subsection, we have the weak convergence results

$$\tilde{c}_\kappa \rightharpoonup \tilde{c},\ \tilde{n}_{k,\kappa} \rightharpoonup \tilde{n}_k \text{ in } L^2(0,T;H^1(\Omega^0)) \cap L^r((0,T)\times\Omega^0),$$

$$\tilde{p}_\kappa \rightharpoonup \tilde{p} \text{ in } L^2(0,T;H^1(\Omega^0)),$$

for any $1 < r < \infty$. Additionally, by the Aubin-Lions Lemma we have the strong convergence results

$$\tilde{c}_\kappa \to \tilde{c},\ \tilde{n}_{k,\kappa} \to \tilde{n}_k \text{ strongly in } L^2(0,T;L^p(\Omega^0))$$

for $1 < p < 6$. Using the uniform bounds $0 \le \tilde{n}_{k,\kappa} \le n_T$ and $0 \le \tilde{c}_\kappa \le c_M$, we have

$$\tilde{c}_\kappa \to \tilde{c},\ \tilde{n}_{k,\kappa} \to \tilde{n}_k \text{ strongly in } L^r((0,T)\times\Omega^0)$$

for any $1 < r < \infty$. Furthermore, by the Arzela-Ascoli Theorem (for example, see Lemmas 2.2 and 2.3 in [23]),

$$\tilde{c}_\kappa(t) \rightharpoonup \tilde{c}(t),\ \tilde{n}_{k,\kappa}(t) \rightharpoonup \tilde{n}_k(t) \text{ in } L^r(\Omega^0) \text{ for a.e. } t \in [0,T].$$

Thus, the limit $(\tilde{n}_k, \tilde{c}, \tilde{p})$ satisfies the weak formulation (2.8) with initial data $n_{k,0}, c_0$. Since we can map $(\tilde{n}_k, \tilde{c}, \tilde{p}) \to (n_k, c, p)$, it follows that (n_k, c, p) is a weak solution to the system (2.1). This proves Theorem 2.1.2.

2.4 Proof of Theorem 2.2.2

In this section we make a simplification of notation and drop the subscripts $\kappa, \Delta t$. We will first prove that, provided Δt is sufficiently small and c^{m-1}, n_k^{m-1} satisfy certain conditions, the update scheme (2.3) has a solution (n_k^m, c^m, p^m). We do this using a fixed point argument in Section 2.4.1. Then, in Section 2.4.2, we prove that the argument in Section 2.4.1 can be iterated over the entire time interval $[0,T]$, and the quantities (n_k^m, c^m, p^m) satisfy the uniform bounds in Theorem 2.1.2.

2.4.1 Existence of solutions to (2.3) for sufficiently small time step

At each time step we must solve the implicit scheme (2.3). We do this via the following steps:

Step 1	Solve for c^m.	Lemma 2.4.1
Step 2	Assume that n_k^m are given and solve for p^m.	Lemma 2.4.2
Step 3	Assume that p^m is given and solve for n_k^m.	Lemma 2.4.3
Step 4	Fixed point argument to close loop in Steps 2 and 3.	Lemma 2.4.4

We start by demonstrating that we can solve $(2.3)_4$ for c^m.

Lemma 2.4.1. *Suppose that $\boldsymbol{v}_s^m \in H^1(\Omega^m)$ and $0 \leq n_k^{m-1}$, $\sum_k n_k^{m-1} = n_T$, $0 \leq c^{m-1} \leq c_M$, and $c^{m-1} \in W^{2,p}(\Omega^{m-1})$. Then, for sufficiently small Δt (how small is dependent only on fixed quantities), there exists a unique solution $c^m \in W^{2,p}(\Omega^m) \cap W^{1,\infty}(\Omega^m)$, for any $3 < p < \infty$, to $(2.3)_4$ satisfying $0 \leq c^m \leq c_M$ with*

$$\|c^m\|_{H^1(\Omega^m)} \leq C,$$

$$\|c^m\|_{W^{1,\infty}(\Omega^m)} + \|c^m\|_{W^{2,p}(\Omega^m)} \leq C\left(1 + \|c^{m-1}\|_{W^{2,p}(\Omega^{m-1})}\right)$$

where C is dependent on $\|c^{m-1}\|_{H^1(\Omega^{m-1})}$.

Proof. We first determine the existence of a solution to $(2.3)_4$. Since $\boldsymbol{v}_s \in H^1(\Omega^m)$ we have, by Sobolev embedding, $\boldsymbol{v}_s \in L^6(\Omega^m)$. Then, it is clear that the bilinear form

$$\mathcal{B}_{\boldsymbol{v}_s^m,\Delta t}[u,v] = \int_{\Omega^m} (1 + r_c\Delta t)uv + \Delta t(\mu\nabla u \cdot \nabla v - v\nabla u \cdot \boldsymbol{v}_s^m)$$

is bounded in $H^1(\Omega^m)$, i.e.

$$\mathcal{B}_{\boldsymbol{v}_s^m,\Delta t}[u,v] \leq \beta \|u\|_{H^1(\Omega^m)}\|v\|_{H^1(\Omega^m)},$$

where β is a constant dependent on $\Delta t, \mu, \|\boldsymbol{v}_s^m\|_{H^1(\Omega^m)}$. Here, we have used a Sobolev embedding estimate. The rest of the proof of this lemma follows as a series of claims.

Claim 2.4.1.1. *If Δt is sufficiently small Δt, then $\mathcal{B}$ is coercive.*

Proof of claim. Taking $v = u$, we have

$$\begin{aligned}\mathcal{B}_{\boldsymbol{v}_s^m,\Delta t}[u,u] &= \int_{\Omega^m} (1 + r_c\Delta t)u^2 + \Delta t(\mu|\nabla u|^2 - u\nabla u \cdot \boldsymbol{v}_s^m) \\ &\geq \|u\|^2_{L^2(\Omega^m)} + \mu\Delta t\|\nabla u\|^2_{L^2(\Omega^m)} - C\Delta t\|\boldsymbol{v}_s^m\|_{L^6(\Omega^m)}\|u\|^{1/2}_{L^2(\Omega^m)}\|\nabla u\|^{3/2}_{L^2(\Omega^m)} \\ &\geq \left(1 - C(\epsilon^{-1})\Delta t\right)\|u\|^2_{L^2(\Omega^m)} + \Delta t\left(\mu - \frac{\epsilon}{2}\right)\|\nabla u\|^2_{L^2(\Omega^m)}.\end{aligned}$$

We take ϵ small enough that $\epsilon = 2(\mu - \alpha_0)$ for some $0 < \alpha_0 < \mu$. Then, for fixed ϵ determined in the previous step, we take Δt small enough that $1 - C(\epsilon^{-1})\Delta t = \alpha_1$ for some $\alpha_1 > 0$. Then, there exists $\alpha > 0$ such that $\alpha\|u\|^2_{H^1(\Omega^m)} \leq \mathcal{B}[u,u]$ for any $u \in H^1(\Omega^m)$. More specifically, $\alpha = \min\{\alpha_1, \alpha_0\Delta t\}$, and we can also write

$$\alpha_1\|u\|^2_{L^2(\Omega^m)} + \alpha_0\Delta t\|\nabla u\|^2_{L^2(\Omega^m)} \leq \mathcal{B}[u,u].$$

■

Therefore, for any $f \in L^2(\Omega^m)$, there exists a unique solution $u \in H^1(\Omega^m)$ to the problem $\mathcal{B}[u,v] = (f,v)$ for all $v \in H^1(\Omega^m)$. Due to the assumptions on $f, c^{m-1}, n_k^{m-1}, c_v$, it follows that

$$(c^{m-1} - \Delta t f(c^{m-1})(n_p^{m-1} + n_q^{m-1})) \circ A^m + r_c\Delta t c_v^m \in L^\infty(\Omega^m).$$

We have the estimate

$$\alpha_1 \|c^m\|^2_{L^2(\Omega^m)} + \alpha_0 \Delta t \|\nabla c^m\|^2_{L^2(\Omega^m)} \leq C\left(1 + \Delta t\right) \|c^m\|_{L^2(\Omega^m)}.$$

Thus, assuming $\Delta t \leq 1$, we can write

$$\|c^m\|^2_{L^2(\Omega^m)} + \Delta t \|c^m\|^2_{H^1(\Omega^m)} \leq C.$$

Claim 2.4.1.2. *The solution c^m satisfies*

$$\Delta t \|D^-_{\Delta t} c^m\|_{L^2(\Omega^m)} + \Delta t \|c^m\|_{H^2(\Omega^m)} + \|c^m\|_{H^1(\Omega^m)} \leq C.$$

Proof of claim. Moving the Laplacian to the left hand side of the equation yields

$$D^-_{\Delta t} c^m - \mu \Delta c^m + r_c c^m_v = \nabla c^m \cdot \boldsymbol{v}^m_s - \left(f(c^{m-1})(n^{m-1}_p + n^{m-1}_q)\right) \circ A^m + r_c c^m_v.$$

Squaring both sides, multiplying by Δt, and integrating over Ω^m yields

$$\begin{aligned}
&\Delta t \|D^-_{\Delta t} c^m\|^2_{L^2(\Omega^m)} + \mu^2 \Delta t \|\Delta c^m\|^2_{L^2(\Omega^m)} + 2\mu(1 + r_c \Delta t) \|\nabla c^m\|^2_{L^2(\Omega^m)} + 2 r_c \|c^m\|_{L^2(\Omega^m)} \\
&\quad \leq 2C \|\nabla c^{m-1}\|_{L^2(\Omega^{m-1})} \|\nabla c^m\|_{L^2(\Omega^m)} + C \|c^m\|_{L^2(\Omega^m)} \\
&\qquad + C \Delta t \left(\|\nabla c^m\|_{L^2(\Omega^m)} \|c^m\|_{H^2(\Omega^m)} + 1\right) \\
&\quad \leq (\epsilon + C \Delta t \delta^{-1}) \|\nabla c^m\|^2_{L^2(\Omega^m)} + \epsilon \|c^m\|^2_{L^2(\Omega^m)} + \delta \Delta t \|c^m\|^2_{H^2(\Omega^m)} + c(\epsilon^{-1}),
\end{aligned}$$

where C is dependent on $\|c^{m-1}\|_{H^1(\Omega^{m-1})}$ and fixed quantities. We note that by Lemma A.1.2, we have

$$\|c^m\|_{H^2(\Omega^m)} \leq C\left(\|\Delta c^m\|_{L^2(\Omega^m)} + \|c^m\|_{H^1(\Omega^m)}\right),$$

so we have

$$\begin{aligned}
&\Delta t\|D^-_{\Delta t}c^m\|^2_{L^2(\Omega^m)} + C\Delta t\|c^m\|^2_{H^2(\Omega^m)} + 2\mu(1+r_c\Delta t)\|\nabla c^m\|^2_{L^2(\Omega^m)} + 2r_c\|c^m\|_{L^2(\Omega^m)} \\
&\quad\le (\epsilon + C\Delta t\delta^{-1})\|\nabla c^m\|^2_{L^2(\Omega^m)} + \epsilon\|c^m\|^2_{L^2(\Omega^m)} + \delta\Delta t\|c^m\|^2_{H^2(\Omega^m)} + c(\epsilon^{-1}).
\end{aligned}$$

. Taking $\delta \le \frac{C}{2}$, $\epsilon \le \frac{\mu}{2}, \frac{r_c}{2}$ and assuming $\Delta t \le \frac{\mu}{2C}$ we can write

$$\Delta t\|D^-_{\Delta t}c^m\|^2_{L^2(\Omega^m)} + C\Delta t\|c^m\|^2_{H^2(\Omega^m)} + \mu\|\nabla c^m\|^2_{L^2(\Omega^m)} + r_c\|c^m\|^2_{L^2(\Omega^m)} \le C,$$

where, again, C is dependent on $\|c^{m-1}\|_{H^1(\Omega^{m-1})}$ and fixed quantities. ■

Claim 2.4.1.3. *The solution* c^m *satisfies*

$$\|c^m\|_{W^{1,\infty}(\Omega^m)} + \|c^m\|_{W^{2,p}(\Omega^m)} \le C\left(1 + \|c^{m-1}\|_{W^{2,p}(\Omega^{m-1})}\right).$$

Proof of claim. Taking $c = c^m - c^{m-1}\circ A^m$, we find c satisfies

$$(\Delta t)^{-1}c - \mu\Delta c = \nabla c^m\cdot \boldsymbol{v}^m_s - (f(c^{m-1})(n^{m-1}_p + n^{m-1}_q))\circ A^m + \Delta(c^{m-1}\circ A^m) + r_c(c^m_v - c^m),$$

so it follows from Lemma A.1.2 that

$$\begin{aligned}
&\|c\|_{W^{2,p}(\Omega^m)} \\
&\quad\le C\|(\Delta t)^{-1}c - \mu\Delta c\|_{L^p(\Omega^m)} \\
&\quad\le C\left(1 + \|\nabla c^m\|_{L^q(\Omega^m)}\|\boldsymbol{v}_s\|_{L^6(\Omega^m)} + \|\Delta(c^{m-1}\circ A^m)\|_{L^p(\Omega^m)} + \|c\|_{W^{1,p}(\Omega^m)}\right),
\end{aligned}$$

where $\frac{1}{q} + \frac{1}{6} = \frac{1}{p}$, and thus

$$\|c^m\|_{W^{2,p}(\Omega^m)} \le C\left(1 + \|c^m\|_{W^{1,q}(\Omega^m)} + \|c^{m-1}\|_{W^{2,p}(\Omega^{m-1})}\right),$$

where C is independent of m. Additionally, since $3 < p$ it follows that $2 < q < \frac{3p}{3-p}$ and we have the interpolation estimate

$$\|c^m\|_{W^{1,q}(\Omega^m)} \leq \|c^m\|_{H^1(\Omega^m)}^{\theta} \|c^m\|_{W^{2,p}(\Omega^m)}^{1-\theta} \leq C\|c^m\|_{W^{2,p}(\Omega^m)}^{1-\theta}.$$

Since $\theta < 1$, we can then use Young's inequality to write

$$\|c^m\|_{W^{1,q}(\Omega^m)} \leq C\|c^m\|_{W^{2,p}(\Omega^m)}^{1-\theta} \leq C(\epsilon^{-1}) + \epsilon\|c^m\|_{W^{2,p}(\Omega^m)}.$$

If we take ϵ small enough that $C\epsilon \leq \frac{1}{2}$, then

$$\|c^m\|_{W^{2,p}(\Omega^m)} \leq C\left(1 + \|c^{m-1}\|_{W^{2,p}(\Omega^{m-1})}\right),$$

where C is dependent only on $\|c^{m-1}\|_{H^1(\Omega^{m-1})}$ and fixed quantities.

Furthermore, since the above holds for any $3 < p < \infty$, it follows that

$$\|c^m\|_{W^{1,\infty}(\Omega^m)} \leq C\|c^m\|_{W^{2,p}(\Omega^m)} \leq C\left(1 + \|c^{m-1}\|_{W^{2,p}(\Omega^{m-1})}\right).$$

where C is dependent on fixed quantities, along with $\|c^{m-1}\|_{H^1(\Omega^{m-1})}$. ■

Claim 2.4.1.4. *The solution c^m satisfies $0 \leq c^m \leq c_M$.*

Proof of claim. We note that

$$c^{m-1} - \Delta t f(c^{m-1})(n_p^{m-1} + n_q^{m-1}) + r_c c_v^m \geq c^{m-1} - C\Delta t\ c^{m-1} + r_c c_v^m \geq 0,$$

provided $\Delta t \leq C$, where $C = Kn_T$, with K the Lipschitz constant for f. Taking $c^m_- = \max(0, -c^m)$ as a test function yields

$$\begin{aligned}
\alpha \|c^m_+\|_{H^1(\Omega^m)} &\leq \mathcal{B}[c^m_-, c^m_-] = -\mathcal{B}[c^m, c^m_-] \\
&= -\int_{\Omega^m} (c^{m-1} - \Delta t f(c^{m-1})(n_p^{m-1} + n_q^{m-1}) + r_c(c_v - c^m))c^m_- \\
&\leq -\int_{\Omega^m} (c^m_-)^2 \leq 0.
\end{aligned}$$

Furthermore, taking $\bar{c}^m = c_M - c^m$, we see that $\bar{c}^m$ satisfies

$$D_t^- \bar{c}^m - \mu \Delta \bar{c}^m + r_c \bar{c}^m = f(c^{m-1})(n_p^{m-1} + n_q^{m-1}) + r_c(c_M - c_v^m) \geq 0.$$

Then, since the right hand side is nonnegative and $\bar{c}^{m-1} \geq 0$, it follows that $\bar{c}^m \geq 0$ by the same argument as for the positivity of c^m. ■

□

Next, we will prove that, for fixed n_k^m, the elliptic equation for p^m has a solution.

Lemma 2.4.2. *Suppose the hypotheses of Lemma 2.4.1 hold. Fix* $n_k^m \in H^1(\Omega^m)$. *Then, there exists a unique solution* $p^m \in H^1(\Omega^m)$ *to* (2.3)$_3$, *such that*

$$\|p^m\|_{H^2(\Omega^m)} \leq C \left[1 + \left(1 + \|c^{m-1}\|_{W^{2,p}(\Omega^{m-1})}\right) \|n_k^m\|_{H^1(\Omega^m)}\right],$$

where C is dependent on fixed quantities, as well as $\|c^{m-1}\|_{H^1(\Omega^{m-1})}$, for some $p > n$. Furthermore, for $n_k^m, \tilde{n}_k^m \in H^1(\Omega^m)$ we have

$$\|p^m[n_k^m] - p^m[\tilde{n}_k^m]\|_{H^2(\Omega^m)} \leq C \left(1 + \|c^{m-1}\|_{W^{2,p}(\Omega^{m-1})}\right) \|n_k^m - \tilde{n}_k^m\|_{H^1(\Omega^m)},$$

$$\|p^m[n_k^m] - p^m[\tilde{n}_k^m]\|_{H^1(\Omega^m)} \leq C \left(1 + \|c^{m-1}\|_{W^{2,p}(\Omega^{m-1})}\right) \|n_k^m - \tilde{n}_k^m\|_{L^2(\Omega^m)}.$$

Proof. We can write $\Delta p^m = f^m$, where, due to (2.6), $\int_{\Omega^m} f^m = 0$. Furthermore,

$$\|f^m\|_{L^2(\Omega^m)} \leq C \left(1 + \|c^{m-1}\|_{W^{2,p}(\Omega^{m-1})}\right) \|n_k^m\|_{H^1(\Omega^m)}.$$

It follows from Lemma A.1.3 that

$$\|p^m\|_{H^2(\Omega^m)} \leq C \left(\|\boldsymbol{v}_s\|_{H^1(\Omega^m)} + \|f^m\|_{L^2(\Omega^m)}\right),$$

which proves the first estimate.

Next, let $n_k^m, \tilde{n}_k^m \in H^1(\Omega^m)$, and define $p^m := p^m[n_k^m]$, $\tilde{p}^m := p^m[\tilde{n}_k^m]$. We see that $q^m := p^m - \tilde{p}^m$ satisfies

$$\Delta q^m = C \left(\operatorname{div}((n_p^m - \tilde{n}_p^m + n_q^m - \tilde{n}_q^m))\psi(c^m)\nabla c^m + n_T^{-1} K_B(c^m)(n_p^m - \tilde{n}_p^m)\right)$$

with homogeneous Neumann boundary conditions. Thus, it follows from Lemma A.1.3 that

$$\|q^m\|_{H^2(\Omega^m)} \leq C \left(1 + \|c^{m-1}\|_{W^{2,p}(\Omega^{m-1})}\right) \|n_k^m - \tilde{n}_k^m\|_{H^1(\Omega^m)},$$

$$\|q^m\|_{H^1(\Omega^m)} \leq C \left(1 + \|c^{m-1}\|_{W^{2,p}(\Omega^{m-1})}\right) \|n_k^m - \tilde{n}_k^m\|_{L^2(\Omega^m)},$$

where C is dependent on $\|c^{m-1}\|_{H^1(\Omega^{m-1})}$ and fixed quantities.

□

Next, we would like to close the loop and solve for n_k^m.

Lemma 2.4.3. *Suppose the hypotheses of Lemma 2.4.1 hold. Fix* $\tilde{n}_k^m \in H^1(\Omega^m)$ *and let* $\boldsymbol{w}^m := -\nabla p^m - \boldsymbol{v}_s^m + \psi(c^m)\nabla c^m$ *where* $p^m := p^m[\tilde{n}_k^m]$*. Then, provided* Δt *is small enough that*

$$\Delta t \leq C \min\left\{1, \frac{1}{\|c^{m-1}\|^4_{W^{2,p}(\Omega^{m-1})}}\right\}, \tag{2.9}$$

where C *is dependent on fixed quantities and* $\|\tilde{n}_k^m\|_{H^1(\Omega^m)}$*, there exist unique* $n_k^m \in H^1(\Omega^m)$ *solving*

$$\frac{n_k^m - n_k^{m-1}}{\Delta t} - d\Delta n_k^m + \operatorname{div}(n_k^m \boldsymbol{w}^m) + n_k^m \operatorname{div} \boldsymbol{v}_s^m = G_k(c^m, \tilde{n}^m).$$

Furthermore, if

$$\Delta t \leq C \min\left\{1, \frac{1}{\|c^{m-1}\|^\gamma_{W^{2,p}(\Omega^{m-1})}}\right\}, \tag{2.10}$$

for some $\gamma > 4$*, then*

$$\|n_k^m\|^2_{H^1(\Omega^m)} + \Delta t \|n_k^m\|^2_{H^2(\Omega^m)} \leq C$$

where C *is is dependent on fixed quantities and* $\|c^{m-1}\|_{H^1(\Omega^{m-1})}$ *and* $\|\tilde{n}_k^m\|_{H^1(\Omega^m)}$ *in a non-decreasing manner.*

Proof. In what follows, we let C denote a constant dependent only on fixed quantities, and $\tilde{C}$ a constant dependent on fixed quantities and on $\|c^{m-1}\|_{H^1(\Omega^{m-1})}$ and $\|\tilde{n}^m\|_{H^1(\Omega^m)}$. Neither C nor $\tilde{C}$ will be dependent on Δt, provided Δt is sufficiently small. Additionally, let $\hat{C} = \tilde{C}\left(1 + \|c^{m-1}\|_{W^{2,p}(\Omega^{m-1})}\right)$, where $\tilde{C}$ is a constant as defined above.

Define the bilinear form

$$\mathcal{B}[u,v] = \int_{\Omega^m} uv + \Delta t(d\nabla u \cdot \nabla v - u\boldsymbol{w}^m \cdot \nabla v + uv \operatorname{div} \boldsymbol{v}_s^m),$$

for $u, v \in H^1(\Omega^m)$. The rest of this proof follows as a series of claims.

Claim 2.4.3.1. *The operator $\mathcal{B}$ is continuous and coercive.*

Proof of claim. From Lemmas 2.4.1, 2.4.2 we have

$$\|\boldsymbol{w}^m\|_{H^1(\Omega^m)} \leq \tilde{C}\left(1 + \|c^{m-1}\|_{W^{2,p}(\Omega^{m-1})}\right) \leq \hat{C}.$$

We first note that

$$\begin{aligned}\mathcal{B}[u,v] &\leq \|u\|_{L^2}\|v\|_{L^2} + d\Delta t\|\nabla u\|_{L^2}\|\nabla v\|_{L^2} + \Delta t\|u\|_{L^6}\|\boldsymbol{w}\|_{L^3}\|\nabla v\|_{L^2} \\ &\quad + \Delta t\|u\|_{L^4}\|v\|_{L^4}\|\operatorname{div} \boldsymbol{v}_s^m\|_{L^2} \\ &\leq \hat{C}\|u\|_{H^1}\|v\|_{H^1}.\end{aligned}$$

Thus, $\mathcal{B}$ is continuous. We now want to show that $\mathcal{B}$ is also coercive. Set $u \in H^1(\Omega^m)$. We have

$$\begin{aligned}\mathcal{B}[u,u] &= \int_{\Omega^m} u^2 + \Delta t(d|\nabla u|^2 - u\boldsymbol{w}^m \cdot \nabla u + u^2 \operatorname{div} \boldsymbol{v}_s^m) \\ &\geq \|u\|_{L^2} + d\Delta t\|\nabla u\|_{L^2} - \Delta t(\|u\|_{L^3}\|\boldsymbol{w}^m\|_{L^6}\|\nabla u\|_{L^2} + \|u\|_{L^4}^2\|\operatorname{div} \boldsymbol{v}_s^m\|).\end{aligned}$$

We note that

$$-\|u\|_{L^3}\|\boldsymbol{w}^m\|_{L^6}\|\nabla u\|_{L^2} \geq -\epsilon^{-3}\hat{C}^4\|u\|_{L^2}^2 - \epsilon C\|\nabla u\|_{L^2}^2$$

by using Sobolev embedding, interpolation, and Young's inequality, and

$$-\|u\|_{L^4}^2\|\operatorname{div}\boldsymbol{v}_s^m\| \geq -C\|u\|_{L^2}^{1/2}\|u\|_{H^1}^{3/2}\|\operatorname{div}\boldsymbol{v}_s^m\|_{L^2} \geq -\epsilon^{-3}C\|u\|_{L^2}^2 - \epsilon C\|\nabla u\|_{L^2}^2.$$

Putting this all together, we have

$$\mathcal{B}[u,u] \geq \|u\|_{L^2} + d\Delta t\|\nabla u\|_{L^2} - \epsilon^{-3}\Delta t\hat{C}^4\|u\|_{L^2}^2 - \epsilon C\Delta t\|\nabla u\|_{L^2}^2. \tag{2.11}$$

Thus, if we take ϵ small enough that $d - C\epsilon = \alpha_0 > 0$ and Δt small enough that $1 - \epsilon^{-3}\Delta t\hat{C}^4 = \alpha_1 > 0$, it follows that there exists $\alpha > 0$ such that

$$\alpha\|u\|_{H^1(\Omega^m)}^2 \leq \mathcal{B}[u,u].$$

We note that $\alpha = \min(\alpha_0\Delta t, \alpha_1)$, and we can also write

$$\alpha_1\|u\|_{L^2(\Omega^m)}^2 + \Delta t\alpha_0\|\nabla u\|_{L^2(\Omega^m)}^2 \leq \mathcal{B}[u,u].$$

■

Thus, it must be that for each $f \in H^{-1}(\Omega^m)$, there exists a unique solution $u \in H^1(\Omega^m)$ to the problem $\mathcal{B}[u,v] = (f,v)$ for all $v \in H^1(\Omega^m)$. Then, noting that $n_k^{m-1} + \Delta t G_k(c^m, \tilde{n}^m) \in L^2(\Omega^m)$, it follows that there exists a unique solution $n_k^m \in H^1(\Omega^m)$.

Claim 2.4.3.2. *If Δt satisfies (2.9), the solution n_k^m satisfies*

$$\|n_k^m\|_{L^2(\Omega^m)}^2 + \Delta t\|\nabla n_k^m\|_{L^2(\Omega^m)}^2 \leq \tilde{C}.$$

Proof of claim. Recalling the estimate (2.11) and taking $u = n_k^m$ yields yielding

$$\begin{aligned}\|n_k^m\|_{L^2}^2 + d\Delta t\|\nabla n_k^m\|_{L^2}^2 &\leq \mathcal{B}[n_k^m, n_k^m] + \epsilon^{-3}\Delta t\hat{C}^4\|n_k^m\|_{L^2}^2 + \epsilon C\Delta t\|\nabla n_k^m\|_{L^2}^2 \\ &\leq \|n_k^{m-1}\|_{L^2}\|n_k^m\|_{L^2} + \Delta t\tilde{C}\|n_k^m\|_{L^2} \\ &\quad + \epsilon^{-3}\Delta t\hat{C}^4\|n_k^m\|_{L^2}^2 + \epsilon C\Delta t\|\nabla n_k^m\|_{L^2}^2.\end{aligned}$$

Taking ϵ small enough that $C\epsilon \leq \frac{d}{2}$ and applying Young's inequality yields

$$\|n_k^m\|_{L^2}^2 + d\Delta t\|\nabla n_k^m\|_{L^2}^2 \leq \Delta t\tilde{C} + \|n_k^{m-1}\|_{L^2}^2 + \Delta t\|n_k^m\|_{L^2}^2 + \Delta t\hat{C}^4\|n_k^m\|_{L^2}^2.$$

Then, if (2.9) is satisfied, it follows that

$$\|n_k^m\|_{L^2}^2 + d\Delta t\|\nabla n_k^m\|_{L^2}^2 \leq \Delta t\tilde{C} + \|n_k^{m-1}\|_{L^2}^2 \leq \tilde{C}.$$

■

Claim 2.4.3.3. *If Δt satisfies (2.10), the solution n_k^m satisfies*

$$\|n_k^m\|_{H^1(\Omega^m)}^2 + \Delta t\|n_k^m\|_{H^2(\Omega^m)}^2 \leq \tilde{C}.$$

Proof of claim. Due to the previous claim, we have $\|n_k^m\|_{L^2} \leq \tilde{C}$ and $\|n_k^m\|_{H^1} \leq \tilde{C}(\Delta t)^{-1/2}$. Using these bounds on n_k^m, we can write

$$\|\operatorname{div}(n_k^m\boldsymbol{w}^m)\|_{L^2} \leq \|\nabla n_k^m\|_{L^3}\|\boldsymbol{w}^m\|_{L^6} + \|n_k^m\|_{L^\infty}\|\boldsymbol{w}^m\|_{H^1} \leq \hat{C}\|n_k^m\|_{W_p^1}$$

for some $p > n$. Next, we take Δn as a test function, which yields

$$\begin{aligned}
&\|\nabla n_k^m\|_{L^2}^2 + d\Delta t\|\Delta n_k^m\|_{L^2}^2 \\
&\quad \leq \|\nabla n_k^{m-1}\|_{L^2}\|\nabla n_k^m\|_{L^2} + \Delta t\|\operatorname{div}(n_k^m \boldsymbol{w}^m)\|_{L^2}\|\Delta n_k^m\|_{L^2} \\
&\qquad + \Delta t\left(\|n_k^m\|_{L^\infty}\|\boldsymbol{v}_s^m\|_{H^1} + C\|\tilde{n}^m\|_{L^2}\right)\|\Delta n_k^m\|_{L^2} \\
&\quad \leq \tilde{C} + \frac{1}{2}\|\nabla n_k^m\|_{L^2} + \frac{d\Delta t}{4}\|\Delta n_k^m\|_{L^2}^2 + \Delta t\hat{C}\|n_k^m\|_{W_p^1}\|\Delta n_k^m\|_{L^2}.
\end{aligned}$$

Thus, we can write

$$\|\nabla n_k^m\|_{L^2}^2 + d\Delta t\|\Delta n_k^m\|_{L^2}^2 \leq \tilde{C} + \Delta t\hat{C}^2\|n_k^m\|_{W_p^1}^2 + \frac{d\Delta t}{2}\|\Delta n_k^m\|_{L^2}^2,$$

Due to interpolation estimates, Sobolev embedding, and Lemma A.1.2, we have

$$\|n_k^m\|_{W_p^1} \leq C\|n_k^m\|_{H^1}^{\theta}\|n_k^m\|_{H^2}^{1-\theta} \leq C\left(\|n_k^m\|_{H^1} + \|n_k^m\|_{H^1}^{\theta}\|\Delta n_k^m\|_{L^2}^{1-\theta}\right).$$

Thus,

$$\begin{aligned}
\|\nabla n_k^m\|_{L^2}^2 + d\Delta t\|n_k^m\|_{H^2}^2 &\leq \tilde{C} + \Delta t\hat{C}^2\|n_k^m\|_{W^{1,p}}^2 + C\Delta t\|n_k^m\|_{H^1}^2 \\
&\leq \tilde{C} + \Delta t\hat{C}^2\|n_k^m\|_{H^1}^{2\theta}\|n_k^m\|_{H^2}^{2(1-\theta)} + C\Delta t\|n_k^m\|_{H^1}^2 \\
&\leq \tilde{C} + \Delta t\hat{C}^{2/\theta}\|n_k^m\|_{H^1}^2 + \frac{d\Delta t}{2}\|n_k^m\|_{H^2}^2.
\end{aligned}$$

Then, if we take Δt sufficiently small that $\Delta t\hat{C}^{2/\theta} \leq \frac{1}{2}$, we have

$$\|n_k^m\|_{H^1}^2 + d\Delta t\|n_k^m\|_{H^2}^2 \leq \tilde{C}.$$

We recall that $0 < \theta < 1$ is such that $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{6}$ with $p > n$, so it follows that $\theta < \frac{1}{2}$ and thus $\frac{2}{\theta} > 4$, it is sufficient that Δt satisfies (2.10). ■

□

Now, we want to prove that the mapping from $\tilde{n}_k^m \to n_k^m$ has a unique fixed point.

Lemma 2.4.4. *Suppose the hypotheses of Lemma 2.4.1 hold and Δt is small enough that (2.10) is satisfied. Define $\mathcal{M}_{\Delta t}^m : H^1(\Omega^m) \to H^1(\Omega^m)$ by $\mathcal{M}_{\Delta t}^m[\tilde{n}_k^m] = n_k^m$, and let*

$$X_{\Delta t}^m := \left\{ n \in H^1(\Omega^m) : \|n - n^{m-1}\|_{H^1(\Omega^m)} \le 4 \max\left\{1, \|n^{m-1}\|_{H^1(\Omega^m)}\right\}\right\}.$$

Then, $\mathcal{M}_{\Delta t}^m$ has a unique fixed point on $X_{\Delta t}^m$.

Proof. We will use Schauder's fixed fixed point theorem. We know that $X_{\Delta t}^m$ is a closed, convex subset of $H^1(\Omega^m)$. Additionally, since $H^1(\Omega^m)$ is compactly embedded in $L^2(\Omega^m)$, it follows that $X_{\Delta t}^m$ is precompact in $L^2(\Omega^m)$. Since $X_{\Delta t}^m$ is closed in $L^2(\Omega^m)$, it follows that $X_{\Delta t}^m$ is a compact subset of $L^2(\Omega^m)$. Therefore, we must show that $\mathcal{M}_{\Delta t}^m$ is a continuous mapping from $X_{\Delta t}^m$ onto itself.

We define $C, \tilde{C}, \hat{C}$ in the same manner as in the proof of Lemma 2.4.3.

Claim 2.4.4.1. *For any $\tilde{n} \in X_{\Delta t}^m$, $n = \mathcal{M}_{\Delta t}^m[\tilde{n}] \in X_{\Delta t}^m$.*

Proof of claim. Defining $\boldsymbol{w}$ as in the proof of Lemma 2.4.3 we have $\|\boldsymbol{w}^m\|_{H^1(\Omega^m)} \le \hat{C}$. Additionally, due to the definition of $X_{\Delta t}^m$, we must have $\|\tilde{n}\|_{H^1(\Omega^m)} \le \tilde{C}$. Due to Lemma 2.4.3, we have $\|n\|_{H^1(\Omega^m)}^2 + \Delta t \|n\|_{H^2(\Omega^m)}^2 \le \tilde{C}$.

We see that $n - n^{m-1}$ satisfies

$$\begin{aligned} n - n^{m-1} - d\Delta t \Delta(n - n^{m-1}) &= \Delta t \left(-\operatorname{div}(n\boldsymbol{w}^m) + n \operatorname{div} \boldsymbol{v}_s^m + G_k(c, \tilde{n}) + d\Delta n^{m-1}\right) \\ &= \Delta t \left(g^m + d\Delta n^{m-1}\right). \end{aligned}$$

Furthermore, we have $\|g^m\|_{L^{3/2}} \leq \tilde{C}\|n\|_{H^1}\left(1+\|\boldsymbol{w}\|_{H^1}\right) \leq \hat{C}$ by Holder's inequality and Sobolev embedding.

Taking $n - n^{m-1}$ as a test function yields

$$\begin{aligned}
&\|n-n^{m-1}\|_{L^2}^2 + d\Delta t\|\nabla(n-n^{m-1})\|_{L^2}^2 \\
&\quad\leq \Delta t\left(\|g^m\|_{L^{3/2}}\|n-n^{m-1}\|_{L^3} + d\|\nabla n^{m-1}\|_{L^2}\|\nabla(n-n^{m-1})\|_{L^2}\right) \\
&\quad\leq \Delta t\hat{C}\|n-n^{m-1}\|_{L^3} + \frac{d\Delta t}{2}\|\nabla n^{m-1}\|_{L^2}^2 + \frac{d\Delta t}{2}\|\nabla(n-n^{m-1})\|_{L^2}^2.
\end{aligned}$$

We can then write

$$\|n-n^{m-1}\|_{L^2}^2 + \frac{d\Delta t}{2}\|\nabla(n-n^{m-1})\|_{L^2}^2 \leq \Delta t\hat{C}\|n-n^{m-1}\|_{L^3} + \frac{d\Delta t}{2}\|\nabla n^{m-1}\|_{L^2}^2.$$

By Sobolev embedding and interpolation estimates, we have

$$\Delta t\hat{C}\|n-n^{m-1}\|_{L^3} \leq \frac{1}{2}\|n-n^{m-1}\|_{L^2}^2 + (\Delta t\hat{C})^{4/3}\|\nabla(n-n^{m-1})\|_{L^2}^{2/3},$$

and

$$(\Delta t\hat{C})^{4/3}\|\nabla(n-n^{m-1})\|_{L^2}^{2/3} \leq (\Delta t)^{3/2}(\hat{C})^2 + \frac{d\Delta t}{4}\|\nabla(n-n^{m-1})\|_{L^2}^2.$$

Thus, after dividing through by $d\Delta t/4$, we have

$$\|n-n^{m-1}\|_{H^1}^2 \leq (\Delta t)^{1/2}(\hat{C})^2 + 2\|\nabla n^{m-1}\|_{L^2}^2.$$

Thus, provided Δt is small enough that (2.9) is satisfied, we have

$$\|n-n^{m-1}\|_{H^1}^2 \leq 4\max\left\{1, \|\nabla n^{m-1}\|_{L^2}^2\right\},$$

and it follows that $n \in X_{\Delta t}^m$. ■

Claim 2.4.4.2. *The operator $\mathcal{M}_{\Delta t}^{m}$ is continuous on $X_{\Delta t}^{m}$.*

Proof of claim. Suppose that $\{\tilde{n}_j\}_{j=1}^{\infty} \subset X_{\Delta t}^{m}$ is a convergent sequence in $H^1(\Omega^m)$ with $\tilde{n}_j \to \tilde{n} \in X_{\Delta t}^{m}$. We know that $\tilde{n}_j$ is uniformly bounded in $H^1(\Omega^m)$, and thus $\{\mathcal{M}[\tilde{n}_j]\}$ is uniformly bounded in $H^2(\Omega^m)$ by Lemma 2.4.3. Therefore, setting $n_j = \mathcal{M}[\tilde{n}_j]$, there exists a subsequence n_{j_i} such that $n_{j_i} \to n$ in $H^1(\Omega^m)$, and $n_{j_i} \rightharpoonup n$ weakly in $H^2(\Omega^m)$. We need to show that $n = \mathcal{M}[\tilde{n}]$.

We note that by Lemma 2.4.2, since $\tilde{n}_j$ converges strongly in $H^1(\Omega^m)$, it follows that p_j converges strongly to p in $H^2(\Omega^m)$, where $p_j = p[\tilde{n}_j]$ and $p = p[\tilde{n}]$. Then, we recall the bilinear form

$$\mathcal{B}_{\boldsymbol{w},\boldsymbol{v}_s^m,\Delta t}[u,v] = \int_{\Omega^m} uv + \Delta t\left(d\nabla u \cdot \nabla v - u\boldsymbol{w} \cdot \nabla v + uv \operatorname{div} \boldsymbol{v}_s^m\right)$$

for any $u, v \in H^1(\Omega^m)$. Taking $\boldsymbol{w}_j = -\frac{K}{\lambda}\nabla p_h - \boldsymbol{v}_s + \psi(c)\nabla c$, we know that

$$\mathcal{B}_{\boldsymbol{w}_{j_i},\boldsymbol{v}_s^m,\Delta t}[n_{j_i}, v] = (n_k^{m-1} + \Delta t G_k(c, \tilde{n}_{j_i}), v)$$

for any $v \in H^1(\Omega^m)$. We also note that

$$\mathcal{B}_{\boldsymbol{w}_{j_i},\boldsymbol{v}_s^m,\Delta t}[n_{j_i}, v] \to \mathcal{B}_{\boldsymbol{w},\boldsymbol{v}_s^m,\Delta t}[n, v],$$

since

$$n_{j_i} \to n \text{ in } H^1(\Omega^m) \cap L^p(\Omega^m) \text{ for } p < 6,$$

$$\boldsymbol{w}_{j_i} \to \boldsymbol{w} \text{ in } L^p(\Omega^m) \text{ for } p < 6.$$

Here, we have used the fact that $H^1(\Omega^m)$ is compactly embedded in $L^p(\Omega^m)$ for any $p < 6$. Additionally, we have

$$(n_k^{m-1} + \Delta t G_k(c, \tilde{n}_{j_i}), v) \to (n_k^{m-1} + \Delta t G_k(c, \tilde{n}), v),$$

due to the strong convergence results for $\tilde{n}$. This implies that $\mathcal{M}[\tilde{n}] = n$, so the mapping $\mathcal{M}$ is continuous on $X_{\Delta t}^m$. ■

□

We now want to prove the result $0 \leq n_k^m \leq n_T$ for the solution of the fixed point problem above.

Lemma 2.4.5. *Suppose the hypotheses of Lemma 2.4.1 are satisfied and* (2.10) *is satisfied. Then, the solution* n_k^m *obtained in Lemma 2.4.3 satisfies*

$$0 \leq n_k^m \leq n_T, \quad \sum_k n_k^m = n_T.$$

Proof. We recall that we have the estimate $0 \leq c^m \leq c_M$ provided $0 \leq c^{m-1} \leq c_M$. Now, we want to demonstrate that $0 \leq n_k^m$ and $\sum_k n_k^m = n_T$ provided $0 \leq n_k^{m-1}$ and $\sum_k n_k^{m-1} = n_T$. We will use a similar method as when proving the bounds on c^m.

First, we prove the positivity of n_k^m. Taking $n_{k,-}^m = \max(0, -n_k^m)$ as a test function, and recalling the estimates in Lemma 2.4.3 (which are valid since Δt is sufficiently small), we obtain

$$\begin{aligned}\alpha_1\|n_{k,-}^m\|^2_{L^2(\Omega^m)} + \alpha_0\Delta t\|n_{k,-}^m\|^2_{L^2(\Omega^m)} &\le \mathcal{B}[n_{k,-}^m, n_{k,-}^m] \\ &= -\int_{\Omega^m}\left(n_k^{m-1}n_{k,-}^m + \Delta t G_k(c^m, n^m)n_{k,-}^m\right) \\ &\le -\Delta t\int_{\Omega^m} G_k(c^m, n^m)n_{k,-}^m.\end{aligned}$$

Noting that

$$G_k(c^m, n^m) = H_k(c^m, n^m)n_k + \sum_{j\neq k} F_{k,j}(c^m)n_j^m,$$

where $F_{k,j}(c^m) \ge 0$, we can write

$$-F_{k,j}(c^m)n_j^m \le F_{k,j}(c^m)n_{j,-}^m, \quad -H_k(c^m, n^m)n_k^m n_{k,-}^m = H_k(c^m, n^m)(n_{k,-}^m)^2$$

so we have

$$\alpha_1\|n_{k,-}^m\|^2_{L^2(\Omega^m)} + \alpha_0\Delta t\|n_{k,-}^m\|^2_{L^2(\Omega^m)} \le \Delta t\int_{\Omega^m} G_k(c^m, n_-^m)n_{k,-}^m.$$

Summing over k yields

$$\alpha_1\sum_k \|n_{k,-}^m\|^2_{L^2(\Omega^m)} + \alpha_0\Delta t\sum_k \|n_{k,-}^m\|^2_{L^2(\Omega^m)} \le C\Delta t\sum_k \|n_{k,-}^m\|^2_{L^2(\Omega^m)}.$$

Thus, by taking Δt small enough that $\alpha_1 - C\Delta t > 0$ we obtain $\sum_k \|n_{k,-}^m\|^2_{H^1(\Omega^m)} \le 0$, which implies that $0 \le n_k^m$.

Next, we define $N^m : n_T - \sum_k n_k^m$, and note that N^m satisfies

$$D_t^- N^m + \operatorname{div}(N^m \boldsymbol{v}^m) = d\Delta N^m.$$

We note that the above problem must have a unique solution for sufficiently small Δt, and since $N^{m-1} \equiv 0$, it follows that $N^m \equiv 0$ is a solution. Thus, we have $\sum_k n_k^m = n_T$. $\square$

We note here that the choices for Δt in the previous lemmas require that

$$\Delta t \leq C \min \left\{ 1, \frac{1}{\|c^{m-1}\|^{\gamma}_{W^{2,p}(\Omega^{m-1})}} \right\},$$

for some $\gamma > 4$. Since C is dependent on $\|n^{m-1}\|_{H^1(\Omega^m)}$, $\|c^{m-1}\|_{H^1(\Omega^{m-1})}$ in a non-increasing manner, if we can demonstrate that

$$\|n\|_{L^\infty(0,T;H^1(\Omega_{\Delta t}))} + \|c\|_{L^\infty(0,T;H^1(\Omega_{\Delta t}))} + \|c\|_{L^q(0,T;W^{2,p}(\Omega_{\Delta t}))} \leq C,$$

where C is independent of Δt and $q > \gamma$, it follows that $(\Delta t)^{1/q}\|c^m\|_{W^{2,p}(\Omega^m)} \leq C$, so we need Δt to satisfy

$$\Delta t \leq C(\Delta t)^{-\gamma/q} \leq C(\Delta t)^{-\gamma/q} \min \left\{ 1, \frac{1}{(\Delta t)^{\gamma/q}\|c^{m-1}\|^{\gamma}_{W^{2,p}(\Omega^{m-1})}} \right\}.$$

Since $\gamma < q$, it follows that $1 - \gamma/q > 0$, so we require $(\Delta t)^{1-\gamma/q} \leq C$, which means that there is a lower bound on the step size Δt. Thus, it remains to demonstrate the required uniform bounds in order to prove that the previous steps can be iterated.

2.4.2 Regularity of the solution

Now that we have a solution for fixed Δt, we would like to prove regularity results. We recall the estimates

$$0 \leq c^m \leq c_M, \quad 0 \leq n_k^m \leq n_T, \quad \sum_k n_k^m = n_T.$$

Additionally, assume that m is such that $t^m = m\Delta t$, and $0 < t^m \leq T$. We note that, due to the mollification of the initial data, we want to prove uniform bounds independent of Δt in order to demonstrate that we can take a uniform step size Δt, as well as uniform bounds independent of κ, the mollification parameter. In the following sections we will be careful to demonstrate which bounds are independent of Δt and which are independent of κ.

2.4.2.1 A general energy estimate

We would like to prove energy estimates for a general discrete-in-time parabolic equation of the form

$$\begin{cases} D_t^- a^m = \Delta a^m + \operatorname{div}(\boldsymbol{w}^m a^m) + g^m, \quad \text{on } \Omega^m, \\ \nabla a^m \cdot \boldsymbol{n} = 0 \text{ on } \Gamma^m, \end{cases} \tag{2.12}$$

with initial data a^0.

Lemma 2.4.6. *Suppose a is a solution to (2.12) such that $0 \leq a^m \leq a_\infty$ for all $0 \leq t^m \leq T$. Furthermore, suppose $\boldsymbol{w} \in L^2(Q_{\Delta t})$, $g \in L^2(Q_{\Delta t})$, and $a^0 \in L^2(\Omega^0)$. Then, $a \in L^2(0,T;H^1(\Omega_{\Delta t}))$ with*

$$\|a\|_{L^\infty(0,T;L^2(\Omega_{\Delta t}))} + \|a\|_{L^2(0,T;H^1(\Omega_{\Delta t}))} \leq C_1.$$

Furthermore, suppose $\boldsymbol{w} \in L^\infty(0,T;L^q(\Omega_{\Delta t}))$ and $\operatorname{div} \boldsymbol{w} \in L^2(Q_{\Delta t})$ for some $q > n$ and $a^0 \in H^1(\Omega^m)$. Then, it follows that $a \in L^\infty(0,T;H^1(\Omega_{\Delta t})) \cap L^2(0,T;H^2(\Omega_{\Delta t}))$

with

$$\|a\|_{L^\infty(0,T;H^1(\Omega_{\Delta t}))} + \|a\|_{L^2(0,T;H^2(\Omega_{\Delta t}))} \le C_2,$$

where C_2 *is dependent on* C_1 *as well as* $\|a^0\|_{H^1(\Omega^0)}$, $\|\boldsymbol{w}\|_{L^\infty(0,T;L^2(\Omega_{\Delta t}))}$, *and* $\|\operatorname{div}\boldsymbol{w}\|_{L^2(Q_{\Delta t})}$.

Proof. First, take a^m as a test function. This yields

$$\begin{aligned}
&\|a^m\|^2_{L^2(\Omega^m)} + \Delta t\|\nabla a^m\|^2_{L^2(\Omega^m)} \\
&\quad\le (\|a^{m-1}\circ A^m\|_{L^2(\Omega^m)} + \Delta t\|g^m\|_{L^2(\Omega^m)})\|a^m\|_{L^2(\Omega^m)} - \Delta t\int_{\Omega^m}\nabla a^m\cdot(a^m\boldsymbol{w}^m) \\
&\quad\le \frac{1}{2}\left(\|a^{m-1}\circ A^m\|^2_{L^2(\Omega^m)} + \Delta t\|g^m\|^2_{L^2(\Omega^m)} + (1+\Delta t)\|a^m\|^2_{L^2(\Omega^m)}\right) \\
&\qquad + \frac{\Delta t}{2}\|\nabla a^m\|^2_{L^2(\Omega^m)} + \frac{\Delta t}{2}\|a^m\|^2_{L^\infty(\Omega^m)}\|\boldsymbol{w}^m\|^2_{L^2(\Omega^m)}.
\end{aligned}$$

After cancellation, we have

$$\begin{aligned}
\|a^m\|^2_{L^2(\Omega^m)} + \Delta t\|\nabla a^m\|^2_{L^2(\Omega^m)} \le \|a^{m-1}\circ A^m\|^2_{L^2(\Omega^m)} + \Delta t\|a^m\|^2_{L^2(\Omega^m)} \\
+ \Delta t\|g^m\|^2_{L^2(\Omega^m)} + \Delta t\|a^m\|^2_{L^\infty(\Omega^m)}\|\boldsymbol{w}^m\|^2_{L^2(\Omega^m)}.
\end{aligned}$$

Then, summing over m and applying a discrete Gronwall inequality yields

$$\begin{aligned}
&\|a\|^2_{L^\infty(0,T;L^2(\Omega_{\Delta t}))} + \|\nabla a\|^2_{L^2(Q_{\Delta t})} \\
&\quad\le C\left(\|a^0\|^2_{L^2(\Omega^0)} + \|g\|^2_{L^2(Q_{\Delta t})} + \|a\|^2_{L^\infty(Q_{\Delta t})}\|\boldsymbol{w}\|^2_{L^2(Q_{\Delta t})}\right) \qquad (2.13) \\
&\quad\le C_1.
\end{aligned}$$

Here, we have used the fact that $\|a^{m-1}\|_{L^2(\Omega^{m-1})} = \|a^{m-1}\circ A^m\|_{L^2(\Omega^m)}$.

Next, we take Δa^m as a test function, yielding

$$\begin{aligned}\|\nabla a^m\|^2_{L^2(\Omega^m)} + \Delta t \|\Delta a^m\|^2_{L^2(\Omega^m)} \le \frac{1}{2}&\left(\|\nabla(a^{m-1}\circ A^m)\|^2_{L^2(\Omega^m)} + \|\nabla a^m\|_{L^2(\Omega^m)}\right)\\ &+ \Delta t\left(\|g^m\|^2_{L^2(\Omega^m)} + \frac{1}{4}\|\Delta a^m\|^2_{L^2(\Omega^m)}\right)\\ &+ \frac{\Delta t}{4}\|\Delta a^m\|^2_{L^2(\Omega^m)} + \Delta t\|\operatorname{div}(a^m \boldsymbol{w}^m)\|^2_{L^2(\Omega^m)}.\end{aligned}$$

After cancellation, this yields

$$\begin{aligned}&\|\nabla a^m\|^2_{L^2(\Omega^m)} + \Delta t\|\Delta a^m\|^2_{L^2(\Omega^m)}\\ &\quad\le \|\nabla(a^{m-1}\circ A^m)\|^2_{L^2(\Omega^m)} + 2\Delta t\left(\|g^m\|^2_{L^2(\Omega^m)} + \|\operatorname{div}(a^m\boldsymbol{w}^m)\|^2_{L^2(\Omega^m)}\right).\end{aligned}$$

We then note that

$$\|\operatorname{div}(a^m\boldsymbol{w}^m)\|_{L^2(\Omega^m)} \le \|a^m\|_{L^\infty(\Omega^m)}\|\operatorname{div}\boldsymbol{w}^m\|_{L^2(\Omega^m)} + \|\nabla a^m\|_{L^{q'}(\Omega^m)}\|\boldsymbol{w}^m\|_{L^q(\Omega^m)}$$

where $\frac{1}{q}+\frac{1}{q'}=\frac{1}{2}$ and we take $2<q'<6$ so that $3<q<\infty$. Additionally, due to Sobolev embedding we have

$$\|\nabla a^m\|_{L^{q'}(\Omega^m)} \le C\|a^m\|^\theta_{H^1(\Omega^m)}\|a^m\|^{1-\theta}_{H^2(\Omega^m)},$$

where $\frac{1}{q'}=\frac{\theta}{2}+\frac{1-\theta}{6}$, and due to Lemma A.1.2 we have

$$\|a^m\|_{H^2(\Omega^m)} \le C\left(\|\Delta a^m\|_{L^2(\Omega^m)} + \|a^m\|_{H^1(\Omega^m)}\right).$$

Thus,

$$\|\nabla a^m\|_{L^{q'}(\Omega^m)} \le C\left(\|a^m\|_{H^1(\Omega^m)} + \|\Delta a^m\|^{1-\theta}_{L^2(\Omega^m)}\|a^m\|^\theta_{H^1(\Omega^m)}\right).$$

It follows that

$$\begin{aligned}
&2\Delta t\|\operatorname{div}(a^m \boldsymbol{w}^m)\|^2_{L^2(\Omega^m)}\\
&\quad\leq C\Delta t\left(\|\operatorname{div}\boldsymbol{w}^m\|^2_{L^2(\Omega^m)} + \|a^m\|^2_{H^1(\Omega^m)}\right) + \frac{\Delta t}{2}\|\Delta a^m\|^2_{L^2(\Omega^m)}.
\end{aligned}$$

Then, substituting this back into the energy inequality and cancelling like terms yields

$$\begin{aligned}
\|\nabla a^m\|^2_{L^2(\Omega^m)} + \frac{\Delta t}{2}\|\Delta a^m\|^2_{L^2(\Omega^m)} &\leq \|\nabla(a^{m-1}\circ A^m)\|^2_{L^2(\Omega^m)} + 2\Delta t\|g^m\|^2_{L^2(\Omega^m)}\\
&\quad + C\Delta t\left(\|\operatorname{div}\boldsymbol{w}^m\|^2_{L^2(\Omega^m)} + \|a^m\|^2_{H^1(\Omega^m)}\right).
\end{aligned}$$

Summing over m yields

$$\begin{aligned}
&\|\nabla a\|^2_{L^\infty(0,T;L^2(\Omega_{\Delta t}))} + \frac{1}{2}\|\Delta a\|^2_{L^2(Q_{\Delta t})}\\
&\quad\leq \|\nabla a^0\|^2_{L^2(\Omega^0)} + \sum_{k=0}^{m-1}\left(\|\nabla(a^k\circ A^{k+1})\|^2_{L^2(\Omega^{k+1})} - \|\nabla a^k\|^2_{L^2(\Omega^k)}\right)\\
&\qquad + C\left(1 + \|g\|^2_{L^2(Q_{\Delta t})} + \|a\|^2_{L^2(0,T;H^1(\Omega_{\Delta t}))}\right).
\end{aligned}$$

Then, by Lemma A.1.5, it follows that

$$\begin{aligned}
&\sum_{k=0}^{m-1}\left(\|\nabla(a^k\circ A^{k+1})\|^2_{L^2(\Omega^{k+1})} - \|\nabla a^k\|^2_{L^2(\Omega^k)}\right)\\
&\quad\leq C(\epsilon^{-1})\|\nabla a\|_{L^2(Q_{\Delta t})} + \epsilon\|a\|_{L^2(0,T;H^2(\Omega_{\Delta t}))},
\end{aligned}$$

and thus, taking ϵ small enough and using a discrete Gronwall inequality, we have

$$\begin{aligned}
&\|\nabla a\|^2_{L^\infty(0,T;L^2(\Omega_{\Delta t}))} + \|\Delta a\|^2_{L^2(Q_{\Delta t})} \\
&\quad \leq C \left(1 + \|\nabla a^0\|^2_{L^2(\Omega^0)} + \|g\|^2_{L^2(Q_{\Delta t})} + \|a\|^2_{L^2(0,T;H^1(\Omega_{\Delta t}))}\right) \\
&\quad \leq C_2,
\end{aligned}$$

where we have also used (2.13). We can again use Lemma A.1.2 to write

$$\|a\|_{L^2(0,T;H^2(\Omega_{\Delta t}))} \leq C \left(\|a\|_{L^2(0,T;H^1(\Omega_{\Delta t}))} + \|\Delta a\|_{L^2(Q_{\Delta t})}\right) \leq C_2.$$

□

Lemma 2.4.7. *Suppose that the term* $\operatorname{div}(a^m \boldsymbol{w}^m)$ *is replaced by* $\boldsymbol{w}^m \cdot \nabla a^m$ *in* (2.12). *Then, Lemma 2.4.6 still holds.*

Proof. The first estimate can be derived in a similar way by noting that in the proof of Lemma 2.4.6, taking a^m as a test function and integrating by parts led to a term of the form

$$\int_{\Omega^m} \nabla a^m \cdot \boldsymbol{w}^m a^m,$$

which we now get from taking a^m as a test function without integrating by parts.

The second estimate follows by noting that

$$\|\boldsymbol{w}^m \cdot \nabla a^m\|_{L^2(\Omega^m)} \leq \|\operatorname{div}(a^m \boldsymbol{w}^m)\|_{L^2(\Omega^m)}.$$

□

2.4.2.2 Regularity for c

We recall that we have

$$D^-_{\Delta t}c^m = \mu\Delta c^m - r_c c^m + \boldsymbol{v}^m_s \cdot \nabla c^m - f^{m-1},$$

where $f^{m-1} = f(c^{m-1})(n^{m-1}_p + n^{m-1}_q) + r_c c^m_v \in L^\infty(\Omega^m)$, independent of m and κ. Due to Lemma 2.4.7, it follows that

$$\|c\|_{L^2(0,T;H^1(\Omega_{\Delta t}))} \le C_1, \quad \|c\|_{L^\infty(0,T;H^1(\Omega_{\Delta t}))} + \|c\|_{L^2(0,T;H^2(\Omega_{\Delta t}))} \le C_2,$$

where C_1, C_2 are dependent on fixed quantities and C_2 is also dependent on $\|c^0\|_{H^1(\Omega^m))}$.

Next, we will demonstrate uniform bounds on c in the space $L^q(0,T;W^{2,p}(\Omega_{\Delta t}))$. These bounds will be dependent on κ.

Lemma 2.4.8. *Suppose* $c^0 \in W^{2,p}(\Omega^0)$ *for some* $3 < p < 6$ *with* $\nabla c^0 \cdot \boldsymbol{n} = 0$ *on* Γ^0, *and* $0 \le n^m_k \le n_T$. *Then,* $c \in L^q(0,T;W^{2,p}(\Omega_{\Delta t})) \cap L^\infty(0,T;W^{1,p}(\Omega_{\Delta t}))$ *for any* $1 < q < \infty$, *and*

$$\|c\|_{L^q(0,T;W^{2,p}(\Omega_{\Delta t}))} + \|c\|_{L^\infty(0,T;W^{1,p}(\Omega_{\Delta t}))} \le C,$$

where C *is dependent on* $\|c^0\|_{W^{2,p}(\Omega^0)}$ *and fixed quantities.*

Proof. We will map all quantities into the space Ω^0. First, we note that $\eta^{-1}(t^m) : \Omega^m \to \Omega^0$. Using the notation $\eta_m = \eta(t^m)$, $\eta^{-1}_m := \eta^{-1}(t^m)$ and $\tilde{c}^m = c \circ \eta^{-1}_m$, we use

Lemma A.1.2 to note that $\tilde{c}$ satisfies

$$(\Delta c^m) \circ \eta = A(t^m)\tilde{c}^m = \sum_{i,j} \partial_{x_i}(a_{ij}^m \partial_{x_j} \tilde{c}^m), \quad \sum_i a_{ij}^m \partial_{x_j} \tilde{c}^m \ n_i^m = 0.$$

We also have

$$\begin{aligned}(\nabla c^m \cdot \boldsymbol{v}_s^m) \circ \eta_m &= \sum_i (\boldsymbol{v}_{s,i}^m \circ \eta_m) \cdot \big(\nabla \tilde{c}^m \cdot \big(\partial_{x_i} \eta_m^{-1} \circ \eta_m\big)\big) \\ &= F(t^m) \cdot \nabla \tilde{c}^m.\end{aligned}$$

Then, $\tilde{c}^m$ satisfies the parabolic equation

$$\begin{aligned}D_{\Delta t}^{-} \tilde{c}^m &= dA(t^m)\tilde{c}^m - r_c \tilde{c}^m + r_c \tilde{c}_v + F(t^m) \cdot \nabla \tilde{c}^m - f(\tilde{c}^{m-1})(\tilde{n}_p^{m-1} + \tilde{n}_q^{m-1}) \\ &= dA(t^m)\tilde{c}^m - r_c \tilde{c}^m + g^m.\end{aligned}$$

Claim 2.4.8.1. *The solution c^m satisfies $\|c^m\|_{L^q(0,T;W^{2,p}(\Omega_{\Delta t}))} \leq C$.*

Proof of claim. For each m we define c_0^m to be the solution to

$$A(t^m)\hat{c}^m = \Delta c_0, \quad \sum_i a_{ij}^m \partial_{x_j} \hat{c}^m \ n_i = 0.$$

Due to Lemma A.1.3, we have

$$\|\hat{c}^m\|_{W^{2,p}(\Omega^0)} \leq C \|c_0\|_{W^{2,p}(\Omega^0)}$$

We see that $\hat{c}^0 = c_0$ and, for $n < m$,

$$A(t^m)(\hat{c}^m - \hat{c}^n) = (A(t^n) - A(t^m))\hat{c}^n = \sum_i \partial_{x_i}((a_{ij}^n - a_{ij}^m)\partial_{x_j} \hat{c}^n),$$

$$\sum_i a_{ij}^m \partial_{x_j}(\hat{c}^m - \hat{c}^n) \ n_i = -\sum_i a_{ij}^m \partial_{x_j} \hat{c}^n \ n_i = \sum_i (a_{ij}^n - a_{ij}^m)\partial_{x_j} \hat{c}^n \ n_i,$$

since $\hat{c}^n$ satisfies homogeneous oblique boundary conditions on Ω^0. Then, by Lemma A.1.3 and the trace theorem, we have

$$\|\hat{c}^m - \hat{c}^n\|_{W^{1,p}(\Omega^0)} \leq C\|(a_{ij}^n - a_{ij}^m)\partial_{x_j}\hat{c}^n\|_{L^p(\Omega^0)}.$$

Furthermore, for $3 < p \leq 6$, we have

$$\begin{aligned}\|\hat{c}^m - \hat{c}^n\|_{L^p(\Omega^0)} &\leq C\|\hat{c}^m - \hat{c}^n\|_{H^1(\Omega^0)} \leq C\|a_{ij}^n - a_{ij}^m\|_{L^2(\Omega^0)}\|\hat{c}^n\|_{W^{1,\infty}(\Omega^0)} \\ &\leq C\|(\nabla(\eta_i^n)^{-1} \cdot \nabla(\eta_j^n)^{-1}) \circ \eta^n - (\nabla(\eta_i^m)^{-1} \cdot \nabla(\eta_j^m)^{-1}) \circ \eta^m\|_{L^2(\Omega^0)}\|\hat{c}^n\|_{W^{2,p}(\Omega^0)},\end{aligned}$$

due to the definition of a_{ij}^m (see Lemma A.1.2). Due to the bound on η^{-1} in $W^{1,\infty}(0,T;H^1(\Omega_t)) \cap L^\infty(0,T;W^{2,\infty}(\Omega_t))$, it follows that

$$\begin{aligned}&\sum_{i,j}\|(\nabla(\eta_i^n)^{-1} \cdot \nabla(\eta_j^n)^{-1}) \circ \eta^n - (\nabla(\eta_i^m)^{-1} \cdot \nabla(\eta_j^m)^{-1}) \circ \eta^m\|_{L^2(\Omega^0)} \\ &\qquad \leq C\sum_i \|\nabla\eta^{-1}\|_{L^\infty}\|(\nabla(\eta_i^m)^{-1}) \circ \eta^n - (\nabla(\eta_i^m)^{-1}) \circ \eta^m\|_{L^2(\Omega^0)} \\ &\leq C(t^m - t^n)\|\nabla\eta^{-1}\|_{L^\infty(Q_T}\|\partial_t(\nabla\eta^{-1} \circ \eta)\|_{L^\infty(0,T;L^2(\Omega_t))} \\ &\leq C(t^m - t^n).\end{aligned}$$

Thus, it follows that

$$\|D_{\Delta t}^- \hat{c}\|_{L^\infty(0,T;L^p(\Omega^0))} \leq C\|c_0\|_{W^{2,p}(\Omega^0))}.$$

Then, we define $d^m = \tilde{c}^m - \hat{c}^m$, which satisfies homogeneous oblique boundary conditions and zero initial data, as well as the equation

$$D_{\Delta t}^- d^m - A(t^m)d^m + r_c d^m = g^m - D_{\Delta t}^- \hat{c}^m + A(t^m)\hat{c}^m - r_c\hat{c}^m = \hat{g}^m \text{ on } \Omega^0,$$

Thus, we can apply Theorem 2.1 in [37] to write

$$\begin{aligned}\|D^-_{\Delta t} d^m\|_{L^q(0,T;L^p(\Omega^0))} + \|d^m\|_{L^q(0,T;W^{2,p}(\Omega^0))} &\le C\left(\|\hat{g}^m\|_{L^q(0,T;L^p(\Omega^0))} + \|c_{\kappa,0}\|_{W^{2,p}(\Omega^0)}\right)\\ &\le C\left(\|g^m\|_{L^q(0,T;L^p(\Omega^0))} + \|c_{\kappa,0}\|_{W^{2,p}(\Omega^0)}\right)\end{aligned}$$

for any $1 < p, q < \infty$. We see that

$$\|g^m\|_{L^q(0,T;L^p(\Omega^0))} \le \|F^m\|_{L^\infty(0,T;L^6(\Omega^0))}\|\nabla\tilde{c}^m\|_{L^q(0,T;L^r(\Omega^0))} + C,$$

where $\frac{1}{r} + \frac{1}{6} \le \frac{1}{p}$. The estimates for η, η^{-1} indicate that $\|F^m\|_{L^\infty(0,T;L^6(\Omega^0))} \le C$ independent of $m, \Delta t$, and we have the interpolation estimate

$$\|\nabla\tilde{c}^m\|_{L^q(0,T;L^r(\Omega^0))} \le \|\nabla\tilde{c}^m\|^\theta_{L^q(0,T;L^2(\Omega^0))}\|\tilde{c}^m\|^{1-\theta}_{L^q(0,T;W^2_p(\Omega^0))},$$

where $\frac{1}{r} = \frac{\theta}{2} + \frac{1-\theta}{p'}$, where $p' = \infty$ if $p > n$ and $p' = \frac{3p}{3-p}$ if $p \le n$. Note that we can select r small enough that $r < p'$. Using the previous result that c is uniformly bounded in $L^\infty(0,T;H^1(\Omega_{\Delta t}))$, along with the fact that $\|\nabla\tilde{c}^m\|_{L^2(\Omega^0)} \le C\|\nabla c^m\|_{L^2(\Omega^m)}$, we obtain

$$\|\nabla\tilde{c}^m\|_{L^q(0,T;L^r(\Omega^0))} \le C\|\tilde{c}^m\|^{1-\theta}_{L^q(0,T;W^{2,p}(\Omega^0)},$$

where C depends on $\|c^0\|_{H^1(\Omega^0)}$. Thus, noting that $r < p'$, it follows that $\theta < 1$ and we can use Young's inequality to write

$$\|\nabla\tilde{c}^m\|_{L^q(0,T;L^r(\Omega^0))} \le C(q,\epsilon) + \epsilon\|\tilde{c}^m\|_{L^q(0,T;W^{2,p}(\Omega^0))}.$$

Taking ϵ sufficiently small (how small depends only on p, q), we obtain

$$\|D^-_{\Delta t}\tilde{c}\|_{L^q(0,T;L^p(\Omega^0))} + \|\tilde{c}\|_{L^q(0,T;W^{2,p}(\Omega^0))} \le C,$$

where C is dependent on $\|c^0\|_{W_p^2(\Omega^0)}$, as well as Ω^0, p, q. It follows from Lemma A.1.1 that

$$\|c^m\|_{L^q(0,T;W^{2,p}(\Omega_{\Delta t}))} \leq C.$$

■

Claim 2.4.8.2. *The solution c^m satisfies $\|c^m\|_{L^\infty(0,T;W^{1,p}(\Omega_{\Delta t}))}$.*

Proof of claim. First, we assume that $p = 2k$, where k is an integer. We use the inequality

$$a^k - b^k = \left(\sum_{i=0}^{k-1} a^{k-i-1} b^{2i}\right)(a-b),$$

and let

$$F_k^m(c) = \sum_{i=0}^{k-1} |\nabla c^m|^{k-i-1} |\nabla c^{m-1}|^{2i}$$

for ease of notation. We can write

$$\begin{aligned} D_t^- |\nabla c^m|^{2k} &= F_k^m(c) D_t^- |\nabla c^m|^2 \\ &= F_k^m(c) \left(2\nabla c^m D_t^- \nabla c^m - \Delta t |D_t^- \nabla c^m|^2\right) \\ &\leq 2F_k^m(c) \nabla c^m D_t^- \nabla c^m. \end{aligned}$$

After integrating over Ω^m and applying integration by parts we obtain

$$\begin{aligned} D_t^- \|\nabla c^m\|_{L^{2k}(\Omega^m)}^{2k} &\leq 2\int_{\Omega^m} \left[F_k^m(c) \nabla c^m D_t^- \nabla c^m\right] \\ &= -2\int_{\Omega^m} \left[D_t^- c^m \ F_k^m(c) \Delta c^m + D_t^- c^m \nabla c^m \cdot \nabla F_k^m(c)\right]. \end{aligned}$$

We note that

$$\|F_k^m(c)\|_{L^{2k}(\Omega^m)} \leq C(k) \left(\|\nabla c^m\|_{L^{2k}(\Omega^m)}^{2(k-1)} + \|\nabla c^{m-1}\|_{L^{2k}(\Omega^m)}^{2(k-1)}\right)$$

while

$$\begin{aligned}
&\|\nabla F_k^m(c)\|_{L^{2k}(\Omega^m)} \\
&\quad \le C(k)\left(\|D^2c^m\|_{L^{2k}(\Omega^m)} + \|D^2c^{m-1}\|_{L^{2k}(\Omega^m)}\right)\left(\|\nabla c^m\|^{2(k-1)}_{L^{2k}(\Omega^m)} + \|\nabla c^{m-1}\|^{2(k-1)}_{L^{2k}(\Omega^m)}\right),
\end{aligned}$$

so it follows that

$$\begin{aligned}
D_t^-\|\nabla c^m\|^{2k}_{L^{2k}(\Omega^m)} \le C(k)\|D_t^-c^m\|_{L^{2k}(\Omega^m)}\left(\|D^2c^m\|_{L^{2k}(\Omega^m)} + \|D^2c^{m-1}\|_{L^{2k}(\Omega^m)}\right)& \\
\cdot\left(\|\nabla c^m\|^{2(k-1)}_{L^{2k}(\Omega^m)} + \|\nabla c^{m-1}\|^{2(k-1)}_{L^{2k}(\Omega^m)}\right).&
\end{aligned}$$

Then, summing over m and using the previous claim yields

$$\|\nabla c^m\|^{2k}_{L^{2k}(\Omega^m)} \le C\left(1 + \|c_h^0\|^{2k}_{W^{2,2k}(\Omega^0)}\right) + \sum_{j=0}^{m-1}\left(\|\nabla c^j\|^{2k}_{L^{2k}(\Omega^{j+1})} - \|\nabla c^j\|^{2k}_{L^{2k}(\Omega^j)}\right).$$

Due to Lemma A.1.5, we can write

$$\sum_{j=0}^{m-1}\left(\|\nabla c^j\|^{2k}_{L^{2k}(\Omega^{j+1})} - \|\nabla c^j\|^{2k}_{L^{2k}(\Omega^j)}\right) \le C\Delta t\sum_{j=0}^{m-1}\|c^j\|^{2k}_{W^{2,2k}(\Omega^j)},$$

and thus,

$$\|\nabla c^m\|^{2k}_{L^{2k}(\Omega^m)} \le C\left(1 + \|c_h^0\|^{2k}_{W^{2,2k}(\Omega^0)}\right).$$

Since this holds for any $p = 2k$, by interpolation it holds for any $1 < p < \infty$. ■

□

2.4.2.3 Regularity for p

We use Lemmas A.1.3 and 2.4.8, along with the L^∞ bounds on n_k^m, to write

$$\|\nabla p^m\|_{L^q(\Omega^m)} \leq C\left(1 + \|c^m\|_{W^{1,q}(\Omega^m)}\right)$$

for $1 < q < \infty$. Since $\|c^m\|_{L^\infty(0,T;W^{1,q}(\Omega_{\Delta t}))} \leq C$, with C dependent only on fixed quantities and $\|c^0\|_{W^{2,q}(\Omega^0)}$, then we also have $\|p^m\|_{L^\infty(0,T;W^{1,q}(\Omega_{\Delta t}))} \leq C$. While this bound is independent of Δt it is dependent on κ, since the L^p estimates for c^m are dependent on κ.

We can also derive estimates independent of κ, by noting that

$$\|(n_p^m + n_q^m)\psi(c^m)\nabla c^m\|_{L^2(\Omega^m)} \leq C n_T c_M \|c^m\|_{H^1(\Omega^m)},$$

so

$$\|p^m\|_{H^1(\Omega^m)} \leq C\left(1 + \|c^m\|_{H^1(\Omega^m)}\right),$$

and thus

$$\|p\|_{L^2(0,T;H^1(\Omega_{\Delta t}))} \leq C\left(1 + \|c^m\|_{L^2(0,T;H^1(\Omega_{\Delta t}))}\right) \leq C,$$

independent of κ.

2.4.2.4 Regularity for n_k, p

Now we are ready to prove a uniform bound on $\|n_k\|_{L^\infty(0,\tau;H^1(\Omega_{\Delta t}))}$. We first notice that $\boldsymbol{w} \in L^2(Q_{\Delta t})$, where $\boldsymbol{w}^m = -\nabla p^m - \boldsymbol{v}_s^m + \psi(c^m)\nabla c^m$. From Lemma 2.4.6 we then have

$$\|n_k\|_{L^2(0,T;H^1(\Omega_{\Delta t}))} \leq C_1,$$

where C_1 is dependent on fixed quantities as well as $\|c^0\|_{L^2(\Omega^0)}$, where we have used the estimates for c^m, p^m. Furthermore, we can then obtain the estimate

$$\|p^m\|_{H^2(\Omega^m)} \leq C(1 + \|n_k^m\|_{H^1(\Omega^m)})\|c^m\|_{W^{1,\infty}(\Omega^m)} + \|c^m\|_{H^2(\Omega^m)}),$$

so it follows that

$$\|p^m\|_{L^2(0,T;H^2(\Omega_{\Delta t}))} \leq C\left(1 + \|n_k^m\|_{L^{q'}(0,T;H^1(\Omega_{\Delta t}))}\right),$$

where C is dependent on $\|c^0\|_{W_q^2(\Omega^0)}$. Thus,

$$\|\boldsymbol{w}\|_{L^2(0,T;H^1(\Omega_{\Delta t}))} \leq C\left(1 + \|n_k^m\|_{L^{q'}(0,T;H^1(\Omega_{\Delta t}))}\right).$$

Plugging this in to the estimate from Lemma 2.4.6, we obtain

$$\begin{aligned}\|n_k\|_{L^\infty(0,T;H^1(\Omega_{\Delta t}))} + \|n_k\|_{L^2(0,T;H^2(\Omega_{\Delta t}))} &\leq C\left(1 + \|n_k^0\|_{H^1(\Omega^0)} + \|\boldsymbol{w}\|_{L^2(0,T;H^1(\Omega_{\Delta t}))}\right) \\ &\leq C\left(1 + \|n_k^0\|_{H^1(\Omega^0)} + \|n_k\|_{L^{q'}(0,T;H^1(\Omega_{\Delta t}))}\right).\end{aligned}$$

We also have the interpolation inequality

$$\begin{aligned}\|n_k\|_{L^{q'}(0,T;H^1(\Omega_{\Delta t}))} &\leq C\|n_k\|^{1-\theta}_{L^\infty(0,T;H^1(\Omega_{\Delta t}))}\|n_k\|^{\theta}_{L^2(0,T;H^1(\Omega_{\Delta t}))} \\ &\leq \epsilon\|n_k\|_{L^\infty(0,T;H^1(\Omega_{\Delta t}))} + c(\theta,\epsilon)\end{aligned}$$

where $\frac{1}{q'} = \frac{\theta}{2}$ and we have used the bound on n_k in $L^2(0,T;H^1(\Omega_{\Delta t}))$. Thus, taking $\epsilon < \frac{1}{2}$ putting this all together yields

$$\|n_k\|_{L^\infty(0,T;H^1(\Omega_{\Delta t}))} + \|n_k\|_{L^2(0,T;H^2(\Omega_{\Delta t}))} \leq C\left(1 + \|n_k^0\|_{H^1(\Omega^0)}\right).$$

2.4.2.5 Summary

We collect the bounds obtained in the previous sections. For $1 < q < \infty$, $2 \leq p < \infty$, we have

$$0 \leq c^m \leq c_M, \quad 0 \leq n_k^m \leq n_T, \quad \sum_k n_k^m = n_T,$$

$$\|c\|_{L^\infty(0,T;W^{1,p}(\Omega_{\Delta t}))} + \|c\|_{L^q(0,T;W^{1,\infty}(\Omega_{\Delta t}))} + \|c\|_{L^q(0,T;W^{2,p}(\Omega_{\Delta t}))} \leq C,$$

$$\|n_k\|_{L^\infty(0,T;H^1(\Omega_{\Delta t}))} + \|n_k\|_{L^2(0,T;H^2(\Omega_{\Delta t}))} \leq C,$$

$$\|\boldsymbol{v}\|_{L^\infty(0,T;L^p(\Omega_{\Delta t}))} + \|\boldsymbol{v}\|_{L^q(0,T;H^1(\Omega_{\Delta t}))} + \|p\|_{L^q(0,T;L^2(\Omega_{\Delta t})} \leq C,$$

where C is dependent only on fixed quantities and κ, $\|c^0\|_{W^{2,p}(\Omega^0)}$, $\|n_k^0\|_{H^2(\Omega^0)}$. Thus, the choices for Δt in the previous sections have a lower bound and the process can be iterated over the interval $[0, T]$ with uniform Δt.

We also collect the uniform bounds that are satisfied independent of κ:

$$0 \leq c^m \leq c_M, \quad 0 \leq n_k^m \leq n_T, \quad \sum_k n_k^m = n_T,$$

$$\|c\|_{L^2(0,T;H^1(\Omega_{\Delta t}))}, \ \|n_k\|_{L^2(0,T;H^1(\Omega_{\Delta t}))}, \ \|p\|_{L^2(0,T;H^1(\Omega_{\Delta t}))} \leq C.$$

The bounds on $D_t^- c$ and $D_t^- n_k$ can be obtained by integrating the functions against a test function $\phi \in L^2(0, T; H^1(\Omega_{\Delta t}))$ and using the above bounds. This proves Theorem 2.2.2.

2.5 Model with Brinkman's Equation

We now replace Darcy's Law with Brinkman's equation, yielding the system

$$
\begin{aligned}
&\partial_t n_k + \operatorname{div}(n_k \boldsymbol{v}) = d\Delta n_k - \operatorname{div}(n_k \psi(c)\nabla c) + G_k(c,n) && \text{in } Q_T,\\
&\partial_t c = \mu\Delta c - (n_p + n_q) f(c) + r_c(c_v - c) && \text{in } Q_T,\\
&\nabla p = -\frac{\lambda}{K}\boldsymbol{v} + \mu\Delta\boldsymbol{v} - n_T\nabla\phi && \text{in } Q_T,\\
&\operatorname{div}\boldsymbol{v} = -n_T^{-1}\left(\operatorname{div}((n_p+n_q)\psi(c)\nabla c) - G_{tot}\right) && \text{in } Q_T,\\
&\boldsymbol{v} = \boldsymbol{v}_s, \quad \partial_{\boldsymbol{n}} n_k = \partial_{\boldsymbol{n}} c = 0 && \text{on } \Gamma_T,\\
&(n_k, c)|_{t=0} = (n_{k,0}, c_0) && \text{in } \Omega(0).
\end{aligned}
\tag{2.14}
$$

We require slightly stricter assumptions on the functions f, ψ than in (2.2). We now assume

$$
\begin{cases}
f : [0,+\infty) \to [0,+\infty) \text{ is a Lipschitz continuous function with } f(0) = 0,\\
\psi : [0,+\infty) \to [0,+\infty) \text{ is a } C^{1+\nu} \text{ function for some } \nu > 0,\\
\nabla\phi \in L^\infty(Q_T) \text{ and } \phi \text{ is independent of time.}
\end{cases}
\tag{2.15}
$$

We define a weak solution in a similar manner as before.

Definition 2.5.1. We say that $(n_p, n_q, n_l, c, \boldsymbol{v}, p)$ is a weak solution of problem (2.14) provided that the following hold:

For $k = (p, q, l)$ and a.e. $t \in [0, T]$,

$$
\begin{aligned}
&\int_0^t \int_{\Omega(s)} \Big(- n_k \Big(\boldsymbol{v} + \psi(c) \nabla c \Big) \cdot \nabla \varphi_k + d \nabla n_k \cdot \nabla \varphi_k - G_k(c, n) \varphi_k \Big) dx ds \\
&\quad = \int_0^t \int_{\Omega(s)} n_k \partial_t \varphi_k dx ds + \int_{\Omega(0)} n_{k0} \varphi_k(0) dx - \int_{\Omega(t)} n_k(t) \varphi_k(t) dx, \\
&\int_0^t \int_{\Omega(s)} \Big(\mu \nabla c \cdot \nabla \varphi_c + (n_p + n_q) f(c) \varphi_c \Big) dx ds \\
&\quad = \int_0^t \int_{\Omega(s)} c \partial_t \varphi_c dx ds + \int_{\Omega(0)} c_0 \varphi_c(0) dx - \int_{\Omega(t)} c(t) \varphi_c(t) dx, \\
&\int_0^t \int_{\Omega} p \operatorname{div} \Phi dx\ ds - \int_0^t \int_{\Omega} \Big(\nu \nabla \boldsymbol{v} : \nabla \Phi + \frac{\lambda}{K} \boldsymbol{v} \Phi + n_T \nabla \phi \cdot \Phi \Big) dx\ ds = 0,
\end{aligned}
$$

for any for any $\varphi_k, \varphi_c \in C^\infty(\overline{Q_T})$, $k \in (p, q, l)$, and any $\Phi \in C^\infty(\overline{Q_T}; \mathbb{R}^n)$. We also require that the divergence constraint $(2.14)_4$ is satisfied weakly in the sense of distributions.

The main result for this chapter is similar to the case for Darcy's Law. However, in order to obtain existence of a solution to Stokes' equations (given by $(2.14)_{3,4}$), we require slightly higher regularity for the right hand side of (1.3). This means that we will need more regularity for c, so we require more regularity for the initial data c_0.

Theorem 2.5.2. *Let $\Omega(t) \subset \mathbb{R}^n$, $n = 2, 3$ be a bounded domain of class $C^{1,1}$ for $0 \le t \le T$ and assume that the initial data $(n_{k,0}, c_0)$ satisfy*

$$
0 \le n_{k,0}, \quad n_{p,0} + n_{q,0} + n_{l,0} = n_T, \quad 0 \le c_0, c_v \le c_M, \quad c_0 \in H^1(\Omega(0)),
$$

and the assumptions in (2.15) hold. Then the problem (2.14) admits a weak solution in the sense of Definition 2.5.1.

2.5.1 Proof of Theorem 2.5.2

The main theorem of this chapter is proven in a similar manner as Theorem 2.1.2. We first define a sequence of regularized, time-discrete approximating problems. Then, we demonstrate convergence of solutions to those problems.

2.5.1.1 Approximating problems

We define the approximating problems for this scheme in a similar manner as Section 2.2.2. Specifically, for $x \in \Omega^m$ we have the time-discrete system for

$$\begin{aligned}
&D_t^- n_k^m + \operatorname{div}(n_k^m \boldsymbol{w}^m) = d\Delta n_k^m - n_k^m \operatorname{div} \boldsymbol{v}_s^m + G_k(c^m, n^m), \\
&D_t^- c^m - \nabla c^m \cdot \boldsymbol{v}_s^m = \mu \Delta c^m - f^{m-1} + r_c(c_v^m - c^m), \\
&\nabla p^m = -\frac{\lambda}{K}\boldsymbol{v}^m + \mu \Delta \boldsymbol{v}^m - n_T \nabla \phi + \frac{\lambda}{K}\boldsymbol{v}_s^m - \mu \Delta \boldsymbol{v}_s^m, \\
&\operatorname{div} \boldsymbol{v}^m = \operatorname{div} \boldsymbol{v}_s^m - n_T^{-1}\left(\operatorname{div}((n_p^m + n_q^m)\psi(c^m)\nabla c^m) - G_{tot}\right).
\end{aligned} \tag{2.16}$$

We have made the substitution $\boldsymbol{v}^m = \boldsymbol{v}^m - \boldsymbol{v}_s^m$ in order to simplify the system. We also let

$$c_v^m = (\Delta t)^{-1} \int_{t^{m-1}}^{t^m} \left(c_v(t) \circ A^m(t)\right),$$

$$\boldsymbol{w}^m = \boldsymbol{v}_s^m + \psi(c^m)\nabla c^m,$$

$$f^{m-1} = (f(c^{m-1})(n_p^{m-1} + n_q^{m-1})) \circ A^m,$$

as before. We supplement this system with the homogeneous boundary conditions

$$\boldsymbol{v}^m = 0, \ \nabla n_k^m \cdot \boldsymbol{n} = 0, \ \nabla c^m \cdot \boldsymbol{n} = 0 \text{ on } \Gamma^m. \tag{2.17}$$

Note that we have homogeneous boundary conditions on $\boldsymbol{v}^m$ since we have taken $\boldsymbol{v}^m = \boldsymbol{v}^m - \boldsymbol{v}_s^m$.

We will also smooth our initial cell and nutrient densities through mollification. We use (2.4) to smooth the initial cell densities, but follow a difference procedure for the initial nutrient density since we must now have $\|c_{\kappa,0}\|_{H^1(\Omega(0))} \leq C\|c_0\|_{H^1(\Omega(0))}$ with C independent of κ. To do this, we note that c_0 can be approximated by $C^\infty(\overline{\Omega(0)})$ functions, the construction of which guarantee that $0 \leq c_{\kappa,0} \leq c_M$ (see, for example, Theorem 3 in [21]) and $c_{\kappa,0} \to c_0$ in $H^1(\Omega(0))$.

Then, we have a similar existence result for this approximating scheme as we had in the previous chapter. Note, however, that we obtain more regularity for c^m due to the assumptions on the initial data, and the regularity for $p^m, \boldsymbol{v}^m$ are different due to switching Darcy's Law for Brinkman's equation.

Theorem 2.5.3. *Under the assumptions of Theorem 2.5.2 for sufficiently small* $\kappa, \Delta t > 0$ *the system* (2.16), (2.17), (2.4) *has a global weak solution* $(n_{k,\kappa,\Delta t}, c_{\kappa,\Delta t}, p_{\kappa,\Delta t}, \boldsymbol{v}_{\kappa,\Delta})$ *satisfying*

$$0 \leq c_{\kappa,\Delta t} \leq c_M, \; 0 \leq n_{k,\kappa,\Delta t} \leq n_T, \; \sum_k n_{k,\kappa,\Delta t} = n_T,$$

where n_T, c_M *are independent of* κ. *Furthermore, the solution satisfies the bounds*

$$\|D_{\Delta t}^- c_{\kappa,\Delta t}\|_{L^2(0,T;L^2(\Omega_{\Delta t}))} + \|c_{\kappa,\Delta t}\|_{L^\infty(0,T;H^1(\Omega_{\Delta t}))} + \|c_{\kappa,\Delta t}\|_{L^2(0,T;H^2(\Omega_{\Delta t}))} \leq C,$$

$$\|D_{\Delta t}^- n_{k,\kappa,\Delta t}\|_{L^2(0,T;H^{-1}(\Omega_{\Delta t}))} + \|n_{k,\kappa,\Delta t}\|_{L^2(0,T;H^1(\Omega_{\Delta t}))} \leq C,$$

$$\|\boldsymbol{v}_{\kappa,\Delta t}\|_{L^\infty(0,T;L^2(\Omega_{\Delta t}))} + \|\boldsymbol{v}_{\kappa,\Delta t}\|_{L^2(0,T;L^6(\Omega_{\Delta t}))} + \|\boldsymbol{v}_{\kappa,\Delta t}\|_{L^q(0,T;W^{1,p}(\Omega_{\Delta t}))} \leq C,$$

$$\|p_{\kappa,\Delta t}\|_{L^q(0,T;L^p(\Omega_{\Delta t})} \leq C$$

for some $p, q > 1$*. Here,* C *is dependent on fixed quantities and* $\|c_\kappa^0\|_{H^1(\Omega^0)}$*,* $\|n_{k,\kappa}^0\|_{L^2(\Omega^0)}$*.*

Then, the proof of Theorem 2.5.2 follows in a similar manner as the proof of Theorem 2.1.2. We can show that $(2.16)_{1-4}$ converge to $(2.14)_{1-4}$ in the same manner as for the model in the previous section.

2.5.2 Proof of Theorem 2.5.3

The proof of Theorem 2.5.3 is similar to the proof of Theorem 2.2.2, except that we must now demonstrate the existence (and regularity) of a solution to Stokes' equations instead of an elliptic equation for p. To do this, we will need higher regularity for c.

2.5.2.1 Fixed point argument

The fixed point argument proceeds in the same way as the previous chapter. The arguments for n_k, c will be the same as before, but we must provide an alternate argument for $p, \boldsymbol{v}$ due to the new governing equations. We first prove an analog of Lemma 2.4.2.

Lemma 2.5.4. *Suppose the hypotheses of Lemma 2.4.1 hold. Fix* $n_k^m \in H^1(\Omega^m)$*. Then, there exists a unique solution* $p^m, \boldsymbol{v}^m \in L^2(\Omega^m) \times H^1(\Omega^m)$ *to* $(2.16)_{5,6}$*, such that*

$$\|p^m\|_{L^2(\Omega^m)} + \|\boldsymbol{v}^m\|_{H^1(\Omega^m)} \leq C\left(1 + \left(1 + \|c^{m-1}\|_{W_p^2(\Omega^{m-1})}\right)\|n_k^m\|_{H^1(\Omega^m)}\right),$$

where C is dependent on fixed quantities, as well as $\|c^{m-1}\|_{H^1(\Omega^{m-1})}$, for some $p > n$. Furthermore, for $n_k^m, \tilde{n}_k^m \in H^1(\Omega^m)$ we have

$$\|\boldsymbol{v}^m[n_k^m] - \boldsymbol{v}^m[\tilde{n}_k^m]\|_{H^1(\Omega^m)} \leq C\left(1 + \|c^{m-1}\|_{W_p^2(\Omega^{m-1})}\right)\|n_k^m - \tilde{n}_k^m\|_{H^1(\Omega^m)},$$

$$\|\boldsymbol{v}^m[n_k^m] - \boldsymbol{v}^m[\tilde{n}_k^m]\|_{L^2(\Omega^m)} \leq C\left(1 + \|c^{m-1}\|_{W_p^2(\Omega^{m-1})}\right)\|n_k^m - \tilde{n}_k^m\|_{L^2(\Omega^m)}.$$

Proof. Due to (2.6) and Lemma A.1.4, we can solve Stokes' equations (i.e. $(2.16)_{5,6}$) and write

$$\|p^m\|_{L^2(\Omega^m)} + \|\boldsymbol{v}^m\|_{H^1(\Omega^m)} \leq C\left(1 + \|K_B(c^m)n_p^m - n_T K_R(c^m, n^m)\|_{L^2(\Omega^m)}\right.$$
$$\left. + \|\operatorname{div}((n_p^m + n_q^m)\psi(c^m)\nabla c^m)\|_{L^2(\Omega^m)}\right),$$

where C is dependent only on fixed quantities, and independent of m. Then, we note that

$$\|\operatorname{div}((n_p^m + n_q^m)\psi(c^m)\nabla c^m)\|_{L^2(\Omega^m)} \leq \|c^m\|_{W_p^2(\Omega^m)}\|n_k^m\|_{H^1(\Omega^m)}\left(1 + \|c^m\|_{H^1(\Omega^m)}\right)$$
$$\leq C\|c^m\|_{W_p^2(\Omega^m)}\|n_k^m\|_{H^1(\Omega^m)}$$

for some $p > n$. Here, we have used the embedding $\|n_k^m\|_{L^p(\Omega^m)} \leq C\|n_k^m\|_{H^1(\Omega^m)}$ for $p \leq 6$, along with the regularity results for c^m. Furthermore,

$$\|K_B(c^m)n_p^m - n_T K_R(c^m, n^m)\|_{L^2(\Omega^m)} \leq \|c^m\|_{L^\infty(\Omega^m)}\|n_p^m\|_{L^2(\Omega^m)},$$

so we have

$$\|p^m\|_{L^2(\Omega^m)} + \|\boldsymbol{v}^m\|_{H^1(\Omega^m)} \leq C\left(1 + \left(1 + \|c^{m-1}\|_{W_p^2(\Omega^{m-1})}\right)\|n_k^m\|_{H^1(\Omega^m)}\right),$$

where C is dependent on $\|c^{m-1}\|_{H^1(\Omega^{m-1})}$ and fixed quantities.

Next, let $n_k^m, \tilde{n}_k^m \in H^1(\Omega^m)$, and define $\boldsymbol{v}^m := \boldsymbol{v}^m[n_k^m]$, $\tilde{\boldsymbol{v}}^m := \boldsymbol{v}^m[\tilde{n}_k^m]$. We see that $\boldsymbol{u}^m := \boldsymbol{v}^m - \tilde{\boldsymbol{v}}^m$ satisfies

$$\begin{cases} \nabla q = -\frac{\lambda}{K}\boldsymbol{u}^m + \nu\Delta\boldsymbol{u}^m, \\ \operatorname{div}\boldsymbol{u}^m = -n_T^{-1}\left[\left(\operatorname{div}((n_p^m - \tilde{n}_p^m + n_q^m - \tilde{n}_q^m))\psi(c^m)\nabla c^m\right) - K_B(c^m)(n_p^m - \tilde{n}_p^m)\right]. \end{cases}$$

Thus, it follows that

$$\|\boldsymbol{u}^m\|_{H^1(\Omega^m)} \le C\left(1 + \|c^{m-1}\|_{W_p^2(\Omega^{m-1})}\right)\|n_k^m - \tilde{n}_k^m\|_{H^1(\Omega^m)},$$
$$\|\boldsymbol{u}^m\|_{L^2(\Omega^m)} \le C\left(1 + \|c^{m-1}\|_{W_p^2(\Omega^{m-1})}\right)\|n_k^m - \tilde{n}_k^m\|_{L^2(\Omega^m)},$$

where C is dependent on $\|c^{m-1}\|_{H^1(\Omega^{m-1})}$ and fixed quantities. $\square$

Since this is an analogous result to Lemma 2.4.2, the rest of the fixed point argument (including the positivity of n_k) follows in the same manner as in Section 2.4.1. It remains to demonstrate the regularity of the solution.

2.5.2.2 Regularity

The regularity of the solution follows in a similar manner as in Section 2.4.2. We recall the estimates

$$0 \le c^m \le c_M, \quad 0 \le n_k^m \le n_T, \quad \sum_k n_k^m = n_T.$$

In the following sections we will be careful to demonstrate which bounds are independent of Δt and which are independent of κ.

2.5.2.3 Regularity for c

As in Section 2.4.2.2, due to Lemma 2.4.7, it follows that

$$\|c\|_{L^2(0,T;H^1(\Omega_{\Delta t}))} + \|c\|_{L^\infty(0,T;H^1(\Omega_{\Delta t}))} + \|c\|_{L^2(0,T;H^2(\Omega_{\Delta t}))} \le C, \tag{2.18}$$

where C is dependent on fixed quantities and $\|c^0\|_{H^1(\Omega^0)}$. Since we now assume that $c_0 \in H^1(\Omega^0)$, this bound is independent of both Δt and κ.

We would also like to use Lemma 2.4.8. However, we do not necessarily have $\nabla c_{\kappa,0} \cdot \boldsymbol{n} = 0$ in this case. Thus, letting $\tilde{c}^1 = c^1 \circ \eta^1$, we note that

$$((\Delta t)^{-1} + r_c)(\tilde{c}^1 - c_{\kappa,0}) - \mu A(\Delta t)(\tilde{c}^1 - c_{\kappa,0}) = \boldsymbol{v}_s^m \cdot B(\Delta t)\nabla \tilde{c}^1 - f^0 + \mu A(\Delta t) c_{\kappa,0} = \hat{f}.$$

Since $B(\Delta t)(\tilde{c}^1 - c_{\kappa,0}) \cdot \boldsymbol{n} = B(\Delta t) c_{\kappa,0} \cdot \boldsymbol{n}$, it follows from Theorem 2.3.3.6 in [30] that

$$\begin{aligned}
&\|\tilde{c}^1 - c_{\kappa,0}\|_{W^{2,p}(\Omega(0))} \\
&\quad \le C\left(\|A(\Delta t)(\tilde{c}^1 - c_{\kappa,0}) + (\Delta t^{-1} + r_c)(\tilde{c}^1 - c_{\kappa,0})\|_{L^p(\Omega(0))} + \|\nabla c_{\kappa,0}\|_{W^{1-1/p,p}(\Gamma(0))}\right) \\
&\quad \le C\left(\|\hat{f}\|_{L^p(\Omega(0))} + \|c_{\kappa,0}\|_{W^{2,p}(\Omega(0))}\right)
\end{aligned}$$

due to the trace theorem. Furthermore,

$$\|\hat{f}\|_{L^p(\Omega(0))} \le C\left(1 + \|\tilde{c}^1\|_{W^{1,p}(\Omega(0))} + \|c_{\kappa,0}\|_{W^{2,p}(\Omega(0))}\right).$$

Using Sobolev embedding, an interpolation estimate, and Young's inequality, as well as (2.18), we have

$$\|\tilde{c}^1 - c_{\kappa,0}\|_{W^{2,p}(\Omega(0))} \le C\left(1 + \|\tilde{c}^1\|_{W^{2,p}(\Omega(0))}\right).$$

Finally, this yields

$$\|\tilde{c}^1\|_{W^{2,p}(\Omega(0))} \leq C\left(1 + \|c_{\kappa,0}\|_{W^{2,p}(\Omega(0))}\right).$$

Then, we apply Lemma 2.4.8 to the problem on $[\Delta t, T]$ with initial data c^1 on Ω^1. This yields

$$\|c\|_{L^q(0,T;W^{2,p}(\Omega_{\Delta t}))} + \|c\|_{L^\infty(0,T;W^{1,p}(\Omega_{\Delta t}))} \leq C, \tag{2.19}$$

where C is dependent on $\|c^0\|_{W^{2,p}(\Omega^0)}$ and fixed quantities. This bound is independent of Δt but dependent on κ.

2.5.2.4 Regularity for v

We use Lemma A.1.4 to write

$$\|\boldsymbol{v}^m\|_{L^p(\Omega^m)} \leq C\left(1 + \|c^m\|_{W^{1,p}(\Omega^m)}\right)$$

for $1 < p < \infty$. Due to (2.19) we also have

$$\|\boldsymbol{v}\|_{L^\infty(0,T;L^p(\Omega_{\Delta t}))} \leq C.$$

While this bound is independent of Δt, it is dependent on κ, since the L^p estimates for c^m are dependent on κ.

We can also derive estimates independent of κ, using (2.18) and Sobolev embedding to obtain

$$\|\boldsymbol{v}\|_{L^\infty(0,T;L^2(\Omega_{\Delta t}))} + \|\boldsymbol{v}\|_{L^2(0,T;L^6(\Omega_{\Delta t})} \leq C \tag{2.20}$$

for $r \leq 6$.

2.5.2.5 Regularity for n_k

Now we are ready to prove a uniform bound on $\|n_k\|_{L^\infty(0,\tau;H^1(\Omega_{\Delta t}))}$. First, letting $\boldsymbol{w}^m = \boldsymbol{v}^m + \psi(c^m)\nabla c^m$, it follows from (2.18), (2.20) that $\boldsymbol{w} \in L^2(Q_{\Delta t})$. Thus, due to Lemma 2.4.6 we have the estimate

$$\|n_k\|_{L^2(0,T;H^1(\Omega_{\Delta t}))} \leq C, \tag{2.21}$$

where C is dependent on fixed quantities, including $\|c^0\|_{H^1(\Omega^0)}$.

Furthermore, we know that $\boldsymbol{w} \in L^\infty(0,T;L^q(\Omega_{\Delta t}))$ with $\operatorname{div}\boldsymbol{w} \in L^2(Q_{\Delta t})$, due to the estimates for c^m and the constraint on $\operatorname{div}\boldsymbol{v}^m$, with

$$\begin{aligned}
\|\operatorname{div}\boldsymbol{w}\|_{L^2(Q_{\Delta t})} &\leq C\left(\|\nabla n\|_{L^{q'}(0,T;L^2(\Omega_{\Delta t}))}\|\nabla c\|_{L^q(0,T;L^\infty(\Omega_{\Delta t}))} + \|\Delta c\|_{L^2((0,T)\times\Omega_{\Delta t})}\right)\\
&\leq C\left(1 + \|\nabla n\|_{L^{q'}(0,T;L^2(\Omega_{\Delta t}))}\right)
\end{aligned}$$

where C is dependent on $\|c^0\|_{W^{2,p}(\Omega^0)}$ and other fixed quantities, while $\frac{1}{q'} + \frac{1}{q} = \frac{1}{2}$. Thus, the estimate

$$\|n_k\|_{L^\infty(0,T;H^1(\Omega_{\Delta t}))} + \|n_k\|_{L^2(0,T;H^2(\Omega_{\Delta t}))} \leq C$$

follows from Lemma 2.4.6 in the same way as the previous chapter. Here, C is dependent on $\|c^0\|_{W^{2,p}(\Omega^0)}$ and thus is dependent on κ but independent of Δt.

2.5.2.6 Regularity v^m, p^m

By Lemma A.1.4, for $p < \infty$ we have

$$\|p^m\|_{L^p(\Omega^m)} + \|\boldsymbol{v}^m\|_{W^{1,p}(\Omega^m)} \leq C\left(1 + \|\operatorname{div}((n_p^m + n_q^m)\psi(c^m)\nabla c^m)\|_{L^p(\Omega^m)}\right).$$

Using (2.19), (2.21) we obtain

$$\|p\|_{L^q(0,T;L^2(\Omega_{\Delta t}))} + \|\boldsymbol{v}\|_{L^q(0,T;H^1(\Omega_{\Delta t}))} \leq C,$$

for $1 < q < \infty$, where C is dependent on q, $\|c^0\|_{W^{2,p}(\Omega^0)}$, $\|n_k^0\|_{H^2(\Omega^0)}$ for some $p > n$, sufficiently large q. Since $\|c^0\|_{W^{2,p}(\Omega^0)}$ is dependent on κ, this bound is also dependent on κ.

We can also derive an estimate that is independent of κ:

$$\begin{aligned}
&\|p^m\|_{L^p(\Omega^m)} + \|\boldsymbol{v}^m\|_{W^{1,p}(\Omega^m)} \\
&\quad \leq C\left(1 + \|\nabla n_k^m\|_{L^2(\Omega^m)}\|\nabla c^m\|_{L^{p'}(\Omega^m)} + \|\nabla c^m\|^2_{L^{p'}(\Omega^m)} + \|\Delta c^m\|_{L^2(\Omega^m)}\right)
\end{aligned}$$

for some $2 < p' \leq 6$ so that $1 < p \leq \frac{3}{2}$. Then, we have

$$\|\nabla c\|_{L^{q'}(0,T;L^{p'}(\Omega_{\Delta t}))} \leq C\|c\|^{\theta}_{L^\infty(0,T;H^1(\Omega_{\Delta t})}\|c\|^{1-\theta}_{L^2(0,T;H^2(\Omega_{\Delta t}))},$$

where $\frac{1}{q'} = \frac{1-\theta}{2}$ and $\frac{1}{p'} = \frac{\theta}{2} + \frac{1-\theta}{6}$. Then, we take $0 < \theta < 1$ so that $2 < p' < 6$ and $2 < q' < \infty$. Thus, we have

$$\|p\|_{L^q(0,T;L^p(\Omega_{\Delta t}))} + \|\boldsymbol{v}\|_{L^q(0,T;W^{1,p}(\Omega_{\Delta t}))} \leq C,$$

where $\frac{1}{q} = \frac{1}{2} + \frac{1}{q'}$ and $\frac{1}{p} = \frac{1}{2} + \frac{1}{p'}$ so that $1 < p < \frac{3}{2}$ and $1 < q < 2$. The constant C is dependent on $\|c^0\|_{H^1(\Omega^0)}$ and $\|n_k^0\|_{L^2(\Omega^0)}$ and independent of $\kappa, \Delta t$.

2.5.2.7 Summary

We collect the bounds obtained in the previous sections. For $1 < q < \infty$, $3 < p < \infty$, we have

$$0 \le c^m \le c_M, \quad 0 \le n_k^m \le n_T, \quad \sum_k n_k^m = n_T,$$

$$\|c\|_{L^\infty(0,T;W^{1,p}(\Omega_{\Delta t}))} + \|c\|_{L^q(0,T;W^{1,\infty}(\Omega_{\Delta t}))} + \|c\|_{L^q(0,T;W^{2,p}(\Omega_{\Delta t}))} \le C,$$

$$\|n_k\|_{L^\infty(0,T;H^1(\Omega_{\Delta t}))} + \|n_k\|_{L^2(0,T;H^2(\Omega_{\Delta t}))} \le C,$$

$$\|\boldsymbol{v}\|_{L^\infty(0,T;L^p(\Omega_{\Delta t}))} + \|\boldsymbol{v}\|_{L^q(0,T;H^1(\Omega_{\Delta t}))} + \|p\|_{L^q(0,T;L^2(\Omega_{\Delta t}))} \le C,$$

where C is dependent only on fixed quantities and κ, $\|c^0\|_{W_p^2(\Omega^0)}$, $\|n_k^0\|_{H^2(\Omega^0)}$. Thus, the choices for Δt in the previous sections have a lower bound and the process can be iterated over the interval $[0,T]$ with uniform Δt.

We also collect the uniform bounds that are satisfied independent of κ:

$$0 \le c^m \le c_M, \quad 0 \le n_k^m \le n_T, \quad \sum_k n_k^m = n_T,$$

$$\|c\|_{L^\infty(0,T;H^1(\Omega_{\Delta t}))} + \|c\|_{L^2(0,T;H^2(\Omega_{\Delta t}))} \le C,$$

$$\|n_k\|_{L^2(0,T;H^1(\Omega_{\Delta t}))} \le C,$$

$$\|\boldsymbol{v}\|_{L^\infty(0,T;L^2(\Omega_{\Delta t}))} + \|\boldsymbol{v}\|_{L^2(0,T;L^r(\Omega_{\Delta t}))} + \|\boldsymbol{v}\|_{L^q(0,T;W_p^1(\Omega_{\Delta t}))} + \|p\|_{L^q(0,T;L^p(\Omega_{\Delta t}))} \le C,$$

for any $1 < r \le 6$, and some fixed $p, q > 1$. This proves Theorem 2.5.3.

Chapter 3: Convergent Finite Element Scheme

In this chapter we construct a convergent numerical scheme that approximates the chemotaxis-fluid problem (either (2.1) or (2.14)). This scheme is constructed by a discretization in time via the backward Euler method and a discretization in space by a finite element approximation. We will first consider the most basic case: the problem with Darcy's Law on a fixed, polygonal domain. We will then expand the scheme to consider the case of the problem with Darcy's Law on a smooth domain and on a moving domain. Finally, we consider the case of the problem with Brinkman's equation on a fixed, polygonal domain.

Much of the work in this chapter in the case of Darcy's Law will appear in the research article [49].

Remark 3.0.1. The additional considerations of the presence of a drug and/or a fourth population of healthy cells discussed in the introduction still hold in this case. For the sake of simplicity we will again ignore the case of drug application.

3.1 Introduction and Main Results: Darcy's Law on a Fixed Polygonal Domain

We first discuss the assumptions placed on the domain and the finite element spaces. We assume that the domain $\Omega \subset \mathbb{R}^2$ is a polygonal domain. For each $h > 0$, let $\mathcal{T}_h$ be a triangulation of Ω that covers Ω exactly, i.e. $\bigcup_{\tau \in \mathcal{T}_h} \tau = \Omega$. For each triangle $\tau \in \mathcal{T}_h$, let h_τ be the diameter of the smallest circumscribed circle of τ and ρ_τ be the diameter of the largest inscribed circle of τ. Let $h = \max_{\tau \in \mathcal{T}_h} h_\tau$. We assume that

(A1) There exists $c_0 > 0$ such that $h_\tau \geq c_0 h$ for each $\tau \in \mathcal{T}_h$.

(A2) There exists $c_1 > 0$ such that $h_\tau \leq c_1 \rho_\tau$ for each $\tau \in \mathcal{T}_h$.

(A3) Each element $\tau \in \mathcal{T}_h$ is an acute or right triangle.

Then, we define the finite element space S_h associated with $\mathcal{T}_h$ by

$$S_h = \left\{ v \in C^0(\Omega) : v|_\tau \in \mathcal{P}_1 \text{ for each } \tau \subset \mathcal{T}_h \right\},$$

where $\mathcal{P}_1$ is the set of linear functions. Thus, any element of S_h is a continuous, piecewise linear function and $S_h \subset H^1(\Omega)$. Furthermore, we set

$$Y_h = \left\{ p \in S_h : \int_\Omega p = 0 \right\}.$$

3.1.1 Finite element formulation

We introduce the following notation:

$$(u,v) = \int_\Omega uv, \quad a(u,v) = \int_\Omega \nabla u \cdot \nabla v, \quad h(u,v,\boldsymbol{w}) = \int_\Omega u\boldsymbol{w} \cdot \nabla v.$$

Letting α, β be the vectors holding the nodal values of $u, v \in S_h$ and $\{\phi_i\}_{i=1}^N$ be the nodal basis functions of S_h, we can write

$$\begin{aligned}
(u,v) &= \alpha^T \boldsymbol{M} \beta = \beta^T \boldsymbol{M} \alpha, \;\; (\boldsymbol{M})_{ij} = m_{ij} = \int_\Omega \phi_i \phi_j, \\
a(u,v) &= \alpha^T \boldsymbol{A} \beta = \beta^T \boldsymbol{A} \alpha, \;\; (\boldsymbol{A})_{ij} = a_{ij} = \int_\Omega \nabla\phi_i \cdot \nabla\phi_j, \\
h(u,v,\boldsymbol{w}) &= \alpha^T \boldsymbol{H}(\boldsymbol{w}) \beta, \;\; (\boldsymbol{H}(\boldsymbol{w}))_{ij} = h_{ij} = \int_\Omega \phi_i \boldsymbol{w} \cdot \nabla\phi_j.
\end{aligned}$$

Here, $\boldsymbol{M}$ is the consistent mass matrix and $\boldsymbol{A}$ is the stiffness matrix. We introduce the lumped-mass matrix $\tilde{\boldsymbol{M}} = \operatorname{diag}\{\tilde{m}_{ii}\}$, where $\tilde{m}_{ii} = \sum_j m_{ij}$, and define

$$(u,v)_h = \alpha^T \tilde{\boldsymbol{M}} \beta = \beta^T \tilde{\boldsymbol{M}} \alpha.$$

Additionally, we define the artificial diffusion matrix $\boldsymbol{D}$ by

$$d_{ij} = \max\{0, -h_{ij}, -h_{ji}\} = d_{ji} \text{ for } i \neq j, \;\; d_{ii} = \sum_{j \neq i} d_{ij}.$$

This ensures that $h_{ij} + d_{ij} \geq 0$ for $i \neq j$, and the matrix $\boldsymbol{D}$ is symmetric with zero row and column sums. Then, we set

$$k(u,v,\boldsymbol{w}) = \alpha^T \boldsymbol{K}(\boldsymbol{w}) \beta, \;\; \boldsymbol{K}(\boldsymbol{w}) = \boldsymbol{H}(\boldsymbol{w}) + \boldsymbol{D}(\boldsymbol{w}).$$

We can then define the weak formulation of the discrete problem as follows:

$$
\begin{aligned}
&(D_t^- n_{k,h}^m, \chi)_h + d\; a(n_{k,h}^m, \chi) = k(n_{k,h}^{m-1}, \chi, \boldsymbol{w}_h^{m-1}) + (G_{k,h}^{m-1}, \chi)_h, \\
&(D_t^- c_h^m, \chi)_h + \mu\; a(c_h^m, \chi) + r_c(c_h^m, \chi)_h = (F_h^m, \chi)_h, \\
&a(p_h^m, \chi) = n_T^{-1} k(n_{p,h}^m + n_{q,h}^m, \chi, \psi(c_h^m)\nabla c_h^m) - n_T^{-1}(G_h^m, \chi)_h
\end{aligned}
\tag{3.1}
$$

for all $\chi \in S_h$, where

$$
\begin{gathered}
\boldsymbol{w}_h^{m-1} = -\nabla p_h^{m-1} + \psi(c_h^{m-1})\nabla c_h^{m-1}, \\
G_{k,h}^{m-1} = G_k(c_h^{m-1}, n_h^{m-1}), \\
F_h^m = -f(c_h^{m-1})(n_{p,h}^{m-1} + n_{q,h}^{m-1}) + r_c c_{v,h}^m, \\
G_h^m = \big(n_T K_B(c_h^m) n_{p,h}^m - K_{R,h}(c_h^m, n_h^m)\big) = \sum_k G_{k,h}^m, \\
K_{R,h} = n_T (K_B(c_h^m) n_{p,h}^m, 1)_h,
\end{gathered}
$$

and $c_{v,h}^m \in S_h$ is a finite element approximation of c_v. Furthermore, this system is supplemented with initial data $n_{k,h}^0, c_h^0$.

We also place several assumptions on f, ψ:

$$
\begin{cases}
f : [0, +\infty) \to [0, +\infty) \text{ is a Lipschitz function with } f(0) = 0, \\
\psi : [0, +\infty) \to [0, +\infty) \text{ is a Lipschitz continuous function.}
\end{cases}
\tag{3.2}
$$

Remark 3.1.1. Strictly speaking, we only require that $(3.1)_3$ holds for $\eta \in Y_h$. However, we note that for any $\chi \in S_h$, we have $\chi = \eta + \gamma$ for some $\eta \in Y_h$ and constant γ. Due to the definition of the removal term $K_{R,h}$ and the transport term $h(\cdot,\cdot,\cdot)$, as well as the matrix $\boldsymbol{D}$, it follows that $a(p_h^m, \gamma) = (G_h^m, \gamma)_h = k(n_h^m, \gamma, \psi(c_h^m)\nabla c_h) = 0$, and thus $(3.1)_3$ is satisfied for all $\chi \in S_h$.

3.1.2 Finite element operators

We define the L^2, H^1 projections $P_h : L^2(\Omega) \to S_h$, $R_h : H^1(\Omega) \to S_h$ by

$$(P_h v, \chi) = (v, \chi) \text{ for all } \chi \in S_h,$$

$$(\nabla R_h v, \nabla \chi) + (R_h v, \chi) = (\nabla v, \nabla \chi) + (v, \chi) \text{ for all } \chi \in S_h.$$

The projection operators have the following properties (see, for example, [3], [45], and (2.6a), (2.6b) in [37]):

$$\begin{aligned} &\|P_h v - v\|_{W^{s,p}} \le Ch^{m-s}\|v\|_{W^{m,p}}, \quad m = 0,1,2,\ 1 \le p \le \infty, \\ &\|R_h v - v\|_{W^{s,p}} \le Ch^{m-s}\|v\|_{W^{m,p}}, \qquad m = 1,2,\ 1 < p < \infty, \end{aligned} \tag{3.3}$$

for $s = 0, 1$ and $v \in W^{m,p}(\Omega)$.

We then turn our attention to the mass-lumped inner product. We have the estimates ([6])

$$\begin{aligned} \|u\|_{L^2}^2 \le (u,u)_h &\le C\|u\|_{L^2}^2, \\ |(u,v)_h| &\le C\|u\|_{L^p}\|u\|_{L^{p'}(\Omega)}, \\ |(u,v)_h - (u,v)| &\le Ch^{m+l}\|u\|_{W^{m,p}}\|v\|_{W^{l,p'}},\ 0 \le l, m \le 1, \end{aligned} \tag{3.4}$$

where $1 \le p, p' \le \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. Furthermore, for $u, v \in S_h$ the mass-lumped inner product satisfies (this is a straightforward generalization of Lemma 1 in [6]),

$$|(u,v)_h - (u,v)| \le Ch^2\|\nabla u\|_{L^p}\|\nabla v\|_{L^{p'}},$$

where $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Due to the assumptions (A1) and (A2), we have the inverse inequality

$$h\|\nabla u\|_{L^p} \leq \|u\|_{L^p}. \tag{3.5}$$

Thus, we can write

$$|(u,v)_h - (u,v)| \leq Ch\|u\|_{L^p}\|\nabla v\|_{L^{p'}} \leq C\|u\|_{L^p}\|v\|_{L^{p'}}.$$

We also define a norm based on the mass-lumped inner product, given by

$$\|v\|_{h,p} = \sup_{v \in S_h} \frac{(u,v)_h}{\|u\|_{L^{p'}}} \text{ for } \frac{1}{p} + \frac{1}{p'} = 1.$$

Letting $\bar{S}_h$ be the space spanned by the barycentric basis functions $\bar{\phi}_i$, we define a projection operator $D_h : L^1(\Omega) \to \bar{S}_h$ by

$$D_h v = \sum_i v_i \bar{\phi}_i, \quad v_i = \frac{1}{|D_i|}\int_\Omega v\bar{\phi}_i = \frac{1}{|D_i|}\int_{D_i} v,$$

where D_i is the barycentric domain associated with node i. Since the basis functions $\bar{\phi}_i$ are orthogonal, it follows that

$$\int_\Omega u_h D_h v = \int_\Omega u_h v \text{ for all } u_h \in \bar{S}_h.$$

Furthermore, we can define a mass-lumping operator $M_h : S_h \to \bar{S}_h$ defined by

$$M_h v_h = \sum_i v_h(P_i)\bar{\phi}_i,$$

where P_i is the node corresponding to $\overline{\phi}_i$, such that

$$(u,v)_h = (M_h u, M_h v) \text{ for all } u, v \in S_h.$$

We also define $K_h = M_h^* M_h$, where M_h^* is the adjoint of M_h. The operators M_h, K_h are invertible. Then, for $p \in [1, \infty]$, we have

$$\begin{aligned} \|D_h v\|_{L^p(\Omega)} &\leq C\|v\|_{L^p(\Omega)}, \quad v \in L^p(\Omega) \\ c\|v\|_{L^p(\Omega)} \leq \|M_h v\|_{L^p(\Omega)} &\leq C\|v\|_{L^p(\Omega)}, \quad v \in S_h \\ \|M_h v - v\|_{L^p(\Omega)} &\leq Ch\|\nabla v\|_{L^p(\Omega)} \quad v \in S_h. \end{aligned} \tag{3.6}$$

The estimates for M_h also hold for K_h.

Finally, we turn our attention to the discrete Laplace operators. We let $L_h : S_h \to S_h$ be the discrete, Neumann Laplacian defined by

$$(L_h u, v) = (\nabla u, \nabla v) + (u, v) \text{ for all } u, v \in S_h,$$

while $A_h : S_h \to S_h$ is the the mass-lumped version of L_h defined by

$$(A_h u, v)_h = (\nabla u, \nabla v) + (u, v)_h \text{ for all } u, v \in S_h.$$

We have the estimates

$$\begin{aligned} c\|A_h u\|_{L^p(\Omega)} \leq \|L_h u\|_{L^p(\Omega)} &\leq C\|A_h u\|_{L^p(\Omega)} \text{ for } p \in [1, \infty], \\ \|u\|_{W^{1,\infty}(\Omega)} &\leq C\|L_h u\|_{L^p(\Omega)} \text{ for } p > n, \\ \|u\|_{W^{1,\infty}(\Omega)} &\leq C\|A_h u\|_{L^p(\Omega)} \text{ for } p > n \end{aligned}$$

for all $u \in S_h$. The first estimate is derived from the fact that $K_h A_h - K_h = K_h - I$, and the estimates for K_h, K_h^{-1}. The second estimate is proven in Lemma 4.5 in [44], for example, and is essentially a Sobolev embedding estimate. The third estimate comes from the first two estimates.

3.1.3 Main Results

One of the main results of this section is the existence and uniform boundedness of solutions to the scheme (3.1).

Theorem 3.1.2 (Existence of Finite Element Solutions)**.** *Suppose the assumptions* (3.2) *are met and*

$$0 \leq n_{k,h}^0, \ \sum_k n_{k,h}^0 = n_T, \ 0 \leq c_h^0, c_{v,h}^m \leq c_M.$$

Then, provided h sufficiently small and $\Delta t \leq Ch^{2+\alpha}$ for some $\alpha > 0$, there exists a unique solution $(n_{k,h}, c_h, \boldsymbol{v}_h, p_h)$ to the system (3.1) *with initial data $n_{k,h}^0, c_h^0$. Furthermore, the solution satisfies the bounds*

$$0 \leq c_h^m \leq c_M, \ 0 \leq n_{k,h}^m, \ \sum_k n_{k,h}^m = n_T \textit{ for each } m,$$

$$\|n_{k,h}\|_{L^2(0,T;H^1(\Omega))} + \|c_h\|_{L^2(0,T;H^1(\Omega))} + \|p_h\|_{L^2(0,T;H^1(\Omega))} \leq C,$$

$$\|D_t^- n_{k,h}\|_{L^2(0,T;H^{-1}(\Omega))} + \|D_t^- c_h\|_{L^2(0,T;H^{-1}(\Omega))} \leq C,$$

where C is dependent on fixed quantities.

Remark 3.1.3. As stated in the next section, in order to guarantee that the scheme is positivity-preserving we require

$$\Delta t \leq C \min \left\{ 1, \frac{h}{\|\boldsymbol{w}_h^{m-1}\|_\infty} \right\}.$$

The L^∞ bound on c_h, along with inverse estimates, implies that

$$\|c_h^{m-1}\|_{W^{1,\infty}} \leq Ch^{-1} \|c_h^{m-1}\|_\infty \leq Ch^{-1}.$$

Then, we note that $(L_h p_h^{m-1}, \eta) = a(p_h^{m-1}, \eta)$ and, for $q < 2$, we have

$$|a(p_h^{m-1}, \eta)| \leq C \left(\|\eta\|_{L^q} + \|\nabla c_h^{m-1}\|_{L^p} \|\eta\|_{W^{1,q}} \right) \leq C(1 + h^{-(2-2/p)}) \|\eta\|_{L^q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. This is because

$$\|\nabla c_h^{m-1}\|_{L^p} \leq C \|\nabla c_h^{m-1}\|_{L^\infty}^{1-\frac{2}{p}} \|c_h^{m-1}\|_{L^2}^{\frac{2}{p}} \leq C h^{-(1-2/p)} \text{ and } \|\eta\|_{W^{1,q}} \leq C h^{-1} \|\eta\|_{L^q}.$$

Thus, it follows that

$$|(L_h p_h^{m-1}, \eta)| \leq C(1 + h^{-(2-2/p)}) \|\eta\|_{L^q},$$

for all η, so

$$\|\nabla p_h^{m-1}\|_{L^\infty} \leq C \|L_h p_h^{m-1}\|_{L^p} \leq C h^{-(2-2/p)} = C h^{-1-\alpha},$$

where α, C are dependent on p and $\alpha \to 0$ as $p \to 2$. This yields the condition $\Delta t \leq C h^{2+\alpha}$ for some $\alpha > 0$. In practice, the condition $\Delta t \leq C h^2$, or potentially even a linear constraint, should be sufficient.

The second main result of this section is the convergence of the solutions of the scheme (3.1) to a solution of the system (2.1).

Theorem 3.1.4 (Convergence of numerical scheme)**.** *Suppose that*

$$0 \leq n_{k,0}, \quad \sum_k n_{k,0} = n_T, \quad 0 \leq c_0, c_v \leq c_M$$

and there exists a sequence $(h_k, \Delta t_k)$ *such that* $\Delta t_k \leq C h^{2+\alpha}$ *and* $h_k \to 0$. *Define*

$$n_{k,h}^0 = M_h^{-1} D_h n_{k,0}, \quad c_h^0 = M_h^{-1} D_h c_0, \quad c_{v,h}^m = (\Delta t)^{-1} M_h^{-1} D_h \int_{t^{m-1}}^{t^m} c_v(t)\, dt.$$

Then, for each k there exists a solution $(n_{h_k}, c_{h_k}, \boldsymbol{v}_{h_k}, p_{h_k})$ to (3.1) with initial data $n^0_{h_k}, c^0_{h_k}$. Furthermore, as $k \to \infty$, $(n_{h_k}, c_{h_k}, \boldsymbol{v}_{h_k}, p_{h_k})$ converges weakly to $(n, c, \boldsymbol{v}, p)$, where $(n, c, \boldsymbol{v}, p)$ is a solution to (2.1) with initial data n_0, c_0.

We note that the operator $M_h^{-1} D_h$ preserves upper and lower bounds, so $n^0_{k,h}, c^0_h, c^m_{v,h}$ satisfy the hypotheses of Theorem 3.1.2.

3.2 Proof of Theorem 3.1.2

First, we rewrite the update schemes $(3.1)_{1,2}$ for $n^m_{k,h}, c^m_h$ using matrix notation. Letting $\alpha^m_{k,h}, \beta^m_h$ denote the nodal values of $n^m_{k,h}, c^m_h$, respectively, we have

$$\begin{aligned}\left(\tilde{\boldsymbol{M}} + d\Delta t\boldsymbol{A}\right)\alpha^m_{k,h} &= \left(\tilde{\boldsymbol{M}} + \Delta t\boldsymbol{K}(\boldsymbol{w}^{m-1}_h)\right)\alpha^{m-1}_{k,h} + \Delta t\boldsymbol{g}^{m-1}_{k,h},\\ \left(\tilde{\boldsymbol{M}} + \mu\Delta t\boldsymbol{A} + r_c\Delta t\tilde{\boldsymbol{M}}\right)\beta^m_h &= \tilde{\boldsymbol{M}}\beta^{m-1}_h - \Delta t\boldsymbol{f}^m_h + \Delta t r_c\tilde{\boldsymbol{M}}\beta^m_{v,h},\end{aligned} \tag{3.7}$$

where

$$\boldsymbol{g}^{m-1}_{k,h} = (g_i), \quad g_i = G^{m-1}_{k,i}\tilde{m}_{ii},$$

$$\boldsymbol{f}^m_{k,h} = (f_i), \quad f_i = (\alpha^{m-1}_{p,i} + \alpha^{m-1}_{q,i})\tilde{m}_{ii}f(c^{m-1}_i),$$

and $\beta^m_{v,h}$ holds the nodal values of $c^m_{v,h}$.

Claim 3.2.1. *There is a unique solution $(n^m_{k,h}, c^m_h)$ to the update scheme (3.7).*

Proof of claim. The matrices

$$\boldsymbol{L}_1 = \tilde{\boldsymbol{M}} + d\Delta t\boldsymbol{A}, \quad \boldsymbol{L}_2 = \tilde{\boldsymbol{M}} + \mu\Delta t\boldsymbol{A} + r_c\Delta t\tilde{\boldsymbol{M}}$$

have strictly positive diagonal elements and non-positive off-diagonal elements. Thus, it follows that $\boldsymbol{L}_1, \boldsymbol{L}_2$ are invertible, and (3.7) has a unique solution. ■

3.2.1 Positivity preservation

We want to demonstrate that the update scheme (3.7) is positivity-preserving. Specifically, we want to show that

$$0 \le c_h^m \le c_M, \quad 0 \le n_{k,h}^m, \quad \sum_k n_{k,h}^m = n_T \tag{3.8}$$

for all m.

Lemma 3.2.2. *Suppose that the initial data satisfy (3.8). Then, if Δt is small enough that*

$$\Delta t \le C \min\left\{1, \frac{h}{\|\boldsymbol{w}_h^{m-1}\|_\infty}\right\}$$

it follows that (3.8) is satisfied for all m with $0 \le t^m \le T$.

Proof. It suffices to show that if (3.8) is satisfied for $n_{k,h}^{m-1}, c_h^{m-1}$ then it is also satisfied for $n_{k,h}^m, c_h^m$. We note that if $0 \le n_{k,h}^m$ and $\sum_k n_{k,h}^m = n_T$, then $0 \le n_{k,h}^m \le n_T$ for each k.

Due to the assumptions on $n_{k,h}^{m-1}, c_h^{m-1}, c_v$, we know that

$$0 \le f_i \le C n_T c_M \tilde{m}_{ii},$$

$$\boldsymbol{g}_{k,h}^{m-1} = \boldsymbol{G}_k(c_h^{m-1}, n_h^{m-1})\alpha_{k,h}^{m-1} + \tilde{\boldsymbol{g}}_{k,h}^{m-1},$$

where $0 \le \tilde{\boldsymbol{g}}_{k,h}^{m-1}$ and $\boldsymbol{G}_k(c_h^{m-1})$ is a diagonal matrix with entries

$$g_{ii} = \tilde{G}_k((\beta_h^{m-1})_i, (\alpha_h^{m-1})_i) \int_\Omega \phi_i^2 = \tilde{G}_k((\beta_h^{m-1})_i, (\alpha_h^{m-1})_i)\tilde{m}_{ii},$$

so $g_{ii} \geq -c_k \tilde{m}_{ii}$. Thus, we can write

$$\alpha_{k,h}^m = \boldsymbol{L}_1^{-1} \left[\left(\tilde{\boldsymbol{M}} + \Delta t \boldsymbol{K}(\boldsymbol{w}_h^{m-1}) + \Delta t \boldsymbol{G}_k(c_h^{m-1}) \right) \alpha_{k,h}^{m-1} + \Delta t \tilde{\boldsymbol{g}}_{k,h}^{m-1} \right],$$

$$\beta_h^m = \boldsymbol{L}_2^{-1} \left[\tilde{\boldsymbol{M}} \beta_h^{m-1} - \Delta t \boldsymbol{f}_h^{m-1} + r_c \Delta t \tilde{\boldsymbol{M}} \beta_{v,h}^m \right].$$

We recall that $\boldsymbol{L}_1, \boldsymbol{L}_2$ both have non-positive off-diagonal elements and strictly positive diagonal elements. Thus, it follows from Theorem 3.1 in [48] that $\boldsymbol{L}_1^{-1}, \boldsymbol{L}_2^{-1} \geq 0$.

Furthermore, if Δt is small enough that $\tilde{m}_{ii} \beta_i^{m-1} - \Delta t f_i \geq 0$, or equivalently $\Delta t \leq \frac{1}{C n_T c_M}$, then

$$(\tilde{\boldsymbol{M}} - \Delta t \boldsymbol{F}_h^{m-1}) \beta_h^{m-1} + r_c \Delta t \tilde{\boldsymbol{M}} \beta_{v,h}^m \geq 0$$

and we have $\beta_h^m \geq 0$. Additionally, provided Δt is small enough that $\tilde{m}_{ii} + \Delta t(k_{ii} + g_{ii}) \geq 0$, it follows that

$$\tilde{\boldsymbol{M}} + \Delta t \boldsymbol{K}(\boldsymbol{w}_h^{m-1}) + \Delta t \boldsymbol{G}_k(c_h^{m-1}) \geq 0,$$

since all off-diagonal entries are nonnegative. This constraint on Δt is satisfied when

$$\Delta t \leq C \frac{h}{\|\boldsymbol{w}_h^{m-1}\|_\infty}$$

because we have

$$h_{ii}(\boldsymbol{w}) \geq -Ch^{-1} \|\boldsymbol{w}\|_\infty m_{ii} \geq -Ch^{-1} \|\boldsymbol{w}\|_\infty \tilde{m}_{ii},$$

$$0 \leq d_{ij}(\boldsymbol{w}) \leq \max\{|h_{ij}|, |h_{ij}|\} \leq Ch^{-1} \|\boldsymbol{w}\|_\infty m_{ij},$$

$$d_{ii}(\boldsymbol{w}) = -\sum_{j \neq i} d_{ij}(\boldsymbol{w}) \geq -\sum_{j \neq i} Ch^{-1} \|\boldsymbol{w}\|_\infty m_{ij} \geq -Ch^{-1} \|\boldsymbol{w}\|_\infty \tilde{m}_{ii}.$$

Thus, since $\alpha_{k,h}^{m-1} \geq 0$ and $\tilde{\boldsymbol{g}}_{k,h}^{m-1} \geq 0$, we have $\alpha_{k,h}^m \geq 0$.

Furthermore, taking $\tilde{\beta}_h^m$ to hold the nodal values of $c_M - c_h^m$, we have

$$((1 + r_c\Delta t)\tilde{\boldsymbol{M}} + \mu\Delta t\boldsymbol{A})\tilde{\beta}_h^m = \tilde{\boldsymbol{M}}\tilde{\beta}_h^{m-1} + \Delta t\boldsymbol{F}_h^{m-1}\beta_h^{m-1} + r_c\Delta t\tilde{\boldsymbol{M}}(\boldsymbol{c}_M - \beta_{v,h}^m),$$

where $\boldsymbol{c}_M$ is a constant vector with each element equal to c_M. Since $c_M - c_h^{m-1} \geq 0$ and $c_M - c_{v,h}^m \geq 0$, we have $\tilde{\beta}_h^{m-1}, \boldsymbol{c}_M - \beta_{v,h}^m \geq 0$ and the right hand side is nonnegative since $f_{ii} \geq 0$. Then, by the same argument as before, it follows that $\tilde{\beta}_h^m \geq 0$, so $\beta_h^m \leq c_M$.

Finally, taking $\tilde{\alpha}_h^m$ to hold the nodal values of $n_T - \sum_k n_{k,h}^m$, we have

$$\begin{aligned}\left(\tilde{\boldsymbol{M}} + \Delta t\boldsymbol{A}\right)^{-1}\tilde{\alpha}_h^m = \left(\tilde{\boldsymbol{M}} + \Delta t\boldsymbol{K}(\tilde{\boldsymbol{v}}_h^{m-1})\right)\tilde{\alpha}_{k,h}^{m-1} - n_T\Delta t\boldsymbol{K}(\boldsymbol{v}_h^{m-1})\boldsymbol{1}\\ - \Delta t\boldsymbol{K}(\psi(c_h^{m-1})\nabla c_h^{m-1})(\alpha_{p,h}^{m-1} + \alpha_{q,h}^{m-1}) - \sum_k \Delta t\boldsymbol{g}_{k,h}^{m-1},\end{aligned}$$

and we can write

$$\boldsymbol{K}(\boldsymbol{v}_h^{m-1})\boldsymbol{1} = \boldsymbol{H}(\boldsymbol{v}_h^{m-1})\boldsymbol{1} + \boldsymbol{D}(\boldsymbol{v}_h^{m-1})\boldsymbol{1} = \boldsymbol{H}(\boldsymbol{v}_h^{m-1})\boldsymbol{1},$$

since $\boldsymbol{D}$ has zero row sums. Additionally, since $\tilde{\alpha}_h^{m-1} = 0$, we have

$$\tilde{\boldsymbol{M}}\tilde{\alpha}_h^m = -n_T\Delta t\boldsymbol{H}(\boldsymbol{v}_h^{m-1})\boldsymbol{1} - \Delta t\boldsymbol{K}(\psi(c_h^{m-1})\nabla c_h^{m-1})(\alpha_{p,h}^{m-1} + \alpha_{q,h}^{m-1}) - \Delta t\boldsymbol{g}_h^{m-1},$$

where $\boldsymbol{g}_h^{m-1}$ holds the nodal values of G_h^{m-1}. We argue that

$$n_T\boldsymbol{H}(\boldsymbol{v}_h^{m-1})\boldsymbol{1} = -\boldsymbol{K}(\psi(c_h^{m-1})\nabla c_h^{m-1})(\alpha_{p,h}^{m-1} + \alpha_{q,h}^{m-1}) - \boldsymbol{g}_h^{m-1}. \tag{3.9}$$

We note that

$$[\boldsymbol{H}(\boldsymbol{v}_h^{m-1})\boldsymbol{1}]_i = -\frac{\lambda}{K}[\boldsymbol{H}(\nabla p_h^{m-1})\boldsymbol{1}]_i = -\frac{\lambda}{K}a(p_h^m, \phi_i),$$

so (3.9) is exactly $(3.1)_3$. This means that we can write $\tilde{\boldsymbol{M}}\tilde{\alpha}_h^m = 0$, and thus $\tilde{\alpha}_h^m = 0$, which means $\sum_k \alpha_{k,h}^m$. □

3.2.2 Estimates for Artificial Diffusion

We would like to obtain bounds on the term $d(\chi, \eta, \boldsymbol{w})$ for $\chi, \eta \in S_h$, $\boldsymbol{w} \in L^p(\Omega)$. The following estimate holds.

Lemma 3.2.3. *Let* $\chi, \eta \in S_h$. *Then, if* $\nabla\chi \in L^{p_1}(\Omega)$, $\nabla\eta \in L^{p_2}(\Omega)$, *and* $\boldsymbol{w} \in L^{p_3}(\Omega)$ *with* $p_i \in [1, \infty]$ *and* $\sum_i \frac{1}{p_i} = 1$,

$$|d(\chi, \eta, \boldsymbol{w})| \leq Ch\|\nabla\chi\|_{L^{p_1}(\Omega)}\|\boldsymbol{w}\|_{L^{p_2}(\Omega)}\|\nabla\eta\|_{L^{p_3}(\Omega)}.$$

and if $\chi \in L^{p_1}$ *instead, then*

$$|d(\chi, \eta, \boldsymbol{w})| \leq C\|\chi\|_{L^{p_1}(\Omega)}\|\boldsymbol{w}\|_{L^{p_2}(\Omega)}\|\nabla\eta\|_{L^{p_3}(\Omega)}.$$

Here, C *is independent of* h. *Furthermore,* $d(\chi, \eta, \boldsymbol{w}) = d(\eta, \chi, \boldsymbol{w})$ *so the above estimates hold when* η, χ *are switched.*

Proof. Letting α, β hold the nodal values of χ, η, respectively, we note that $\boldsymbol{D}$ is symmetric, so we can write

$$\begin{aligned}
d(\chi, \eta, \boldsymbol{w}) &= \sum_{i,j} \alpha_i \beta_j d_{ij} \\
&= \sum_{i \neq j} \alpha_i d_{ij} (\beta_j - \beta_i) \\
&= \sum_{i<j} d_{ij} (\alpha_i - \alpha_j)(\beta_j - \beta_i) \\
&\leq \sum_{i<j} |h_{ij}| (\alpha_i - \alpha_j)(\beta_j - \beta_i) \\
&= \sum_{i<j} \left| \int_\Omega \phi_j \boldsymbol{w} \cdot \nabla \phi_j \right| (\alpha_i - \alpha_j)(\beta_j - \beta_i).
\end{aligned}$$

We note that $|\phi_j \boldsymbol{w} \cdot \nabla \phi_j| \leq C h^{-1} |\boldsymbol{w}|$, so

$$\begin{aligned}
d(\chi, \eta, \boldsymbol{w}) &\leq h^{-1} \sum_{i<j} \int_{\Omega_{ij}} |\boldsymbol{w}| |\alpha_i - \alpha_j| |\beta_j - \beta_i| \\
&\leq h^{-1} \sum_{i,j} \int_{\Omega_{ij}} |\boldsymbol{w}| |\alpha_i - \alpha_j| |\beta_j - \beta_i|,
\end{aligned}$$

where Ω_{ij} is the domain of $\phi_i \phi_j$, so that

$$\sum_{i,j} \int_{\Omega_{ij}} f \leq C \int_\Omega f,$$

since $\bigcup_{i,j} \Omega_{i,j} = \Omega$ and Ω_{ij} are finitely overlapping, independent of h. Thus, if $\nabla\chi \in L^{p_1}(\Omega)$, $\nabla\eta \in L^{p_2}(\Omega)$, and $\boldsymbol{w} \in L^{p_3}(\Omega)$ with $\sum_i \frac{1}{p_i} = 1$, then

$$\begin{aligned}
|d(\chi, \eta, \boldsymbol{w})| &\le h^{-1} \sum_{i,j} \int_{\Omega_{ij}} |\boldsymbol{w}||\alpha_i - \alpha_j||\beta_j - \beta_i| \\
&\le Ch \sum_{i,j} \int_{\Omega_{ij}} |\nabla\chi||\boldsymbol{w}||\nabla\eta| \\
&\le Ch \|\nabla\chi\|_{L^{p_1}(\Omega)} \|\boldsymbol{w}\|_{L^{p_2}(\Omega)} \|\nabla\eta\|_{L^{p_3}(\Omega)}.
\end{aligned}$$

The inverse inequality (3.5) proves the last inequality. □

3.2.3 Energy Estimates

We derive energy estimates in much the same way as in Section 2.4.2. However, there are certain difficulties to be addressed.

Firstly, we are working with the lumped mass matrix $\tilde{\boldsymbol{M}}$ and the associated inner product $(\cdot,\cdot)_h$ instead of the consistent mass matrix $\boldsymbol{M}$ and inner product $(\cdot,\cdot)$. However, the properties of the lumped inner product $(\cdot,\cdot)_h$ stated previously allow us to use this inner product in a similar way as the standard L^2 inner product.

The next difficulty lies with the added artificial diffusion terms in the update rules for $n_{k,h}^m$. However, these terms can be dealt with in a similar way as other terms using Lemma 3.2.3, so should not pose too much of a challenge.

3.2.3.1 Estimates for c

We start with energy estimates for c_h, which are more straightforward. We recall the formulation

$$(D_t^- c_h^m, \chi)_h + \mu \; a(c_h^m, \chi) + r_c(c_h^m, \chi)_h = (F_h^m, \chi)_h,$$

with $\|F_h^m\|_\infty \leq C$. Taking c_h^m as a test function yields

$$\begin{aligned}
&(c_h^m, c_h^m)_h + \mu\Delta t \; a(c_h^m, c_h^{m-1}) + r_c\Delta t(c_h^m, c_h^{m-1})_h \\
&\quad = (c_h^m, c_h^{m-1})_h + \Delta t(F_h^m, c_h^m)_h \\
&\quad \leq \frac{1}{2}\left(1 + r_c\Delta t\right)\|c_h^m\|_{2,h}^2 + \frac{1}{2}\|c_h^{m-1}\|_{2,h}^2 + C.
\end{aligned}$$

Thus, after cancellation and summing over m, it follows that

$$\|c_h\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla c_h\|_{L^2((0,T)\times\Omega)} \leq C\left(1 + \|c_h^0\|_{L^2(\Omega)}^2\right) \leq C. \tag{3.10}$$

3.2.3.2 Estimates for p

Next we gather energy estimates for p_h^m. Taking p_h^m as a test function yields

$$a(p_h^m, p_h^m) = \frac{\lambda}{Kn_T} k(n_{p,h}^m + n_{q,h}^m, p_h^m, \psi(c_h^m)\nabla c_h^m) - \frac{\lambda}{Kn_T}(G_h^m, p_h^m)_h.$$

Thus, using Lemma 3.2.3, we have

$$\begin{aligned}
\|\nabla p_h^m\|_{L^2(\Omega)}^2 &\leq C\left(n_T c_M \|\nabla c_h^m\|_{L^2(\Omega)} \|\nabla p_h^m\|_{L^2(\Omega)} + C n_T c_M \|p_h^m\|_{L^2(\Omega)}\right) \\
&\leq C\left(1 + \|\nabla c_h^m\|_{L^2(\Omega)}\right)\|\nabla p_h^m\|_{L^2(\Omega)},
\end{aligned}$$

where we have used a Poincare estimate for p_h. Summing over m yields

$$\|p_h^m\|^2_{L^2(0,T;H^1(\Omega))} \leq C\left(1 + \|\nabla c_h^m\|_{L^2((0,T)\times\Omega)}\right) \leq C.$$

This estimate, combined with (3.10), implies that $\|\boldsymbol{w}_h^m\|_{L^2(0,T;L^2(\Omega))} \leq C$.

3.2.3.3 Estimates for n

The update scheme for $n_{k,h}^m$ reads

$$(D_t^- n_{k,h}^m, \chi)_h + d\ a(n_{k,h}^m, \chi) = k(n_{k,h}^{m-1}, \chi, \boldsymbol{w}_h^{m-1}) + (G_{k,h}^{m-1}, \chi)_h$$

which is an implicit- explicit scheme. Similar to the case for c, we take n_h^m as a test function. We know that

$$h(n_{k,h}^{m-1}, n_{k,h}^m, \boldsymbol{w}_h^{m-1}) = \int_\Omega n_{k,h}^{m-1}\boldsymbol{w}_h^{m-1}\cdot\nabla n_{k,h}^m \leq n_T\|\boldsymbol{w}_h^{m-1}\|_{L^2(\Omega)}\|\nabla n_{k,h}^m\|_{L^2(\Omega)}$$

while, due to Lemma 3.2.3,

$$d(n_{k,h}^{m-1}, n_{k,h}^m, \boldsymbol{w}_h^{m-1}) \leq Cn_T\|\nabla n_{k,h}^m\|_{L^2(\Omega)}\|\boldsymbol{w}_h^{m-1}\|_{L^2(\Omega)} \leq C\|\nabla n_{k,h}^m\|_{L^2(\Omega)}.$$

Therefore, we have

$$\begin{aligned}
&\|n_{k,h}^m\|^2_{2,h} + d\Delta t\|\nabla n_{k,h}^m\|^2_{L^2(\Omega)} \\
&\quad \leq \|n_{k,h}^m\|_{2,h}\|n_{k,h}^{m-1}\|_{2,h} + C\Delta t\left(1 + \|\nabla n_{k,h}^m\|_{L^2(\Omega)}\right) \\
&\quad \leq C\Delta t + \frac{1}{2}\left(\|n_{k,h}^m\|^2_{2,h} + \|n_{k,h}^{m-1}\|^2_{2,h}\right) + \frac{d\Delta t}{2}\|\nabla n_{k,h}^m\|^2_{L^2(\Omega)}.
\end{aligned}$$

After cancellation and summing over m we have

$$\|n_h^m\|_{L^2(\Omega)} + d\Delta t \sum_{k=1}^{m} \|\nabla n_h^k\|^2_{L^2(\Omega)} \leq C\left(1 + \|n_h^0\|^2_{L^2(\Omega)}\right).$$

3.3 Proof of Theorem 3.1.4

The proof of this theorem relies on the bounds from Theorem 3.1.2. Those bounds imply that

$$c_{h_k} \rightharpoonup c,\; n_{h_k} \rightharpoonup n \text{ in } L^2(0,T;H^1(\Omega)) \cap L^r((0,T)\times\Omega),$$
$$p_{h_k} \rightharpoonup p \text{ in } L^2(0,T;H^1(\Omega)),$$

where $1 < r < \infty$. Furthermore, as in Section 2.3.0.1, we can use Lemma 2.3.1 to prove that

$$c_{h_k} \to c,\; n_{h_k} \to n \text{ in } L^r((0,T)\times\Omega)$$

for any $r < \infty$. We also note here, that for any $\phi \in C^\infty([0,T]\times\overline{\Omega})$, we can take

$$\phi^m_{h_k} = R_{h_k}\phi^m \in S_{h_k},$$

where $\phi^m = \phi(t^m)$ for $t^{m-1} < t \leq t^m$. It follows from the properties of the operator R_h (see Lemma A.3.1) that

$$\begin{aligned} &\phi^m_{h_k} \to \phi \text{ strongly in } L^r(0,T;W^{1,r}(\Omega)), \\ &D_t^-\phi^m_{h_k} \to \partial_t\phi \text{ strongly in } L^r((0,T)\times\Omega) \end{aligned} \tag{3.11}$$

for any $1 < r < \infty$. Furthermore, we have

$$\begin{aligned}\int_0^T |(f,\phi_{h,k})_h - (f,\phi_{h,k})| &\le Ch\int_0^T \|f\|_{L^s(\Omega)}\|\nabla\phi_{h,k}\|_{L^{s'}(\Omega)}\\ &\le Ch\|f\|_{L^r(0,T;L^s(\Omega))}\|\nabla\phi_{h,k}\|_{L^r(0,T;L^{s'}(\Omega))}\\ &\to 0,\end{aligned}$$

where $\frac{1}{s}+\frac{1}{s'}=1$ and $\frac{1}{r}+\frac{1}{s'}=1$. Thus, we have the convergence

$$\int_0^T (f_k,\phi_{h_k})_h \to \int_0^T (f,\phi), \quad \int_0^T (f_k, D_t^-\phi_{h_k})_h \to \int_0^T (f,\partial_t\phi).$$

We can also use summation by parts to write

$$\begin{aligned}\int_0^T (D_t^- u_h, \phi_h^m)_h &= \sum_{k=1}^m \Delta t (D_t^- u_h^k, \phi_h^k)_h\\ &= (u_h^m,\phi_h^m)_h - (u_h^0,\phi_h^0)_h - \sum_{k=0}^{m-1}\Delta t (u_h^k, D_t^+\phi_h^k)_h.\end{aligned}$$

Additionally, we note that

$$\sum_{j=1}^m \Delta t (c_{v,h}^j, \phi_h^j)_h = \int_\Omega\int_0^T c_v M_h \phi_{h,\Delta t} \to \int_\Omega\int_0^T c_v\phi,$$

due to (3.6), (3.11). Similarly, we have

$$(c_h^0,\phi_h^0)_h = \int_\Omega c_0 M_h\phi_h^0 \to \int_\Omega c_0\phi(0)$$

by the convergence estimates for ϕ_h. The same holds for $n_{k,h}^0$.

We also need to show that

$$(c_{h_k}^m, \phi_{h_k}^m)_h,\ (n_{h_k}^m,\phi_{h_k}^m)_h \to \int_\Omega c(t)\phi(t),\ \int_\Omega n(t)\phi(t),$$

provided $t^m \to t$. This follows from Lemma A.2.1, the bounds on n_h, c_h, and noting that

$$|(c_{h_k}^m, \phi_{h_k}^m) - (c_{h_k}^m, \phi_{h_k}^m)_h| \leq Chc_M \|\phi\|_{L^\infty(\Omega)} \to 0,$$

$$|(n_{h_k}^m, \phi_{h_k}^m) - (n_{h_k}^m, \phi_{h_k}^m)_h| \leq Chn_T \|\phi\|_{L^\infty(\Omega)} \to 0.$$

Finally, due to Lemma 3.2.3 we have the estimate

$$|d(n_h^{m-1}, \boldsymbol{w}_h^{m-1}, R_h\phi)| \leq Ch\|\nabla\phi\|_{L^\infty(\Omega)} \|\boldsymbol{w}_h^{m-1}\|_{L^2(\Omega)} \|\nabla n_h^{m-1}\|_{L^2(\Omega)}.$$

This implies that the artificial diffusion term vanishes as $h \to 0$.

3.4 Model with Darcy's Law on a Smooth Domain

We now consider the problem (3.1) on a smooth domain Ω. In this case, the triangulation will be defined on a polygonal approximation Ω_h of Ω, so that any node of a triangle that lies on $\Gamma_h = \partial\Omega_h$ will also lie on $\Gamma = \partial\Omega$. We assume that Ω is convex, so that $\Omega_h \subset \Omega$ is also convex. We again require assumptions (A1)-(A3).

The main results for this section are that Theorem 3.1.2 and 3.1.4 still hold on a smooth, convex domain $\Omega \subset \mathbb{R}^2$. We need to prove that the estimates (3.3), (3.4), (3.6) still hold uniformly on Ω_h. This is straightforward for (3.4), and thus (3.6), as it holds on each triangular element. For the interpolation and projection operators I_h, P_h, we have from Lemma 3.1 of [33] that

(i) P_h is a bounded operator from $L^p(\Omega_h) \to L^p(\Omega_h)$ and from $W^{1,p}(\Omega_h) \to W^{1,p}(\Omega_h)$ for $p \in [1, \infty]$,

(ii) for integer $0 \leq l \leq 2$, the interpolation operator I_h satisfies

$$|I_h v - v|_{W^{l,\infty}(\Omega_h)} \leq Ch^{2-l}|v|_{W^{2,\infty}(\Omega_h)},$$

(iii) there exists a quasi-interpolation operator $\tilde{I}_h : W^{1,1}(\Omega_h) \to S_h$ with

$$\|\tilde{I}_h v - v\|_{W^{l,p}(\Omega_h)} \leq Ch^{m-l}\|v\|_{W^{m,p}(\Omega_h)}$$

for $l = 0, 1$, $m = 1, 2$, and $p \in [1, \infty]$.

The last estimate also holds quasi-elementwise. Furthermore, we note that since P_h is the L^2 projection onto S_h, it must be that

$$\|P_h v - v\|_{L^2(\Omega_h)} \leq \|\tilde{I}_h v - v\|_{L^2(\Omega)} \leq Ch^m\|v\|_{H^m(\Omega)}$$

for $m = 0, 1, 2$. By the same argument, since R_h is the H^1 projection onto S_h, it must be that

$$\|R_h v - v\|_{H^1(\Omega_h)} \leq \|\tilde{I}_h v - v\|_{H^1(\Omega)} \leq Ch^{m-1}\|v\|_{H^m(\Omega)}.$$

We note here that we can replace R_h by $\tilde{I}_h$ in Lemma A.3.1 and the lemma is still valid. These estimates are sufficient to prove Theorem 3.1.2 on a polygonal approximation Ω_h of a smooth domain Ω.

The convergence estimates are slightly trickier, since $n_{k,h}, c_h, p_h$ are defined on Ω_h, not on Ω, while the test functions ϕ are all defined on Ω. We would like to define extensions of the finite element functions on Ω.

For a boundary element $\tau \in \mathcal{T}_h$, we can extend τ to a curved 'pie-shaped' element $\hat{\tau}$ such that $\hat{\tau}$ matches Γ locally. Otherwise, we set $\hat{\tau} = \tau$. It follows that

$\Omega = \bigcup_{\tau \in \mathcal{T}_h} \hat{\tau}$. For each function $v_h \in S_h$, we can define $\hat{v}_h = \pi_h v_h$, such that v_h is defined on Ω, with v_h extended linearly from τ to $\hat{\tau}$ for each $\tau \in \mathcal{T}_h$. We argue that the mapping π_h is bounded from $L^2(\Omega_h) \to L^2(\Omega)$ and from $H^1(\Omega_h) \to H^1(\Omega)$.

Lemma 3.4.1. *Suppose that $u_h, v_h \in S_h$. Then, the operator $\pi_h : S_h \to \hat{S}_h$ satisfies*

$$\begin{aligned} |(\pi_h u_h, \pi_h v_h)_\Omega - (u_h, v_h)_{\Omega_h}| &\leq Ch \|u_h\|_{L^2(\Omega_h)} \|v_h\|_{H^1(\Omega_h)}, \\ c\|u_h\|_{L^2(\Omega_h)} \leq \|\pi_h u_h\|_{L^2(\Omega)} \leq C\|u_h\|_{L^2(\Omega_h)}, \\ c\|\nabla u_h\|_{L^2(\Omega_h)} \leq \|\nabla \pi_h u_h\|_{L^2(\Omega)} \leq C\|\nabla u_h\|_{L^2(\Omega_h)}. \end{aligned}$$

Furthermore, π_h is invertible.

The proof is in the appendix. We then arrive at the following result.

Lemma 3.4.2. *Fix $h > 0$ sufficiently small, and assume that $\Delta t > 0$ satisfies $\Delta t \leq Ch^{2+\alpha}$ for some $\alpha, C > 0$ independent of h. Suppose $0 \leq n_{k,h}^0 \leq n_T$ and $0 \leq c_h^0 \leq c_M$ on Ω_h. Then, there exists a unique solution $(n_{k,h}, c_h, p_h)$ to the problem on the domain Ω_h, and the extensions $\hat{n}_{k,h} := \pi_h n_{k,h}$, $\hat{c}_h := \pi_h c_h$, $\hat{p}_h : \pi_h c_h$ satisfy*

$$\begin{aligned} \|D_t^- \hat{c}_h\|_{L^2(0,T;H^{-1}(\Omega))} + \|\hat{c}_h\|_{L^2(0,T;H^1(\Omega))} &\leq C, \\ \|D_t^- \hat{n}_{k,h}\|_{L^2(0,T;H^{-1}(\Omega))} + \|\hat{n}_{k,h}\|_{L^2(0,T;H^1(\Omega))} &\leq C, \\ \|\hat{p}_h\|_{L^\infty(0,T;H^1(\Omega))} &\leq C. \end{aligned}$$

Most of these estimates follow directly from Lemma 3.4.1. We show how we arrive at the estimates for $D_t^- \hat{n}_{k,h}$ and $D_t^- \hat{c}_h$ in more detail. First, we note that

$$\|D_t^- n_{k,h}\|_{L^2(0,T;H^{-1}(\Omega_h))} \leq C$$

via the same estimates used in the case of a polygonal domain. Then, for any $v_h \in \hat{S}_h$, it follows that $v_h = \pi_h w_h$ for some $w_h \in S_h$, since π_h is invertible. We have

$$
\begin{aligned}
|(D_t^- \hat{n}_{k,h}, v_h)_\Omega| &\leq C|(D_t^- n_{k,h}, w_h)_{\Omega_h}| + |(D_t^- \hat{n}_{k,h}, w_h)_\Omega - (D_t^- n_{k,h}, w_h)_{\Omega_h}| \\
&\leq C|(D_t^- n_{k,h}, w_h)_{\Omega_h}| + Ch\|D_t^- n_{k,h}\|_{L^2(\Omega_h)}\|w_h\|_{H^1(\Omega_h)}
\end{aligned}
$$

by Lemma 3.4.1 and the inverse inequality. Additionally, since $D_t^- n_{k,h}$ is uniformly bounded in $L^2(0,T;H^{-1}(\Omega_h))$, it follows that

$$
|(D_t^- n_{k,h}, w_h)_{\Omega_h}| \leq C\|w_h\|_{H^1(\Omega_h)} \leq C\|v_h\|_{H^1(\Omega)},
$$

where we have again used Lemma 3.4.1, and furthermore, $h\|D_t^- n_{k,h}\|_{L^2(\Omega_h)} \leq C$. This proves the bound on $D_t^- \hat{n}_{k,h}$. The bound on $D_t^- \hat{c}_{k,h}$ is proven in the same manner

The uniform bounds in Lemma 3.4.2 are sufficient to show convergence in the same manner as in Section 3.3. We note here that when using the discrete Aubin-Lions lemma, we must take $X = L^r(\Omega)$ for $1 \leq r < 6$. However, we must show that if $\hat{f}_h \rightharpoonup f$ in $L^r(\Omega)$ and $\hat{\phi}_h \to \phi$ in $L^s(\Omega)$ with $\frac{1}{r} + \frac{1}{s} < 1$, it follows that

$$
\int_{\Omega_h} f_h \phi_h \to \int_\Omega f\phi.
$$

To do this, we note that

$$
\int_{\Omega_h} f_h \phi_h + \int_{\Omega\backslash\Omega_h} \hat{f}_h \hat{\phi}_h = \int_\Omega \hat{f}_h \hat{\phi}_h \to \int_\Omega f\phi,
$$

Due to the convergence estimates determined above, we know that

$$
\|\hat{f}_h \hat{\phi}_h\|_{L^q(\Omega\cup\Omega_h)} \leq C,
$$

where $\frac{1}{q} = \frac{1}{r} + \frac{1}{s} < 1$. Then, we can use Holder's inequality to write

$$\left| \int_{\Omega \backslash \Omega_h} f_h \phi_h \right| \le |\Omega \backslash \Omega_h|^{1/q'} \|f_h \phi_h\|_{L^q(\Omega \backslash \Omega_h)} \le C |\Omega \backslash \Omega_h|^{1/q'}.$$

Due to the construction of Ω_h, it must be that $|\Omega \backslash \Omega_h| \le Ch$ with C independent of h. Therefore,

$$\left| \int_{\Omega \backslash \Omega_h} f_h \phi_h \right| \le C h^{1/q'} \to 0$$

as $h \to 0$. This proves the desired result.

3.5 Model with Darcy's Law on a Moving Domain

Next, we would like to consider the problem on a moving domain where the motion of the domain is prescribed, like we did in the functional analytic case. Again, we use the Arbitrary Lagrangian-Eulerian, or ALE, framework.

3.5.1 Moving mesh

We have the non-cylindrical domain

$$Q_T = \{(x,t) : x \in \Omega(t) \text{ for } t \in [0,T]\}$$

We assume that for each $t \in [0,T]$, $\Omega(t) \subset \mathbb{R}^2$ is a convex, polygonal domain. Then, we split the interval $[0,T]$ into N sub-intervals of length Δt, as before. For $t^m = m\Delta t$, we set $\Omega^m = \Omega(t^m)$. Thus, we will have a fixed domain and fixed mesh over each subinterval.

We define an initial triangulation $\mathcal{T}_h^0$ of Ω^0 that satisfies the assumptions (A1)-(A3), and let S_h^0 be the space of continuous, piecewise linear finite element functions on $\mathcal{T}_h^0$. Then, we let $\eta_h \in [S_h^0]^2$ be a finite element approximation of η, given by $\eta_h(t) = I_h\eta(t)$ for all t. We define $\eta_h^m = \eta_h(t^m)$. For each $m > 0$, we let $\mathcal{T}_h^m$ be the image of $\mathcal{T}_h^0$ under the mapping η_h^m, so that $\mathcal{T}_h^m$ is a triangulation of Ω^m. We let S_h^m be the space of continuous, piecewise linear functions on the triangulation $\mathcal{T}_h^m$ for $0 \leq t^m \leq T$. We define $Q_T^h = \bigcup_m (t^{m-1}, t^m] \times \Omega^m$, so that Q_T^h is an approximation of Q_T.

3.5.1.1 Properties of the mapping

Since $\eta_h(t)$ is a nodal interpolant of $\eta(t)$ and $\eta \in L^\infty(0,T;W^{2,\infty}(\Omega(t)))$, it follows that

$$\|\eta_h(t) - \eta(t)\|_{L^\infty(\Omega^0)} + h\|\nabla(\eta_h(t) - \eta(t))\|_{L^\infty(\Omega^0)} \leq Ch^2|\log h|\|\eta(t)\|_{W^{2,\infty}(\Omega^0)} \quad (3.12)$$

for each $t \in [0,T]$. For details, see [46], [28], or standard properties of the interpolation operator.

Furthermore, since we assumed that η is bi-Lipschitz we have

$$c_L|x-y| \leq |\eta_h(x) - \eta_h(y)| = |\eta(x) - \eta(y)| \leq C_L|x-y|,$$

for nodal points $x, y \in \Omega^0$. Since η_h is linear on τ for each $\tau \in \mathcal{T}_h^0$, it must be that η_h is also bi-Lipschitz, with the same constants as η. Additionally, (3.12) implies that

$$|\det(\nabla\eta_h) - \det(\nabla\eta)| = |\det(\nabla\eta_h) - 1| \leq Ch^2|\log h|^2, \quad (3.13)$$

so provided h is sufficiently small, $0 < c \leq \det(\nabla \eta_h^m) \leq C$ uniformly in h, m, and thus η_h^m is invertible.

Remark 3.5.1. It is important to note here that we *cannot* guarantee that η_h^m is a volume-preserving mapping like we can with η. Thus, we have to be slightly careful. Luckily, most of the estimates will be done via energy methods, which do not require mapping between domains.

Then, we note that since $\eta_h^m \in S_h^0$ and is thus linear on each triangular element in $\mathcal{T}_h^0$, it follows that $(\eta_h^m)^{-1} \in S_h^m$ since it must also be continuous and linear on each $\tau \in \mathcal{T}_h^m$. Furthermore, since $(\eta_h^m)^{-1} = \eta^{-1}(t^m)$ on each node of $\mathcal{T}_h^m$, it follows that $(\eta_h^m)^{-1} = I_h^m \eta^{-1}(t^m)$.

We examine the spaces S_h^m. If $v \in S_h^m$, then it follows that $\tilde{v} = v \circ \eta_h^m \in S_h^m$, and similarly $v = \tilde{v} \circ (\eta_h^m)^{-1}$. Thus, we can think of S_h^m as the image of S_h^0 under η_h^m. Additionally, letting $\overline{S}_h^m$ be the space spanned by the barycentric basis functions (defined previously), we also see that, for $v \in \overline{S}_h^m$, we can write $v = \tilde{v} \circ (\eta_h^m)^{-1}$ for some $\tilde{v} \in \overline{S}_h^0$, since η_h^m is an affine mapping on each $\tau \in \mathcal{T}_h^0$ (note: since η_h^m is affine on each τ, it will map barycentric domains to barycentric domains). This means that, for $v_h^m = v_h \circ (\eta_h^m)^{-1}$, it follows that $M_h^m v_h^m = (M_h^0 v_h) \circ (\eta_h^m)^{-1}$, where M_h^m is the mass-lumping operator on $\mathcal{T}_h^m$.

Then, we let $\boldsymbol{v}_{s,h,\Delta t} = \partial_t \eta_{h,\Delta t}$ so that $\boldsymbol{v}_{s,h,\Delta t}$ is piecewise constant in time and $\boldsymbol{v}_{s,h}^m = \boldsymbol{v}_{s,h,\Delta t}(t^m) = \left(\frac{\eta_h^m - \eta_h^{m-1}}{\Delta t}\right) \circ (\eta_h^m)^{-1} = \frac{I - \eta_h^{m-1} \circ (\eta_h^m)^{-1}}{\Delta t}$ for m such that $0 < t^m \leq T$. It follows that $\boldsymbol{v}_{s,h}^m \in [S_h^m]^2$.

3.5.1.2 Properties of the mesh

We must prove that $\mathcal{T}_h^m$ satisfies the assumptions (A1)-(A3) independent of h and m, provided T is sufficiently small. Suppose that the initial triangulation $\mathcal{T}_h^0$ satisfies(A1)-(A3). Then, there exists $t_0 > 0$ such that the triangulations $\mathcal{T}_h^m$, for $0 \leq t^m \leq t_0$, satisfy (A1), (A2) (see, for example, [46]). This is due to the assumptions on η.

However, it remains to show that (A3) is satisfied. Given certain initial triangulation $\mathcal{T}_h^0$ which satisfy (A1)-(A3), it is possible to show that certain mappings η will preserve the condition (A3). For example, any translation or rotation will preserve (A3).

Additionally, if the initial triangulation $\mathcal{T}_h^0$ is acute, then it can be shown that there exists $t_1 > 0$ such that the triangulations $\mathcal{T}_h^m$, for $0 \leq t \leq t_1$, are also acute. Furthermore, we note that any convex polygon has an acute triangulation, as proved in Theorem 1 in [39]. Thus, we can generate an acute triangulation $\mathcal{T}^0$ of the domain Ω^0, and create the mesh $\mathcal{T}_h^0$ by subdividing the acute triangles in $\mathcal{T}^0$ until the desired mesh size is reached. The triangulation $\mathcal{T}_h^0$ will satisfy conditions (A1)-(A3) since all subdivisions of a triangle are congruent to the original triangle, and will have a maximal interior angle less than $\pi/2$ since the triangulation is acute.

Therefore, it is always possible to choose an initial mesh $\mathcal{T}_h^0$ so that the assumptions (A1)-(A3) are satisfied on a sufficiently short time interval. This means that domain can be remeshed a finite number of times. Between remeshings, the mesh simply moves with the motion of the domain, η_h, without the need to redefine nodes or edges. Without loss of generality, we will assume that $[0, T]$ is such that no remeshings are needed.

3.5.2 Finite element scheme on a polygonal domain

Before we define the finite element formulation for this problem, we need to determine how to deal with the ALE advection terms $\nabla c \cdot \boldsymbol{v}_s, \nabla n_k \cdot \boldsymbol{v}_s$ arising in the governing equation for the nutrient and cell densities. We must define an approximation of this term that allows us to prove that the scheme is positivity preserving and preserves the upper bound c_M, n_T on c, n_k. We do this in a similar way as for the artificial diffusion for the advection term in the governing equations for the nutrient density. Letting

$$\hat{j}_{ij}(\boldsymbol{v}) = -\int_\Omega (\nabla \phi_i \cdot \boldsymbol{v})\phi_j,$$

we see that $\hat{j}_{ij}(\boldsymbol{v}) = h_{ji}(\boldsymbol{v})$, so $\hat{\boldsymbol{J}}(\boldsymbol{v}) = \boldsymbol{H}^T(\boldsymbol{v})$. Since $\boldsymbol{D}(\boldsymbol{v})$ is symmetric, it follows that $\boldsymbol{J}(\boldsymbol{v}) = \hat{\boldsymbol{J}}(\boldsymbol{v}) + \boldsymbol{D}(\boldsymbol{v})$ has nonnegative off-diagonal entries, just like $\boldsymbol{K}$. In fact, $\boldsymbol{J}(\boldsymbol{v}) = \boldsymbol{K}^T(\boldsymbol{v})$.

For the cell densities, we again proceed in a similar way as before by writing $\nabla n_k \cdot \boldsymbol{v}_s = \operatorname{div}(n_k \boldsymbol{v}_s) - n_k \operatorname{div} \boldsymbol{v}_s$, so that we need to discretize the term $n_k \operatorname{div} \boldsymbol{v}_s$ in a manner which will preserve positivity.

Furthermore, we approach the inhomogeneous Neumann boundary conditions on the pressure by splitting the pressure into two components, p, q with

$$\Delta q = -\operatorname{div} \boldsymbol{v}_s, \quad \nabla q \cdot \boldsymbol{n} = -\boldsymbol{v}_s \cdot \boldsymbol{n},$$

$$\Delta p = \operatorname{div} \boldsymbol{v}_s + n_T^{-1} \operatorname{div}((n_p + n_q)\psi(c)\nabla c) - n_T^{-1} \sum_k G_k(c, n), \quad \nabla p \cdot \boldsymbol{n} = 0.$$

The term $\operatorname{div} \boldsymbol{v}_s$ arises here, which we need to handle in the same way as the $n_k \operatorname{div} \boldsymbol{v}_s$ term from the governing equations for the cell densities.

We want these terms to be represented by a diagonal matrix in the same way as for the lumped mass matrix, so we define

$$m_{ij}(f) = \int_\Omega \phi_i \phi_j f.$$

Then, we set

$$\tilde{m}_{ii}(f) = \sum_j m_{ij}(f), \quad \tilde{\boldsymbol{M}}(f) = \operatorname{diag}\{\tilde{m}_{ii}(f)\}$$

so that $\tilde{\boldsymbol{M}}(f)$ is a diagonal matrix. If u, v have nodal values α, β, respectively, we set

$$(u, v, f)_h^m = \alpha^T \tilde{\boldsymbol{M}}^m \beta.$$

Then, we have the finite element formulation

$$
\begin{aligned}
&(D_t^- n_{k,h}^m, \chi)_h^m + d\ a^m(n_{k,h}^m, \chi) \\
&\quad = k^{m-1}(n_{k,h}^{m-1}, \chi \circ (A_h^m)^{-1}, \boldsymbol{w}_h^{m-1}) - (n_{k,h}^{m-1}, \chi \circ (A_h^m)^{-1}, \operatorname{div} \boldsymbol{v}_{s,h}^{m-1})_h^{m-1} \\
&\qquad + (G_{k,h}^{m-1}, \chi \circ (A_h^m)^{-1})_h^{m-1}, \\
&(D_t^- c_h^m, \chi)_h^m + \mu\ a^m(c_h^m, \chi) + r_c(c_h^m, \chi)_h = k^m(\chi, c_h^m, \boldsymbol{v}_{s,h}^m) + (F_h^m \circ A_h^m, \chi)_h^m, \\
&a^m(q_h^m, \chi) = -(\boldsymbol{v}_{s,h}^m, \nabla \chi)_{\Omega^m} \\
&a^m(p_h^m, \chi) = -(1, \chi, \operatorname{div} \boldsymbol{v}_{s,h}^m)_h^m + n_T^{-1} k^m(n_{p,h}^m + n_{q,h}^m, \chi, \psi(c_h^m)\nabla c_h^m) + n_T^{-1}(G_h^m, \chi)_h^m
\end{aligned}
\tag{3.14}
$$

for all $\chi \in S_h^m$, where $A_h^m = \eta_h^{m-1} \circ (\eta_h^m)^{-1} : \Omega^m \to \Omega^{m-1}$ and

$$
\boldsymbol{w}_h^{m-1} = -\nabla q_h^{m-1} - \nabla p_h^{m-1} + \psi(c_h^{m-1})\nabla c_h^{m-1} - \boldsymbol{v}_{s,h}^{m-1},
$$

$$
K_{R,h} = n_T(K_B(c_h^m) n_{p,h}^m, 1)_h,
$$

and $G_{k,h}^{m-1}, F_h^{m-1}, G_h^m$ are the same as in prior sections. We have used the notation

$$
(u, v)_h^m = \int_{\Omega^m} M_h^m u M_h^m v, \quad a^m(u, v) = \int_{\Omega^m} \nabla u \cdot \nabla v,
$$

$$
k^m(u, v, \boldsymbol{w}) = \int_{\Omega^m} u \nabla v \cdot \boldsymbol{w}.
$$

We need to verify that the estimates proven in the case of the fixed domain are also valid here.

We note that we will define $c_{v,h}^m$ similarly as before, by

$$
c_{v,h}^m = (\Delta t)^{-1} (M_h^m)^{-1} D_h^m \int_{t^{m-1}}^{t^m} c_v(t) \circ A^m(t)\ dt,
$$

where we have accounted for the fact that c_v is defined on a non-cylindrical domain.

3.5.3 Positivity preservation

Denoting by $\alpha_{k,h}^m, \beta_h^m$ the vectors holding the nodal values of functions $n_{k,h}^m, c_h^m$ we can write the update schemes for $n_{k,h}^m, c_h^m$ as

$$\begin{aligned}
&\left(\tilde{\boldsymbol{M}}^m + d\Delta t\boldsymbol{A}^m\right)\alpha_{k,h}^m \\
&\qquad = \left(\tilde{\boldsymbol{M}}^m + \Delta t\boldsymbol{K}^{m-1}(\boldsymbol{w}_h^{m-1}) - \Delta t\tilde{\boldsymbol{M}}^{m-1}(\operatorname{div}\boldsymbol{v}_{s,h}^{m-1})\right)\alpha_{k,h}^{m-1} + \Delta t\boldsymbol{g}_{k,h}^{m-1}, \\
&\left(\tilde{\boldsymbol{M}}^m + \mu\Delta t\boldsymbol{A}^m + r_c\Delta t\tilde{\boldsymbol{M}}^m - \Delta t(\boldsymbol{K}^m(\boldsymbol{v}_{s,h}^m))^T\right)\beta_h^m \\
&\qquad = \left(\tilde{\boldsymbol{M}}^m - \Delta t\boldsymbol{F}_h^{m-1}\right)\beta_h^{m-1} + \Delta t\tilde{\boldsymbol{M}}^m\boldsymbol{r}_c.
\end{aligned}$$

Most of the argument for positivity-preservation follows in the same method as before due to the construction of $\boldsymbol{K}$ and $\tilde{\boldsymbol{M}}(f)$. The addition of the terms $\boldsymbol{K}(\boldsymbol{v}_{s,h})$, $\tilde{\boldsymbol{M}}^{m-1}(\boldsymbol{v}_{s,h}^{m-1})$ requires us to have Δt small enough that

$$\Delta t \leq C\frac{h}{\|\nabla c_h^{m-1}\|_{L^\infty} + \|\nabla p_h^{m-1}\|_{L^\infty} + \|\boldsymbol{v}_{s,h}\|_{L^\infty(Q_T)}}.$$

In order to demonstrate $c_h^m \leq c_M$, we take $\tilde{\beta}_h^m$ to hold the nodal values of $c_M - c_h^m$ which yields

$$\begin{aligned}
&((1 + r_c\Delta t)\tilde{\boldsymbol{M}}^m + \mu\Delta t\boldsymbol{A}^m - \Delta t(\boldsymbol{K}^m(\boldsymbol{v}_{s,h}^m))^T)\tilde{\beta}_h^m \\
&\qquad = \tilde{\boldsymbol{M}}^m\tilde{\beta}_h^{m-1} - \Delta t(\boldsymbol{K}^m(\boldsymbol{v}_{s,h}^m))^T\boldsymbol{c}_M + \Delta t\boldsymbol{F}_h^{m-1}\beta_h^{m-1} + \Delta t\tilde{\boldsymbol{M}}^m(\boldsymbol{c}_M - \boldsymbol{r}_c),
\end{aligned}$$

where $\boldsymbol{c}_M$ is a constant vector with each element equal to $r_c c_M$. We note that, since $\boldsymbol{c}_M$ is a constant vector, $(\boldsymbol{K}^m(\boldsymbol{v}_{s,h}^m))^T\boldsymbol{c}_M = \hat{\boldsymbol{J}}^m(\boldsymbol{v}_{s,h}^m)\boldsymbol{c}_M$ since $\boldsymbol{D}^m$ has zero row and column sums. Furthermore,

$$[(\boldsymbol{K}^m(\boldsymbol{v}_{s,h}^m))^T\boldsymbol{c}_M]_i = -\int_{\Omega^m} \nabla c_M \cdot \boldsymbol{v}_{s,h}^m \phi_i = 0.$$

Since $c_M - c_h^{m-1} \geq 0$ and $r_c(c_M - c_v) \geq 0$, we have $\tilde{\beta}_h^{m-1}, \boldsymbol{c}_M - \boldsymbol{r}_c \geq 0$ and the right hand side is nonnegative since $f_{ii} \geq 0$. Then, by the same argument as before, it follows that $\tilde{\beta}_h^m \geq 0$, so $\beta_h^m \leq c_M$.

Additionally, in order to demonstrate $\sum_k n_{k,h}^m = n_T$, we take $\tilde{\alpha}_h^m$ to hold the nodal values of $n_T - \sum_k n_{k,h}^m$. This yields

$$\begin{aligned}\left(\tilde{\boldsymbol{M}}^m + \Delta t \boldsymbol{A}^m\right)^{-1} \tilde{\alpha}_h^m = {} & \left(\tilde{\boldsymbol{M}}^m + \Delta t \boldsymbol{K}^{m-1}(\tilde{\boldsymbol{v}}_h^{m-1}) - \Delta t \tilde{\boldsymbol{M}}^{m-1}(\operatorname{div} \boldsymbol{v}_{s,h}^{m-1})\right) \tilde{\alpha}_{k,h}^{m-1} \\ & - n_T \Delta t \left(\boldsymbol{K}^{m-1}(\boldsymbol{v}_h^{m-1}) - \tilde{\boldsymbol{M}}^{m-1}(\operatorname{div} \boldsymbol{v}_{s,h}^{m-1}))\right) \boldsymbol{1} \\ & - \Delta t \boldsymbol{K}^{m-1}(\psi(c_h^{m-1}) \nabla c_h^{m-1})(\alpha_{p,h}^{m-1} + \alpha_{q,h}^{m-1}) - \sum_k \Delta t \boldsymbol{g}_{k,h}^{m-1},\end{aligned}$$

and we can write

$$\boldsymbol{K}^{m-1}(\boldsymbol{v}_h^{m-1})\boldsymbol{1} = \boldsymbol{H}^{m-1}(\boldsymbol{v}_h^{m-1})\boldsymbol{1} + \boldsymbol{D}^{m-1}(\boldsymbol{v}_h^{m-1})\boldsymbol{1} = \boldsymbol{H}^{m-1}(\boldsymbol{v}_h^{m-1})\boldsymbol{1},$$

since $\boldsymbol{D}^{m-1}$ has zero row sums. Then, since $\tilde{\alpha}_h^{m-1} = 0$, we have

$$\begin{aligned}(\tilde{\boldsymbol{M}}^m + \Delta t \boldsymbol{A}^m)\tilde{\alpha}_h^m = {} & -n_T \Delta t \left(\boldsymbol{H}^{m-1}(\boldsymbol{v}_h^{m-1}) - \tilde{\boldsymbol{M}}^{m-1}(\operatorname{div} \boldsymbol{v}_{s,h}^{m-1})\right) \boldsymbol{1} \\ & - \Delta t \boldsymbol{K}^{m-1}(\psi(c_h^{m-1}) \nabla c_h^{m-1})(\alpha_{p,h}^{m-1} + \alpha_{q,h}^{m-1}) - \Delta t \boldsymbol{g}_h^{m-1},\end{aligned}$$

where $\boldsymbol{g}_h^{m-1}$ holds the nodal values of G_h^{m-1}. Using the fact that $\boldsymbol{v}_h^{m-1} = -\nabla q_h^{m-1} - \nabla p_h^{m-1} - \boldsymbol{v}_{s,h}^{m-1}$ and the update schemes for q_h^m, p_h^m we have

$$\begin{aligned}-n_T \boldsymbol{H}^{m-1}(\boldsymbol{v}_h^{m-1})\mathbf{1} = -n_T \tilde{\boldsymbol{M}}^{m-1}(\operatorname{div} \boldsymbol{v}_{s,h}^{m-1})\mathbf{1} \\ + \boldsymbol{K}^{m-1}(\psi(c_h^{m-1})\nabla c_h^{m-1})(\alpha_{p,h}^{m-1} + \alpha_{q,h}^{m-1}) + \boldsymbol{g}_h^{m-1}.\end{aligned}$$

This means that we can write $\tilde{\boldsymbol{M}}^{m-1}\tilde{\alpha}_h^m = 0$, and thus $\tilde{\alpha}_h^m = 0$, which means $\sum_k \alpha_{k,h}^m$.

3.5.4 Energy Estimates

Next, we need to demonstrate that the solution satisfies the same energy estimates as in the previous sections. We that $\boldsymbol{v}_{s,h} \in L^\infty(0, T; W^{1,\infty}(\Omega))$ uniformly in h and the mesh $\mathcal{T}_h^m$ is uniformly non-degenerate in time. Then, the regularity estimates for c_h^m follow in a similar manner as before, using the estimate

$$k^m(c_h^m, c_h^m, \boldsymbol{v}_{s,h}^m) \leq C\|\nabla c_h^m\|_{L^2(\Omega^m)}\|c_h^m\|_{L^2(\Omega^m)} \leq C\|\nabla c_h^{m-1}\|_{L^2(\Omega^m)}.$$

We can derive a bound on q_h^m by taking q_h^m as a test function and obtaining

$$\|q_h\|_{L^\infty(0,T;L^2(\Omega_{\Delta t}))} \leq C\|\boldsymbol{v}_s\|_{L^\infty(0,T;L^2(\Omega_{\Delta t}))}.$$

The estimate for p_h^m follows in a similar manner as before by noting that

$$(1, p_h^m, \operatorname{div} \boldsymbol{v}_{s,h}^m)_h^m = \int_{\Omega^m} p_h^m \operatorname{div} \boldsymbol{v}_{s,h}^m \leq \|p_h^m\|_{L^2(\Omega^m)}\|\operatorname{div} \boldsymbol{v}_{s,h}^m\|_{L^2(\Omega^m)}.$$

Taking these estimates together yields a uniform bound on $\boldsymbol{w}_h$ in $L^2(Q_T)$. Combining this with the estimates

$$k^m(n_h^m \circ (A_h^m)^{-1}, n_h^{m-1}, \boldsymbol{v}_{s,h}^{m-1}) \leq C\|\nabla n_h^{m-1}\|_{L^2(\Omega^{m-1})}$$

$$(n_{k,h}^{m-1}, n_{k,h}^m \circ (A_h^m)^{-1}, \operatorname{div} \boldsymbol{v}_{s,h}^{m-1})_h^{m-1} \leq Cn_T^2\| \operatorname{div} \boldsymbol{v}_{s,h}^{m-1}\|_{L^2(\Omega^{m-1})} \leq C$$

yields the desired bounds on $n_{k,h}^m$.

3.5.5 Convergence

It remains to demonstrate convergence. We define $\tilde{n}_{k,h}^m,\ \tilde{c}_h^m,\ \tilde{p}_h^m,\ \tilde{\boldsymbol{v}}_{s,h}^m \in S_h^0$ in the same manner as Chapter 2. The regularity of η_h^m implies that the same uniform bounds are satisfied by these modified quantities on Ω_0, and thus weak convergence of the modified quantities is obtained on Ω_0. We note that, while $\det(\eta_h^m) \neq 1$, we do have $0 < c \leq \det(\eta_h^m) \leq C$ uniformly in h, m. Furthermore, since

$$(D_t^- a_h^m) \circ \eta^m = \frac{\tilde{a}_h^m - \tilde{a}_h^{m-1}}{\Delta t} = D_t^- \tilde{a}_h^m,$$

we obtain the same bounds on the discrete time derivatives of $\tilde{n}_{k,h}$ and $\tilde{c}_h$ as before.

We define

$$(B_h^m)_{ij} = (b_{ij})_h^m, \quad (b_{ij})_h^m = (\partial_{x_i}(\eta_h^m)_j^{-1}) \circ \eta_h^m.$$

Lemma 3.5.2. *The modified quantities $\tilde{n}^m_{k,h}, \tilde{c}^m_h, \tilde{p}^m_h$ satisfy the weak formulation*

$$
\begin{align*}
&\int_{\Omega(0)} \tilde{n}^m_{k,h}\chi^m - \int_{\Omega(0)} \tilde{n}^0_{k,h}\chi^0 - \sum_{j=0}^{m-1} \Delta t \int_{\Omega(0)} \tilde{n}^j_{k,h} D_t^+ \chi^j \\
&\quad = \sum_{j=0}^{m-1} \Delta t \int_{\Omega(0)} \chi^j \left(-\tilde{n}^{j-1}_{k,h} B^{j-1}_h : \nabla \tilde{\boldsymbol{v}}^{j-1}_{s,h} + G_k(\tilde{c}^{j-1}_h, \tilde{n}^{j-1}_h) \right) \\
&\qquad + \sum_{j=1}^{m} \Delta t \int_{\Omega(0)} \Big((-dB^j_h \nabla \tilde{n}^j_{k,h} + \tilde{n}^{j-1}_{k,h} \tilde{\boldsymbol{w}}^{j-1}_h) \cdot B^j_h \nabla \chi^j \Big) + \mathcal{E}_1 \\
&\sum_{j=1}^{m} \Delta t \int_{\Omega(0)} \chi^j \Big(D_t^- \tilde{c}^j_h - B^j_h \nabla \tilde{c}^j_h \cdot \tilde{\boldsymbol{v}}^j_{s,h} + \mu B^j_h \nabla \tilde{c}^j_h \cdot B^j_h \nabla \chi^j \Big) \\
&\quad = -\sum_{j=1}^{m} \Delta t \int_{\Omega(0)} \Big((\tilde{n}^{j-1}_{p,h} + \tilde{n}^{j-1}_{q,h}) f(\tilde{c}^{j-1}_h) \chi^j \Big) + \mathcal{E}_2 \tag{3.15} \\
&\sum_{j=1}^{m} \Delta t \int_{\Omega(0)} B^j_h \nabla \tilde{q}^j_h \cdot B^j_h \nabla \chi^j = -\sum_{j=1}^{m} \Delta t \int_{\Omega(0)} \tilde{\boldsymbol{v}}^j_{s,h} \cdot B^j_h \nabla \chi^j + \mathcal{E}_3 \\
&\sum_{j=1}^{m} \Delta t \int_{\Omega(0)} \left(B^j_h \nabla \tilde{p}^j_h \cdot B^j_h \nabla \chi^j + \chi^j B^j_h : \nabla \tilde{\boldsymbol{v}}^j_{s,h} \right) \\
&\quad = n_T^{-1} \sum_{j=1}^{m} \Delta t \int_{\Omega(0)} (\tilde{n}^j_{p,h} + \tilde{n}^j_{q,h}) \psi(\tilde{c}^j_h) B^j_h \nabla \tilde{c}^j_h \cdot B^j_h \nabla \chi^j \\
&\qquad + n_T^{-1} \sum_{j=1}^{m} \Delta t \int_{\Omega(0)} \left(K_B(\tilde{c}^j_h) \tilde{n}^j_{p,h} - n_T K_R(\tilde{c}^j_h, \tilde{n}^j_h) \right) \chi^j + \mathcal{E}_4,
\end{align*}
$$

where $\mathcal{E}_i \to 0$ as $h \to 0$. Here

$$
\tilde{\boldsymbol{w}}^m_h = -B^m_h \nabla \tilde{p}^m_h - B^m_h \nabla \tilde{q}^m_h + B^m_h \nabla \tilde{c}^m_h - \tilde{\boldsymbol{v}}^m_{s,h}.
$$

Proof. There are several components of the error terms $\mathcal{E}_i$. We have to account for the fact that $\det \eta^m_h \neq 1$, the difference between the mass-lumped and consistent inner products, and the artificial diffusion terms.

We first show how we deal with the discrete time derivative term. For example, for c_h^m, we have

$$\begin{aligned}
&\sum_{j=1}^{m} \Delta t (D_t^- c_h^j, \chi^j)_h^j \\
&\quad = \sum_{j=1}^{m} \Delta t (M_h^j D_t^- c_h^j, M_h^j \chi^j)_{\Omega^j} \\
&\quad = \sum_{j=1}^{m} \Delta t \int_{\Omega^0} M_h^0 (D_t^- \tilde{c}_h^j) M_h^0 \tilde{\chi}^j + \sum_{j=1}^{m} \Delta t \int_{\Omega^0} M_h^0 (D_t^- \tilde{c}_h^j) M_h^0 \tilde{\chi}^j (\det \eta_h^j - 1) \\
&\quad = \int_{\Omega^0} M_h^0 \tilde{c}_h^m M_h^0 \tilde{\chi}^m - \int_{\Omega^0} M_h^0 \tilde{c}_h^0 M_h^0 \tilde{\chi}^0 - \sum_{j=0}^{m-1} \Delta t \int_{\Omega^0} M_h^0 \tilde{c}_h^j M_h^0 D_t^+ \tilde{\chi}^j \\
&\qquad + \sum_{j=1}^{m} \Delta t \int_{\Omega^0} M_h^0 (D_t^- \tilde{c}_h^j) M_h^0 \tilde{\chi}^j (\det \eta_h^j - 1).
\end{aligned}$$

A similar argument holds for $n_{k,h}^m$. Then, we can write

$$\mathcal{E}_1 = \mathcal{E}_1^0 + \sum_{j=1}^{m} \Delta t \mathcal{E}_1^j,$$

with

$$\mathcal{E}_1^0 = \big[(\tilde{n}_{k,h}^m, \chi^m)_{\Omega^0} - (\tilde{n}_{k,h}^m, \chi^m)_h^0\big] - \big[(\tilde{n}_{k,h}^0, \chi^0)_{\Omega^0} - (\tilde{n}_{k,h}^0, \chi^0)_h^0\big]$$

and

$$\begin{aligned}
\mathcal{E}_1^m = &\int_{\Omega(0)} F_k(\det \eta_h^m - 1) + \big((\tilde{n}_{k,h}^j, D_t^+ \chi^j)_h - (\tilde{n}_{k,h}^j, D_t^+ \chi^j)\big) \\
&- \big(G_k(\tilde{c}_h^{j-1}, \tilde{n}_h^{j-1}), \chi^j)_h - (G_k(\tilde{c}_h^{j-1}, \tilde{n}_h^{j-1}), \chi^j)\big) \\
&+ \big((n_{k,h}^{m-1}, \chi^j \circ (\eta_h^{m-1})^{-1}, \operatorname{div} \boldsymbol{v}_{s,h}^{m-1})_h^{m-1} - (n_{k,h}^{m-1}, \chi^j \circ (\eta_h^{m-1})^{-1}, \operatorname{div} \boldsymbol{v}_{s,h}^{m-1})_{\Omega^{m-1}}\big) \\
&+ d^{m-1}(n_{k,h}^{m-1}, \chi \circ (\eta_h^{m-1})^{-1}, \boldsymbol{w}_h^{m-1}) \\
= &\ \mathcal{E}_{1,1}^m + \mathcal{E}_{1,2}^m + \mathcal{E}_{1,3}^m + \mathcal{E}_{1,4}^m + \mathcal{E}_{1,5}^m,
\end{aligned}$$

with $\mathcal{E}_1 = \sum_{j=1}^m \Delta t \mathcal{E}_1^j$. Here, F_k is uniformly bounded in $L^1(\Omega(0))$ and the term $\mathcal{E}_{1,1}^m$ accounts for the fact that $\det \eta_h^m \neq 1$. The terms $\mathcal{E}_{1,i}^m$ for $i = 2, 3, 4$ account for the error due to mass lumping, while $\mathcal{E}_{1,5}^m$ is the artificial diffusion term. Then, due to the estimates (3.12), (3.13), we have

$$\sum_{j=1}^m \Delta t |\mathcal{E}_{1,1}^j| \to 0.$$

Due to the estimate (3.6), it follows that

$$\sum_{j=1}^m \Delta t |\mathcal{E}_{1,2}^j| + |\mathcal{E}_{1,3}^j| + |\mathcal{E}_{1,4}^j| \leq Ch \to 0.$$

and due to Lemma 3.2.3,

$$\sum_{j=1}^m \Delta t |\mathcal{E}_{1,5}^j| \leq Ch \to 0.$$

Finally, we note that $|\mathcal{E}_1^0| \leq Ch \to 0$ due to the L^∞ bounds on $n_{k,h}^m$. It follows that $|\mathcal{E}_1| \to 0$. The other error terms $\mathcal{E}_i$ for $i = 2, 3, 4$ are less complicated, and the same argument can be used to show that $|\mathcal{E}_i| \to 0$. □

Next, we argue that $B_h^m \to B$ in $C([0, T] \times \Omega_0)$, where

$$b_{ij}(t) = (\partial_{x_i} \eta_j^{-1}(t)) \circ \eta(t), \quad B(t) = (b_{ij}(t)).$$

We note that

$$|b_{ij}(t^m) - (b_h^m)_{ij}| \leq |(\partial_{x_i} (\eta_h^m)_j^{-1}) \circ (\eta_h^m - \eta(t))| + |(\partial_{x_i} \eta_j^{-1} - \partial_{x_i} (\eta_h^m)_j^{-1}) \circ \eta(t)|.$$

For the first term we have the estimate

$$|(\partial_{x_i}(\eta_h^m)_j^{-1}) \circ (\eta_h^m - \eta(t))| \leq \|\partial_{x_i}(\eta_h^m)_j^{-1}\|_{L^\infty((0,T)\times\Omega^m)}\|\eta_h^m - \eta(t)\|_{L^\infty((0,T)\times\Omega)}$$
$$\leq Ch^2|\log h|$$

due to the regularity assumptions on η, η_h^m. The second inequality is due to noting $\eta_h^m = \eta_h(t^m)$ and

$$\|\eta_h(t^m) - \eta(t)\|_{L^\infty((0,T)\times\Omega)} \leq \|\eta_h(t^m) - \eta(t^m)\|_{L^\infty((0,T)\times\Omega)} + \|\eta(t^m) - \eta(t)\|_{L^\infty((0,T)\times\Omega)},$$

where

$$\|\eta_h(t^m) - \eta(t^m)\|_{L^\infty((0,T)\times\Omega)} \leq Ch^2|\log h|,$$
$$\|\eta(t^m) - \eta(t)\|_{L^\infty((0,T)\times\Omega)} \leq C\Delta t\|\partial_t\eta(t)\|_{L^\infty((0,T)\times\Omega)} \leq Ch^2$$

due to the assumptions on Δt and the fact that $\partial_t\eta \in L^\infty([0,T]\times\Omega)$. Furthermore, we can write

$$|(\partial_{x_i}\eta_j^{-1} - \partial_{x_i}(\eta_h^m)_j^{-1}) \circ \eta(t)| \leq \|\partial_{x_i}\eta_j^{-1} - \partial_{x_i}(\eta_h^m)_j^{-1}\|_{L^\infty((0,T)\times\Omega^m)}\|\eta\|_{L^\infty(0,T;\Omega^0)}$$
$$\leq Ch|\log h|,$$

where we have assumed that the error estimate for the approximation η_h also holds for η_h^{-1}. Thus, since $Bh|\log h| \to 0$, it follows that $B_h^m \to B$ in $L^\infty((0,T)\times\Omega_0)$. This is sufficient to prove convergence in the same manner as the functional analytic problem.

Thus, the quantities (n_k, c, p) defined by $n_k = \tilde{n}_k \circ \eta^{-1}$, $c = \tilde{c} \circ \eta^{-1}$ and $p = \tilde{p} \circ \eta^{-1}$ satisfy the original weak formulation of the problem (2.1).

3.6 Model with Brinkman's Equation on a Fixed Polygonal Domain

3.6.1 Formulation and main results

We assume the same definitions for $\mathcal{T}_h, S_h, Y_h$ as in Section 3.1. Then, we let X_h be a finite element space such that the inf-sup condition

$$\sup_{\boldsymbol{v}\in X_h} \frac{(p, \operatorname{div} \boldsymbol{v})}{\|\boldsymbol{v}\|_{H^1(\Omega)}} \geq \beta \|p\|_{L^2(\Omega)}$$

is satisfied for all $p \in Y_h$, with $\beta > 0$ fixed. For example, X_h could be the MINI or Taylor-Hood elements. Then, we look for $n_{k,h}^m, c_h^m \in S_h$, $p_h^m \in Y_h$, $\boldsymbol{v}_h^m \in X_h$ satisfying

$$\begin{aligned}
&(D_t^- n_{k,h}^m, \chi)_h + d\ a(n_{k,h}^m, \chi) = k(n_{k,h}^{m-1}, \chi, \boldsymbol{w}_h^{m-1}) + (G_{k,h}^{m-1}, \chi)_h, \\
&(D_t^- c_h^m, \chi)_h + \mu\ a(c_h^m, \chi) + r_c(c_h^m, \chi)_h = (F_h^{m-1}, \chi)_h, \\
&b(p_h^m, \Phi) + \nu\ a(\boldsymbol{v}_h^m, \Phi) + \frac{\lambda}{K}(\boldsymbol{v}_h^m, \Phi) = -n_T(\nabla\phi, \Phi), \\
&b(\chi, \boldsymbol{v}_h^m) = n_T^{-1} k(n_{p,h}^m + n_{q,h}^m, \chi, \psi(c_h^m)\nabla c_h^m) + n_T^{-1}(G_h^m, \chi)_h
\end{aligned} \tag{3.16}$$

for all $\chi \in S_h$, $\Phi \in X_h$, where

$$\boldsymbol{w}_h^{m-1} = \boldsymbol{v}_h^{m-1} + \psi(c_h^{m-1})\nabla c_h^{m-1}.$$

Remark 3.6.1. As in Remark 3.1.1, $(3.16)_4$ is technically valid for $\chi \in Y_h$. However, by the same reasoning as before, we can prove that it also holds for $\chi \in S_h$.

The results of this section are similar to those in the case of Darcy's Law. The main difference is that we require more regularity of the initial data. In fact, we require even more regularity of the initial data than we did for the functional analytic problem with Brinkman's equation. This is because we do not have the L^p,

$p < 2$, estimates for Stokes equations that we used in the functional analytic case. Thus, we need higher regularity for c to account for this.

Before we introduce the main existence result for this scheme, we need to define a discrete interpolation space. Let

$$X_{h,p} = \left\{ v \in S_h : \|v\|_{L^p(\Omega)} < \infty \right\}, \quad D(A_h^p) = \left\{ v \in S_h^0 : \|A_h v\|_{L^p(\Omega)} < \infty \right\}.$$

Then, we can define a real interpolation space $(X_{h,p}, D(A_h^p))_{1-1/p,p}$, and we denote by $\|\cdot\|_{1-1/p,p}$ the norm on this space.

Theorem 3.6.2 (Existence of Finite Element Solutions)**.** *Suppose*

$$0 \leq n_{k,h}^0, \ \sum_k n_{k,h}^0 = n_T, \ 0 \leq c_h^0, c_{v,h}^m \leq c_M, \quad c_h^0 \in (X_{h,p}, D(A_h^p))_{1-1/p,p}, \quad p > 2.$$

Then, provided h sufficiently small and $\Delta t \leq Ch^2$, there exists a unique solution $(n_{k,h}, c_h, \boldsymbol{v}_h, p_h)$ to the system (3.1) with initial data $n_{k,h}^0, c_h^0$. Furthermore, the solution satisfies the bounds

$$0 \leq c_h \leq c_M, \ 0 \leq n_{k,h}, \ \sum_k n_{k,h}^m = n_T \text{ for each } m,$$

$$\|n_{k,h}\|_{L^2(0,T;H^1(\Omega))} + \|\boldsymbol{v}_h\|_{L^q(0,T;H^1(\Omega))} + \|p_h\|_{L^q(0,T;L^2(\Omega))} \leq C,$$

$$\|c_h\|_{L^\infty(0,T;W^{1,p}(\Omega))} + \|c_h\|_{L^p(0,T;W^{2,p}(\Omega))} \leq C,$$

$$\|D_t^- n_{k,h}\|_{L^2(0,T;H^{-1}(\Omega))} + \|D_t^- c_h\|_{L^2(0,T;H^{-1}(\Omega))} \leq C,$$

where C is dependent on fixed quantities including $\|c_h^0\|_{1-1/p,p}$.

The second main result of this section is the convergence of the solutions of the scheme (3.1) to a solution of the system (2.1).

Theorem 3.6.3 (Convergence of numerical scheme)**.** *Suppose that*

$$0 \leq n_{k,0}, \quad \sum_k n_{k,0} = n_T, \quad 0 \leq c_0, c_v \leq c_M, \quad c_0 \in W^{2-2/p,p}(\Omega), \quad p > 2,$$

and define

$$n^0_{k,h} = M_h^{-1} D_h n_{k,0}, \quad c^0_h = M_h^{-1} D_h c_0, \quad c^m_{v,h} = (\Delta t)^{-1} M_h^{-1} D_h \int_{t^{m-1}}^{t^m} c_v(t).$$

Additionally, suppose there exists a sequence $(h_k, \Delta t_k)$ *such that* $\Delta t_k \leq Ch^2$ *and* $h_k \to 0$*. Then, for each* k *there exists a solution* $(n_{h_k}, c_{h_k}, \boldsymbol{v}_{h_k}, p_{h_k})$ *to* (3.1) *with initial data* $n^0_{h_k}, c^0_{h_k}$*. Furthermore, as* $k \to \infty$*,* $(n_{h_k}, c_{h_k}, \boldsymbol{v}_{h_k}, p_{h_k})$ *converges weakly to* $(n, c, \boldsymbol{v}, p)$*, where* $(n, c, \boldsymbol{v}, p)$ *is a solution to* (2.14) *with initial data* n_0, c_0*.*

3.6.2 Proof of Theorem 2.5.2

We introduce several results related to regularity of Stokes' equations and L^p discrete parabolic regularity.

3.6.2.1 Estimates for discrete Stokes' equations

We need to demonstrate sufficient regularity of $\boldsymbol{v}_h, p_h$ in order to prove convergence, and to determine appropriate bounds on $n_{k,h}$. We recall the formulation

$$\begin{aligned} b(p^m_h, \Phi) + \nu\ a(\boldsymbol{v}^m_h, \Phi) + \frac{\lambda}{K}(\boldsymbol{v}^m_h, \Phi) &= -n_T(\nabla\phi, \Phi) \\ b(\chi, \boldsymbol{v}^m_h) &= n_T^{-1} k(n^m_{p,h} + n^m_{q,h}, \chi, \psi(c^m_h)\nabla c^m_h) + n_T^{-1}(G^m_h, \chi)_h \end{aligned} \tag{3.17}$$

for all $\chi \in S_h, \Phi \in X_h$.

Lemma 3.6.4. *Suppose that* $0 \le n_{k,h}^m \le n_T$ *and* $c_h^m \in H^1(\Omega)$ *with* $0 \le c_h^m \le c_M$. *Then, there exists a unique solution* $(p_h^m, \boldsymbol{v}_h^m) \in Y_h \times X_h$ *to* (3.17) *satisfying*

$$\|\boldsymbol{v}_h^m\|_{L^2(\Omega)} \le C(1 + \|c_h^m\|_{H^1(\Omega)}),$$

where C *is dependent on fixed quantities but independent of* h, m. *Furthermore, if* $n_{k,h}^m \in H^1(\Omega)$ *and* $c_h^m \in W^{1,\infty}(\Omega)$ *with* $A_h c_h^m \in L^2(\Omega)$, *then*

$$\begin{aligned} &\|p_h^m\|_{L^2(\Omega)} + \|\boldsymbol{v}_h^m\|_{H^1(\Omega)} \\ &\quad \le C\left(1 + \|A_h c_h\|_{L^2(\Omega)} + \|c_h\|_{W^{1,\infty}(\Omega)}\left(\|n_{k,h}^m\|_{H^1(\Omega)} + \|c_h^m\|_{H^1(\Omega)}\right)\right), \end{aligned}$$

where again C *is independent of* h, m.

Proof. Let $g_h^m \in S_h$ have nodal values

$$g_i^m = (\alpha^T \boldsymbol{K}(\psi(c_h^m)\nabla c_h^m)\boldsymbol{M}^{-1})_i + (K_h G_h^m)_i,$$

where α holds the nodal values of $n_{p,h}^m + n_{q,h}^m$. Then, we can write (3.17) in the form

$$\begin{aligned} &b(p_h^m, \Phi) + \nu\, a(\boldsymbol{v}_h^m, \Phi) + \frac{\lambda}{K}(\boldsymbol{v}_h^m, \Phi) = -n_T(\nabla\phi, \Phi) \\ &b(\chi, \boldsymbol{v}_h^m) = (g_h^m, \chi) \end{aligned} \tag{3.18}$$

for all $\chi \in S_h, \Phi \in X_h$. Then, since the operator $b(q, \boldsymbol{v}) = (q, \operatorname{div} \boldsymbol{v})$ satisfies the inf-sup (or Brezzi) condition, there exists a unique solution $(p_h^m, \boldsymbol{v}_h^m) \in Y_h \times X_h$.

Next, we demonstrate that $|(g_h^m, q)| \le C\|q\|_{H^1(\Omega)}$. We note that

$$\begin{aligned} n_T^{-1} h(n_{p,h}^m + n_{q,h}^m, q, \psi(c_h^m)\nabla c_h^m) &= n_T^{-1}\int_\Omega (n_{p,h}^m + n_{q,h}^m)\psi(c_h^m)\nabla c_h^m \cdot \nabla q \\ &\le C\|\nabla c_h^m\|_{L^2(\Omega)}\|\nabla q\|_{L^2(\Omega)} \end{aligned}$$

and $n_T^{-1}(G_h^m, \tilde{q})_h \le C\|\tilde{q}\|_{L^2(\Omega)}$. Furthermore, due to Lemma 3.2.3, it follows that

$$|d(n_{p,h}^m + n_{q,h}^m, q, \psi(c_h^m)\nabla c_h^m)| \le C n_T \|\psi(c_h^m)\nabla c_h^m\|_{L^2(\Omega)}\|\nabla q\|_{L^2(\Omega)}.$$

This demonstrates that $|(g_h^m, q)| \le C(1 + \|c_h^m\|_{H^1(\Omega)})\|q\|_{H^1(\Omega)}$.

Next, we argue that if $(\boldsymbol{v}, p)$ satisfy

$$\begin{aligned} (p, \operatorname{div} \boldsymbol{u}) + \frac{\lambda}{K}(\boldsymbol{v}, \boldsymbol{u}) + \mu(\nabla \boldsymbol{v}, \nabla \boldsymbol{u}) &= (f, \boldsymbol{u}), \\ (q, \operatorname{div} \boldsymbol{v}) &= (g, q), \end{aligned} \tag{3.19}$$

with $\|f\|_{L^2(\Omega)} \le C$ and $|(g, q)| \le C\|q\|_{H^1(\Omega)}$ then $\|\boldsymbol{v}\|_{L^2} \le C$. We can do this by a similar duality argument as presented in [22], noting (see [31]) that if $(\boldsymbol{w}, r)$ is a solution to (3.19) with $f \in L^2$ and $g = 0$, then

$$\|\nabla r\|_{L^2} + \|\Delta_h \boldsymbol{w}\|_{L^2} + \|\boldsymbol{w}\|_{H^1} \le C\|f\|_{L^2}.$$

To prove the final estimates for $p_h^m, \boldsymbol{v}_h^m$, we recall

$$(g_h^m, q_h) = n_T^{-1} k(n_{p,h}^m + n_{q,h}^m, q, \psi(c_h^m)\nabla c_h^m) + n_T^{-1}(G_h^m, q)_h,$$

with $(G_h^m, q)_h \le C\|q\|_{L^2(\Omega)}$, and, by Lemma 3.2.3,

$$\begin{aligned} d(n_{p,h}^m + n_{q,h}^m, q, \psi(c_h^m)\nabla c_h^m) &\le h\|\nabla(n_{p,h}^m + n_{q,h}^m)\|_{L^2(\Omega)}\|\psi(c_h^m)\nabla c_h^m\|_{L^\infty(\Omega)}\|\nabla q\|_{L^2(\Omega)} \\ &\le C\|q\|_{L^2(\Omega)} \end{aligned}$$

due to the energy estimates on n_h^m, c_h^m and the inverse inequality. Next, we examine the term $h(n_{p,h}^m + n_{q,h}^m, q, \psi(c_h^m)\nabla c_h^m)$.

We take $\psi_h^m = (n_{p,h}^m + n_{q,h}^m)\psi(c_h^m)$. Due to Lemma A.3.1 and (3.5), we have

$$\begin{aligned}
\|R_h(\psi_h^m q)\|_{L^2(\Omega)} &\leq C\left(\|q\|_{L^2(\Omega)} + h\|\nabla q\|_{L^2(\Omega)} + h\|q\nabla(\psi_h^m)\|_{L^2(\Omega)}\right)\\
&\leq C\left(\|q\|_{L^2(\Omega)} + \|q\|_{L^2(\Omega)}\|\nabla n_h^m\|_{L^2(\Omega)}\|\nabla c_h^m\|_{L^\infty(\Omega)}\right)\\
&\leq C\|q\|_{L^2(\Omega)},
\end{aligned}$$

where C is dependent on fixed quantities along with $\|c_h^m\|_{W^{1,\infty}(\Omega)}, \|n_h^m\|_{H^1(\Omega)}$. Then, we can write

$$\begin{aligned}
(\psi_h^m \nabla c_h^m, \nabla q) &= (\nabla c_h^m, \nabla(q\psi_h^m)) - (\nabla c_h^m, q\nabla\psi_h^m)\\
&= (L_h c_h^m, R_h(q\psi_h^m)) - (\nabla c_h^m, q\nabla\psi_h^m) + (c_h^m, R_h(q\psi_h^m)) - (c_h^m, q\psi_h^m).
\end{aligned}$$

It follows that

$$|(\psi_h^m \nabla c_h^m, \nabla q)| \leq C\left(1 + \|L_h c_h^m\|_{L^2(\Omega)} + \|\nabla c_h^m\|_{L^\infty(\Omega)}\|\nabla\psi_h^m\|_{L^2(\Omega)}\right)\|q\|_{L^2(\Omega)}.$$

Thus,

$$\|g_h\|_{L^2(\Omega)} \leq C\left(1 + \|L_h c_h\|_{L^2(\Omega)} + \|\nabla c_h\|_{L^\infty(\Omega)}\|\nabla\psi_h\|_{L^2(\Omega)}\right).$$

The desired inequality follows from the definition of ψ_h, the fact that $\|L_h c_h\|_{L^2} \leq C\|A_h c_h\|_{L^2}$, and noting that

$$\|p_h^m\|_{L^2(\Omega)} + \|\boldsymbol{v}_h^m\|_{H^1(\Omega)} \leq C\left(1 + \|g_h\|_{L^2(\Omega)}\right).$$

$\square$

3.6.2.2 L^p discrete parabolic regularity

In order to demonstrate convergence of the fluid velocity and pressure (using Lemma 3.6.4), we need to obtain a bound on c_h in $L^p(0, T; W^{1,\infty}(\Omega))$ for $p > 2$. This estimate can be obtained by recalling the discrete Sobolev inequality

$$\|c_h\|_{L^p(0,T;W^{1,\infty}(\Omega))} \leq \|A_h c_h\|_{L^p((0,T)\times\Omega)},$$

and proving an L^p bound on $A_h c_h$.

Specifically, we would like to prove that the L^p regularity estimate in Theorem IV of [35], which was proven in the case of homogeneous Dirichlet boundary conditions and the mass-lumped discrete Dirichlet Laplacian, also holds in the case of Neumann boundary conditions.

Theorem 3.6.5 (Theorem IV from [35]). *Suppose c_h satisfies*

$$(D_t^- c_h, \chi)_h + (A_h c_h, \chi)_h = (f, \chi)_h \text{ for all } \chi \in S_h,$$

with $f \in L^p((0,T)\times\Omega)$ and $c_h^0 \in (X_{h,p}, D(A_h^p))_{1-1/p,p}$. Then, there exists C independent of $h, \Delta t$ such that

$$\begin{aligned}\|D_t^- c_h\|_{L^p((0,T)\times\Omega)} + \|c_h\|_{L^p(0,T;W^{1,p}(\Omega))} + \|A_h c_h\|_{L^p((0,T)\times\Omega)} \\ \leq C\left(\|f\|_{L^p((0,T)\times\Omega)} + \|c_h\|_{1-1/p,p}\right).\end{aligned}$$

Define the mass-lumped discrete Dirichlet Laplace operator A_h^D by

$$(A_h^D u_h, v_h)_h = (\nabla u_h, \nabla v_h) \text{ for all } v_h \in S_h.$$

The proof in [35] relies on several properties of the operator A_h^D:

- A_h^D is invertible,
- A_h^D is positivity-preserving, and
- A_h^D generate an analytic and contraction semigroup.

These facts can be used to prove that A_h^D has bounded imaginary powers, and thus maximal L^p regularity. It is possible to show that the Neumann Laplacian A_h also satisfies these properties. We note that A_h is invertible due to the definition. It is possible to show that A_h is positivity-preserving and generates an analytic and contraction semigroup in the same way as for A_h^D. Thus, this result also holds for the operator A_h.

3.6.2.3 Uniform bounds

Much of the proof of this theorem follows in the same manner as the proof of Theorem 3.1.2. Lemma 3.2.2 holds in the same manner as before, with minor adjustments necessary when demonstrating that $\sum_k n_{k,h}^m = n_T$. Then, these estimates are sufficient to apply the energy estimates used in the case of Darcy's Law, and to use Theorem 3.6.5 to determine that

$$\|c_h\|_{L^p(0,T;W^{1,\infty}(\Omega))} + \|A_h c_h\|_{L^p((0,T)\times\Omega)} \leq C\left(1 + \|c_h^0\|_{1-1/p,p}\right).$$

It follows from Lemma 3.6.4 that

$$\|\boldsymbol{v}_h\|_{L^\infty(0,T;L^2(\Omega))} \leq C\left(1 + \|c_h\|_{L^\infty(0,T;L^2(\Omega))}\right) \leq C,$$

where C is dependent on fixed quantities. Then, this result is sufficient to use the energy estimates from Section 3.2.3.3, yielding

$$\|n_{k,h}\|_{L^2(0,T;H^1(\Omega))} \leq C.$$

Finally, we can use this estimate for $n_{k,h}$ in Lemma 3.6.4 to write

$$\begin{aligned} &\|p_h^m\|_{L^2(\Omega)} + \|\boldsymbol{v}_h^m\|_{H^1(\Omega)} \\ &\quad \leq C\left(1 + \|A_h c_h^m\|_{L^2(\Omega)} + \|c_h^m\|_{W^{1,\infty}(\Omega)} \|c_h^m n_{k,h}^m\|_{H^1(\Omega)}\right), \end{aligned}$$

and thus

$$\begin{aligned} &\|p_h^m\|_{L^q(0,T;L^2(\Omega))} + \|\boldsymbol{v}_h^m\|_{L^q(0,T;H^1(\Omega))} \\ &\quad \leq C\left(1 + \|A_h c_h^m\|_{L^2((0,T)\times\Omega)} + \|c_h^m\|_{L^p(0,T;W^{1,\infty}(\Omega))} \|c_h^m n_{k,h}^m\|_{L^2(0,T;H^1(\Omega))}\right) \leq C, \end{aligned}$$

where C is dependent on fixed quantities including $\|c_h^0\|_{1-1/p,p}$ and $\frac{1}{q} = \frac{1}{2} + \frac{1}{p}$, so $1 < q < 2$.

Finally, the inverse inequality yields

$$\|\boldsymbol{v}_h\|_{L^\infty} + \|\nabla c_h\|_{L^\infty} \leq Ch^{-1}\left(\|\boldsymbol{v}_h\|_{L^\infty(0,T;L^2(\Omega))} + \|c_h\|_{L^\infty(0,T;H^1(\Omega))}\right) \leq Ch^{-1}.$$

This combined with the assumption on Δt in Lemma 3.2.2 yields the estimate $\Delta t \leq Ch^2$ that we find in Theorem 3.6.2. This proves the theorem.

3.6.3 Proof of Theorem 3.6.3

We first argue that if the hypotheses of Theorem 3.6.3 are satisfied, so are the hypotheses of Theorem 3.6.2. As in the case for Darcy's Law, since $M_h^{-1} D_h$

preserves upper and lower bounds, the upper and lower bounds on $n_{k,h}^0, c_h^0, c_{v,h}^m$ are satisfied. It remains to show that $c_h^0 \in (X_{h,p}, D(A_h^p))_{1-1/p}$.

The norm on the real interpolation space $(X_{h,p}, D(A_h))_{1-1/p,p}$, defined by

$$\|v_h\|_{1-1/p,p} = \left(\int_0^\infty t^{-p+1} (K^h(v_h, t))^p \frac{dt}{t} \right)^{1/p}$$

where

$$K^h(v_h, t) = \inf \{ \|a_h\|_{h,p} + t\|A_h b_h\|_{h,p} : v_h = a_h + b_h,\ a_h, b_h \in S_h \}$$

for any $v_h \in S_h$. Additionally, the real interpolation space $(L^p(\Omega), D(A_p))_{1-1/p,p}$ is embedded in the fractional Sobolev space $W^{2-2/p,p}(\Omega)$, so there exists $C \geq 0$ such that

$$\left(\int_0^\infty t^{-p+1} (K(v, t))^p \frac{dt}{t} \right)^{1/p} \leq C\|v\|_{W^{2-2/p,p}(\Omega)}$$

where

$$K(v, t) = \inf \left\{ \|a\|_{L^p(\Omega)} + t\|Ab\|_{L^p(\Omega)} : v = a + b,\ a \in L^p(\Omega), b_h \in D(A_p) \right\}.$$

Then, for each a, b with $c_0 = a+b$, $a \in L^p(\Omega)$, $b \in D(A_p)$, we note that $c_0 \in W^{1,p}(\Omega)$, and since $W^{1,p}(\Omega)$ is embedded in $D(A_p)$, it must be that $a, b \in W^{1,p}(\Omega)$. Then, we can define $a_h = M_h^{-1} D_h a$, $b_h = M_h^{-1} D_h b$, and since M_h^{-1}, D_h are linear, it follows that $c_h^0 = a_h + b_h$. Thus,

$$K^h(c_h, t) \leq \|a_h\|_{h,p} + t\|A_h b_h\|_{h,p}.$$

We have

$$\|a_h\|_{h,p} \leq C\|a\|_{L^p(\Omega)} \text{ and } \|A_h b_h\|_{h,p} \leq C\|Ab\|_{L^p(\Omega)}$$

due to the properties of the discrete operators, and thus $K^h(c_h, t) \leq CK(c, t)$ for all t, with C independent of t. It follows that

$$\|c_h^0\|_{1-1/p,p} \leq C \left(\int_0^\infty t^{-p+1} (K(v,t))^p \frac{dt}{t} \right)^{1/p} \leq C \|c_0\|_{W^{2-2/p,p}(\Omega)},$$

as desired.

Then, the proof of Theorem 3.6.3 relies on the bounds from Theorem 3.6.2. Those bounds imply that

$$\begin{aligned}
&c_{h_k} \rightharpoonup c \text{ in } L^p(0,T;W^{1,r}(\Omega)) \cap L^r(0,T;H^1(\Omega)) \cap L^r((0,T)\times\Omega),\\
&n_{h_k} \rightharpoonup n \text{ in } L^2(0,T;H^1(\Omega)) \cap L^r((0,T)\times\Omega),\\
&\boldsymbol{v}_{h_k} \rightharpoonup \boldsymbol{v} \text{ in } L^2((0,T)\times\Omega) \cap L^q(0,T;H^1(\Omega)),\\
&p_{h_k} \rightharpoonup p_h \text{ in } L^q(0,T;L^2(\Omega)),
\end{aligned}$$

where $2 < p < \infty, 1 < q < 2$ are as in Theorem 3.6.3, and $1 < r < \infty$. The rest of the proof follows in the same manner as Section 3.3.

Chapter 4: Results of Numerical Experiments

4.1 Implementation of Finite Element Schemes

There are a variety of packages available for use in the implementation of finite element schemes in Python. Project FEniCS is probably the most sophisticated package, and requires a very small amount of code to implement a variety of relatively standard finite element schemes. However, since we need to implement mass-lumping and define an artificial diffusion matrix, we are working with a slightly non-standard scheme. This is still doable in FEniCS, but removes a lot of the user-friendly aspects that are important. Furthermore, FEniCS is a very large package, and more difficult to install than many other Python packages. We choose to use the more lightweight Scikit-FEM package. This package is primarily used to build matrices used in finite element methods, such as mass and stiffness matrices. Thus, we use Scikit-FEM, or skfem, to build the matrices described in Section 3.1.1. Once we have the mass matrix $\boldsymbol{M}$ and the transport matrix $\boldsymbol{H}$, it is straightforward to build the lumped mass matrix $\tilde{\boldsymbol{M}}$ and the artificial diffusion matrix $\boldsymbol{D}$.

We note that, in order for the update scheme for p to be well-defined, we need to restrict p and the test functions to the space with zero mean value over Ω. There are several ways to do this.

The most basic way is to note that any finite element function in M_h has one fewer degree of freedom than a finite element function in S_h. Thus, we can set one of the nodal values of p to zero, and eliminate that row and column in the matrix vector problem we are solving. Then, after we determine p, we subtract the mean value to end up with a function with zero mean value. This method is sufficient in certain cases, but in others it creates an undesirably large numerical error.

In the case where the method related to degrees of freedom is insufficient, we approach the problem using the method of Lagrange multipliers. For an elliptic problem

$$\Delta p = \operatorname{div} \boldsymbol{g} + f \text{ in } \Omega, \quad \nabla p \cdot \boldsymbol{n} = \boldsymbol{g} \cdot \boldsymbol{n} \text{ on } \Gamma, \quad \int_\Omega p = 0$$

the weak formulation with Lagrange multipliers requires us to find $p \in H^1(\Omega), c \in \mathbb{R}$ such that

$$\int_\Omega \nabla p \cdot \nabla q + \int_\Omega cq + \int_\Omega dq = \int_\Omega \boldsymbol{g} \cdot \nabla q + \int_\Omega fq \text{ for all } q \in H^1(\Omega),\ d \in \mathbb{R}.$$

This method seems to work sufficiently well for all of the test cases, although it does still introduce some numerical error.

In the next sections we present the results of several different numerical experiments.

4.2 Two Tumor Regions Experiment

We will focus primarily on a two tumor experiment, similar to an example presented in [52]. We will use this example to explore the effects of drug application and chemotaxis, as well as the choice of certain parameters. In this experiment, we fix $d = 10^{-4}$ to ensure only a small amount of diffusion of the cells. We consider initial data consisting of two small tumors centered at $(0.4, 0.5)$ and $(0.65, 0.65)$ inside the domain $\Omega = [0,1] \times [0,1]$. The initial proliferating cell density is given by

$$n_{p,0} = \frac{1}{2}e^{-400((x-0.4)^2+(y-0.5)^2)} + \frac{1}{2}e^{-700((x-0.65)^2+(y-0.65)^2)},$$

so that both tumors are taken to be Gaussian distributions. We take $\bar{c} = 0.25$, $K_B = 15$, $K_A = 13$, $f(c) = 20c$, and $u_0 = c_0 = c_v = 1$ uniformly on Ω. Furthermore, we take $h = 1/128$ and $\Delta t = 1/1448 \approx 128^{-3/2}$. In practice, a time step of $\Delta t \approx h^{3/2}$ seems to obtain similar accuracy as $\Delta t \approx h^2$, so we take the larger time step to ensure better efficiency. We take $h = 1/128$ to reduce the effect of artificial diffusion, which has a diffusion coefficient on the order of h.

Case 1: base case no drug application or chemotaxis, $\mu = 10^{-3}$

Case 2: drug application $g(u) = 15u$, $K_1 = K_2 = 5$, $K = 0$, $\mu = 10^{-3}$

Case 3: chemotaxis $\psi = 0.02$, $\mu = 10^{-3}$

Case 4: chemotaxis and drug application

Case 5: diffusive nutrient no drug application or chemotaxis, $\mu = 0.1$

Case 6: diffusive nutrient with chemotaxis, $\psi = 0.02$, $\mu = 0.1$

There are several interesting takeaways from these results. First, we see that in the base case (case 1 in Figures 4.1 and 4.2), the tumor initially grows (see time $t = 0.5$) and then begins to develop a necrotic core at $t = 0.5$. At this time most of the nutrient near the center of the tumor has been consumed, so new cells are not begin born and more cells are dying off.

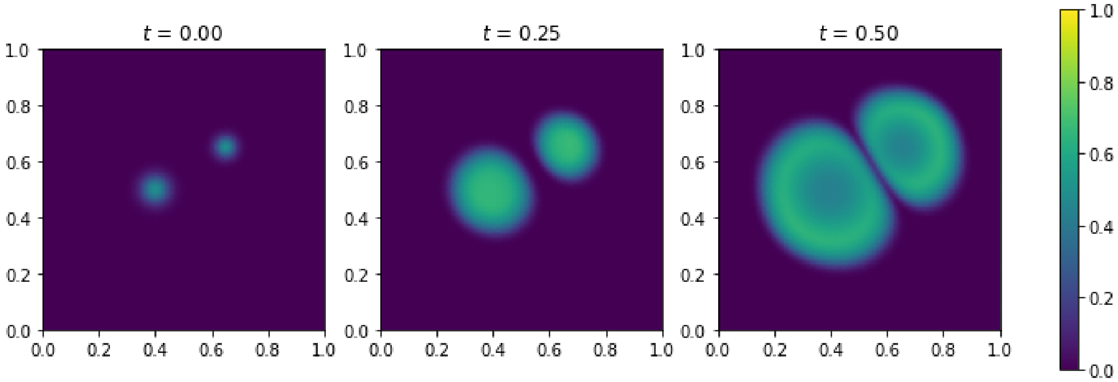

Figure 4.1: Case 1: evolution of n_p without drug or chemotaxis.

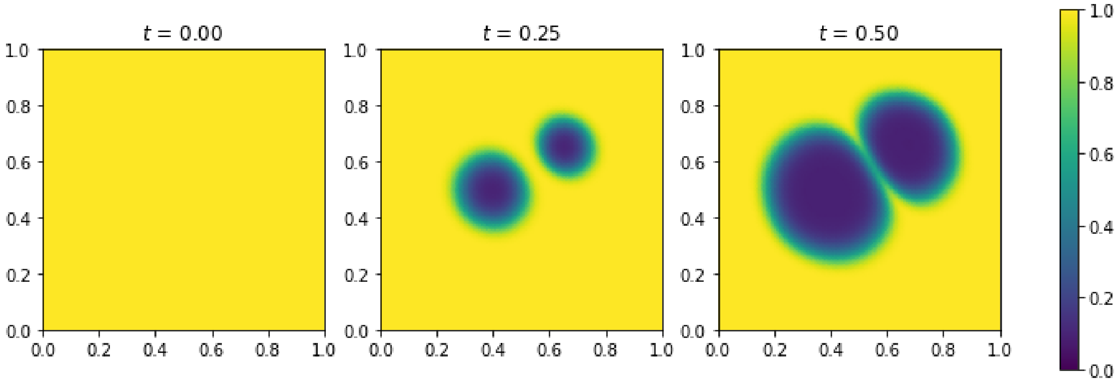

Figure 4.2: Case 1: evolution of c without drug or chemotaxis.

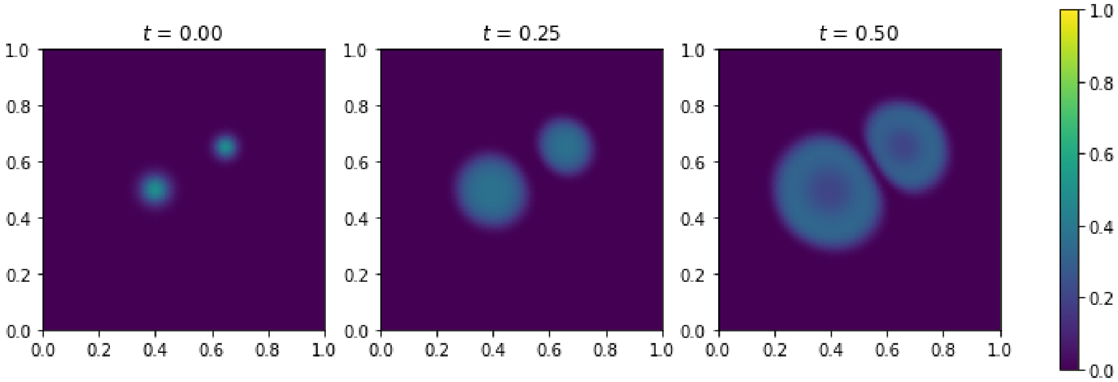

Figure 4.3: Case 2: evolution of n_p with drug application.

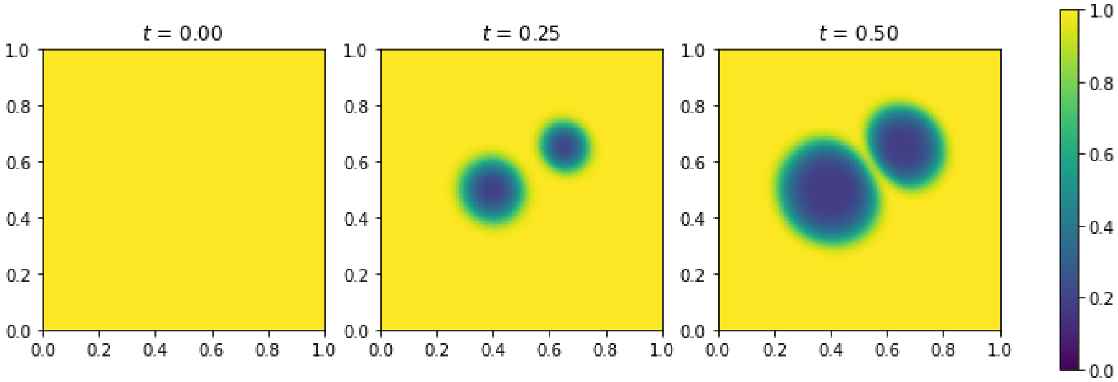

Figure 4.4: Case 2: evolution of c with drug application.

We also note that the effect of drug application (Figures 4.3 and 4.4) is as expected: since the drug inhibits the growth of cells, the tumors don't reach the same size as in the case without a drug. The tumors do still grow slightly initially, but develop a more significant necrotic core by time $t = 0.5$ and have lower density of proliferating cells overall. Additionally, because the density of proliferating cells is lower, less of the nutrient is consumed.

Chemotactic effects have a significant impact on the evolution of the tumor, especially in case 3 when $\mu = 10^{-3}$ (see Figures 4.5 and 4.6). The tumors have larger volumes at times $t = 0.25$ and $t = 0.5$ than in the case without chemotaxis, although the overall density of proliferating cells is lower. This is due to the fact

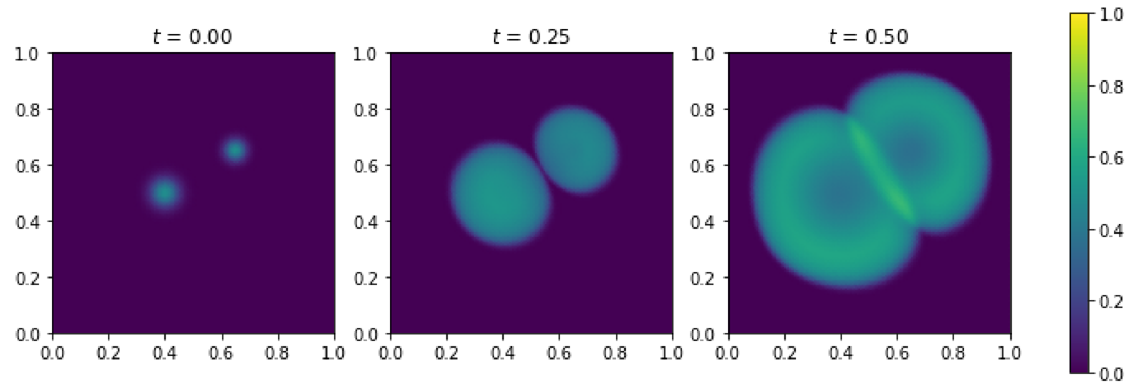

Figure 4.5: Case 3: evolution of n_p with chemotaxis.

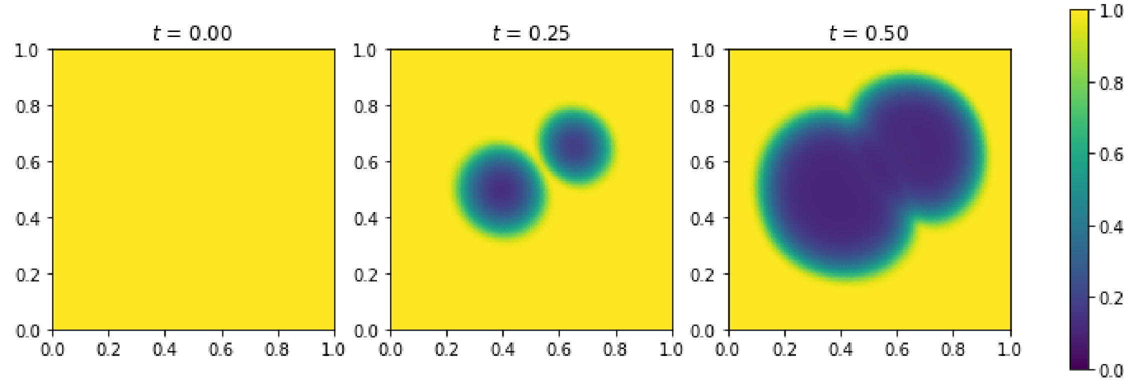

Figure 4.6: Case 3: evolution of c with chemotaxis.

that, as the nutrient is consumed within the tumor, proliferating cells move away from the center of the tumor towards higher nutrient densities, which causes the tumor to expand more quickly. However, this also lowers the density of proliferating cells since they spread out more quickly, which leads to a lower proliferation rate. Furthermore, it is interesting to note that in case 3, the two tumors actually start to overlap, unlike in case 1 where there was a gap between the cells. This overlap is because proliferating cells from each tumor start to move into the region between the two tumors, where the gradient of the nutrient is quite sharp.

In case 4 with both drug application and chemotaxis, the results are essentially what we would expect due to the previous examples. The necrotic core developed

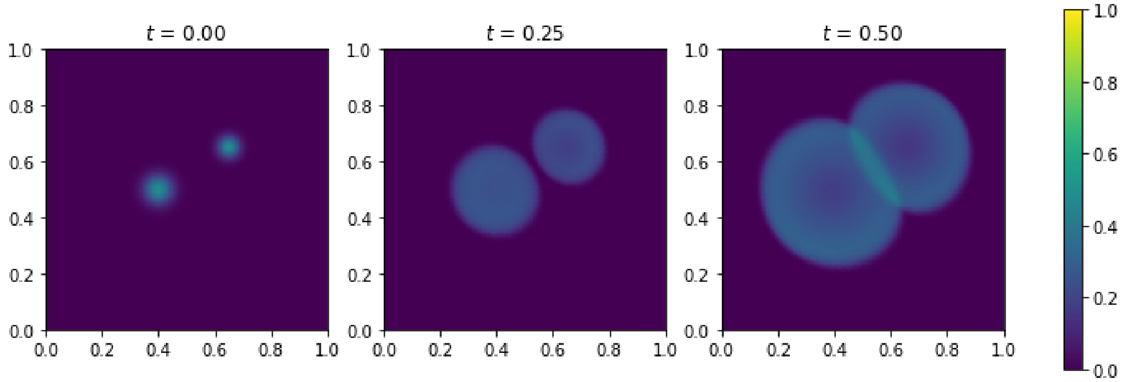

Figure 4.7: Case 4: evolution of n_p with chemotaxis and drug application.

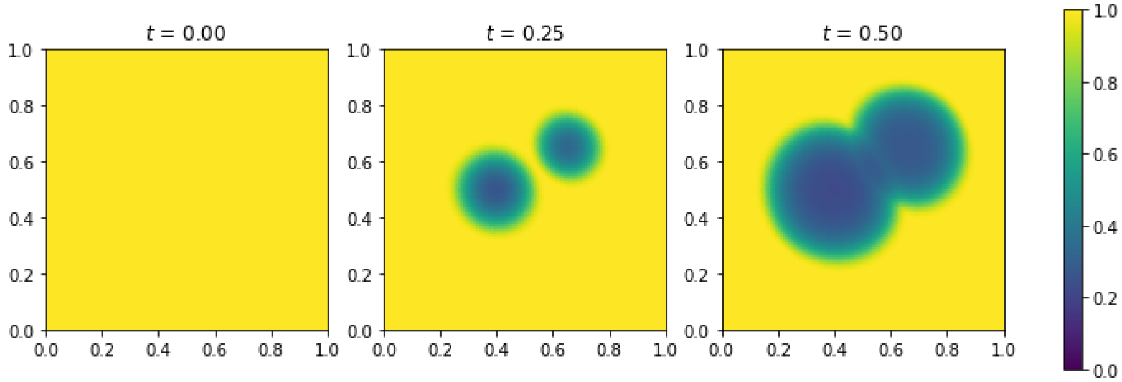

Figure 4.8: Case 4: evolution of c with chemotaxis and drug application.

in the tumors is significant, and the two tumors do overlap. The tumors are larger than in case 2, where no chemotactic effects were considered.

Finally, we examine the effect of an increased diffusion coefficient for the nutrient, taking $\mu = 0.1$ instead of $\mu = 10^{-3}$. We see that in case 5, without chemotaxis, the tumors are much more dense than in case 1, and no necrotic core has developed by time $t = 0.5$. Examining Figure 4.10, we can see that this is due to a higher density of the nutrient within each tumor region. Since μ is larger than in case 1, more nutrient moves into the tumor regions from outside of the tumor regions. Thus, the density of the nutrient outside of where the tumors lie is lower than in case 1, but it is higher within the tumorous regions.

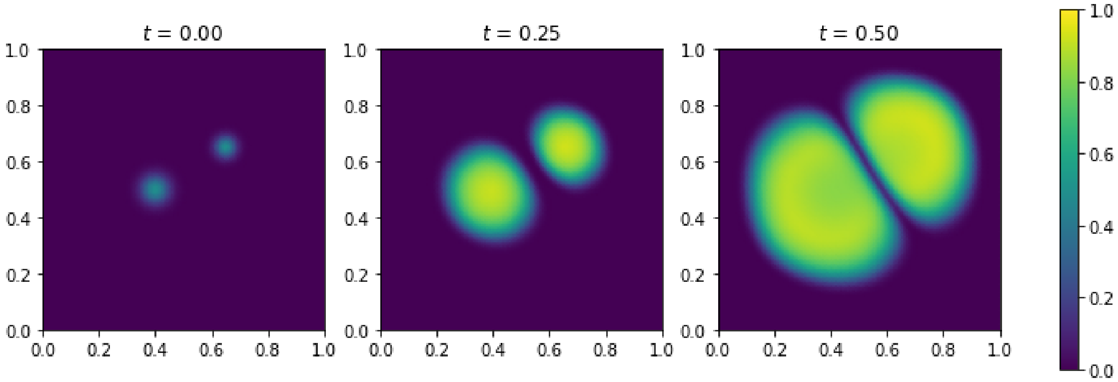

Figure 4.9: Case 5: evolution of n_p with $\mu = 0.1$.

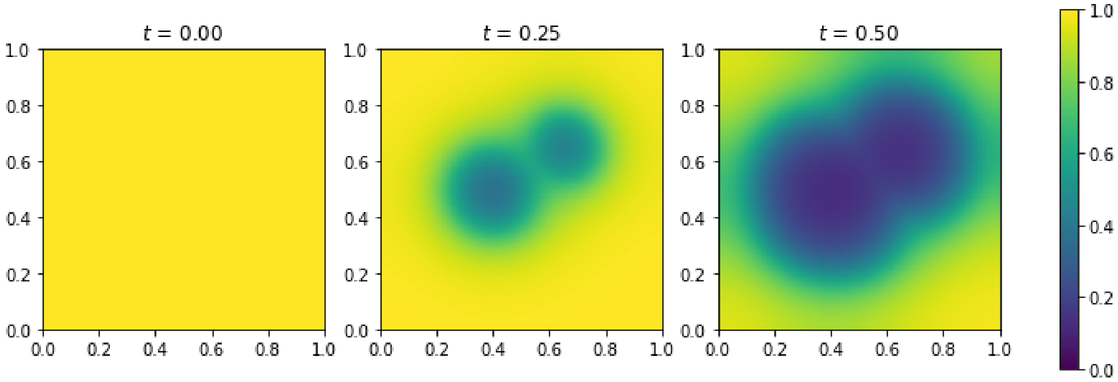

Figure 4.10: Case 5: evolution of c with $\mu = 0.1$.

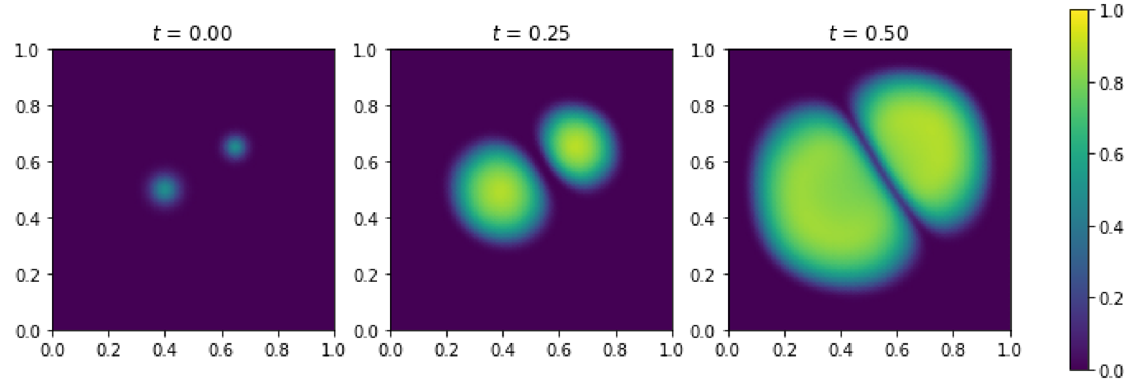

Figure 4.11: Case 6: evolution of n_p with $\mu = 0.1$ and chemotaxis.

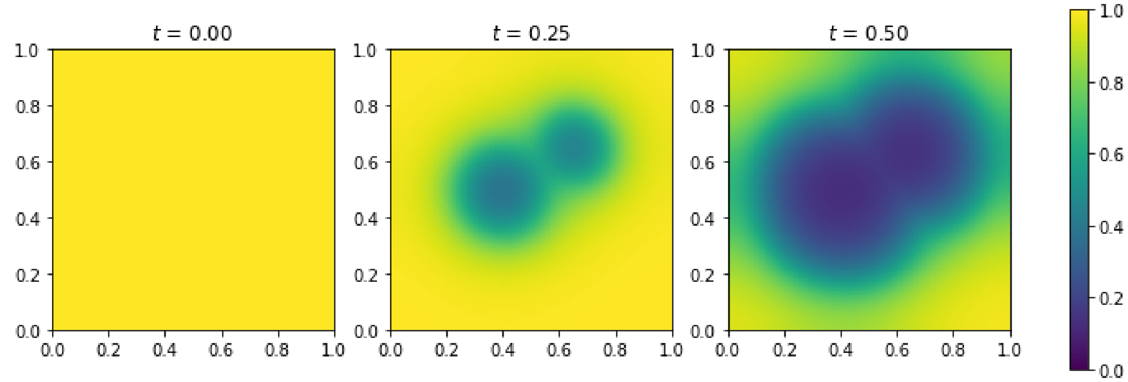

Figure 4.12: Case 6: evolution of c with $\mu = 0.1$ and chemotaxis.

In case 6, where chemotaxis is present, the results are quite similar to case 5, the tumors are just slightly large and less dense. This is in contrast to the comparison between cases 1 and 3, which produced quite different results. Since the nutrient has a higher diffusion coefficient, the gradients of the nutrient density are less sharp than in case 3, so the effects of chemotaxis, which are proportional to ∇c, will be less significant. In fact, we see that in this case the two tumors do not overlap like they did in case 6.

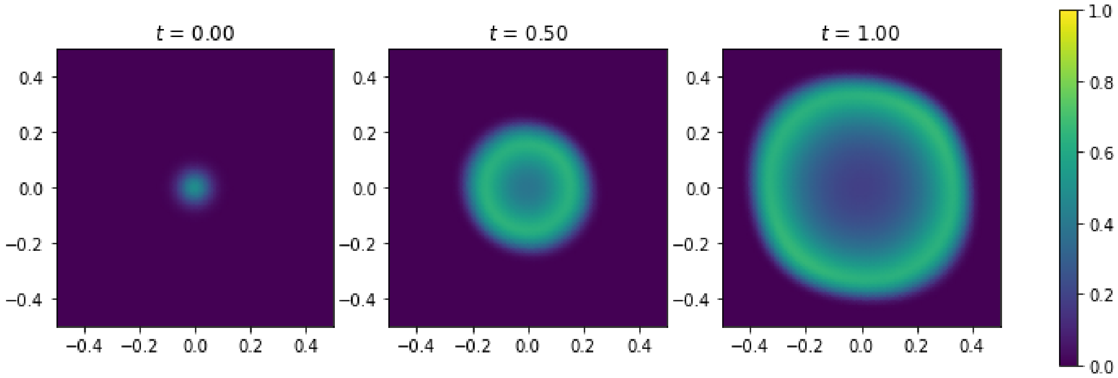

Figure 4.13: Evolution of n_p on a square domain.

4.3 Effects of the Boundary

We now examine the effect that the shape of the domain has on the evolution of the tumor. We consider two separate domains: a square domain $\Omega_1 = [-0.5, 0.5] \times [-0.5, 0.5]$, and a circular domain $\Omega_2 = B(0, 0.5)$. In both cases, we start with a small Gaussian tumor, given by

$$n_{p,0} = \frac{1}{2}\exp\{-300(x^2 + y^2)\}.$$

We set $f(c) = 10c$, $K_B = K_A = 10$, $d = 10^{-4}$, $\mu = 10^{-3}$, $\bar{c} = 0.5$, $c_0 = c_v = 1$. We ignore drug application and chemotaxis for this problem.

In addition, we consider the problem on a domain Ω_3 which is time-dependent. The initial domain is $\Omega_3(0) = \Omega_2 = B(0, 0.5)$, and the mapping of the domain is given by

$$\eta(t, x, y) = \left(x(1 + 0.2 * \sin(2\pi t)),\ \frac{y}{1 + 0.2 * \sin(2\pi t)} \right).$$

This experiment is also similar to an example in [52]. The results are shown in Figures 4.13-4.15.

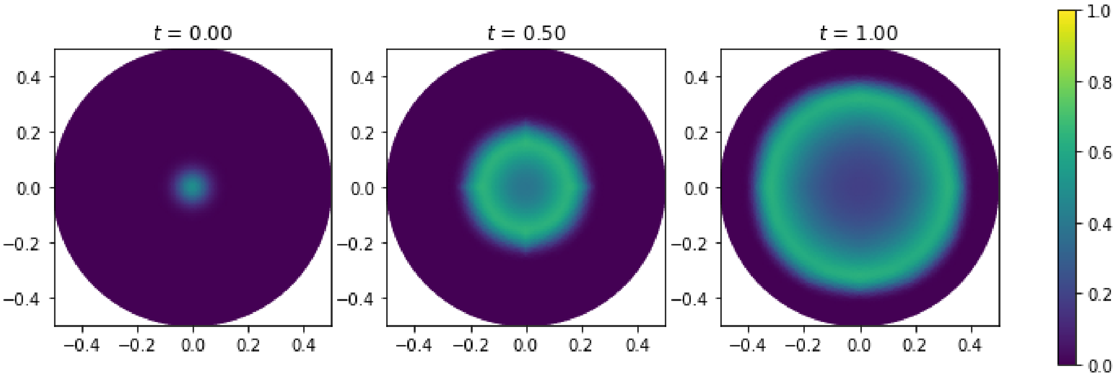

Figure 4.14: Evolution of n_p on a circular domain.

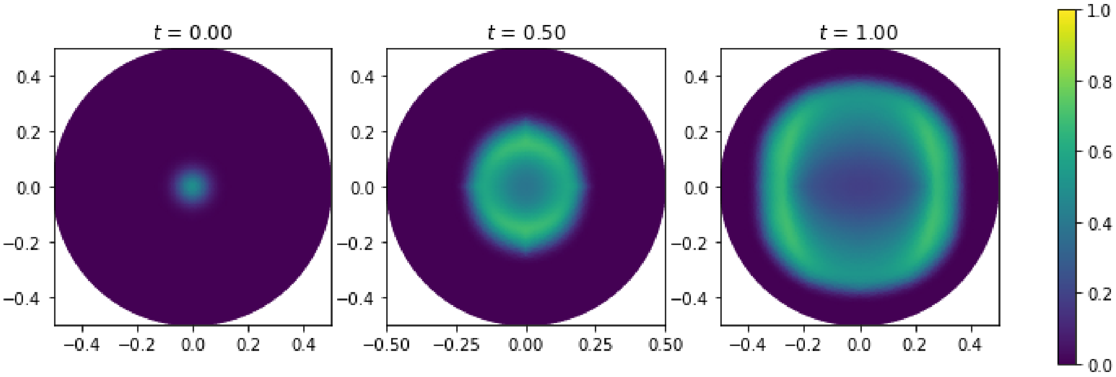

Figure 4.15: Evolution of n_p on time-dependent domain.

We see that the shape of the domain affects how the tumor evolves. In the case of the circular domain, the initial tumor is radially symmetric and remains so (aside from a few artifacts from the mesh) at future times. On the other hand, on the square mesh the tumor becomes more oblong, and is no longer radially symmetric. Furthermore, in the case of the time-dependent domain, the domain is actually circular at times $t = 0, 0.5, 1$, which helps to compare the evolution of the tumor to the circular domain case. We see that the tumor does not evolve in the same way on the time-dependent domain. At time $t = 0.5$ the tumor is slightly stretched vertically, while at $t = 1$ the tumor has a more square shape than it does on the fixed, circular domain and also has a region of lower density vertically across the middle of the tumor.

Chapter 5: Conclusion

In recent years, many varieties of mathematical models for tumors have been studied by mathematicians, computer scientists, biologists, and others. These models consider many different factors for tumor evolution, including interactions with drugs or nutrients. However, many of these models neglect the effect of cell chemotaxis. Certain types of cancerous cells are known to undergo chemotaxis in the presence of different nutrients and chemicals [42], so it is important that these effects can be incorporated into a model.

Models which do include chemotaxis either do not include the effects of a flow field within the tumor (such as [41] and the Cahn-Hilliard models studied in [10], [27], [47]), or use a simpler divergence constraint than (1.3) (see, for example [26]), often specifically examining the case of an incompressible flow ([20], [32], [36]). Alternatively, the free boundary problem for tumor growth examined in [7] introduces a regularization of the divergence constraint in order to produce strong solutions.

Other researchers have looked at models for chemotaxis outside of cancer modeling, such as in Keller-Segel models for the aggregations of slime molds ([34], [45]) or in the case of general chemotaxis-fluid models with incompressible flow ([12],

[18], [38]). Since these models either neglect coupling with a flow field or assume incompressibility, the nonlinearity of the divergence constraint (1.3) is avoided.

We build upon these previous models by examining a chemotaxis fluid model for tumor growth in which the cells within the tumor are transported by a flow field. The flow of cells through the extracellular matrix can be modeled either by Darcy's Law or by Brinkman's equation, both of which are governing equations for flow through a porous medium. Existence of weak solutions can be proven by defining a sequence of convergent approximating schemes. One main challenge in proving this existence result is the nonlinearity in the divergence constraint (1.3).

The use of the Arbitrary Lagrangian Eulerian (ALE) framework is a key part of the analysis. This method allows us to consider the problem on a moving (time-dependent) domain and is also especially useful in the case of a finite element scheme. While the ALE method has previously been used mostly for fluid-structure interaction problems [4] but has also been used increasingly in a variety of other problems in fluid dynamics [46]. To our knowledge this method has not been used for tumor growth models.

We are also able to construct a convergent, positivity-preserving finite element scheme for this model. Related schemes have been studied for chemotaxis models in the absence of a flow ([45], [48]), and for chemotaxis-fluid models in the case of incompressible flow ([5], [11], [19]). We are not aware of the existence of a similar finite element scheme for a chemotaxis fluid model without incompressibility.

On a fixed domain a scheme for both the case of Darcy's Law and the case of Brinkman's equation can be defined, while we are only able to define a scheme on a moving domain for the case of Darcy's Law. Numerical experimentation shows that the effects of chemotaxis *are* significant. Specifically, tumors tend to expand more quickly and become less dense in the presence of chemotaxis.

Further experimentation could be done to examine the numerical results for the case of Brinkman's equation and compare these results to the case of Darcy's Law. Additionally, we plan to be able to compare these simulated results to actual tumor growth observed in experiments. This would allow us to determine more appropriate parameters for different types of tumors, and to determine whether the chemotactic effects are observed in real data.

5.1 Future Work: A Free Boundary Problem

The current model has several limitations, primarily due to the difficulties of solving either Darcy's Law or Brinkman's equation on a moving domain given the divergence constraint (1.3). In order to solve these equations, we

- imposed homogeneous Neumann boundary conditions on c,
- defined the removal term $K_R(c,n) = -n_T^{-1} \int_{\Omega(t)} K_B(c) n_p$, and
- assumed that the mapping $\eta(t) : \Omega(0) \to \Omega(t)$ was volume-preserving.

These assumptions limit the applications of the model. For example, we want to consider the case where the nutrient is constantly supplied through the boundary of the tumor, which would correspond to Dirichlet boundary conditions for c. Ideally, we would allow the tumor to expand or contract, which we cannot do if η is volume-preserving, unless we consider the case of healthy cells surrounding a tumorous region as in the previous chapter. However, in this case there are no forces (like surface tension) holding the tumor together, so the tumors expand more quickly and are less dense than we would likely see in a physical setting.

To address these limitations, we plan to examine the case of a free boundary problem. Such a problem is defined in a similar matter as in Chapter 1, but with different boundary conditions. In this case we do not prescribe the domain $\Omega(t)$ a priori but rather solve for the velocity of the domain $\boldsymbol{v}_s$ as part of the problem and construct $\Omega(t)$ from $\boldsymbol{v}_s$. Specifically, we consider the boundary conditions

$$\nabla n_k \cdot \boldsymbol{n} = 0, \quad c = c_B, \quad p = \gamma\kappa, \text{ on } \Gamma(t),$$

where κ is the mean curvature of $\Gamma(t)$ and γ is a surface tension coefficient, and let the velocity of the domain $\boldsymbol{v}_s$ be given by

$$\boldsymbol{v} \cdot \boldsymbol{n} = \boldsymbol{v}_s \cdot \boldsymbol{n} \text{ on } \Gamma(t).$$

These conditions are similar to the free boundary conditions defined by Friedman in [25] and other related works (for example, [7], [8], [54]). This means that the boundary of the domain is allowed to move in response to the surface tension of the tumor.

However, if the boundary $\Gamma(t)$ is not sufficiently smooth, we run into problems with the regularity of the pressure p along the boundary, and thus the regularity of the velocity $\boldsymbol{v}_s$ of the domain. Thus, a simpler problem is to consider a radially symmetric model, where $\Omega(t)$ is a ball of radius $R(t)$ and the quantities are dependent only on the radial position r and the time t. In this case, $\kappa = \frac{1}{R(t)}$ on $\Gamma(t)$. We expect that we can solve this problem in a similar way as before, after reducing the model to a problem in space r, t.

Appendix

A.1 Estimates for mapping

Lemma A.1.1. *Suppose that* $\tilde{u} \in W^{1,p}(\Omega(t))$*,* $1 \leq p \leq \infty$*, and define* $u = \tilde{u} \circ \eta(t)$*. Then,* $u \in W^{1,p}(\Omega(0))$ *with*

$$c_L \|\tilde{u}\|_{W^{1,p}(\Omega(t))} \leq \|u\|_{W^{1,p}(\Omega(0))} \leq C_L \|\tilde{u}\|_{W^{1,p}(\Omega(t))}.$$

Furthermore, if $\tilde{u} \in W^{2,p}(\Omega(t))$*, then there exist constants* c, C *independent of* $\Omega(t), t$ *such that*

$$c\|u\|_{W^{2,p}(\Omega(0))} \leq \|\tilde{u}\|_{W^{2,p}(\Omega(t))} \leq C\|u\|_{W^{2,p}(\Omega(0))}.$$

Proof. Denote by $\boldsymbol{x}, \boldsymbol{y}$ the coordinates of $\Omega(0)$ and $\Omega(t)$, respectively. We note that

$$\partial_{x_i} u = \partial_{x_i} (\tilde{u} \circ \eta) = (\nabla \tilde{u} \circ \eta) \cdot (\partial_{x_i} \eta),$$

$$\partial_{y_i} \tilde{u} = \partial_{y_i} \left(u \circ \eta^{-1}\right) = (\nabla u \circ \eta^{-1}) \cdot (\partial_{y_i} \eta^{-1}).$$

Thus, it follows that

$$|\nabla u|^2 \leq C_L^2 |\nabla \tilde{u} \circ \eta|^2 \text{ and } |\nabla \tilde{u}|^2 \leq c_L^{-2} |\nabla u \circ \eta^{-1}|^2.$$

Using the fact that $\det(\nabla\eta) = 1$, we have

$$c_L \|\tilde{u}\|_{W^{1,p}(\Omega(t))} \leq \|u\|_{W^{1,p}(\Omega(0))} \leq C_L \|\tilde{u}\|_{W^{1,p}(\Omega(t))},$$

under the assumption that $c_L \leq 1 \leq C_L$.

Next, we see that

$$\partial_{x_i}\partial_{x_j} u = (((D^2\tilde{u}) \circ \eta) \cdot (\partial_{x_j}\eta)) \cdot (\partial_{x_i}\eta) + ((\nabla\tilde{u}) \circ \eta) \cdot (\partial_{x_i}\partial_{x_j}\eta).$$

It follows that

$$\|D^2 u\|_{L^p(\Omega(0))} \leq C_L^2 \|D^2\tilde{u}\|_{L^p(\Omega(t))} + \|\nabla\tilde{u}\|_{L^p(\Omega(t)}\|D^2\eta\|_{L^\infty(\Omega(0))} \leq C\|\tilde{u}\|_{W_p^2(\Omega(t))}.$$

The other direction of the inequality follows in the same manner, by using the bounds on η^{-1}. □

Lemma A.1.2. *Suppose $\tilde{u} \in W^{2,p}(\Omega(t))$ and define $u = \tilde{u} \circ \eta$. Then,*

$$\Delta\tilde{u} = (A(t)u) \circ \eta^{-1} = \left(\partial_{x_i}(a_{ij}\partial_{x_j} u)\right) \circ \eta^{-1}.$$

where $A(t)$ is a strongly elliptic operator on $W^{2,p}(\Omega(0))$. If $\tilde{u}$ satisfies homogeneous Neumann or Dirichlet boundary conditions, then $\tilde{u}$ also has homogeneous boundary conditions (either an oblique condition of the form $\sum_{ij} a_{ij}\partial_{x_j} u \cdot n_i = 0$ or Dirichlet conditions), and there exist $\lambda_0 > 0, C > 0$ such that

$$\|\tilde{u}\|_{W^{2,p}(\Omega(t))} \leq C\left(\|\Delta\tilde{u}\|_{L^p(\Omega(t))} + \|\tilde{u}\|_{W^{1,p}(\Omega(t))}\right),$$

$$\|\tilde{u}\|_{W^{2,p}(\Omega(t))} \leq C\left(\|\Delta\tilde{u} + \lambda\tilde{u}\|_{L^p(\Omega(t))}\right),$$

for all $\lambda > \lambda_0$. Here, C is independent of t and λ.

Proof. Denote by $\boldsymbol{x}, \boldsymbol{y}$ the coordinates of $\Omega(0)$ and $\Omega(t)$, respectively. Since $\tilde{u} = u \circ \eta^{-1}$, we can write

$$\Delta \tilde{u} = \sum_i (((D^2 u) \circ \eta^{-1}) \cdot (\partial_{y_i} \eta^{-1})) \cdot (\partial_{y_i} \eta^{-1}) + ((\nabla u) \circ \eta^{-1}) \cdot (\Delta \eta^{-1}),$$

so it follows that

$$\Delta \tilde{u} = (A(t)u) \circ \eta^{-1} + (B(t)u) \circ \eta^{-1},$$

where

$$A(t)u = \sum_{ij} \partial_{x_i}(a_{ij} \partial_{x_j} u), \quad a_{ij} = (\nabla \eta_i^{-1} \cdot \nabla \eta_j^{-1}) \circ \eta,$$

$$B(t)u = \sum_i b_i \partial_{x_i} u, \quad b_i = \Delta \eta_i^{-1} \circ \eta - \sum_j \partial_{x_j} a_{ij}.$$

Claim A.1.2.1. *If* $\det(\nabla \eta) = 1$*, then* $B(t)u = 0$*, and thus* $\Delta \tilde{u} = (A(t)u) \circ \eta^{-1}$.

Proof of claim. It is possible to write

$$b_i = -\sum_j (\nabla \eta_i^{-1} \circ \eta) \cdot (\partial_{x_j} (\nabla \eta_j^{-1} \circ \eta)) = -\sum_k (\partial_{y_k} \eta_i^{-1} \circ \eta) \sum_j \partial_{x_j} (\partial_{y_k} \eta_j^{-1} \circ \eta).$$

Furthermore, Jacobi's formula gives $\partial_x \det(A) = \operatorname{tr}(\operatorname{adj} A \partial_x A)$, and since $A^{-1} = (\det(A))^{-1} \operatorname{adj} A$, it follows that

$$\partial_x \det(A) = \det(A) \sum_i [A^{-1}]_{row_i} [\partial_x A]_{col_i}.$$

In the case that $A = \nabla \eta^{-1}$, this yields

$$0 = \partial_{y_j} \det(\nabla \eta^{-1}) = \det(\nabla \eta^{-1}) \sum_i (\partial_{y_i} \partial_{y_j} \eta^{-1}) \cdot (\nabla \eta_i \circ \eta^{-1}).$$

We also notice that

$$\sum_j \partial_{x_j}(\partial_{y_k}\eta_j^{-1}\circ\eta) = \sum_{i,j}\left((\partial_{y_i}\partial_{y_k}\eta_j^{-1})\circ\eta\right)\partial_{x_j}\eta_i = \sum_i\left((\partial_{y_i}\partial_{y_k}\eta^{-1})\circ\eta\right)\cdot\nabla\eta_i = 0.$$

Thus, $b_i = 0$. ■

Claim A.1.2.2. *The operator $A(t)$ is a strongly elliptic operator.*

Proof of claim. Taking $\xi \in \mathbb{R}^n$ with $\|\xi\| = 1$, we have

$$\begin{aligned}\sum_{j,k} a_{jk}\xi_i\xi_j &= \sum_i\left(\sum_j \xi_j\partial_{x_i}\eta_j\sum_k \xi_k\partial_{x_i}\eta_k\right) = \sum_i\left(\sum_j(\xi_j\partial_{x_i}\eta_j)^2\right)\\ &= \sum_j \xi_j^2|\nabla\eta_j|^2 \geq c_L^2\sum_j \xi_j^2 = c_L^2|\xi|^2.\end{aligned}$$

Thus, $A(t)$ is strongly elliptic. ■

Claim A.1.2.3. *If $\tilde{u}$ satisfies homogeneous boundary conditions, so does u.*

Proof of claim. It is obvious that if $\tilde{u}$ satisfies homogeneous Dirichlet conditions, so does u. Then, we examine the case of Neumann boundary conditions. It can be shown that if the vector $\boldsymbol{n}$ is a normal vector at $\boldsymbol{x} \in \Gamma(0)$, then the vector $\tilde{\boldsymbol{m}}$ given by

$$\tilde{m}_j = \sum_i (n_i\circ\eta^{-1})\partial_{y_j}\eta_i^{-1}$$

is a normal vector at $\boldsymbol{y} = \eta(\boldsymbol{x}) \in \Gamma(t)$.

Then, we have

$$\begin{aligned}(\nabla\tilde{u}\cdot\tilde{\boldsymbol{m}})\circ\eta &= \sum_i(\partial_{y_i}\tilde{u}\circ\eta)\,\tilde{m}_i\circ\eta = \sum_{i,j}(\partial_{y_i}\tilde{u}\circ\eta)\,n_j((\partial_{y_i}\eta_j^{-1})\circ\eta)\\ &= \sum_{i,j,k}(\partial_{x_k}u)\left(\partial_{y_i}\eta_k^{-1}\circ\eta\right)n_j((\partial_{y_i}\eta_j^{-1})\circ\eta) = \sum_{j,k} a_{j,k}^m\,(\partial_{x_k}u)\,n_j.\end{aligned}$$

Thus, u satisfies the homogeneous oblique boundary condition

$$\sum_{ij} a_{ij} \partial_{x_j} u \cdot n_i = 0.$$

■

Finally, we need to prove the desired regularity estimates. Then, since u has homogeneous boundary conditions it follows from Theorem 2.3.3.2 in [30] that

$$\|u\|_{W^{2,p}(\Omega(0))} \leq C \left(\|A(t)u\|_{L^p(\Omega(0))} + \|u\|_{W^{1,p}(\Omega(0))} \right),$$

where C is independent of t. Furthermore, by Theorem 2.3.3.6 in [30], there exist $\lambda_0, C > 0$ such that for all $\lambda \geq \lambda_0$,

$$\|u\|_{W^{2,p}(\Omega(0))} \leq C \|A(t)u + \lambda u\|_{L^p(\Omega(0))},$$

Additionally, we have

$$\|A(t)u\|_{L^p(\Omega(0))} \leq \|\Delta \tilde{u}\|_{L^p(\Omega(t))}, \quad \|A(t)u + \lambda u\|_{L^p(\Omega(0))} \leq \|\Delta \tilde{u} + \lambda \tilde{u}\|_{L^p(\Omega(t))}.$$

Therefore, combining the above estimates with Lemma A.1.1, we obtain

$$\begin{aligned} \|\tilde{u}\|_{W^{2,p}(\Omega(t))} &\leq \|u\|_{W^{2,p}(\Omega(0))} \\ &\leq C \left(\|A(t)u\|_{L^p(\Omega(0))} + \|u\|_{W^{1,p}(\Omega(0))} \right) \\ &\leq C \left(\|\Delta \tilde{u}\|_{L^p(\Omega(t))} + \|\tilde{u}\|_{W^{1,p}(\Omega(t))} \right), \end{aligned}$$

and the constant C is independent of t. Similarly, it follows that

$$\|\tilde{u}\|_{W^{2,p}(\Omega(t))} \leq C \left(\|\Delta \tilde{u} - \lambda \tilde{u}\|_{L^p(\Omega(t))} \right),$$

for any $\lambda \geq \lambda_0$, with C independent of t, λ. $\square$

Lemma A.1.3. *Let $0 \leq t \leq T$, $1 < p < \infty$, and fix $\boldsymbol{f}, g \in L^p(\Omega(t))$ with $\int_{\Omega(t)} g = 0$. Then, there exists a unique solution $u \in W^{1,p}(\Omega(t))$ to the problem*

$$\begin{cases} \Delta u = \operatorname{div} \boldsymbol{f} + g \ \text{ in } \Omega(t), \\ \nabla u \cdot \boldsymbol{n} = \boldsymbol{f} \cdot \boldsymbol{n} \ \text{ on } \Gamma(t) \\ \int_{\Omega(t)} u = 0, \end{cases}$$

satisfying the bound

$$\|u\|_{W^{1,p}(\Omega(t))} \leq C \left(\|\boldsymbol{f}\|_{L^p(\Omega(t))} + \|g\|_{L^p(\Omega(t))} \right)$$

with C independent of t. Furthermore, if $\boldsymbol{f} \in W^{1,p}(\Omega(t))$, we have the estimate

$$\|u\|_{W^{2,p}(\Omega(t))} \leq C \left(\|\boldsymbol{f}\|_{W^{1,p}(\Omega(t))} + \|g\|_{L^p(\Omega(t))} \right).$$

Proof. Suppose the hypotheses of the lemma are satisfied. We define the Bogovskii operator $\mathcal{B}_{\Omega(t)}$ on $\Omega(t)$ (see for example Section 3.3 in [40]) so that

$$\operatorname{div} \mathcal{B}_{\Omega(t)} g = g \text{ in } \Omega(t) \text{ and } \mathcal{B}_{\Omega(t)} g = 0 \text{ on } \partial\Omega(t)$$

for any $g \in L^p(\Omega(t))$ with $\int_{\Omega(t)} g = 0$. The operator $\mathcal{B}_{\Omega(t)}$ is bounded from L^p to L^p. Thus, taking $\hat{\boldsymbol{f}} = \boldsymbol{f} + \mathcal{B}_{\Omega(t)} g$, the elliptic problem can be rewritten as

$$\begin{cases} \Delta u = \operatorname{div} \hat{\boldsymbol{f}} \text{ in } \Omega(t), \\ \nabla u \cdot \boldsymbol{n} = \hat{\boldsymbol{f}} \cdot \boldsymbol{n} \text{ on } \Gamma(t). \end{cases}$$

Then, by Theorem 1 in [2], there exists a solution u with

$$\|\nabla u\|_{L^p(\Omega(t))} \leq C(t)\|\hat{\boldsymbol{f}}\|_{L^p(\Omega(t))} \leq C(t)\left(\|\boldsymbol{f}\|_{L^p(\Omega(t))} + \|g\|_{L^p(\Omega(t))}\right)$$

It remains to show that $C(t)$ can be taken to be independent of t. We do prove this by contradiction. Suppose this is not true. Then, there exists a sequence $\{u_k, \boldsymbol{f}_k, g_k, t_k\}$ such that u_k solves

$$\begin{cases} \Delta u_k = \operatorname{div} \boldsymbol{f}_k + g_k \text{ in } \Omega(t_k), \\ \nabla u_k \cdot \boldsymbol{n} = \boldsymbol{f}_k \text{ on } \Gamma(t_k), \end{cases}$$

and $\|\nabla u_k\|_{L^p(\Omega(t_k))} = c$ with $\|\boldsymbol{f}_k\|_{L^p(\Omega(t_k))}, \|g_k\|_{L^p(\Omega(t_k))} \to 0$. As shown in [2], we consider the weak formulation of the problem, given by

$$\int_{\Omega(t_k)} \nabla u_k \cdot \nabla \phi = \int_{\Omega(t_k)} (\boldsymbol{f}_k \cdot \nabla \phi - g_k \phi),$$

so that we avoid having to define $\nabla u_k, \boldsymbol{f}_k$ on the boundary.

We note that since $t_k \in [0, T]$, the sequence $\{t_k\}$ is bounded, and there exists an accumulation point t. Without loss of generality, assume that $t_k \to t$. Then, we define $\eta_k = \eta(t_k) \circ \eta^{-1}(t)$, and let $\tilde{u}_k = u_k \circ \eta_k$, $\tilde{\boldsymbol{f}}_k = \boldsymbol{f}_k \circ \eta_k$, $\tilde{g}_k = g_k \circ \eta_k$. It follows that $\nabla \tilde{u}_k = B_k (\nabla u_k) \circ \eta_k$, where $(B_k)_{ij} = \partial_{x_j} (\eta_k)_i$. Thus, we have

$$\int_{\Omega(t)} B_k \nabla \tilde{u}_k \cdot B_k \nabla \tilde{\phi} = \int_{\Omega(t)} \tilde{\boldsymbol{f}}_k \cdot B_k \nabla \tilde{\phi} - \int_{\Omega(t)} \tilde{g}_k \tilde{\phi},$$

where now $\tilde{\phi}$ is a test function on $\Omega(t)$. Thus, noting that

$$\alpha \cdot B_k \beta = \alpha^T B_k \beta = (B_k^T \alpha)^T \beta = B_k^T \alpha \cdot \beta,$$

we can write

$$\begin{aligned}\int_{\Omega(t)} \nabla\tilde{u}_k\cdot\nabla\tilde{\phi} &= \int_{\Omega(t)} \Big(\tilde{\boldsymbol{f}}_k + (B_k^T\tilde{\boldsymbol{f}}_k - \tilde{\boldsymbol{f}}_k) + (\nabla\tilde{u}_k - B_k^T B_k\nabla\tilde{u}_k)\Big)\cdot\nabla\tilde{\phi} - \int_{\Omega(t)} \tilde{g}_k\tilde{\phi}_k \\ &= \int_{\Omega(t)} \tilde{F}_k\cdot\nabla\tilde{\phi} - \int_{\Omega(t)} \tilde{g}_k\tilde{\phi}_k.\end{aligned}$$

It follows that

$$\begin{cases}\Delta\tilde{u}_k = \operatorname{div}\tilde{F}_k + \tilde{g}_k \text{ in } \Omega(t),\\ \nabla\tilde{u}_k\cdot\boldsymbol{n} = \tilde{F}_k\cdot\boldsymbol{n} \text{ on } \Gamma(t),\end{cases}$$

and thus

$$\|\nabla\tilde{u}_k\|_{L^p(\Omega(t))} \le C\left(\|\tilde{F}_k\|_{L^p(\Omega(t))} + \|\tilde{g}_k\|_{L^p(\Omega(t))}\right),$$

where C is dependent on $\Omega(t)$. Then, we note that

$$\begin{aligned}\|\tilde{F}_k\|_{L^p(\Omega(t))} \le \|\tilde{\boldsymbol{f}}_k\|_{L^p(\Omega(t))} &+ \|B_k^T - I\|_{L^\infty(\Omega(t))}\|\tilde{\boldsymbol{f}}_k\|_{L^p(\Omega(t))} \\ &+ \|B_k^T B_k - I\|_{L^\infty(\Omega(t))}\|\nabla\tilde{u}_k\|_{L^p(\Omega(t))}.\end{aligned}$$

Recalling the assumption that $\|\nabla u_k\|_{L^p(\Omega(t_k))} = c$ and $\|\boldsymbol{f}_k\|_{L^p(\Omega(t_k))}, \|g_k\|_{L^p(\Omega(t))} \to 0$, it follows from Lemma A.1.1 that

$$0 < c_1 \le \|\nabla\tilde{u}_k\|_{L^p(\Omega(t))} \le c_2, \quad \|\tilde{\boldsymbol{f}}_k\|_{L^p(\Omega(t))}, \|\tilde{g}_k\|_{L^p(\Omega(t))} \to 0.$$

Then, we note that the assumptions on $\eta(t)$ imply that $\|B_k\|_{L^\infty}, \|B_k^T\|_{L^\infty} \le C$, so it follows that

$$\begin{aligned}\lim_{k\to\infty}\|\tilde{F}_k\|_{L^p(\Omega(t))} &= \lim_{k\to\infty}\|B_k^T B_k - I\|_{L^\infty(\Omega(t))}\|\nabla\tilde{u}_k\|_{L^p(\Omega(t))} \\ &\le c_2\lim_{k\to\infty}\|B_k^T B_k - I\|_{L^\infty(\Omega(t))}.\end{aligned}$$

Then,

$$(B_k^T B_k)_{i,j} = \nabla(\eta_k)_j \cdot \nabla(\eta_k)_i,$$

and since $\partial_{x_i}(\eta_k)_j \to \delta_{ij}$ in $L^\infty(\Omega(t))$, it follows that $B_k^T B_k \to I$ in $L^\infty(\Omega(t))$ and thus

$$\lim_{k\to\infty} \|\tilde{F}_k\|_{L^p(\Omega(t))} = 0.$$

However, this is a contradiction because we have

$$0 < c_1 \le \|\nabla \tilde{u}_k\|_{L^p(\Omega(t))} \le C\left(\|\tilde{F}_k\|_{L^p(\Omega(t))} + \|\tilde{g}_k\|_{L^p(\Omega(t))}\right),$$

so it must be that the first estimate holds.

Next, we assume that $\boldsymbol{f} \in W^{1,p}(\Omega(t))$. Then, as in the proof of Lemma A.1.2, we have that $\tilde{u} = u \circ \eta(t)$ satisfies

$$\begin{cases} A(t)\tilde{u} = (\operatorname{div} \boldsymbol{f}) \circ \eta(t) & \text{in } \Omega(0), \\ B\nabla\tilde{u} \cdot \boldsymbol{m} = (\boldsymbol{f}) \circ \eta \cdot \boldsymbol{m} & \text{on } \Gamma(0), \end{cases}$$

where $B_{ij} = a_{ij}$ and a_{ij} is as defined in the proof of Lemma A.1.2, and $\boldsymbol{m}$ is the outward normal vector on $\Gamma(0)$. Thus, by Theorem 2.3.3.2 in [30], we have

$$\|\tilde{u}\|_{W^{2,p}(\Omega(0))} \le C\left(\|\operatorname{div} \boldsymbol{f}\|_{L^p(\Omega(t))} + \|\boldsymbol{f} \cdot \boldsymbol{m} \circ \eta\|_{W^{1-1/p,p}(\Gamma(0))} + \|\tilde{u}\|_{W^{1,p}(\Omega(0))}\right).$$

We also have

$$\|(\boldsymbol{f} \circ \eta) \cdot \boldsymbol{m}\|_{W^{1-1/p,p}(\Gamma(0))} \le C\|\operatorname{div}(\boldsymbol{f} \circ \eta)\|_{L^p(\Omega(0))}$$

by the Trace Theorem. Combining all of these estimates with Lemma A.1.1 we have

$$\|u\|_{W^{2,p}(\Omega(t))} \leq C \left(\|\boldsymbol{f}\|_{W^{1,p}(\Omega(t))} + \|u\|_{W^{1,p}(\Omega(t))}\right).$$

Combining this with the previous estimate in $W^{1,p}$ proves the desired bound. □

Lemma A.1.4. *For $\tilde{f} \in L^p(\Omega(t))$, $\tilde{g} \in L^p_0(\Omega(t))$ with $1 < p < \infty$, and constant $\lambda > 0$, there exists a unique solution $(\tilde{u}, \tilde{p})$ to the problem*

$$\begin{cases} \nabla \tilde{p} = -\lambda \tilde{u} + \Delta \tilde{u} + \tilde{f}, \\ \\ \operatorname{div} \tilde{u} = \tilde{g} \end{cases}$$

on $\Omega(t)$, with $\tilde{u}$ satisfying homogeneous Dirichlet boundary conditions on $\partial\Omega(t)$, provided $\int_{\Omega(t)} \tilde{g} = 0$. Furthermore, the solution satisfies

$$\|\tilde{p}\|_{L^p(\Omega(t))} + \|\tilde{u}\|_{W^1_p(\Omega(t))} \leq C \left(\|\tilde{f}\|_{L^p(\Omega(t))} + \|\tilde{g}\|_{L^p(\Omega(t))}\right) \tag{A.1}$$

with C independent of t.

In addition, if $\tilde{f} \in L^q(\Omega(t))$ and $\tilde{g} \in W^{-1}_q(\Omega(t))$ for $1 < q < \infty$, we have

$$\|\tilde{u}\|_{L^q(\Omega(t))} \leq C \left(\|\tilde{f}\|_{L^q(\Omega(t))} + \|\tilde{g}\|_{W^{-1}_q(\Omega(t))}\right), \tag{A.2}$$

with C independent of t.

Proof. The existence of a unique solution $(\tilde{u}, \tilde{p}) \in W^1_p(\Omega(t)) \times L^p(\Omega(t))$ is guaranteed by standard estimates for the Stokes equations (for example, Proposition 3.2 in [29]).

We argue the inequalities via contradiction. Suppose $1 < p < \infty$. Then, we argue the inequality (A.1) holds by contradiction. Assume that (A.1) does not hold. Then, there exists some sequence $(t^n, u^n, p^n, f^n, g^n)$ such that (u^n, p^n) solves Stokes equations with f^n, g^n on $\Omega^n = \Omega(t^n)$, and

$$\|p^n\|_{L^p(\Omega^n)} + \|u^n\|_{W^1_p(\Omega^n)} = C \text{ and } \lim_{n\to\infty} \|f^n\|_{L^p(\Omega^n)} + \|g^n\|_{L^p(\Omega^n)} = 0.$$

Then, it must be that, up to a subsequence, $t^n \to t$. Define $A^n = \eta^n \circ \eta^{-1}(t) : \Omega(t) \to \Omega^n$, and $\tilde{a}^n = u^n \circ A^n$, for $a = p, u, f, g$. Due to previous estimates in this section, we must have, for some constants $c, C > 0$,

$$c \le \|\tilde{p}^n\|_{L^p(\Omega(t))} + \|\tilde{u}^n\|_{W^1_p(\Omega(t))} \le C,$$

$$\lim_{n\to\infty} \|\tilde{f}^n\|_{L^p(\Omega(t))} + \|\tilde{g}^n\|_{L^p(\Omega(t))} = 0$$

Next, we note that

$$\Delta\tilde{u}^n - (\Delta u^n) \circ A^n = \Delta\tilde{u}^n - \sum_{i,j} \partial_{x_i}(a^n_{ij}\partial_{x_j}\tilde{u}^n)$$

where

$$a^n_{ij} = (\nabla(A^n_i)^{-1} \cdot \nabla(A^n_j)^{-1}) \circ A^n.$$

Then, we note that, due to the smoothness of $A^n, (A^n)^{-1}$ and the fact that $(A^n)^{-1} \to I$, the identity operator, it follows that

$$a^n_{ij} \to \delta_{ij}, \quad b^n_i \to 0 \text{ in } C(\Omega(t)).$$

Additionally, we note that

$$\begin{aligned}\nabla\tilde{p}^n - (\nabla p^n) \circ A^n &= (\partial_{x_i}\tilde{p}^n) - \sum_j c^n_{ij}\partial_{x_j}\tilde{p}^n \\ &= (\partial_{x_i}\tilde{p}^n) - \sum_j \partial_{x_j}(c^n_{ij}\tilde{p}^n) - (\partial_{x_j}c^n_{ij})\tilde{p}^n,\end{aligned}$$

where

$$c_{ij}^n = (\partial_{x_i}(A_j^n)^{-1}) \circ A_j^n \to \delta_{ij} \text{ and } \partial_{x_j} c_{ij}^n \to 0$$

by the same argument as above. Finally,

$$\operatorname{div} \tilde{u}^n - (\operatorname{div} u^n) \circ A^n = \sum_i \left(\partial_{x_i} \tilde{u}_i^n - \left(\sum_j c_{ij}^n \partial_{x_j} \tilde{u}_i^n \right) \right).$$

where c_{ij}^n is the same as above. Then, $(\tilde{u}^n, \tilde{p}^n)$ satisfy

$$\nabla \tilde{p}^n = -\lambda \tilde{u}^n + \Delta \tilde{u}^n + \tilde{f}^n + F^n,$$

$$\operatorname{div} \tilde{u}^n = \tilde{g}^n + G^n,$$

where

$$F^n = -\Delta \tilde{u}^n + (\Delta u^n) \circ A^n + \nabla \tilde{p}^n - (\nabla p^n) \circ A^n,$$

$$G^n = \operatorname{div} \tilde{u}^n - (\operatorname{div} u^n) \circ A^n.$$

Then, we have the estimate

$$\begin{aligned} &\|\tilde{u}^n\|_{W_p^1(\Omega(t))} + \|\tilde{p}^n\|_{L^p(\Omega(t))} \\ &\quad \le C(t) \left(\|\tilde{f}^n\|_{L^p(\Omega(t))} + \|F^n\|_{W_p^{-1}(\Omega(t))} + \|\tilde{g}^n\|_{L^p(\Omega(t))} + \|G^n\|_{L^p(\Omega(t))} \right). \end{aligned}$$

Due to the original assumptions,

$$\|\tilde{f}^n\|_{L^p(\Omega(t))} + \|\tilde{g}^n\|_{L^p(\Omega(t))} \to 0.$$

Furthermore, due to the previous analysis, we note that we can write

$$\begin{aligned}\|F^n\|_{W_p^{-1}(\Omega(t))} &\leq \|a_{ij} - \delta_{ij}\|_{L^\infty(\Omega(t))}\|\nabla \tilde{u}^n\|_{L^p(\Omega(t))} \\ &\quad + \left(\|c_{ij}^n - \delta_{ij}\|_{L^\infty(\Omega(t))} + \|\partial_{x_j} c_{ij}^n\|_{L^\infty(\Omega(t))}\right)\|\tilde{p}\|_{L^p(\Omega(t))} \\ &\leq C\left(\|a_{ij} - \delta_{ij}\|_{L^\infty(\Omega(t))} + \|c_{ij}^n - \delta_{ij}\|_{L^\infty(\Omega(t))} + \|\partial_{x_j} c_{ij}^n\|_{L^\infty(\Omega(t))}\right) \\ &\to 0,\end{aligned}$$

and

$$\begin{aligned}\|G^n\|_{L^p(\Omega(t))} &\leq \|c_{ij} - \delta_{ij}\|_{L^\infty(\Omega(t))}\|\nabla \tilde{u}^n\|_{L^p(\Omega(t))} \\ &\leq C\|c_{ij} - \delta_{ij}\|_{L^\infty(\Omega(t))} \\ &\to 0.\end{aligned}$$

Thus, it follows that

$$\|\tilde{u}^n\|_{W_p^1(\Omega(t))} + \|\tilde{p}^n\|_{L^p(\Omega(t))} \to 0.$$

However, this is a contradiction, since

$$0 < c \leq \|\tilde{u}^n\|_{W_p^1(\Omega(t))} + \|\tilde{p}^n\|_{L^p(\Omega(t))}.$$

Thu, it must be that (A.1) holds.

Next, we prove a stricter estimate, again by contradiction. There exists a constant $C \geq 0$, independent of t, such that if (u, p) solves Stokes equation for (f, g) on $\Omega(t)$, with $0 \leq t \leq T$, then

$$\|u\|_{W_p^2(\Omega(t))} + \|p\|_{W_p^1(\Omega(t))} \leq C\left(\|f\|_{L^p(\Omega(t))} + \|g\|_{W_p^1(\Omega(t))}\right). \tag{A.3}$$

We again prove this result by contradiction. Assume that (A.3) does not hold. Then, there exists some sequence $(t^n, u^n, p^n, f^n, g^n)$ such that (u^n, p^n) solve Stokes equation for (f^n, g^n) on $\Omega^n = \Omega(t^n)$ and

$$\|u^n\|_{W_p^2(\Omega^n)} + \|p^n\|_{W_p^1(\Omega^n)} = c > 0,$$

$$\lim_{n\to\infty} \|f^n\|_{L^p(\Omega^n)} + \|g^n\|_{W_p^1(\Omega^n)} = 0.$$

Using similar arguments as before, we know that, up to a subsequence, $t^n \to t$, and we see that $(\tilde{u}^n, \tilde{p}^n)$ solve

$$\nabla \tilde{p}^n = -\lambda \tilde{u}^n + \Delta \tilde{u}^n + \tilde{f}^n + F^n,$$

$$\operatorname{div} \tilde{u}^n = \tilde{g}^n + G^n.$$

Then, we have the estimate

$$\begin{aligned} &\|\tilde{u}^n\|_{W_p^2(\Omega(t))} + \|\tilde{p}^n\|_{W_p^1(\Omega(t))} \\ &\quad \le C\left(\|\tilde{f}^n\|_{L^p(\Omega(t))} + \|F^n\|_{L^p(\Omega(t))} + \|\tilde{g}^n\|_{W_p^1(\Omega(t))} + \|G^n\|_{W_p^1(\Omega(t))}\right). \end{aligned}$$

We also have

$$\begin{aligned} \|F^n\|_{L^p(\Omega(t))} &\le \|a_{ij}^n - \delta_{ij}\|_{L^\infty(\Omega(t))}\|D^2\tilde{u}^n\|_{L^p(\Omega(t))} + \|\partial_{x_i} a_{ij}^n\|_{L^\infty(\Omega(t))}\|\nabla \tilde{u}^n\|_{L^p(\Omega(t))} \\ &\quad + \|c_{ij}^n - \delta_{ij}\|_{L^\infty(\Omega(t))}\|\tilde{p}\|_{W_p^1(\Omega(t))} \\ &\le C\left(\|a_{ij}^n - \delta_{ij}\|_{L^\infty(\Omega(t))} + \|\partial_{x_i} a_{ij}^n\|_{L^\infty(\Omega(t))} + \|c_{ij}^n - \delta_{ij}\|_{L^\infty(\Omega(t))}\right) \\ &\to 0, \end{aligned}$$

and

$$\begin{aligned}\|G^n\|_{W^1_p(\Omega(t))} &\le \|c_{ij} - \delta_{ij}\|_{W^1_\infty(\Omega(t))} \|\tilde{u}^n\|_{W^2_p(\Omega(t))} \\ &\le C \|c_{ij} - \delta_{ij}\|_{W^1_\infty(\Omega(t))} \\ &\to 0.\end{aligned}$$

This draws a contradiction in the same manner as before.

Then, (A.2) follows by the same duality argument used in [22].

□

Lemma A.1.5. *Fix $0 \le s < t \le T$, and suppose $u \in W^{2,p}(\Omega(t))$ for some $2 \le p < \infty$ with $\eta \in W^{1,\infty}(0,T; H^1(\Omega(0)))$. Define $A = \eta(t) \circ \eta^{-1}(s) : \Omega(s) \to \Omega(t)$ and $A^{-1} = \eta(s) \circ \eta^{-1}(t) : \Omega(t) \to \Omega(s)$ and $\tilde{u} = u \circ A$. Then,*

$$\left| \|\nabla u\|^p_{L^p(\Omega(t))} - \|\nabla \tilde{u}\|^p_{L^p(\Omega(s))} \right| \le (t-s) \left(C(\epsilon^{-1}, p) \|\nabla u\|^p_{L^p(\Omega(t))} + \epsilon \|u\|^p_{W^{2,p}(\Omega(t))} \right),$$

where $C(\epsilon^{-1}, p)$ is independent of s, t.

Proof. We first prove this lemma in the case $p = 2$. We first recall that Lemma A.1.1 gives the bound

$$c_L \|\tilde{u}\|_{H^1(\Omega(s))} \le \|u\|_{H^1(\Omega(t))} \le C_L \|\tilde{u}\|_{H^1(\Omega(s))}.$$

Furthermore, we have the Sobolev embedding estimate

$$\|u\|_{W^{1,q}(\Omega(t))} \le C \|u\|_{H^2(\Omega(t))},$$

for C independent of t, and $q \leq 6$. Then, we have

$$\left| \|\nabla u\|^2_{L^2(\Omega(t))} - \|\nabla \tilde{u}\|^2_{L^2(\Omega(s))} \right| = \int_{\Omega(s)} ((\nabla u) \circ A - \nabla \tilde{u}) \cdot ((\nabla u) \circ A + \nabla \tilde{u})$$

$$\leq C \|(\nabla u) \circ A - \nabla \tilde{u}\|_{L^{\frac{6}{5}}(\Omega(s))} \|\nabla u\|_{L^6(\Omega(t))}$$

$$\leq C \|(\nabla u) \circ A - \nabla \tilde{u}\|_{L^{\frac{6}{5}}(\Omega(s))} \|u\|_{H^2(\Omega(t))}.$$

Then, we examine the term $\|(\nabla u) \circ A - \nabla \tilde{u}\|_{L^{\frac{6}{5}}(\Omega(s)}$. We estimate

$$\partial_{x_i} \tilde{u} = \partial_{x_i}(u \circ A) = (\nabla u \circ A) \cdot (\partial_{x_i} A).$$

Thus,

$$(\nabla u) \circ A - \nabla \tilde{u} = ((\nabla u) \circ A) \cdot (I - \nabla A),$$

where I is the identity matrix. We know that

$$\partial_{x_i} A = \partial_{x_i}(\eta(t) \circ \eta(s)^{-1}) = ((\nabla \eta(t)) \circ \eta(s)^{-1}) \cdot (\partial_{x_i} \eta(s)^{-1}),$$

$$((\nabla \eta(s)) \circ \eta(s)^{-1}) \cdot (\partial_{x_i} \eta(s)^{-1}) = \partial_{x_i} \boldsymbol{x} = (\delta_{ij})_{j=1}^n,$$

so it follows that

$$\nabla A^k - I = \left(((\nabla \eta(t) - \nabla \eta(s)) \circ \eta(s)^{-1}) \cdot (\partial_{x_i} \eta(s)^{-1}) \right)_{i=1}^n.$$

Under the assumption that $\eta \in W^{1,\infty}(0, T; H^1(\Omega(0)))$, it follows that $\nabla \eta$ is Lipschitz in $L^2(\Omega^0)$ with respect to t, so

$$\|\nabla \eta(t) - \nabla \eta(s)\|_{L^2(\Omega(0))} \leq C(t - s),$$

and thus

$$\|\nabla A - I\|_{L^2(\Omega(s))} \leq C(t - s).$$

Therefore, we have

$$\|(\nabla u)\circ A - \nabla\tilde{u}\|_{L^{\frac{6}{5}}(\Omega(s))} \le C\|\nabla u\|_{L^3(\Omega(t))}\|\nabla A - I\|_{L^2(\Omega(s))} \le C\|\nabla u\|_{L^3(\Omega(t))},$$

and the desired estimate follows by noting that

$$\begin{aligned} C\|\nabla u\|_{L^3(\Omega(t))}\|u\|_{H^2(\Omega(t))} &\le \|\nabla u\|^{1/2}_{L^2(\Omega(t))}\|u\|^{3/2}_{H^2(\Omega(t))} \\ &\le C(\epsilon^{-1})\|\nabla u\|^2_{L^2(\Omega(t))} + \epsilon\|u\|^2_{H^2(\Omega(t))}. \end{aligned}$$

Next, we examine the case of even p with $p > 2$, and note that it follows for all $2 \le p < \infty$ by interpolation. In this case, since $p > n$, we have the Sobolev embedding estimate

$$\|u\|_{W^{1,\infty}(\Omega(t))} \le C\|u\|_{W^{2,p}(\Omega(t))},$$

with C independent of t. Using the algebraic estimate

$$a^k - b^k = (a-b)(a^{k-1}b^0 + a^{k-2}b^1 + ... + a^1b^{k-2} + a^0b^{k-1}),$$

and taking $p = 2k$, we have

$$\begin{aligned} \left|\|\nabla u\|^{2k}_{L^{2k}(\Omega(t))} - \|\nabla\tilde{u}\|^{2k}_{L^{2k}(\Omega(s))}\right| &= \int_{\Omega(s)} \left||(\nabla u)\circ A|^{2k} - |\nabla\tilde{u}|^{2k}\right| \\ &\le C\|\nabla u\|^{2k-2}_{L^\infty(\Omega(t))}\left|\|\nabla u\|^2_{L^2(\Omega(t))} - \|\nabla\tilde{u}\|^2_{L^2(\Omega(s))}\right| \\ &\le C\|u\|^{p-2}_{W^{2,p}(\Omega(t))}\left|\|\nabla u\|^2_{L^2(\Omega(t))} - \|\nabla\tilde{u}\|^2_{L^2(\Omega(s))}\right|. \end{aligned}$$

Then, by the argument in the case of $p = 2$, it follows that

$$\left|\|\nabla u\|^2_{L^2(\Omega(t))} - \|\nabla\tilde{u}\|^2_{L^2(\Omega(s))}\right| \le C(t-s)\|\nabla u\|_{L^3(\Omega(t))}\|u\|_{H^2(\Omega(t))},$$

and the desired estimate follows by noting that

$$
\begin{aligned}
C\|\nabla u\|_{L^3(\Omega(t))}\|u\|_{H^2(\Omega(t))}\|u\|^{p-2}_{W^{2,p}(\Omega(t))} &\leq C\|\nabla u\|^{1+\theta}_{L^p(\Omega(t))}\|u\|^{p-1-\theta}_{W^{2,p}(\Omega(t))} \\
&\leq C(\epsilon^{-1},p)\|\nabla u\|^p_{L^p(\Omega(t))} + \epsilon\|u\|^p_{W^{2,p}(\Omega(t))}
\end{aligned}
$$

by Sobolev embedding and Young's inequality. □

A.2 Other estimates

Lemma A.2.1 (Semidiscrete Version of Arzela Ascoli). *Suppose* $u_{\Delta t}$ *is piecewise constant in time with* $u_{\Delta t} = u^m_{\Delta t}$ *on* $(t^{m-1}, t^m]$, *and that* $u^m_{\Delta t} \in H \subset\subset B$, *where either* $H = L^2(\Omega), B = H^1(\Omega)$ *or* $H = H^{-1}(\Omega), B = L^2(\Omega)$, *and* $\Delta t = t^m - t^{m-1}$. *Furthermore, suppose*

$$
\|D_t^- u_{\Delta t}\|_{L^2(0,T;H)} \leq C, \quad \|u_{\Delta t}\|_{L^\infty(0,T;B)} \leq C
$$

independent of $\Delta t > 0$. *Then, up to a subsequence,*

(i) $u_{\Delta t} \to u$ *in* $L^\infty(0,T;H)$ *with* $u \in C(0,T;H)$,

(ii) if $H = H^{-1}$, $B = L^2$, *then* $u_{\Delta t}(t) \rightharpoonup u(t)$ *in* B *for a.e.* $t \in [0,T]$.

Proof. Property (i) is a simplification of the more general Theorem 6.2 in [16]. We note that $[0,T]$ is a compact metric space, and H is complete metric space compactly embedded in B. Thus, the uniform bound $\|u_{\Delta t}\|_{L^\infty(0,T;B)} \leq C$ implies that $u_{\Delta t}$ is relatively compact in H for all $s \in [0,T]$. Furthermore, we note that

$$
\|u^m_{\Delta t} - u^n_{\Delta t}\|_H \leq \sum_{j=n+1}^{m} \|u^j_{\Delta t} - u^{j-1}_{\Delta t}\|_H = \Delta t \sum_{j=n+1}^{m} \|D_t^- u^j_{\Delta t}\|_H,
$$

so

$$\|u_{\Delta t}^m - u_{\Delta t}^n\|_H^2 \leq (m-n)(\Delta t)^2 \sum_{j=n+1}^{m} \|D_t^- u_{\Delta t}^j\|_H^2,$$

by Jensen's inequality. Thus,

$$\|u_{\Delta t}^m - u_{\Delta t}^n\|_H^2 \leq (m-n)\Delta t \|D_t^- u_{\Delta t}\|_{L^2(0,T;H)}^2 \leq C(t^m - t^n).$$

We can then use Remark 6.3 in [16] to determine that we have met the hypotheses of Theorem 6.2 in [16]. This means that $\{u_{\Delta t}\}$ is relatively compact in $L^\infty(0,T;H)$ and, up to a subsequence, $u_{\Delta t} \to u$ in $L^\infty(0,T;H)$ with $u \in C(0,T;H)$.

We now prove property (ii) in a manner similar to the proof of Lemma 6.2 in [40]. From property (i) we know that $u_{\Delta t} \to u$ in $L^\infty(0,T;H^{-1}(\Omega))$. Then, for fixed $\epsilon > 0$ there exists $\Delta t_0 > 0$ such that for any $\Delta t_j, \Delta t_k \leq \Delta t_0$, it follows that

$$\left| \int_\Omega (u_{\Delta t_k}(t) - u_{\Delta t_j}(t))\eta \right| \leq \|u_{\Delta t_k}(t) - u_{\Delta t_j}(t)\|_{H^{-1}(\Omega)} \|\eta\|_{H^1(\Omega)} \leq \epsilon \|\eta\|_{H^1(\Omega)}$$

for any test function $\eta \in \mathcal{D}(\Omega)$. Thus, it must be that

$$\int_\Omega u_{\Delta t_k}(t) \to l_\eta(t)$$

for each $\eta \in \mathcal{D}$, $t \in (0,T]$, with

$$|l_\eta(t)| \leq \left| \limsup_{k\to\infty} \int_\Omega u_{\Delta t_k}(t)\eta \right| \leq c\|\eta\|_{L^2(\Omega)}$$

due to the uniform bounds on $u_{\Delta t_k}$ in $L^\infty(0,T;L^2(\Omega))$. Following the same argument as in the proof of Lemma 6.2 in [40], l_η can be represented as

$$l_\eta(t) = \int_\Omega g(t)\eta$$

with $g(t) \in L^2(\Omega)$, and we also have the estimate $\|g(t)\|_{L^\infty(0,T;L^2(\Omega))} \leq c$. We also have

$$\left|\int_\Omega (g(t) - g(s))\eta\right| \leq \|g(t) - g(s)\|_{H^{-1}(\Omega)} \|\eta\|_{H^1(\Omega)} \to 0$$

since $g \in C(0,T;H^{-1}(\Omega))$, and thus the map $t \mapsto \int_\Omega g(t)\eta$ is continuous for each $\eta \in \mathcal{D}$. Then, we can write

$$\left|\int_\Omega (g(t) - g(s))\eta\right| \leq \left|\int_\Omega (g(t) - g(s))\overline{\eta}\right| + \left|\int_\Omega (g(t) - g(s))(\eta - \overline{\eta})\right|,$$

where now $\eta \in L^2(\Omega)$ and $\overline{\eta} \in \mathcal{D}(\Omega)$. For any $\eta \in L^2(\Omega)$ the right hand side can be made sufficiently small by taking $\overline{\eta} \in \mathcal{D}(\Omega)$ such that

$$\|\eta - \overline{\eta}\|_{L^2(\Omega)} \leq \frac{\epsilon}{2},$$

and taking $|t - s|$ small enough that

$$\left|\int_\Omega (g(t) - g(s))\overline{\eta}\right| \leq \frac{\epsilon}{2},$$

which is possible due to the previously proven continuity of the mapping $t \mapsto \int_\Omega g(t)\eta$ for $\eta \in \mathcal{D}$. Thus, this mapping is also continuous for $\eta \in L^2(\Omega)$.

Finally, we write

$$\begin{aligned}\left|\int_\Omega (g_n(t) - g(t))\eta\right| &\leq \left|\int_\Omega (g_n(t) - g(t))\overline{\eta}\right| + \left|\int_\Omega (g_n(t) - g(t))(\eta - \overline{\eta})\right| \\ &\to 0\end{aligned}$$

where the second term on the right hand side vanishes because we can take $\overline{\eta}$ sufficiently close to η in $L^2(\Omega)$ and the first term vanishes due to the convergence of

$g_n \to g$ in $C(0,T;H^{-1}(\Omega))$. Thus,

$$\int_\Omega g_n(t)\eta \to \int_\Omega g(t))\eta$$

for any $\eta \in L^2(\Omega)$, which proves property (ii). $\square$

A.3 Estimates for finite element schemes

Lemma A.3.1. *Suppose $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, and $\mathcal{T}_h$ is a triangulation of Ω which exactly fits the boundary, and is quasiuniform with spatial step size $h > 0$. The operator R_h has the following properties for $s = 1, 2$:*

(i) for all $u \in W^{s,p}(\Omega)$, $\|u - R_h u\|_{W^{s-l,p}(\Omega)} \leq Ch^{s-l}\|u\|_{W^{s,p}(\Omega)}$ for $l = 0, 1$ and $1 < p < \infty$,

(ii) for all $u \in W^{1,p}(\Omega)$, $\|R_h u\|_{W^{1,p}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$ for $1 < p \leq \infty$,

(iii) if $h_k \to 0$ and $u \in W^{s,p}(\Omega)$ then $R_{h_k}u \to u$ strongly in $W^{s-1,p}$.

(iv) If $u \in L^p(0,T;W^{s,p}(\Omega))$ with $\partial_t u \in L^p(0,T;W^{1,p}(\Omega)) \cap W^{1,p}(0,T;L^p(\Omega))$, then $D_t^- R_h u^m \to \partial_t u$ in $L^p((0,T)\times\Omega)$, where $u^m(t) = u(t^m)$ for $t^{m-1} < t \leq t^m$, where $\Delta t = t^m - t^{m-1} \leq h$.

Proof. Property (i) is due to (2.6b) in [37]. If $p = 2$, property (i) is due simply to the fact that $R_h u$ is the H^1 projection of u onto S_h. Property (ii) is due to (22) in [35], and property (iii) follows immediately from property (i), by noting that

$$\lim_{k\to\infty} \|R_{h_k}u - u\|_{W^{s-1,p}(\Omega)} \leq C\|u\|_{W^{s,p}(\Omega)} \lim_{k\to\infty} h_k = 0.$$

Property (iv) is slightly more complicated. We can write

$$\|D_t^- R_h u^m - \partial_t u\|_{L^p((0,T)\times\Omega)} \leq \|D_t^-(R_h u^m - u^m)\|_{L^p((0,T)\times\Omega)} + \|D_t^- u^m - \partial_t u\|_{L^p((0,T)\times\Omega)}.$$

We note that $D_t^- R_h u^m = R_h D_t^- u^m$, so

$$\begin{aligned}\|D_t^- R_h u^m - D_t^- u^m\|_{L^p((0,T)\times\Omega)} &= \|R_h D_t^- u^m - D_t^- u^m\|_{L^p((0,T)\times\Omega)} \\ &\leq Ch\|D_t^- u^m\|_{L^p(0,T;W^{1,p}(\Omega))}.\end{aligned}$$

Additionally, we know that

$$\begin{aligned}D_t^- u^m &= \frac{u(t^m) - u(t^{m-1})}{\Delta t} = (\Delta t)^{-1} \int_{t^{m-1}}^{t^m} \partial_t u(t)\ dt, \\ \nabla(D_t^- u^m) &= \frac{\nabla u(t^m) - \nabla u(t^{m-1})}{\Delta t} = (\Delta t)^{-1} \int_{t^{m-1}}^{t^m} \partial_t \nabla u(t)\ dt,\end{aligned}$$

so

$$\begin{aligned}&\|D_t^- u^m\|^p_{L^p(0,T;W^{1,p}(\Omega))} \\ &\quad= \sum_m (\Delta t)^{-p+1} \int_\Omega \left[\left(\int_{t^{m-1}}^{t^m} \partial_t u(t)\ dt \right)^p + \left(\int_{t^{m-1}}^{t^m} \partial_t \nabla u(t)\ dt \right)^p \right] \\ &\quad\leq \int_0^T \int_\Omega \left[(\partial_t u(t))^p + (\partial_t \nabla u(t))^p \right]\ dt \\ &\quad= \|\partial_t u\|^p_{L^p((0,T)\times\Omega)} + \|\partial_t \nabla u\|_{L^p((0,T)\times\Omega)} \\ &\quad= \|\partial_t u\|^p_{L^p(0,T;W^{1,p}(\Omega))}\end{aligned}$$

by Jensen's inequality. Thus,

$$\|D_t^- u^m\|_{L^p(0,T;W^{1,p}(\Omega))} \leq \|\partial_t u\|_{L^p(0,T;W^{1,p}(\Omega))}.$$

Furthermore,

$$\begin{aligned} D_t^- u^m - \partial_t u &= (\Delta t)^{-1} \int_{t^{m-1}}^{t^m} \left(\partial_s u(s) - \partial_t u(t)\right)\, ds \\ &= (\Delta t)^{-1} \int_{t^{m-1}}^{t^m} \int_t^s \partial_{t'}^2 u(t')\, dt'\, ds. \end{aligned}$$

Thus, we have

$$\begin{aligned} |D_t^- u^m - \partial_t u|^p &= (\Delta t)^{-p} \left| \int_{t^{m-1}}^{t^m} \int_t^s \partial_{t'}^2 u(t')\, dt'\, ds \right|^p \\ &\leq (\Delta t)^{p-2} \int_{t^{m-1}}^{t^m} \int_t^s \left|\partial_{t'}^2 u(t')\right|^p \, dt'\, ds \\ &\leq (\Delta t)^{p-1} \int_{t^{m-1}}^{t^m} \left|\partial_t^2 u(t)\right|^p \, ds \end{aligned}$$

by Jensen's inequality. Thus,

$$\|D_t^- u^m - \partial_t u\|_{L^p((0,T)\times\Omega)}^p = (\Delta t)^p \|\partial_t^2 u(t)\|_{L^p((0,T)\times\Omega)}^p,$$

so

$$\|D_t^- u^m - \partial_t u\|_{L^p((0,T)\times\Omega)} \leq \Delta t \|\partial_t^2 u(t)\|_{L^p((0,T)\times\Omega)}.$$

Putting these inequalities together yields

$$\|D_t^- R_h u^m - \partial_t u\|_{L^p((0,T)\times\Omega)} \leq Ch\left(\|\partial_t u\|_{L^p(0,T;W^{1,p}(\Omega))} + \|\partial_t^2 u\|_{L^p((0,T)\times\Omega)}\right).$$

Therefore, it follows that

$$\begin{aligned} \lim_{h\to 0} \|D_t^- R_h u^m - \partial_t u\|_{L^p((0,T)\times\Omega)} &\leq C\left(\|\partial_t u\|_{L^p(0,T;W^{1,p}(\Omega))} + \|\partial_t^2 u\|_{L^p((0,T)\times\Omega)}\right) \lim_{h\to 0} h \\ &= 0. \end{aligned}$$

A.3.1 Proof of Lemma 3.4.1

Suppose that

$$|(\pi_h u_h, \pi_h v_h)_\Omega - (u_h, v_h)_{\Omega_h}| \leq Ch\|u_h\|_{L^2(\Omega_h)}\|v_h\|_{H^1(\Omega_h)} \tag{A.4}$$

for all $u_h, v_h \in S_h$. This result, along with the inverse inequality, implies the estimate

$$c\|u_h\|_{L^2(\Omega_h)} \leq \|\pi_h u_h\|_{L^2(\Omega)} \leq C\|u_h\|_{L^2(\Omega_h)}.$$

Furthermore, we note that $\nabla \pi_h u_h$ is constant on $\hat{\tau}$, with $\nabla \pi_h u_h = \nabla u_h$ on τ. Thus, under the assumption that $|\hat{\tau}| \leq 2|\tau|$, it follows that

$$\|\nabla \pi_h u_h\|^2_{L^2(\hat{\tau})} \leq 2\left|(\nabla u_h)|_\tau\right|^2|\tau| = 2\|\nabla u_h\|^2_{L^2(\tau)}.$$

Summing over all τ proves the estimate

$$\|\nabla \pi_h u_h\|^2_{L^2(\Omega)} = \left|(\nabla u_h)|_\tau\right|^2|\hat{\tau}| \leq 2\|\nabla u_h\|^2_{L^2(\Omega_h)}.$$

Furthermore, since Ω is convex, $|\hat{\tau}| \geq |\tau|$, which yields the estimate

$$\|\nabla \pi_h u_h\|^2_{L^2(\Omega)} \geq \|\nabla u_h\|^2_{L^2(\Omega_h)}.$$

We now set out to prove that (A.4) holds. Let $\tau \in \mathcal{T}_h$ be a boundary element with extension $\hat{\tau}$. Without loss of generality, we assume that the x-axis lies along $\Gamma_h \cap \tau$, so that $y = 0$ on $\Gamma_h \cap \tau$. Furthermore, we assume that h is sufficiently small that Γ can be represented by a Lipschitz continuous function $h(x)$ on $\hat{\tau} \cap \Gamma$.

We note that $\mathrm{dist}(x, \Gamma_\tau) \le h$, where $\Gamma_\tau = \Gamma_h \cap \tau$, due to the construction of Ω_h. Then, for $u_h, v_h \in S_h$ we write

$$\begin{aligned}\int_{\hat{\tau}\setminus\tau} \pi_h u_h, \pi_h v_h &= \int_{x_0}^{x_1} \int_0^{h(x)} (\pi_h u_h(x,y))(\pi_h v_h(x,y))\ dy\ dx \\ &= \int_{x_0}^{x_1} \int_0^{h(x)} (\pi_h u_h(x,0) + y\nabla u_h)(\pi_h v_h(x,0) + y\nabla v_h)\ dy\ dx\end{aligned}$$

since $\pi_h u_h, \pi_h v_h$ are linear on $\hat{\tau}$. Thus,

$$\begin{aligned}&\int_{x_0}^{x_1} \int_0^{h(x)} (\pi_h u_h(x,0) + y\nabla u_h)(\pi_h v_h(x,0) + y\nabla v_h)\ dy\ dx \\ &\quad = \int_{x_0}^{x_1} \int_0^{h(x)} \Big[(\pi_h u_h(x,0))(\pi_h v_h(x,0)) + y(\pi_h v_h(x,0))(\nabla u_h) \\ &\qquad\qquad + y(\pi_h u_h(x,0))(\nabla v_h) + y^2(\nabla u_h)(\nabla v_h)\Big]\ dy\ dx \\ &\quad \le h \int_{x_0}^{x_1} |\pi_h u_h(x,0)||\pi_h v_h(x,0)|\ dx \\ &\qquad + \frac{h^2}{2} \int_{x_0}^{x_1} \left(|\pi_h v_h(x,0)||\nabla u_h| + |\pi_h u_h(x,0)||\nabla v_h|\right)\ dx + Ch^4|\nabla u_h||\nabla v_h| \\ &\quad \le h\|\pi_h u_h\|_{L^2(\Gamma_\tau)}\|\pi_h v_h\|_{L^2(\Gamma_\tau)} + Ch^{5/2}\left(\|\pi_h v_h\|_{L^2(\Gamma_\tau)}|\nabla u_h| + \|\pi_h u_h\|_{L^2(\Gamma_\tau)}|\nabla v_h|\right) \\ &\qquad + Ch^4|\nabla u_h||\nabla v_h|\end{aligned}$$

Recalling that $|\tau| \ge ch^2$, we then have $Ch|\nabla u_h| \le C\|\nabla u_h\|_{L^2(\tau)}$. Furthermore, we have the trace inequality for τ_h (see Lemma 3.2 in [53])

$$\|u_h\|_{L^2(\Gamma_\tau)} \le C\left(\|u_h\|_{L^2(\tau)} + h\|\nabla u_h\|_{L^2(\tau)}\right),$$

so it follows that

$$\begin{aligned}|(\pi_h u_h, \pi_h v_h)_{\hat{\tau}} - (u_h, v_h)_\tau| &\leq C \left(h\|u_h\|_{L^2(\tau)}\|v_h\|_{L^2(\tau)} + h^2\|u_h\|_{L^2(\tau)}\|\nabla v_h\|_{L^2(\tau)} \right. \\ &\qquad \left. + h^2\|v_h\|_{L^2(\tau)}\|\nabla u_h\|_{L^2(\tau)} + h^2\|\nabla u_h\|_{L^2(\tau)}\|\nabla v_h\|_{L^2(\tau)} \right) \\ &\leq Ch\|u_h\|_{L^2(\tau)}\|v_h\|_{H^1(\tau)}\end{aligned}$$

Summing over all τ yields (A.4).

□

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