ABSTRACT

Title of Dissertation: MUTATION INVARIANT FUNCTIONS ON CLUSTER ENSEMBLES ASSOCIATED WITH SURFACES

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We define the notion of an invariant function on a cluster ensemble with respect to a group action of the cluster modular group on its associated function fields. We realize many examples of previously studied functions as elements of this type of invariant ring and give many new examples. We construct invariants for cluster algebras associated with surfaces using hyperbolic geometry, Teichmúller theory and skein algebras of surfaces. We complete a classification of them for surface ensembles for the action of Dehn twists, and generalize this classification to the non-surface mutation finite setting. We use this classification to answer some questions about the structure of affine cluster algebras, to construct a correspondence between $\mathcal{A}$ and $\mathcal{X}$ invariants, and to propose an explanation for why many different computations of canonical bases of cluster algebras agree.

MUTATION INVARIANT FUNCTIONS ON CLUSTER ENSEMBLES ASSOCIATED WITH SURFACES

by

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List of Mathematical Symbols

$\mathbb{F}$	A fixed coefficient field of characteristic 0
$(\mathbb{R}^{>0})$	The positive real numbers
$\mathrm{GL}(n, \mathbb{Z})$	The general linear group with elements in $\mathbb{Z}$
$\mathrm{PSL}(n, \mathbb{Z})$	The projective linear group
$\mathbb{H}^2$	Hyperbolic space
$\mathrm{Sk}(S)$	The skein algebra of the surface S
$Gr(n, k)$	The Grassmannian of k planes in n space
$\widetilde{A_n}$	The affine root system of type A
$A_1^{(1,1)}$	The affine extended root system of type A
$\mathbf{i}$	The seed of a cluster ensemble
$\mathrm{MCG}\,(S)$	The mapping class group of a surface
$\mathrm{PMCG}\,(S)$	The puncture preserving mapping class group
Dev	The developing map
Aut	The group of automorphisms of an object
Pos	The category of positive varieties
Grp	The category of groups
Tr	The trace of a matrix
$\lfloor x \rfloor$	The floor function
$\lceil x \rceil$	The ceiling function
sgn	The sign of a real number
$Gr(n, k)$	The Grassmannian of k planes in n space

Chapter 1: Introduction

Cluster algebras were first defined in the early 2000s by S. Fomin and A. Zelevinsky [1–4] as an explanation of certain functional recurrences called Y-systems and as a way of organizing particular bases in the theory of quantum groups. They have since enjoyed many applications to other areas such as Teichmüller theory [5,6], representation theory [7,8], supersymmetric Yang-Mills theory [9], and recently they were used in a vital way in a proof of Zagier's conjecture in weight 4 [10].

Essentially, a cluster algebra provides an atlas of coordinate charts on a variety, called a "cluster variety", along with transition maps which satisfy a number of interesting and important properties. The coordinate variables of each chart are called the "cluster variables", and the transition maps are called "mutations" between the clusters. The data that controls the mutations is encoded by a "quiver" which underlies each cluster. The cluster algebra is the $\mathbb{Z}$ algebra generated by all of the cluster variables, and the cluster variety is the algebraic variety obtained by gluing an algebraic torus for each cluster along the mutation maps.

One often finds that there are sequences of mutations which returns you to a cluster with the same underlying quiver, but with new cluster variables. The collection of such transformations forms a group called the "cluster modular group".

When the cluster algebra at hand has finitely many clusters, this group is finite and one may easily compute all of the information that the cluster algebra encapsulates directly. However, most cluster algebras are not "finite type" and have infinitely many clusters.

This thesis is an attempt to introduce and study a collection of functions assigned to any cluster variety that are invariant under specified subgroups of the cluster modular group. It is our hope that these functions will encode important information about infinite sequences of mutations, and that they will in some ways behave like cluster variables which live outside of the reach of regular mutation.

1.1 Motivation

Important examples of cluster algebras arise in a natural way from coordinate rings of double Bruhat Cells, flag varieties and Grassmannians [3, 11]. These typically give cluster algebras of "Geometric Type" meaning that the combinatorial data underlying the algebra has a simple description in terms of quivers and mutations.

In their seminal papers [5, 12, 13], Fock and Goncharov construct algebraic versions of higher Teichmüller spaces consisting of particular types of representations of surface groups into semi simple split real Lie groups. They explicitly construct all such representations and show that they may be realized as the positive real valued points of a pair of cluster varieties called a cluster ensemble. These cluster ensembles have a number of novel properties, e.g. there is a duality between a cluster ensemble associated with the surface S and Lie group G and the ensemble associated with the same surface and the Langlands dual group.

Most of these examples are not associated with cluster algebras that have finitely many clusters.

Example 1.1.1. The coordinate rings of the affine cones over the Grassmannians of k planes in n space, $Gr(k, n)$, have a cluster algebra structure on them, as shown by Scott [11]. These algebras only have finitely many clusters when $(k-2)(n-k-2) < 4$.

One notices from this example, that there is not an obvious change in the structure of the Grassmannian when we consider one associated with a finite type cluster algebra and one associated with an infinite type cluster algebra. This leads us to the problem of determining what information encoded in the structure of finite type cluster algebras carry over into the infinite type setting, and what information is lost or modified. Mutation invariant functions are examples of new phenomenon appearing in the infinite type setting, and we find that they appear naturally as a part of the solution to this problem.

1.1.1 Canonical basis

An important problem which arises in the study of all forms of cluster algebras, is the task of finding a "canonical basis" of a cluster algebra. Such a basis gives a basis of the cluster algebra as a vector space and satisfies some particularly nice positivity properties. For cluster algebras with finitely many clusters, this basis simply consists of the cluster variables [14]. When the cluster algebra has infinitely many clusters, there are usually basis elements that do not appear in any cluster;

this is our primary example of information encoded in finite type cluster algebras that must be modified in the infinite type setting.

Canonical bases have been studied by Zelevinsky in [15] for an "affine $\widetilde{A_1}$ type" cluster algebra, by Muskier, Shiffler, and Williams in [16] for cluster algebras associated with surfaces, and by Geiss, Leclerc and Schröer in [17] for more general algebras.

Through their duality map, Fock and Goncharov construct a canonical basis for the cluster ensemble associated with surface group representations into $G = \mathrm{PSL}(n, \mathbb{R})$. Gross, Hacking, Kontsevich and Keel construct canonical basis for cluster ensembles satisfying some technical conditions in [18]. Their basis, called the "theta basis", is constructed using entirely different machinery than that of Fock and Goncharov.

Our notion of mutation invariant function produces elements of canonical bases in many cases.

Example 1.1.2. Let C be the cluster algebra associated with an affine $\widetilde{A_1}$ quiver, as studied in [15]. We describe this algebra in examples 1.2.1 and 3.1.1. We write $\{a_i\}$ for the collection of cluster variables found by mutations. The invariant $F = \frac{a_0^2+a_1^2+1}{a_0 a_1}$ is an example of an element of a canonical basis of a cluster algebra which is not a cluster variable. Let $C_k(x)$ be the normalized Chebychev polynomial defined by $C_k(\lambda + \lambda^{-1}) = \lambda^k + \lambda^{-k}$. A canonical basis of this cluster algebra is given by

$$\mathbf{B} = \big\{\{a_i | i \in \mathbb{Z}\}, \{C_k(F) | k \in \mathbb{Z}^{>0}\}\big\}. \tag{1.1}$$

This same basis is computed in many different ways, and it is not clear why the exact same function F appears each time. This function is realized as a cluster character in [8, 17], a trace function in [5], a theta function in [18], an element of the bracelet basis of a surface skein algebra in [16, 19] among others. Viewing the functions associated with each of these constructions individually, we may see that they always produce manifestly mutation invariant functions. Our classification theorem essentially says that they must all be functions of the same invariant. We conjecture (see 6.6.1) that there is a canonical basis of mutation invariants which unifies these different constructions.

1.1.2 Other examples

Several other examples of cluster ensemble invariants have already been noted in the literature, but they have not been studied in a unified way. Of these examples, the invariant of the Markov quiver, shown in example 3.1.3, is probably the most well known and studied. This function is essentially the classical Markov Diophantine equation $x^2+y^2+z^2 = 3xyz$, and its mutation invariance encodes the transformation of Markov triples $(x, y, z) \to (x, y, \frac{x^2+y^2}{z})$.

Some examples of similar Diophantine equations arising from cluster algebras were studied in [20], and we recall such an equation in example 3.2.2. The invariants of the Somos 4 and 5 sequences studied in [21] are further examples, shown in example 3.3.1. The invariants in each of these examples provide useful tools for understanding the number theoretical properties of each sequence.

We also generalize the invariant of the Markov quiver in a different way. This quiver can be seen to be related to the Dynkin diagram of an "affine-extended" or "Elliptic" root system of type $A_1^{(1,1)}$ introduced by Saito [22]. We call quivers associated to these types of root systems "doubly extended" type quivers and we show invariants for types $D_4^{(1,1)}$ and $G_2^{(3,3)}$ in examples 3.2.4 and 3.2.3.

1.1.3 New ingredients and their inspiration

All of these previously studied functions are invariant under the action of a single mutation. The primary new ingredient of this paper is the introduction of an action of cluster modular group on the function fields of the ensemble. We show that each of the previously known examples can be realized as elements of invariant rings in regards to this group action. This will allow us to generalize these examples to invariants associated to more complicated mutation sequences and subgroups of the cluster modular group.

We hope that this theory provides a method to understand and tame the infiniteness of certain cluster ensembles. When the cluster modular group is infinite, we naturally have some infinite repeating structures underlying the exchange complex of the associated ensemble. This is often encoded by the cluster modular group; when there are cluster modular group elements of infinite order, we have sequences of mutations which send us to clusters which have the same underlying quiver, but new cluster variables. Invariants for these elements provide us with rational functions whose value and formula is the same on each cluster obtained along this mutation sequence.

Often, we may use the invariance of a function to study the limiting behavior of the cluster variables found along a mutation path. Recently, a cluster algebraic interpretation of some of the "symbols" of 8 particle $\mathcal{N} = 4$ SYM scattering amplitudes has been studied in [23–25]. These symbols are related to the limiting structure of an affine cluster algebra and were studied using an invariant of the $\widetilde{A}_1$ affine cluster algebra. We briefly show a similar analysis of this limiting behavior in example 3.1.1.

This analysis can be rephrased in terms of hyperbolic geometry, as example 3.1.1 shows. The $\widetilde{A_1}$ cluster ensemble is associated with a hyperbolic structure on an annulus, and the relevant mutation path corresponds to a Dehn twist on the annulus. We generalize this analysis to a pair of pants in chapter 4, and this becomes a critical ingredient in the proof of our main theorems.

The key ingredient towards understanding the limiting behavior is the unquantized skein algebra associated with a marked surface studied in the context of cluster algebras by [19, 26]. We find that sequences of cluster variables which appear along mutation paths corresponding to Dehn twists always satisfy linear recurrences, and we write these recurrences explicitly in terms of elements of the skein algebra.

1.2 Summary of Main Results

Our primary result is the classification of mutation invariant functions associated with some particularly simple mutation sequences on cluster varieties which can be built from surfaces. The main tools for this classification will come from the geometry and topology underlying these surfaces.

Teichmüler spaces of surfaces provide some of the simplest and most important examples of cluster varieties. For technical reasons the surfaces we consider are allowed to have boundary components and punctures, but they must have at least one marked point on each boundary and the total number of marked points and punctures must be more than 1.

Given such a surface, S, one may consider the Teichmüler space $T(S)$, which we think of as the moduli space of hyperbolic structures on S. These structures are required to have cusps at each puncture and marked point. Hyperbolic structures on a surface encode a representation $\rho : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{Z})$ called the "monodromy representation". Each element $\delta \in \pi_1(S)$ is mapped to a matrix called the monodromy operator of δ.

One may construct a cluster variety associated with S. The main topological ingredients are "arcs" on S which are non self intersecting isotopy classes of paths between marked points and punctures on S. We can construct a cluster variety, $\mathcal{A}_S$, where the cluster variables correspond to arcs, the clusters correspond to triangulations of S and the mutations corresponds to "flips" of triangulations. The positive real valued points of $\mathcal{A}_S$ parameterize points in $T(S)$ along with a "decoration" consisting of horocycles around each cusp. The mapping class group is naturally a subgroup of the cluster modular group of this cluster variety. Moreover, the monodromy representations associated with points of $T(S)$ can be expressed rationally in terms of the cluster variables, and we call $Tr(\rho(\delta))$ the "trace function" associated with δ.

We will focus on classifying invariants for particular types of mutation sequences, namely those that correspond to Dehn Twists. There is a natural collection of arcs and closed curves on S which are invariant after applying a Dehn twist about δ, namely those arcs and closed curves which do not intersect δ. The cluster variables and trace functions associated with these arcs and curve are examples of invariant functions on A_S. Our main theorem 5.1.1 implies in this case that

Theorem 1.2.1. *The ring of invariant rational functions for the action of a Dehn twist about δ on $\mathcal{A}_S$ is generated by traces of monodromy operators on S associated with closed curves that do not intersect δ, and cluster variables associated with arcs that do not intersect δ.*

This theorem is easily rephrased in terms of the skein algebra of the surface. The primary method for this will arise from the geometric interpretation of many cluster ensembles coming from Teichmüller theory, hyperbolic geometry, and skein algebras of surfaces.

Corollary 1.2.1. *The invariant ring for a Dehn twist about δ on $\mathcal{A}$ is exactly the the subalgebra of the skein algebra of S consisting of elements corresponding to skeins which do not intersect δ*

Example 1.2.1. We will illustrate this theorem with an example. We let S be an annulus with one marked point on each boundary component, as shown in figure 1.1. There are infinitely many clusters and cluster variables in $\mathcal{A}_S$. There are two cluster variables associated with the two boundary arcs, and these appear in every cluster. There are infinitely many possible interior arcs, and each cluster contains two of them which do not cross. These arcs may be identified by their winding number around the closed curve δ.

We write a_n for the cluster variable with winding number n, and write c, d for the cluster variables for the outer and inner arcs. Our clusters and mutations (omitting the variables c, d) look like

$$\ldots \longleftrightarrow (a_{-1}, a_0) \longleftrightarrow (a_0, a_1) \longleftrightarrow (a_1, a_2) \longrightarrow (a_2, a_3) \longleftrightarrow \ldots . \quad (1.2)$$

The formula for cluster variable mutation relates the new cluster variable obtained by flipping an arc to the cluster variables in the original cluster. Explicitly, we have

$$a_{n+1} = \frac{a_n^2 + cd}{a_{n-1}} \quad (1.3)$$

and the cluster algebra is given by

$$\mathbb{Z}[a_i]. \quad (1.4)$$

Every rational function in the cluster variables can be written as a rational function in $\{a_0, a_1, c, d\}$.

The cluster modular group of this cluster algebra is the group $\Gamma_S = \mathbb{Z}$. This corresponds to the mapping class group of S, and we write γ for the generator that is a clockwise Dehn twist about the arc δ on S. There is an action of Γ_S on the rational functions of $\mathcal{A}_S$. This is given by

$$\gamma(\{a_0, a_1, c, d\}) = \{a_1, \frac{a_1^2 + cd}{a_0}, c, d\}. \quad (1.5)$$

One may compute the trace function associated with δ and find that

$$Tr(\rho(\delta)) = \frac{a_0^2 + a_1^2 + cd}{a_0 a_1} := F(a_0, a_1, c, d). \quad (1.6)$$

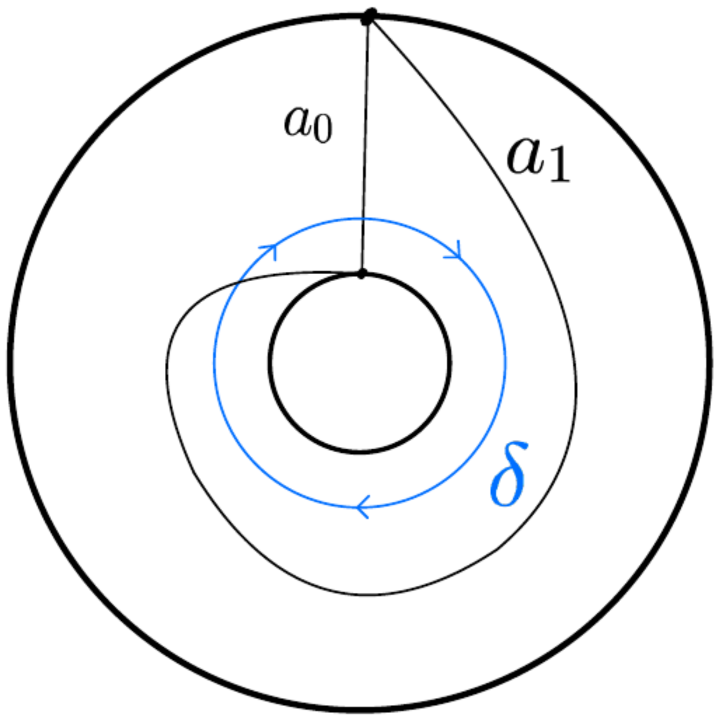


Figure 1.1: An annulus with one marked point on each boundary component.

This function is invariant under the action of Γ_S since

$$\gamma(F) = \frac{a_1^2 + (\frac{a_1^2+cd}{a_0})^2 + cd}{a_1 \frac{a_1^2+cd}{a_0}} = \frac{(a_1^2+cd)}{(a_1^2+cd)} \frac{1+\frac{a_1^2+cd}{a_0^2}}{\frac{a_1}{a_0}} = \frac{a_0^2+a_1^2+cd}{a_0 a_1} = F. \tag{1.7}$$

Theorem 1.3.1 implies that

$$\mathbb{F}(a_0, a_1, c, d)^{\Gamma_S} = \mathbb{F}(F, c, d) \tag{1.8}$$

where $\mathbb{F}$ is our fixed coefficient field.

1.3 Structure of this Thesis and Main Results

Chapter 2 reviews the technical details behind cluster ensembles and Teichmüler theory, and introduces the notion of a mutation invariant function. Chapter 3 shows many examples of invariant functions in detail. In chapter 4, we prove some foundational results about invariants, show geometric constructions of them, and use them to compute sequences of cluster variables.

Chapter 5 contains the proof of our main theorem:

Theorem 1.3.1 (Theorem 1)**.** *Let S be a marked surface, and let $(\mathcal{A}_S, \mathcal{X}_S)$ be the cluster ensemble associated with S. Let γ be the cluster modular group element corresponding to a Dehn twist about δ, a simple closed curve on S.*

1. *The ring $\mathbb{F}(\mathcal{X}_S)^{\langle\gamma\rangle}$ is generated by traces of monodromy operators of excised closed curves on S and invariant $\mathcal{X}$ coordinates for an excising triangulation of δ.*

2. *The ring $\mathbb{F}(\mathcal{A}_S)^{\langle\gamma\rangle}$ is generated by traces of monodromy operators of excised closed curves on S and invariant $\mathcal{A}$ coordinates of an excising triangulation of δ.*

We also extend this to cluster ensembles associated with general mutation finite quivers and trivial coefficients:

Theorem 1.3.2. *Let Q be a $T_{p,q,r}$, X_6 or X_7 Quiver of figures 5.9 or 2.3 and let $(\mathcal{A}, \mathcal{X})$ be the cluster ensemble associated with Q. Let $\gamma = \{1, (12)\} \in \Gamma_Q$ be a cluster Dehn twist in cluster modular group of $(\mathcal{A}, \mathcal{X})$. Let $x_3, \ldots, x_m$ be the $\mathcal{X}$ coordinates associated with nodes that are connected to nodes 1 and 2.*

1. *The invariant ring $\mathbb{F}(\mathcal{X}_S)^{\langle\gamma\rangle}$ is generated by the function $G(x_1, x_2) = \frac{(x_2(x_1+1)+1)^2}{x_1 x_2}$, $x_3(x_2(x_1+1)+1), \ldots, x_m(x_2(x_1+1)+1)$ and the remaining $\mathcal{X}$ coordinates not connected to nodes 1 and 2.*

2. *The Invariant ring* $\mathbb{F}(\mathcal{A}_S)^{\langle\gamma\rangle}$ *is generated by* $\rho^*\sqrt{G}$ *and* $\mathcal{A}$ *coordinates associated to nodes other than node 1 and 2.*

Chapter 6 discusses some corollaries of the main theorems and some conjectures about the general nature of mutation invariants. We prove a conjecture of [27] on the structure of "affine $\mathcal{A}$ coordinates"

We give strong evidence through an abundance of examples of a deeper theory underlying the existence and structure of cluster ensemble invariants. We show in each example a correspondence between invariants on the $\mathcal{A}$ and $\mathcal{X}$ spaces via denominator vectors. We prove this correspondence in the surface case. We also show evidence that there should be a basis of $\mathcal{A}$ invariants such that the cluster modular group acts on this basis by positive Laurent polynomials. This is a generalization of the Laurent phenomenon of a cluster algebra. These conjectures together should be related to the duality conjectured in [5] and proved in [18].

We strongly suspect that the existence of invariants will always be related to some other important manifestation of the ensemble, be it geometric or number theoretic.

This thesis is primarily a continuation and restructuring of the author's previous work initiated in [27].

Chapter 2: Preliminaries on Cluster Ensembles

We will recall the basic notions of a cluster ensemble introduced by Fock and Goncharov in [12]. This consists of a pair of positive spaces ($\mathcal{A}$,$\mathcal{X}$) along with a map ρ from $\mathcal{A}$ to $\mathcal{X}$, a notion of seeds and mutations, and a group Γ that acts by automorphisms of the entire structure. We will, however, simplify these definitions to emphasize a more concrete and computational framework in which to introduce and study invariants. We introduce an action of Γ on the field of rational functions on each of the $\mathcal{A}$ and $\mathcal{X}$ spaces. Our notion of an invariant function will be in regards to this group action. We will also review the notions of a hyperbolic structure on a surface, the Teichmüller space of a surface, and the skein algebra of a surface and show how these are naturally related to cluster ensembles.

2.1 Quivers and Mutations

Our first simplification is to only consider cluster ensembles of "geometric" type, meaning that we can use quivers to define our ensembles.

2.1.1 Quivers

Definition 2.1.1. A *quiver* is a directed and weighted graph with no self loops or 2 cycles. We think of a quiver as a graphical representation of a matrix, M, called the *exchange matrix* which has entries $[\epsilon_{ij}]$ equal to the number of arrows from node i to node j. We denote the set of nodes of Q by $N(Q)$ and we usually refer to them by their index in this set.

We will allow non-symmetrically weighted arrows between nodes, to account for non skew-symmetric exchange matrices. These arrows are labeled with a pair of weights. In this case we require that the exchange matrix is skew-symmetrizable, meaning that there is a diagonal matrix, D, such that MD^{-1} is skew-symmetric. The matrix D associates to each node a multiplier, d_i.

In the various diagrams of quivers in the paper, we label the multipliers of our nodes as superscripts in brackets and label the edges by the weights, where no label means weight 1, a double arrow means weight 2, a single label means a symmetric weight and a pair of weights for an arrow from node i to node j is $-\epsilon_{ji}, \epsilon_{ij}$[1].

Definition 2.1.2. We will call a quiver *simply laced*, in analogy with Dynkin diagrams, if its exchange matrix is skew symmetric and non simply laced otherwise.

Two quivers are isomorphic if there is a map between their nodes that makes their associated exchange matrices identical. In other words, their exchange matrices are identical after conjugation by a permutation matrix.

[1]This ordering is meant to agree with Berhnard Keller's java applet Quiver Mutation in JavaScript.

2.1.2 Mutation of quivers

Quivers will underlie the coordinate atlases of our cluster ensembles and mutations will give the transition maps between coordinate charts.

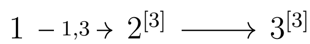
Definition 2.1.3. A *mutation* of a quiver Q at node i, written $\mu_i(Q)$, generates a new quiver by the following two operations

1. For every pair of nodes j, k with weighted arrows $j \xrightarrow{a,b} i \xrightarrow{c,d} k$, add an arrow of weight ac, bd from j to k.
2. Swap the direction and weights of all the arrows coming into and out of node i.

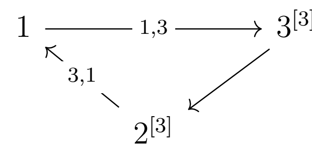
Definition 2.1.4. The set of all quivers up to quiver isomorphism obtained by all possible sequences of mutations of a particular quiver is called the *mutation class* of the quiver, denoted by $\mu(Q)$.

Definition 2.1.5. It is useful to include nodes in a quiver that we do not allow mutations at. These nodes are called *frozen* and we write Q^μ for the subquiver of Q consisting of non frozen nodes and the arrows between them. We call this subquiver the *mutable portion* of Q. The *rank* of a quiver is equal to $\#N(Q^\mu)$.

Remark 2.1.1. When considering an isomorphism of quivers, we generally ignore their frozen nodes and only consider maps between their mutable portions. In this way, we have that two quivers with the same mutable portions have the same mutation class. We will call an isomorphism including a map on frozen nodes a *frozen isomorphism* or quivers.



(a) $Q_{\widetilde{G}_2}$.



(b) $\mu_2(Q_{\widetilde{G}_2})$.

Figure 2.1: Two quivers of type $\widetilde{G}_2$

Example 2.1.1. Let $Q_{\widetilde{G}_2}$ be the quiver shown in figure 2.1a. We read the pair of arrow weights as "1 in , 3 out" meaning that the matrix associated to this quiver is $\begin{bmatrix} 0 & 3 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$. We may take $D = \text{diag}(1, 3, 3)$ so that nodes 2 and 3 get multipliers of 3. This quiver is associated to a root system of affine type $\widetilde{G}_2$, where node 1 is associated with the larger root and nodes 2 and 3 are associated with the smaller roots. $\mu_2(Q_{\widetilde{G}_2})$ is shown in figure 2.1b. The exchange matrix changes to $\begin{bmatrix} 0 & -3 & 3 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$ after mutating.

2.1.3 Folding of quivers

The relationship between the classical "folding" of the simply laced Dynkin diagrams to form the non simply laced diagrams can be extended to quivers. We refer to [28] for full details about folding of quivers.

Essentially, to fold a quiver we will group its nodes into disjoint sets and perform mutations by requiring that we mutate all the nodes of a given set together. We call the operation of mutating each of the elements of a set of nodes in turn a "group mutation". This does not depend on the order of mutations when there are no arrows between any two nodes in the set, i.e. each of the mutations of nodes in the set commute with each other.

With this in mind, we have the following definition:

Definition 2.1.6. A *folding* of a quiver, Q, with n nodes is a choice of k non empty and disjoint sets of nodes whose union contains all of the nodes of Q satisfying the following conditions:

1. The nodes contained in a given set have no arrows between themselves.

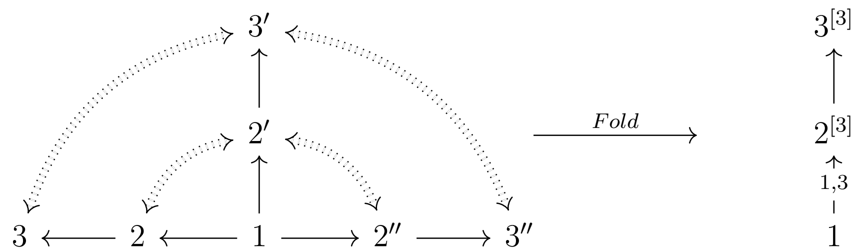
2. Condition 1 is satisfied after any number of group mutations of these fixed sets.

These properties will always be satisfied when Q is the quiver which is an orientation of a simply laced finite, affine or doubly extended Dynkin diagram and the sets are given by the orbits of nodes under an automorphism of Q.

In this situation, all of the mutation structures we will be interested in studying will be the same for the folding of Q and for a particular skew-symmetrizable quiver. Let $K_1, \ldots, K_k$ be the groups of nodes in this folding and let m_{ij} be the total number of arrows from nodes in K_i to nodes in K_j. Then we can construct a skew-symmetrizable quiver Q_{fold} with k nodes respectively and arrows of weight $\frac{m_{ij}}{|K_i|}, \frac{m_{ij}}{|K_j|}$

from node K_i to node K_j.

Example 2.1.2. We may obtain the quiver $Q_{\widetilde{G_2}}$ of example 2.1.1 by folding a quiver associated with an $\widetilde{E_6}$ Dynkin diagram in the following way:



(2.1)

2.1.4 Important mutation classes of quivers

The combinatorial properties of a cluster ensemble are controlled by the mutation class of the quiver, $\mu(Q)$, underlying its seeds. Before we give the definitions of a cluster ensemble, we will briefly discuss some important mutation classes of quivers.

Quivers which have a finite mutation class are called "finite mutation type". For the most part, all of the examples and situations we will consider will be with finite mutation type quivers. These quivers are classified in [28, 29]. Importantly, all of the non simply laced finite mutation class quivers can be obtained by folding a simply laced one.

The classification of finite mutation type quivers can be split in two ways. First, we find that these quivers are either associated with Dynkin diagrams, called Dynkin-type, or are of non-Dynkin type. The Dynkin-type quivers are then split into finite, affine or doubly extended types. These mutation classes contain quivers which are orientations of the Dynkin diagrams of finite, affine, or affine-extended

Finite	Affine	Doubly Extended	Exotic
E_6	$\widetilde{E_6}$	$E_6^{(1,1)}$	X_6
E_7	$\widetilde{E_7}$	$E_7^{(1,1)}$	X_7
E_8	$\widetilde{E_8}$	$E_8^{(1,1)}$	

Figure 2.2: The exceptional (non surface) quivers.

(a) X_6 (b) X_7

Figure 2.3: The exotic non-surface finite mutation quivers

root systems respectively.

We refer to [22] for a background on our notations used in regards to affine extended root systems. These Dynkin diagrams are shown in Appendix A.

Secondly, we may see that every simply laced finite mutation class quiver can either be obtained as the mutation class of a quiver associated with a triangulation of a marked surface (see section 2.4) or is one of 11 exceptional examples listed in figure 2.2. The non-simply laced finite mutation class quivers can all be obtained by folding a simply laced one, as shown in [28].

2.2 Positive spaces and Cluster Ensembles

We will simplify the notion of positive space of [12] in hopes of making computations more concrete.

Definition 2.2.1. A *positive chart* will simply mean an algebraic torus, $(\mathbb{G}_m)^n$, along with a distinguished coordinate chart consisting of its n characters. A map between positive charts is called a *positive map* and satisfies the property that pullback of the distinguished coordinate variables of the target is a rational function with positive integral coefficients.

A *positive variety* is a variety obtained by gluing positive charts together with positive maps. We write *Pos* for the category of positive varieties.

Remark 2.2.1. Each positive variety, X, has a well defined set of positive real valued points denoted by $X(\mathbb{R}^{>0})$. This has a structure of a real manifold. Positive maps $\psi : X \to Y$ induce homeomorphisms $X(\mathbb{R}^{>0}) \to Y(\mathbb{R}^{>0})$.

Definition 2.2.2. A positive space is a collection of positive varieties along with invertible positive maps between them. Equivalently, A positive space is a functor from a groupoid to the category of positive varieties.

Let Q be a quiver with mutable nodes $N_1, \cdots, N_m$. Let $\alpha = \{\alpha_1, \cdots, \alpha_m\}$ be algebraically independent generators of the function field $\mathbb{F}(\alpha_1, \cdots, \alpha_m)$. We associate these variables variables to the nodes of Q in an obvious way.

Definition 2.2.3. The pair of a quiver along with variables associated to its nodes is called a seed, denoted by $\mathbf{i} = (Q, \alpha)$. The field $\mathbb{F}(\alpha)$ is the function field of the seed. The positive chart with characters given by the seed variables is called the *Seed Torus* of $\mathbf{i}$.

Definition 2.2.4. Two seeds are *isomorphic* if there is a quiver isomorphism which induces an isomorphism of the associated function fields. When there are frozen node, we also require that the fields generated by the frozen variables are isomorphic.

Definition 2.2.5. We can *mutate* seeds as follows: Given a seed $\mathbf{i} = (Q, \alpha)$, we construct a seed $\mu_i(\mathbf{i}) = (\mu_i(Q), \alpha')$ where the new quiver is given by mutating Q at node i and we associated a new collection of variables $\alpha' = \{\alpha'_1, \cdots, \alpha'_m\}$.

Equipped with the notions of quivers, mutations and seeds, we can define the pair $(\mathcal{A}_Q, \mathcal{X}_Q)$ of positive spaces associated to an initial seed with quiver Q. Both of these spaces will be defined by generating a collection of seeds by mutating an initial seed generated from Q, but in each case we define different positive maps between the seed torus of $\mathbf{i}$ and $\mu(\mathbf{i})$.

2.2.1 The $\mathcal{A}$ space

Let Q be a quiver with mutable nodes $N_1, \cdots, N_n$ and frozen nodes $F_{n+1}, \cdots, F_m$. Let $\mathbf{a} = \{a_1, \cdots, a_n, f_{n+1}, \cdots, f_m\}$ be algebraically independent generators of the function field $\mathbb{F}(a_1, \cdots, a_n, f_{n+1}, \cdots, f_m)$ called the initial variables. We associate these variables to the nodes of Q in the obvious way to make a seed $\mathbf{i} = (Q, \mathbf{a})$.

Definition 2.2.6. We call these particular seed variables $\mathcal{A}$ *coordinates*. The $\mathcal{A}$ coordinates associated to the frozen nodes are called *coefficients*. We write $\mathcal{A}_\mathbf{i}$ for the positive chart with coordinate variables given by the seed $\mathbf{i} = (Q, \mathbf{a})$. This seed is called the *initial seed* associated with Q.

Mutation of Q at a node i produces a new seed consisting of a new quiver $\mu_i(Q)$ and a new collection of variables $\mathbf{a}'$. We define a map $\mu_i : \mathcal{A}_{\mathbf{i}} \to \mathcal{A}_{\mu_i(\mathbf{i})}$ by

$$\mu_i^*(a_j') = \begin{cases} \frac{1}{a_i}\left(\prod_{\epsilon_{ik}>0} {a_k}^{\epsilon_{ik}} + \prod_{\epsilon_{ik}<0} {a_k}^{\epsilon_{ik}}\right) & i = j \\ a_j & i \neq j \end{cases}. \tag{2.2}$$

We can thereby write all of the new $\mathcal{A}$ coordinates obtained by mutations in terms of the initial $\mathcal{A}$ coordinates.

Definition 2.2.7. Each collection of $\mathcal{A}$ coordinates generated in this way is called a *cluster*, and the $\mathbb{F}$-subalgebra of $\mathbb{F}(a_1, \cdots, a_n)[f_{n+1}, \cdots, f_m]$ generated by the clusters of $\mathcal{A}$ coordinates obtained from all possible sequences of mutations at non-frozen nodes is the *cluster algebra* associated to the quiver Q, denoted by $\mathcal{C}(Q)$.

Definition 2.2.8. $\mathcal{A}_Q$ is the positive space obtained from the positive charts associated to every seed generated by mutation of $\mathbf{i}$ and maps given by the mutation isomorphism for each mutation. We describe this space as a functor in section 2.3.

2.2.2 The $\mathcal{X}$ space

The $\mathcal{X}$ space will have a definition similar to that of the $\mathcal{A}$ space, but with a different exchange rule. In the case that Q has frozen nodes, we will not include extra variables, and we may essentially ignore these nodes in the definition.

Given a quiver, Q, with mutable nodes $N_1, \cdots, N_n$, we associate independent variables $\mathbf{x} = \{x_1, \cdots, x_n\}$ called $\mathcal{X}$ coordinates. This pair is called an $\mathcal{X}$ seed, also denoted by $\mathbf{i}$. Mutation of Q at node i produces a new $\mathcal{X}$ seed, $\mathbf{i}'$ consisting of a quiver $\mu_i(Q)$ and a new collection of $\mathcal{X}$ coordinates, $\mathbf{x}'$. We define a map of positive varieties $\mu_i : \mathcal{X}_{\mathbf{i}} \to \mathcal{X}_{\mu_i(\mathbf{i})}$ by

$$\mu_i^*(x_j') = \begin{cases} x_i^{-1} & i = j \\ x_j(1 + x_i^{-\operatorname{sgn} \epsilon_{ji}})^{-\epsilon_{ji}} & i \neq j \end{cases}. \tag{2.3}$$

We note that we do not assign variables for the frozen nodes in Q.

Definition 2.2.9. $\mathcal{X}_Q$ is the positive space consisting of the positive varieties associated to every seed generated by mutation and maps given by the above mutation isomorphism for each mutation.

2.2.3 Cluster Ensembles

It is worth viewing these positive spaces as two different functors from the same groupoid.

Definition 2.2.10. The *Seed groupoid* $\mathcal{G}_{\mathbf{i}}$ is the groupoid with one object for each seed obtained from mutations of $\mathbf{i}$ and maps given by mutation paths. We do not include any notion of seed isomorphism in this definition.

Remark 2.2.2. This notion of seed groupoid is not the same as that of Fock and Goncharov in [5]. Their notion is the same as our notion of cluster modular groupoid in section 2.3.

We may view the $\mathcal{A}$ and $\mathcal{X}$ spaces associated with the same seed as two different functors $\mathcal{A}, \mathcal{X} : \mathcal{G}_{\mathbf{i}} \to \mathit{Pos}$, where we assign positive charts to each seed and maps according to the appropriate mutation rules.

Definition 2.2.11. Given an initial quiver, Q, we write $(\mathcal{A}_Q, \mathcal{X}_Q)$ for the pair of positive spaces generated by using Q along with independent variables for each node as a seed for each space, where $\mathcal{X}_Q = \mathcal{X}_{Q^\mu}$ since we do not assign $\mathcal{X}$ coordinates to the frozen nodes. This pair of spaces is the *cluster ensemble* associated to Q. The *rank* of this ensemble is $\#N(Q^\mu)$. We often drop the subscripts when Q is implied.

Remark 2.2.3. The positive varieties $\mathcal{A}_{\mathbf{i}}$ and $\mathcal{X}_{\mathbf{i}}$ associated to the initial seed Q provides a base point of each space. Changing Q to any other quiver in $\mu(Q)$ gives the same positive space, just with a different base point.

The final important property of a cluster ensemble is a natural transformation $\rho : \mathcal{A}_Q \to \mathcal{X}_Q$. ρ has a simple definition in terms of the initial coordinates. If $(a_1, \cdots, a_n, a_{n+1}, \cdots, a_{n+f})$ and $(x_1, \ldots, x_n)$ are the initial $\mathcal{A}$ and $\mathcal{X}$ coordinates and the exchange matrix of Q is $[\epsilon_{ij}]$, then we have that $\rho^*(x_i) = \prod_i a_i^{\epsilon_{ij}}$. ρ commutes with mutations and provides a map of positive spaces.

Example 2.2.1. We illustrate these definitions with a well known example. Let Q be a quiver with two mutable nodes, 1 and 2, with a single arrow from node 1 to node 2. Each mutation of Q gives an isomorphic quiver, and so all of the seeds we obtain by mutation are isomorphic.

We compute cluster variables using the $\mathcal{A}$ mutation rule. This is shown in figure 2.4. We see that after 5 mutations we return to the original cluster variables. Thus this seed is not identical to the original seed since the variables have switched places. After 5 more mutations we return to a seed which is identical to the initial seed.

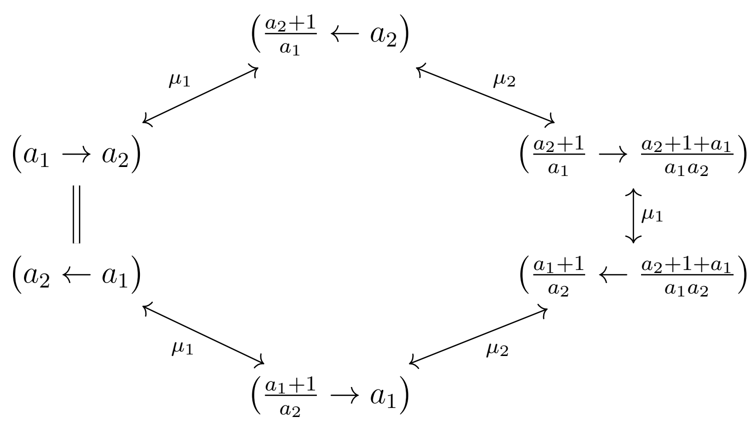


Figure 2.4: The five $\mathcal{A}$ seeds with different cluster variables.

The cluster algebra $\mathcal{C}(Q)$ is the algebra

$$\mathcal{C}(Q) = \mathbb{F}[a_1, a_2, \frac{a_2+1}{a_1}, \frac{a_1+1}{a_2}, \frac{a_2+1+a_1}{a_1 a_2}] \tag{2.4}$$

The positive spaces $\mathcal{A}_Q$ and $\mathcal{X}_Q$ each consist of 10 distinct positive charts with positive maps given by the mutation rules. For example, if $\mathbf{i}$ is the initial $\mathcal{X}$ seed with variables x_1, x_2 and $\mu_1(\mathbf{i}) = \mathbf{i}'$ with associated variables x_1', x_2' then we have the positive map given by

$$\mu_i : \mathcal{X}_{\mathbf{i}} \to \mathcal{X}_{\mathbf{i}'}, \quad \mu_i^*(x_1', x_2') = (x_1^{-1}, x_2(1+x_1)). \tag{2.5}$$

The map $\rho : \mathcal{A}_Q \to \mathcal{X}_Q$ is given in terms of the initial coordinates by

$$\rho^*(x_1, x_2) = (a_2, a_1^{-1}) \tag{2.6}$$

2.3 The Mutation Structures of a Cluster Ensemble

The mutation class of a quiver underlying a seed of a cluster ensemble controls many of that ensembles properties. The most well known example of this is the fact that a cluster ensemble has finitely many clusters if and only if its quiver is of finite type, see [2]. We call these cluster ensembles "finite type" and refer to them by the name of their Dynkin diagram.

We will be generally interested in "mutation finite" cluster ensembles, i.e. ensembles for which their underlying quiver has a finite mutation class. When this quiver is of Dynkin type, we will refer to that ensemble as being that particular type.

Mutation finite cluster ensembles which are not of finite type have infinitely many clusters, but only finitely many quivers up to isomorphism underlying their seeds. A natural way to study such a situation is to study the group of transformations which act by permuting different clusters associated with isomorphic seeds. Such a transformation is determined entirely by where it sends a single cluster, since the other clusters may all be obtained by mutations from a given one. We call this type of transformation an automorphism of the mutation structure of our cluster ensemble.

2.3.1 The cluster modular group

Our current notion of seed groupoid has different objects for all seeds obtained by mutations. It is natural to define a groupoid with objects corresponding to seeds up to seed isomorphism; This will allow us to define the cluster modular group.

Definition 2.3.1. A seed $\mathbf{i}'$ obtained by a sequence finite of mutations, $P = \mu_{i_1} \circ \mu_{i_2} \circ \ldots$, from an initial seed $\mathbf{i}$ is *cluster equivalent* to $\mathbf{i}$ if the map $P : \mathcal{A}_{\mathbf{i}} \to \mathcal{A}_{\mathbf{i}'}$ is simply a permutation of the $\mathcal{A}$ coordinate variables. In other words, the seeds $\mathbf{i}$ and $\mathbf{i}'$ have the same unordered collection of cluster variables.

Remark 2.3.1. We defined this notion in terms of the $\mathcal{A}$ mutation rule, but it is equivalent to the condition that $P : \mathcal{X}_{\mathbf{i}} \to \mathcal{X}_{\mathbf{i}'}$ is a permutation of the $\mathcal{X}$ coordinate variables. It is always the case that two cluster equivalent seeds are isomorphic.

A seed isomorphism $\sigma : \mathbf{i} \to \mathbf{i}'$ induces a map on the varieties associated with each seed by $\sigma^*(\alpha'_{\sigma(i)}) = \alpha_i$.

Definition 2.3.2. A pair $\{P, \sigma\}$ of a mutation path and seed isomorphism is called a *trivial cluster transformation* if the composition of the mutation path and seed isomorphism induces the identity map on the $\mathcal{A}$ and $\mathcal{X}$ varieties.

A trivial cluster transformation must consist of a path to a cluster equivalent seed along with a seed isomorphism which induces the identity permutation on the cluster variables.

Definition 2.3.3. The cluster modular groupoid, $\mathcal{G}_Q$, is the groupoid with objects given by seeds obtained from mutating the seed $\mathbf{i}$ up to seed isomorphism. The maps are given by pairs of mutation sequences and seed isomorphisms up to those which are trivial cluster transformations. The group associated with this groupoid is called the *Cluster Modular Group*.

This group can be considered as the group of symmetries of the "exchange complex" of the cluster ensemble, see section 2.4 of [12], but we will not need this notion here.

We will give a concrete description of the cluster modular group in terms of sequences of mutations and isomorphisms of quivers.

Fix a seed $\mathbf{i}$ with quiver Q. Let $\tilde{\Gamma}_{\mathbf{i}} = \big\{\{P, \sigma\}\big\}$ where P is a sequence of mutations and σ is an isomorphism $Q^{\mu} \to P(Q)^{\mu}$. Because P transforms Q into an isomorphic quiver, we may compose these mutation paths. We write paths of mutations as a sequence of the node names read from left to right. The action of σ as an element of the symmetric group is written with the standard left action.

Since each mutation is an involution, we have that $\tilde{\Gamma}_{\mathbf{i}}$ is a subgroup of $(\mathbb{Z}/2\mathbb{Z}^{*k}) \rtimes S_k$ where $\mathbb{Z}/2\mathbb{Z}^{*k}$ is a free product of cyclic groups and the right hand side of the semidirect product acts by permuting the factors.

Remark 2.3.2. $\tilde{\Gamma}_{\mathbf{i}}$ acts on the elements of the seeds of $\mathcal{A}$ and $\mathcal{X}$ coordinates whose underlying quiver is isomorphic to Q. P provides a path to a new seed and σ provides a map between the initial seed variables and the final variables. We note that σ is only an isomorphism of quivers, not of seeds, which is why this action is nontrivial.

Definition 2.3.4. The *cluster modular group based at* $\mathbf{i}$, $\Gamma_{\mathbf{i}}$ is defined to be $\tilde{\Gamma}_{\mathbf{i}}$ modulo the subgroup generated by trivial cluster transformations.

Our definition currently depends on the choice of initial seed $\mathbf{i}$. If $\mathbf{i}'$ is any other seed mutation equivalent to $\mathbf{i}$, then $\Gamma_{\mathbf{i}'} \simeq \Gamma_{\mathbf{i}}$, with the isomorphism given by conjugation of paths by a mutation path between $\mathbf{i}$ and $\mathbf{i}'$.

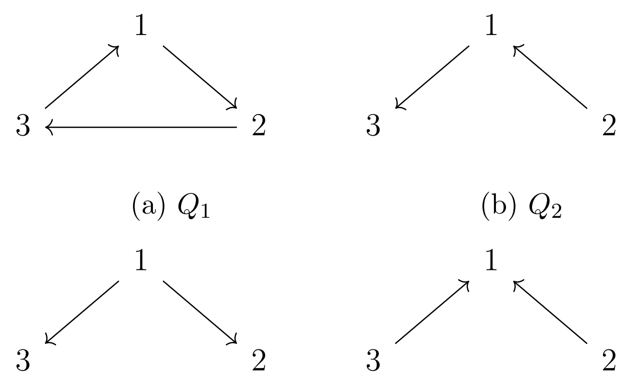
Definition 2.3.5. The *cluster modular group*, $\Gamma_Q : \mathcal{G}_{\mathcal{Q}} \to Grp$, is defined to be the functor from the cluster modular groupoid which assigns the group $\Gamma_{\mathbf{i}}$ for each seed $\mathbf{i}$ along with group isomorphisms given by conjugation by mutation paths. An element of the cluster modular group is a collection of elements $\gamma_{\mathbf{i}} \in \Gamma_{\mathbf{i}}$ which are mapped to each other by the group isomorphisms defined for each mutation path.

Remark 2.3.3. We will often simply refer to the group $\Gamma_{\mathbf{i}}$ as *the* cluster modular group, knowing that we really have a collection of isomorphic groups for each seed isomorphism class. We will also simply refer to elements of the abstract group $\Gamma_{\mathbf{i}}$ as elements of the cluster modular group, knowing that these elements take on different presentations depending on which seed we are interested in.

Example 2.3.1. Let's examine the cluster modular groupoid and group of a type A_2 cluster ensemble, continuing example 2.2.1. The quiver of a A_2 cluster ensemble consists of two nodes, 1 and 2, with a single arrow between them. As we saw, each of the new seeds we obtained by mutations are isomorphic to the original seed. Thus the cluster modular groupoid only has one object. Therefore, each map is a cluster modular group element.

Mutation at node 1 produces a quiver that is isomorphic to the starting quiver (after permuting the nodes). We can represent this group element by $\gamma = \{1, (12)\}$. This element clearly generates the cluster modular group. The only relation is that $\gamma^5 = e$. This relation comes from the fact that 5 consecutive applications of γ reproduces the original cluster variables. Thus the cluster modular group is $\mathbb{Z}/(5\mathbb{Z})$.

We can see the distinction between the seed groupoid and cluster modular groupoid quite clearly in this example. The seed groupoid has many more objects than we really need.



(a) Q_1 (b) Q_2

(c) Q_3 (d) Q_4

Figure 2.5: Choices of quiver isomorphism classes in the mutation class of a quiver of type A_3.

Example 2.3.2. If we take a quiver of type A_3 as our initial seed, it is easy to compute the cluster modular groupoid and cluster modular group. We can do this by taking a representative for each of the quivers in the mutation class of our seed and then writing all of the possible pairs of mutation paths and quiver isomorphisms between them. A choice of possible representatives for the quiver isomorphism classes is shown in figure 2.5 and a diagram of the groupoid is shown in figure 2.6. It is not too difficult to compute the cluster modular group by checking that every map from Q_1 to itself is generated by the two maps shown in the figure. These two elements commute and the top has order 2 and the bottom has order 3. Thus the cluster modular group is $\mathbb{Z}/6\mathbb{Z}$.

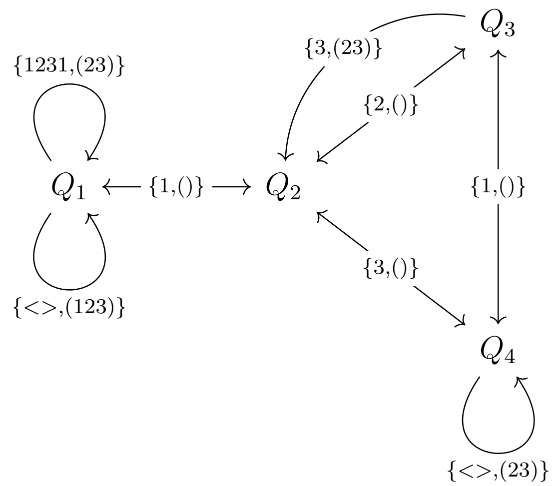


Figure 2.6: The cluster modular groupoid of the A_3 cluster ensemble. A selection of non-identity maps are shown.

2.3.2 Action of the cluster modular group on functions

Given $(\mathcal{A}, \mathcal{X})$ a cluster ensemble, we wish to define a notion of a field of rational functions on the $\mathcal{A}$ and $\mathcal{X}$ spaces.

There is a natural functor from the category of positive varieties to the category of function fields over $\mathbb{F}$ obtained by associating a positive variety its function field. We write $\mathbb{F}(\mathcal{A})$ for the composition of the functor realizing our positive space $\mathcal{A}$ with this functor. This will give us a notion of rational functions on a positive space.

Concretely, $\mathbb{F}(\mathcal{A})$ provides us with a function field, $\mathbb{F}(\mathcal{A}_{\mathbf{i}})$, for each seed $\mathbf{i}$. When we change to another quiver, $Q' \in \mu(Q)$ we can pull back functions along $P : A_{\mathbf{i}} \to A_{\mathbf{i}'}$ to obtain an isomorphism

$$\mathbb{F}(\mathcal{A}_{\mathbf{i}}) \simeq \mathbb{F}(\mathcal{A}_{\mathbf{i}'}). \tag{2.7}$$

We will occasionally write $\mathbb{F}(\mathcal{A}, \mathcal{X})$ for the functions on a cluster ensemble.

Again fix a starting seed $\mathbf{i}$ with quiver Q. We can now define an action of $\Gamma_{\mathbf{i}}$ on the rational functions on the $\mathcal{A}_Q$ or $\mathcal{X}_Q$ space.

Let $\gamma \in \Gamma_{\mathbf{i}}$. Given $f \in \mathbb{R}(\mathcal{A}_{\mathbf{i}})$ we can define $\gamma(f)(a_1, \ldots, a_n) = P(\sigma(f))$, where σ acts as a map from the cluster variables on the initial seed $\mathbf{i}$ to the seed $P(\mathbf{i})$ and P acts by pullback along $P : \mathcal{A}_{\mathbf{i}} \to \mathcal{A}_{P(\mathbf{i})}$. We define an action on elements of $\mathbb{F}(\mathcal{X}_{\mathbf{i}})$ in an analogous way.

Elements in Γ_Q act via natural transformations $\mathbb{F}(\mathcal{A}) \to \mathbb{F}(\mathcal{A})$ and $\mathbb{F}(\mathcal{X}) \to \mathbb{F}(\mathcal{X})$ where we act on each function field as above.

Definition 2.3.6. Given $f \in \mathbb{F}(\mathcal{A}_{\mathbf{i}})$ or $f \in \mathbb{F}(\mathcal{X}_{\mathbf{i}})$ we call the set $\Gamma_{\mathbf{i}}(f)$ the *exchange class* of f. $\Gamma_Q(f)$ provides a set of exchange classes for each quiver isomorphism class in the mutation class of Q.

Example 2.3.3. Let $Q = Q_1$ of figure 2.5. Let $f \in \mathbb{F}(\mathcal{A}_Q)$ be given by

$$f(a_1, a_2, a_3) = \frac{a_1 + a_2 + a_3}{a_1 a_2 a_3}. \tag{2.8}$$

Then we may compute that $\{1231, (23)\}(f) = f^{-1}$. Lets write this out in detail. We write a_i, b_i, c_i, d_i, e_i for the variables on each of the five seeds found along the path. The important exchange relations are

$$\mu_1^*(b_1) = \frac{a_2 + a_3}{a_1} \qquad \mu_2^*(c_1) = \frac{1 + b_1}{b_2} \tag{2.9}$$

$$\mu_3^*(d_3) = \frac{1 + c_1}{c_2} \qquad \mu_1^*(e_1) = \frac{d_2 + d_3}{d_1} \tag{2.10}$$

and all other exchange relations are trivial.

The isomorphism (23) provides the map $\{a_1, a_3, a_2\} \xrightarrow{(23)} \{e_1, e_2, e_3\}$ and the mutation path acts on $(23)(f)$ by pullback. Writing the value of f by abuse of notation, we compute that

$$\{1231,(23)\}(f) = \mu_1 \circ \mu_2 \circ \mu_3 \circ \mu_1 \left((23)(\frac{a_1+a_2+a_3}{a_1a_2a_3})\right) =$$
$$\mu_1 \circ \mu_2 \circ \mu_3 \circ \mu_1 \left(\frac{e_1+e_3+e_2}{e_1e_3e_2}\right) = \mu_1 \circ \mu_2 \circ \mu_3 \left(\frac{d_1+1}{d_2d_3}\right) =$$
$$\mu_1 \circ \mu_2 \left(\frac{c_2}{c_3}\right) = \mu_1 \left(\frac{b_2b_3}{b_1+1}\right) = \frac{a_1a_2a_3}{a_1+a_2+a_3} = f^{-1}.$$

We clearly have that $\{<>,(123)\}(f) = f$. Therefore, the cluster modular group $\Gamma_Q = \mathbb{Z}/6\mathbb{Z}$ acts on f via the quotient group $\mathbb{Z}/2\mathbb{Z}$.

Definition 2.3.7. Let $\Gamma^\circ \subset \Gamma_Q$ be a functor which assigns a subgroup $\Gamma^\circ_{\mathbf{i}} \subset \Gamma_{\mathbf{i}}$ for each seed $\mathbf{i}$. The invariant field for Γ° on $\mathcal{A}_Q$ or $\mathcal{X}_Q$ is a collection of invariant fields $\mathbb{F}(\mathcal{A}_{\mathbf{i}})^{\Gamma^\circ_{\mathbf{i}}}$ or $\mathbb{F}(\mathcal{X}_{\mathbf{i}})^{\Gamma^\circ_{\mathbf{i}}}$ respectively for each seed $\mathbf{i}$.

It is not clear from this definition that this collection of invariant rings is non empty. It is most likely not true that there are invariants for any cluster ensemble with non-trivial cluster modular group, but we will not discuss this here.

Remark 2.3.4. We generally write elements of these rings in terms of the initial seed, $\mathbf{i}$, associated with Q.

2.3.3 Folding of cluster ensembles

When our cluster ensemble is associated with a non-skew symmetric quiver which can be obtained by folding, there is a natural relationship between the $\mathcal{A}$ and $\mathcal{X}$ coordinate variables which gives a map between the functions defined on each space. Let R be a quiver obtained by folding Q and let $\mathbf{j}, \mathbf{i}$ be seeds associated with R, Q respectively. Let K_i be the sets of nodes of Q which are folded to obtain R.

Picking an order to mutate the nodes in each set realizes the seed groupoid associated with R as a subgroupoid of the seed groupoid of Q. Pulling the functors $\mathcal{A}_Q, \mathcal{X}_Q : \mathcal{G}_{\mathbf{i}} \to Pos$ back along this inclusion gives functors $\hat{\mathcal{A}}_Q, \hat{\mathcal{X}}_Q : \mathcal{G}_{\mathbf{j}} \to Pos.$ There is a natural transformation, $\eta : \mathcal{A}_R, \mathcal{X}_R \to \hat{\mathcal{A}}_Q, \hat{\mathcal{X}}_Q$ which maps the positive charts $(\mathcal{A}_{\mathbf{j}}, \mathcal{X}_{\mathbf{j}})$ to $(\mathcal{A}_{\mathbf{i}}, \mathcal{X}_{\mathbf{i}})$ defined on coordinates by assigning all of the cluster variables associated to nodes in K_i to the cluster variable associated to the folded node in $\mathbf{j}$. We can think of this map as being similar to a diagonal embedding of the varieties $(\mathcal{A}_{\mathbf{j}}, \mathcal{X}_{\mathbf{j}})$ in $(\mathcal{A}_{\mathbf{i}}, \mathcal{X}_{\mathbf{i}})$.

Pull back along this embedding gives a map between the functions $\mathbb{F}(\mathcal{A}_{\mathbf{i}}, \mathcal{X}_{\mathbf{i}}) \xrightarrow{fold} \mathbb{F}(\mathcal{A}_{\mathbf{j}}, \mathcal{X}_{\mathbf{j}})$ for each seed $\mathbf{i}$ which folds to a seed for the cluster ensemble associated with $\mathbf{j}$. This map does not depend on the order of the nodes chosen for each set. Since elements of $\mathbb{F}(\mathcal{A}_Q, \mathcal{X}_Q)$ are collections of functions on each positive variety which agree along mutation paths, we can define a map

$$\mathbb{F}(\mathcal{A}_Q, \mathcal{X}_Q) \xrightarrow{fold} \mathbb{F}(\mathcal{A}_R, \mathcal{X}_R) \tag{2.11}$$

by folding the functions on the positive varieties coming from seeds in $\mathcal{G}_{\mathbf{i}}$ appearing in the subgroupoid associated with $\mathbf{j}$. This map is clearly surjective.

We can also realize the cluster modular group $\Gamma_{\mathbf{j}}$ as a sub-quotient of the cluster modular group $\Gamma_{\mathbf{i}}$, where the "sub" comes from only considering sequences which mutate the folding sets together, and "quotient" being that we ignore automorphisms which map the folding sets into themselves. Thus, if $\gamma \in \Gamma_{\mathbf{i}}$ is an element which can be mapped into $\Gamma_{\mathbf{j}}$, then we get a map

$$\mathbb{F}(\mathcal{A}_Q, \mathcal{X}_Q)^{\langle\gamma\rangle} \xrightarrow{fold} \mathbb{F}(\mathcal{A}_R, \mathcal{X}_R)^{\langle\gamma\rangle} \tag{2.12}$$

on the invariant rings.

2.4 Preliminaries on Hyperbolic Geometry and Teichmüller spaces

We will recall some background about hyperbolic structures on surfaces and their moduli spaces

2.4.1 Hyperbolic space

Recall that the hyperbolic place, $\mathbb{H}^2$, has two important models, the Poincaré disk, $\{z \in \mathbb{C}, |z| < 1\}$ and upper half plane , $\{z \in \mathbb{C}, \Im(z) > 0\}$. These two models are related by a Mobius transformation mapping the upper half plane into the unit disk.

The boundary of the hyperbolic plane, $\partial\mathbb{H}^2$, is identified with $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$. This identification is quite natural in the upper half plane model, but we may also label points on the boundary in the disk model with elements of $\mathbb{R} \cup \{\infty\}$. Points on the boundary are called cusps. There is a natural cyclic order on the points of $\partial\mathbb{H}^2$. There are two choices for this, which we may fix by deciding if a given triple of points is positively or negatively orientated. As a convention we will fix $(\infty, 1, 0)$ as a positive triple.

Each of these models comes equipped with a complete Reimannian metric, g with constant curvature equal to -1. The group of isometries of $\mathbb{H}^2$ is identified with the group of Mobius transformations which fix the upper half plane, and is isomorphic to $\mathrm{PSL}(2, \mathbb{R})$.

Name	Example	Trace	Fixed Points
Elliptic	$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$	$Tr < 2$	One point in $\mathbb{H}^2$
Parabolic	$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$	$Tr = 2$	One point on $\partial\mathbb{H}^2$
Hyperbolic	$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$	$Tr > 2$	Two points on $\partial\mathbb{H}^2$

Figure 2.7: The conjugacy classes of non-identity elements in $\mathrm{PSL}(2,\mathbb{R})$

The elements of $\mathrm{PSL}(2,\mathbb{R})$ are in 3 distinct non identity conjugacy classes. They are shown in figure 2.7

We will also need to recall the notion of a "horocycle" about a boundary point on $\mathbb{H}^2$. These are curves on $\mathbb{H}^2$ with constant curvature and which only touch touch the boundary at a single point. In the disk model, they are given by circles which are tangent to a boundary point, and in the upper half plane model, they are given by tangent circles or horizontal lines (these lines are the horocycles tangent to the point $\infty \in \partial\mathbb{H}^2$). Each parabolic element of $\mathrm{PSL}(2,\mathbb{R})$ fixes a single boundary point and translates points in $\mathbb{H}^2$ along the horocycles touching this point.

2.4.2 Hyperbolic structures on surfaces

Let S be a surface of genus g with b boundary components, p punctures, and n marked points on the boundary. Let B be the set of boundary curves, P be the set of punctures and M be the set of marked points on the boundary. Let S° be $S - \{B \cup P\} = \partial S$, the interior of S.

Definition 2.4.1. We call S a *marked surface* if it satisfies that each boundary component has at least 1 marked point on it and that $n - 3\chi(S) = 6g - 6 + 3b + 3p + n > 0$.

Definition 2.4.2. An *arc* on S is an isotopy class of paths between two marked points or punctures on S. A *closed curve* on S is an isotopy class of embeddings of the circle on S. A closed curve is called *simple* if it has a representative isotopy class which does not intersect itself. An *Ideal Triangulation* of S is a maximal collection of pairwise non-intersecting arcs on S.

Each marked surface admits an ideal triangulation, and every such triangulation has the same cardinality, $N = 6g - 6 + 3b + 3p + 2n$.

Definition 2.4.3. A *hyperbolic structure* on S is a Riemannian metric, g, on S satisfying the following properties:

1. g has constant curvature equal to -1.

2. g has finite area.

3. g takes punctures and marked points to cusps.

4. the arcs of $B - M$ are geodesics of g.

There are several important equivalent ways of defining a hyperbolic structure. The hyperbolic metric on S locally defines a map $S \to \mathbb{H}^2$. In other words, for any point q on S° there exists $U \subset S$, $q \in U$ with U isometric to an open set of $\mathbb{H}^2$.

Definition 2.4.4. Let $\tilde{S}$ be the universal covering space of S and let π be the covering map $\tilde{S} \to S$. This local isometry can be extended to an immersion

$$\text{Dev} : \tilde{S} \to D \subset \mathbb{H}^2 \tag{2.13}$$

where D is some subset of $\mathbb{H}^2$, called the *developing map*. Any element $\delta \in \pi_1(S)$ corresponds to a deck transformation of $\tilde{S}$ over S, and the developing map takes such transformations to hyperbolic isometries of $\mathbb{H}^2$. Thus, the developing map gives a representation

$$\psi : \pi_1(S) \to \text{Isom}(\mathbb{H}^2) = \text{PSL}(2, \mathbb{R}) \tag{2.14}$$

called the *monodromy representation*.

Let Γ be the image of $\pi_1(S)$ under ψ. Then Γ is a discrete subgroup of $\text{PSL}(2, \mathbb{Z})$, and ψ takes loops homotopic to punctures to parabolic elements in Γ. Γ acts on D by isometries, so D/Γ naturally has a hyperbolic structure which is isometric to the structure on S.

This situation is encompassed in the following commutative diagram:

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{Dev} & D \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ S & \xrightarrow{\simeq} & D/\Gamma \end{array} . \tag{2.15}$$

Essentially, the developing map allows us to view a hyperbolic structure on a surface as directly coming from the canonical hyperbolic metric on $\mathbb{H}^2$.

Example 2.4.1. The surface $S = S_{0,1,1,2}$, a once punctured digon is shown in figure 2.8. The monodromy representation takes the element of $\pi_1(S)$ corresponding to δ to the parabolic element $\gamma \in \mathrm{PSL}(2, \mathbb{R})$ which fixes $p \in \mathbb{H}^2$ and moves n_1 to γn_1.

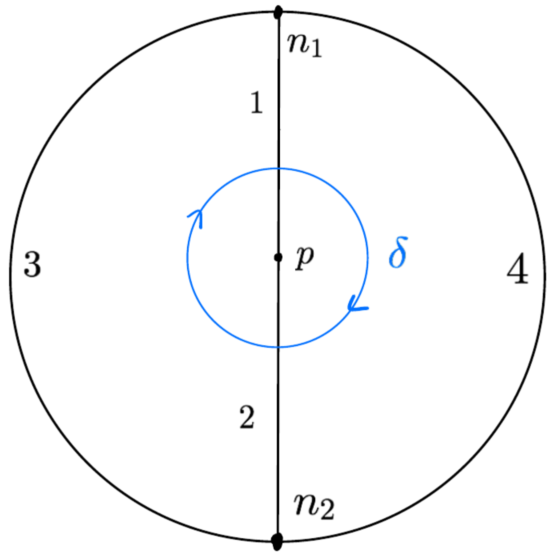
2.4.3 Teichmüller space

The notion of Teichmüller spaces comes from several different perspectives. For us, we will consider the Teichmüller space of a surface, $T(S)$ as the moduli space of hyperbolic structures on S up to an equivalence relation. To aid with our definitions, we need the notion of a marked hyperbolic structure on S.

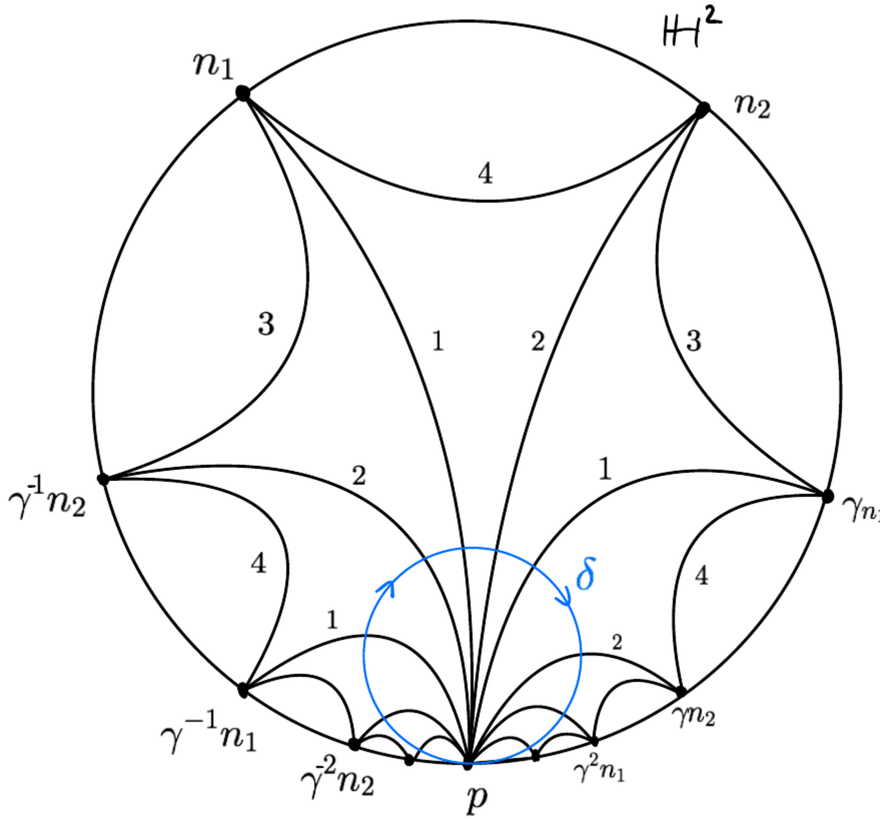
Definition 2.4.5. A *marked hyperbolic structure* on S is a triple (M, g, ϕ) such that $\phi : S \to M$ is a homeomorphism and (M, g) is a hyperbolic structure on the surface M.

Definition 2.4.6. Two marked hyperbolic structures on S, (M, g, ϕ) and (M', g', ϕ') are equivalent if there is a homeomorphism, $\psi : M \to M'$ such that the pullback of g' along ψ is g and $\phi'^{-1} \circ \psi \circ \phi : S \to S$ is isotopic to the identity map. The *Teichmüller space* of S, $T(S)$ is defined to be the set of marked hyperbolic structures on S up to equivalence.

There are several important equivalent ways of describing $T(S)$. In view of developing maps and monodromy representations, we can see that two equivalent marked hyperbolic structures gives rise to an isometry, $\tilde{\psi}$ of $\mathbb{H}^2$ such that $\mathrm{Dev}(\tilde{M}) =$



(a) S



(b) Dev(S)

Figure 2.8: The surface S along with its developing image in $\mathbb{H}^2$.

$\tilde{\psi}(\mathrm{Dev}(\tilde{M}')$ which descends to the map ψ on the quotients. Furthermore, the two groups $\Gamma = \psi(\pi_1(M))$ and $\Gamma' = \psi'(\pi_1(M'))$ are conjugate in $\mathrm{PSL}(2, \mathbb{R})$ by the element $\tilde{\psi}$. Thus, we have the following proposition:

Proposition 2.4.1. In the absence of marked points, the Teichmüller space $T(S)$ is equivalent to the set of representations, $\psi : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{R})$, up to conjugation by $\mathrm{PSL}(2, \mathbb{R})$ such that ψ is discrete, faithfull, and sends loops homotopic to punctures to parabolic elements.

2.5 Coordinates on Teichmüller Space

We will review three different coordinate systems on several variations of Teichmüller space.

Let S be a marked surface. For the first two coordinate systems, we will fix an ideal triangulation Δ of S.

2.5.1 Teichmüller $\mathcal{X}$ space

Let $N^\circ = 6g - 6 + 3b + 3p + n$ be the number of interior arcs of Δ.

Definition 2.5.1. The *Teichmüller $\mathcal{X}$ space* of S, $T_{\mathcal{X}}(S)$, is defined to be the real manifold homeomorphic to $(\mathbb{R}^{>0})^{N^\circ}$ with a positive real valued coordinate for each interior arc of Δ. We call these coordinates the $\mathcal{X}$ coordinates of $T_{\mathcal{X}}(S)$.

The main ingredient of our first set of coordinates will be the cross ratio of four points on $\mathbb{RP}^1$.

Definition 2.5.2. Let (z_1, z_2, z_3, z_4) be four points in $\mathbb{R} \cup \{\infty\}$. The *cross ratio*, $r(z_1, z_2, z_3, z_4)$ is defined to be

$$r(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_1 - z_4)} \tag{2.16}$$

This particular choice of cross ratio has the property that it is a positive real number for any positively oriented quadruple of points.

Identifying $\partial\mathbb{H}^2$ with $\mathbb{RP}^1$, we see that the cross ratio is a $\mathrm{PSL}(2, \mathbb{R})$ invariant for four points on $\partial\mathbb{H}^2$. There is a natural way to assign a cross ratio to each interior (non boundary) edge in our ideal triangulation Δ.

We may lift Δ to a triangulation, $\tilde{\Delta}$ of $\tilde{S}$ and embed this in $\mathbb{H}^2$ via the developing map. Thus each interior edge, $\tilde{e}$, in $\tilde{\Delta}$ is identified with the four points in $\partial\mathbb{H}^2$ of the square containing the edge after developing. Picking z_1 to be one of the endpoints of $\tilde{e}$ and the other three points z_2, z_3, z_4 to cyclically oriented from z_1 allows us to assign the cross ratio $r(z_1, z_2, z_3, z_4)$ to the edge $\tilde{e}$. The particular choice of endpoint for z_1 does not change the assigned cross ratio.

We assign cross ratios to the interior edges, denoted x_e of Δ by choosing a lift of them to $\tilde{\Delta}$ and assigning the cross ratio of the lift as before. This does not depend on the choice of lift since the difference of lifts is described by an isometry of $\mathbb{H}^2$.

Definition 2.5.3. An opening, $\hat{S}$ of a marked surface, S, is a hyperbolic surface where the punctures on S have been replaced with geodesic boundary components, along with a choice of orientation of each new boundary.

An ideal triangulation of S can be lifted to a triangulation of $\hat{S}$ where the arcs which touch the punctures are lifted to arcs which spiral around the boundaries in the direction which agrees with their chosen orientation. The collection of surfaces which are openings of S are parameterized by signed lengths of their geodesic boundaries, $\{l_p\}$ for p a puncture, where the sign indicates whether the orientation of the boundary component agrees with the orientation of S. The situation when the boundary has length 0 is equivalent to there being a puncture on the opened surface

Proposition 2.5.1 (Thurston-Fock)**.** Any collection of $6g-g+3b+3p+n$ positive real numbers can be realized as the cross ratio coordinates of an opening of S. In other words $T_{\mathcal{X}}(S)$ parameterizes hyperbolic surfaces which are openings of S.

Part of the proof of this proposition is an explicit construction of a representation $\pi_1(S) :\to \mathrm{PSL}(2,\mathbb{R})$ leading to a hyperbolic structure on an opened surface. We recall the basic principals of this construction

For each puncture p on S there is an important associated product of $\mathcal{X}$ coordinates given by

$$p_{\mathcal{X}} = \prod_{e|p\in\partial e} x_e = e^{l_p} \tag{2.17}$$

which is equal to the exponential of the signed boundary length.

Proposition 2.5.2. The Teichmüler space $T(S)$ embeds in $T_{\mathcal{X}}(S)$ as the submanifold where $p_{\mathcal{X}} = 1$ for each puncture of S.

2.5.2 Decorated Teichmüler space

Decorated Teichmüller space, denoted $T_{\mathcal{A}}(S)$, introduced by Penner in [30] parameterizes hyperbolic structures on S along with a "decoration" of S, consisting of a choice of horocycle around each puncture and marked point. Choices of horocycles on S correspond via the developing map to collections of horocycles around the cusps of $D \subset \mathbb{H}^2$ which are fixed by the group Γ.

Again we fix Δ an ideal triangulation of S. Let $N = 6g - g + 3b + 3p + 2n$ be the number of arcs in Δ.

Definition 2.5.4. The *decorated Teichmüller space*, $T_{\mathcal{A}}(S)$, is defined to be the real manifold homeomorphic to $(\mathbb{R}^{>0})^N$ with a positive real valued coordinate for each arc of Δ. We call these coordinates the $\mathcal{A}$ coordinates of $T_{\mathcal{A}}(S)$.

There is a natural way to assign a positive real number for every (including boundary arcs) edge, $e \in \Delta$ using our decoration. Let l_e be the signed length of e measured between the horocycles on the endpoints of e, where the sign is positive if the lifts of the horocycles do not intersect in $\mathbb{H}^2$ after developing and negative otherwise. We assign the "λ length"

$$a_e = \exp(l_e/2) \tag{2.18}$$

to each arc in Δ.

Proposition 2.5.3 (Penner [31])**.** Every collection of $N = 6g - g + 3b + 3p + 2n$ positive real numbers appears as the lambda lengths of a decorated hyperbolic structure on S. In other words, $T_{\mathcal{A}}(S)$ parameterizes decorated hyperbolic structures on S

Proposition 2.5.4 (Penner [31])**.** The natural map $T_{\mathcal{A}}(S) \to T(S)$ forgetting the decoration is a fibration, with fibers homeomorphic to $(\mathbb{R}^{>0})^{n+p}$ parameterized by the lengths of the decorating horocycles

There is a natural relationship between the lambda lengths of a decorated hyperbolic structure and cross ratios. We have a map $\rho : T_{\mathcal{A}}(S) \to T_{\mathcal{X}}(S)$ defined on coordinates by fixing Δ an ideal triangulation of S and defining

$$\rho^*(x_e) = \frac{\lambda_{12}\lambda_{34}}{\lambda_{23}\lambda_{14}} \tag{2.19}$$

when $x_e = r(z_1, z_2, z_3, z_4)$ and λ_{ij} is the lambda length of the arc connecting the marked points corresponding to z_i and z_j. This map is the inspiration of the map $\rho : \mathcal{A} \to \mathcal{X}$ of a cluster ensemble.

2.5.3 Representations from points in Teichmüller spaces

Points in $T_{\mathcal{A}}(S)$ and $T_{\mathcal{X}}(S)$ naturally give rise to representations $\pi_1(S) \to \mathrm{PSL}(2, \mathbb{R})$ with desired properties. Part of the proofs of the propositions in the previous two sections is an explicit calculation of this representation. We will review these ideas here.

Given S a marked hyperbolic surface with ideal triangulation Δ we can construct a "fatgraph" on S as follows. Let $\tilde{F}$ be the dual graph to Δ, i.e, F has one vertex for each triangle in Δ and edges between vertices whenever two triangles share an edge, and edges to the boundary whenever a triangle contains a boundary component . Then we generate the fatgraph, F from $\tilde{F}$ by replacing each vertex with a triangle of three "short edges" connecting the three "long edges" edges at its vertices, see figure 2.9 for an example.

This fatgraph can be used to construct a representation $\psi : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{R})$ for a point in $T_{\mathcal{X}}(S)$ in the following way. We will first assign matrices to the edges of F. To each long edge of F which crosses an interior arc $e \in \Delta$, we assign the matrix $X_e = \begin{bmatrix} 0 & -\sqrt{x_e} \\ (\sqrt{x_e})^{-1} & 0 \end{bmatrix}$ where x_e is the $\mathcal{X}$ coordinate associated with the arc e. The three short edges will be assigned the matrices

$$R = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \tag{2.20}$$

Let $\gamma \in \pi_1(S)$. We may homotope γ to our fat so that γ may be obtained by following along segments of F. We then obtain $\psi(\gamma)$ by first picking a segment to begin and then multiplying the matrices for each consecutive segment obtained from following γ, using R whenever we use a short edge to turn right and L whenever we turn left.

Example 2.5.1. Continuing example 2.4.1, the fatgraph for Δ is shown in figure

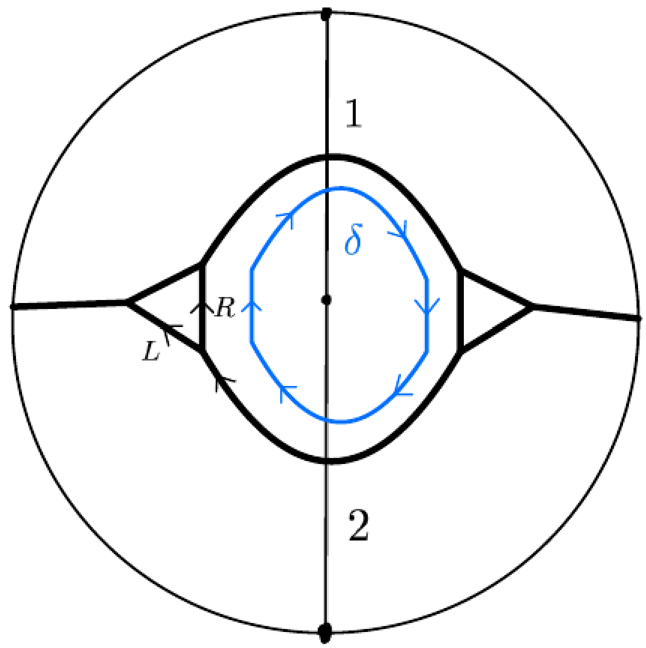


Figure 2.9: The fatgraph for a triangulation of a punctured digon.

2.9. We can see that the element $\delta \in \pi_1(S)$ can be represented by the matrix

$$X_1 R X_2 R = \begin{bmatrix} \sqrt{x_1 x_2} & 0 \\ \frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} & \frac{1}{\sqrt{x_1 x_2}} \end{bmatrix}. \tag{2.21}$$

This matrix has trace equal to 2 exactly when $x_1 x_2 = 1$

We may use this to compute the traces of monodromy operators associated with a hyperbolic structure on a surface in terms of $\mathcal{X}$ coordinates, and we may use the map $\rho : T_{\mathcal{A}}(S) \to T_{\mathcal{X}}(S)$ to write these formulas in terms of the $\mathcal{A}$ coordinates.

2.6 Cluster Ensembles associated with Surfaces

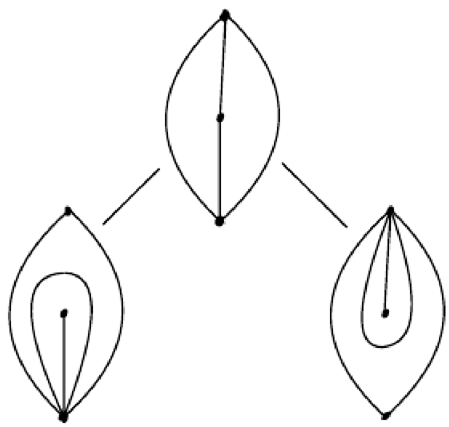
In many cases, cluster ensemble invariants have a geometric interpretation coming from the relationship between cluster ensembles and the Teichmüller theory of surfaces. We refer to [13] and [32, 33] for background on the ingredients of a cluster ensemble associated to a hyperbolic surface. We recall some of these ideas here.

2.6.1 Quivers from triangulations

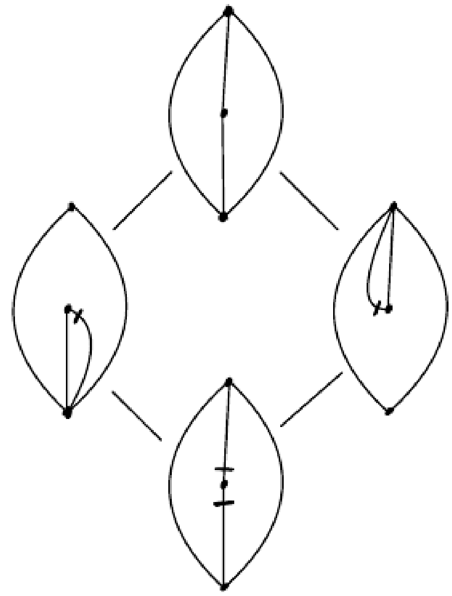
Let S be a marked surface. Given Δ a triangulation of S, we associate a quiver, Q_Δ, as follows: For each edge $e \in \Delta$ we add a node N_e and for each triangle $t \in \Delta$ we add a clockwise oriented cycle of arrows between the nodes associated with the edges of t. In the situation where we have arrows between two nodes in opposite directions, we cancel them, as shown in example 2.6.2. The nodes associated to boundary edges are frozen. There are $-3\chi(S) + 2n$ total nodes and n frozen nodes.

There is one caveat to this construction when S has punctures. In this case it may be possible to have a "self folded" triangle in an ideal triangulation of S see figure 2.10a. In this case, the construction mentioned above does not produce the correct quiver. However, we can always find a triangulation of S with no self folded triangles, and use this to construct a quiver associated with the triangulation.

Mutation of nodes in Q_Δ corresponds to a "flip" or "Whitehead Move" in Δ at the corresponding arc. A flip is the operation of removing the specified arc, and replacing it with the unique new arc which forms a new triangulation. Again, there is a caveat to this when S has punctures. The interior arc of a self folded triangle cannot be flipped, but the corresponding node in the quiver can be. This is addressed in [32] by the addition of "tagged" arcs. Essentially, we replace the



(a) Arcs in a punctured digon.



(b) The tagged arc flip graph.

Figure 2.10: Untagged vs tagged arcs in a punctured digon.

outside arc of a self folded triangulation with a tagged arc as shown in figure 2.10. There is then a rule for flipping tagged arcs which agrees with the mutation rule for quivers. With this addition, we may always flip any arc and this always agrees with mutation of corresponding quivers. We do not need the details of this in general.

We can generate a graph associated to S where the nodes correspond to triangulations (possibly with tagged arcs) and the edges correspond to flips of triangulations. We call this graph the "flip graph" of S.

Let $\mathbf{i}$ be the seed obtained by using Q_Δ as its underlying quiver.

Proposition 2.6.1 ([32]). The flip graph of S agrees with the cluster modular groupoid of $\mathbf{i}$ in the following way: If the triangulation of S, Δ', is obtained from Δ by two different paths of flips, then the two seeds obtained from $\mathbf{i}$ by the corresponding paths of mutations are isomorphic.

The cluster ensemble generated by using Q_Δ as a seed is closely related to the Teichmüller theory of S. We will usually refer to this seed simply as Δ when it is understood that we mean the seed associated with the triangulation Δ.

The positive real valued points of the positive varieties $\mathcal{A}_\Delta$ and $\mathcal{X}_\Delta$ are exactly the Teichmüller spaces $T_{\mathcal{A}}(S)$ and $T_{\mathcal{X}}(S)$ defined in the previous section. We write $(\mathcal{A}_S, \mathcal{X}_S)$ for the cluster ensemble generated from any triangulation of S.

Importantly, one checks that the associated $\mathcal{A}$ and $\mathcal{X}$ exchange relations correspond exactly to the change in the lambda lengths and cross ratios after flipping. Thus, the cluster ensemble encodes the spaces $T_{\mathcal{A}}(S)$ and $T_{\mathcal{X}}(S)$ along with specified coordinate systems for each ideal triangulation.

Example 2.6.1. Let S be a disk with n marked points on the boundary. The cluster ensemble encodes some well known algebraic objects. The cluster algebra associated with $\mathcal{A}_S$ is the affine cone coordinate ring of the Grassmannian $Gr(2, n)$, see [11]. The space $\mathcal{X}_S$ is essentially the moduli space $M_{0,n}$ of genus zero curves with n distinguished points see example 1.3 of [12].

For any n, there are finitely many triangulations, and thereby finitely many clusters. We can also see that there is always a triangulation for which the associated quiver is an orientation of an A_n Dynkin diagram. Thus this cluster ensemble is of type A_n.

Example 2.6.2. Following example 2.4.1, we can construct the quiver for this surface and triangulation. Figure 2.11 shows the construction of the quiver. Notice

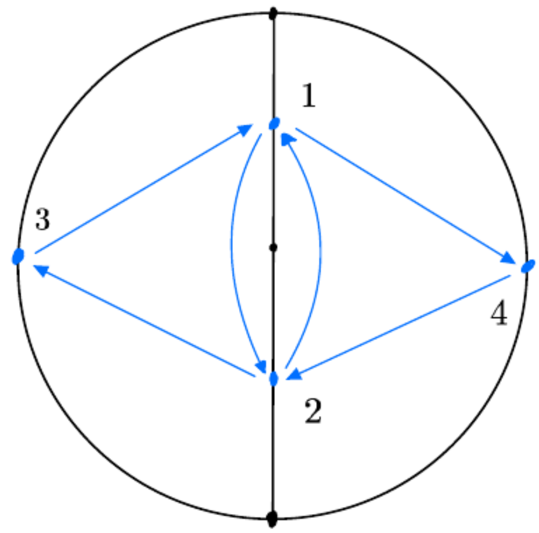


Figure 2.11: Generating the quiver for the surface S and Δ. The two middle arrows are removed.

that we cancel the arrows between nodes 1 and 2. The quiver, Q_Δ, we obtain is

$$\begin{array}{ccccc} & & 1 & & \\ & \nearrow & & \searrow & \\ [3] & & & & [4] \\ & \nwarrow & & \swarrow & \\ & & 2 & & \end{array} . \tag{2.22}$$

We can see that flips at arcs 1 or 2 will generate self folded triangles. We can see what the quivers associated to these triangulations should be by mutating the corresponding nodes of Q_Δ. We cannot flip arc 1 and then flip arc 2 without introducing tagged arcs. This is necessary since we can mutate node 1 and then node 2 of Q_Δ without issue.

The cluster ensemble associated with S has four clusters and the flip graph of S has four nodes. This is shown in figure 2.11.

2.6.2 Action of the mapping class group

We can define an action of the mapping class group, MCG(S), on the triangulations of S and hence identify the mapping class group as a subgroup of the cluster modular group, Γ_S of our cluster ensemble $(\mathcal{A}_S, \mathcal{X}_S)$. We give an explicit construction of this subgroup here. We refer to [34] section 2 for computations involving the mapping class group of selected surfaces.

Given $f \in \mathrm{MCG}(S)$ we can define $\gamma_f \in \Gamma_S$ as follows: f gives a new triangulation of S and hence by [32] there is a path of flips, P_f, taking Δ to $f(\Delta)$. f defines a map between the edges of Δ and $f(\Delta)$ and it preserves the adjacency relations between the triangles of Δ. This means that Δ and $f(\Delta)$ have the same associated quivers. P defines a map between the nodes of $Q_{f(\Delta)}$ and $P(Q_\Delta)$ since these quivers come from the same triangulation. Let $\sigma_{f,P}$ be the isomorphism of quivers Q_Δ to $P(Q_\Delta)$ defined by the composition

$$\sigma_{f,P} : Q_\Delta \xrightarrow{f} Q_{f(\Delta)} \xrightarrow{P} P(Q_\Delta). \tag{2.23}$$

Thus to f we associate $\gamma_f = \{P_f, \sigma_{f,P}\}$.

It is not immediately clear that this does not depend on the choice of path, P. Let $\{P, \sigma\}$ and $\{R, \tau\}$ be two possible representatives of γ_f. Then we have

$$\{P, \sigma\}\{R, \tau\}^{-1} = \{P, \sigma\}\{\tau^{-1}(R^{-1}), \tau^{-1}\} = \{P\sigma\tau^{-1}(R^{-1}), \sigma\tau^{-1}\}. \tag{2.24}$$

We need to show that this element is a trivial cluster transformation. $\sigma\tau^{-1}$ is the quiver isomorphism from $R(Q_\Delta)$ to $P(Q_\Delta)$ coming from the fact that these both correspond to the same triangulation of S. The composite mutation path,

$P\sigma\tau^{-1}(R^{-1})$, consists of following P and then following R^{-1} back to our initial cluster. This introduces a permutation on the cluster variables determined by the map $\tau\sigma^{-1} : P(Q_\Delta) \to R(Q_\Delta)$. Together these permutations act trivially on the cluster variables, and γ_f is well defined in the cluster modular group.

For all but finitely many quivers associated with surfaces, the cluster modular group is essentially equal to the mapping class group, see [35] proposition 8.5. For the remaining surfaces, one may check case by case that $\mathrm{MCG}(S)$ is always a finite index normal subgroup of Γ.

2.6.3 Skein algebras of surfaces

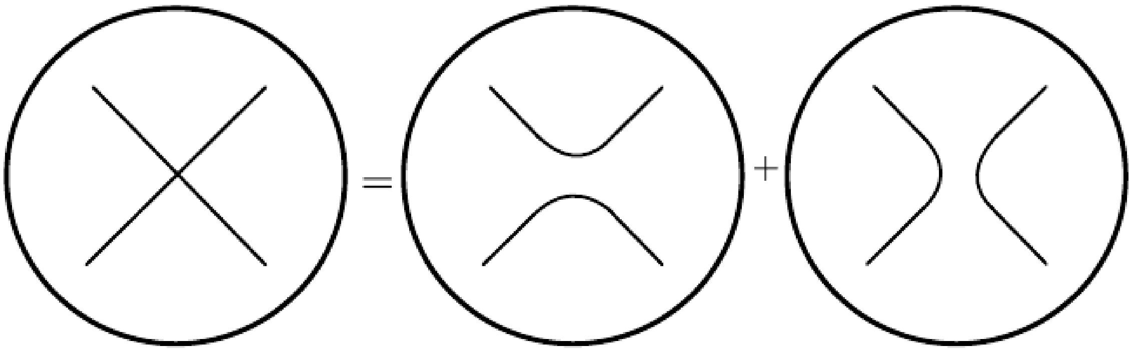
The skein algebra associated with a marked surface will play a key role in understanding the relationships between invariant functions on a cluster ensemble associated with a surface. We refer to [26] for full details on the relationship between skein algebras and cluster algebras associated with surfaces. We note that we will always be considering the unquantized version of the skein algebra.

Let S be a marked surface.

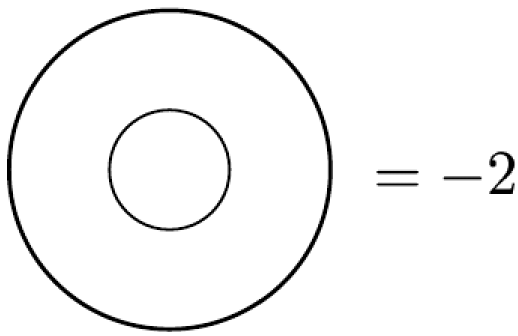
Definition 2.6.1. A *Skein* on S is any collection of arcs on S or closed curves on S. We do not require that these arcs or closed curves do not intersect each other or themselves.

Definition 2.6.2. The *Skein Algebra*, $\mathrm{Sk}(S)$ is the algebra with generators given by the skeins on S subject to the Skein relations, see figure 2.12

The cluster algebra associated with S, $\mathcal{A}(S)$, is naturally contained in the



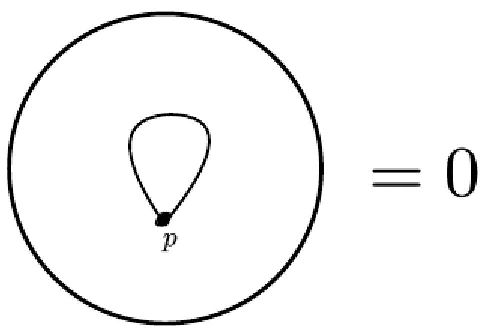
(a) The skein relation for a crossing.



,

(b) A contractible closed curve is equal to

−2.



(c) A contractible arc is equal to zero.

Figure 2.12: The skein relations for arcs and curves on a marked surface.

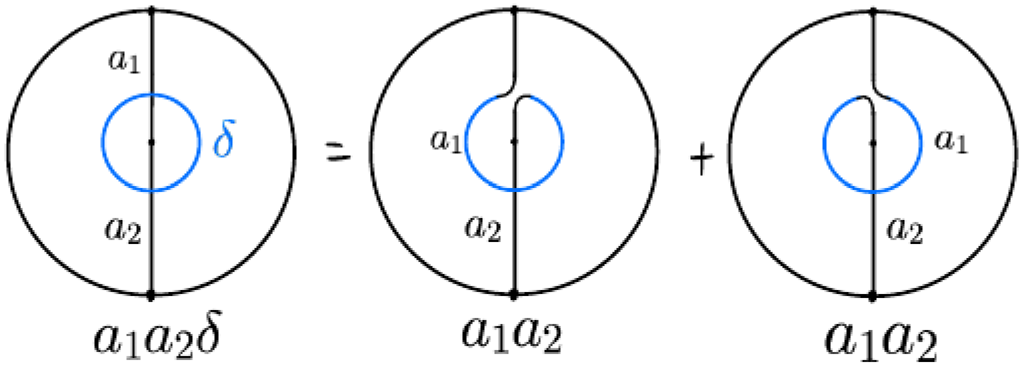


Figure 2.13: Computing the element $\delta a_1 a_2$ in $\mathrm{Sk}(S)$.

skein algebra. The skein relation is simply another description of the $\mathcal{A}$ exchange relation.

Let T be the collection of trace functions for closed curves on S. These functions also naturally form a subalgebra of the skein algebra associated with skeins which do not include marked points. The following proposition is a natural description of the skein algebra of a surface, see [6].

Proposition 2.6.2. The skein algebra of S is generated by the elements of $\mathcal{A}(S)$ and T.

This proposition essentially means that we may use skein relations to write any skein as products and sums of skeins with no crossings.

Example 2.6.3. We can compute the element of the skein algebra corresponding to the arc δ on the surface S from example 2.4.1. We can see that

$$\delta a_1 a_2 = a_1 a_2 + a_1 a_2 \tag{2.25}$$

as shown in figure 2.13. Thus $\delta = 2$ in the skein algebra. This is expected, since this element must be the same as the trace of the monodromy operator associated with δ which is 2, since this arc is homotopic to a puncture and we are computing this from a point in $T_{\mathcal{A}}(S)$.

Chapter 3: Examples of Mutation Invariant Functions

We now compile many interesting examples of mutation invariants.

3.1 First Examples

Example 3.1.1. First we cover example 1.2.1 in a bit more detail.

Let Q be a quiver of type $\widetilde{A}_1$ shown in figure 3.1. Lets first understand the cluster ensemble associated with this quiver. The seed groupoid looks essentially as follows, with pairs of $\mathcal{A}$ and $\mathcal{X}$ coordinates shown on the nodes of Q to represent the seeds $\mathbf{i}, \mu_1(\mathbf{i}) = \mathbf{i}'$ and $\mu_2(\mathbf{i}) = \mathbf{i}''$:

$$\begin{array}{ccccccccc} & & (a_1'', x_1'') & & (a_1, x_1) & & (a_1', x_1') & & \\ & & \Uparrow & & \Downarrow & & \Uparrow & & \\ \xleftrightarrow{\mu_1} & & & \xleftrightarrow{\mu_2} & & \xleftrightarrow{\mu_1} & & \xleftrightarrow{\mu_2} & . \\ & & (a_2'', x_2'') & & (a_2, x_2) & & (a_2', x_2') & & \end{array} \tag{3.1}$$

Each of these seeds has the same underlying quiver so the seed groupoid only has one object and has maps indexed by $\mathbb{Z}$. Thus, the cluster modular group is isomorphic to $\mathbb{Z}$ and has the same presentation at every seed. A generator of this group is given by

$$\gamma = \{1, (12)\} \in \Gamma_Q \simeq \mathbb{Z}. \tag{3.2}$$

The positive varieties $\mathcal{A}_{\mathbf{i}}$ and $\mathcal{X}_{\mathbf{i}}$ are both homeomorphic to $(\mathbb{G}_m)^2$. The mu-

$$1 \Longrightarrow 2$$

Figure 3.1: Quiver of type $\widetilde{A}_1$.

tations induce birational maps according to the mutation rules. For example, we have

$$\mu_1^*(a_1', a_2') = (\frac{1+a_2^2}{a_1}, a_2) \tag{3.3}$$

$$\mu_1^*(x_1', x_2') = (x_1^{-1}, x_2(1+x_1)^2). \tag{3.4}$$

The map $\rho : \mathcal{A}_{\mathbf{i}} \to \mathcal{X}_{\mathbf{i}}$ is given on coordinates by

$$\rho^*(x_1, x_2) = (a_2^2, a_1^{-2}). \tag{3.5}$$

The action of Γ_Q on $\mathbb{F}(\mathcal{A}_{\mathbf{i}})$ and $\mathbb{F}(\mathcal{X}_{\mathbf{i}})$ is given by

$$\gamma(a_1, a_2) = (a_2, \frac{1+a_2^2}{a_1}) \tag{3.6}$$

$$\gamma(x_1, x_2) = (x_2(1+x_1)^2, x_1^{-1}). \tag{3.7}$$

It is not too difficult to write an invariant for Γ_Q on $\mathcal{A}_Q$. The function

$$F(a_1, a_2) = \frac{1+a_1^2+a_2^2}{a_1 a_2} \tag{3.8}$$

is invariant. We also have an invariant for this group on $\mathcal{X}_Q$:

$$G(x_1, x_2) = \frac{(x_2(x_1+1)+1)^2}{x_1 x_2}. \tag{3.9}$$

It is easy to check that $\rho^*(G) = F^2$

Let us interpret these invariants in terms of hyperbolic geometry. The $\widetilde{A}_1$ affine ensemble is associated to an annulus with one marked point on each boundary component (see figure 3.2). The quiver for this triangulation is shown in figure 3.3. There are two frozen nodes and coefficients to account for the boundary arcs on S, call the variables on the $\mathcal{A}$ space associated with them c, d. The cluster modular group corresponds exactly to the mapping class group of S and the generator $\gamma = \{1, (12)\}$ corresponds to a Dehn twist about δ. If we take the trace of the monodromy operator, $\psi(\delta)$, associated with the closed curve δ, we find that on the $\mathcal{X}$ space

$$Tr(\psi(\delta)) = \frac{x_2(x_1+1)+1}{\sqrt{x_1 x_2}}. \tag{3.10}$$

This is exactly the square root of $G(x_1, x_2)$ from before. Furthermore, we have

$$\rho^*(\sqrt{G(x_1, x_2)}) = \frac{a_1^2 + a_2^2 + cd}{a_1 a_2} = F(a_1, a_2, c, d) \tag{3.11}$$

which is the previous invariant we found, taking into account the new coefficients.

We can also investigate the limiting behavior of the cluster variables via the hyperbolic geometry of S and the skein algebra of S.

Let us explicitly compute the element of $\mathrm{Sk}(S)$ corresponding to the simple closed curve δ on S. Call this element F_{sk}. We find using a skein relation that

$$a_1 F_{sk} = a_2 + \mu_2(a_2) = a_2 + \frac{a_1^2 + cd}{a_2} = \frac{a_1^2 + a_2^2 + cd}{a_2}. \tag{3.12}$$

Thus $F_{sk} = F$ is simply another incarnation of the same invariant found earlier. The invariance of F can be seen since the closed curve δ does not change when we do a Dehn twist.

Let $\{a_n\} = \gamma^{n-1}(a_1) = \{a_1, a_2, \frac{1+a_2^2}{a_1}, \dots\}$ be the sequence of cluster variables obtained after repeated applications of γ. Then, applying γ^{n-1} to our previous skein

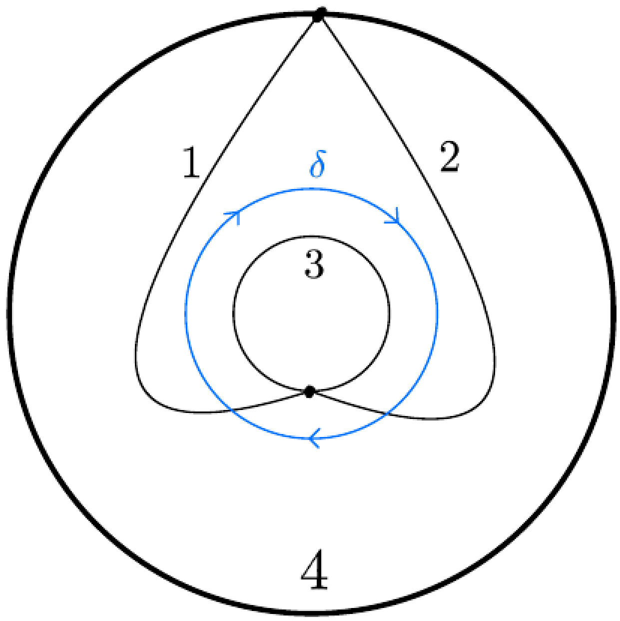


Figure 3.2: The surface S, along with a choice of triangulation. δ is the generator of the fundamental group and the mapping class γ corresponds to a Dehn twist about δ.

relation and using the invariance of F, we find the linear recurrence

$$a_{n+1} = a_n F - a_{n-1}. \tag{3.13}$$

We can explicitly write a solution for this linear recurrence in terms of the initial

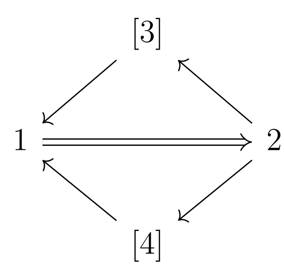


Figure 3.3: The quiver associated with a marked annulus.

two cluster variables a_1, a_2 and the coefficients c, d. Let

$$\lambda = \frac{F + \sqrt{F^2 - 4}}{2} = \exp(\operatorname{arccosh}(F/2)) \tag{3.14}$$

be such that $\lambda + \lambda^{-1} = F$. λ is the eigenvalue of a matrix representing $\rho(\delta)$. Let

$$w = a_2 - a_1\lambda^{-1} \quad w' = -a_2 + a_1\lambda. \tag{3.15}$$

Then we have that

$$a_n = \frac{1}{\lambda - \lambda^{-1}}(w\lambda^n + w'\lambda^{-n}). \tag{3.16}$$

The $\mathcal{A}$ coordinates correspond to lambda lengths associated to the two interior arcs on S. After many applications of γ, their lambda lengths grow like multiples of λ. One may check that there is exactly 1 closed geodesic on S with the same homotopy class as δ and that its hyperbolic length is equal to $2\operatorname{arccosh}(F/2)$. Thus, the multiplier λ is essentially the lambda length of the arc δ. This should make intuitive sense, since each time we twist, we are adding approximately the length of this geodesic to each arc, so in terms of lambda-lengths, we multiply by the exponential of half of the length.

Example 3.1.2. The mutation class of a quiver of type D_n has an element that looks like a directed n-cycle, as shown in figure 3.4. Let Q be this quiver. The cluster modular group for Q is $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ generated by r and t in each of the factors[1]. The function

$$F(a_1, a_2, \ldots, a_n) = \frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \cdots + \frac{1}{a_n a_1} \tag{3.17}$$

[1]This is not quite true when $n = 4$, but the given function is still an invariant.

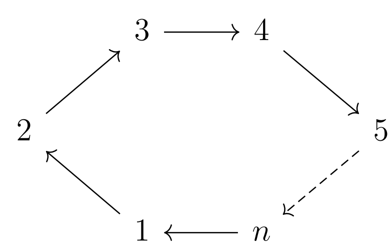


Figure 3.4: $Q_{n-cycle}$. This quiver is simply an n cycle of nodes and arrows.

is an element of $\mathbb{F}(\mathcal{A}_Q)^{<r>}$ and satisfies $t(F) = F^{-1}$. This is a generalization of example 2.3.3.

Example 3.1.3. Let $(\mathcal{A}_Q, \mathcal{X}_Q)$ be the cluster ensemble with trivial coefficients associated with the Markov quiver, Q, of figure 3.6a. The function

$$F(a_1, a_2, a_3) = \frac{a_1^2 + a_2^2 + a_3^2}{a_1 a_2 a_3} \tag{3.18}$$

is an invariant function for all of $\Gamma_Q = \mathrm{PSL}(2, \mathbb{Z})$ on $\mathcal{A}_Q$. The function

$$G(x_1, x_2, x_3) = x_1 x_2 x_3 \tag{3.19}$$

is an invariant for Γ_Q on $\mathcal{X}_Q$.

The function F encapsulates the Diophantine properties of the Markov numbers. The Markov numbers are generated by evaluating the $\mathcal{A}$ coordinates at $(1, 1, 1)$. We write these as triples of integers (x, y, z). Since $F(1, 1, 1) = 3$, we have that the Markov numbers all satisfy $x^2 + y^2 + z^2 - 3xyz = 0$.

If we freeze any one of the nodes in Q, then remaining mutable portion is a quiver of type $\widetilde{A}_1$. We can use the invariant in the exact same way as example 3.1.1 to study the limiting behavior of mutations on this modified quiver. In this way we recover similar analysis of [36].

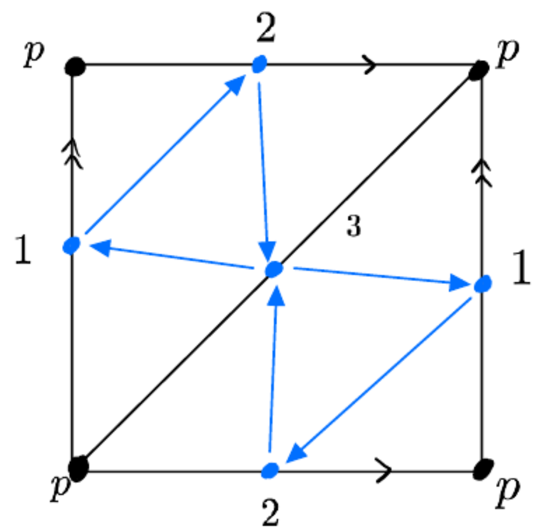
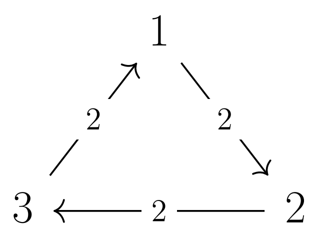


Figure 3.5: The surface $S_{1,0,1,0}$ with a choice of triangulation and associated quiver.

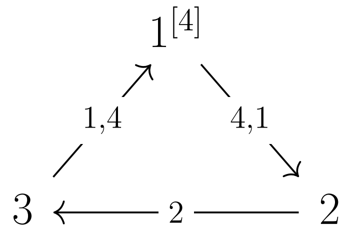
There is a simple geometric interpretation of the invariant of the Markov quiver. If $S = S_{1,1}$ is a torus with one puncture then the cluster ensemble associated to S is of type $A_1^{(1,1)}$ see figure 3.5. If we have a hyperbolic metric on S with the puncture at infinity and a horocycle around the puncture, then the theory of lambda lengths implies that the function F is simply the formula for half the length of the horocycle in terms of the $\mathcal{A}$ coordinates. Since the length of the horocycle is independent on the triangulation and there is only one topological type of triangulation, F must be invariant under exchanges.

3.2 Invariants for Doubly Extended Quivers

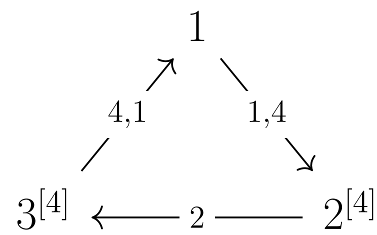
Quivers associated with doubly extended Dynkin diagrams all have similar properties to the Markov quiver, which is of type $A_1^{(1,1)}$. We will show several examples of invariants which are analogous to the Markov invariant.



(a) The Markov quiver. Also associated with a punctured torus and of doubly extended type $A_1^{(1,1)}$.



(b) $BC_1^{(2,1)}$.



(c) $BC_1^{(2,4)}$.

Figure 3.6: Quivers associated with doubly extended Dynkin diagrams with 3 nodes.

Example 3.2.1. Let Q be a quiver of type $BC_1^{(2,1)}$ Then the function

$$F(a_1, a_2, a_3) = \frac{a_1^4 + (a_2 + a_3)^2}{a_1^2 a_2 a_3} \tag{3.20}$$

is an element of $\mathbb{F}(\mathcal{A}_Q)^{\Gamma_Q}$.

Example 3.2.2. Let Q be a quiver of type $BC_1^{(2,4)}$ The function

$$F(a_1, a_2, a_3) = \frac{a_1^2 + 2a_1(a_2^2 + a_3^2) + a_2^4 + a_3^4}{a_1 a_2^2 a_3^2} \tag{3.21}$$

is an element of $\mathbb{F}(\mathcal{A}_Q)^{\Gamma_Q}$. This function and its Diophantine properties were studied by Lampe in [20]. It was shown that all solutions for the equation $F(x, y, z) = 7$ could be obtained from the initial solution $F(1, 1, 1) = 7$ by cluster mutations. It would be interesting to study this type of Diophantine problem for other invariant functions.

Example 3.2.3. A quiver of type $G_2^{(3,3)}$ has two quiver isomorphism classes. Let Q be the isomorphism class shown in figure 3.7. we can see that the element $\gamma = \{1, (12)\}$ is in Γ_Q. It is not hard to compute that

$$1 \to N(\gamma) \to \Gamma_Q \to D_{12} \to 1 \tag{3.22}$$

where $N(\gamma)$ is the normalizer of the element γ and D_{12} is the dihedral group with 12 elements. The functions

$$F_1(a_1, a_2, a_3, a_4) = \frac{(a_1^3 + a_2^3)(a_1 + a_2) + a_3a_4(2a_1^2 + a_1a_2 + 2a_2^2) + a_3^2a_4^2}{a_1^2a_2^2a_3a_4} \tag{3.23}$$

and

$$F_2(a_1, a_2, a_3, a_4) = \frac{(a_1 + a_2)^2 + a_3a_4}{a_1a_2a_3^2} \tag{3.24}$$

are elements of $\mathbb{F}(\mathcal{A}_Q)^{N(\gamma)}$.

Example 3.2.4. A quiver of type $D_4^{(1,1)}$ has 4 mutation classes, call them Q_1, Q_2, Q_3, Q_4, see figure 3.8. Let Q_4 be our seed. The cluster modular group can be written as an extension

$$1 \to \mathbb{Z} * \mathbb{Z} \to \Gamma_{Q_4} \to \mathrm{Aut}(F_4)^+ \to 1 \tag{3.25}$$

where $\mathrm{Aut}(F_4)^+$ is orientation preserving group of automorphisms of the F_4 root

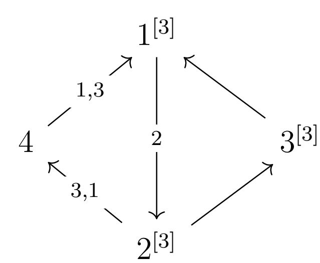


Figure 3.7: Quiver of type $G_2^{(3,3)}$.

system of order 1152. This group is generated by the orientation preserving elements of the Weyl group $W(F_4)$ along with the duality autormorphism.

We will see below why $\mathbb{Z} * \mathbb{Z}$ is a normal subgroup. $W(F_4)$ is generated in the quotient by the elements of $\mathrm{Aut}(Q_4)$ and the paths $\{1231, (23)\}$ and $\{14, (2356)\}$.

An element of $\mathbb{F}(\mathcal{A}_{Q_4})^{\mathbb{Z}*\mathbb{Z}}$ is

$$F(a_1, a_2, a_3, a_4, a_5, a_6) = \frac{(a_1 a_4 + a_2 a_5 + a_3 a_6)^2}{a_1 a_2 a_3 a_4 a_5 a_6} \tag{3.26}$$

There is also an $\mathcal{X}$ invariant

$$G(x_1, \ldots, x_6) = x_1 x_2 x_3 x_4 x_5 x_6. \tag{3.27}$$

These are not the only functions in its exchange class, that is to say that $\mathrm{Aut}(F_4)^+$ acts non trivially on this function. After applying the element $\{1231, (23)\}$, we obtain the functions

$$F_{456} = \frac{a_1 a_4 + a_2 a_5 + a_3 a_6}{a_4 a_5 a_6} G_{456} = x_4 x_5 x_6 \tag{3.28}$$

There are 24 different functions in the exchange class of F. They are

$$F, \quad F^{-1}, \quad F_{456}, \quad F_{456}^{-1}, \quad \frac{a_1}{a_4} \tag{3.29}$$

along with each of their images under the automorphism group of Q_4.

There is a relationship with these functions and the invariant of the Markov quiver. If we fold Q_4 by associating nodes 1 and 4, 2 and 5, 3 and 6, we obtain a quiver of type $A_1^{(1,1)}$. Under this folding, each of these functions becomes an invariant function for a cluster algebra of type $A_1^{(1,1)}$. In other words, if we set $a_1 = a_4, a_2 = a_5, a_3 = a_6$ then our functions become either $1, F'$ or F'^2, where F' is the invariant of example 3.1.3.

There is also a relationship between these invariants and the invariants of the $BC_1^{(2,1)}$ ensemble of example 3.2.1. We can fold Q_1 to obtain a quiver of type $BC_1^{(2,1)}$ by associating nodes 2, 3, 5, and 6. We can find invariants for $\mathcal{A}_{Q_1}$ by pulling our set of invariants on $\mathcal{A}_{Q_4}$ back along a mutation path between these two quivers. In doing so, we obtain the functions

$$F_{1425} = \frac{(a_1 + a_4)^2 + a_2 a_3 a_5 a_6}{a_1 a_4 a_2 a_5}, \quad F_{1425}^{-1}, \quad F_{23} = \frac{a_2}{a_3} \tag{3.30}$$

and each of their images under the action of $\mathrm{Aut}(Q_1)$, as elements of $\mathbb{F}(\mathcal{A}_{Q_1})^{\mathbb{Z}*\mathbb{Z}}$. Each of these clearly folds to an invariant of the $BC_1^{(2,1)}$ algebra by setting $a_2 = a_3 = a_5 = a_6$.

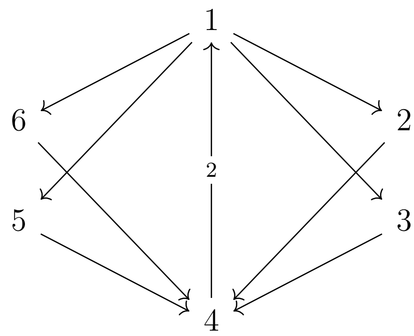
Example 3.2.5. The Weyl group, $W(D_4) = \mathbb{Z}_2^3 \rtimes S_4$, is a normal subgroup of the modular group of the $D_4^{(1,1)}$ cluster ensemble. For a simple presentation for this, take Q_1 as our initial quiver and look at the subgroup generated by the $\mathrm{Aut}(Q_1) = S_4$ and the path $\{214, (142)\} * \{314, (143)\}^{-1}$. This generates $W(D_4)$ as a subgroup of $\mathbb{Z}_2^4 \rtimes S_4$ generated by the elements of S_4 and the element $((1, -1, 0, 0), id)$. We have an exact sequence:

$$1 \to W(D_4) \to \Gamma \to \mathrm{PSL}(2, \mathbb{Z}) \to 1 \tag{3.31}$$

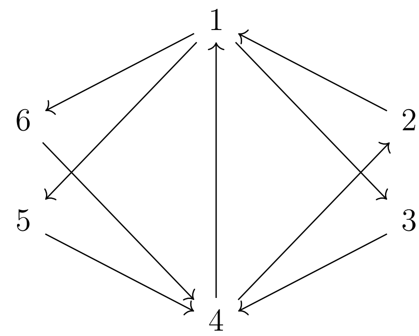
Consider Q_4 as our seed. The function

$$F(a_1, a_2, a_3, a_4, a_5, a_6) = \frac{(a_1 a_4 + a_2 a_5 + a_3 a_6)^3}{a_1 a_2 a_3 a_4 a_5 a_6} \tag{3.32}$$

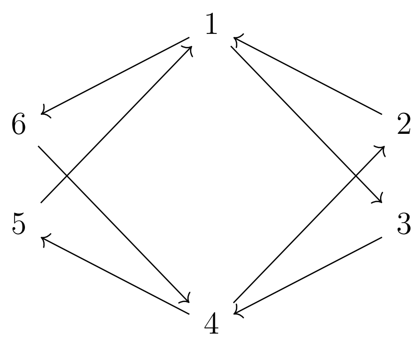
is an element of $\mathbb{F}(\mathcal{A}_{Q_4})^{W(D_4)}$.



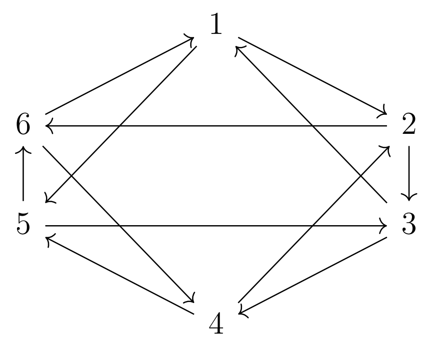
(a) Q_1



(b) Q_2



(c) Q_3



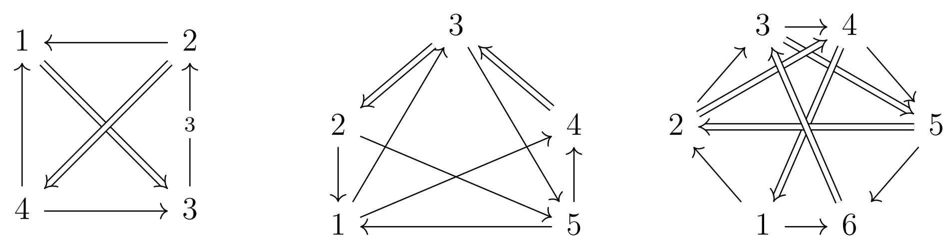
(d) Q_4

Figure 3.8: Quivers of type $D_4^{(1,1)}$.

3.3 Examples from Somos Sequences

Example 3.3.1. The Somos 4, 5 and 6 sequences can be associated to cluster algebras in a simple way[2]. If we look at the variables on the $\mathcal{A}$ space generated by starting with quivers Q_{s4} , Q_{s5} or Q_{s6} and following the mutation paths $\gamma_4 = \{1, (1234)\}$, $\gamma_5 = \{1, (12345)\}$ or $\gamma_6 = \{1, (123456)\}$, we obtain the respective Somos sequences by evaluating all of the initial cluster variables at 1. Explicitly,

[2]This Somos 6 sequence is not the most famous instance, see [37] for a discussion of the example shown here.



(a) Q_{s4}. (b) Q_{s5}. (c) Q_{s6}.

Figure 3.9: Quivers for the Somos 4, 5 and 6 sequences.

these sequences can be obtained by the recurrence

$$X_{n+k}X_n = X_{n+1}X_{n+k-1} + X_{n+\lfloor k/2 \rfloor}X_{n+\lceil k/2 \rceil} \tag{3.33}$$

with $\{X_1, X_2, \ldots, X_k\}$ all equal to 1 for $k = 4, 5, 6$ respectively. This gives the sequences

$$S_4 = \{1, 1, 1, 1, 2, 3, 7, 23, 59, 314, \ldots\} \tag{3.34}$$

$$S_5 = \{1, 1, 1, 1, 1, 2, 3, 5, 11, 37, 83, \ldots\} \tag{3.35}$$

$$S_6 = \{1, 1, 1, 1, 1, 1, 2, 3, 4, 8, 17, 50, 107 \ldots\}. \tag{3.36}$$

There are invariant functions on each of these cluster ensembles for these mutation paths. The functions

$$F_4(a_1, a_2, a_3, a_4) = \frac{a_1^2a_4^2 + a_1a_3^3 + a_4a_2^3 + a_2^2a_3^2}{a_1a_2a_3a_4} \tag{3.37}$$

$$F_5(a_1, a_2, a_3, a_4, a_5) = \frac{a_1^2a_4^2a_5 + a_1a_2^2a_5^2 + a_1a_3^2a_4^2 + a_2^2a_3^2a_5 + a_2a_3^3a_4}{a_1a_2a_3a_4a_5} \tag{3.38}$$

and

$$F_6(a_1,a_2,a_3,a_4,a_5,a_6) = \frac{a_1^2a_2a_5a_6^2 + a_1^2a_4a_5^3 + a_2^3a_3a_6^2 + a_1a_3^2a_4a_5^2 + a_2^2a_3a_4^2a_6 + a_1a_3a_4^3a_5 + a_2a_3^3a_4a_6 + a_3^3a_4^3}{a_1a_2a_3a_4a_5a_6} \tag{3.39}$$

are elements of $\mathbb{F}(\mathcal{A}_{Q_{s4}})^{<\gamma_4>}$, $\mathbb{F}(\mathcal{A}_{Q_{s5}})^{<\gamma_5>}$ and $\mathbb{F}(\mathcal{A}_{Q_{s6}})^{<\gamma_6>}$ respectively.

Explicitly, this means that each of these functions has the same value when applied to a consecutive collection of numbers from its respective sequence. For example, we see that

$$F_6(1,1,1,1,1,1) = 8 \stackrel{?}{=} F_6(1,2,3,4,8,17) \tag{3.40}$$

$$= \frac{4624 + 2048 + 6936 + 2304 + 3264 + 1536 + 3672 + 1728}{3264} = 8 \tag{3.41}$$

The functions F_4 and F_5 have appeared in [21] in relation to the number theoretic properties of the Somos sequences e.g. one may compute the $j-$invariant of an elliptic curve associated with the sequences using the values of these functions. While the cluster algebra interpretation of the Somos sequences has been studied, the mutation invariant formulation of these functions is new.

These functions for general periodic quivers are studied in [38].

Chapter 4: First Properties and Construction of Invariants

In this chapter we prove some theorems about the abstract structure of mutation invariants, and show some basic constructions of them. We also do some computations which will be help full in the proof of our main theorem.

4.1 Basic properties of mutation invariant functions

First, we consider properties and constructions of invariants that can be used in general.

4.1.1 Trivial Invariants

Since it respects mutations, the map $\rho : \mathcal{A} \to \mathcal{X}$ gives a map

$$\rho^* : \mathbb{F}(\mathcal{X})^{\Gamma_\circ} \cup \{\infty\} \to \mathbb{F}(\mathcal{A})^{\Gamma_\circ} \cup \{\infty\} \tag{4.1}$$

by pullback. This is very useful for studying the invariants on the $\mathcal{A}$ space when there are nontrivial coefficients. However, since ρ is not surjective or injective in general, we cannot say much about the image of the set of $\mathcal{X}$ invariants under ρ^*.

Definition 4.1.1. A rational function in the coefficients of $\mathcal{A}$ is called a *trivial* $\mathcal{A}$ *invariant.*

These functions are clearly unchanged by the action of cluster modular group elements.

We can understand what functions pull-back to trivial $\mathcal{A}$ invariants via ρ^* as follows: any vector, $v = (v_1, \ldots, v_n)$ of the kernel of the exchange matrix gives such a function since $\rho^*(\prod x_i^{v_i})$ is a function just of frozen variables, see section 2.3 of [12]. These functions are called "Casimir" functions.

Definition 4.1.2. A $\mathcal{X}$ invariant that is also a Casimir function will be called a *Casimir $\mathcal{X}$ invariant.*

Example 4.1.1. The type A_n cluster ensembles have an interesting Casimir $\mathcal{X}$ invariant when n is odd. The cluster modular group of the type A_n cluster ensemble is $\mathbb{Z}/(n+3)\mathbb{Z}$, generated by γ. Take Q to be the quiver in figure 4.1 and let $G(x_1, \ldots, x_{2n+1}) = x_1 x_3 \ldots x_{2n+1}$. Then G is a trivial Casimir in $\mathbb{F}(\mathcal{X}_Q)^{\mathbb{Z}/m\mathbb{Z}}$, where $m = \frac{n+3}{2}$. G satisfies $\gamma(G) = G^{-1}$.

$$1 \longrightarrow 2 \longrightarrow \ldots \longrightarrow 2n+1$$

Figure 4.1: Quiver of type A_{2n+1}.

4.1.2 Invariants of Subensembles

The invariants of the $\mathcal{A}$ space and $\mathcal{X}$ space of an ensemble respect inclusions of "subensembles" in different ways.

Definition 4.1.3. Let $R^\mu \subset Q^\mu$ be a subquiver of the mutable portion of Q. Let R be the quiver obtained from Q by freezing all the nodes in $N(Q) - N(R^\mu)$. We consider $(\mathcal{A}_R, \mathcal{X}_R)$ to be a *subensemble* of $(\mathcal{A}_Q, \mathcal{X}_Q)$.

Let $\gamma_R = \{P_R, \sigma_R\}$ be an element of the cluster modular group of R that extends trivially to an element of the modular group of Q, i.e. there exists $\gamma_Q = \{P_R, \sigma_Q\}$ where $\sigma_Q|_R = \sigma_R$ and σ_Q does not permute any of the frozen nodes in R.

Remark 4.1.1. In all our applications, we will need to construct a unique extension of cluster modular group elements in Γ_R.

We can now state the following pair of propositions:

Proposition 4.1.1. $\mathbb{F}(\mathcal{A}_Q)^{<\gamma_Q>}$ is exactly $\mathbb{F}(\mathcal{A}_R)^{<\gamma_R>}$. Furthermore, evaluating the variables associated to the nodes in $N(Q) - N(R^\mu)$ at 1 gives a map of sets $\mathbb{F}(\mathcal{A}_Q)^{<\gamma_Q>} \cup \{\infty\} \to \mathbb{F}(\mathcal{A}_{R^\mu})^{<\gamma_R>} \cup \{\infty\}$.

Proposition 4.1.2. We have a natural inclusion $\mathbb{F}(\mathcal{X}_R)^{<\gamma_R>} \subset \mathbb{F}(\mathcal{X}_Q)^{<\gamma_Q>}$. Furthermore, evaluating the variables associated to nodes in $N(Q^\mu) - N(R^\mu)$ at 0 gives a surjection of sets $\mathbb{F}(\mathcal{X}_Q)^{<\gamma_Q>} \cup \{\infty\} \to \mathbb{F}(\mathcal{X}_R)^{<\gamma_R>} \cup \{\infty\}$.

Proof. The first proposition follows from the fact that the mutation rule on the $\mathcal{A}$ space only changes the variable being mutated. The first part of the second proposition follows since the $\mathcal{X}$ coordinates are unchanged when frozen variables are added. The only thing left to prove is that evaluating the nodes in $N(Q^\mu) - N(R^\mu)$ at 0 gives a map from $\mathbb{F}(\mathcal{X}_Q)^{<\gamma_Q>}$ to $\mathbb{F}(\mathcal{X}_R)^{<\gamma_R>}$. The surjection is obvious because of the inclusion of rings.

Let $r = \#N(R^\mu)$ and renumber the nodes of Q so that the first r nodes are the nodes in $N(R^\mu)$. Let $G_i(x_1, \ldots, x_n) = x_i$ and let $j = \sigma^{-1}(i)$. Then for any $i > r$, we have

$$\gamma_Q(G_i) = x_j h_j(x_1, \ldots, x_r) \tag{4.2}$$

, where h_j is some rational function that does not depend on any $x_i, i > r$. This follows since we never mutate at node i in γ_Q, and the $\mathcal{X}$ mutation rule only multiplies coordinates by functions of those which are mutated.

Let $K = \mathbb{F}[x_{r+1}, \ldots, x_n]$ and $L = \mathbb{F}(x_1, \ldots, x_r) = \mathbb{F}(\mathcal{X}_R)$. Then, γ_Q acts on any monomial term in K by multiplication by an element of L. Now we write any function, $g = \frac{p}{q} \in \mathbb{F}(\mathcal{X}_Q) = \mathbb{F}(x_1, \ldots, x_n)$, as a ratio of two polynomials in K with coefficients in L. Let $p_0, q_0 \in L$ be the constant terms of p and q. Now we may see that if g is invariant then we have

$$p\gamma_Q(q) = q\gamma_Q(p) \tag{4.3}$$

which implies by comparing constant terms that

$$p_0\gamma_Q(q_0) = q_0\gamma_Q(p_0) \tag{4.4}$$

and so we have that $\frac{p_0}{q_0} \in L$ is invariant.

□

Corollary 4.1.1. *The dimension of the field extension* $\mathbb{F}(\mathcal{X}_Q)^{<\gamma_Q>}/\mathbb{F}(\mathcal{X}_R)^{<\gamma_R>} \leq n - r$.

Proof. This follows from the proof of the previous theorem. Since γ_Q acts on each monomial in K independently by multiplication by elements in L, any basis of $\mathbb{F}(\mathcal{X}_Q)^{\langle\gamma_Q\rangle}/\mathbb{F}(\mathcal{X}_R)^{<\gamma_R>}$ can be written as a basis of monomials in K with coefficients in L. Now we may see that any particular monomial in K can only be associated with one independent invariant, since the ratio of two such invariants is in $\mathbb{F}(\mathcal{X}_R)^{\langle\gamma_Q\rangle}$. The corollary now follows from realizing that there are at most $n-r$ algebraically independent monomials in K.

□

Remark 4.1.2. The evaluation maps should be considered to be related to projective maps of varieties which have the corresponding invariant rings as function fields. However, it is not clear that these varieties actually exist.

Combining these two propositions together, we may state the following useful theorem.

Theorem 4.1.1. *Let Q be a quiver and let R be a quiver obtained by deleting some subset $D \subset N(Q)$ of the nodes of Q. Let $\mathcal{A}(D)$ and $\mathcal{X}(D)$ be the collections of $\mathcal{A}$ and $\mathcal{X}$ coordinates associated with the nodes in D. Suppose that there is an element $\gamma \in \Gamma_R$ such that γ does not mutate at any node in R adjacent to a node in D. Then there exists $\gamma_Q \in \Gamma_Q$ which restricts to γ_R and we have*

1. $\mathbb{F}(\mathcal{A}_Q)^{\langle\gamma_Q\rangle} = \mathbb{F}(\mathcal{A}_R)^{\langle\gamma_R\rangle}(\mathcal{A}(D))$
2. $\mathbb{F}(\mathcal{X}_Q)^{\langle\gamma_Q\rangle} = \mathbb{F}(\mathcal{X}_R)^{\langle\gamma_R\rangle}(\mathcal{X}(D))$

or to write simply

$$\mathbb{F}(\mathcal{A}_Q, \mathcal{X}_Q)^{\langle\gamma_Q\rangle} = \mathbb{F}(\mathcal{A}_R, \mathcal{X}_R)^{\langle\gamma_R\rangle}(D). \tag{4.5}$$

Proof. Our element γ_Q can be constructed from γ_R by extending the quiver isomorphism of γ_R to all of Q.

Since γ_R does not mutate at any node adjacent to the nodes in D, we may construct a new quiver R' where we freeze every node of R which is adjacent to any node in D and naturally think of γ_R as an element in $\Gamma_{R'}$. Then adding the nodes of D to R' as frozen nodes does not change the mutation structure of the ensemble associated with $\mathbb{F}'$ since the new nodes are only connected to frozen nodes. Adding these nodes simply adds coefficients which appear in every cluster, but do not effect the exchange relations in any way.

Now, since γ_R returns R to an isomorphic quiver, it also does the same for R'. Finally, we may find our desired isomorphism of Q by attaching the nodes in D to R' before and after applying γ_R and choosing the map which gives an isomorphism between these two quivers.

Denote by Q' the quiver obtained from Q by freezing all the nodes in D along with nodes adjacent to those in D. We have $\mu(Q') = \mu(R')$. Now we have

$$\mathbb{F}(\mathcal{A}_Q)^{\langle\gamma_Q\rangle} = \mathbb{F}(\mathcal{A}_{Q'})^{\langle\gamma_Q\rangle} = \mathbb{F}(\mathcal{A}_{R'})^{\langle\gamma_R\rangle}(\mathcal{A}(D)) = \mathbb{F}(\mathcal{A}_R)^{\langle\gamma_R\rangle}(\mathcal{A}(D)) \tag{4.6}$$

where the first and third equality follow from proposition 4.1.1 the second equality follows from our realization that the $\mathcal{A}$ coordinates for D are trivially invariant and that $A_{\tilde{R}}$ and $A_{R'}$ have the same mutation structure.

For the $\mathcal{X}$ space, we have by proposition 4.1.2

$$\mathbb{F}(\mathcal{X}_R)^{\langle\gamma_R\rangle} \subset \mathbb{F}(\mathcal{X}_Q)^{\langle\gamma_Q\rangle} \tag{4.7}$$

is a field extension of degree less than $\#D$ by corollary 4.1.1. We can also can see that

$$\mathbb{F}(\mathcal{X}_R)^{\langle\gamma_R\rangle}(\mathcal{X}(D)) \subset \mathbb{F}(\mathcal{X}_Q)^{\langle\gamma_Q\rangle}. \tag{4.8}$$

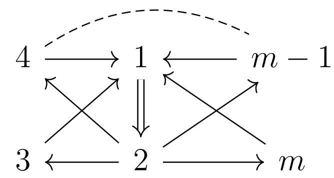


Figure 4.2: Quiver with double edge and oriented cycles.

This must be an equality since we may write a basis of $R(\mathcal{X}_Q)^{\langle\gamma_Q\rangle}$ over $\mathbb{F}(\mathcal{X}_R)^{\langle\gamma_R\rangle}$ in terms of monomials in $\mathcal{X}(D)$, but every such monomial is contained in $\mathbb{F}(\mathcal{X}_R)^{\langle\gamma_R\rangle}(\mathcal{X}(D))$.

□

Example 4.1.2. Let R be the quiver shown in figure 4.2, and let Q be any quiver which is obtained by adding nodes to R which are only connected to nodes $3, \ldots, m$. Let D be the set of these new nodes. We have $\gamma = \{1, (12)\} \in \Gamma_R, \Gamma_Q$. Then theorem 4.1.1 implies that

$$\mathbb{F}(\mathcal{A}_Q, \mathcal{X}_Q)^{\langle\gamma\rangle} = \mathbb{F}(\mathcal{A}_R, \mathcal{X}_R)^{\langle\gamma\rangle}(D). \tag{4.9}$$

4.2 Invariants Of Ensembles Associated With Surfaces

Our main source of invariants of cluster ensembles associated with surfaces will be from trace functions associated to closed curves on S. Our formulas in section 2.5.3 show how we may write the monodromy operator associated with traversing a closed curve, δ, in terms of the $\mathcal{A}$ and $\mathcal{X}$ coordinates of a triangulation of S. This allows us to assign a trace function of the $\mathcal{A}$ and $\mathcal{X}$ coordinates for each closed curve δ on S. The formula for a trace function only depends on the order that δ touches the arcs of the triangulation. Therefore, this formula must remain the same if we

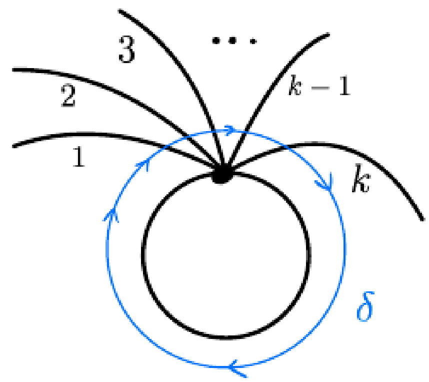


Figure 4.3: The arcs near a boundary with one marked point.

apply a mapping class to S which fixes δ, and it thereby gives rise to an invariant function.

4.2.1 Calculating trace functions

It will be important to know what the exact formula for the trace of a monodromy operator for a closed loop on S is in terms of the $\mathcal{A}$ and $\mathcal{X}$ coordinates in a special case.

Let S be a marked surface and assume that S has a boundary component with exactly one marked point. Let δ be a closed curve on S homotopic to this boundary component and let $\{e_1, e_2, \ldots, e_k\}$ be the arcs in an ideal triangulation of S which touch the marked point on this boundary, see figure 4.3.

We can write an explicit formula for the trace of the monodromy operator associated with δ in terms of the $\mathcal{X}$ coordinates. Recall the matrices associated to arcs in Δ and the matrices R and L from section 2.5.3. Call the matrices associated to $\{e_1, e_2, \ldots, e_k\}$ $\{X_1, \ldots, X_k\}$. Choosing the orientation of δ shown in figure 4.3, we compute that

$$\rho(\delta) = X_1 R X_2 R \dots X_k. \tag{4.10}$$

With a little bit of work, we find

$$\mathrm{Tr}\left(\psi(\delta)\right) = \frac{1}{\sqrt{x_1 x_2 \dots x_k}}\Big(\sum_{i=1}^{i=k}\prod_{j=1}^{j=i} x_j + 1\Big). \tag{4.11}$$

This is a rational function in the $\mathcal{X}$ coordinates exactly when the product under the square root is a square.

4.2.2 Other types of surface invariants

We can produce $\mathcal{A}$ invariants by clever use of the following fact about lambda lengths. Let S be a hyperbolic surface decorated with horocycles, $t \in \Delta$ be a triangle, and let h_3 be the segment of the horocycle bounded between edges 1 and 2, see figure 4.4. Then the hyperbolic length h_3 is $\frac{\lambda_3}{\lambda_1\lambda_2}$, see lemma 4.4 of [31].

We can use this formula to write a function that evaluates to the length of a horocycle about a marked point on our surface in terms of the $\mathcal{A}$ coordinates of a given triangulation. This function is preserved under the action of mapping classes since the total length of the horocycle and the topology of the triangulation are preserved. It is important to notice that this expression will not be invariant under any mapping class that permutes this marked point with another.

We can construct an invariant for the entire mapping class group by summing all of the lengths of horocycle segments on S. Let e_{t1}, e_{t2}, e_{t3} be the three edges of

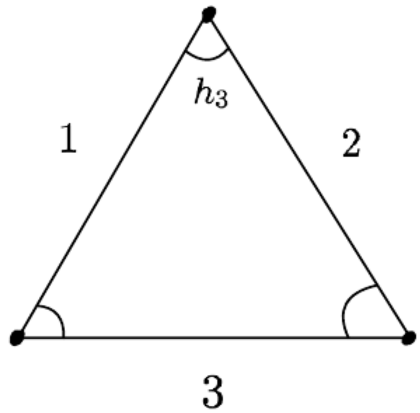


Figure 4.4: An ideal hyperbolic triangle with horocycle segments around its three vertices.

a triangle $t \in \Delta$. This invariant is given by the formula

$$\sum_{t\in\Delta} \frac{a_{e_{t1}}^2 + a_{e_{t2}}^2 + a_{e_{t3}}^2}{a_{e_{t1}} a_{e_{t2}} a_{e_{t3}}}. \tag{4.12}$$

We can easily produce Casimir $\mathcal{X}$ invariants associated to each puncture on S by taking the product of the $\mathcal{X}$ coordinates associated to edges touching the puncture. These are the functions $p_{\mathcal{X}}$ mentioned in 2.17.

Example 4.2.1. We can interpret the invariants of example 3.2.4 using a surface. The cluster ensemble associated to $S = S_{0,0,4,0}$, a four punctured sphere, is of type $D_4^{(1,1)}$. If we take a triangulation of S that topologically looks like a tetrahedron, then the quiver associated with this triangulation is Q_4 from figure 3.8 and the length of a particular horocycle around a puncture is given by a function like F_{456}. The function G_{456} is the $\mathcal{X}$ invariant associated a puncture on S. We have that

$$\mathrm{PMCG}(S) = \mathbb{Z} * \mathbb{Z} \tag{4.13}$$

and hence we have that these functions are invariant under the subgroup claimed.

Remark 4.2.1. We note that specifying a values for horocycle lengths and fixin $p_{\mathcal{X}} = 1$ for every puncture are the relations between $T_{\mathcal{A}}(S), T_{\mathcal{X}}(S)$ and $T(S)$. We may think of these functions as being the "gluing equations" for a hyperbolic structure on S.

One can interpret the invariance of these expressions as a way of stating that the action of the mapping class group does not change the topological structure of the surface and hence does not change the conditions that must be satisfied for there to be a hyperbolic structure on it.

4.3 Sequences of $\mathcal{A}$ coordinates from Dehn twists

As we saw in example 3.1.1, the invariant function (the trace function) of the $\widetilde{A}_1$ affine ensemble was essential for understanding the sequence of cluster variables obtained along the mutation path we were considering. We will need to understand the sequences of $\mathcal{A}$ coordinates found by doing Dehn twists on general surfaces. In most cases, we can arrange that the closed arc which we are twisting lives in a cylinder on S and thereby simply apply the analysis of 3.1.1. The only new case we will need to consider is when our surface is a pair of pants with one marked point on one of its boundaries. This surface is not a marked surface, since there are no marked points on two of its boundaries, but we may still consider its skein algebra.

Let S be this surface, shown in figure 4.5. Two arcs from the marked point to itself is shown, call these arcs a_0 and b_0. Let A, B, K be the elements of the skein algebra of S associated with closed curves homotopic to the three boundaries as

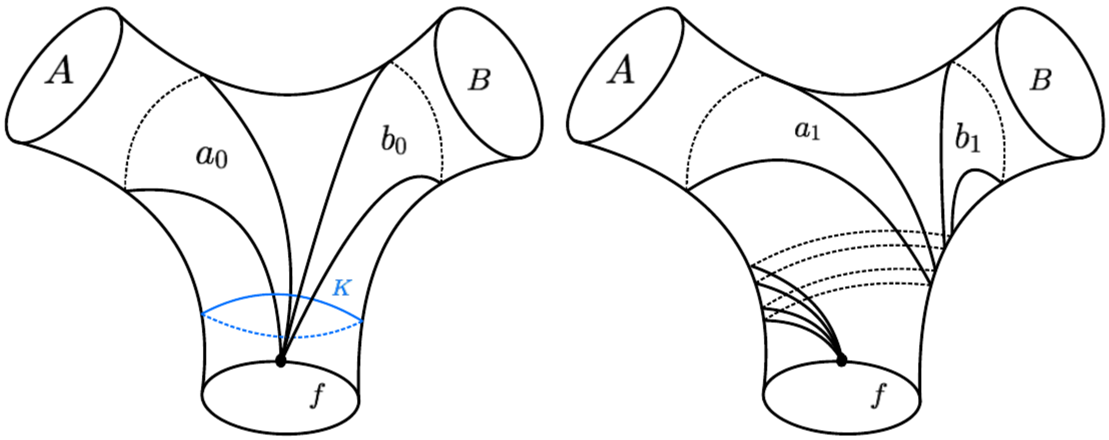


Figure 4.5: A pair of pants with one marked point before and after performing a Dehn twist about the bottom boundary

shown in the figure. Furthermore let f be the arc from the marked point to itself which is homotopic to K. Let $\{a_n\}, \{b_n\}$ be the sequence of arcs obtained after performing n clockwise Dehn twists about K. Using several Skein relations, we may first compute that

$$a_0 K = b_0 + b_1 + fA \tag{4.14}$$

see figure 4.6. Multiplying through by K and using similar skein relations, we find

$$a_0 K^2 = b_0 K + b_1 K + fAK = a_{-1} + a_0 + fB + a_0 + a_1 + fB + fAK. \tag{4.15}$$

Thus, we find that $\{a_n\}$ again satisfies a linear recurrence:

$$a_{n+1} = a_n(K^2 - 2) - a_{n-1} - f(AK + 2B). \tag{4.16}$$

We may solve this recurrence explicitly. Let $K = \lambda + \lambda^{-1}$ and let

$$w = a_1 - a_0\lambda^{-2} + \frac{f(AK + 2B)}{K^2 - 4}(\lambda^{-2} - 1) \tag{4.17}$$

$$w' = -a_1 + a_0\lambda^2 - \frac{f(AK + 2B)}{K^2 - 4}(\lambda^2 - 1). \tag{4.18}$$

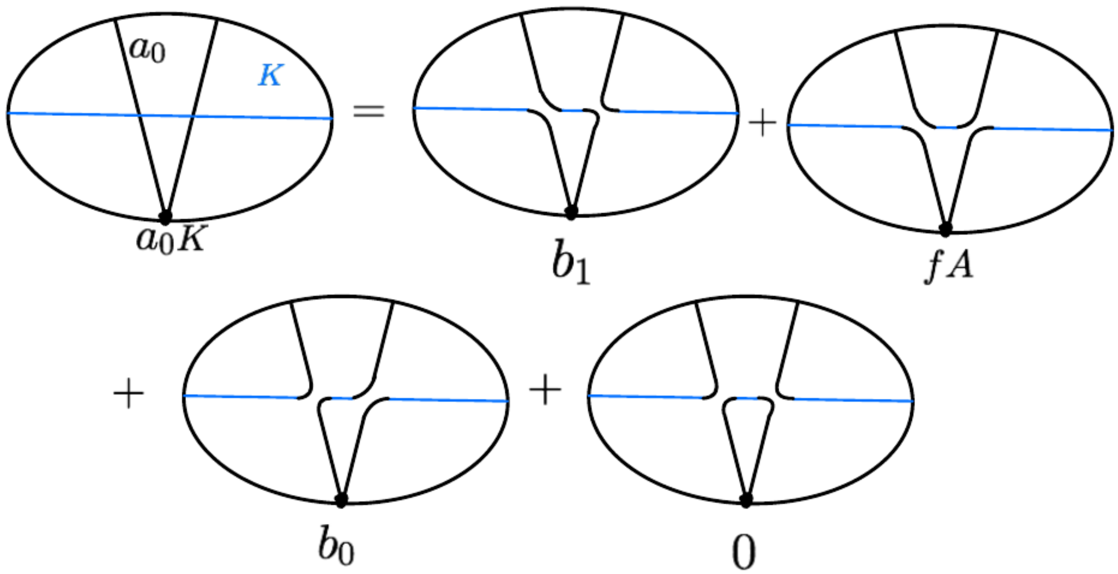


Figure 4.6: The Skein relations obtained from computing a_0K. This is a zoomed in view near the points where the crossings occur

Then we have

$$a_n = \frac{1}{\lambda^2 - \lambda^{-2}}(w\lambda^{2n} + w'\lambda^{-2n}) + \frac{f(AK+2B)}{K^2-4}. \tag{4.19}$$

Furthermore, using more skein relations we may compute that

$$ww' = \frac{K^2(A^2+B^2+K^2+ABK-4)}{K^2-4} \tag{4.20}$$

$$w - w' = 2a_1 - a_0(K^2-2) + f(AK+2B) = a_1 - a_{-1}. \tag{4.21}$$

Thus w and $-w'$ are roots of the following polynomial equation

$$w^2 - (a_1 - a_{-1})w - \frac{K^2(A^2+B^2+K^2+ABK-4)}{K^2-4} = 0. \tag{4.22}$$

This analysis can be applied whenever we are in a situation where we have an $\mathcal{A}$ coordinate in a surface cluster algebra which lives in a pair of pants on S. This will allow us to compute sequences of $\mathcal{A}$ coordinates for Dehn twists which do not live in cylinders on S. We simply need to evaluate the trace functions for K, A, B in terms of the other $\mathcal{A}$ coordinates, then we may use our explicit formula to compute the sequence.

Chapter 5: Surface Invariants

Now we will state and prove our main theorems. We concentrate first on ensembles associated with surfaces, but we show that the same proof ideas work for mutation finite cluster ensembles as well.

5.1 Invariants of Surface Cluster Ensembles

In this section, we will classify the invariants in the situation where our cluster ensemble is associated with any surface and we are considering invariants with respect to the action of a single Dehn Twist.

Let $S = S_{g,b,p,n}$ be a marked surface. We will construct invariants rings for the cluster ensembles associated with ideal triangulations of S by using particular initial triangulations obtained by "excising" our given Dehn twist.

Definition 5.1.1. We say a simple closed curve, δ on S is *genus 0* on S if cutting along δ leaves S connected, or if we can find a marked point or puncture on both connected components of S after cutting.

When δ is genus 0, we may pick an ideal triangulation of S which cuts excises a cylinder containing δ in the following way: pick any two, not necessarily distinct marked points or punctures on S, p_1, p_2 along with curves c_1 from p_1 to one side of

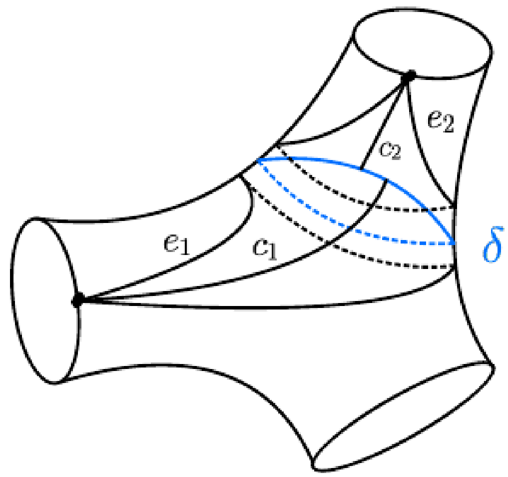


Figure 5.1: The construction of arcs which excise a given closed curve.

δ and c_2 to the other side of δ. We can always find these curves since δ is excisable. We then begin a triangulation of S by first using the arc e_1 obtained by following c_1, traversing δ and then following c_1 back to p_1 and a similarly constructed arc e_2. This procedure is shown in figure 5.1.

When δ is not genus 0, we may still find one marked point and curve coming to δ from one side. In this case we can construct e_1 and we cut out a surface of genus g from S with one boundary. We say that δ is genus g, and we may again complete

Definition 5.1.2. A triangulation of S which contains the arcs e_i arcs is called an *excising triangulation* for δ The arcs of an excising triangulation which intersect δ are called the *interior arcs* and the arcs who's associated quiver nodes are connected to the nodes of the interior arcs are called *boundary arcs.*

See figure 5.2 for a examples of various genus curves on a surface.

Every simple closed curve on S and choice of excising triangulation determines

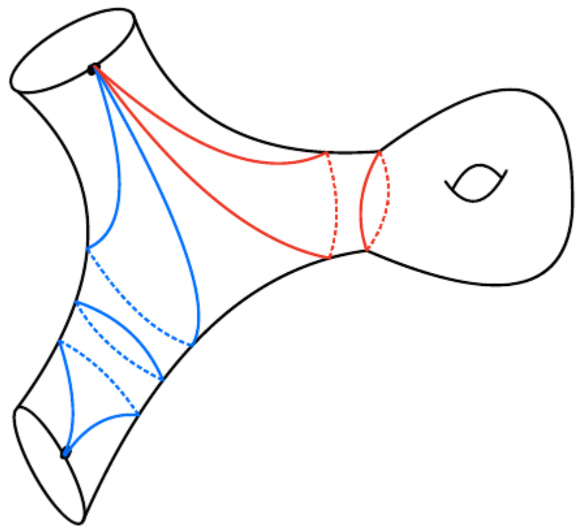


Figure 5.2: The blue closed curve is genus 0, and the red closed curve is genus 1.

two important subensembles. We denote by $(\mathcal{A}_\delta, \mathcal{X}_\delta)$ the subensemble obtained by freezing all of the nodes other than the interior arcs. When delta is genus 0, this subensemble is of type $\widetilde{A_1}$, and we say that it is of genus g type otherwise.

We denote the subensemble obtained by freezing all of the nodes other than the interior arcs and boundary arcs by $(\mathcal{A}_\delta^b, \mathcal{X}_\delta^b)$. We have

$$(\mathcal{A}_\delta, \mathcal{X}_\delta) \subset (\mathcal{A}_\delta^b, \mathcal{X}_\delta^b) \subset (\mathcal{A}, \mathcal{X}) \tag{5.1}$$

.

Definition 5.1.3. We call the coefficients of $\mathcal{A}_\delta$ for any choice of excising triangulation the invariant $\mathcal{A}$ coordinates for δ

Let m be the total number of interior and boundary arcs. Suppose that δ is genus 0 on S. Then there are 2 interior arcs of our triangulation and between 2 and 4 boundary arcs. Number nodes of the quiver associated with our excising triangulation with the first being the interior arc on the source of the double edge, then the second interior arc, then the boundary arcs, then the rest.

Definition 5.1.4. The functions $x_3(x_2(x_1+1)+1), \ldots, x_m(x_2(x_1+1)+1), x_{m+1} \ldots, x_N$ are called *invariant $\mathcal{X}$ coordinates* for our excising triangulation when δ has genus 0 on S

Suppose now that δ is genus $g > 0$. There are now 1 or 2 boundary arcs. let x_I be the product of the $\mathcal{X}$ coordinates associated with the interior arcs. Recall that the total number of interior arcs is $6g-2$, and number the nodes of the quiver with the boundary arcs first, then interior, then the rest.

Definition 5.1.5. The functions $x_I x_1, \ldots, x_I x_{m-(6g-2)}, x_{m+1}, \ldots x_N$ are called the *invariant $\mathcal{X}$ coordinates* for our excising triangulation when δ has genus g on S.

Definition 5.1.6. In any case, the $\mathcal{A}$ coordinate functions for arcs of our excising triangulation which do not intersect δ (non interior arcs) are the *invariant $\mathcal{A}$ coordinates* for our excising triangulation.

Definition 5.1.7. The closed curves that have representatives that are fully contained in the excised surface are called the *excised closed curves*.

In figure 5.2, the excised curve for the blue curve are simply curves which wrap around the excised cylinder a number of times. The excised curves for excising the red curve are curves which are contained in the genus 1 surface.

Let γ be the cluster modular group element corresponding to a Dehn twist about δ. Our main theorem can be stated as follows:

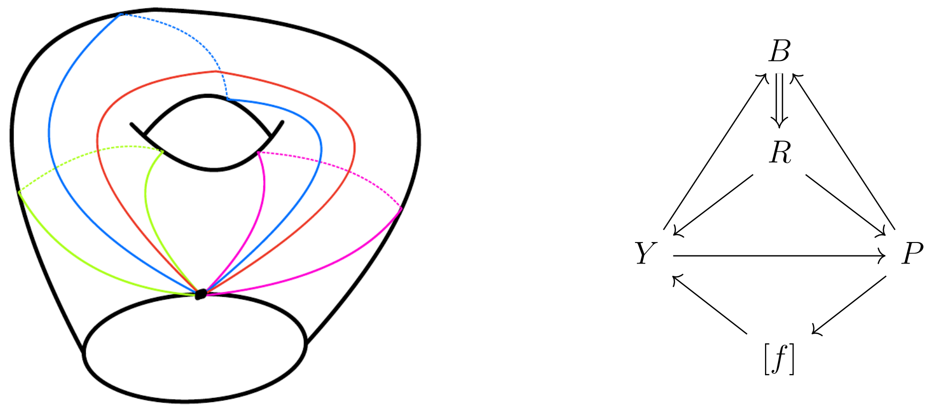


Figure 5.3: The triangulation and associated quiver when $g = 1$ quiver

Theorem 5.1.1. *1. The ring $\mathbb{F}(\mathcal{X}_S)^{\langle\gamma\rangle}$ is generated by traces of monodromy operators of excised closed curves on S and invariant $\mathcal{X}$ coordinates for an excising triangulation of δ.*

2. The ring $\mathbb{F}(\mathcal{A}_S)^{\langle\gamma\rangle}$ is generated by traces of monodromy operators of excised closed curves on S and invariant $\mathcal{A}$ coordinates of an excising triangulation of δ.

We note that in the $\mathcal{X}$ case, not every trace function is itself a rational function because of the square roots that we have seen 4.2.1. However, the generators we need are simply those trace functions which are rational.

Before tackling the main theorem, we will treat the case where S is either a cylinder or surface of genus g with one boundary component. This surface for $g = 0, g = 1, g > 1$ is shown in figure 5.3. We also fix a triangulation Δ and associated quiver Q of each of these surfaces, shown in figures 5.3,5.5 and 5.4

Lemma 5.1.1. *Let $S = S_{0,2,0,2}$ be an annulus with one marked point on each boundary or let $S = S_{g,1,0,1}$ be a surface of genus g with one boundary component and a*

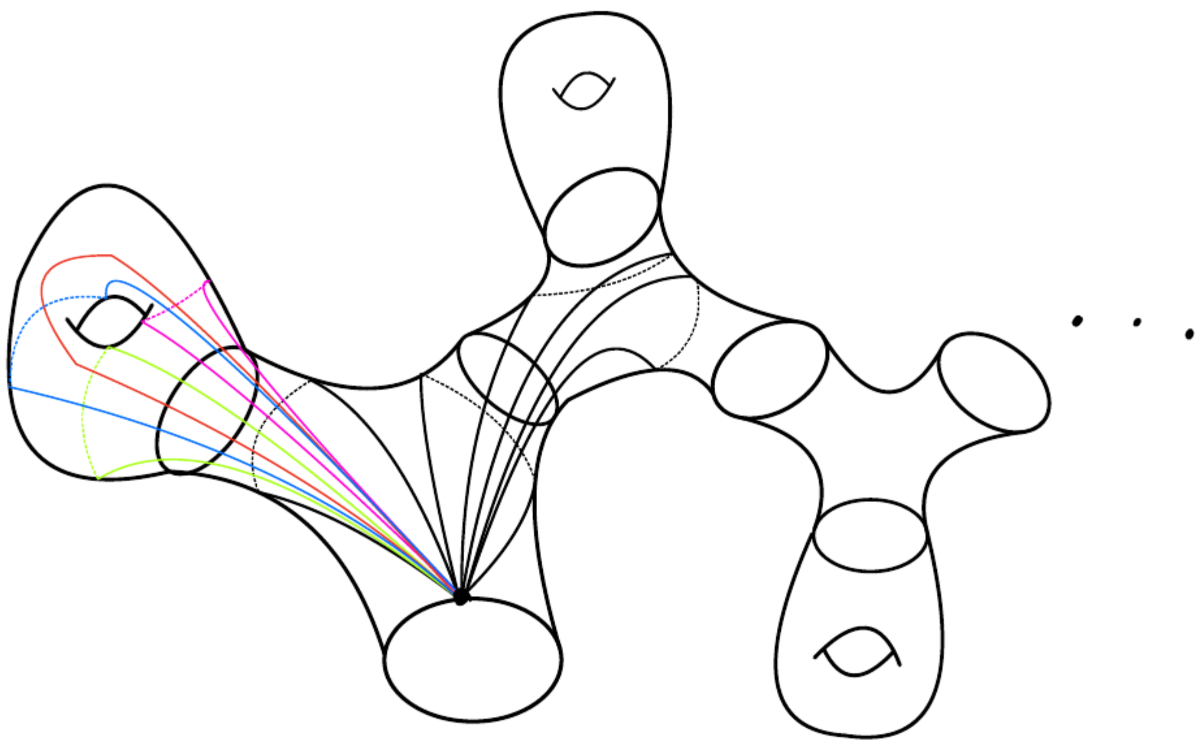


Figure 5.4: The triangulation for when $g > 1$. Each of the g handles is associated with 4 arcs of the triangulation as figure 5.3, and each of non-handle pairs of pants is associated with 2 arcs if the triangulation.

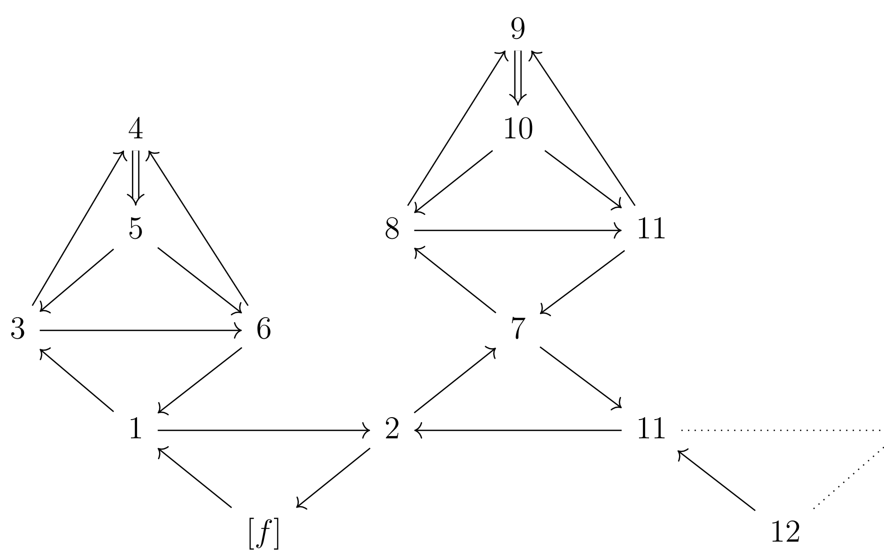


Figure 5.5: The quiver Q when $g > 1$

single marked point on the boundary. Let δ be a simple closed curve on S homotopic to the boundary component. Let $\gamma \in \Gamma_S$ be the cluster modular group element corresponding to a Dehn twist about δ.

1. *The ring $\mathbb{R}(\mathcal{X}_S)^{\langle\gamma\rangle}$ is generated by traces of of monodromy operators about simple non-excisable closed curves and by squares of traces of monodromy operators about simple excisable closed curves on S.*

2. *Then the ring $\mathbb{R}(\mathcal{A}_S)^{\langle\gamma\rangle}$ is generated by traces of monodromy operators about simple closed curves on S and frozen variables.*

Proof. We will first treat the $\mathcal{A}$ case with $g > 0$. Let $m = 6g - 2$ and let $h(x_1, \ldots, x_{m+1}) \in \mathbb{R}(\mathcal{A}_S)^{\langle\gamma\rangle}$ be an invariant function. Let $\{a_1, \ldots, a_{6g-2}, f\}$ be our collection of $\mathcal{A}$ coordinates for Δ triangulation of S show in figure. The coordinate f is the frozen variable.

Then let $\{a_{i_n}\}$ be the sequence of coordinates obtained from acting on our initial set by γ^n. The invariance of h implies that

$$h(a_1, \ldots, a_m, f) = h(a_{1_n}, \ldots, a_{m_n}, f). \tag{5.2}$$

. Let $\tilde{h}$ be defined by

$$\tilde{h}(\frac{1}{y_1}, \frac{y_2}{y_1}, \frac{y_3}{y_1}, \ldots, \frac{y_m}{y_1}, y_{m+1}) = h(y_1, \ldots, y_{m+1}). \tag{5.3}$$

Then from the invariance of h we have

$$h(a_1, \ldots, a_m, f) = \lim_{n\to\infty} h(a_{1_n}, \ldots, a_{m_n}, f) = \tilde{h}(0, \lambda_2, \ldots, \lambda_m, f) \tag{5.4}$$

where $\lambda_i = \lim_{n\to\infty} \frac{y_i}{y_1}$. We can compute λ_i explicitly using the analysis of section 4.3.

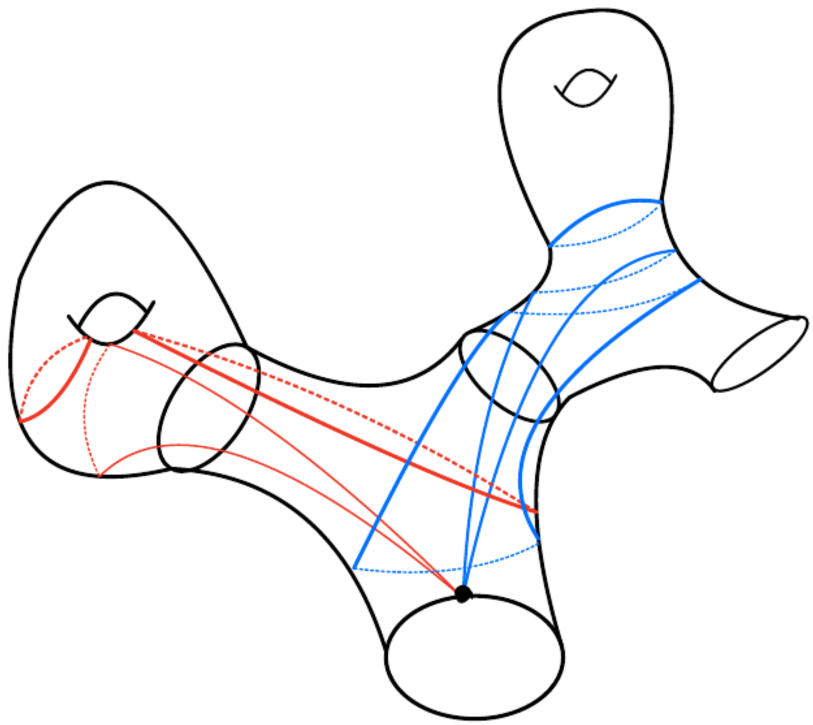


Figure 5.6: Two arcs in Δ along with the pants that they live in.

Each of the arcs in Δ can be seen to live in a pair of pants in the following way. We may generate each arc by following a path from the marked point to a closed curve, A_i, then traveling around this closed curve and finally going back to the marked point. This arc lives in a pair of pants with boundaries A_i, B_i, K where $K = \partial S$ and B_i is the closed curve which is homotopic to the generator of $H_1(S)$ equal to $[A] + [K]$. This situation is shown in figure 5.6.

In the continuation, we will use A_i, B_i, K to refer interchangeably to the elements of the skein algebra of S or the traces of monodromy operators associated with their respective closed curves. Let $K = \lambda + \lambda^{-1}$ as before. By our analysis of sequences of $\mathcal{A}$ coordinates we have

$$a_{i_n} = \frac{1}{\lambda^2 - \lambda^{-2}}(w_i\lambda^{2n} + w_i'\lambda^{-2n}) + \frac{f(A_iK + 2B_i)}{K^2 - 4} \tag{5.5}$$

where

$$w_i = a_{i_1} - a_{i_0}\lambda^{-2} + \frac{f(A_iK + 2B_i)}{K^2 - 4}(\lambda^{-2} - 1) \tag{5.6}$$

$$w_i' = -a_{i_1} + a_{i_0}\lambda^{2} - \frac{f(A_iK + 2B_i)}{K^2 - 4}(\lambda^{2} - 1). \tag{5.7}$$

Thus, we find that

$$\lambda_i = \frac{w_i}{w_1} \tag{5.8}$$

Thus, any invariant function only depends on $\{\lambda_2, \dots, \lambda_m, f\}$. Thus our field of invariants contains all of the rational functions in the lambda's which are also rational functions in the $\mathcal{A}$ coordinates. Clearly, w_i is an element in the field extension $\mathbb{R}(\mathcal{A}_S)(\lambda)$ over $\mathbb{R}(\mathcal{A}_S)$. Let $T \subset \mathbb{R}(\mathcal{A}_S)$ be the field generated by traces of closed curves on S and f. We will show that each λ_i is an element in $T(\lambda)$. The following tower of fields shows the situation:

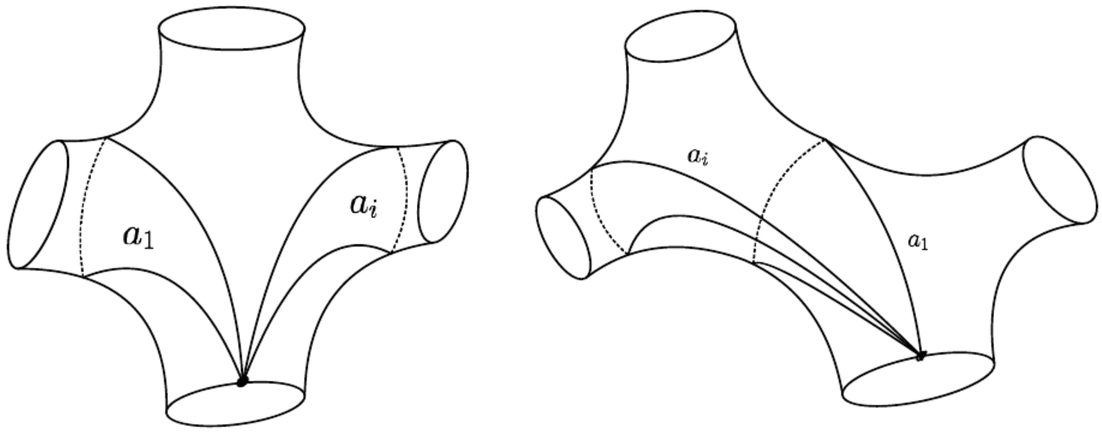
$$\begin{array}{ccccc} & & \mathbb{R}(\mathcal{A}_S)(\lambda) & & \\ & \overset{\infty}{\diagup} & & \overset{2}{\diagdown} & \\ \lambda_i \overset{?}{\in} T(\lambda) & & & & h \in \mathbb{R}(\mathcal{A}_S) \\ & \underset{2}{\diagdown} & & \underset{\infty}{\diagup} & \\ & & T & & \end{array} \tag{5.9}$$

Let $\sigma : \mathbb{R}(\mathcal{A}_S)(\lambda) \to \mathbb{R}(\mathcal{A}_S)(\lambda)$ be the field automorphism which sends λ to λ^{-1}. Then we have

$$\sigma(w_i) = -w_i' = -w_i^{-1}\left(\frac{K^2(A_i^2 + B_i^2 + K^2 + A_iB_iK - 4)}{K^2 - 4}\right) \tag{5.10}$$

and so

$$\sigma(\lambda_i) = \frac{w_i'}{w_1'} = \lambda_i^{-1}\left(\frac{A_i^2 + B_i^2 + K^2 + A_iB_iK - 4}{A_1^2 + B_1^2 + K^2 + A_1B_1K - 4}\right). \tag{5.11}$$



(a) a_i is separate from a_1

(b) a_i is contained in a_1

Figure 5.7: The two possibilities for the arrangement of arcs a_1 and a_i

Thus $\lambda_i\sigma(\lambda_i) \in T$. We wish to now prove that $\lambda_i + \sigma(\lambda_i) \in T$. This calculation is more involved. There are two possibilities for how the arcs a_1 and a_i are situated on S. a_i is either contained in a_1 or is separate, as figure 5.7 shows

We treat the separate case here. The non separate case follows since if a_i is contained in a_1 then we can find an arc which is separate from both, b, and write

$$\frac{a_i}{a_1} = \frac{a_i}{b}\frac{b}{a_1}. \tag{5.12}$$

For the sake simplifying notation lets write $a_n = a_{1_n}$,$c_n = a_{i_n}$, $w_1 = w_a, w_i = w_c$ and $A_1 = A, B_1 = B, A_i = C, B_i = D$, as figure 5.8 shows.

We need to compute $W := w_a w_c' + w_a' w_c$. From our formulas we first find

$$W = -2a_1c_1 - 2a_0c_0 + (K^2-2)(a_1c_0 + a_0c_1) - (a_1 + a_0)f(CK + 2D)$$
$$-(c_1 + c_0)f(AK + 2B) + \frac{f^2(AK + 2B)(CK + 2D)}{K^2 - 4}.$$

Next we will calculate $K^2(a_1c_0 + a_0c_1)$ using skein relations. There are essentially two generic possibilities for how the arcs $a_0$and c_0 are situated on S. The arc

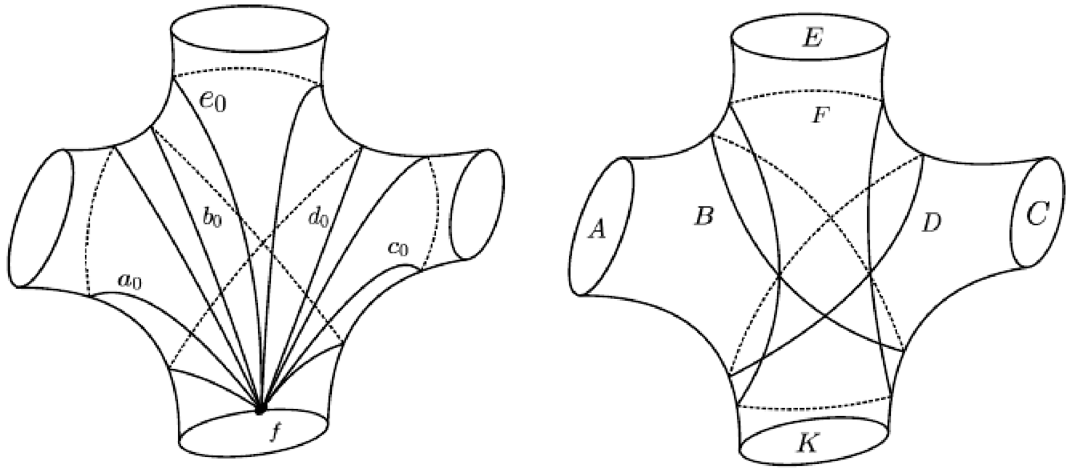


Figure 5.8: The collections of arc and closed curves for the calculation of W in the separate case

c_0 is either contained in a_0, like the four colored arc of figure 5.4, or they are sep We will treat the first case shown in figure 5.8

For the first case, we compute

$$K(a_0c_1) = b_1c_1 + a_0d_0 + f(c_1A + e_1 + a_0C) \tag{5.13}$$

$$K(a_1c_0) = a_1d_0 + b_1c_0 + f(a_1C + e_0 + c_0A + a_0C + b_1D) \tag{5.14}$$

$$+ f^2(AD + CB + E + KF + KAC). \tag{5.15}$$

So we have

$$K(a_1c_0 + a_0c_1) = (c_0 + c_1)fA + (a_0 + a_1)fC \tag{5.16}$$

$$+ b_1(c_1 + c_0 + fD) + d_0(a_0 + a_1) + a_0fC + (e_0 + e_1)f \tag{5.17}$$

$$+ f^2(AD + CB + E + KF + KAC). \tag{5.18}$$

Using skein relations for Kd_0 and Kb_1 we find

$$K(a_1c_0 + a_0c_1) = (c_0 + c_1)fA + (a_0 + a_1)fC \tag{5.19}$$

$$+ 2b_1d_0K - d_0fB + a_0fC + (e_0 + e_1)f \tag{5.20}$$

$$+ f^2(AD + CB + E + KF + KAC). \tag{5.21}$$

And using a different skein relation for d_0fB we find

$$d_0fB = a_0fC + (e_0 + e_1)f + f^2E. \tag{5.22}$$

So now multiplying through by K and using the above, we have

$$K^2(a_1c_0 + a_0c_1) = (c_0 + c_1)fKA + (a_0 + a_1)fCK \tag{5.23}$$

$$+ 2b_1d_0K^2 + f^2(AD + CB + KF + KAC). \tag{5.24}$$

Lastly we have

$$b_1d_0K^2 = (a_0 + a_1 + fB)(c_0 + c_1 + fD) \tag{5.25}$$

$$= a_0c_0 + a_1c_1 + a_0c_1 + a_1c_0 + (c_0 + c_1)fB + (a_0 + a_1)fD + f^2BD. \tag{5.26}$$

So finally we have

$$K^2(a_1c_0 + a_0c_1) = (c_0 + c_1)(fKA + 2B) + (a_0 + a_1)(fCK + 2D) \tag{5.27}$$

$$+ 2a_0c_0 + 2a_1c_1 + 2a_0c_1 + 2a_1c_0 \tag{5.28}$$

$$+ f^2(AD + CB + KF + KAC + 2BD). \tag{5.29}$$

At long last we have

$$W = f^2(AD + CB + KF + KAC + 2BD) + \frac{2f^2(AK + 2B)(CK + 2D)}{K^2 - 4}. \tag{5.30}$$

Thus λ_i satisfies a quadratic polynomial equation with coefficients in T, and is not fixed by σ. Moreover, any other automorphism which moves λ must also move λ_i. Thus $T(\lambda_i) = T(\lambda)$ for all i and so $h \in T(\lambda)$. Then since $h \in \mathbb{R}(\mathcal{A}_S)$ is fixed by σ, we have $h \in T$.

The $\mathcal{A}$ case when $g = 0$ follows by essentially the same argument, only simpler.

Now we will treat the $\mathcal{X}$ case. We can see that in each case the map $\rho^* : \mathbb{R}(\mathcal{X}) \to \mathbb{R}(\mathcal{A})$ is injective since there are are no punctures and there is only one marked point on each boundary. Furthermore, one may check that this map is still injective even if we remove the variables associated with the frozen nodes.

Thus we may compute the invariant ring by determining which functions pull back to $\mathcal{A}$ invariants after evaluating the frozen variables at 1. Clearly, we may obtain all such functions by pulling back trace functions in the $\mathcal{X}$ coordinates, since the only other elements of the $\mathcal{A}$ invariant ring are generated by the frozen variables.

Now, we will determine which closed curves have traces which are rational functions in the $\mathcal{X}$ coordinates. Let θ be a closed curve on S which is excisable. Since there is only one marked point on S, these curves are exactly the closed curves which leave S connected after cutting. We may pick a triangulation of S which only touches θ once, since we may find two paths from the marked point to ether side of θ and pick the arc which concatenates these paths. Based on the analysis of section 4.2.1, the trace of the monodromy operator associated with θ must have a square root in the denominator when computed with a triangulation which has an arc which only touches it once.

Thus the only closed skeins which have trace functions that are rational functions in the $\mathcal{X}$ coordinates are those for which have even intersection number with every arc. This completes the theorem on the $\mathcal{X}$ space.

□

Now we are ready to prove the general case.

Proof. First, we pick an excising triangulation of δ. We will first compute the ring of invariants for $(\mathcal{A}^b_\delta, \mathcal{X}^b_\delta)$ associated with this triangulation. Let $(\mathcal{A}', \mathcal{X}')$ be the cluster ensemble obtained by using the same quiver associated with $(\mathcal{A}^b_\delta, \mathcal{X}^b_\delta)$ but with all of the frozen node removed. Call by D the set of these nodes. We can find a mutation sequence representing γ which does not mutate at any of the boundary arcs, since we could cut along the boundary arcs to make a surface with them as actual boundaries and consider γ in this surface.

Thus by theorem 4.1.1 we have

$$\mathbb{R}(\mathcal{A}_S, \mathcal{X}_S)^{\langle\gamma\rangle} = \mathbb{R}(\mathcal{A}^b_\delta, \mathcal{X}^b_\delta)^{\langle\gamma\rangle}(D). \tag{5.31}$$

We only need to compute $\mathbb{R}(\mathcal{A}^b_\delta, \mathcal{X}^b_\delta)^{\langle\gamma\rangle}$

Lemma 5.1.1 exactly computes $\mathbb{R}(\mathcal{A}_\delta, \mathcal{X}_\delta)^{\langle\gamma\rangle}$. The theorem on the $\mathcal{A}$ space now follows from proposition 4.1.1 applied to $\mathcal{A}_\delta \subset \mathcal{A}^b_\delta$.

The invariant $\mathcal{X}$ functions for δ defined in 5.1.5 can be checked to be invariant under the action of γ. Since these invariant $\mathcal{X}$ coordinates are degree 1 in the boundary $\mathcal{X}$ coordinates, theorem 4.1.2 implies that these functions together with the elements of $\mathbb{R}(\mathcal{X}_\delta)^{\langle\gamma\rangle}$ must generate all of $\mathbb{R}(\mathcal{X}^b_\delta)^{\langle\gamma\rangle}$. This completes the theorem on the $\mathcal{X}$ space □

The following corollary follows from direct inspection.

Corollary 5.1.1. *The invariant ring for a Dehn twist about δ on $\mathcal{A}_S$ is exactly the the subalgebra of the skein algebra of S consisting of elements corresponding to skeins which do not intersect δ*

5.2 Invariants on Affine, Doubly Extended and Exotic Ensembles

We will give a classification of the invariants for analogues of Dehn twists on affine type and doubly extended type cluster ensembles with trivial coefficients. These classifications will all follow from the following corollary of theorem 5.1.1

Following example 4.1.2, let R be the quiver shown in figure 4.2, and let Q be any quiver which is obtained by adding nodes to R which are only connected to nodes $3, \ldots, m$. Let D be the set of these new nodes. We have $\gamma = \{1, (12)\} \in \Gamma_R, \Gamma_Q$. Let

$$G(x_1, x_2) = \frac{(x_2(x_1+1)+1)^2}{x_1 x_2} \tag{5.32}$$

$$F(a_1, a_2, a_3, \ldots, a_m) = \rho^*\left(\sqrt{G}\right) = \frac{a_1^2 + a_2^2 + a_3 \cdots a_m}{a_1 a_2} \tag{5.33}$$

Then the proof of theorem 5.1.1 and theorem 4.1.1 implies the following corollary:

Corollary 5.2.1.

$$\mathbb{F}(\mathcal{A}_R)^{\langle\gamma\rangle} = \mathbb{F}(F, a_3, \ldots, a_m) \tag{5.34}$$

$$\mathbb{F}(\mathcal{X}_R)^{\langle\gamma\rangle} = \mathbb{F}\left(G, x_3(x_2(x_1+1)+1), \ldots, x_m(x_2(x_1+1)+1)\right) \tag{5.35}$$

$$\mathbb{F}(\mathcal{A}_Q, \mathcal{X}_Q)^{\langle\gamma\rangle} = \mathbb{F}(\mathcal{A}_R, \mathcal{X}_R)^{\langle\gamma\rangle}(D). \tag{5.36}$$

The $\widetilde{A_{p,q}}$, $\widetilde{D_n}$, $A_1^{(1,1)}$ and $D_4^{(1,1)}$ affine and doubly extended type cluster ensembles are associated to an annulus with p and q marked points on each boundary and a twice punctured disk with $n-2$ marked points on the boundary respectively, see [32] examples 6.9 and 6.10. The E type affine and doubly extended cluster ensembles are not associated to surfaces, but there is an analogue coming from the notion of a "cluster Dehn Twist" of Ishibashi in [39].

5.2.1 $T_{p,q,r}$ Quivers

Each of the affine ADE mutation classes contain the quiver $T_{p,q,r}$, where $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1$, see figure 5.9 , [40]. For $\widetilde{A_{p,q}}$, we have $(p,q,r) = (p,q,1)$, for $\widetilde{D_n}$ we have $(p,q,r) = (n-2,2,2)$, and for $\widetilde{E}_n$ we have $(p,q,r) = (n-3,3,2)$. We will use this quiver as our initial seed. We associate variables $(a_1, a_2, b_2, \ldots, b_p, c_2, \ldots, c_q, d_2, \ldots, d_r)$ for the variables on the $\mathcal{A}$ space and variables $(x_1, x_2, y_2, \ldots, y_p, z_2, \ldots, z_q, w_2, \ldots, w_r)$ for the variables on the $\mathcal{X}$ space in an obvious way.

The $E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}$ cluster ensembles are associated with $T_{p,q,r}$ quivers with (p,q,r) equal to $(3,3,3), (4,4,2), (6,3,2)$ respectively. The A and D type doubly extended ensembles are not quite explicitly $T_{p,q,r}$ quivers, but have quivers with analogous properties, see figures 3.6a and 3.8a

The cluster modular groups of the cluster ensembles associated with $T_{p,q,r}$ quivers and the X_6, X_7 (see figure 2.3) quivers have the element $\gamma = \{1, (12)\} \in \Gamma_Q$. In the case that this ensemble is associated with a surface, this element corresponds to a Dehn twist. In the non surface cases, we refer to this element as a cluster Dehn

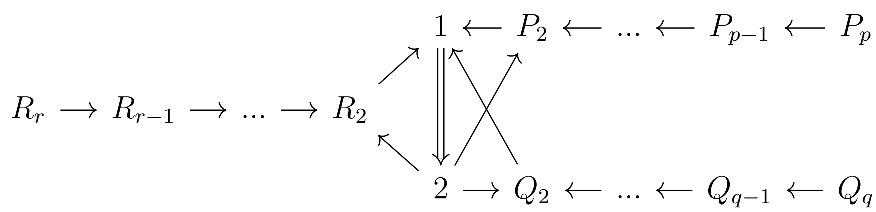


Figure 5.9: General form of a $T_{p,q,r}$ quiver.

twist.

Let Q be a non-surface type mutation finite quiver with, as in table 2.2.

Definition 5.2.1. An element $\gamma \in \Gamma_Q$ is called a *cluster Dehn twist* if there is some seed $\mathbf{i}$ for which $\gamma \in \Gamma_{\mathbf{i}}$ is represented by a single mutation at a node on a double edge. An element $\tau \in \Gamma_Q$ is called a *partial cluster Dehn twist* if some power of it is a cluster Dehn twist.

Remark 5.2.1. Ishibashi does not make a distinction between partial and non-partial cluster Dehn twists.

Remark 5.2.2. A finite index subgroup of the cluster modular group Γ_Q for any mutation finite quiver is always generated by partial cluster Dehn twists as shown in [35] for surfaces, [40] for doubly-extended types and [41] for the affine and exotic types. In most cases, we actually obtain the full cluster modular group this way.

When we have a cluster Dehn twist, $\gamma = \{1, (12)\}$, represented by mutation at a node on a double edge, we may define $\mathcal{A}$ and $\mathcal{X}$ trace functions for γ.

Definition 5.2.2. The function $G(x_1, x_2) = \frac{(x_2(x_1+1)+1)^2}{x_1 x_2}$ is the *cluster trace function* for γ on the $\mathcal{X}$ space and $\rho^* \sqrt{G}$ is the cluster trace function for γ on the $\mathcal{A}$ space.

Again, there is a notion of invariant $\mathcal{A}$ and $\mathcal{X}$ coordinates for γ

Definition 5.2.3. The collection of $\mathcal{A}$ coordinates associated to nodes other than node 1 and 2 are the invariant $\mathcal{A}$ coordinates for γ. The functions $\{x_k(x_2(x_1+1)+1), x_l\}$ when the node for x_k is connected to nodes 1 and 2, and the node for x_l is not connected to nodes 1 or 2 are the invariant $\mathcal{X}$ coordinates for γ.

We will classify the invariants for these ensembles for the subgroup generated by $\gamma = \{1, (12)\}$

Theorem 5.2.1. *The invariant functions for any cluster Dehn twist are generated by the cluster trace function for γ and the invariant $\mathcal{A}$ and $\mathcal{X}$ coordinate functions.*

This follows from corollary 5.2.1.

5.2.2 Invariants for partial Dehn twists

Suppose that $(\mathcal{A}, \mathcal{X})$ is a finite mutation type cluster ensemble and $\tau \in \Gamma$ is a partial cluster Dehn twist satisfying $\tau^n = \gamma$ for some cluster Dehn twist γ and n minimal.

The cyclic group $C_n = \mathbb{Z}/n\mathbb{Z}$ acts on these γ invariant $\mathcal{A}$ and $\mathcal{X}$ functions in an obvious way: since $\tau^n = \gamma$ we can see that the action of τ on these coordinates descends to an action of $C_n = \langle\tau\rangle/\langle\gamma\rangle$. This gives us the following proposition

Proposition 5.2.1. $\mathbb{R}(\mathcal{A}, \mathcal{X})^{\langle\tau\rangle} = (\mathbb{R}(\mathcal{A}, \mathcal{X}))^{\langle\gamma\rangle})^{C_n}$.

This can be used to compute the invariant rings for any partial Dehn twist by studying the action of C_n on the invariant ring associated with γ. These invariant rings will simply consist of symmetric functions in the elements of $(\mathbb{R}(\mathcal{A}, \mathcal{X}))^{\langle\gamma\rangle})$.

Chapter 6: Further Problems and Applications

We will discuss some various applications, properties and conjectures about mutation invariant functions.

6.1 Affine $\mathcal{A}$ coordinates

We can use the theorems at the end of chapter 5 to answer a question posed in [27] about the $\mathcal{A}$ coordinates appearing on the tails of a $T_{p,q,r}$ quiver. Let Q be a $T_{p,q,r}$ quiver which is of affine type (so $1/p + 1/q + 1/r - 1 > 0$). Then by the calculation of the cluster modular group of affine cluster algebras done in [40], we have that the cluster modular group is generated by the commuting partial cluster Dehn twists τ_p, τ_q, τ_r satisfying $\tau_i^i = \gamma = \{1, (12)\}$ along with the elements of $\operatorname{Aut} Q$. Thus, there is only one distinct cluster Dehn twist for any affine cluster ensemble.

We call the $\mathcal{A}$ coordinates which appear on nodes other than 1 and 2 for any seed with quiver isomorphic to Q "affine $\mathcal{A}$ coordinates". Let F be the cluster trace function for γ, $\{\alpha\}$ be the set of initial affine $\mathcal{A}$ coordinates and let $R = \mathbb{F}[\alpha^{\pm 1}, F]$ be the ring generated by F and Laurent polynomials in $\{\alpha\}$

Theorem 6.1.1. *The affine $\mathcal{A}$ coordinates are elements of R*

Proof. We know from the Laurent phenomenon that all of the $\mathcal{A}$ coordinates are

Laurent polynomials in the initial $\mathcal{A}$ coordinates. Furthermore, we know that all of the affine $\mathcal{A}$ coordinates are invariant functions under the action of γ. Thus, these affine coordinates are elements of $\mathbb{F}(\alpha, F)$. Let A be the ring of Laurent polynomials in the initial cluster variables.

Now we will show that $\mathbb{F}(\alpha, F) \cap A = R$. Let $x = \frac{a_1}{a_2}$. First, we notice that

$$F = x + x^{-1} + \frac{b_2 c_2 d_2}{a_1 a_2} \tag{6.1}$$

and

$$F^n = (x)^n + (x)^{-n} + \ldots. \tag{6.2}$$

Now we show that the only elements of $\mathbb{F}(F) \cap A$ are polynomials in F. Suppose that

$$\frac{P(F)}{Q(F)} = \frac{h}{b} \in A \tag{6.3}$$

is a Laurent polynomial in A with Q and P relatively prime. Then if $Q(F)$ is not a monomial in $\{\alpha\}$, then there must be a cancellation between the numerator and denominator. This cannot happen if Q and P are relatively prime, since F is algebraically independent over $\mathbb{F}$. Thus we have that $Q(F)$ must be a monomial in A. This can not happen unless Q is constant since $F = x + x^{-1}$ modulo b_2 and we see that every non constant polynomial in F must have $x^n + x^{-n}$ as its leading term in x. Thus $\mathbb{F}(\alpha, F) \cap A = R$.

Now we see that the elements of $\mathbb{F}(F) \cap A$ and $\mathbb{F}(\alpha) \cap A$ generate $\mathbb{F}(\alpha, F) \cap A$ since the elements in α are algebraically independent with a_1 and a_2.

□

This theorem naturally has an analogue for surface cluster algebras. Let S be a marked surface and let $\gamma \in \Gamma_S$ be the cluster modular group element corresponding to a Dehn twist about a simple closed curve δ and let F be the trace function for this curve. Let $\{\alpha\}$ be the set of $\mathcal{A}$ coordinates in an excising triangulation for δ that do not intersect δ and let $R = \mathbb{F}[\alpha^{\pm 1}, F]$. Then we have

Theorem 6.1.2. *The $\mathcal{A}$ coordinates for arcs on S that do not intersect δ are elements of R*

The proof of this theorem follows from the same argument as the previous theorem.

6.2 Invariants for Cluster Dehn Twists on General Mutation Finite Types via folding

We have only completed our classification for simply laced finite mutation type ensembles. We will not treat the general case here, but it should be easy to obtain from what we have already completed. We may conjecture that the folding map of invariants is always surjective.

Conjecture 6.2.1. Let R be a quiver obtained by folding a quiver Q. Then the map:

$$\mathbb{F}(\mathcal{A}_Q, \mathcal{X}_Q)^{\langle\gamma\rangle} \xrightarrow{fold} \mathbb{F}(\mathcal{A}_R, \mathcal{X}_R)^{\langle\gamma\rangle} \tag{6.4}$$

is surjective.

We believe that this should follow from our classification in the case where Q is a surface or mutation finite quiver, and γ is a cluster Dehn twist.

6.3 Further Geometric structures related to invariants

We suspect that the analysis of section 5 extends to cluster ensembles associated to the higher Teichmüller spaces of Fock and Goncharov in [5]. The mapping class group of the surface will again be a canonical subgroup of the cluster modular group of these ensembles. As before, it is possible to compute traces of monodromy operators about loops on the surface in terms of the $\mathcal{X}$ coordinates, and these will give $\mathcal{X}$ invariants for mapping classes that preserve the given loop.

There is also evidence that invariant functions may aid in the analysis of the gluing equations for more general geometric structures on manifolds, generalizing remark 4.2.1. The work of Nagao, Terashima, and Yamazaki in [42] gives indications of this via their notion of a "parameter periodicity equation".

6.4 Correspondence between $\mathcal{A}$ and $\mathcal{X}$ Invariants

We will give some evidence for a correspondence between the invariants of the $\mathcal{A}$ and $\mathcal{X}$ spaces via "denominator vectors".

Definition 6.4.1. The *denominator vector* of a Laurent polynomial is the vector of exponents appearing in the denominator of the polynomial written as a single fraction. If there are monomial factors in the numerator, we include these as negative valued exponents in the denominator.

Definition 6.4.2. We call a pair of bases or partial bases for $\mathcal{A}$ and $\mathcal{X}$ invariants that correspond via denominator vectors a *correspondence basis*.

Through all of the examples of section 3, we can explicitly see examples of correspondences between the $\mathcal{A}$ and $\mathcal{X}$ invariants. In the cases where no $\mathcal{X}$ invariants are given, there are trivial $\mathcal{X}$ invariants corresponding via denominator vectors to each of the $\mathcal{A}$ invariants.

Example 6.4.1. Explicitly, this correspondence for the invariants of example 3.2.4 are as follows:

$$\frac{(a_1a_4 + a_2a_5 + a_3a_6)^2}{a_1a_2a_3a_4a_5a_6} \longleftrightarrow (x_1x_2x_3x_4x_5x_6)^{-1}$$
$$\frac{(a_1a_4 + a_2a_5 + a_3a_6)}{a_4a_5a_6} \longleftrightarrow (x_4x_5x_6)^{-1}$$
$$\frac{a_1}{a_4} \longleftrightarrow \frac{x_1}{x_4}$$

We can show this correspondence when our cluster ensemble is associated with a surface, S, and we are considering invariants for a Dehn twist about δ on S. This correspondence is very clear when δ has genus zero on S. Let Δ be an excising triangulation of δ and number the arcs as definition 5.1.5.

In this case, the correspondence basis is given by the following table:

6.5 Laurent Property of Invariants

Many of the examples of $\mathcal{A}$ invariants are Laurent polynomials in the $\mathcal{A}$ coordinates. This is not be true in every case, as the functions of example 3.2.4 include inverses of Laurent polynomials too. However we may conjecture the following:

Type of invariant	$\mathcal{A}$	$\mathcal{X}$
Trace function of δ	$\rho^*(\sqrt{G(x_1,x_2)})$	$G(x_1,x_2)=\frac{x_2(x_1+1)+1}{x_1x_2}$
Boundary arcs	$a_3,\dots,a_m$	$x_3(x_2(x_1+1)+1),\dots,x_m(x_2(x_1+1)+1)$
Nonboundary arcs	$a_{m+1},\dots,a_N$	$x_{m+1},\dots,x_N$

Conjecture 6.5.1. There is a basis of $\mathbb{F}(\mathcal{X}_Q)^{\Gamma_\circ}$ with corresponding basis of $\mathbb{F}(\mathcal{A}_Q)^{\Gamma_\circ}$ such that Γ acts on the basis of $\mathbb{F}(\mathcal{A}_Q)^{\Gamma_\circ}$ by positive Laurent polynomials.

The examples of section 3, other than example 3.2.5, give ample evidence for this conjecture. This conjecture also implies, following theorem 6.1, That all of the $\mathcal{A}$ coordinates appearing on the tails of the $T_{p,q,r}$ quiver are Laurent polynomials in the the initial invariants. This is easy to verify case by case, and is probably not too difficult to prove in general.

It seems that in some cases that there is a stronger version of this phenomenon. We may occasionally find a correspondence basis such that the action of the cluster modular group sends the basis to Laurent *monomials* in the basis. In other words, we can pick a basis that gives a representation

$$\Gamma/\Gamma_\circ \to \mathrm{GL}(2,\mathbb{Z}) \tag{6.5}$$

Surprisingly, we have that the action of Γ on the $\mathcal{X}$ invariants is also by Laurent monomials and the induced representation seems to be identical in this case.

Example 6.5.1. Continuing example 3.2.3, we can take (F_1, F_2) as a possibly partial basis of invariants. Then one may check that the paths $\tau = \{34, (12)\}$ and $r = \{414, (132)\}$ generate D_{12} and we have that $\tau((F_1, F_2)) = (F_1, F_1/F_2)$ and $r((F_1, F_2)) = (F_1/F_2, F_1)$. Thus we have the representation

$$\pi : D_{12} \to \mathrm{GL}(2, \mathbb{Z}) \tag{6.6}$$

given by

$$\pi(\tau) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \quad \pi(r) = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \tag{6.7}$$

6.6 Coefficients and Canonical Bases

Throughout our study of mutation invariants, we have mostly ignored coefficients on the $\mathcal{A}$ space. It is natural to wish to study invariants for more general coefficients, but there are some immediate problems one encounters. For example, take the Markov quiver, example 3.1.3, and add "principal coeffiecents" i.e. one frozen node for each unfrozen connected with a single arrow. Mutations return us to a quiver with isomorphic mutable portion, but the frozen nodes are connected differently. We therefore, cannot expect to construct a mutation invariant function which is identical after each mutation.

There should still be some more general notion of invariant which addresses this problem. If we are in a situation where there are non-trivial $\mathcal{X}$ invariants, one may pull them back to the $\mathcal{A}$ space and obtain functions which are invariant in some

Figure 6.1: $\widetilde{A_1}$ quiver with principal coefficients before and after applying γ.

sense. We expect that these functions will be elements of the "Canonical basis" of the cluster algebra associated with the $\mathcal{A}$ space.

Example 6.6.1. For example, if we take an $\widetilde{A_1}$ cluster ensemble with principal coefficients, we can pull our $\mathcal{X}$ invariant, $\sqrt{G(x_1, x_2)}$, of example 3.1.1 back to the $\mathcal{A}$ space. We find

$$\rho^*(G(x_1, x_2)) = F(a_1, a_2, f_1, f_2) = \frac{a_2^2 f_1 f_2 + f_2 + a_1^2}{a_1 a_2 \sqrt{f_1 f_2}} \tag{6.8}$$

Mutations will change how each mutable node is connected with the frozen nodes, so this function will only be invariant up to multiplying its arguments by products of the coefficients. The element $\gamma = \{1, (12)\}$ is still an element of the cluster modular group. We find

$$\gamma(F) = \frac{a_2^2 f_2 + 1 + a_1^2 f_1}{a_1 a_2 \sqrt{f_1 f_2}} = F(a_1 \sqrt{f_1 f_2}, a_2 \sqrt{f_2 / f_1}) \tag{6.9}$$

Thus F is invariant up to multiplying its arguments by functions of the frozen variables. This should be related to the notion of a "cluster quasi-homorphism" of [43].

The function F appears as theta function in the sense of [18]. The computation of the theta function of the limiting ray in the cluster scattering diagram associated with the $\widetilde{A_1}$ cluster algebra in [44], shows

$$\theta(a_1, a_2, x_1, x_2) = a_1 a_2^{-1}(x_2(x_1 + 1) + 1) \tag{6.10}$$

which after applying ρ^* to the $\mathcal{X}$ coordinates we have

$$\theta(a_1, a_2, f_1, f_2) = \frac{a_2^2 f_1 f_2 + f_2 + a_1^2}{a_1 a_2} \tag{6.11}$$

which is essentially the function F.

Remark 6.6.1. The expression of the theta function in equation 6.10 is essentially the form of the invariant $\mathcal{X}$ coordinates of 5.1.5, imagining that there is a third mutable node connected to nodes 1 and 2 in an oriented cycle.

The fact that a theta function associated with a limiting ray of the scattering diagram is in some way an invariant function should make intuitive sense. Whenever the scattering diagram is mutation invariant under the mutation path which takes us towards a limiting ray, the theta function associated with this ray will be invariant too. Our classification theorem for surface cluster algebras shows us that the theta functions associated with Dehn twists should be related to the trace function for the curve we are twisting. This gives a natural relation between theta functions and trace functions.

It is conjectured in [18] that the "theta basis" of a surface cluster algebra should be the same as the basis of trace functions constructed in [5]. Our classification of mutation invariants should act as a stepping stone between these two constructions. Interestingly, the construction theta basis makes no reference to any surface underlying the combinatorial structure of the algebra. Similarly, our notion of mutation invariant does not require any data which is not intrinsic to the cluster algebra, but the classification of invariants is only visible when considering surfaces.

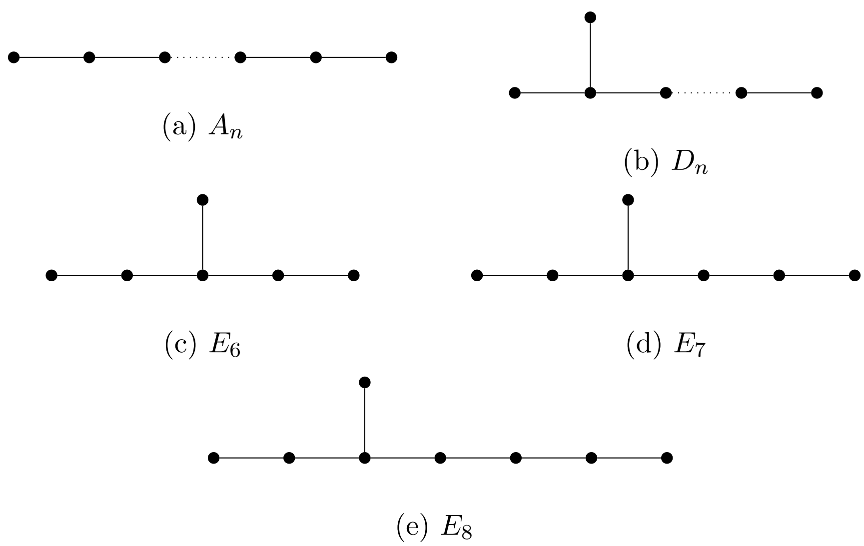
There are several other instances of the same invariant function. In [8], we can see the appearance of the cluster trace function of the affine cluster algebras appearing as the "cluster character" of a module with dimension vector given by the longest root in the affine root system in the associated cluster category. This cluster character can be seen to be mutation invariant by theorem 3.1 of [45]. Again, our classification of mutation invariants helps explain why we see the same function appear.

We can consider the following conjecture as a way of organizing these ideas:

Conjecture 6.6.1. Let Q be a mutation finite quiver. There is a canonical basis of invariant functions for any cluster Dehn twist. The union of these bases over every cluster Dehn twist is the canonical basis of the cluster algebra $C(Q)$.

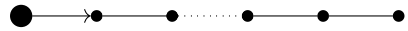
Appendix A: Dynkin Diagrams

A.1 Finite Dynkin Diagrams

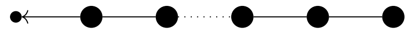


(a) A_n (b) D_n (c) E_6 (d) E_7 (e) E_8

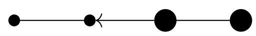
Figure A.1: Simply Laced Finite Dynkin Diagrams



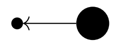
(a) B_n



(b) C_n



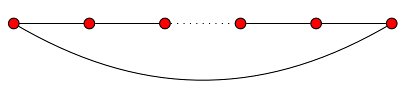
(c) F_4



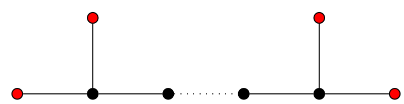
(d) G_2

Figure A.2: Folded Finite Dynkin Diagrams

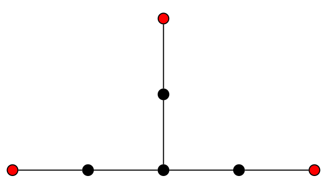
A.2 Affine Dynkin Diagrams



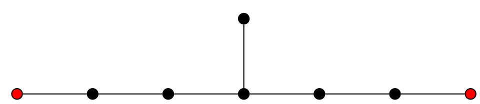
(a) $\widetilde{A_n}$



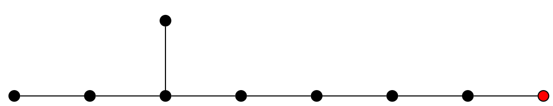
(b) $\widetilde{D_n}$



(c) $\widetilde{E_6}$

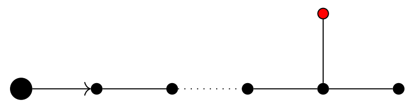


(d) $\widetilde{E_7}$



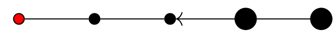
(e) $\widetilde{E_7}$

Figure A.3: Simply Laced Affine Dynkin Diagrams

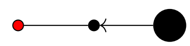


(a) $\widetilde{B_n}$

(b) $\widetilde{C_n}$



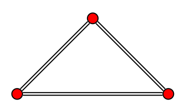
(c) $\widetilde{F_4}$



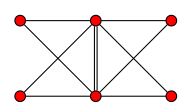
(d) $\widetilde{G_n}$

Figure A.4: Folded Affine Dynkin Diagrams

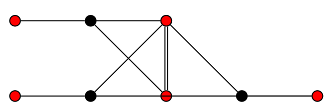
A.3 Doubly-Extended Dynkin Diagrams



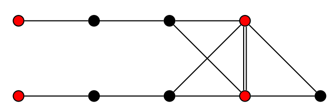
(a) $A_1^{(1,1)}$



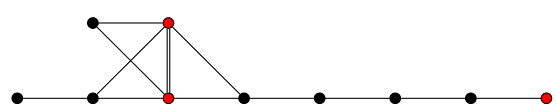
(b) $D_4^{(1,1)}$



(c) $E_6^{(1,1)}$



(d) $E_7^{(1,1)}$



(e) $E_8^{(1,1)}$

Figure A.5: Simply Laced Doubly Extended Dynkin Diagrams

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