ABSTRACT

Title of Dissertation: ESSAYS ON MULTI-DIMENSIONAL OBVIOUSLY REGRET-PROOF MECHANISMS

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This dissertation study strategically simple mechanisms for problems of multi-dimensional allocation. Strategic simplicity is crucial to make mechanisms robust to avoid mistakes and manipulation. However, the current literature on the analysis and implementation of strategically simple mechanisms is limited, especially for markets with many goods and services. This dissertation fills this gap with three complementary but stand-alone chapters. In Chapter 1, we identify and formalize the criteria for strategic simplicity. Then we combine those criteria into a new solution concept and propose a new class of mechanism that satisfies these criteria. In Chapter 2, we analyze the cost of enforcing this new notion of simplicity. In Chapter 3, we show that the new mechanism is a good candidate in an application with substantial welfare implications.

The solution concept we propose in Chapter 1 is called obvious regret-proofness (ORP). It describes conditions that regulate both the extensive form of a dynamic game and the communication between the auctioneer and the bidders. Those conditions make sure that there is a simple rule for bidders to determine his best action

each time he is called to play. Also, it is easy for the bidder to understand and to verify that he will not regret choosing this action because the optimality of this action does not depend on the choices of other bidders. We then translate those requirements into auction rules and propose a new class of mechanisms called Persistent Exit Descending (PED) mechanisms.

Then in Chapter 2, we analyze the cost of pursuing strategic simplicity by implementing the PED mechanism. We first show that for an efficient strategy-proof mechanism, the allocation and payment to a bidder can be dependent on the reports of other bidders. This influence is monotonic and mutual. Therefore, the externality of a bidder's choice can be internalized. In contrast, the influence in PED implementable mechanisms is restricted once the reports of other bidders exceed some certain thresholds. Moreover, the influence can only be one-sided, which means that if a bidder has influence over the other bidder, only if that bidder cannot influence him. Lacking the channel to influence, the decision of a bidder cannot take into account his externality to other bidders. This is the primary source of welfare loss in a PED mechanism.

In Chapter 3, we show that the PED mechanism proposed in Chapter 1 is a good candidate for land assembly problems. It has the properties that most of the land assembly mechanisms in the literature fail to have, but are fundamental to land assembly problems. First, it is strategically simple. Second, its allocation rule is combinatorial. Third, it can assign non-monetary compensations to a bidder. Finally, it fully respects the owners' property rights, and it is ex-post individual rational. We then tailor the PED mechanism to the land assembly problem and apply the analytical framework from Chapter 2 to discuss the advantages and limitations of using the PED mechanism in land assembly problems.

ESSAYS ON MULTI-DIMENSIONAL OBVIOUSLY REGRETPROOF MECHANISMS

by

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Dedication

To my family.

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1.1 Introduction

Strategic simplicity is an important design goal for centralized market mechanisms. If the equilibrium of a mechanism is difficult for bidders to find or to verify, then the mechanism is prone to bidders' strategic mistakes, espionage, and manipulations. If these things happen, then the intended allocation rule cannot be implemented. Also, because figuring out the optimal strategy is difficult, participating in this kind of auction becomes burdensome for bidders. Even strategy-proofness might not be sufficient for strategic simplicity. Because the dominant strategy may not be easy for the bidders to understand or to verify. Moreover, profitable deviations can still exist in strategy-proof mechanisms. To characterize strategically simple mechanisms, Li (2017) propose a dominant strategy solution concept called Obviously Strategy-proofness (OSP). It formalizes the condition that the optimality of the dominant strategy is easy to verify in the extensive form. A strategy S_i is said to be obviously dominant for bidder i, if for all information set I_i which S_i visits, the best-case payoff of any deviation starting from I_i is no better than the worst-case payoff of following S_i. A mechanism is said to be OSP if every type of bidder has an obviously dominant strategy. Compared to a weakly dominant strategy, the optimality of an obviously dominant strategy is easier to verify because bidders only need to compute the outcomes in the best and the worst cases, instead of in all possible contingencies. Li then shows that mechanisms that are OSP are also weakly group strategy-proof (WGSP), which means that there are no group deviations where every member in the group is strictly better off.

Even though OSP appears to be a good criterion for strategic simplicity, this solution concept is still subject to several limitations. First, the OSP implementation of a multi-dimensional allocation problem does not exist in the literature. Even though the solution concept is defined for mechanisms with arbitrary outcome space. However, at the time this paper is written, OSP implementation exists only for allocation problems with unit-demand (like the English Auction; Li (2017)) unit-supply (Milgrom and Segal, 2020), and allocation problems without payments (Bade and Gonczarowski, 2017). These are restricted classes of allocation problems, as many applications in market design require the auctioneer to allocate multi-dimensional and multi-unit goods and services. Second, the definition of OSP does not regulate all essential aspects of strategic simplicity. Thus, even if the one-dimensional OSP implementation is indeed simple for bidders, it is not guaranteed that all the elements of simplicity can be extended to multi-dimensional and more complex allocation problems. For example, to verify that a strategy is obviously dominant, bidders need to compute the best-case and worst-case payoffs, but the simplicity of solving those payoffs is not guaranteed in the definition of OSP. Next, even if it were easy to verify the optimality of a given strategy, that does not necessarily imply that the strategy itself is easy to find and to execute. Notice that in the definition of obviously dominant strategy, the optimality of a given action becomes obvious only when the bidder knows what to do in all future contingencies. Thus, if there exist OSP mechanisms for multi-dimensional allocation problems, it is unclear whether it would be strategically simple for bidders.

In this paper, we identify and formalize the elements that make the one-dimensional OSP mechanisms simple but are missing from the definition of OSP. Then we extend those concepts to multi-dimensional mechanisms and propose a new criterion for strategic simplicity by integrating all these elements. In short, the criterion requires that there exists a simple rule for a bidder to choose the optimal action each time he or she is called to play. Also, the auction rules and the information revealed to the bidder should make it easy to verify that he/she will not regret choosing this action regardless of what other bidders do. We call this criterion obviously regret-proof (ORP) and show that it is a refinement of OSP. Next, we show that the ORP mechanism can be implemented for multi-dimensional problems by proposing a new mechanism called the Persistent Exit Descending (PED) mechanism. We show that the PED mechanism is a multi-dimensional generalization of the English Auction and the personal clock auction in Milgrom and Segal (2020). Finally, we show that a mechanism is ORP if and only if it is a PED mechanism and that all well-behaved OSP-implementable direct mechanisms can be PED-implemented. That is, enforcing stricter requirements of strategic simplicity does not lose the generality of a mechanism.

We also develop the necessary and sufficient conditions for a multi-dimensional mechanism to be OSP. We will see that those conditions are technical. Also, showing that the cautiously optimistic strategy is obviously dominant for an arbitrary OSP mechanism is tedious. The technicality echoes the critique that multi-dimensional OSP mechanisms are not guaranteed to be strategically simple. Nevertheless, the necessary and sufficient conditions provide the theoretical foundation for the results mentioned above. For logical order, this theoretical foundation should come first, but for the readability of the paper, we move these results to the last section of this paper. Therefore, nearly all the proofs are relegated to the Appendix.

Before we dive into the results, let us look at some applications where the dimensionality of the strategically simple mechanisms has substantial welfare implications. The first application is the land assembly problem, which is the main topic for Chapter 3. In a land assembly problem, one or more developers try to acquire multiple contiguous parcels of land from different owners for redevelopment. It is usually costly and time-consuming to make a deal for this kind of transaction in decentralized bargaining. That is because the complementarity among adjacent owners provides them incentives to delay entering the negotiation so they can receive a higher surplus. This is known in the literature as the holdout problem. In that case, implementing a centralized mechanism is a good way to avoid a costly back-and-forth bargaining process. However, since the most of the target participants in this mechanism are small homeowners with little experience with game theory and auctions, if the mechanism is too difficult and burdensome, they would not have an incentive to participate at all. Therefore, strategic simplicity is a very important design goal for this allocation problem.

Another important design goal is to allow non-monetary compensation to the participating bidders, which implies the allocation rule should be multi-dimensional. At the time this paper is written, most of the market design solutions for land assembly treat the problem as a purely monetary transaction—the mechanism determines whether they sell their land, and if they do, the monetary compensation they receive. However, it is very common in the practice that owners are offered newly built apartments as non-monetary compensation. Another type of non-monetary compensation is to assign one of the existing apartments as the replacement home, provided that the owner of that apartment agrees to sell. If the preference of bidders is diverse enough, there would be some bidders who are willing to trade only if they are offered such non-monetary compensations. Thus, mechanisms with diverse types of compensation expand the possibilities of transactions and hence increases the probability of trade and the expected surplus. However, allowing non-monetary compensation to a bidder implies that more goods would be allocated to this bidder. Thus, the allocation problem becomes multi-dimensional and incompatible with the OSP mechanisms that exist in the current literature. In Chapter 3, we will get into detail how the PED mechanism developed in this chapter can be adapted to the land assembly problem why the PED mechanism is particularly desirable for this application.

Another application is the FCC Incentive Auction, where the unit-supply OSP design was eventually discarded in favor of the one that can implement a multi-dimensional allocation rule. The FCC Incentive Auction is structurally parallel to the land assembly problem. It assembles geographically and spectrally contiguous TV radio stations as opposed to geographically contiguous parcels of land in a land assembly. The assembled spectrum will then be repurposed and auctioned off to telecommunication companies. The personal clock mechanism (Milgrom and Segal, 2020) that we mentioned earlier was actually proposed for this application. However, what was modeled in Milgrom and Segal (2020) is a simplified problem, with only bidders that own a single UHF station. In that case, the allocation problem can be reduced to a unit-supply problem. However, in the full model, bidders may have multiple and different types of stations, i.e. UHF and VHF ones. Also, the FCC needed an allocation rule that could move bidders on a UHF band to a VHF band, provided that the VHF station owner clears his station. Again, this type of allocation rule is multi-dimensional and thus not compatible with the design in Milgrom and Segal (2020). Therefore, the OSP personal clock mechanism was not adopted in the real auction that was held in 2017.

From the these applications, we know that multi-dimensional strategically simple mechanisms are needed in real life and have substantial welfare implications. Thus, it is important for market designers to know how to implement multi-dimensional strategically simple mechanisms and make sure that they are indeed simple. That is the reason for us to characterize the elements of strategic simplicity and to propose the mechanism that implements these elements.

We need to make it clear that strategic simplicity for bidders does not imply computational simplicity for the auctioneer. For a PED mechanism, it can be computationally difficult for the auctioneer to find the optimal, feasible, and budget balanced descending policy. This is because the nature of choosing a descending policy is a dynamic decision process, where the state space explodes with the dimensionality of the problem. That means that, the computation of the PED mechanism becomes more difficult as the number of bidders and the dimensionality of their allocation spaces get large. This is called the Curse of Dimensionality in the literature of dynamic decision making. For that kind of problem, there might still be numerical methods that can approximate the optimal solution. However, that is outside the scope of this paper. Therefore, we only introduce the auction rules and requirements for PED mechanisms in this paper. We will leave the topics about optimization and computation for future research.

As for the organization of this paper, Section 1.2 introduces the model that formalizes the multi-dimension and multi-unit allocation problem. We also restate the definitions of the obviously dominant strategy and the OSP mechanism. Section 1.3 identifies and formalizes the elements of strategic simplicity, and it proposes the new solution concept of an obviously regret-proof mechanism. Section 1.4 introduces the generic ORP implementation, the Persistent Exit Descending (PED) mechanism, and contrasts it with one-dimensional OSP mechanisms. Finally, in Section 1.5, we state the necessary and sufficient condition for mechanisms to be OSP, which is the theoretical building block of this paper.

1.2 The Model

We consider an allocation problem between a non-strategic bidder and a finite set of strategic bidders, N. The client is the agent who hires the auctioneer to host the mechanism. He offers to each bidder $i \in N$ an individualized allocation space

$Y_i \subseteq \mathbb{R}^{K_i}$ for some $K_i \in \mathbb{N}$. An allocation $y_i = (y_{i1}, y_{i2}, \ldots, y_{iK_i}) \in Y_i$ specifies a K_i dimensional vector of goods and services assigned to bidder i. Let $y_i^0 \in Y_i$ denote the endowment allocation of bidder i. An outcome of a mechanism is a pair $x_i = (y_i, p_i)$, which specifies an allocation $y_i \in Y_i$ and a payment $p_i \in \mathbb{R}$ from the client to bidder i. The payment p_i can take negative values, which means bidder i makes a net payment of $|p_i|$ to the client. Let $X_i \equiv Y_i \times \mathbb{R}$ be the outcome space for bidder $i \in N$. We assume that each bidder has a weakly convex, weakly increasing, continuous preference on X_i that is quasi-linear in payment p_i. Let $\mathcal{U}_i$ be the set of utility functions that represent those preferences such that the endowment outcome is normalized to zero, that is, $u_i(x_i^0) = u_i(y_i^0, 0)$. Let $Y_N \equiv \Pi_{i \in N} Y_i$ and $X_N \equiv \Pi_{i \in N} X_i$ be the set of all allocation profiles and outcomes, respectively. The client specifies a subset of allocation profiles $\mathcal{F} \subseteq Y_N$ that are feasible, and his willingness to pay $w(y_N) \in \mathbb{R}$ for each feasible allocation profile $y_N \in \mathcal{F}$. We assume that the client is budget-constrained to pay at most $w(y_N)$ for each $y_N \in \mathcal{F}$. Given an outcome profile $x_N = (y_N, p_N)$, the total payment the client must pay is $\sum_{i \in N} p_i$. Thus, the payoff for the client is $w(y_N) - \sum_{i \in N} p_i$. A market is then a tuple that summarizes all the above elements, $\Omega = \langle N, X_N, \mathcal{U}_N, \mathcal{F}, w \rangle$.

1.2.1 Examples

Our multi-dimensional multi-quantity allocation model is general enough to represent a large class of market design problems. In particular, we show by example that the allocation problem modeled in this paper is a superset of one in Milgrom and Segal (2020).

Example 1 (Single Object English Auction) *We can model the allocation problem for the English Auction as follows. The client is the seller. He has one object to sell. Let $N = \{1, 2, 3, \dots\}$ be the set of bidders. Each bidder either gets 1 unit of the good, $y_i = 1$, or 0 unit, $y_i = 0$. Thus, $Y_i = \{0, 1\} \subseteq \mathbb{R}$. The endowment for each i is $y_i^0 = 0$. The allocation profile $y_N \in \mathcal{F}$ if and only if at most one $y_i = 1$. The outcome for i loses is represented by $x_i = (y_i^0, 0)$. If i wins at price p_i, the outcome is $x_i = (1, -p_i)$. The negative sign indicates that the payment is made from the bidder to the client (seller). This problem can be OSP implemented by an English Auction.*

Example 2 (The FCC Incentive Auction) *We consider a simplified FCC Incentives Auction. In this example, the client is the Federal Communication Committee (FCC), who wants to buy radio spectrum from station owners. Suppose that there are just two station owners, $N = \{1, 2\}$. Each of these owners is endowed with one indivisible station. He can either relinquish his rights to the station to FCC or keep it. Thus, $Y_i = \{0, 1\} \subseteq \mathbb{R}$, where $y_i = 0$ if and only if he relinquishes his station. The endowment for each i is $y_i^0 = 1$. The outcome $x_i = (0, 100)$ implies i sells his spectrum for \$100; while $x_i = (1, 0)$ implies i keeps his station and receives no payments. Suppose that the buyer is indifferent between these two stations. The value of getting either one is w.*

This allocation problem can be implemented by a simple personal clock auction. The auctioneer alternately offers a descending price to each owner starting from w.

Once called to play, the bidder observes the clock price and chooses either to exit and keep his spectrum or to continue. The auction concludes when one of the bidders chooses to exit, and the remaining bidder sells his spectrum at the most recent clock price; or when both bidders choose to exit at the initial offer at w. As shown in Milgrom and Segal (2020), this mechanism is OSP.

In these examples, there are only two elements in the allocation set Y_i. That is, a bidder is either allocated a good or not. Li (2017) calls this kind of problem a binary allocation problem. He also shows that these problems can be OSP-implemented by the personal clock auction. We can also interpret the binary allocation problem as the unit-demand or unit-supply allocation problem.

In the following example, we will depart from the unit-demand/supply assumption and see the real generality of this model.

Example 3 (Multi-Objects Forward Auction) *Now, we consider that there is a seller who has 2 indivisible objects to sell. Let $K = \{k_1, k_2\}$ be the set of goods. There are 2 bidders, $N = \{1, 2\}$. Because there are 2 goods to be allocated, the outcome space is now 2 dimensional. $Y_i = \{(0,0), (1,0), (0,1), (1,1)\} \subseteq \mathbb{R}$, where $y_{ik} = 1$ if and only if good $k \in K$ is assigned to bidder i. The endowment for each i is $y_i^0 = (0,0)$. The allocation is feasible, $y_N \in \mathcal{F}$, if and only if $y_1 + y_2 \leq (1,1)$.*

We can see that the allocation space Y_i is no longer binary. This type of allocation problems is outside the scope discussed in the papers of Milgrom and Segal (2020) and Li (2017).

Example 4 (The FCC Incentive Auction) *Consider a more general version of Example 2. Suppose that the two previously mentioned stations belong to a type called UHF stations. Now, there are two more owners, each of them owns a station of an inferior type, the VHF station. Suppose that the buyer still wants to buy only one UHF station. However, it is possible that some UHF station owners would be willing to move to one of the VHF stations for some compensation.*

Thus, we can consider multilateral transactions where one of the UHF owners is paid to move to one of the VHF stations, and the VHF owner is paid to go off the air. Thus, the buyer still gets the UHF spectrum. This opens a new possibility to make a deal that clears a UHF spectrum. If the subjective cost for the UHF owner moving to a VHF station is low, and the VHF owner does not value its station very much, this kind of transaction may have more surplus than having one of the UHF bidders selling his spectrum.

For simplicity, we assume that bidder 1 can move only to the station of bidder 3; and bidder 2 can move only to the station of 4. That is, these 4 bidders are divided into two coalitions, competing for the right to clear the UHF station for some compensation.

In this example, we simply introduce how to model the allocation problem. We will revisit this example and show how a PED mechanism can implement this kind of transaction when we have introduced the PED mechanism in Section 1.4.

Let $N = \{1, 2, 3, 4\}$*, where bidder 1 and 2 are UHF owners and 3 and 4 are VHF owners. Then for each of the UHF bidders, his allocation space is two-dimensional, where the first dimension specifies whether he is allocated the UHF* *station, while the second dimension specifies the associated VHF one. Thus, there are in total three possible allocations,* $Y_i = \{(0,0),(1,0),(0,1)\}$*. Where* $(0,0)$ *means he is not allocated both stations,* $(1,0)$ *means he is allocated the UHF one, which is his endowment, and* $(0,1)$ *means that he is moved to the VHF one. The allocation space for each of the VHF bidders remains to be one-dimensional,* $Y_i = \{0,1\}$*, as the endowed VHF station is the only good to be allocated.*

Consider a multilateral transaction where bidder 1 moves to the VHF station of bidder 3. Then the outcome for bidder 1 is $x_1 = (0,1,p_1^m)$ *for some payments* $p_1^m \geq 0$*. The outcome for bidder 3 is* $x_3 = (0,p_3)$ *for some* $p_3 \geq 0$*. The remaining bidders then receive their endowment outcomes. This is the kind of transaction that cannot be fulfilled in the personal clock mechanism in Milgrom and Segal (2020).*

1.2.2 The Mechanisms

A direct mechanism on market Ω is a pair of functions $M = (\alpha_N, \rho_N)$ such that $\alpha_N : \mathcal{U}_N \to \mathcal{F}$ and $\rho_N : \mathcal{U}_N \to \mathbb{R}^{|N|}$. A dynamic mechanism on market Ω is an extensive form game where each of the terminal nodes $z \in Z$ is mapped to an outcome $\chi_i(z) \in X_i$ assigned to each $i \in N$. As the audience of this paper may already be familiar with the definitions of an extensive form game, we leave the full definition in the Appendix. Table 1.1 lists the key notations that we use in the rest of the paper.

A dynamic mechanism D is said to be ex-post budget balanced if for all possible outcomes, $x_N = (y_N, p_N) \in \chi_N(Z)$, we have $\sum_{i \in N} p_i \leq w(y_N)$.

Name	Notation	Representative element
Histories	H	h
Histories i is called to play	H_i	h_i
Precedence relation over H	$\prec$	
Initial history	$h_\emptyset$	
Terminal histories	Z	z
Information sets for i	$\mathcal{I}_i$	I_i
Actions available at I_i	$A(I_i)$	a

Table 1.1: Notation for Extensive Game Form

1.2.3 Weakly Dominant

The outcome of a bidder is fixed when we specify an initial history $h \in H$ and the strategies each player plays according to S_N. Thus, we use $u_i(h_\emptyset, S_i, S_{-i})$ to denote the payoff when the game starts at $h_\emptyset$ and when bidders play S_N.

Definition 1 (Weakly Dominant) *A strategy S_i is said to be weakly dominant if, for all S_i' and S_{-i}, we have*

$$u_i(h_\emptyset, S_i, S_{-i}) \geq u_i(h_\emptyset, S_i', S_{-i}) \tag{1.1}$$

A mechanism is said to be strategy-proof (SP) if there exists a weakly dominant strategy for each $u_i \in \mathcal{U}_i$.

1.2.4 Obviously Dominant

Obvious dominance is defined only on dynamic strategies. A strategy for bidder i in a dynamic mechanism D is a function S_i that maps each information set $I_i \in \mathcal{I}_i$ to an available action $S_i(I_i) \in A(I_i)$. Because the mechanism is nonstochastic, given a strategy profile $S_N = (S_i)_{i \in N}$ and a starting history $h \in H$, the path of play and the terminal node are fixed. We denote the terminal node in this case by $z(h, S_N)$, sometimes written as $z(h, S_i, S_{-i})$.

Definition 2 (Achievable Outcome) *We say an outcome $x_i \in X_i$ is achievable by strategy S_i at $I_i \in \mathcal{I}_i$ if there exist $h \in I_i$ and a strategy profile of other bidders, S_{-i}, such that*

1. $h_\emptyset \prec h \prec z(h_\emptyset, S_i, S_{-i})$ *and*

2. $x_i = \chi_i(z(h_\emptyset, S_i, S_{-i}))$.

In words, x_i is achievable by S_i at I_i means the information set I_i will be visited and x_i will be assigned to bidder i if all players play according to (S_i, S_{-i}) for some strategy profile S_{-i}. Let $X_i(S_i, I_i)$ denote all the outcomes that are achievable by S_i at I_i. Notice the definition does not require any optimality or equilibrium condition. x_i can be achievable by S_i under some off-equilibrium or non-optimal strategies S_{-i}. We say x_i is achievable by an action $a \in A(I_i)$ if there exists $S_i : S_i(I_i) = a$ such that x_i is achievable by S_i at I_i. Let $X_i(a)$ be the set of all achievable outcomes by action a. Similarly, we say x_i is achievable at I_i if there exists S_i such that $x_i \in X_i(S_i, I_i)$. Let $X_i(I_i)$ denote all achievable outcomes at I_i.

We say $I_i \in \mathcal{I}_i$ is on the path of strategy S_i if there exist S_{-i} and $h \in I_i$ such that $h_\emptyset \prec h \prec z(h_\emptyset, S_i, S_{-i})$.

$I_i' \prec I_i$ if there exists $h' \in I_i'$, $h \in I_i$ such that $h' \prec h$.

Definition 3 (Earliest Points of Departure, Li (2017)) *Let $I_i \in \mathcal{I}_i$ be an information set, I_i is said to be an earliest point of departure of strategy S_i and S_i' if*

1. $S_i(I_i) \neq S_i'(I_i)$;
2. I_i *is on the path of both strategies. (Implies $S_i = S_i'$ on all predecessors of I_i.)*

Definition 4 (Obviously Dominant, Li (2017)) *Let $\mathcal{D}$ be a dynamic mechanism on market Ω. Let S_i be a strategy for $i \in N$ and $u_i \in \mathcal{U}_i$. We say S_i is obviously dominant for u_i if for all alternative strategy S_i' and earliest point of departure I_i of S_i and S_i',*

$$\min\{u_i(x_i)|x_i \in X_i(S_i, I_i)\} \geq \max\{u_i(x_i)|x_i \in X_i(S_i', I_i)\} \tag{1.2}$$

A dynamic mechanism is said to be obviously strategy-proof (OSP) if there exists an obviously dominant strategy for all $u_i \in \mathcal{U}_i$ and all $i \in N$.

Is it obvious to bid in an OSP mechanism? When a bidder is called to play at I_i, he needs to know three pieces of information in order to make sure his choice at this period dominates other alternative actions: first, what would be available in each of the future contingencies; second, what he would do in those contingencies; third, the best-case outcome of other actions. The first two are responsible for figuring out the LHS of Equation (1.2), while the third is responsible for the RHS. If any of the above is difficult to find out, the "obviously dominant" strategy might not be that obvious to bidders. In the next section, we identify and formalize the characteristics of a mechanism that makes it easy for bidders to figure these pieces of information.

1.3 Elements of Strategic Simplicity

In this section, we identify and formalize the elements of strategic simplicity that exist in the one-dimensional OSP mechanisms but are missing in the definition of OSP.

We argue that any criterion of strategic simplicity should cover at least the following four aspects of simplicity. First, there should be a simple rule for bidders to select the optimal action at each period given the information revealed by the auctioneer. Second, it should be simple for the bidder to understand that the optimality of the strategy does not depend on the choices of other bidders. Third, the auction rules and the information revealed by the auctioneer should make it easy for bidders to execute their optimal strategy. Finally, the auction rule should be simple and predictable such that it is easy for bidders to verify the optimality of their strategies. The definition of OSP does not explicitly characterize all the above aspects of strategic simplicity. However, we can observe these elements in the English Auction and the Single-dimensional mechanism proposed by Milgrom and Segal (2020).

In this section, we first go over the one-dimensional mechanism and identify those desirable properties. Then we go over each of these criteria and formalize them for multi-dimensional problems. Finally, we propose a new solution concept called obviously regret-proof by combining all the elements we formalize.

1.3.1 Example: The One-Dimensional OSP Mechanism

Now we describe the descending clock mechanism in Milgrom and Segal (2020). Consider the allocation problem for the FCC Incentive Auction described in Example 2. We initialize the auction with every bidder being active, $A_t = N$, where A_t is the set of active bidders at period t. For each $t \geq 1$, the auctioneer calls an active bidder $i \in A_t$ to play. When i is called, the auctioneer announces the offered outcome $x_{it} = (0, p_{it})$ to bidder i, which specifies that the bidder to relinquish his rights to his endowment and to receive a payment p_{it}. The bidder can choose either to continue or to exit. When he exits, he secures his endowment outcome with zero payment, $x_i^0 = (1, 0)$ and he becomes inactive. The auction proceeds to $t+1$ with $A_{t+1} = A_t \backslash \{i\}$. If he chooses to continue, two things may occur. First, he could be called again at some future period, $t' > t$. In which case, the announced offer drops to $x_{it'} = (0, p_{it'})$, where $p_{it'} < p_{it}$. Once again, the bidder can choose to exit or to continue once again. Because the announced payment only descends, it is often referred to as the clock price. Second, the bidder could win outcome x_{it} and become inactive. Again, if this happens, the auction proceeds to $t+1$ with $A_{t+1} = A_t \backslash \{i\}$. The auction concludes at T when $A_T = \emptyset$.

We discuss the four aspects of strategic simplicity in this game.

1. The simplicity of the strategy: The dominant strategy in this game is simple for bidders to follow. To follow the rule, the bidder first compute the most

desirable terminal outcomes that can be reached from the current period. We refer to this kind of outcomes as the fantasy outcomes, as those outcomes are not necessarily fulfillable in an equilibrium. The rule instructs the bidder to secure one of the fantasy outcomes whenever possible; otherwise, choose any action that is needed to achieve one of the fantasy outcomes, as if he only looks at the bright side and waiting for the best case to happen. In the English auction, if a bidder prefers to win at the clock price to lose, then the winning outcome with the current clock pricing is the unique fantasy outcome for the bidder. This outcome is fulfilled only if the bidder chooses to continue and all other bidders exit right away. The rule requires the bidder to choose to continue to pursue his fantasy outcome. If at some point the clock prices rises such that losing is weakly better than winning, then the losing outcome becomes the new fantasy outcome. The rule then requires the bidder to secure this fantasy outcome by choosing to exit. We call strategies following this simple rule the cautiously optimistic strategies.

2. The simplicity of the optimality concept: It suffices for the bidder to understand that whenever he is called to play, he will not regret choosing the action that pursues his fantasy outcome. Bidders don't need to understand the concept of weakly or obviously strategic dominance to know that the optimality of this strategy does not depend on the choices of other bidders.

3. The simplicity of executing the dominant strategy: In this auction, it is easy for bidders to accurately execute the cautiously optimistic strategy. Because

the auctioneer explicitly reveals the best-case outcome achievable by each of the actions. Also, it is very straightforward to secure an outcome. Therefore, executing the cautiously optimistic strategy is simple in this one dimensional mechanism.

4. The simplicity of verifying the optimality: It is easy to verify that a bidder does not regret playing the cautiously optimistic strategy. Because there exists a clear pattern about the future availability of the outcomes a bidder forgoes. When a bidder chooses to continue in order to pursue the winning outcome, he knows that if the best-case did not occur, he will still be offered the chance to secure the same exit outcome. Therefore, the worst case that can happen when he plays the cautiously optimistic strategy is that he chooses to exit in some future period. We call this clear pattern about the availability of forgone outcome the strong persistence of exits. It makes it easy for bidders to verify that their cautiously optimistic strategies are indeed regret-free.

In the rest of this section, we will extend and formalize the above concepts for multi-dimensional mechanisms. Then we combine all the elements and arrive at a new criterion of strategic simplicity, obviously regret-proofness (ORP).

1.3.2 Simplicity of the Dominant Strategy

To make it easy for bidders to participate in a mechanism, there should be a clear rule for the bidder to select the best action according to the information revealed by the auctioneer. Also, the rule should be applicable to all periods that he is called to play. In general, a strategy can be a very large mapping if the game has a large number of contingencies. Even if the strategy is already solved and given to the bidder, without a simple pattern, the strategy is merely an overwhelmingly long list of what to do in each of the contingencies. Then it would not be easy for bidders to handle this list and to look up what to do when he is in the real auction. Therefore, to formalize a criterion for strategic simplicity, we need to be explicit about the rule for bidders to choose their optimal actions.

The simple strategy that we are about to formalize is called cautiously optimistic strategy. We first assume that the bidder knows his fantasy outcomes whenever he is called to play. Fantasy outcomes are the most desirable outcomes achievable in the game tree from the current period. Those outcomes need not to be assurable in an equilibrium. However, when some of them are, the cautious part requires the bidder to secure one of the assurable outcomes. Otherwise, the optimistic part requires the bidder to pursue one of the fantasy outcomes by choosing the action needed to achieve it.

Below, we first introduce the optimistic strategy, then refine it with the notion of cautious. We show that even the notion of optimistic strategy may appear to be myopic, it is necessary for all weakly dominant strategies. Then we show that once we refine it with caution, then it suffices to be obviously dominant in an OSP mechanism.

Optimistic Strategies

Now we introduce the optimistic strategy. An optimistic strategy always selects an action with the most preferred best-case outcome. It is called optimistic because the strategy is formed as if the bidder is choosing naively, without considering whether the best-case outcome can be realized in an equilibrium. Therefore, we call the best-case outcome *fantasy outcome*. We formalize the above concepts as follows.

Let $I_i \in \mathcal{I}_i$, an outcome x_i is said to be a fantasy outcome of action $a \in A(I_i)$ for $u_i \in \mathcal{U}_i$ if $x_i \in \arg\max\{u_i(x_i')|x_i' \in X_i(a)\}$. That is, x_i is the most preferred outcome achievable by action a. Similarly, we say an outcome x_i is a fantasy outcome at I_i for u_i if $x_i \in \arg\max\{u_i(x_i')|x_i' \in X_i(I_i)\}$. Let $X_i^*(I_i, u_i)$ denote the set of fantasy outcomes at I_i for u_i.

Definition 5 (Optimistic Strategy) *A strategy S_i for bidder $i \in N$ is said to be optimistic for $u_i \in \mathcal{U}_i$ if for all I_i on the path of S_i, if $S_i(I_i) = a$, then*

$$\max\{u_i(x_i)|x_i \in X_i(a)\} \geq \max\{u_i(x_i)|x_i \in X_i(a')\} \tag{1.3}$$

for all $a' \in A(I_i)$.

The condition described in Equation (1.3) echoes the condition in Equation (1.4). That is, if a worst-case outcome of following strategy S_i and choosing a is better than the fantasy outcome of any other a', then a must have the most preferred fantasy outcome among all available actions. It turns out that this is not only a necessary condition for obviously dominant strategy, it is also a necessary condition for all weakly dominant strategy.

Theorem 1 *Let D be a dynamic mechanism on Ω. If a strategy S_i on D is weakly dominant for $u_i \in \mathcal{U}_i$, then S_i is optimistic for u_i.*

The intuition behind the necessary part is simple. If a more desirable outcome is achievable by another action, it implies that there exists some strategy profile (S'_i, S'_{-i}) such that when other bidders play S'_{-i}, the better outcome x'_i will be assigned to i if i deviates to S'_i. This is a violation of weak dominance as well as obvious dominance.

However, the optimistic strategy is not a definite guide for bidders in OSP mechanisms. Because it is possible for some fantasy outcome $x^*_i \in X^*_i(I_i, u_i)$ to be achievable by more than one action. In that case, choosing an action that has the most preferred fantasy outcome might not be obviously dominant. This condition is prevalent in OSP mechanisms. For example, in the English Auction with one good to be sold, when the clock price is already above the bidder's value for that object, the fantasy outcome for that bidder is to lose, as there is no way he could win and get a positive payoff. However, he could achieve this outcome either by exiting at the current period or by continuing in the current period and exiting at some future period. Thus, choosing both actions in the current period is optimistic. As it should be clear in this example, the bidder should secure his fantasy outcome whenever he is able to. That is exactly the refinement we are about to introduce.

Cautiously Optimistic Strategies

To refine the concept of optimistic, we introduce the concept of cautiously optimistic. First, we formalize the idea of securing an outcome. We say a strategy S_i secures outcome $x_i \in X_i$ at information set $I_i \in \mathcal{I}_i$ if $X_i(S_i, I_i) = \{x_i\}$. Intuitively, this means that starting from all histories in I_i, playing according to S_i implies his outcome is fixed at x_i regardless of what other players do in future periods. We say x_i is assurable at I_i if there exists some strategy that secures x_i at I_i. When a bidder secures an outcome, it is like the mechanism has concluded for that bidder.

It should be clear that in the English Auction, to secure the losing outcome is equivalent to exiting the auction. In a multi-dimensional and multi-unit OSP mechanism, we can generalize the concept and offer bidders multiple assurable outcomes. Then, a cautious way to bid in this kind of mechanism is to exit and secure a fantasy outcome whenever some of the most preferred outcomes become assurable.

Definition 6 (Cautiously Optimistic Strategy) *We say a strategy S_i is cautiously optimistic for u_i if the following conditions hold:*

1. *If $X_i^*(I_i, u_i)$ contains some assurable outcomes, then S_i secures one of the assurable outcomes in $X_i^*(I_i, u_i)$ at I_i.*

2. *Otherwise, if $S_i(I_i) = a$ then $X_i(a) \cap X_i^*(I_i, u_i) \neq \emptyset$.*

The cautiously optimistic strategies always exist. Because we only consider games with finite length, the set of all possible outcomes is finite. Therefore, the set of achievable outcomes for each action also is finite. Because there always exist some most preferred outcomes in a finite set, the optimistic strategy always exits. Because the "cautious" part refines the strategy only when there is a tie, it does not affect the existence of the strategy.

Note that the cautiously optimistic strategies are not unique. If there are multiple assurable fantasy outcomes at I_i, since all of them are payoff equivalent, securing any of them is cautiously optimistic.

Even with the cautious part, the cautiously optimistic strategy may still sound somewhat naive. Because it pursues the fantasy outcome without caring how the outcome could be fulfilled, and whether it is rational for other bidders to play the strategies to fulfill this outcome. If the fantasy outcome is not guaranteed to be fulfilled, any rational bidder may question whether he would regret forgoing some of the second best outcomes while pursuing the first best. We argue that if the mechanism is OSP, bidders will never regret playing a cautiously optimistic strategy. But before we arrive at that result, we first have to formalize what do we mean by having no regret.

Regret-free Strategies

In this part, we propose a dominant strategy concept that makes it easy for a bidder to understand that the optimality of his action does not depend on what other people do. We call this condition regret-free, as the bidder will not regret all of his choices. This condition is equivalent to being obviously dominant, but is conceptually simpler.

Notice that the definition of an obviously dominant strategy describes a condition that a strategy dominates all other alternative strategies. To fully understand the optimality of the strategy, bidders need to conceptually construct alternative strategies and to consider the earliest points of departure. We argue that this is an unnecessary complication. For bidders with little training in game theory, it is hard for them to grasp the full concept behind alternative strategies the earliest points of departure.

However, what is really defined in obviously dominant strategy is that contingent on being called to play, all the outcomes achievable by the deviating actions are not better than the worst-case outcome of not deviating. Here, we emphasize the dominance relations between actions, rather than entire strategies. Thus, it is easier for bidders to comprehend and to verify the optimality. Also, this interpretation does not require bidders to construct alternative strategies at all, because the achievable outcomes are characteristics of the game tree, and they should be intuitive enough. We now formalize this concept as follows:

Definition 7 (Regretfree Strategy) *A strategy S_i is said to be regret-free in a dynamic mechanism D if for all I_i that is on the path of S_i, if $S_i(I_i) = a$, then for all $a' \in A(I_i)\backslash\{a\}$, we have*

$$\min\{u_i(x_i)|x_i \in X_i(S_i, I_i)\} \geq \max\{u_i(x_i)|x_i \in X_i(a')\} \tag{1.4}$$

The above definition is equivalent to Li's (2017) definition for an obviously dominant strategy.

Theorem 2 *A strategy S_i in mechanism D is obviously dominant if and only if it is regret-free.*

Regret-Proof Mechanisms

Now we formalize the notion of the strategic simplicity of a mechanism where all cautiously optimistic strategies are regret-free. This condition guarantees bidders that it is safe for them to pursue their fantasy outcomes and that they will not regret each of their decisions.

Definition 8 *A dynamic mechanism D is said to be regret-proof if for all $i \in N$ and $u_i \in \mathcal{U}_i$, if S_i is cautiously optimistic for u_i, then S_i is regret-free for u_i.*

This condition turns out to be equivalent to OSP. That means that for all OSP mechanisms, the cautiously optimistic strategy is the simple rule for the bidders to follow. Conversely, because regret-free is equivalent to obviously dominant, any mechanisms satisfying this condition is OSP.

Theorem 3 *Let D be a dynamic mechanism on market Ω. Then D is OSP if and only if it is regret-proof.*

Even though regret-free and OSP are equivalent, there are some conceptual differences in their definitions. First, regret-proofness explicitly indicates the simple rule for bidders to follow, and it emphasizes its desirable incentive property. Therefore, it should be easier for bidders to execute the strategy and comprehend its optimality. While OSP is intended to be a stronger version of strategy-proofness.

It better connects and contrasts with the concept of strategic dominance. Second, for regret-proofness, we do not claim that it is obvious for bidders to "verify" that the cautiously optimistic strategy is regret-free. Just as verifying obviously dominant strategies, the bidder still needs to compute the best- and worst-case outcomes. Therefore, especially for mechanisms of higher dimensionality, the regret-proof may be a better term and a better interpretation of the condition defined in OSP.

In the following parts, we then focus on the elements of a mechanism that simplifies the execution and the verification of the cautiously optimistic strategy.

1.3.3 Simplicity of Executing the Optimal Strategy

To accurately execute the cautiously optimistic strategy, there are still a few things that bidders need to determine. First, bidders need to know the best-case outcomes achievable by each of the available actions. Second, they need to know which of them is assurable, and how to secure the outcome. In an arbitrary multi-dimensional mechanism, the set of achievable outcomes from an action may be large. However, the auctioneer can help to simplify the computation by summarizing the relevant information for bidders. Just as in the personal clock mechanism, the auctioneer explicitly announces the best-case outcome to continue and the best-case outcome to exit for bidders. In this part, we generalize this idea to multi-dimensional problems.

Continuation and Exit Outcomes

The first step to summarize and simplify the information is to filter out the outcomes that are irrelevant for bidders to execute their cautiously optimistic strategies. Roughly speaking, it suffices for the auctioneer to let bidders know the outcomes that could be the fantasy outcome to some $u_i \in \mathcal{U}_i$. In the following, we formalize this idea for both assurable and nonassurable outcomes.

Given $x_i, x_i' \in X_i$, we say x_i' dominates x_i if $x_{ik}' \geq x_{ik}$ for all $k \in K_i$. We say x_i' strictly dominates x_i if x_i' dominates x_i and $p_i' > p_i$. Let $X_i' \subseteq X_i$, we say x_i is undominated if there is no other $x_i' \in X_i'$ such that x_i' dominates x_i; we say x_i is weakly undominated in X_i' if there exists no $x_i' \in X_i'$ such that x_i' strictly dominates x_i.

Lemma 1 *1. There exists $u_i \in \mathcal{U}_i$ such that $u_i(x_i) > u_i(x_i')$ for all $x_i' \in X_i(I_i) \cup X_i^G(I_i) \backslash \{x_i\}$ if and only if x_i is undominated in $X_i(I_i) \cup X_i^G(I_i)$.*

2. There exists $u_i \in \mathcal{U}_i$ such that $u_i(x_i) \geq u_i(x_i')$ for all $x_i' \in X_i(I_i) \backslash \{x_i\}$ if and only if x_i is weakly undominated in $X_i(I_i)$.

We say $x_i^g \in X_i$ is a forgone outcome at I_i if there exists $I_i' \in \psi_i(I_i)$ and $a' \in A(I_i')$ such that $a' \notin \psi_i(I_i)$ and $x_i^g \in X_i(a')$. In words, x_i^g is an outcome achievable by some action that was not chosen in a previously visited information set. Let $X_i^G(I_i)$ denote the set of all forgone outcomes at I_i.

We say an outcome x_i is an exit outcome if x_i is assurable and weakly undominated in $X_i(I_i) \cup X_i^G(I_i)$. Let $X_i^E(I_i)$ be the set of all exit outcomes in $X_i(I_i)$. An outcome $x_i \in X_i(I_i)$ is said to be a continuation outcome at I_i if the following three conditions hold: (1) x_i is nonassurable at I_i, (2) x_i is undominated in $X_i(I_i) \cup X_i^G(I_i)$ and (3) $x_i \notin X_i^G(I_i)$. Let $X_i^C(I_i)$ denote the set of continuation outcomes at I_i.

Intuitively, an exit outcome is an achievable outcome that some bidders may choose to secure. A continuation outcome is an outcome that can induce some bidders to visit the current information set and still choose to continue.

The following result shows that it is sufficient for the auctioneer to reveal just the exit and continuation outcomes to bidders for them to play their cautiously optimistic strategy.

Theorem 4 *Let D be a regret-proof mechanism. Then for all $i \in N$, $u_i \in \mathcal{U}_i$, S_i cautiously optimistic for u_i and I_i on the path of S_i, we have*

1. *if S_i secures a fantasy outcome $x_i \in X_i^*(I_i, u_i)$ at I_i, then x_i is an exit outcome.*
2. *if S_i does not secure any outcome, there exists a continuation outcome in $X_i(S_i(I_i)) \cap X_i^*(I_i, u_i)$.*

Conversely, if x_i is a continuation outcome at I_i, then there exists $u_i \in \mathcal{U}_i$ such that for all cautiously optimistic strategy S_i for u_i, I_i is on the path of S_i and $X_i^*(I_i, u_i) = \{x_i\}$.

In words, the first part of the above theorem says that a bidder either chooses to pursue a continuation outcome or choose to secure an exit outcome. That is, only the exit and continuation outcomes are the relevant candidates that a bidder may pursue with his cautiously optimistic strategy. Thus, it suffices for the auctioneer to reveal just the set of continuation and exit outcomes associated with each action. The bidders can identify the fantasy outcome to pursue from the announced outcomes.

The second part of the theorem says that it is necessary for the auctioneer to reveal all continuation outcomes. This is because that for any continuation outcome, there exist some bidders who visit this information set only to pursue this outcome and that he will continue to pursue it in this period. If the auctioneer did not reveal this outcome, the bidder will not be able to execute his cautiously optimistic strategy accurately.

Explicit Exits

We now characterize the structure of the available actions that make it simple for bidders to secure exit outcomes. First, we require that for all exit outcomes, there exists a unique dedicated action that secures and only secures the exit outcome. Also, once it is chosen, the bidder will never be called again. Second, we requires that the mechanism has no redundant actions.

We say an action $a \in A(I_i)$ is an exit action if $X_i(a)$ is a singleton and i is not called to play after choosing a. Let $E_i(I_i)$ be the set of exit actions at I_i. We say that an action is a continuation action if $X_i(a)$ contains some continuation actions.

Because the exit action achieves only one outcome, the worst- and best-case payoff of choosing it is obvious to bidders. Thus, it should be the default action a bidder chooses when he needs to secure an exit outcome. Once every exit outcome has a dedicated exit action, we claim that any action $a \in A(i_i)$ that is neither a continuation action nor an exit action is redundant. By Theorem 4, a bidder either chooses to pursue a continuation outcome or secure an exit outcome. Because a is not a continuation action, $X_i(a)$ contains no continuation outcomes, bidders choosing to pursue continuation outcomes will not choose it. For bidders choosing to secure an exit outcome, there is a dedicated exit action that secures the same exit outcome. Therefore, action a is redundant.

We now state the full requirement:

Definition 9 *We say D has explicit exits if for all $i \in N$, $I_i \in \mathcal{I}_i$, the following conditions hold:*

1. *for all exit outcomes $x_i^e \in X_i^E(I_i)$, there exits an unique exit action $a \in E_i(I_i)$ with $X_i(a) = \{x_i^e\}$.*

2. *for all $a \in A(I_i)$, a is either an exit action or a continuation action.*

In short, mechanisms with explicit exits enforce a clear way for bidders to secure an exit outcome. They keep the essential continuation actions and exit actions, and remove all redundant ones.

The Clock Announcement

Now we are ready to formalize the multi-dimensional clock announcement for regret-proof mechanisms with explicit exits. In a clock announcement, the auctioneer tells each bidder the set of exit actions and the set of continuation actions. For continuation action, the auctioneer tells the bidder only the set of continuation outcomes. If a is an exit action, the auctioneer reveals the unique exit outcome it secures. According to Theorem 4, to ensure bidders can accurately identify their cautiously optimistic strategy, it suffices for the auctioneer to reveal just the continuation outcomes and the exit outcomes to bidders. Also, since all exit outcomes have dedicated exit actions, there is no need to announce the achievable exit outcomes for a continuation action.

Formally, let D be a regret-proof mechanism with explicit exits. The clock announcement at I_i is a tuple, $(C_i(I_i), E_i(I_i), X_i^A)$, such that

1. $E_i(I_i)$ is the set of exit actions at I_i.

2. $C_i(I_i)$ is the set of continuation actions at I_i.

3. $X_i^A(a)$ denotes the set of announced outcomes for an announced action $a \in E_i(I_i) \cup C_i(I_i)$, such that

   - if $a \in E_i(I_i)$, then $X_i^A(a) = X_i(a)$;

   - if $a \in C_i(I_i)$, then $X_i^A(a) = X_i^C(I_i) \cap X_i(a)$.

Notice that an announcement is purely a way for the auctioneer to summarize and communicate the current auction state to bidders. It does not impose any restrictions on the extensive form.

Descending Clock

The reason that this is called the clock announcement is that the announced continuation outcomes become inferior over time. This is

Lemma 2 *Let D be a regret-proof mechanism and let $I_i \prec I_i'$ such that $X_i^C(I_i') \neq \emptyset$, then for all $x_i' \in X_i^C(I_i')$, there exists $x_i \in X_i^C(I_i)$ such that $x_i \geq x_i'$.*

Definition 10 *Let $I_i \in \mathcal{I}_i$ and let I_i' be the immediate predecessor of I_i. Let $a' \in A(I_i)$ such that a' is the action taken to reach a. We say I_i is a descended node if there exists a continuation outcome x_i' at I_i' such that $x_i' \in X_i(a')$ but $x_i' \notin X_i(I_i)$.*

1.3.4 Simplicity for Verifying Optimality

To verify that the cautiously optimistic strategy is regret-free, a bidder needs to know the best-case outcome of an action and the worst-case payoff of following the cautiously optimistic strategy. The best-case outcomes are easy to see thanks to the explicit exits and the clock announcements. Therefore, what is left is to make it simple for bidders to know the worst-case outcome of following the cautiously optimistic strategy.

What we do is to make it clear for bidders that all of the current exit outcomes will be assurable again whenever some of their fantasy outcomes are no longer achievable. Therefore, the bidder always has the chance to secure one of those exit outcomes when the first best did not occur. Hence, the worst case that can happen by playing the cautiously optimistic strategy is that the bidder chooses to secure one of the exit outcomes in some future period. Therefore, the bidder will not regret forgoing the exit outcome in the current period in pursuit of his fantasy outcomes.

For bidder i, there are two possible scenarios in which his fantasy outcomes may disappear after he chooses a continuation action $a \in A(I_i)$. In the first case, bidder i is called again at some future information set I'_i and some of the continuation outcomes in $X_i(a)$ are no longer achievable in $X_i(I'_i)$. This is the generalized notion of the descending clock. In this case, we require that all the exit outcomes at I_i are assurable at I'_i. That is, $X_i^E(I_i) \subseteq X_i(I'_i)$.

For the second case, i is not called again before the auction reaches some terminal node $z \in Z$, where bidder i is assigned the outcome $\chi_i(z)$. If $X_i(a)$ contains more than one continuation outcome, in bidder i's perspective, all continuation outcomes other than $\chi_i(z)$ disappears. The core idea behind the Strong Persistence of Exits is that whenever some continuation outcomes disappear, all the previous exit outcomes should be assurable. But that conflicts with z being a terminal node. Therefore, we should prevent this case to make it clear for bidders that his cautiously optimistic strategy is regret-free. That means that we require that when a bidder chooses an action with more than one continuation outcome, he must be called again in some future information sets.

The above requirement is relevant only in multi-dimensional mechanisms. For example, in the English Auction, there is only one continuation outcome associated with the continuation action. Therefore, if a bidder chooses to continue, we know that he prefers the continuation outcome over the exit outcome. Thus, when all other bidders leave, the auctioneer can assign the unique continuation outcome to the winning bidder without calling him to action again.

The above two conditions are formalized as follows:

Definition 11 *We say D satisfies the property of strong persistence of exits if the* *following two conditions hold:*

1. *if I_i is a descended node, let I_i' be the immediate predecessor of I_i, then $X_i^E(I_i) \supseteq X_i^E(I_i')$.*

2. *for all $I_i \in \mathcal{I}_i$ and $a \in A(I_i)$, if $X_i(a)$ contains more than one continuation outcomes, then for all $z \in Z$, if $I_i \prec z$, then there exists I_i' such that $I_i \prec I_i' \prec z$.*

In words, the first condition says that the set of exit outcomes does not shrink whenever the clock descends. The second condition guarantees that when there are multiple continuation outcomes achievable by the same action, the auctioneer will always call him to play in some future period. Because he is called to play, the first condition can kick in to ensure that the previous exit outcomes are again available, making sure the bidder has no regret. In short, the combination of the two conditions makes it easy for a bidder to know that he will not regret forgoing any of the exit outcomes.

The Single Continuation Property

One may ask whether the bidders might regret forgoing other continuation outcomes. Because we may have multiple continuation actions, choosing one of them also implies forgoing other continuation outcomes. In the following result, we show that in any regret-proof mechanism, if a bidder chooses a continuation action a, all the continuation outcomes achievable by other actions will be less desirable than some exit outcome x_i^e. Therefore, if a bidder does not regret forgoing x_i^e, nor will he regret forgoing the continuation outcome.

Theorem 5 *Let D be an regret-proof dynamic mechanism. Let $u_i \in \mathcal{U}_i$, let S_i be a cautiously optimistic strategy for u_i and let I_i on the path of S_i. If $X_i^*(I_i, u_i)$ contains no assurable outcomes, then there exists $a \in A(I_i)$ such that*

- $X_i^*(I_i, u_i) \subseteq X_i(a)$ *and*
- $X_i^*(I_i, u_i) \cap X_i(a') = \emptyset$ *for all* $a' \in A(I_i) \backslash \{a\}$.
- *If x_i is a continuation outcome at I_i but $x_i \notin X_i(a)$, then there exist x_i^e assurable at I_i such that $u_i(x_i^e) \geq u_i(x_i)$.*

In words, this result says that at each information set, there is at most one continuation action that is relevant to the bidder. All the fantasy outcomes are achievable only by that action. Furthermore, all the other continuation actions are dominated in the sense that each of the outcomes they can achieve is worse than some exit outcome. Therefore, if a bidder does not regret forgoing that exit outcome, he will not regret forgoing the continuation outcome. Thus, we do not have to maintain the persistence of the forgone continuation outcomes. It suffices just to enforce the persistence of "exits".

The conditions of the extensive form that are needed to enforce this property are stated in Definition 15. It might seem a little be technical. However, bidders do not have to check if the extensive form or the auction rule satisfies this condition. Because as long as the mechanism is regret-proof, the bidder can directly observe the announced outcomes and easily verify that there is only one relevant continuation action. Therefore, it is still easy for a bidder to see that he will not regret following the cautiously optimistic strategy.

1.3.5 The Obviously Regret-Proof Mechanism

Now we are ready to propose the new solution concept by combining all the aspects of simplicity formalized in previous parts.

Definition 12 (Obviously Regret-Proof Mechanism) *Let D be a dynamic mechanism on market Ω, we say D is obviously regret-proof if the following conditions are met:*

1. *D is regret-proof,*
2. *D has explicit exits,*
3. *D implements the descending clock announcement,*
4. *D satisfies the property of the strong persistence of exits.*

To summarize, obviously regret-proofness (ORP) requires a mechanism to be easy for a bidder to understand that he will not regret selecting actions cautiously and optimistically whenever he is called to play. The regret-proof part is equivalent to being OSP. But it does not guarantee obviousness. The obviousness comes from the requirements of explicit exits, clock announcements, and the strong persistence of exits. That means that, we require that securing an outcome is simple and immediate; the auctioneer announces summarized information that reveals only the outcomes that bidders would pursue and all of the available exits; and finally, the set of exit outcomes does not shrink whenever the best-case did not happen and the clock descends.

Now we have a criterion for strategic simplicity. Then the next question is whether this notion of strategic simplicity can be implemented in a multi-dimensional mechanism. The next section confirms it by proposing a generic implementation of the ORP mechanism.

1.4 The PED Mechanism

In this section, we present the Persistent Exit Descending (PED) mechanism, which is designed to be ORP. Or we can also interpret the PED mechanism as a reformulation of ORP mechanisms in the language of auction rules, as opposed to the characteristics of extensive forms. We first present the auction rules, and then we use some examples to demonstrate how they work and why it is a meaningful generalization of the personal clock mechanism and the English Auction. Finally, we present the results of its generality. That is, the PED mechanisms are the generic implementation of all ORP mechanisms. Also, the PED mechanisms also implement all well-behaved OSP implementable allocation rules.

1.4.1 Auction Rules

We introduce the PED mechanism in the following order. First, we introduce the flow of the auction; second, we introduce the requirement for what the auctioneer should announce to bidders. Third, we present the requirements for the evolution of the announcements. Finally, we state the desirable properties of the PED mechanisms.

The Flow of the Auction

At $t = 0$, the auctioneer initializes the auction with the set of active bidders $A_0 = N$. At each period $t \geq 0$, the auctioneer calls a bidder $i \in A(t)$ to play. When bidder i is called to play, the auctioneer offers him two sets of actions: (E_{it}, C_{it}). Actions in E_{it} are called exit actions, while actions in C_{it} are called continuation actions. For each action $a \in E_{it} \cup C_{it}$, the auctioneer also announces a set of outcomes achievable by this action, $X_i^A(a) \subseteq X_i$. Bidder i observes (E_{it}, C_{it}) and $X_i^A(a)$ for all offered actions, and then submits one action a_{it} in $E_{it} \cup C_{it}$ to the auctioneer. Then the auction enters the next period, $t + 1$.

Let $X_{it}^E = \cup_{a \in E_{it}} X_i^A(a)$ be the set of all outcomes specified in the exit actions. Similarly, let $X_{it}^C = \cup_{a \in C_{it}} X_i^A(a)$ be the set of outcomes specified in the continuation actions. Let $X_{it} = X_{it}^C \cup X_{it}^E$.

We require that for all exit actions, $a \in E_{it}$, $X_i^A(a)$ is a singleton. If $a \in E_{it}$, $X_i^A(a) = \{x_i\}$, and bidder i chooses a at period t, bidder i will be assigned x_i at the end of the auction. Also, bidder i becomes inactive, that is, $A_{t+1} = A_t \backslash \{i\}$.

If bidder i chooses to continue with $a_{it} \in C_{it}$, one of the following two cases will happen in some future period t'. First, bidder i wins $X_i^A(a_{it})$ and must secure one of the winning outcomes in $X_i^A(a_{it})$. If $X_i^A(a_{it})$ is a singleton, then i is directly assigned the winning outcome and will not be called to play. Otherwise, i is called at t', and is offered $X_{it}^C = \emptyset$ and $X_{it}^E = X_i^A(a_{it})$. That means that, i has to choose which one in $X_i^A(a_{it})$ to secure. After t', i becomes inactive.

In the second case, the clock descends in the sense that for all $x_i' \in X_{it'}$, there exists $x_i \in X_{it}$ such that $x_i' \leq x_i$. That is to say, each of the offered outcomes in $X_i^A(a_{it})$ becomes either weakly inferior or it disappears. However, in this case, we require the exit outcomes to be persistent, $X_{it}^E \subseteq X_{it'}^E$. That means that, we allow some of the weakly descended outcomes to turn into exit outcomes. But once they become exit outcomes, they will always be exit outcomes whenever the clock descends.

The Requirements of the Announced Outcomes

There are a few requirements for the announced outcomes. We require that each $x_i \in X_{it}^E$ is weakly undominated in X_{it} and that each $x_i \in X_{it}^C$ is undominated in X_{it}. Also, we require that each of the announced outcomes in X_{it}^C is achievable but nonassurable in this mechanism. This requirement implies two guidelines for the auctioneer. First, if there is no way that an outcome can be fulfilled, it should not be announced at all. Second, if an outcome will not descend or disappear in any case, announce it as the exit outcome, not the continuation outcome. The purpose of these requirements is to guarantee all outcomes in X_{it}^E are exit outcomes and X_{it}^C are continuation outcomes.

Another more technical requirement is that, for all $a, a' \in C_{it}$ and all $x_i \in$

$X_i^A(a), x_i' \in X_i^A(a')$, there exists $x_i^e \in X_{it}^E$ and $\lambda \in (0,1)$ such that $x_i^e \geq \lambda x_i + (1-\lambda)x_i'$. This condition is needed to satisfy the single continuation property defined in Definition 15. It is necessary to guarantee that there is at most one continuation action relevant to all types of bidders. This condition, together with the strong persistence of exits, is sufficient to guarantee the mechanism is regret-free.

1.4.2 PED is a Meaningful Multi-Dimensional Generalization

We now argue that the PED mechanism is a meaningful generalization of the English Auction and the personal clock auction. The PED mechanism collapses to the personal clock mechanism if there is only one good to be allocated to i, and that good is his endowment. In that case, we have $X_{it}^C = \{(0, p_{it})\}$, and $X_{it}^E = \{x_i^0\} = \{(1,0)\}$ in each period. Because there is only one continuation outcome and one exit outcome, there is only one continuation action and one exit action, as in the personal clock mechanism.

The PED mechanism collapses to the English Auction if there is only one common good to allocate to all bidders, and they are alternately called to offer the same outcome. In this case, $X_{it}^E = \{(0,0)\}$ and $X_{it}^C = \{(1, p_{it})\}$ with $p_{it} \leq 0$. The payment $p_{it} \leq 0$ means that the bidder should pay $|p_{it}|$ to the client if he wins. The mechanism is still "descending" in a sense that $p_{it'} < p_{it}$ if $t' > t$. It is called an "ascending clock" auction in the literature because usually the auctioneer announces $|p_{it}| \geq 0$. However, the nature of the PED mechanism, including the English Auction, is that the outcome achievable by choosing to continue becomes less and less desirable.

The Simplified but Multi-Dimensional FCC Incentive Auction Example

Let's revisit Example 4 to see how a PED mechanism works for this simplified, but multi-dimensional version of the FCC Incentive Auction. In this example, we have two UHF station owners and two VHF station owners, but the buyer wants only one of the UHF stations. We model this problem in a way such that these four bidders are divided into two competing groups. Each group contains one UHF owner and one VHF owner, that is $\{1,3\}$ and $\{2,4\}$. Then the two groups must compete for the right to clear the UHF station for some compensation in a PED mechanism.

At $t = 0, 1$, offer the VHF bidders $X_{it}^C = \{(0, p_{it})\}$ and $X_{it}^E = \{(1, 0)\}$. That is, ask them if they are willing to clear their station at the clock price p_{it}. Based on their response, make conditional offers to the paired UHF owners. If bidder 3 chooses to continue at $t = 1$, then at $t = 3$, we offer bidder 1 one continuation action and one exit action, such that $X_{1t}^C = \{(0, 0, p_{1t}), (0, 1, p_{1m})\}$ and $X_{1t}^E = \{(1, 0, 0)\}$. That is, if bidder 1 chooses to continue, he might have the chance to sell his station at p_{1t} or move to bidder 3's station and receive payment p_{1m}. Or he could choose to exit and just keep his endowment. If bidder 3 chooses to exit at $t = 1$, do not offer bidder 1 the chance to move at $t = 3$, that is, $X_{1t}^C = \{(0, 0, p_{1t})\}$. Offer similarly to bidder 2 at $t = 4$. Then we can alternately lower the clock prices p_{it} for $i = 1, 2, 3, 4$

for the two groups to compete the right to clear the UHF station.

If one of the VHF bidders chooses to exit, the problem for the paired UHF bidder falls back to the one-dimensional one in Example 2. But if the paired VHF station chooses to continue, he has an additional offer to move. Therefore, even if the clock price to sell p_{it} may have gotten too low, if the option to move is still acceptable, the UHF bidder will still choose to continue. Therefore, the cost of moving to a VHF station becomes another dimension to compete with bidder 2. Also, we can have bidders 3 and 4 compete by lowering their clock prices. If one of them exits, we must immediately remove the moving outcome from the continuation outcomes of the paired UHF bidder. The winning UHF station is determined when his UHF opponent chooses to exit. When that happens, he has the chance to secure one of the outcomes in his previously chosen continuation action. If the paired VHF station has exited, that means he will sell his station at the most recent clock price. If the VHF station is still active, he will be called again and he must choose between selling and moving. If the winning UHF bidder chooses to move, the paired VHF owner wins his continuation outcome and will have to transfer his station to the UHF owner. If the UHF winner chooses to sell, the VHF owner will be forced to exit.

In this example, we see that the competition intensifies when we expand the possibility of transactions compared to the unit-supply model. The two UHF owners are not only competing on the reserve value of their endowed spectrum, but they are also competing on the cost to move to a VHF station. The PED mechanism can determine multi-lateral transactions that have a higher social surplus, or it can increase the probability of having a transaction. Therefore, the PED mechanism is a generalization with meaningful welfare implications.

1.4.3 PED is the Generic Implementation of ORP

We now state that, not only is PED designed to be ORP, but also all ORP mechanism are PED mechanisms. Therefore, we can say that PED is the way ORP mechanisms are implemented in the language of auction rules, instead of the characteristics of extensive form. Thus, if a market designer wishes to implement an ORP mechanism, the PED mechanism is the mechanism of choice.

Theorem 6 *A dynamic mechanism is ORP if and only if it is a PED mechanism.*

Because ORP implies OSP, it also means that the PED mechanism is a multi-dimensional OSP implementation.

Corollary 7 *A PED mechanism is OSP.*

Verifying Regret-Proofness

If we claim the PED mechanism is obviously regret-proof, then at least it should not be too hard for us to understand that it is regret-proof from the above rules and the observed announcement. First, we restate the cautiously optimistic strategy.

When bidder i is called to play given the announced outcomes, X_{it}, the bidder first computes the most desirable announced outcomes X_{it}^*. If some of the most desirable outcomes are exit outcomes, then arbitrarily choose one and select the corresponding action to secure it. Otherwise, the bidder will observe that all his most favorite outcomes are achievable by only one continuation action in C_{it}, select that action.

We can verify that the strategy is indeed regret-free. First, if i chooses to secure an exit outcome. Then the worst-case outcome of this strategy trivially equals the secured outcome. Since it is the most preferred outcome, it is better than all other exits. Also, because the clock only descends, there is no way he can get a more desirable outcome by choosing to continue. Therefore, he will not regret choosing to secure this outcome. Next, if the bidder chooses to continue, then whenever any of his fantasy outcome in X^*_{it} becomes unachievable, the previously available exits will again be assurable. Therefore, the worst-case to happen is that he chooses one of the exits in some future period. Therefore, the worst-case payoff of following this strategy is not worse than the best-case payoff of any other deviation.

1.4.4 No Loss of Generality

Now we formalize what we mean when we say that the PED mechanism implements all well-behaved OSP-implementable allocation rules.

A direct mechanism is a pair of functions $M = (\alpha_N, \rho_N)$, such that $\alpha_N : \mathcal{U}_N \to \mathcal{F}$ and $\rho_N : \mathcal{U}_i \to \mathbb{R}^{|N|}$. We say a bidder is a winner in allocation $y_N \in \mathcal{F}$ if $y_i \neq y_i^0$. Let $n(y_N) \equiv \{i \in N | y_i \neq y_i^0\}$ denotes the set of winners for y_N.

Definition 13 *We say D OSP-implements M if for all $u_N \in \mathcal{U}_N$, there exists S_N*

such that for u_i, we have

1. *S_i is obviously dominant for all $i \in N$, and*

2. *$\chi_i(z(h_0, S_N)) = (\alpha_i(u_N), \rho_i(u_N))$.*

In which case, we say M is OSP-implementable.

When we mentioned a *well-behaved* allocation rule earlier, we referred to monotonicity. Intuitively, it means that when all bidders report an alternative preference where the currently assigned outcome becomes more attractive, they should all be assigned the same outcome. This condition also implies that, if a coalition of bidders values their allocation more, then not only the assignment to this coalition should stay unchanged, but also the assignment to other bidders should be the same.

To formalize it, we first must define a set of partial orders on $\mathcal{U}_i$ based on the relative attractiveness of an outcome x_i. Let $\bar{U}(x_i, u_i) \equiv \{x_i' \in X_i | u_i(x_i) \geq u_i(x_i)\}$ denote the upper contour set. For each $x_i \in X_i$, we define a relation on $\mathcal{U}_i$: $u_i \geq_{x_i} u_i'$ if $\bar{U}(x_i, u_i) \subseteq \bar{U}(x_i, u_i')$ Intuitively, if the upper contour set of x_i shrinks, then fewer outcomes are more desirable than x_i. Hence, the smaller the upper contour set, the more valuable x_i is relative to other outcomes. On the other hand, suppose u_i and u_i' have the same upper contour sets at x_i, quasi-linearity requires that u_i and u_i' represent the same preference.

Lemma 3 *For all $x_i \in X_i$, the relation $\geq_{x_i}$ is transitive, reflexive, and anti-symmetric on $\mathcal{U}_i$. That is, $\geq_{x_i}$ is a partial order on $\mathcal{U}_i$.*

Definition 14 *A direct mechanism $M = (\alpha_N, \rho_N)$ is said to be monotonic if for all $u_N \in \mathcal{U}_N$, if $\alpha_N(u_N) = y_N$, $\rho_N(u_N) = p_N$, $x_N = (y_N, p_N)$, then for all $u'_N \in \mathcal{U}_N$ such that $u'_i \geq_{x_i} u_i$ for all $i \in N$, we have $\alpha_N(u'_N) = y_N$.*

Finally, we are ready to formally state the result:

Theorem 8 *Let M be a monotonic direct mechanism, then M is OSP-implementable if and only if M is PED-implementable and therefore ORP-implementable.*

The reason we need monotonicity is to prevent cases like the following: Suppose there are two different actions at I_i, and both of them secure the same outcome x_i. Let $u_i \preceq_{x_i} u'_i$ and both preferences are supposed to secure x_i at I_i. Then there could be an equilibrium where u_i and u'_i take different actions, and the auctioneer is able to make different arrangements to the rest of the bidders based on the choice of actions. Then when we remove the duplicated action, we inevitably change the equilibrium outcomes. Then this OSP mechanism cannot be reconstructed to be a PED mechanism. We can exclude this scenario because it does not make sense to make different arrangements based on an arbitrary choice between the actions securing the same outcomes.

1.5 Necessary and Sufficient Conditions of OSP Mechanisms

Now we formally define the technical conditions for the necessary and sufficient conditions for a mechanism to be OSP.

Definition 15 (Single Continuation Property) *We say D satisfies the Single Continuation Property if for all $I_i \in \mathcal{I}_i$, the following conditions are met:*

1. *if x_i is a continuation outcome at I_i, then x_i is achievable by only one action in $A(I_i)$.*

2. *if x_i, x_i' are distinct continuation outcomes at I_i, and $x_i \in X_i(a)$, $x_i' \in X_i(a')$ for some distinct actions $a, a' \in A(I_i)$, then (x_i, x_i') is convex-dominated in $X_i(I_i) \cup X_i^G(I_i)$.*

Let $h \in H$, $I_i \in \mathcal{I}_i$, we say I_i is the most recent node of h for bidder i if

1. there exists $h' \in I_i$ such that $h' \prec h$,

2. for all h'' such that $h' \prec h'' \prec h$, $\iota(h'') \neq i$.

We say I_i is the most recent node of I_i' if there exists $h \in I_i'$ such that I_i is the most recent node of h for i. We say an action a is the most recent action of h for i if $a \in \psi_i(h) \cap A(I_i)$, where I_i is the most recent node of h for i.

Definition 16 (Disappearing Outcome) 1. *Let $I_i \in \mathcal{I}$ and a be the most recent action of I_i. We say x_i is a disappearing outcome at I_i if $x_i \in X_i(a)$ and $x_i \notin X_i(I_i)$.*

2. *Let $z \in Z$, and let a be the most recent action of z. We say x_i is a disappearing outcome for bidder i at z if $x_i \in X_i(a)$ and $\chi_i(z) \neq x_i$.*

3. *Let $X_i^D(I_i)$ and $X_i^D(z)$ denote the sets of disappearing outcomes for I_i and z, respectively.*

Definition 17 (Persistent Exit) *We say D satisfies the Persistent Exits Property if the following conditions hold:*

1. *for all* $I_i \in \mathcal{I}_i$, $x_i^g \in X_i^G(I_i)$ *and* $x_i^d \in X_i^D(I_i)$, *if* (x_i^d, x_i^g) *is convex-undominated in* $X_i(I_i) \cup X_i^G(I_i)$, *then* x_i^g *is assurable at* I_i.

2. *for all* $z \in Z$, *if* $X_i^D(z) \neq \emptyset$, *then for all* $x_i^d \in X_i^D(z)$ *and* $x_i^g \in X_i^G(z)$, (x_i^d, x_i^g) *is convex-dominated by* $\chi_i(z)$.

Theorem 9 *Let* D *be a dynamic mechanism on market* Ω, *then* D *is OSP if and only if* D *satisfies the Persistent Exit Property and the Single Continuation Property.*

1.6 Chapter Conclusion

In this paper, we see that the definitions of the obviously dominant strategy and the obvious strategy-proofness do not immediately imply strategic simplicity in multi-dimensional and multi-unit allocation problems. To improve upon the concept of OSP, we proposed a refined solution concept called obviously regret-proof (ORP). ORP requires the auction rules and announcement to make it easy for any bidder to see that whenever he is called to play, he will not regret playing the cautiously optimistic strategy. This strategy simply tells bidders always to pursue their most desirable achievable outcomes and to secure one of them whenever possible. We then translated the extensive form characteristics of ORP mechanisms into the language of auction rules and proposed its generic implementation, the PED mechanism. We then showed that the PED mechanism is a meaningful extension of the one-dimensional personal clock auction and the English Auction. Also, the PED mechanism can implement any monotonic and OSP-implementable allocation rule. Therefore, enforcing stricter simplicity requirements does not sacrifice the generality of the implementable allocation rules.

Chapter 2: The Cost of Obviousness

2.1 Introduction

Market designers pursue mechanisms that have good incentive properties. They craft the allocation and payment rules to prevent bidders from manipulating the mechanism and making mistakes. However, that also implies that we lose some flexibility in those rules, which may reduce the welfare and the chance of transactions. The impact on welfare from those restrictions may depend on the structure of the allocation problem. Thus, whenever there is a new mechanism available, it is important for market designers to be aware of the restrictions and the welfare implications before fully adopting it.

In the previous chapter, we proposed both a new solution concept and a new mechanism to reduce strategic errors and manipulation by bidders. We also extended the implementation of strategic simple mechanisms to the new domain of multi-dimensional and multi-unit allocation problems. This raises new questions: what do we lose from pursuing this notion of strategic simplicity? What are the restrictions and welfare implications that are imposed on the allocation and payment rules? These questions are important for market designers to test whether a mechanism is a good fit for a given allocation problem. Given that the multi-dimensional allocation problem is very general, these questions should be relevant to many market designers.

To review, the solution concept proposed in the previous chapter is called Obviously Regret-Proofness (ORP), which is a refinement and an improvement upon the Obviously Strategy-proofness by Li (2017). OSP is proposed because the dominant strategy in a strategy-proof mechanism might not be easy to comprehend. Moreover, strategy-proof mechanisms might not be able to prevent profitable group deviations. Therefore, mistakes and manipulations still occur in strategy-proof mechanisms (Rees-Jones, 2018). In an OSP mechanism, it is easier for the bidders to verify that the dominant strategy out-performs all the others. Also, OSP implies weak group strategy-proofness (WGSP). This means that there is no group deviation that strictly benefits all the deviators. However, as criticized in the previous chapter, if we extend OSP mechanisms to the domain of multi-dimensional and multi-unit allocation problems, even OSP mechanisms cannot guarantee that the dominant strategy is easy to find and verify. Therefore, the concept of ORP was proposed to cover the elements of strategic simplicity that are not captured in the definition of OSP.

Concisely, an ORP mechanism is an OSP mechanism with more explicit rules and announcements to help bidders pursue and secure their most desirable best-case outcomes. It guarantees that there is a simple rule for each bidder to select the optimal action whenever he is called, a rule that will instruct him to choose the action that can achieve his most desirable achievable outcome and secure it whenever it is possible. The ORP mechanisms have auction rules and announcements that make it easier for bidders to determine the best-case outcomes and the ways to secure them. The chances to exit are persistent. Therefore a bidder will not regret forgoing these outcomes to pursue of his most desirable ones. In short, the ORP mechanism not only enhances the characteristics of the extensive form of a mechanism, but it also formalizes the requirements for the way that the auctioneer should filter information and communicate with bidders. All these efforts are designed to make it easier for bidders to participate and verify the optimality of their actions.

The PED mechanism is a translation of extensive form characteristics of ORP mechanisms to the language of auction rules. It is also a multi-dimensional extension of the personal clock auctions in Milgrom and Segal (2020) and Li (2017). The descending structure is similar—the longer the bidder stays in the mechanism, the less desirable the outcomes offered will be. What makes them different is that in PED mechanisms, bidders can be offered multiple ways to exit or to continue in the auction. The persistence of exits is the key property that makes the mechanism ORP, as it enforces a clear and predictable pattern about the availability of outcomes that bidders had previously forgone.

Thus, compared to weakly strategy-proof (SP) mechanisms, the PED mechanisms are more robust to avoid bidder mistakes and manipulations. However, for economists to make decisions, knowing the marginal benefits of using PED mechanisms is not enough. We also need to know the marginal costs. By marginal costs, we mean the restrictions on the allocation rules added to the requirements for strategy-proofness. Understanding these restrictions and their welfare implications, we can better evaluate the trade-off between strategic simplicity and the loss of welfare.

In this Chapter, we study the cost of the class of "obvious" mechanisms (both ORP, OSP). It suffices to consider the class of PED-implementable mechanisms only. This is because all well-behaved OSP-implementable mechanisms are also PED-implementable, as pointed out in the previous chapter. That means that even though ORP is a stricter notion of strategic simplicity compared to OSP, there is no loss of generality of the allocation rules it can implement.

To study the restrictions, we first establish our analytical framework by characterizing all the strategy-proof mechanisms. We show that a mechanism is strategy-proof if and only if it is a pricing mechanism. In a pricing mechanism, once the reports of other bidders are fixed, the bidder cannot change the payments of all the possible allocations with his own report. The mechanism then assigns the allocation that maximizes the bidder's payoff as if the bidder is doing so by himself given the schedule of prices. This result is comparable to the Groves mechanism (Groves, 1973). We call this payment the pricing of the allocation. Pricing not only determines the payment of an allocation contingent on being assigned, but it also determines the allocation rule, because the higher the pricing, the more likely it becomes the most preferred allocation for a bidder. Thus, the payment and allocation rule of a mechanism can be characterized by its pricing functions.

Next, we characterize the welfare-maximizing pricing mechanism, the Vickrey (1961)-Clarke (1971)-Groves (1973) (VCG) mechanism. We treat it as a baseline model to compare the differences in the pricing functions between the efficient ones and the PED-implementable ones. We show that an important feature of VCG pricing is the mutual influence of pricing between bidders. When the values of complementary allocations rise or the values of competing allocations fall, the pricing should increase to encourage the bidder to choose it. Therefore, the pricing of an allocation can reflect its marginal contribution to the welfare of the rest of the agents. Also, this influence is mutual as both competition and complementarity are mutual relationships.

In contrast, in PED-implementable mechanisms, the ability for other bidders to influence the pricing is restricted and the influence is one-sided. We show that it is inevitable that once the values of some other allocations exceed some thresholds, the channel to influence the pricing function is be blocked. We call that region where the influence halted the *constant pricing space*. Therefore, the larger the constant pricing space of an allocation, the more inefficient the allocation becomes. We then show that the constant pricing spaces form a hierarchical structure in the sense that the larger ones contain the smaller ones. This makes it inevitable for some allocations to have larger constant pricing spaces than others. Moreover, we show that a winning bidder can exert his influence only within the constant pricing space of his winning allocation, where his winning outcome cannot be influenced by the bidders he can influence. That implies that mutual influence between winners is not possible. Even though it is a restriction of the allocation rule, it also prevents profitable group deviations. This can be considered the cost of weak group strategy-proofness.

The rest of the chapter is organized as follows. First, in Section 2.2, we restate the multi-dimensional and multi-unit allocation problem and the PED mechanism introduced in Chapter 1. Then in Section 2.3, we build the analytical foundation by introducing the generic strategy-proof mechanism: the pricing mechanism. In Section 2.4, we characterize VCG pricing as a baseline for us to compare with the PED-implementable one. In Section 2.5, we present our major results. We show for all PED-implementable mechanism the influence between bidders is restricted and can be only one-way. We also characterize those restrictions and highlight the trade-off between the ability to influence and being influenced.

2.2 The Model

2.2.1 The Allocation Problem

The allocation problem we consider in this chapter is the same as the model in the previous chapter. It is a problem between a non-strategic bidder and a finite set of strategic bidders, N. The client is the agent who hires the auctioneer to host the mechanism. He offers to each bidder $i \in N$ an individualized allocation space $Y_i \subseteq \mathbb{R}^{K_i}$ for some $K_i \in \mathbb{N}$. An allocation $y_i = (y_{i1}, y_{i2}, \ldots, y_{iK_i}) \in Y_i$ specifies a K_i dimensional vector of goods and services assigned to bidder i. Let $y_i^0 \in Y_i$ denote the endowment allocation of bidder i. An outcome of a mechanism is a pair $x_i = (y_i, p_i)$, which specifies an allocation $y_i \in Y_i$ and a payment $p_i \in \mathbb{R}$ from the client to bidder i. The payment p_i can take negative values, which means bidder i makes a net payment of $|p_i|$ to the client. Let $X_i \equiv Y_i \times \mathbb{R}$ be the outcome spaces for bidders $i \in N$. We assume that each bidder has a weakly convex, weakly increasing, continuous preference on X_i that is quasi-linear in the payment p_i. Let $\mathcal{U}_i$ be the set of utility functions that represent those preferences such that the endowment outcome is normalized to zero, that is, $u_i(x_i^0) = u_i(y_i^0, 0)$. Let $Y_N \equiv \Pi_{i \in N} Y_i$ and $X_N \equiv \Pi_{i \in N} X_i$ be the set of all allocation profiles and outcomes, respectively. The client specifies a subset of allocation profiles that are feasible $\mathcal{F} \subseteq Y_N$, and his willingness to pay $w(y_N) \in \mathbb{R}$ for each feasible allocation profile $y_N \in \mathcal{F}$. We assume that the client is budget constrained to pay at most $w(y_N)$ for each $y_N \in \mathcal{F}$. A market is then a tuple that summarizes all the above elements, $\Omega = \langle N, X_N, \mathcal{U}_N, \mathcal{F}, w \rangle$.

2.2.2 The Direct Mechanism

A direct mechanism on market Ω is a pair of functions $M = (\alpha_N, \rho_N)$ such that $\alpha_N : \mathcal{U}_N \to \mathcal{F}$ and $\rho_N : \mathcal{U}_N \to \mathbb{R}^{|N|}$. We denote $\chi_i(u_N) = (\alpha_i(u_N), \rho_i(u_N))$.

A mechanism is said to be finite if the image $\alpha_N(\mathcal{U}_N)$ is finite.

Definition 18 *Let M be a direct mechanism on market Ω, we say M is strategy-proof if, for all $u_N \in \mathcal{U}_N$ and all $u_i' \in \mathcal{U}_i$, we have*

$$u_i(\chi_i(u_i, u_{-i})) \geq u_i(\chi_i(u_i', u_{-i})) \tag{2.1}$$

2.2.3 The PED Mechanism

To define a PED-implementable direct mechanism, we first need to restate the PED mechanism auction rules from Chapter 1.

The Flow of the Auction

At $t = 0$, the auctioneer initializes the auction with the set of active bidders $A_0 = N$. At each period $t \geq 0$, the auctioneer calls an active bidder $i \in A(t)$ to play. When bidder i is called to play, the auctioneer offers him two sets of actions: (E_{it}, C_{it}). Actions in E_{it} are called exit actions and actions in C_{it} are called continuation actions. For each action $a \in E_{it} \cup C_{it}$, the auctioneer also announces a set of outcomes achievable by this action, $X_i^A(a) \subseteq X_i$. Bidder i observes (E_{it}, C_{it}) and $X_i^A(a)$ for all offered actions, and then submits one action a_{it} in $E_{it} \cup C_{it}$ to the auctioneer. Then the auction enters the next period, $t+1$.

Let $X_{it}^E = \cup_{a \in E_{it}} X_i^A(a)$ be the set of all outcomes specified in the exit actions. Similarly, let $X_{it}^C = \cup_{a \in C_{it}} X_i^A(a)$ be the set of outcomes specified in the continuation actions. Let $X_{it} = X_{it}^C \cup X_{it}^E$.

We require that for all exit actions, $a \in E_{it}$, $X_i^A(a)$ is a singleton. If $a \in E_{it}$, $X_i^A(a) = \{x_i\}$, and bidder i chooses a at period t, bidder i will be assigned x_i at the end of the auction. Also, bidder i becomes inactive, that is, $A_{t+1} = A_t \backslash \{i\}$.

If bidder i chooses to continue with $a_{it} \in C_{it}$, one of the following two cases will happen in some future period t'. First, bidder i wins $X_i^A(a_{it})$ and must secure one of the winning outcomes in $X_i^A(a_{it})$. If $X_i^A(a_{it})$ is a singleton, then i is directly assigned the winning outcome and will not be called to play. Otherwise, i is called at t', and is offered $X_{it}^C = \emptyset$ and $X_{it}^E = X_i^A(a_{it})$. That means that, i has to choose which one in $X_i^A(a_{it})$ to secure. After t', i becomes inactive.

In the second case, the clock descends in the sense that for all $x_i' \in X_{it'}$, there exists $x_i \in X_{it}$ such that $x'_i \leq x_i$. That is to say, each of the offered outcomes in $X_i^A(a_{it})$ becomes either weakly inferior or it disappears. However, in this case, we require the exit outcomes to be persistent, $X_{it}^E \subseteq X_{it'}^E$. That means that, we allow some of the weakly descended outcomes to turn into exit outcomes. But once they become exit outcomes, they will always be exit outcomes whenever the clock descends.

The Requirements of the Announced Outcomes

There are a few requirements for the announced outcomes. We require that each $x_i \in X_{it}^E$ is weakly undominated in X_{it} and that each $x_i \in X_{it}^C$ is undominated in X_{it}. Also, we require that each of the announced outcomes in X_{it}^C is achievable but nonassurable in this mechanism. This requirement implies two guidelines for the auctioneer. First, if there is no way that an outcome can be fulfilled, it should not be announced at all. Second, if an outcome will not descend or disappear in any case, announce it as the exit outcome, not the continuation outcome. The purpose of these requirements is to guarantee all outcomes in X_{it}^E are exit outcomes and X_{it}^C are continuation outcomes.

Another more technical requirement is that, for all $a, a' \in C_{it}$ and all $x_i \in X_i^A(a), x'_i \in X_i^A(a')$, there exists $x_i^e \in X_{it}^E$ and $\lambda \in (0,1)$ such that $x_i^e \geq \lambda x_i + (1 - \lambda)x'_i$. This condition is needed to satisfy the single continuation property defined in Definition 15. It is necessary to guarantee that there is at most one relevant continuation action to all types of bidders. This condition together with the strong persistence of exits property is sufficient to guarantee the mechanism is regret-free.

2.2.4 Verifying the PED Mechanism is Regret-Proof

In the previous chapter, we showed that the PED mechanism is obviously regret-proof (ORP). Since we do not intend to dive into the definition of ORP, we just verify from the bidder's perspective that it is easy for him to choose an action and to see the optimality of his action. We can verify this without stating the formal definition of regret-free. We intentionally use the less formal language to verify this condition to show the simplicity of the concept behind ORP and PED mechanisms.

The Cautiously Optimistic Strategy

When bidder i is called to play, given the announced outcomes, X_{it}, the bidder first computes the most desirable announced outcomes X^*_{it}. If some of the most desirable outcomes are exit outcomes, the bidder could arbitrarily choose one and select the corresponding action to secure it. Otherwise, the bidder will observe that all his most favorite outcomes are achievable by only one continuation action in C_{it}, according to Theorem 5 in Chapter 1. So, the bidder selects that action.

The Cautiously Optimistic Strategy Is Regret-free

We can verify that the strategy is regret-free. First, if i secures an exit outcome, then the worst-case outcome of this strategy trivially equals the secured outcome. Since it is the most preferred outcome, it is better than all other exits. Also, because the clock only descends, there is no way he can get a more desirable outcome by choosing to continue. Therefore, he will not regret choosing to secure this outcome. Next, if the bidder chooses to continue, then whenever any of his fantasy outcomes in X_{it}^* becomes unachievable, the previously available exits will again be assurable. Therefore, the worst-case to happen is that he chooses one of the exits in some future period. Therefore, the worst-case payoff of following this strategy is not worse than the best-case payoff of any other deviation. Therefore, the bidder will not regret pursuing his most desirable outcomes.

2.2.5 PED-Implementability

In this paper, our focus is on the direct mechanisms that can be implemented by a PED mechanism. Formally, a direct mechanism is said to be PED-implementable if there exist a PED mechanism D and a cautiously optimistic strategy S_i for all $u_i \in \mathcal{U}_i$ and $i \in N$ such that $\chi_N(h_\emptyset, S_N) = \chi_N(u_N)$.

2.3 The Pricing Mechanism

Let $Y_i(u_{-i}) \equiv \alpha_i(\mathcal{U}_i, u_{-i})$ denote the set of possible allocation to i when other bidders report u_{-i}.

Definition 19 *Let M be a direct mechanism on market Ω. We say M is a pricing mechanism if for all $i \in N$, there exists a pricing function $P_i : Y_i \times \mathcal{U}_{-i} \to \mathbb{R}$ such that for all $u_N \in \mathcal{U}_N$*

1. $\alpha_i(u_N) \in \arg\max\{u_i(y_i', P_i(y_i', u_{-i}))|y_i' \in Y_i(u_{-i})\}$.

2. $\rho_i(u_N) = P_i(\alpha_i(u_N), u_{-i})$.

In this case, we say M is supported by the pricing function P_N.

Theorem 10 *Let M be a finite mechanism. Then M is strategy-proof if and only if M is a pricing mechanism.*

Given a pricing mechanism M, let $X_i(u_{-i}) \equiv \{(y_i, P_i(y_i, u_{-i}))|y_i \in Y_i(u_{-i})\}$ be the set of outcomes available to bidder i for a given report profile from the rest of the bidders.

Lemma 4 *Let M be a finite pricing mechanism. Then M is ex-post individual rational if and only if for all $u_N \in \mathcal{U}_N$ and all $i \in N$, there exists $x_i \in X_i(u_{-i})$ such that $x_i \geq x_i^0$.*

2.4 VCG Pricing as the Baseline

In this section, we characterize the Vickrey (1961)-Clarke (1971)-Groves (1973) (VCG) mechanism and the associated VCG pricing functions. Then we characterize the mutual and monotonic influence between bidders in VCG pricing functions.

2.4.1 The VCG Pricing Mechanism

The VCG mechanism is a mechanism that implements the efficient allocation rule by internalizing a bidder's marginal effect to total welfare into his payment rule. Therefore, the optimal choice of this bidder is align with the optimal choice for the economy. We formalize this concept as follows.

First, we decompose u_i into $u_i(y_i, p_i) = v_i(y_i) + p_i$. The term $v_i(y_i)$ can be considered as the value of an allocation y_i. The total welfare is the sum of values of the bidders and the client

$$W(y_N, u_N) = \sum_{i \in N} v_i(y_i) + w(y_N) \tag{2.2}$$

An allocation rule α_N is said to be efficient if it maximizes the total welfare among the feasible allocations at all $u_N \in \mathcal{U}_N$. That is

$$\alpha_N(u_N) \in \arg\max\{W(y'_N, u_N) | y'_N \in \mathcal{F}\} \tag{2.3}$$

We now formalize this payment rule, which should take into account the marginal effect of his assignment to the rest of the agents. Let $V_i(y_i, u_{-i}) \equiv \max\{\sum_{j \neq i} v_j(y'_j) + w(y'_N) | y'_i = y_i, y'_N \in \mathcal{F}\}$ be the maximal welfare from the rest of the agents given that bidder i is assigned y_i. Then the marginal effect of i being assigned y_i instead of not participating can be characterized by $V_i(y_i, u_{-i}) - V_i(y_i^0, u_{-i})$, which is payment rule for the VCG mechanism.

Definition 20 (The VCG Mechanism) *A mechanism $M = (\alpha_N, \rho_N)$ is said to be a VCG mechanism on market Ω if α_N is efficient and for all $i \in N$, $u_i \in \mathcal{U}_i$, we have*

$$\rho_i(u_N) = V_i(\alpha_i(u_N), u_{-i}) - V_i(y_i^0, u_{-i}) \tag{2.4}$$

The following theorem confirms that the VCG mechanism is strategy-proof and ex-post individual rational.

Theorem 11 *A VCG mechanism $M = (\alpha_N, \rho_N)$ is strategy-proof, ex-post individual rational and efficient. Also, M is supported by the following pricing function:*

$$P_i(y_i, u_{-i}) = V_i(y_i, u_{-i}) - V_i(y_i^0, u_{-i}) \tag{2.5}$$

for each $i \in N$, $y_i \in Y_i(u_{-i})$ and $u_{-i} \in \mathcal{U}_{-i}$.

2.4.2 Monotonic and Mutual Influence of VCG Pricing

We can observe a notion of monotonicity in VCG pricing. In the RHS of Equation (2.5), pricing is increasing with $V_i(y_i, u_{-i})$ and decreasing with $V_i(y_i^0, u_{-i})$. Roughly speaking, the bidder will be encouraged with higher pricing to choose y_i if the value of the complementary allocations gets higher; and he will be discouraged with lower pricing if the competing allocation has a higher value. We now formalize this monotonicity concept.

First, we define the order on the preference set $\mathcal{U}_i$ based on how the preference value a particular outcome $x_i \in X_i$. Intuitively, x_i is more valuable in preference u_i than in u_i' if x_i is preferred to a larger set of outcomes. Define $u_i \geq_{x_i} u_i'$ if for all $p_i \in \mathbb{R}$, the upper contour set of x_i is smaller for u_i, that is

$$\{x_i' \in X_i | u_i(x_i') \geq u_i(x_i)\} \subseteq \{x_i' \in X_i | u_i'(x_i') \geq u_i'(x_i)\}. \tag{2.6}$$

Because all the preferences in $\mathcal{U}_i$ are quasi-linear, the shape of the upper contour sets will be the same for all outcomes with the same allocation y_i. Thus, if the upper contour set of $x_i = (x_i, p_i)$ is smaller in u_i than in u'_i, then the upper contour set of other $x'_i = (y_i, p'_i)$ also is smaller.

Lemma 5 *Let $x_i = (y_i, p_i) \in X_i$. If $u_i \geq_{x_i} u'_i$, then for all $x'_i = (y_i, p'_i)$, we have $u_i \geq_{x'_i} u'_i$.*

Thus, what we are really comparing is the relative value of $v_i(y_i)$. With this property, we can define $u_i \geq_{y_i} u'_i$ if there exists $x_i = (y_i, p_i) \in X_i$ such that $u_i \geq_{x_i} u'_i$.

Lemma 6 *If $u_i \geq_{y_i} u'_i$, then for all $y'_i \in Y_i$, we have $v_i(y_i) - v_i(y'_i) \geq v'_i(y_i) - v'_i(y'_i)$. In particular, when $y'_i = y^0_i$, we have $v_i(y_i) \geq v'_i(y_i)$.*

We now define what we mean by complementary and competing allocation. Let $\alpha^*_N(y_i, u_{-i}) \equiv \arg\max\{\sum_{j \neq i} v_j(y'_j) + w(y'_N) | y'_i = y_i, y'_N \in \mathcal{F}\}$. We say y'_N is a complementary allocation to y_i at u_{-i} if $y'_N \in \alpha^*_N(y_i, u_{-i})$. We say y'_N is a substitutable allocation to y_i at u_{-i} if $y'_N \in \alpha^*_N(y^0_i, u_{-i})$.

Theorem **??** states that in a VCG mechanism, the pricing function is monotonic:

Theorem 12 *If M is a VCG mechanism, then for all $i \in N$, $u_{-i} \in \mathcal{U}_{-i}$ and $y_i \in Y_i(u_{-i})$,*

1. *if y'_N is a complementary allocation to y_i at u_{-i}, then for all $u'_{-i} \in \mathcal{U}_{-i}$ such that $u'_j \geq_{y'_j} u_j$ for all $j \neq i$, we have $P_i(y_i, u'_{-i}) \geq P_i(y_i, u_{-i})$.*

2. *if y'_N is a substitutable allocation to y_i at u_{-i}, then for all $u'_{-i} \in \mathcal{U}_{-i}$ such that $u'_j \geq_{y'_j} u_j$ for all $j \neq i$, we have $P_i(y_i, u'_{-i}) \leq P_i(y_i, u_{-i})$.*

This result states that in an efficient mechanism, there are two types of influence on the pricing of an allocation: one from the complementary allocation and one from the competing allocation. Because the reports of other bidders can influence the pricing of an allocation, the price can reflect the marginal effect of a bidder's choice and guide the bidder to make an efficient choice.

2.5 Restricted Influence in PED-Implementable Mechanisms

In this section, we show that for a pricing mechanism, to be PED-implementable, the ability of pricing functions to adjust according to the values of other bidders is restricted. We show that it is inevitable for the winning allocations to have a *constant pricing space*. This means that the pricing of an allocation cannot change after the values of other allocations from other bidders exceed some certain thresholds. Even though we may wish that the restrictive area is small, however, we show that there exists a hierarchy of these constant pricing spaces across bidders, and, therefore, it is inevitable for some allocations to have larger spaces than others. Or more precisely, if some allocation has a small constant pricing space, then there must exist a larger constant pricing space for other allocations of other bidders. Also, we show that a winning bidder can exert influence only in the constant pricing space of his winning outcome. That implies that if the value of an allocation can influence the pricing of other allocation, then the pricing itself cannot be influenced. Thus, there is a trade-off between influencing others and being influenced.

2.5.1 Constant Pricing Spaces

The Threshold Space

We first characterize the constant pricing space as a threshold space. The intuition behind a threshold space is that it collects all preferences where the value of some specified allocations exceed certain thresholds. Formally, given $i \in N$, a threshold for i is the pair $\tau_i = (X_i^C, X_i^E)$, where X_i^C and X_i^E are non-empty finite subsets of X_i. Given a threshold τ_i, define $x_i^*(\tau_i, u_i) \equiv \arg\max\{u_i(x_i)|x_i \in X_i^C \cup X_i^E\}$. Define the one-dimensional threshold space supported by τ_i as:

$$\mathcal{U}_i(\tau_i) = \{u_i \in \mathcal{U}_i | x_i^*(\tau_i, u_i) \subseteq X_i^C\} \tag{2.7}$$

That is, u_i is collected if it values some allocation in X_i^C high enough so that u_i prefers this outcome to all others in $X_i^C \cup X_i^E$. We can define an N-dimensional threshold space from the product of one-dimensional ones. Let $u_N \in \mathcal{U}_N$, $S \subseteq N$, and τ_i for $i \in S$ such that $u_i \in \mathcal{U}_i(\tau_i)$ for all $i \in S$. Define the N-dimensional threshold space as:

$$\mathcal{U}_N(u_N, \tau_S) = \Pi_{i \notin S}\{u_i\} \times \Pi_{i \in S}\mathcal{U}_i(\tau_i) \tag{2.8}$$

The following lemma highlights the threshold property. Intuitively, the lemma says that given a threshold $\tau_i = (X_i^C, X_i^E)$, if u_i values $x_i \in X_i^C$ high enough such that x_i is the most preferred outcome in $X_i^C \cup X_i^E$, then all the preferences u_i' that value x_i more than u_i are also included in the threshold space $\mathcal{U}_i(\tau_i)$.

Lemma 7 *Suppose $u_i \in \mathcal{U}_i(\tau_i)$, let $x_i \in x_i^*(\tau, u_i)$. Then for all $u_i' \geq_{x_i} u_i$, we have $u_i' \in \mathcal{U}_i(\tau_i)$.*

The following lemma builds the connection between the threshold space and the PED mechanisms. It says that the set of preferences that choose to continue in a cautiously optimistic strategy is a threshold space. That is because u_i chooses to continue with $a_{it} \in X_{it}^C$ only if all of his most preferred achievable outcomes are in a_{it}.

Lemma 8 *Let D be a PED mechanism. For any given $u_{-i} \in \mathcal{U}_{-i}$, let $\mathcal{U}_i'(u_{-i}) \subseteq \mathcal{U}_i$ be the set of preferences such that*

1. *the profile (u_i, u_{-i}) reaches period t if everyone plays a cautiously optimistic strategy,*
2. *u_i chooses to continue with some continuation action $a_{it} \in X_{it}^C$ with a cautiously optimistic strategy,*

Then $\mathcal{U}_i'$ is a threshold space with threshold $\tau_i = (a_{it}, X_{it}^E)$, that is, $\mathcal{U}_i'(u_{-i}) = \mathcal{U}_i(\tau_i)$

Constant Pricing Space

The constant pricing space is a threshold space where an allocation y_i has a fixed pricing. Formally, a threshold space $\mathcal{U}_N(u_N, \tau_S)$ is said to be a constant pricing space for x_i at u_N if $P_i(y_i, \cdot) = p_i$ on $\mathcal{U}_N(u_N, \tau_S)$.

In contrast to the monotonicity property in the VCG mechanism, if y_i has a constant pricing space $\mathcal{U}_i(u_N, \tau_S)$, then it means that once the values of allocations in X_j^C are higher than the specified threshold, then the pricing of y_i becomes a constant and no longer adjusts according to the values of other bidders. This is particularly restrictive if X_j^C contains some complementary or competing outcomes because efficient pricing should increase with the values of complementary allocation, and decrease with the values of competing allocations.

As a constant pricing space is restrictive, we may wish it to be small, if it has to exist. The following lemma shows the conditions to have smaller threshold spaces. Intuitively, the threshold space is small means that only a small subset of $\mathcal{U}_i$ prefers outcomes in X_i^C to the ones in X_i^E. That happens when the outcomes in X_i^C are not desirable and the set X_i^E is large.

Lemma 9 *Let* $\tau_i = (X_i^C, X_i^E)$ *and* $\hat{\tau}_i = (\hat{X}_i^C, \hat{X}_i^E)$ *such that*

1. *for all* $x_i^c \in X_i^C$*, there exists* $\hat{x}_i^c \in \hat{X}_i^C$ *such that* $\hat{x}_i^c \geq x_i^c$.
2. $X_i^E \supseteq \hat{X}_i^E$

Then $\mathcal{U}_i(\tau) \subseteq \mathcal{U}_i(\hat{\tau})$.

Thus, we can define $\tau_i \leq \hat{\tau}$ if the two conditions in Lemma 9 are met. Then we can generalize this concept to an N-dimensional threshold spaces. The following result says that a N-dimensional threshold space is smaller if the set of bidders regulated by the threshold is smaller, and each of the thresholds in effect also is smaller.

Lemma 10 *Let* $\mathcal{U}_N(u_N, \tau_S)$ *and* $\mathcal{U}_N(u_N, \hat{\tau}_{\hat{S}})$ *be two threshold spaces. If* $S \subseteq \hat{S}$ *and* $\tau_i \leq \hat{\tau}_i$ *for all* $i \in S$*, then* $\mathcal{U}_N(u_N, \tau_i) \subseteq \mathcal{U}_N(u_N, \hat{\tau}_{\hat{S}})$.

Now we present the main result. The following theorem says that even though we wish that the constant pricing space were small if it has to exit, the existence and the size of the constant pricing spaces are regulated by a hierarchical structure.

Theorem 13 (Hierarchy of Constant Pricing) *If* M *is PED-implementable, then for all* $u_N \in \mathcal{U}_N$ *and* $i \in N$*, there exists a subset of outcomes* $\bar{X}_i(u_N) \subseteq X_i(u_{-i})$ *and constant pricing spaces, denoted* $\mathcal{U}_N(x_i, u_N)$ *for all* $x_i \in \bar{X}_i(u_N)$ *such that:*

1. $\chi_i(u_N) \in \bar{X}_i(u_N)$ *for all* $i \in N$.

2. *There is a reordering on* $\cup_{i \in N} \bar{X}_i(u_N) = \{x_{(1)}, x_{(2)}, \ldots, x_{(k)}\}$ *such that* $\mathcal{U}_N(x_{(1)}, u_N) \supseteq \mathcal{U}_N(x_{(2)}, u_N) \supseteq \cdots \supseteq \mathcal{U}_N(x_{(k)}, u_N)$.

3. *For* $i \in N$ *and* $x_i \in \cup_{j \in N} \bar{X}_j(u_N)$*, if* $\mathcal{U}_N(x_i, u_N) = \mathcal{U}_N(u_N, \tau_S)$*, and* $x_i = x_{(n)}$*, then for all* $j \neq i$

   (a) *if* $j \in S$*, then* $X_j^E = \{x_{(m)} \in \bar{X}_j(u_N) | m < n\}$.

   (b) *if* $j \in S$*, then for all* $x_j \in X_j(u_{-j}) \backslash X_j^E$*, there exists* $x_j^c \in X_j^C$ *such that* $x_j^c \geq x_j$.

   (c) *if* $j \notin S$*, then* $\chi_j(u_N) = x_{(m)}$ *for some* $m < n$.

The theorem says several things. First, it is inevitable that the constant pricing spaces exist for all the assigned allocations. Second, the constant pricing spaces are hierarchical in the sense that the larger ones contain the smaller ones. So, the higher the constant pricing space in the hierarchy, the larger the set X_j^C must be in the thresholds τ_j for $j \in S$, and therefore the larger the constant pricing space. Third, having small constant pricing spaces come at the cost of having larger ones for other allocations. If the constant pricing space of y_i is small because X_j^E is large, then there exist constant pricing spaces for each $x_j^e \in X_j^E$, and all of them are larger in the hierarchy. If the constant pricing space is small because the S is small, this implies that there are larger ones for all the assigned outcomes for $j \notin S$. In short, with this hierarchical structure of constant pricing spaces, it is inevitable to have large threshold spaces that restrict the pricing of the allocations.

The intuition behind this result is that the constant pricing space for each x_i is the subset $\mathcal{U}_j' \subseteq \mathcal{U}_j$, which chooses to continue when x_i is announced to be an exit outcome. Because by Lemma 8, $\mathcal{U}_j'$ is a threshold space, and once x_i is announced to be an exit outcome, any variation within $\mathcal{U}_j'$ cannot change the payment of y_i anymore. Because in the evolution of the announcements, the exit outcomes are persistent.

The following example shows the connection between constant pricing spaces and implementing PED mechanisms, and why constant pricing spaces are inefficient.

Example 5 *Consider that there is a seller who has two objects, A and B, to sell. There are two bidders, $N = \{1, 2\}$. Each of them demands either A or B, that is $Y_i = \{y_i^0 = (0,0), y_i^A = (1,0), y_i^B = (0,1)\}$.*

In a PED mechanism, we can initially offer both types of outcomes as continuation outcomes: $E_{it} = \{a_i^0\}$ where $a_i^0 = \{x_i^0\}$, and $C_{it} = \{a_{ti}\}$, where $a_{it} =$

$\{(y_i^A, p_{it}^A), (y_i^B, p_{it}^B)\}$. That means, when bidder i is called to play, he can either choose a_i^0 to exit and secure x_i^0, or he can choose to continue. According to the auction rules, the next time he is called, either he wins and is forced to choose one outcome from a_{it}, or the outcomes in a_{it} become inferior and he will be offered the chance to secure x_i^0.

However, it is not possible for the auctioneer to close the auction with both allocations still offered as continuation outcomes for both bidders. This is because we cannot prevent both bidders from simultaneously choosing conflicting allocations, such as $(x_1, x_2) = ((y_1^A, p_{it}^A), (y_2^A, p_{2t}^B))$. Therefore, the auctioneer either needs to remove some of them or convert some of them to exit outcomes. One way to do that is to announce some $x_1^B = (y_1^B, p_i^B)$ and $x_2^B = (y_2^B, p_2^B)$ to be exit outcomes for both bidders. That is, $E_{it} = \{a_i^0, a_i^B\}$ where $a_i^B = \{x_i^B\}$ and $C_{it} = \{a_{it}\}$ where $a_{it} = \{(y_i^A, p_{it}^A)\}$. Then the p_{it} can be interpreted as the clock price. If both of them choose to continue, then the auctioneer iteratively lowers the clock prices, p_{it}^A of y_1^A and y_2^A. This way, when one bidder exits, he either chooses to be assigned his endowment x_i^0 or good B, x_i^B. Either way, we can force the remaining bidder to choose the most recent $x_{it}^A = (y_i^A, p_{it}^A)$. Thus we can prevent conflicts.

The cost of doing this is that we need to fix the pricing for y_1^B and y_2^B before we know the preferences of the two bidders. Since y_1^B and y_2^B are competing allocations, in an efficient mechanism, the pricing of y_1^B and y_2^B should decrease with each other when the values of these two allocations are high enough. However, when we set x_i^B as an exit outcome, we create a constant pricing space for x_i^B. Then the value of his opponent cannot influence the pricing of y_i^B in this constant pricing space.

Therefore, the allocation of y_i^B will be less efficient.

2.5.2 No Mutual Influence

Now we show that there is no mutual influence in PED-implementable mechanisms. That is, if the value of y_i can affect the availability or the pricing of y_j, then the pricing of y_i itself cannot be affected by the value of y_j.

Let $\mathcal{U}_N(y_S, u_{N\backslash S}) \equiv \{u'_N \in \mathcal{U}_N | \alpha_S(u'_N) = y_S, u'_j = u_j \ \forall j \notin S\}$

Definition 21 *Given $u_N \in \mathcal{U}_N$, we say y_i has winner influence over $y_j \in Y_j(u_{-j})$ at u_N if $u_i \in \mathcal{U}_i(y_i, u_{-i})$, and there exists $u'_i \in \mathcal{U}_i(y_i, u_{-i})$ such that*

1. *$y_j \notin Y_j(u'_i, u_{-ij})$, or*

2. *$P_j(y_j, u'_i, u_{-ij}) \neq P_j(y_j, u_i, u_{-ij})$.*

Theorem 14 *Let M be a PED-implementable direct mechanism. Suppose $S \subseteq N$ and $i \in S$, $u_N \in \mathcal{U}_N$, $\chi_N(u_N) = x_N = (y_N, p_N)$. If y_i has winner influence over y_j for the rest of $j \in S$, then there exists a constant pricing space $\mathcal{U}_N(x_i, u_N)$ for x_i at u_N such that $\mathcal{U}_N(y_S, u_{-N\backslash S}) \subseteq \mathcal{U}_N(x_i, u_N)$*

The take away from this result is that there is no mutual influence in PED-implementable mechanisms. If we consider a special case where $S = \{i, j\}$, then the theorem would state that if y_i has influence over the availability or the pricing of y_j, then the value of y_j cannot influence the pricing of y_i. Since all the influence is one-directional, if bidder i can improve the payoff of bidder j, then bidder j cannot improve the payoff of bidder i. In contrast, in a VCG mechanism, the influence works both ways. When goods are complements, the mutual influence creates room for profitable deviation. Thus, the restriction on mutual influence can be interpreted as the cost of preventing group deviation, or the cost of enforcing WGSP.

This result also highlights the trade-off between influencing others and being influenced. If the value of y_i can affect the pricing of a large set of allocations, then it has to imply that it has a large constant pricing space. This would mean that the pricing function for y_i itself is more restrictive. This can be considered the cost of having influence. With this result, the hierarchy of constant pricing also can be interpreted as a hierarchy of the ability to influence. As the constant pricing space becomes larger, it asserts more influence, but the assignment of this allocation also becomes less efficient.

2.6 Chapter Conclusion

This chapter showed that the persistence of exits is a double-edged sword. On the one hand, it guarantees that the PED mechanism is strategically simple, but it causes the allocation rule to be less efficient. It limits the ability of the values of a bidder to influence the pricing of other bidders. While this ability is what makes the VCG mechanism efficient. We also showed that it is inevitable that some allocations will have larger constant pricing spaces than others. The hierarchy of constant pricing spaces resembles the sequential order in which these outcomes become exit offers in the PED mechanism. Finally, there is a trade-off between the ability to influence and the ability to be influenced. The greater influence an allocation has, the less it can be influenced, and hence the less efficient its assignment becomes. Market designers who are evaluating whether the PED mechanism is a good fit should first consider whether the hierarchy of constant pricing spaces makes the allocation rule too undesirable.

Chapter 3: Obviously Regret-Proof Combinatorial Land Assembly

3.1 Introduction

Land assembly is a process in which developers gain ownership of multiple contiguous parcels of land to create one larger tract. It is particularly valuable when the structures on the smaller parcels are too depreciated and fragmented to support the growth of a city. In such cases, replacing those structures with bigger and more modern ones is often a better utilization of scarce urban land resources. (Brooks and Lutz, 2016)

However, the complementarity among the owners makes it difficult to reach a transaction through the decentralized multilateral bargaining (Menezes and Pitchford, 2004; Miceli and Segerson, 2012; Kominers and Weyl, 2012b). Once we have a target set to assemble, we need to acquire consent from all the owners. In that case, each owner has a form of monopoly power and has the incentive to hold out—to delay entering the negotiation—in order to get a higher surplus. Thus, multilateral and sequential negotiation becomes costly and time consuming. Expecting this problem, the developer may be deterred from entering the market in the first place, or he may exit the bargaining early once he realizes someone is holding out.

Can centralized market mechanisms be used to solve the holdout problem?

In a centralized mechanism, we have a structured process to aggregate information about bidders' value to the auctioneer. Therefore, we can avoid the time-consuming, back-and-forth process between the developer and the owners. The literature documents the ways that market designers attempted land assembly (Kominers and Weyl, 2012b,a; Plassmann and Tideman, 2007, 2010). However, those mechanisms are restrictive in the sense that they fail to address the some fundamental features of the land assembly problem.

First, those mechanisms are restricted to use unanimity rules. It means that the auctioneer is exogenously given a target set to assemble, such that either the transaction occurs for everyone in the set or no transaction at all. However, there are many other buildings in the neighborhood and therefore many subsets of contiguous buildings. The unanimity rule is restricting the developer to only try to reach a deal with a particular one. It is restrictive because we lose the opportunity to discover transaction possibilities with other subsets. In particular, the larger sets will less likely to be chosen, because the probability of trade decreases sharply as the number of bidders increases (Mailath and Postlewaite, 1990).

Second, these mechanisms overlooked the importance of non-monetary compensations. It is very common in land assembly practice that the land owners are offered newly built apartments as non-monetary compensations. This is because getting cash payments does not settle down the where-to-live problem for the owners. When home owners are deciding whether to sell their properties to the developer, they will also consider the cost and the location of their next home if they do. If the developer can offer them a new apartment in the new building, then not only it is an upgrade of their homes, but it also implies that they can stay in the same neighborhood. This is good for the developer, too. Because having a set of initial buyers of the new building also reduces the uncertainty in sales in the future. Thus, the probability of reaching a deal and the surplus of a transaction would increase if we allow this type of property exchange. However, the challenge for having non-monetary compensation is that the outcome space must be multi-dimensional, as there is more than one good to be allocated. Moreover, if the new apartments come with different sizes and design, then the dimensionality increases.

Third, these mechanisms are not strategically simple. Because most of the participants will be small home owners who have little to no experience in game theories and auctions. It is also not likely for them to hire auction consultants during the auction. Thus, even if we carefully craft the allocation and payment rules to induce them to tell the truth, they may fail to understand the rules and solve for the equilibrium strategy. Then the auctioneer cannot really implement the intended allocation and payment rules.

Finally, some of the mechanisms compromise the individual rationality or the property right protection (Kominers and Weyl, 2012b; Plassmann and Tideman, 2010). The motivation is to increase the efficiency of the mechanism under the unanimity rule. But given that the stake is high, enforcing the outcome that is worse than the status quo inevitably involves some government power, which can be controversial.

In this chapter, we aim to propose market design solutions that address all the above problems. We show that the Persistent Exit Descending (PED) mechanism we developed in the first chapter is a good candidate. It implements the combinatorial allocation rule on a multi-dimensional, multi-unit outcome space. It is obviously regret-proof, which means that it is easy for a bidder to find and understand the dominant strategy. Plus, the ex-post individual rationality can be easily enforced. What we need to do is just clean up some nonessential features and simplify it for land assembly problems. We show that for the feasibility structure in land assembly problems, the coexistence of multiple exit outcomes and continuation outcomes is not meaningful and creates fairness concerns. Therefore, we introduce the degenerated version of the PED mechanism that is tailored to the land assembly problem.

The analytical framework in chapter 2 also provides a tool for us to understand the limitation of using this mechanism. First, we show that the mutual influence is completely shut down between winners. Second, the mechanism is not flexible for bidders who own more than one apartment. Given the reports of other bidders, the mechanism can only determine whether to clear a fixed set of his properties. Finally, we also address that the combinatorial allocation rule does not fully escape the Impossibility theorem by Mailath and Postlewaite (1990). For a given target set, the probability to clear that set is still low if the number of bidders in the set is large. Thus, if we are dealing with buildings with a large number of owners, then we would not expect the probability of a trade to be much higher.

The rest of this chapter is organized as follows. In Section 3.2, we present the land assembly allocation problem, which is slightly modified from the multi-dimensional and multi-unit allocation problem. Next, in Section 3.3, we introduce the degenerated PED mechanism, and we explain why we need this simplified and seemingly more restrictive version. In Section 3.4, we show by example how the degenerated PED mechanism works in different types of transactions, and the advantages of using them in land assembly problems. In Section 3.5, we show by examples the restrictions of using PED mechanisms and the challenges for it really to be implemented in the real world.

3.2 The Model

3.2.1 The Allocation Problem

We consider a market with a real estate developer and a finite set of bidders. The bidders can be apartment owners or some people who want to buy the to-be-developed apartments. For simplicity, we assume that there is no co-ownership in this model. That is, each apartment is owned by one owner. However, it is possible that one bidder may own multiple apartments. We also allow that in some buildings, only a subset of the apartment owners participate in the mechanism. This implies that it is not possible to clear those buildings. However, their participation can be meaningful since their apartments could be on the market for the owners in other clearable buildings. Some of these owners may prefer those existing apartments over the new ones. We assume that the real estate developer is non-strategic, and all the information he submits to the auctioneer is authentic.

Let N be a finite set of bidders. Suppose that for all $i \in N$, there exists a set of apartments K_i that i owns and puts on the market. Let $K^E = \cup_{i \in N} K_i$

denote the entire set of apartments on the market. $K_i = \emptyset$ implies that i has no endowed properties or he is not willing to sell his properties. He participates in this mechanism just to buy some apartments. A clearable building B is a building where all its apartments are on the market, that is $B \subseteq K^E$. Let $\mathcal{B}$ denote the set of clearable buildings. Notice that we do not require $\cup\mathcal{B} = K^E$, because even if apartment $k \in K^E$ is in a non-clearable building, we could have a transaction where some owners in the cleared building move to existing apartment k and the original owner of k gets a newly built one of type $k' \in K^N$.

If the developer clears a subset of the buildings, then he can tear down the old structures and build new ones on the plot of land where the old ones stand. Suppose that the developer can build a finite set of types of new apartments, K^N. The type is an abstraction of the heterogeneity of the newly built properties, which can represent apartments with different numbers of bedrooms and bathrooms, on different floors, etc. In addition, within each type, the sizes of the apartments might differ. That is, within the type, One-Bedroom on the Terrace Level, it is possible for the developer to build 1,200 ft^2 ones and 1,000 ft^2 ones.

For each $k \in K^E \cup K^N$, let y_{ik} specify the ownership for apartment k that is allocated to bidder i. If $k \in K^E$, then $y_{ik} \in \{0, 1\}$, where $y_{ik} = 1$ if and only if i gains ownership of the existing apartment k. If $k \in K^N$, then $y_{ik} \in \mathbb{R}$ denotes the total size of the new apartment of type k that i gets. If i is assigned 2 units of 1,000 ft^2 One-Bedroom apartments, then $y_{ik} = 2,000$. Then an allocation for i is a profile of ownership $y_i = (y_{ik})$ for all $k \in K^E \cup K^N$. Let y_i^0 denote the endowment of i before the auction starts. Let Y_i be the set of all possible allocations. Let

$y_N = (y_i)_{i \in N}$ denote the allocation for all bidders. An apartment $k \in K^E$ is said to be cleared in y_N if it is not assigned to any bidder. That is, $y_{ik} = 0$ for all $i \in N$. A clearable building B is said to be cleared in y_N if all of its apartments $k \in B$ are cleared.

An allocation is feasible if the demand for an apartment is less or equal to the supply of the apartment. For the existing apartments, it is straightforward to require that $\sum_{i \in N} y_{ik} \leq 1$. For the new apartments, the supply is more flexible, as the developer might be able to customize the design for a new building based on the demand elicited in the mechanism. Nevertheless, the feasibility of a demand profile is still expected to depend on the set of cleared buildings. The more pieces of land that are cleared, the larger the new buildings can be and the more flexible the supply can be. Because it is not our purpose in this paper to address the technology of land assembly, we abstract this layer away by assuming that the developer has submitted a set $\mathcal{F} \subseteq Y_N$ containing all the feasible allocation profiles.

An outcome for i is the combination of an allocation and a payment from the developer to bidder i, that is $x_i = (y_i, p_i) \in Y_i \times \mathbb{R}$. We allow $p_i < 0$, which means the net payment is made from i to the developer. This can happen when bidder i is buying some new apartments from the developer.

Assume that each bidder $i \in N$ has a linear preference over the outcome space. That is, for each $k \in K^E \cup K^N$, there exists $v_{ik} \in \mathbb{R}$, such that the utility function evaluated at outcome $x_i = (y_i, p_i)$ is given by $u_i(x_i) = u_i(y_i, p_i) = \sum_k (y_{ik} - y_{ik}) v_{ik} + p_i = (y_i - y_i^0) \cdot v_i + p_i$. If $k \in K^E$, v_{ik} can be interpreted as the monetary value for apartment k. For $k \in K^N$, v_{ik} is the per-square-foot value for type k

apartments. We subtract $\sum_k y_{ik}^0 v_{ik}$ to normalize the utility, such that $u_i(x_i^0) = 0$. Let $F_i(v_i) = \Pr(v'_{ik} \leq v_{ik} | k \in K^E \cup K^N)$ be the joint cumulative distribution function.

3.3 The Mechanism

We introduce the degenerated version of the PED mechanism. In a full PED mechanism, bidders may have multiple continuation actions and exit actions to choose from in each period. However, in this degenerated version, each bidder has only one continuation action and one exit action unless he is forced to exit. The reasons for using this restricted version are feasibility and simplicity. That is, concerning feasibility, the full version does not provide much additional value than this degenerated version. Therefore, to make it simple for bidders and for the auctioneer, we use the degenerated version. We will revisit this point after we introduce the mechanism.

3.3.1 Auction Rule

The auction is initialized with $t = 0$ and the set of active bidders $A(0) = N$. At each $t \geq 0$, the auctioneer call an active bidder in $A(t)$ to play. The auctioneer offers to this bidder a set of outcomes $X_{it} \subseteq X_i$ and tells him whether the auction is closing for i. If the mechanism is closing for i at t, then bidder i has to choose one outcome x_i from the offered set X_{it}. He will be assigned the outcome x_i he has chosen at the end of the auction. The auction continues with $A(t+1) = A(t) \backslash \{i\}$.

This means that bidder i is no longer active and will not be called to play in future periods. We require that the auctioneer not to offer outcomes that dominates one another. That is, for all $x_i, x_i' \in X_{it}$, neither $x_i \geq x_i'$ nor $x_i' \geq x_i$. If the mechanism is closing for i at t, and t is the first period i is called to play, then we require that $x_i^0 \in X_{it}$. Therefore, bidder i has the chance to keep his endowment. If t is not closing for i, then bidder i can choose either to continue or to exit. If i chooses to exit, then he becomes inactive and will be assigned his endowment outcome x_i^0 at the end of the auction. The auction enters the next period with $A(t+1) = A(t)\backslash\{i\}$. If bidder i chooses to continue, then the auction continues with $A(t+1) = A(t)$. When bidder i is called to play next time in some $t' > t$, the new offered outcomes is regulated by the following rules:

1. If t' is not closing, then the offered outcomes become inferior: for all $x_i' \in X_{it'}$, there exists $x_i \in X_{it}$ such that $x_{ik} \geq x_{ik}'$ for all $k_i \in K^E \cup K^N$.

2. If t' is closing, then $X_{it'} = X_{it}$.

The auction concludes at T when all bidders have exited. That is, $A(T) = \emptyset$. For notational convenience, if i is not called to play at t, and t' is the last period i is called to play, then set $X_{it} = X_{it'}$.

Proposition 1 *The degenerated PED mechanism is a PED mechanism.*

Notice that for all active bidders $i \in A(t)$, the auctioneer can always force i to exit to x_i^0 by removing all outcomes in X_{it}. This would happen when some bidder j

in the same building chooses to keep his apartment, making it meaningless for the developer to clear i's apartment.

On the other hand, the auctioneer can also prevent bidder i from choosing x_i^0 by closing the auction for him. This happens when the auctioneer is in the process of clearing a building. When some owners have already chosen to clear their houses, or have even been promised a new apartment, then the auctioneer should also clear all the other owners in the same building. In that case, we should not allow those bidders to choose x_i^0 anymore. Thus, the auctioneer should let them chooses only the outcomes in X_{it}. Of course, the auctioneer should initiate the clearing process only when all the owners have chosen to continue in the most recent period before t, and all the outcomes in X_{it} requires i to clear his apartments in the building.

3.3.2 Reasons for the Degenerated Version

In a clearable building, clearing an apartment in that building is meaningful only if all the other owners also clear theirs. If any of them choose to keep their apartments, then it would be meaningless for other bidders in the building to clear theirs. Therefore, when a rejection occurs, the auctioneer should be able to remove the outcomes that require an owner to clear his apartments. Thus, those outcomes are not suitable to be exit outcomes, as exit outcomes cannot be removed once offered. Even though it is possible to offer coexisting exit and continuation outcomes to only one bidder in a building, having a single bidder that is treated differently may raise fairness concerns.

In addition, if we do not have coexisting exit outcomes, then we cannot have more than one continuation offer either. Because we need an exit outcome to convex-dominate the continuation outcomes in different continuation actions. Thus, for simplicity, we just remove the possibility for coexisting exit and continuation offers and multiple continuation actions altogether.

3.4 The Advantages of PED-LA

We will use several examples to demonstrate the advantages of using the PED mechanism for land assembly problems.

3.4.1 Exploring Trading Possibilities

The following example highlights the advantage of the combinatorial allocation rule over the unanimity rule. To focus only on the combinatorial nature of the problem, we first shut down the multi-option feature of this mechanism. Then this model collapses to the binary allocation problem discussed in Milgrom and Segal (2017) and Li (2017).

Example 6 (The Power of the Combinatorial Allocation Rule) *Suppose that there is one row of M contiguous old buildings to be redeveloped. Each building contains one apartment. Each apartment is owned by a distinct owner. Thus, $N = \{1, \ldots, M\}$. For simplicity, we assume that only monetary compensations are offered in this example. Therefore, no new apartment is on the market $K^N = \emptyset$. Let $K_i = \{k_i\}$, where k_i denotes the apartment i owns. Then the allocation set con-*

tains only 2 elements, $Y_i = \{y_i^0, y_i\}$, where y_i^0 is the allocation to keep his endowed apartment, $y_{ik}^0 = 1$ if $k = k_i$, $y_{ik}^0 = 0$ otherwise; and y_i denotes that i clears his apartment and gets no other apartments, $y_i = (0, \ldots, 0)$.

Let v_{ik_i} be the value for i to his apartment, which is uniformly distributed in $[0, 1]$. Given an outcome $x_i = (y_i, p_i)$, the utility of i is given by $u_i(x_i) = u_i(y_i, p_i) = v_i \cdot (y_i - y_i^0) + p_i = -v_{ik} + p_i$.

Suppose that there is a minimal clearing target m for the developer. That is, the developer should clear at least m consecutive buildings. Otherwise, the acquired land will be too small for him to build anything and the cleared buildings have no value to him. If the developer clears $n \geq m$ contiguous buildings, then he gets the value of $n\bar{v}$, for some constant $\bar{v} \in [0, 1]$. Assume that he needs only one set of contiguous buildings.

In a unanimity-rule-only mechanism, the auctioneer is given a fixed subset of $n \leq M$ participants with contiguous buildings. The mechanism proceeds as follows. The auctioneer calls each bidder i in turn to play and offers him $X_{it} = \{(y_i, \bar{v})\}$. If anyone chooses to exit, then the auctioneer forces everyone else to exit as well. If everyone has chosen to continue, then the auctioneer closes for all bidders with X_{it} unchanged. This is equivalent to offering a contingent take-it-or-leave-it offer $(y_i, \bar{v})$ to every bidder, such that the offer is valid if and only if everyone accepts it.

We now calculate the expected welfare for this mechanism. Given that i accepts

$(y_i, \bar{v})$, the conditional expected welfare is

$$\mathbb{E}(u_j(y_i, \bar{v})|v_{ik} \leq \bar{v}) \tag{3.1}$$

$$=\mathbb{E}(-v_{ik} + \bar{v}|v_{ik} \leq \bar{v}) \tag{3.2}$$

$$=\frac{1}{2}\bar{v} \tag{3.3}$$

The net welfare that the developer gets is 0, as he pays full $n\bar{v}$ to all bidders. Then the total conditional expected welfare is $\frac{n}{2}\bar{v}$. The probability for the transaction to be successful is $(\frac{\bar{v}}{1})^n$. Thus, the total expected welfare is $\frac{1}{2}n\bar{v}^{n+1}$. It should be straightforward for readers to verify that if $\frac{n}{n+1} > \bar{v}$, then for all $n' \geq n$, we have:

$$\frac{1}{2}n\bar{v}^{n+1} > \frac{1}{2}n'\bar{v}^{n'+1} \tag{3.4}$$

That means that, when n is large enough, even though the welfare conditioned on a successful transaction is increasing with n, the probability of having a successful transaction drops faster than the welfare increases. Therefore, the total welfare decreases with n when n is sufficiently large.

Also, the welfare drops to zero as n gets large, as predicted by Mailath and Postlewaite (1990). This implies that when we have a large clearing target, the probability of having a successful transaction is inevitably small for a given set of bidders.

However, if we release ourselves from the restriction of the unanimity rule, then if somebody rejects, we can still try to reach a deal with other subsets of bidders.

The probability of having no trade will drop as the number of participants increases and the number of subsets increases. To see this point, we can offer the same outcome, $X_{it} = \{(y_i, \bar{v})\}$, to all the M bidders. Then find the largest number $\bar{m}$ of consecutive bidders who choose to continue. If $\bar{m} \geq n$, then close the auction with these bidders. Otherwise, force everyone to quit. Then for any given clearing target n, the probability of having no consecutive n bidders to continue drops to 0 when M gets large.

Thus, in a combinatorial-allocation-rule mechanism, we mitigate the low probability problem indicated by the impossibility theorem by considering more ways to trade. In contrast to the impossibility theorem, the more bidders that participate, the higher the probability for a transaction to occur. This is possible only when we adopt combinatorial allocation rules.

Also note that the combinatorial allocation rule mentioned in this example is not welfare-maximizing. This simple rule is used to highlight the difference between unanimity and combinatorial rules.

One caveat of this example is that we do not see the role of the descending process in the mechanism. In the following example, we will see how the descending process helps to explore the possibilities of transactions and facilitates competition.

Example 7 *Consider the pure monetary compensation model as in Example 6, with $M = 3$, $N = \{1, 2, 3\}$, and the clearing threshold $\bar{m} = 2$. Suppose the developer gets w_2 if he gets 2 contiguous buildings and w_3 for 3 contiguous buildings. For all other allocations, the values are all 0. Consider the following cases:*

1. $\frac{1}{3}w_3 \geq \frac{1}{2}w_2$: *We can consecutively offer bidders* $X_{it} = \{(y_i, \frac{1}{3}w_3)\}$. *If all three choose to continue, we clear all of their apartments. If bidder* 1 *or bidder* 3 *rejects, we lower the offers for the remaining bidders to* $\{(y_i, \frac{1}{2}w_2)\}$. *If both remaining bidders choose to continue, we have a deal. Otherwise, there is no trade. In this example, the descending clock can be interpreted as a method of trial and error. We first try to reach a deal with a higher value. If we fail and some bidder exits, we try to reach less valuable ones with the remaining bidders. When the clearing target becomes smaller, the willingness to pay of the developer may decrease. Therefore, the offer to the remaining bidders will have to adjust.*

2. *Consider* $w_3 = 0$. *That is, the developer wants only 2 contiguous buildings. Now we can see that bidder* 1 *and* 3 *are in competing positions. If either one of them chooses to quit, it is still possible to reach a deal with the remaining two bidders. But if bidder 2 quits, there is no trade at all. Thus, in this case, we can first alternatively lower the offers to bidder* 1 *and bidder* 3 *starting from* $X_{it} = \{(y_i, \frac{1}{2}w_2)\}$, *until one of them exits. We start from* $\frac{1}{2}w_2$ *because that is the highest possible payment to these bidders. Suppose that bidder* 3 *exits at* t *and* $X_{1t} = \{(y_1, p)\}$ *and* $X_{3t} = \{(y_3, p)\}$, *we offer bidder 2* $X_{2t} = \{(y_2, w_2 - p)\}$. *In this case, the descending clock is a device of competition. In the process of lowering the offer alternatively, bidder 1 and bidder 3 are competing for the right to be in the transaction. Notice that the offer to bidder 2 also depends on the competition result. Because the payment to bidder 1 or bidder 3 has*

been lowered in the competition, after one bidder quits, we can offer a higher payment to bidder 2. Thus, compared to a mechanism with just 2 bidders and when each of them is necessary, having an additional bidder introduces competition. With competition among some subset of bidders, we can make better offers to the essential bidders and increase the probability of trade.

Now we look at an example to see why having more options can increase the probability of trade.

Example 8 *Suppose that there are two buildings and there is one apartment in each building. Suppose that the developer wants only to redevelop building* 1*. We will show that having the owner of building 2 participate can increase the probability of clearing building 1.*

Let $Y_1 = \{y_1^0, y_1^1, y_1^2\} = \{(1,0),(0,0),(0,1)\}$. $y_1^1 = (0,0)$ *is the allocation without any non-monetary compensation.* $y_1^2 = (0,1)$ *is the allocation where bidder* 1 *is allocated the apartment of bidder 2. Let* $Y_2 = \{y_2^0, y_2^1\} = \{(0,1),(0,0)\}$*. In this example, if bidder 1 gets the apartment of bidder 2, bidder 2 must clear his apartment. Thus, the auctioneer should be choosing from the following set of allocations:* $\{(y_1^0, y_2^0),(y_1^1, y_2^0),(y_1^2, y_2^1)\}$*. Suppose that the value to the developer of clearing building 1 is* $\bar{v} \in (0,1)$*. The following two cases compare the differences between mechanisms with and without non-monetary compensation.*

If the non-monetary compensation outcome cannot be offered as in the previous example, the best the auctioneer can do is offer the bidder in building 1: $X_{it} = \{(y_i^1, \bar{v})\}$*. The probability of transaction is then* $\bar{v}$*.*

Now suppose that the non-monetary compensation is offered. Then one way to run this auction is as follows. Let $p_1, p_2 \geq 0$ *such that* $p_1 + p_2 = \bar{v}$. *First, offer to bidder 2:* $X_{2t} = \{(y_2^1, p_2)\}$. *If bidder 2 chooses to continue, offer to bidder 1* $X_{1t} = \{(y_1^1, \bar{v}), (y_1^2, p_1)\}$. *If bidder 1 also chooses to continue, call bidder 1 again with the closing outcomes* X_{1t}. *At this point, we are certain that building 1 is cleared. What is not certain yet is the way it is cleared. If bidder 1 chooses to get the apartment of bidder 2,* (y_1^2, p_1), *we call bidder 2 and forcing him to choose* (y_2^1, p_2). *On the other hand, if bidder 1 chooses to clear his apartment, we force bidder 2 to quit.*

However, if bidder 2 exits at the beginning, then offer to bidder 1 $X'_{1t} = \{(y_1^1, \bar{v})\}$ *as before. Therefore, in this scenario, having the option to move to apartment 2 does not harm bidder 1 at all. If bidder 2 does not accept* (y_2^1, p_2), *things go as usual. But if bidder 2 accepts, bidder 1 is offered a larger set of choices and the choices are not inferior. Thus, the probability of bidder 1 rejecting both options decreases. The total payment conditioned on the developer getting building 1 does not change either. Therefore, the expected welfare increases if we allow this kind of transaction.*

This example shows that having an additional option increases the probability of having a transaction. There will be a similar effect if we add other options like getting a new apartment. In the land assembly market, the probability of getting consent from a fixed group of people can be low when the group is large. Therefore, any way to increase the probability of people saying yes will help. In the market of land assembly, bidders' preferences and budget constraints are diverse. Monetary compensation might not be enough for everyone.

3.5 The Limitations of PED-LA

In this section, we discuss some limitations that are relevant to the land assembly use cases.

3.5.1 The Existing Apartment Offers

Example 9 *Suppose there is 1 clearable building, which contains two existing apartments $B = \{1, 2\}$. There is another apartment 3 in the nearby building which, however, is not clearable. The owner of apartment is willing to participate in the mechanism, hoping to get a new apartment in the newly developed building. In this case, the set of existing apartment is $K^E = \{k_1, k_2, k_3\}$. We assume that the developer builds only one type of new apartment, and we assume that the supply of the apartments will be sufficient once the building is built, $K^N = \{k_n\}$.*

Thus, for $i \in N = \{1, 2, 3\}$, an allocation is a 4-tuple, $y_i = (y_{ik_1}, y_{ik_2}, y_{ik_3}, y_{ik_n})$. For $k_j \in K^E$, $y_{ik_j} \in \{0, 1\}$, and $y_{ik_n} \in \mathbb{R}_+$.

Take the bidder 1 as an example. The endowment for bidder 1 is $y_1^0 = (1, 0, 0, 0)$. The allocation for him to get an existing apartment as compensation is $y_i^e = (0, 0, 1, 0)$. The allocation for him to get a 1,000 ft^2 apartment as compensation is $y_1^n = (0, 0, 0, 1000)$. The allocation for him not to get any non-monetary compensation is $y_1^m = (0, 0, 0, 0)$. We assume that similar notations apply to bidder

2.

For bidder 3, we offer him only the endowment allocation y_3^0, the new apartment offer y_3^n and the pure monetary compensation allocation y_3^m.

Then we can offer the following set for $i = 1, 2$ in the early periods in the auction, $X_{it} = \{x_{it}^0, x_{it}^e, x_{it}^n, x_{it}^m\} = \{(y_i^0, 0), (y_i^e, p_{it}^e), (y_i^n, p_{it}^n), (y_{it}^m, p_{it}^n)\}$. The prices have time subscripts to indicate that they can decrease with time. The size of the new apartment also can decrease with time.

Suppose that in the early stage of this auction, everyone is active. In particular, the outcome to move to apartment 3 are still offered, which are $x_{it}^e \in X_{it}$. Then we show that the auctioneer cannot feasibly clear the market. The reason is as follows:

1. *Suppose we close for bidder 1. In that case, we cannot prevent him from choosing to move to the apartment of bidder 3. If that happens, we must clear the apartment for bidder 3 and we cannot prevent him from choosing a new apartment.*

2. *For the developer to supply a new apartment, we must clear both bidder 1 and bidder 2.*

3. *Then we also must clear bidder 2. To prevent him from choosing the offer to move to apartment 3, we need to remove it from the continuation offers. But by doing so, the offer set becomes inferior and therefore—as required by the property of persistence of exits—we need to offer the bidder the chance to exit to his endowment.*

4. *If bidder 2 chooses to exit to his endowment, the new building cannot be built*

and the promise to bidder 3 cannot be fulfilled.

Readers can verify that a similar argument would apply when we first clear bidder 2 and bidder 3. Therefore, at some state of the auction, we need to remove the offer of moving to apartment 3 from either bidder 1 or bidder 2.

In a more general setting, an existing apartment can be associated with only one bidder in a clearable building. It is not a good that can be generally offered to a large set of bidders.

3.5.2 No Mutual Influence within a Building

We continue with the previous example. We assume that we remove the offer of moving to apartment 3 from bidder 2's continuation offers. Let's suppose that after some periods of lowering prices, the sum of the compensations are below the willingness to pay of the developer. Let's look at how the auctioneer will close the auction.

We show that because bidder 1 and bidder 2 are in the same building, they must always be closed together. When we close bidder 1 and force him to choose a winning outcome in X_{1t}, because all the allocation in X_{it} requires bidder 1 to clear his apartment, it is certain that the building must be cleared. Therefore we should force bidder 2 to clear his as well. To meet the condition that we are clearing a building, bidder 1's value and his choice within X_{1t} will not have any effect on the outcome offered to bidder 2.

In the analytical framework of Chapter 2, this means that on the condition that the building is cleared, none of the bidders in the same building will have an influence on one another. The outcomes offered to each of the bidders in the same building will share the same constant pricing space.

On the contrary, the choice of bidder 1 can have some effect on bidder 3. If bidder 1 chooses to move to apartment 3, then of course the auctioneer has to close bidder 3 immediately and force him to clear his apartment for bidder 1. But If bidder 1 chooses not to move to apartment 3. Then bidder 3 becomes nonessential in the mechanism. The auctioneer can further lower the offers to bidder 3, and close only if he accepts one of them.

3.5.3 No Incompatible Multilateral Choices

Bidders in the same building must be closed together, and no mutual influences could exist. This implies that it is not possible to implement multi-lateral choices that are mutually incompatible.

Example 10 *Consider Example 9. In this case, the developer can offer two designs of new apartments.* $K^N = \{k_H, k_L\}$*, where* K_H *represents luxury apartments and* k_L *represents economic apartments. However, he either supplies an entire building of luxury apartments or an entire building of economic apartments. Thus, if the 2 bidders are choosing a new apartment, they must agree on the design. It is not feasible for one to choose luxury and the other to choose economic.*

However, the bidders in the same building have no influence on each other. When one bidder has made a choice, this choice has no effect on the other. There-

fore, there is no way the auctioneer could prevent the bidders from choosing the infeasible outcome combinations. Thus, the absence of mutual influence can also be interpreted as an inability to coordinate the allocations. In general, this situation of mutually incompatible choices is impossible to implement in a PED mechanism. At some point in the mechanism, we will have to remove all the choices of a luxury apartment or all the choices of an economic apartment to guarantee the feasibility of outcomes.

3.5.4 Degree of Complementarity

This mechanism does not fully escape the impossibility theorem of Mailath and Postlewaite (1990). The mechanism only mitigate the problem by considering a larger set of possible transactions. However, for a particular allocation, the probability of finding an acceptable transaction still decreases sharply as the number of participants gets large. This limits the mechanism's performance when the building to be cleared is large.

3.6 Chapter Conclusion

This chapter shows that the PED mechanism is a good candidate for use in land assembly problems. It implements combinatorial allocation rules on a multi-dimensional outcome space. Compared to the unanimity rule, which is common in the literature, the PED mechanism can consider a larger set of clearing targets. Also, the multi-dimensional outcome space is flexible enough to offer bidders non-monetary compensations like new apartments or existing ones. We also show the limitation of using the PED mechanism. The complementarity within a building shuts down the mutual influence of bidders on each other, which reduces the allocation efficiency and disables the ability to coordinate mutually incompatible allocations.

Appendix A: Definition of Dynamic Mechanisms

Definition 22 (Dynamic Mechanism) *Given a market* $\Omega = \langle N, X_N, \mathcal{U}_N, \mathcal{F}, w\rangle$*, a dynamic mechanism* $\mathcal{D}$ *on market* Ω *is a tuple* $\mathcal{D} = \langle H, \prec, A, \mathcal{A}, \iota, \mathcal{I}_N, \chi_N\rangle$*, where*

1. *H is the set of histories and $\prec$ is a partial order on H, such that*

   (a) *The graph of histories is a tree.*

   (b) *$h_\emptyset$ denotes the initial history.*

   (c) *H has finite depth, i.e.:*

$$\exists d \in \mathbb{N} : \forall h \in H : |\{h' \in H : h' \prec h\}| \leq d \tag{A.1}$$

   (d) *$\sigma(h)$ denote the set of immediate successors of $h \in H$.*

   (e) *$z \in H$ is said to be a terminal history if $\sigma(z) = \emptyset$. Let $Z \subseteq H$ denote the set of all terminal histories.*

2. *A is a set of actions.*

3. *$\mathcal{A} : H \backslash h_\emptyset \to A$ maps each non-initial history to the last action taken to reach it.*

   (a) *For all $h \in H$, $\mathcal{A}$ is one-to-one on $\sigma(h)$.*

(b) $A(h)$ *denotes the actions available at* h*, that is,*

$$A(h) \equiv \bigcup_{h' \in \sigma(h)} \mathcal{A}(h') \tag{A.2}$$

4. $\iota : H \backslash Z \to N$ *is the player function.* $\iota(h)$ *denotes the player (seller) who is called to action at history* h*.*

5. $\mathcal{I}_i$ *is a partition on* $H_i \equiv \{h \in H | \iota(h) = i\}$*, such that*

   (a) *For any* $I_i \in \mathcal{I}_i$ *and* $h, h' \in I_i$*,* $A(h) = A(h')$*.*

   (b) *For any* $I_i \in \mathcal{I}_i$*, we denote* $\iota(I_i) \equiv \iota(h)$ *for any* $h \in I_i$*. Similarly,* $A(I_i) \equiv A(h)$ *for any* $h \in I_i$*.*

   (c) *Each action available only at one information set, that is, if* $a \in A(I_i)$*,* $a' \in A(I_i')$ *and* $I_i \neq I_i'$*, then* $a \neq a'$*.*

6. $\chi_i : Z \to X_i$ *maps each terminal history to an outcome for seller* i*, where*

   (a) $\chi_i = (\alpha_i^G, \rho_i^G)$*,*

   (b) $\alpha_i^G : Z \to K_i$ *is the option assignment function for seller* i*, and*

   (c) $\rho_i^G : Z \to \mathbb{R}$ *is the payment assignment function for seller* i*.*

Appendix B: Proofs of Chapter 1

B.0.1 Fundamental Results for Preference Space

We denote the upper contour set of outcome x_i for utility function u_i as $U(x_i, u_i) \equiv \{x_i' \in X_i | u_i(x_i') \geq u_i(x_i)\}$.

Lemma 11 *Let $U \subseteq X_i$, such that*

1. *U is closed,*

2. *U is weakly convex, and*

3. *for all $x_i \in X_i$, if $x_i \in U$, then for all $x_i' \geq x_i$, we have $x_i' \in U$*

4. *for all $y_i \in Y_i$, there exists $p_i \in \mathbb{R}$ such that $(y_i, p_i) \in U$.*

Then there exists exit unique $u_i \in \mathcal{U}_i$ such that if $x_i \in \partial U$, then $U(x_i, u_i) = U$. In this case, we say u_i represents U.

Proof of Lemma 11. Let $v_i(y_i) \equiv -\inf\{p \in \mathbb{R} | \forall p' \geq p, (y_i, p') \in U\}$ and $u_i(x_i) = u_i(y_i, p_i) \equiv v_i(y_i) - v_i(y_i^0) + p_i$. Thus, the utility function u_i is constructed to be quasi-linear in the monetary payment. Because U is closed, we have $(y_i, v_i(y_i)) \in \partial U \subseteq U$. Let $x_i = (y_i, p_i) \in \partial U$, then $v_i(y_i) = -p_i$ and $u_i(x_i) = v_i(y_i) - v_i(y_i^0) + p_i = -v_i(y_i^0)$. Also, by construction, $u_i(x_i^0) = u_i(y_i^0, 0) = 0$. Claim the following:

1. $U(x_i, u_i) = U$: Let $x'_i \in U(x_i, u_i)$, then

$$u_i(x'_i) \geq u_i(x_i) \tag{B.1}$$

$$\Rightarrow v_i(y'_i) - v_i(y^0_i) + p'_i \geq u_i(x_i) = -v_i(y^0_i) \tag{B.2}$$

$$\Rightarrow v_i(y_i) \geq -p'_i \tag{B.3}$$

$$\Rightarrow \inf\{p \in \mathbb{R} | \forall p' \geq p, (y_i, p') \in U\} \leq p'_i \tag{B.4}$$

$$\Rightarrow x'_i = (y'_i, p'_i) \in U \tag{B.5}$$

Thus $U(x_i, u_i) \subseteq U$. Now let $x'_i \in U$, then

$$p'_i \geq \inf\{p \in \mathbb{R} | \forall p' \geq p, (y_i, p') \in U\} \tag{B.6}$$

$$\Rightarrow -p'_i \leq v_i(y'_i) \tag{B.7}$$

$$\Rightarrow v_i(y'_i) + p'_i \geq 0 \tag{B.8}$$

$$\Rightarrow v_i(y'_i) - v_i(y^0_i) + p'_i \geq -v_i(y^0_i) \tag{B.9}$$

$$\Rightarrow u_i(x'_i) \geq u_i(x_i) \tag{B.10}$$

Therefore $x'_i \in U(x_i, u_i)$ and $U \subseteq U(x_i, u_i)$. Thus we have $U = U(x_i, u_i)$.

2. $U_i(\bar{x}_i, u_i)$ is weakly convex for all $\bar{x}_i \in X_i$.

   Let $x^1_i, x^2_i \in \mathcal{U}_i(\bar{x}_i, u_i)$ and $\lambda \in [0, 1]$. Let $\Delta p = -v_i(\bar{y}) - \bar{p}_i$. Then $v_i(\bar{v}_i) = -(\bar{p}_i + \Delta p)$. Thus $\hat{x}_i \equiv (\bar{y}_i, p_i + \Delta p) \in \partial U$. By the previous claim, $U(\hat{x}_i, u_i) = U$.

   Let $\hat{x}^k_i = (y^k_i, p^k_i + \Delta p)$ for $k = 1, 2$. Then we have $u_i(\hat{x}^k_i) \geq u_i(\hat{x}_i)$ and

$\hat{x}_i \in U(\hat{x}, u_i) = U$ for both $k = 1, 2$. Since U is convex, we have $\hat{x}_i^3 \equiv \lambda\hat{x}_i^1 + (1-\lambda)\hat{x}_i^2 \in U = U(\hat{x}_i, u_i)$. Then $u_i(\lambda\hat{x}_i^1 + (1-\lambda)\hat{x}_i^2) \geq u_i(\hat{x}_i)$. Because u_i is quasi-linear in the payment, we have

$$u_i(\lambda\hat{x}_i^1 + (1-\lambda)\hat{x}_i^2 - \Delta p) \geq u_i(\hat{x}_i - \Delta p) \tag{B.11}$$

$$u_i(\lambda x_i^1 + (1-\lambda)x_i^2) \geq u_i(\bar{x}_i) \tag{B.12}$$

Thus, $U(\bar{x}_i, u_i)$ is weakly convex.

3. u_i is weakly increasing.

   Let $x_i^1 >> x_i^2$. Let $\Delta p = -v_i(y_i^2) - p_i^2$. Then as shown in the previous claim, we have $(y_i^2, p_i^2 + \Delta p) \in \partial U \subseteq U$. Then we also have $(y_i^1, p_i^2 + \Delta p) \geq (y_i^2, p_i^2 + \Delta p)$ and therefore $(y_i^1, p_i^2 + \Delta p) \in U = U((x_i^2, p_i^2 + \Delta p), u_i)$. Thus, $u_i(y_i^1, p_i^2 + \Delta p) \geq u_i(y_i^2, p_i^2 + \Delta p)$. Also, since $p_i^1 > p_i^2$, we have $u_i(y_i^1, p_i^1 + \Delta p) > u_i(y_i^2, p_i^2 + \Delta p)$. Then by quasi-linearity, we arrive $u_i(x_i) > u_i(x_i^2)$. Therefore, u_i is weakly increasing.

4. u_i is continuous.

   We show that for all $x_i' \in X_i$, $U(x_i', u_i)$ is closed. Again, let $\Delta p = -v_i(y_i') - p_i'$. Then as shown in the previous claim, we have $\hat{x}_i \equiv (y_i', p_i' + \Delta p) \in \partial U \subseteq U$. Consider a converging series $x_i^n \to x_i''$ such that $x_i^n \in U(x_i', u_i)$. Construct another series by $\hat{x}_i^n = (y_i^n, p_i^n + \Delta p)$. Then we have $\hat{x}_i^n \to (y_i'', p_i'' + \delta p)$. By quasi-linearity, $\hat{x}_i^n \in U(\hat{x}_i, u_i) = U$. Because U is closed, we have $(y_i'', p_i'' + \Delta p) \in U(\hat{x}_i, u_i)$. Thus, $u_i(y_i'', p_i'' + \Delta p) \geq u_i(\hat{x}_i)$. By quasi-linearity, $u_i(x_i'') \geq$

$u_i(x_i')$. Therefore, $U(x_i', u_i)$ is closed.

Thus, u_i represents a quasi-linear, weakly convex, weakly increasing and continuous preference. Therefore, $u_i \in \mathcal{U}_i$. The uniqueness comes from the fact that we require for all $u_i \in \mathcal{U}_i$, $u_i(x_i^0) = 0$. ■

Lemma 12 *Let D be a dynamic mechanism with perfect recall, if $I_i' \prec I_i$ and I_i is on the path of S_i, then I_i' is on the path of S_i and $X_i(S_i, I_i) \subseteq X_i(S_i, I_i')$.*

Proof of Lemma 12. Let $x_i \in X_i(S_i, I_i)$, then there exists $h \in I_i$, S_{-i} such that $h \prec z(h_\emptyset, S_i, S_{-i})$ and $\chi_i(z(h_\emptyset, S_i, S_{-i})) = x_i$. Because $I_i' \prec I_i$, there exists $h^1 \in I_i'$, $h^2 \in I_i$ such that $h^1 \prec h^2$ and thus $I_i' \in \psi_i(h^2)$. Also, because i has perfect recall, we have $\psi_i(h) = \psi_i(h^2)$, and therefore $I_i' \in \psi_i(h)$. That is, there exists $h'' \in I_i'$ such that $h'' \prec h \prec z(h_\emptyset, S_i, S_{-i})$. Therefore, $x_i \in X_i(S_i, I_i')$ and $X_i(S_i, I_i) \subseteq X_i(S_i, I_i')$.

■

Lemma 13 *Let X_i' be a finite subset of X_i and $x_i^1, x_i^2 \in X_i$. If (x_i^1, x_i^2) is not convex-dominated in X_i', then there exits $u_i \in \mathcal{U}_i$ such that $u_i(x_i^1) > u_i(x_i^2) > u_i(x_i')$ for all $x_i' \in X_i' \backslash \{x_i^1, x_i^2\}$.*

Proof of Lemma 13. Define

$$U(x_i^1, x_i^2, \delta) \equiv \bigcup_{\lambda \in [0,1]} U(\lambda x_i^1 + (1-\lambda) x_i^2, \delta) \tag{B.13}$$

Claim: There exits $\delta > 0$ such that for all $\delta' < \delta$, $x_i \notin U(x_i^1, x_i^2, \delta')$.

Let $\lambda \in [0,1]$, because x_i does not convex-dominate (x_i^1, x_i^2), we have $x_i \not\geq$

$\lambda x_i^1 + (1-\lambda)x_i^2$. Thus, there exits $\delta(\lambda)$ such that for all $\delta' \leq \delta(\lambda)$, $x_i \notin U(\lambda x_i^1 + (1-\lambda)x_i^2, \delta')$. Let $\bar{\delta} = \frac{1}{2}\inf\{\delta(\lambda)|\lambda \in [0,1]\}$. Then $x_i \notin U(x_i^1, x_i^2, \bar{\delta})$.

Because $U(x_i^1, x_i^2, \delta)$ is a union of closed and convex sets, it is also closed and convex. The remaining conditions in Lemma 11 are again straightforward to verify. Thus, there exits a unique $u_i \in \mathcal{U}_i$ such that $U(x_i^1, u_i) = U(x_i^1, x_i^2, \delta)$.

For each $x_i' \in X_i$, let $\Delta p(x_i') \in \mathbb{R}$ such that $(y_i', p_i' + \Delta p(x_i')) \in \partial U(x_i^1, x_i^2, \delta)$. Let $\epsilon = \frac{1}{2}\min\{\Delta p(x_i')|x_i' \in X_i'\}$. Let $\hat{x}_i^2 = (y_i^2, p_i^2 + \epsilon)$ and $\hat{U} = U(x_i^1, \hat{x}_i^2, \delta)$. Then there exits $\hat{u}_i \in \mathcal{U}_i$ such that $U(x_i^1, \hat{u}_i) = \hat{U}$. Then for this $\hat{u}_i$, $\hat{u}_i(x_i^1) > \hat{u}_i(x_i^2) > \hat{u}_i(x_i')$ for all other $x_i' \in X_i'$. ■

Proof of Lemma 1.

First, for the undominated part:

($\Leftarrow$): Suppose that x_i is undominated in X_i', for $\delta > 0$, construct

$$U(x_i, \delta) \equiv \left\{\bar{x}_i \in X_i | \bar{p}_i \geq p_i + \frac{1}{\delta}\min\{0, \min\{\bar{y}_{ik} - y_{ik}|k \in K_i\}\}\right\} \tag{B.14}$$

For all $x_i' \in X_i$, if not $x_i' \geq x_i$, then there exits δ such that for all $\delta' \leq \delta$, $x_i' \notin U(x_i, \delta')$. Because X_i' is finite, and x_i is not dominated in X_i', then there exits δ small enough such that $U(x_i, \delta) \cap X_i' = \emptyset$.

It is straightforward to verify that $U(x_i, \delta)$ satisfy all conditions specified in Lemma 11. Therefore, there exits unique $u_i \in \mathcal{U}_i$ such that $U(x_i, u_i) = U(x_i, \delta)$. Because the preference is strictly increasing in payment and $U(x_i, u_i) \cap (X_i' \setminus \{x_i\}) = \emptyset$. We have $u_i(x_i) > u_i(x_i')$ for all $x_i' \in X_i' \setminus \{x_i\}$.

($\Rightarrow$): Suppose that there exists $u_i \in \mathcal{U}_i$ such that $u_i(x_i) > u_i(x_i')$ for all

$x_i' \in X_i' \backslash \{x_i\}$. If there exists $x_i'' \in X_i'$ such that $x_i'' \neq x_i$ and x_i' dominates x_i, then since u_i is weakly increasing, we have $u_i(x_i'') \geq u_i(x_i)$, which is a contradiction. ($\Rightarrow$): Suppose that x_i is not weakly undominated in $X_i(I_i)$. Then x_i is strictly dominated by some $x_i' \in X_i(I_i)$. Because u_i is strictly increasing in payment and weakly increasing in all other dimensions, we have $u_i(x_i') > u_i(x_i)$. Therefore, $x_i \notin X_i(I_i, u_i)$. Therefore, if $x_i \in X_i^*(I_i, u_i)$, then x_i is weakly undominated in $X_i(I_i)$.

Second, for the weakly undominated part:

($\Leftarrow$): Suppose that x_i is weakly undominated in $X_i(I_i)$. For $\delta > 0$, construct

$$U(x_i, \delta) \equiv \left\{ \bar{x}_i \in X_i | \bar{p}_i \geq p_i + \frac{1}{\delta} \min\{0, \min\{\bar{y}_{ik} - y_{ik} | k \in K_i\}\} \right\} \tag{B.15}$$

If $x_i' \geq x_i$, since x_i is not strictly dominated, we must have $p_i' = p_i$. Then $x_i' \in \partial U(x_i, \delta)$.

For all $x_i' \in X_i$, if not $x_i' \geq x_i$, then there exits δ such that for all $\delta' \leq \delta$, $x_i' \notin U(x_i, \delta')$. Because X_i' is finite, there exits δ small enough such that $\mathring{U}(x_i, \delta) \cap X_i'$ is empty.

It is straightforward to verify that $U(x_i, \delta)$ satisfy all conditions specified in Lemma 11. Therefore, there exits unique $u_i \in \mathcal{U}_i$ such that $U(x_i, u_i) = U(x_i, \delta)$. Because $\mathring{U}(x_i, \delta) \cap X_i'$ is empty, $x_i \in X_i^*(I_i, u_i)$. ∎

B.0.2 Elements of Strategic Simplicity

Proof of Theorem 1. If not, then there exists $I_i \in \mathcal{I}_i$, an alternative action

$a' \in A(I_i)\backslash\{S_i(I_i)\}$ and $x_i' \in X_i(a')$ such that $u_i(x_i') > u_i(x_i)$ for all $x_i \in X_i(a)$ where $a = S_i(I_i)$. Because $x_i' \in X_i(a')$, there exist S_i' such that $x_i \in X_i(I_i, S_i')$. That implies there are S_{-i}, such that $x_i' = \chi_i(z(h_0, S_i', S_{-i}))$. Also, because all the achievable outcomes by a are less desirable then x_i', we have $u_i(h_\emptyset, S_i, S_{-i}) < u_i(h_\emptyset, S_i', S_{-i})$. This equation contradicts with Equation (1.1). Thus the strategy is not weakly dominant. Hence, if a strategy is weakly dominant, it has to be optimistic. ■

Proof of Theorem ??. Let $a \in A(I_i)$, and let $\mathcal{S}_i(a)$ be the set of strategies S_i such that $S_i(I_i) = a$ and that I_i is on the path of S_i. It should be clear that $x_i \in X_i(a)$ if and only if it is achievable by some $S_i \in \mathcal{S}_i(a)$. That is, $x_i \in X_i(I_i, S_i)$. Thus, it follows that $X_i(a') = \cup\{X_i(I_i, S_i')|S_i \in \mathcal{S}_i(a')\}$ for all $a' \in A(I_i)$. Therefore, the RHS of Equation (1.2) and Equation (1.4) are equivalent. ■

Proof of Theorem 3. The sufficient part is trivial. Because the cautiously optimistic strategy S_i always exists. Then by Theorem **??**, S_i is obviously dominant for u_i. Thus, D is OSP.

We now show the necessary part. Suppose D is OSP. Let $i \in N$, $u_i \in \mathcal{U}_i$, S_i be cautiously optimistic for u_i. By Theorem 9, D satisfies the Single Continuation Property and the Persistence of Exit Property. By Lemma 24, S_i is regret-free. ■

Proof of Theorem 4. Let $\psi_i(I_i) = \{I_i^0, a^0, I_i^1, a^1, \ldots, I_i^{n-1}, a^{n-1}\}$, and let I_i be denoted as I_i^n and $S_i(I_i) = a^n$. Because I_i is on the path of S_i, we have $S_i(I_i^m) = a^m$ for all $m = 0, \ldots, n$. We prove by induction that for all $m = 0, \ldots, n$, we have

1. if S_i secures a fantasy outcome $x_i \in X_i^*(I_i^m, u_i)$ at I_i^m, then x_i is an exit outcome at I_i^m.

2. if S_i does not secure any outcome, there exists a continuation outcome in $X_i(S_i(I_i^m)) \cap X_i^*(I_i^m, u_i)$.

First, let $m = 0$, then $X_i^G(I_i^0) = \emptyset$. First, we consider that S_i secures some fantasy outcome $x_i \in X_i^*(I_i^0, u_i)$. Thus, x_i is assurable at I_i^0. By Lemma 1, x_i^e is weakly undominated in $X_i(I_i^0)$. Since $X_i^G(I_i^0)$ is empty, x_i is also weakly undominated in $X_i(I_i^0) \cup X_i^G(I_i^0)$. Therefore, x_i is an exit outcome. Next, we consider the case that if S_i does not secure any outcome. Because D is regret-free, by Theorem 5, there exists an action $a \in A(I_i^0)$ such that $X_i^*(I_i^0, u_i) \subseteq X_i(a)$ and $X_i^*(I_i^0, u_i) \cap X_i(a') = \emptyset$. Also, since S_i is a cautiously optimistic strategy, all outcomes in $X_i^*(I_i^0, u_i)$ are nonassurable. Therefore there exists an undominated outcome $x_i^* \in X_i^*(I_i^0, u_i)$. Then x_i is also undominated in $X_i(I_i^0)$, because if not, then x_i^* is dominated by some x_i'. Because u_i is weakly increasing, $u_i(x_i') \geq u_i(x_i^*)$, then $x_i' \in X_i^*(I_i^0, u_i)$. We then arrive a contradiction since that implies x_i^* is not undominated in $X_i^*(I_i^0, u_i)$. Then we also have that x_i^* is undominated in $X_i(I_i^0) \cup X_i^G(I_i^0)$ since $X_i^G(I_i^0)$ is empty. Therefore, x_i^* is a continuation outcome at I_i^0. Because $X_i^*(I_i^0, u_i) \subseteq X_i(a) = X_i(S_i(I_i))$, $x_i \in X_i(S_i(I_i^m)) \cap X_i^*(I_i^m, u_i)$. Therefore, the statements are true for $m = 0$.

Next, suppose that the statements are true for all $m \leq n-1$, we now show that they are also true for $m = n$. Suppose that $X_i^*(I_i^n, u_i)$ contains no assurable outcome: just as we argued in the previous bullet, there exists an outcome $x_i^* \in X_i^*(I_i^n, u_i)$ which is undominated in $X_i(I_i^n)$. Now we argue that it is also not dominated in $X_i^G(I_i^n)$. Suppose the opposite. Then there exists $x_i^g \in X_i^G(I_i^n)$ such that x_i^g dominates x_i^*. That is, $u_i(x_i^g) \geq u_i(x_i^*)$. Because $x_i^g \in X_i^G(I_i^n)$, we have $x_i^g \in X_i(a')$ for some $a \in A(I_i')$ and $I_i' \in \psi_i(I_i)$. Because D is regret-proof, by the Persistence of Exists Property, x_i^g should be assurable at the earliest I_i'' such that $I_i \prec I_i''$ and that $u_i(x_i^g) \geq \max\{u_i(x_i)|x_i \in X_i(I_i'')\}$. Then, a cautiously optimistic strategy will secure some of the assurable outcome at I_i''. Because $X_i(S_i, I_i^m) \subseteq X_i(S_i, I_i'')$, therefore, S_i must secure that outcome at I_i^n, contradicting our assumption in the beginning. This argument also applies for $x_i^g = x_i$, therefore, not only is x_i undominated in $X_i^G(I_i^n)$, but also $x_i \notin X_i^G(I_i^n)$. Therefore, x_i is a continuation outcome. Again, by the single continuation property, $x_i \in X_i^*(I_i^n, u_i) \cap X_i(S_i(I_i^n))$.

Now suppose $X_i^*(I_i^n, u_i)$ contains some assurable outcome x_i^e. As previously argued, x_i^e is weakly undominated in $X_i(I_i^n)$. Now we argue that it is also weakly undominated in $X_i^G(I_i^n)$. Suppose the opposite, which implies that there exists $x_i^g \in X_i^G(I_i^n)$ such that x_i^g strictly dominates x_i^e. Then we have $u_i(x_i^g) > u_i(x_i^e)$. Since $x_i^e \in X_i^*(I_i^n, u_i)$, we have $x_i^g \notin X_i(I_i^n)$. This is a condition where bidder i regrets that he didn't choose x_i^g, which contradicts with that the mechanism is regret-free and that S_i is cautiously optimistic before I_i^n. Therefore, x_i^e is weakly undominated in $X_i(I_i^n) \cup X_i^G(I_i^n)$ and is an exit outcome. Therefore, both statemenst are true for $m = n$.

Finally, we show the coverse part. Suppose that x_i is a continuation outcome at I_i. Then x_i is undominated in $X_i(I_i) \cup X_i^G(I_i)$. Then by Lemma 1, there exists $u_i \in \mathcal{U}_i$, such that $u_i(x_i) > u_i(x_i')$ for all $x_i \in X_i(I_i) \cup X_i^G(I_i)$. Because $x_i \in X_i(a^m)$ for all $m < n$, then the best-case outcome achievable by a^m must be no worse than x_i, and therefore is strictly better than all the outcomes achievable by other actions.

Thus, for a cautiously optimistic strategy, $S_i(I_i^m) = a^m$, which implies that I_i is on the path of S_i and $X_i^*(I_i, u_i) = \{x_i\}$. ■

Proof of Lemma 2. Let $x_i' \in X_i^C(I_i')$. Because $I_i \prec I_i'$, we have $X_i(I_i') \subseteq X_i(I_i)$ and hence $x_i' \in X_i(I_i)$. If $x_i' \in X_i^C(I_i)$, then it is trivial that $x_i' \geq x_i'$. If $x_i' \notin X_i^C(I_i)$, then it implies that it is dominated in $X_i(I_i) \cup X_i^G(I_i)$. But because $X_i^G(I_i') \supseteq X_i^G(I_i)$ and that x_i' is undominated in $X_i^G(I_i')$, it is not dominated in $X_i^G(I_i)$. Thus, it is dominated in $X_i(I_i)$. Then there must exists

if x_i is a continuation outcome at I_i, then either x_i is also a continuation outcome at I_i' or there exists some continuation outcome x_i' that dominates x_i. Therefore, in a bidder's perspective, the announced outcomes are becoming inferior overtime. ■

Proof of Theorem 5. Because D is OSP, then D satisfies the Persistence of Exit Property and the Single Continuation Property. Then the first and second bullet are the special case of Lemma 22 with $k = n + 1$. The third bullet is the direct implication of the Single Continuation Property. ■

B.0.3 Necessary Conditions for OSP

Lemma 14 *Let D be an OSP mechanism, $u_i \in \mathcal{U}_i$, S_i obviously dominant for u_i, if $x_i \in X_i(I_i)$ and that $u_i(x_i) > u_i(x_i')$ for all $x_i' \in X_i(I_i) \cup X_i^G(I_i) \backslash \{x_i\}$, then I_i is on the path of S_i.*

Proof of Lemma 14. It suffice to show that for all $I_i' \in \psi_i(I_i)$, $S_i(I_i') \in \psi_i(I_i)$. Suppose there exists $I_i' \in \psi_i(I_i)$ such that $S_i(I_i') = a' \notin \psi_i(I_i)$. Because S_i is optimistic, there exists $x_i^* \in X_i^*(I_i', u_i)$ such that $x_i^* \in X_i(a')$. Then $x_i^* \in X_i^G(I_i)$. Because x_i is a continuation outcome, therefore $x_i \notin X_i^G(I_i)$ and $x_i^* \neq x_i$. Then $u_i(x_i) > u_i(x_i^*)$. However, since $I_i' \prec I_i$, we must have $X_i(I_i) \subseteq X_i(I_i')$, which implies $x_i \in X_i(I_i')$ and contradicts that $x_i^* \in X_i^*(I_i', u_i)$. Therefore, $S_i(I_i') \in \psi_i(I_i)$ for all $I_i' \in \psi_i(I_i)$ and I_i is on the path of S_i. ■

Lemma 15 *Let D be an OSP dynamic mechanism, if x_i is a continuation outcome at I_i, then x_i is achievable by only one action in $A(I_i)$.*

Proof of Lemma 15. Suppose the claim is not true, that is, there exists $A' \subseteq A(I_i)$ such that $|A'| \geq 2$ and $x_i \in X_i(a')$ for all $a' \in A$. Because x_i is a continuation outcome, x_i is nonassurable and undominated in I_i. There exists $u_i \in \mathcal{U}_i$ such that $u_i(x_i) \geq u_i(x_i')$ for all $x_i' \in X_i(I_i) \cup X_i^G(I_i)$. Let S_i be an obviously dominant strategy for u_i. By Lemma 14, I_i is on the path of S_i. Also, because S_i is optimistic for u_i, $S_i(I_i) = a$ for some $a \in A'$. Because x_i is also achievable by other actions in A', obvious dominance requires that

$$\min\{u_i(x_i') | x_i' \in X_i(S_i, I_i)\} \geq u_i(x_i) \tag{B.16}$$

However, since x_i is nonassurable at I_i, $X_i(S_i, I_i) \neq \{x_i\}$. That implies there exists $x_i'' \neq x_i$ such that $x_i'' \in X_i(S_i, I_i)$. Since $u_i(x_i'') < u_i(x_i)$, we have $\min\{u_i(x_i') | x_i' \in X_i(S_i, I_i)\} < u_i(x_i)$, which contradicts with equation (B.16) and completes the proof. ■

Lemma 16 *Let D be an OSP dynamic mechanism, if x_i, x_i' are distinct continuation* *outcomes at* I_i*, and* $x_i \in X_i(a)$*,* $x'_i \in X_i(a')$ *for some distinct actions* $a, a' \in A(I_i)$*, then* (x_i, x'_i) *is convex-dominated in* $X_i(I_i) \cup X_i^G(I_i)$*.*

Proof of Lemma 16. Suppose not, that is, (x_i, x'_i) is not convex-dominated in $X_i(I_i) \cup X_i^G(I_i)$, then by Lemma 13, there exists $u_i \in \mathcal{U}_i$ such that $u_i(x_i) > u_i(x'_i) \geq u_i(x''_i)$ for all $x''_i \in X_i(I_i) \cup X_i^G(I_i)$ and $x''_i \neq x_i, x'_i$. Because D is OSP, then there exists an obviously dominant strategy S_i for u_i. By Theorem 1, S_i is optimistic. Therefore $S_i(I_i) = a$.

By Lemma 14, we have I_i is on the path of S_i. Also, because S_i is obviously dominant, we have

$$\min\{u_i(x_i)|x_i \in X_i(I_i, S_i)\} \geq u_i(x'_i) \tag{B.17}$$

By Lemma 15, x'_i is not achievable by a, therefore $x'_i \notin X_i(I_i, S_i)$. Also, because x_i is nonassurable, $X_i(I_i, S_i) \neq \{x_i\}$, and that implies $X_i(I_i, S_i)$ contains some $x''_i \neq x_i, x'_i$. However, the fact that $u_i(x''_i) < u_i(x'_i)$ contradicts with Equation (B.17) and completes the proof. ■

Lemma 17 *Let* D *be an OSP dynamic mechanism, then for all* $I_i \in \mathcal{I}_i$*,* $x_i^g \in X_i^G(I_i)$ *and* $x_i^d \in X_i^D(I_i)$*, if* (x_i^d, x_i^g) *is convex-undominated in* $X_i(I_i) \cup X_i^G(I_i)$*, then* x_i^g *is assurable at* I_i*.*

Proof of Lemma 17. Suppose x_i^g is nonassurable at I_i. By Lemma 13, there exists $u_i \in \mathcal{U}_i$ such that

$$u_i(x_i^d) > u_i(x_i^g) > u_i(x_i) \tag{B.18}$$

for all $x_i \in X_i(I_i) \cup X_i^G(I_i) \backslash \{x_i^d, x_i^g\}$. Because D is OSP, there exists an obviously dominant strategy S_i for u_i.

Claim that I_i is on the path of S_i. Let $\psi_i(I_i) = \{I_i^1, a^1, \ldots, I_i^n, a^n\}$. It suffice to show that for all $I_i^k \in \psi_i(I_i)$, $S_i(I_i^k) \in \psi_i(I_i)$. First, we consider the case for $k = n$. Because x_i^d is a disappearing outcome at I_i, therefore x_i^d is a continuation outcome at I_i^n and $x_i^d \in X_i(a^n)$. By Equation (B.18), $u_i(x_i^d) > u_i(x_i')$ for all $x_i' \in X_i(a')$, $a' \in A(I_i^n) \backslash \{a^n\}$. Therefore, the optimistic strategy S_i should choose a^n. Also, because $x_i^d \in X_i(I_i^n)$, we must have $x_i^d \in X_i(I_i^k)$ for all $k \leq n$. Next, consider the case for $k < n$. Suppose there exists $k < n$ such that $S_i(I_i^k) = a' \notin \psi_i(I_i)$. Because S_i is optimistic, there exists $x_i^* \in X_i^*(I_i^k, u_i)$ such that $x_i^* \in X_i(a')$. Then $x_i^* \in X_i^G(I_i)$. Because x_i^d is a continuation outcome at I_i^n, therefore $x_i^d \notin X_i^G(I_i)$ and $x_i^* \neq x_i^d$. Then, by Equation (B.18), we have $u_i(x_i^d) > u_i(x_i^*)$. However, since $x_i^d \in X_i(I_i^k)$, $x_i^* \notin X_i^*(I_i^k, u_i)$, reaching a contradiction. Therefore, $S_i(I_i^k) \in \psi_i(I_i)$ for all $I_i^k \in \psi_i(I_i)$ and I_i is on the path of S_i.

Because $x_i^g \in X_i^G(I_i)$, there exists $I_i^k \in \psi_i(I_i)$ such that $x_i^g \in X_i(a')$ for some $a' \in A(I_i) \backslash \{a^k\}$. Since S_i is obviously dominant, we must have

$$\min\{u_i(x_i) | x_i \in X_i(I_i^k, S_i)\} \geq u_i(x_i^g) \tag{B.19}$$

By Lemma 12, $X_i(I_i, S_i) \subseteq X_i(I_i^k, S_i)$. Because x_i^g is nonassurable at I_i, there exists $x_i' \neq x_i^g$ such that $x_i' \in X_i(I_i, S_i)$. Also, because $x_i^d \notin X_i(I_i)$, we must have $x_i' \neq x_i^d$. Therefore, $u_i(x_i') < u_i(x_i^g)$ and $x_i' \in X_i(I_i^k, S_i)$, contradicting Equation B.19. Thus, x_i^g must be assurable at I_i. ■

Lemma 18 *Let D be an OSP dynamic mechanism and let $z \in Z$. If $X_i^D(z) \neq \emptyset$, then for all $x_i^d \in X_i^D(z)$ and $x_i^g \in X_i^G(z)$, (x_i^d, x_i^g) is convex-dominated by $\chi_i(z)$.*

Proof of Lemma 18. This Lemma is a special case for Lemma 17. In a normal information set I_i, (x_i^d, x_i^g) is not convex-dominated in X_i ■

B.0.4 Sufficient Condition for OSP

Lemma 19 *Let X_i' be a finite subset of X_i, let $x_i, x_i' \in X_i$ and $x_i \neq x_i'$. Let $V(x_i, x_i') \equiv \{\bar{x}_i \in X_i' \backslash \{x_i, x_i'\} | (x_i, x_i')$ is convex-dominated by $\bar{x}_i\}$. Then if $x_i'' \in V(x_i, x_i')$, then $V(x_i, x_i'') \subset V(x_i, x_i')$.*

Lemma 20 *Suppose D satisfies the Persistent Exit Property, let $(x_i^d, x_i^g) \in X_i^D(I_i) \times X_i^G(I_i)$, if (x_i^d, x_i^g) is convex-undominated in $X_i(I_i)$, then x_i^g is assurable at I_i.*

Proof of Lemma 20. Suppose x_i^g is not assurable at I_i. Because D satisfies the persistent exit property, we have (x_i^d, x_i^g) is convex-dominated in $X_i(I_i) \cup X_i^G(I_i)$. For $(x_i, x_i') \in X_i^2$, $x_i \neq x_i'$, we define $V(x_i, x_i') \equiv \{\bar{x}_i \in X_i(I_i) \cup X_i^G(I_i) | \bar{x}_i$ convex-dominates $(x_i, x_i')\}$. Then we have $V(x_i^d, x_i^g) \cap X_i(I_i) = \emptyset$ and $V(x_i^d, x_i^g) \subseteq X_i^G(I_i)$. Because the game has finite length, $X_i^G(I_i)$ is finite and therefore so is $V(x_i^d, x_i^g)$. By Lemma 19, if we choose any $x_i^1 \in V(x_i^d, x_i^g)$, we have $V(x_i^d, x_i'') \subset V(x_i^d, x_i^g) \subseteq X_i^G(I_i)$. Then for $k \geq 2$, if $V(x_i^d, x_i^{k-1}) \neq \emptyset$, we choose $x_i^k \in V(x_i^d, x_i^{k-1})$. The process should stop within finite steps as $V(x_i^d, x_i^g)$ is finite, and at each step $V(x_i^d, x_i^k)$ strictly shrinks. Let $\hat{x}_i^g = x_i^K$ when the process stops. Then we have $(x_i^d, \hat{x}_i^g)$ is not convex-dominated in $X_i(I_i) \cup X_i^G(I_i)$. Because D satisfies the persistent exit property, $\hat{x}_i^g$ is assurable at I_i. Then $\hat{x}_i^g \in X_i(I_i)$ which contradicts with the assumption that (x_i^d, x_i^g) is not convex-dominated in $X_i(I_i)$. ■

Lemma 21 *Let D be a dynamic mechanism that satisfies the Persistent Exit Property and the Single Continuation Property. Let $u_i \in \mathcal{U}_i$, S_i be cautiously optimistic for u_i, $I_i \in \mathcal{I}_i$ on the path of S_i, if there exists $x_i^d \in X_i^D(I_i)$, $x_i^g \in X_i^G(I_i)$ such that $u_i(x_i^d) > \max\{u_i(x_i)|x_i \in X_i(I_i)\}$ and $u_i(x_i^g) \geq \max\{u_i(x_i)|x_i \in X_i(I_i)\}$, then x_i^g is assurable at I_i.*

Proof. Suppose x_i^g is nonassurable at I_i. Let $v = \max\{u_i(x_i)|x_i \in X_i(I_i)\}$. According to the Persistent Exit Property and Lemma 20, there exists $\bar{x}_i \in X_i(I_i)$ such that $\bar{x}_i$ convex-dominates (x_i^d, x_i^g). Thus, there exists $\lambda \in [0, 1]$ such that $\bar{x}_i \geq \lambda x_i^d + (1 - \lambda)x_i^g$. Because u_i represents an increasing and convex preference, we have:

$$u_i(\bar{x}_i) \geq u_i(\lambda x_i^d + (1 - \lambda)x_i^g) \tag{B.20}$$

$$\geq \lambda u_i(x_i^d) + (1 - \lambda)u_i(x_i^g) \tag{B.21}$$

$$> \lambda v + (1 - \lambda)v \tag{B.22}$$

$$= v = \max\{u_i(x_i)|x_i \in X_i(I_i)\} \tag{B.23}$$

Because $\bar{x}_i \in X_i(I_i)$, Equation B.23 is a contradiction. Thus, x_i^g is assurable at I_i.

■

Let $X_i^C(I_i^k)$ denote the set of continuation outcomes at I_i^k.

Lemma 22 *If D satisfies the Persistent Exit Property and the Single Continuation*

Property, then for all $u_i \in \mathcal{U}_i$, S_i cautiously optimistic for u_i, and I_i on the path of S_i. Let $\psi_i(I_i) = \{I_i^1, a^1, \ldots, I_i^n, a^n\}$, and denote $I_i = I_i^{n+1}$. If $X_i^*(I_i, u_i)$ contains no assurable outcomes, then for all $k = 1, \ldots n+1$, we have

1. $X_i^*(I_i^k, u_i) \subseteq X_i(a^k) \cap X_i^C(I_i^k)$
2. $X_i^*(I_i^k, u_i) \cap X_i(a') = \emptyset$ *for all* $a' \neq a^k$, $a' \in A(I_i^k)$.

Proof of Lemma 22. The proof proceeds with a series of labeled claims.

C1 for all $k = 1, \ldots, n$, $X_i(I_i^k, S_i)$ is not a singleton. That is, S_i does not assure any outcomes at I_i^k.

Because S_i is cautiously optimistic, if S_i assures x_i at I_i^{n+1} for some $x_i \in X_i(I_i^{n=1})$, then that implies x_i is assurable at I_i^{n+1} and $x_i \in X_i^*(I_i^{n+1}, u_i)$, contradicting the assumption that $X_i(I_i^{n+1})$ contains no assurable outcomes. Thus, $X_i(I_i^{n+1})$ is not a singleton. By Lemma 12, for all $I_i^k \in \{1, \ldots, n\}$, we have $X_i(I_i, S_i) \subseteq X_i(I_i^k, S_i)$, therefore $X_i(I_i^k, S_i)$ is not a singleton, either. Hence S_i does not assure any outcome at I_i^k.

The rest of the proof proceeds with mathematical induction. Suppose $k = 1$, we first show that $X_i^*(I_i^1, u_i) \subseteq X_i^C(I_i^1)$. Let $x_i^* \in X_i^*(I_i^1, u_i)$. Because I_i^1 is the first time that i is called to play, $X_i^G(I_i) = \emptyset$. Therefore, $x_i^* \notin X_i^G(I_i^1)$. Furthermore, $x_i^* \in X_i^*(I_i^1, u_i)$ implies that x_i^* is undominated in $X_i(I_i^1)$, and hence also undominated in $X_i^*(I_i) \cup X_i^G(I_i)$. Finally, by Claim C1, S_i does not assure any outcomes at I_i^1. Thus, $X_i^*(I_i^1, u_i)$ contains no assurable outcomes and x_i^* is nonassurable at I_i^1. Therefore x_i^* is a continuation outcome at I_i^1.

Next, we show that for all $a' \in A(I_i^1)\backslash\{a^1\}$, $X_i^*(I_i^1, u_i) \cap X_i(a') = \emptyset$. Suppose the opposite, that is, there exists $a' \in A(I_i^1)$, $x_i' \in X_i^*(I_i^1, u_i)$ such that $a' \neq a^1$ and $x_i' \in X_i(a')$. By the definition of the Single Continuation Property, we must have $x_i' \neq x_i^*$, and that there exists $\bar{x} \in X_i(I_i^1) \cup X_i^G(I_i^1)$ such that $\bar{x}$ convex-dominates (x_i^*, x_i') and x_i' is not a continuation outcome. That is, there exists $\lambda \in [0, 1]$ such that $\bar{x}_i \geq \lambda x_i^* + (1-\lambda)x_i'$. Because u_i is convex and nondecreasing, we have

$$u_i(\bar{x}_i) \geq u_i(\lambda x_i^* + (1-\lambda)x_i') \tag{B.24}$$

$$\geq \lambda u_i(x_i^*) + (1-\lambda)u(x_i') \tag{B.25}$$

$$= u_i(x_i^*) \tag{B.26}$$

Because $X_i^G(I_i^1) = \emptyset$, $\bar{x}_i \in X_i(I_i^1)$. That implies $\bar{x}_i \in X_i^*(I_i^1, u_i)$ and that $\bar{x}_i$ is a continuation outcome at I_i^1, which leads to a contradiction. Therefore, we cannot have $a' \in X_i(I_i^1)$ such that $a' \neq a^1$ and $X_i(a') \cap X_i^*(I_i^1, u_i) \neq \emptyset$. Therefore all outcomes in $X_i^*(I_i^1, u_i)$ are achievable only by a^1. Thus, we proved both claims for $k = 1$.

Suppose the statement is true for all $k \leq m$, and consider the case for $k = m + 1$.

We first prove that $X_i^*(I_i^{m+1}, u_i) \subseteq X_i^C(I_i^{m+1})$. Suppose it is not true, that is, there exists $x_i^* \in X_i(I_i^{m+1}, u_i)$ such that x_i^* is not a continuation outcome at I_i^{m+1}.

C2 $x_i^* \in X_i^G(I_i^{m+1})$ or x_i^* is dominated in $X_i^G(I_i^{m+1})$.

Because x_i^* is achievable but nonassurable at I_i, to make it not a continuation outcome, we must have either $x_i^* \in X_i^G(I_i^{m+1})$ or x_i^* is dominated in $X_i(I_i^{m+1}) \cup$

$X_i^G(I_i^{m+1})$. Then the claim follows from the fact that x_i^* is not dominated in $X_i(I_i^{m+1})$.

C3 $x_i^* \notin X_i^*(I_i^m, u_i)$.

If so, our assumption for $k = m$ requires that x_i^* is a continuation outcome at I_i^m. Then by the definition of continuation outcome, we have $x_i^* \notin X_i^G(I_i^m)$ and that x_i^* is undominated in $X_i^G(I_i^m)$. Also, by the Single Continuation Lemma, x_i^* is not achievable by any actions other than $S_i(I_i^m)$. Hence we have $x_i^* \notin X_i^G(I_i^{m+1})$. Next, because $X_i(I_i^{m+1}) \cup X_i^G(I_i^{m+1}) \subseteq X_i(I_i^m) \cup X_i^G(I_i^m)$, we also have x_i^* is undominated in $X_i(I_i^{m+1}) \cup X_i^G(I_i^{m+1})$, contradicting Claim C2.

Now we let

$$x_i^g = \begin{cases} x_i^*, & \text{if } x_i^* \in X_i^G(I_i^{m+1}) \\ x_i' \in X_i^G(I_i^{m+1}) : x_i' \geq x_i^* & \text{if } x_i^* \text{ is dominated in } X_i^G(I_i^{m+1}) \end{cases} \tag{B.27}$$

and,

$$x_i^d \in X_i^*(I_i^m, u_i) \tag{B.28}$$

In either case, we have $u_i(x_i^g) \geq u_i(x_i^*) = \max\{u_i(x_i) | x_i \in X(I_i^{m+1})\}$ and $x_i^g \in X_i^G(I_i^{m+1})$.

C4 $x_i^d \in X_i^D(I_i^{m+1})$ and $u_i(x_i^d) > u_i(x_i^*)$.

By the induction assumption for $k = m$, we have that x_i^d is a continuation outcome at I_i^m. By the Single Continuity Property, $x_i^d \in X_i(a^m)$. Because

$x_i^* \notin X_i^*(I_i^m, u_i)$ but $x_i^* \in X_i(I_i^m)$, we must have $u_i(x_i^d) > u_i(x_i^*)$. That also implies that $x_i^d \notin X_i(I_i^m)$. Thus, $x_i^d \in X_i^D(I_i^{m+1})$ and $u_i(x_i^d) > u_i(x_i^*)$.

By Lemma 21, x_i^g should be assurable at I_i. That also implies that $x_i^g \in X_i^*(I_i^{m+1})$. Because S_i is cautiously optimistic, S_i must assure some outcome in $X_i^*(I_i^{m+1})$, which contradicts Claim C1 and therefore x_i^* should be a continuation outcome and $X_i^*(I_i^{m+1}) \subseteq X_i^C(I_i^{m+1})$.

Next, we prove that for all $a' \in A(I_i)\backslash\{a\}$, $X_i^*(I_i^{m+1}, u_i) \cap X_i(a') = \emptyset$. Suppose that there exists $a' \in A(I_i^{m+1})$, $a' \neq a^{k+1}$ and $x_i' \in X_i^*(I_i^{m+1}) \cap X_i(a')$. Then by the definition of the Single Continuation Property, there exists $x_i^g \in X_i(I_i^{m+1}) \cup X_i^G(I_i^{m+1})$ such that x_i^g convex-dominates (x_i^*, x_i') and that x_i^g is not a continuation outcome at I_i^{m+1}. Because u_i is convex, we have $u_i(x_i^g) \geq u_i(x_i^*)$. If $x_i^g \in X_i(I_i^{m+1})$, then $x_i^g \in X_i^*(I_i^{m+1}, u_i)$ and therefore $x_i^g \in X_i^G(I_i^{m+1})$.

Because $x_i^g \in X_i^G(I_i^{m+1})$, there exists $1 \leq \bar{k} \leq m$ such that $x_i^g \in X_i(a')$ for some $a' \in A(I_i^{\bar{k}})$ and $a' \neq a^{\bar{k}}$. Also, $x_i^g < \max\{u_i(x_i)|x_i \in X_i(I_i^{\bar{k}})\}$. Because if not, $x_i^g \in X_i^*(I_i^{\bar{k}}, u_i)$ and x_i^g can only be achievable by $a^{\bar{k}}$.

Let $v_k = \max\{u_i(x_i)|x_i \in X_i(I_i^k)\}$ for $k = 1, \ldots, m+1$. Because the set $X_i(I_i^k)$ is shrinking in k, we have v_k is nonincreasing in k. Thus, if $u_i(x_i^g) \geq v_k$, then for all $k' \geq k$, we also have $u_i(x_i^g) \geq v_k$. Thus, there exists $\bar{k} < \hat{k} \leq m+1$ such that $\hat{k}$ is the least k to satisfy $u_i(x_i^g) \geq v_k$.

Then by Lemma 23, x_i^g is assurable at $I_i^{\hat{k}}$, which implies $x_i^g \in X_i^*(I_i^{\hat{k}}, u_i)$. Then because S_i is cautiously optimistic, $X_i(I_i^{\hat{k}}, S_i)$ is a singleton, contradicting Claim C1. Therefore, we have $X_i^*(I_i^{m+1}) \cap X_i(a') = \emptyset$ for all $a' \in A(I_i^{m+1})$ such that $a' \neq a^{m+1}$.

■

Lemma 23 *Suppose that D satisfies the Persistent Exit Property and the Single Continuation Property. Let $u_i \in \mathcal{U}_i$, S_i cautiously optimistic for u_i and I_i on the path of S_i. Let $\psi_i(I_i) = \{I_i^1, a^1, \ldots, I_i^n, a^n\}$ and let $x_i^g \in X_i^G(I_i)$. Suppose that for all $I_i^k \in \psi_i(I_i)$, we have*

1. $X_i^*(I_i^k, u_i) \subseteq X_i(a^k) \cap X_i^C(I_i^k)$
2. $X_i^*(I_i^k, u_i) \cap X_i(a') = \emptyset$ *for all* $a' \neq a^k$, $a' \in A(I_i^k)$.
3. $u_i(x_i^g) < V_i(I_i^k, u_i)$

Then if $u_i(x_i^g) \geq V_i(I_i, u_i)$, then x_i^g is assurable at I_i.

Proof. Let $x_i^d \in X_i^*(I_i^n, u_i)$, then x_i^d is a continuation outcome, $x_i^d \in X_i(a^n)$ and $u_i(x_i^d) > u_i(x_i^g)$. Because $u_i(x_i^g) \geq V_i(I_i, u_i)$, therefore $x_i^d \notin X_i(I_i)$. Then $x_i^d \in X_i^D(I_i^{\bar{k}})$ and $u_i(x_i^d) > V_i(I_i, u_i)$. By Lemma 21, we have x_i^g is assurable at I_i. ■

Lemma 24 *Suppose D satisfies the Persistent Exit Property and the Single Continuation Property. Let $u_i \in \mathcal{U}_i$. If a strategy S_i is cautiously optimistic for u_i, then S_i is obviously dominant for u_i.*

Proof of Lemma 24. Let $u_i \in \mathcal{U}_i$, S_i cautiously optimistic for u_i. Suppose S_i is not obviously dominant, then there exists I_i on the path of S_i, $a' \in A(I_i)$, $x_i^g \in X_i(a')$ such that $a' \neq S(I_i)$ and

$$\min\{u_i(x_i) | x_i \in X_i(I_i, S_i)\} < u_i(x_i^g) \tag{B.29}$$

That implies there exists $h \in I_i$, S_{-i}, $z' = z(h_\emptyset, S_i, S_{-i})$ and $x'_i = \chi_i(z')$, such that $h \prec z'$, and $u_i(x'_i) < u_i(x^g_i)$. Let $\psi_i(z') = \{I^0_i, a^0, \ldots, I^n_i, a^n\}$.

We first show that $u_i(x^g_i) \geq \max\{u_i(x_i) | x_i \in X_i(I^n_i)\}$. Suppose the opposite and let $x^d_i \in X^*_i(I^m_i, u_i)$. Then we must have $u_i(x^d_i) > u_i(x^g_i) > u_i(x'_i)$. Claim that $X^*_i(I^n_i, u_i)$ contains no assurable outcomes. If there is an assurable outcome in $X^*_i(I^n_i, u_i)$, then the cautiously optimistic S_i should assure some $x^*_i \in X^*_i(I^n_i, u_i)$. That is, $X_i(I^n_i, S_i) = \{x^*_i\}$. However, we have $x'_i \in X_i(I^m_i, S_i)$ and $u_i(x^*_i) > u_i(x'_i)$, which is a contradiction. Therefore, $X^*_i(I^n_i, u_i)$ contains no assurable outcomes. By Lemma 22, x^d_i is a continuation outcome at I^n_i, and $x^d_i \in X_i(a^n)$. Therefore, $x^d_i \in X^D_i(z')$. Then by the definition of Persistent Exit property, x'_i convex-dominates (x^d_i, x^g_i). Therefore, there exists $\lambda \in [0,1]$ such that $x'_i \geq \lambda x^d_i + (1-\lambda)x^g_i$.

$$u_i(x'_i) \geq u_i(\lambda x^d_i + (1-\lambda)x^g_i) \tag{B.30}$$

$$\geq \lambda u_i(x^d_i) + (1-\lambda)u_i(x^g_i) \tag{B.31}$$

$$> \lambda u_i(x'_i) + (1-\lambda)u_i(x'_i) \tag{B.32}$$

$$= u_i(x'_i) \tag{B.33}$$

Therefore, we arrive a contradiction and we must have $u_i(x^g_i) \geq \max\{u_i(x_i) | x_i \in X_i(I^n_i)\}$.

Because $x^g_i \in X^G_i(I^n_i)$, there exists $1 \leq \bar{k} \leq n-1$ such that $x^g_i \in X_i(a')$ for some $a' \in A(I^{\bar{k}}_i)$ and $a' \neq a^{\bar{k}}$. Also, $x^g_i < \max\{u_i(x_i) | x_i \in X_i(I^{\bar{k}}_i)\}$. Because if not, $x^g_i \in X^*_i(I^{\bar{k}}_i, u_i)$ and by Lemma 22, x^g_i can only be achievable by $a^{\bar{k}}$.

Let $v_k = \max\{u_i(x_i) | x_i \in X_i(I^k_i)\}$ for $k = 1, \ldots, n$. Because the set $X_i(I^k_i)$ is shrinking in k, we have v_k is nonincreasing in k. Thus, if $u_i(x_i^g) \geq v_k$, then for all $k' \geq k$, we also have $u_i(x_i^g) \geq v_k$. Thus, there exists $\bar{k} < \hat{k} \leq n$ such that $\hat{k}$ is the least k to satisfy $u_i(x_i^g) \geq v_k$.

Then by Lemma 23, x_i^g is assurable at $I_i^{\hat{k}}$, which implies $x_i^g \in X_i^*(I_i^{\hat{k}}, u_i)$. Then because S_i is cautiously optimistic, S_i assures some $x_i^* \in X_i^*(I_i^{\bar{k}}, u_i)$. That is, $X_i(I_i^{\bar{k}}, S_i) = \{x_i^*\}$. Thus, we have $u_i(x_i^*) = u_i(x_i^g) > u_i(x_i')$, and $x_i^* \neq x_i'$. Therefore, $x_i' \notin X_i(I_i^{\bar{k}}, S_i)$. However, by Lemma 12, we must have $X_i(I_i, S_i) \subseteq X_i(I_i^{\bar{x}}, S_i)$, which implies $x_i' \in X_i(I_i^{\bar{x}}, S_i)$ and reaches a contradiction. Therefore, S_i should be obviously dominant.

■

B.0.5 The PED Mechanism

Proof of Theorem 6. Let D be an ORP dynamic mechanism. For any $z \in Z$, $i \in N$, consider the experience $\psi_i(z)$. For all $I_i \in \psi_i$, since D implements the clock announcement, the auctioneer announces the set of continuation and exit actions to bidder i. Also, the auctioneer reveals the continuation outcomes associated with each of the continuation action and the exit outcome for each exit action. Because continuation outcomes are undominated in $X_i(I_i)$ and exit outcomes are weakly undominated in $X_i(I_i)$. Therefore thay are also undominated and weakly undominated in the announced outocmes X_{it}, respectively. Because D is regret-proof, and therefore OSP, D satisfies the single continuation property. Therefore, the announcement at I_i satisfies all the requirements that we mentioned in the PED auction rules. Also, because D satisfies the strong persistence of exits property, whenever the set of exit outcomes does not shrink when the clock descends. Also, if the previously chosen continauation action can achieve more than one continuation outcomes, the auctioneer will have to call him to action again. Also, D is regret-proof and hence OSP, then D satisfies the single continuation property. The above conditions fully characterize a PED mechanism.

Let D be a PED mechanism. We prove by induction that the outcomes in X_{it}^E and X_{it}^C are indeed exit and continuation outcomes and therefore the annoucement satisfies the single continuation property and the strong persistence of exits property. Let $t = 0$, because there is no forgone outcomes, then being nonassurable and undominated in X_{it} is sufficient for an outcome to be an continuation outcome. Also, being weakly undominted in X_{it} is sufficient for being an exit outcome. Now suppose the statement is true for all $t' < t$. Now we show it is true for t. Let $x_i \in X_{it}^C$, then by the auction rule it is achievable but unassurable. If it is dominated by some forgone outcome x_i^g at some $t'' < t$, then by the single continuation property, the forgone outcome x_i^g is an exit outcome at t. By the persistence of exit property, x_i^g is also an exit outcome at t and therefore dominates x_i. Therefore, the outcome x_i is an continuation outcome. This is a contradiction. Similarly, we can use the same logic to show that all the outcomes in X_{it}^E are weakly undominated in X_{it} and in $X_i^G(I_i)$. Therefore, they are all exit outcomes. Therefore, the PED mechanism implements explicit exits and the clock announcement. It is also obvious now that the PED mechanism satisfies the strong persistence of exits property and the single continuation property. Therefore, by Theorem 9, D is OSP and hence regret-free.

Combining all the above properties, D is ORP. ■

Proof of Theorem 8.

Let $M = (\alpha_N, \rho_N)$ be OSP-implementable, then there exists an OSP mechanism D such that for all $i \in N$ $u_i \in \mathcal{U}_i$, there exists $S_i(u_i)$ such that for u_i, we have

1. $S_i(u_i)$ is obviously dominant for u_i,
2. $\chi_i(z(h_0, S_N(u_N))) = (\alpha_i(u_N), \rho_i(u_N))$,

where $S_N(u_N)$ is a short hand notation of $(S_i(u_i))_{i \in N}$.

Because D is OSP, it satisfies the Single Continuation Property and the Persistence of Exits Property. To modify D so that it becomes a PED mechanism, we first need to make it satisfy the Strong Persistence of Exits and then we enforce the explicit exits. The descending clock announcement only regulates how the auctioneer reveals informations to bidder, it doesn't change the extensive form game of D. Before we start to modify D and the dominant strategy equilibrium, we first define the following operation on a dynamic game D:

1. Remove direct assignment:

   For each $i \in N$, and for each terminal node $z \in Z$, let I_i be the last information set before z and let $a_i = S_i(I_i)$ be the action taken to arrive z. If $X_i(a)$ is not a singleton, then insert a history h right before z with only one action a' that reaches z. Call i to play at h, that is $\iota(h) = i$, and add an information set $I_i' = \{h\}$ to $\mathcal{I}_i$. Also, set $A(I_i') = \{a'\}$. Iterate this process through all $i \in N$, then the new game D' has no direct assignment.

2. Remove an action $a \in A(I_i)$:

   Let $H(h) \equiv \{h' \in H | h \prec h'\}$ and $H(a) \equiv \cup\{H(h)|\mathcal{A}(h) = a\}$. Thus $H(a)$ is the set of all $h' \in H$ such that h' is reached only if i chooses action a at I_i. Remove all $H(a)$.

3. Given a mechanism without direct assignment, remove an outcome x_i from $a \in A(I_i)$:

   Because $x_i \in X_i(a)$, there exists $z \in Z$ such that $h \prec z$ for some $h \in I_i$ and $\chi_i(z) = x_i$. Because D has no direct assignment, the last action $\bar{a}$ taken to reach z should have $X_i(\bar{a}) = \{x_i\}$. Then for each such z, there exists an earliest action $a(z)$ taken by i to reach z such that $X_i(a(z)) = \{x_i\}$. Then for all such z remove $a(z)$. Then after the removal, $x_i \notin X_i(a)$.

4. Given a mechanism without direct assignment, if x_i is assurable by a but $X_i(a)$ contians other achievable outcomes, then we can isolate x_i as follows:

   For each $h \in I_i$, get $h' : \mathcal{A}(h') = a$ duplicate the entire subgame $H(h')$ as $H(h'')$ such that the precedence structure is isomorphic to $H(h')$, and $h \prec h''$. Add a new action a' to $A(I_i)$ such that a' is the action to reach the new h'' for each $h \in I_i$. Thus, $X_i(a') = X_i(a)$. Now remove all outcomes in $X_i(a')$ other than x_i. Therefore, we now have $X_i(a') = \{x_i\}$.

We say $a \in A(I_i)$ is an exit action if $X_i(a)$ does not contain any continuation outcome.

With the above defined operations, we can modify D and the strategies $S_i(u_i)$ as follows:

1. Remove direct assignment. Then set $S_i(u_i)(I_i') = a'$ for all the new information set and new action, then the allocation rule is not changed at each of the step in the process. Therefore, D' still implements M.

2. Remove redundant exit outcomes: for all $i \in N$, and $I_i \in \mathcal{I}_i$, if there exists $x_i \in X_i(I_i)$ such that x_i is assurable by more than one exit actions, then arbitrarily choose one exit action a that secures x_i and remove x_i from all other exit actions. Set for all u_i such that $S_i(u_i)$ secures x_i at I_i, change $S_i(u_i)$ such that $S_i(u_i)$ secures x_i with a. This will not change the allocation rule. Because M is monotonic, therefore contingent on i secures x_i at I_i, the contingent allocation for the rest of bidders should be the same for all choices of i.

3. Isolate exit outcomes: for all $i \in N$, $I_i \in \mathcal{I}_i$, and each of the assurable outcomes x_i, isolate x_i such that x_i is assurable by a dedicated action a such that $X_i(a) = \{x_i\}$. Now for other actions a' such that a' is not continuation action and $X_i(a')$ is not singleton, remove a'. For all u_i such that $S_i(u_i)$ secures x_i at I_i, modify $S_i(u_i)$ such that $S_i(u_i)$ secures x_i with the new action a' that $X_i(a) = \{x_i\}$. The modification also doesn't change the allocation and payment rules.

4. After the above operations, when i secures x_i at I_i with action $a \in A(I_i)$, x_i is the only possible outcome. Then if i is called at I_i' after I_i, he only has one sigle action available $a' \in A(I_i')$ with $X_i(a') = \{x_i\}$. Then for all $h' \in I_i'$, let h^1 be the immediate predecessor of h' and h^2 be the unique immediate successor of

h'. We can remove h' and keep all $H(h')$ such that h^2 becomes the immediate successor of h^1. After removing all such histories, i is not called to play at all after choosing a. That implies that i becomes inactive and secures x_i after the first time he chooses an exit action.

5. Now we have a mechanism with 1-1 mapping between exit outcomes and exit actions, and once i secures an exit outcome, he becomes inactive. Now for all $i \in N$, descending nodes $I_i \in \mathcal{I}_i$, if $x_i^g \in X_i^G(I_i)$ is not assurable at I_i, there exits a previous information set I_i' and x_i^g is assurable by a' at I_i' such that $X_i(a') = \{x_i^g\}$. Now duplicate one of the subgame $H(h')$ for some $\mathcal{A}(h') = a'$ and attach to each $h \in I_i$. Modify the strategies $S_i(u_i)$ for each $u_i \in \mathcal{U}_i$ such that $S_i(u_i)$ points to actions that are isomorphic to the original subgame $H(h')$. But keep that $S_i(u_i)(I_i)$ unchanged.

   Because D is OSP, if x_i^g is not assurable at I_i, it means that for all $u_i \in \mathcal{U}_i$, x_i^g is not the fantasy outcome at I_i, therefore, even if x_i^g becomes assurable at I_i, $S_i(u_i)$ not choosing to secure it is still obviously dominant. Therefore, the new subgame will not be visited at all and all of the information sets are not on the path. Therefore, we don't need to verify the obviously dominance condition on those information sets.

Now after these modifications, D now satisfies the strong Persistence of Exits, and enforces 1-1 mapping between the exit actions and exit outcomes. Now if the auctioneer implements the descending clock announcement, D becomes a PED mechanism. In all the above modifications, the allocation and payment rules don't change. Therefore, the new mechanism still implements M. That is, M is PED-implementable. ■

Appendix C: Proofs of Chapter 2

Proof of Theorem 10. ($\Rightarrow$): Suppose M is finite and strategy-proof. First of all, $Y_i(u_{-i})$ is finite because the entire outcome space $\chi_i(\mathcal{U}_i)$ is finite. Second, given $u_{-i} \in \mathcal{U}_{-i}$, the payment $\rho_i(\cdot, u_{-i})$ should be constant on $\mathcal{U}_i(y_i, u_{-i})$. If not, then there exists $u_i', u_i'' \in \mathcal{U}_i(y_i, u_{-i})$ such that $\rho_i(u_i', u_{-i}) > \rho_i(u_i'', u_{-i})$. Then for u_i'', deviating to u_i' is a profitable deviation which violates strategy-proofness. Thus, we set $P_i(y_i, u_{-i})$ to this constant. Thus, the third condition is met. Finally, we show that $\alpha_i(u_N) \in \arg\max\{u_i(y_i', P_i(y_i', u_{-i}))|y_i' \in Y_i(u_{-i})\}$. Suppose not, then there exists $y_i' \in Y_i(u_{-i})$ such that $u_i(y_i, P_i(y_i', u_{-i})) > u_i(y_i, P_i(y_i, u_{-i}))$. Because $y_i' \in Y_i(u_{-i})$, the set $\mathcal{U}_i(y_i', u_{-i}) \neq \emptyset$. Thus, a deviation to $u_i' \in \mathcal{U}_i(y_i', u_{-i})$ will result in an outcome that is strictly better for u_i, again violating strategy-proofness. Therefore, all the three conditions are satisfied.

($\Leftarrow$): Suppose that M is a pricing mechanism and $u_N \in \mathcal{U}_N$. Suppose $\alpha_N(u_N) = y_N$. For any $i \in N$, if bidder i deviates to some reports in $\mathcal{U}_i(y_i, u_{-i})$, then the payment stays the same, therefore it is not a profitable deviation. If he deviates to $U_i(y_i', u_{-i})$ for some other $y_i' \in Y_i(u_{-i})$, then by the definition of the pricing mechanism, we have $\alpha_i(u_N) \in \arg\max\{u_i(y_i', P_i(y_i', u_{-i}))|y_i' \in Y_i(u_{-i})\}$. Thus, this is not a profitable deviation. Thus, the mechanism is strategy-proof. ■

Proof of Lemma 4. The reversed direction should be obvious enough. If the endowment outcome or something dominates it is always available, then the outcome cannot be worse than that. We now show the forward direction. Suppose that M is ex-post IR, and for all $x_i \in X_i(u_{-i})$, $x_i \ngeq x_i^0$. Therefore, x_i^0 is undominated in $X_i(u_{-i})$. Because M is finite, $X_i(u_{-i})$ is finite. Then by Lemma 13, there exists $u_i \in \mathcal{U}_i$ such that $u_i(x_i^0) > u_i(x_i')$ for all $x_i' \in X_i(u_{-i})\backslash\{x_i\}$. Because $X_i(u_{-i})$ is the set of possible outcomes. Then the outcome for u_i is for sure worse then x_i^0. Therefore, the mechanism violates ex-post IR. ■ **Proof of Theorem 11.** Efficiency is implied by the definition of the allocation rule. It suffices for us to just show that the mechanism is strategy-proof and ex-post individual rational. For strategy-proofness, we show that the provided pricing function supports the VCG mechanism.

Let $u_N \in \mathcal{U}_N$, $i \in N$, suppose that $\alpha_N(u_N) = y_N$

$$W(y_N, u_N) \geq W(y_N', u_N) \ \forall y_N' \in \mathcal{F} \tag{C.1}$$

then for all $y'_i \in Y_i(u_{-i})$

$$v_i(y_i) + \max\{\sum_{j \neq i} v_j(y''_j) + w(y''_N) | y''_i = y_i, y''_N \in \mathcal{F}\} \tag{C.2}$$

$$\geq v_i(y'_i) + \max\{\sum_{j \neq i} v_j(y''_j) + w(y''_N) | y''_i = y'_i, y''_N \in \mathcal{F}\} \tag{C.3}$$

$$\Rightarrow v_i(y_i) + V_i(y_i, u_{-i}) \geq v_i(y'_i) + V_i(y'_i, u_{-i}) \tag{C.4}$$

$$\Rightarrow v_i(y_i) + V_i(y_i, u_{-i}) - V_i(y^0_i, u_{-i}) \geq v_i(y'_i) + V_i(y'_i, u_{-i}) - V_i(y^0_i, u_{-i}) \tag{C.5}$$

$$\Rightarrow v_i(y_i) + P_i(y_i, u_{-i}) \geq v_i(y'_i) + P_i(y'_i, u_{-i}) \tag{C.6}$$

Thus, M is supported by the pricing function P_N. Therefore, by Theorem 10, M is strategy-proof.

In Equation C.5, plug $y'_i = y^0_i$, then we have $v_i(\alpha_i(u_N)) + \rho_i(u_N) \geq 0$ for all $u_N \in \mathcal{U}_N$. Therefore, M is ex-post individual rational. ■

Proof of Lemma 6. Let $\bar{U}_i(x_i, u_i) \equiv \{x'_i \in X_i | u_i(x'_i) \geq u_i(x_i)\}$. Since $u_i \geq_{y_i} u'_i$, there exists $x_i = (y_i, p_i) \in X_i$ such that $u_i \geq_{x_i} u'_i$. Let $y'_i \in Y_i$, and $x'_i = (y'_i, v_i(y_i) - v_i(y'_i) + p_i) \in X_i$, then $u_i(x'_i) = v_i(y'_i) + [v_i(y_i) - v_i(y'_i) + p_i] = v_i(y_i) + p_i = u_i(x_i)$. Thus, $x'_i \in \bar{U}_i(x_i, u_i)$. Because $u_i \geq_{x_i} u'_i$, $\bar{U}_i(x_i, u_i) \subseteq \bar{U}_i(x_i, u'_i)$ and $x'_i \in \bar{U}_i(x_i, u'_i)$. Therefore, $v'_i(y'_i) + [v_i(y_i) - v_i(y'_i) + p_i] \geq v'_i(y_i) + p_i$, and finally $v_i(y_i) - v_i(y'_i) \geq v'_i(y_i) - v'_i(y'_i)$. ■

Proof of Theorem 12. Let $y'_N \in \alpha^*_N(y_i, u_{-i})$, $u'_{-i} \in \mathcal{U}_{-i}$ such that $u'_j \geq_{y'_j} u_j$ for all $j \neq i$. Let $y''_N \in \mathcal{F}$ such that $y''_i = y_i$, then we have:

$$\sum_{j \neq i} v_j(y'_j) + w(y'_N) \geq \sum_{j \neq i} v_i(y''_j) + w(y''_N) \tag{C.7}$$

$$\Rightarrow \sum_{j \neq i} [v_j(y'_j) - v_j(y''_j)] + w(y'_N) - w(y''_N) \geq 0 \tag{C.8}$$

$$\Rightarrow \sum_{j \neq i} \Big[[v_j(y'_j) - v_j(y''_j)] - [v'_i(y'_i) - v'_i(y''_i)] + [v'_i(y'_i) - v'_i(y''_i)] \Big] + w(y'_N) - w(y''_N) \geq 0 \tag{C.9}$$

Because $u'_j \geq_{y'_j} u_j$, by Lemma 6, $[v_j(y'_j) - v_j(y''_j)] - [v'_i(y'_i) - v'_i(y''_i)] \leq 0 \; \forall j \neq i$. Thus,

$$\sum_{j \neq i} [v'_i(y'_i) - v'_i(y''_i)] + w(y'_N) - w(y''_N) \geq 0 \tag{C.10}$$

$$\Rightarrow \sum_{j \neq i} v'_j(y'_j) + w(y'_N) \geq \sum_{j \neq i} v'_i(y''_j) + w(y''_N) \tag{C.11}$$

$$\Rightarrow y'_N \in \alpha^*_N(y_i, u'_{-i}) \tag{C.12}$$

Now suppose the pricing decreases. That is, $P_i(y_i, u'_{-i}) < P_i(y_i, u_{-i})$. Suppose

$y_N^1 \in \alpha_N^*(y_i^0, u_{-i})$ and $y_N^2 \in \alpha_N^*(y_i, u'_{-i})$. Then

$$\sum_{j \neq i}[v'_j(y_j) - v'_j(y_j^2)] + w(y_N) - w(y_N^2) \tag{C.13}$$

$$- \Big[\sum_{j \neq i}[v_j(y_j) - v_j(y_j^1)] + w(y_N) - w(y_N^1) \Big] < 0 \tag{C.14}$$

$$\Rightarrow \sum_{j \neq i}[v'_j(y_j) - v'_j(y_j^2) - v_j(y_j) + v_j(y_j^1)] - w(y_N^2) + w(y_N^1) < 0 \tag{C.15}$$

$$\Rightarrow \sum_{j \neq i}[v'_j(y_j) - v'_j(y_j^2) - v_j(y_j) + v_j(y_j^2) - v_j(y_j^2) + v_j(y_j^1)] - w(y_N^2) + w(y_N^1) < 0 \tag{C.16}$$

By Lemma 6, $v'_j(y_j) - v'_j(y_j^2) - v_j(y_j) + v_j(y_j^2) \geq 0 \ \forall j \neq i$

$$\Rightarrow \sum_{j \neq i}[-v_j(y_j^2) + v_j(y_j^1)] - w(y_N^2) + w(y_N^1) < 0 \tag{C.17}$$

$$\Rightarrow \sum_{j \neq i} v_j(y_j^2) + w(y_N^2) > \sum_{j \neq i} v_j(y_i^1) + w(y_N^1) \tag{C.18}$$

The last equation contradicts with that $y_N^1 \in \alpha_N^*(y_i^0, u_{-i})$. Hence, we reject the assumption that the pricing decreases and complete the proof of the first claim. The proof of the second claim uses the same technique as the first one. We leave it for the readers to verify. ■

Proof of Lemma 7. Suppose the claim is not true. That is, there exists $x_i^e \in x_i^*(\tau_i, u'_i) \cap X_i^E$. Then we have $u'_i(x_i^e) \geq u'_i(x_i)$, which implies $u'_i(x_i^e) \in \bar{U}_i(x_i, u'_i) \subseteq \bar{U}_i(x_i, u_i)$. Thus, $u_i(x_i^e) \geq u_i(x_i)$ and $x_i^e \in x_i^*(\tau_i, u_i)$. This contradicts that $u_i \in \mathcal{U}_i(\tau_i)$ and completes the proof. ■

Proof of Lemma 8. Let $\tau_i = (a_{it}, X_{it}^E)$. Let $u_i \in \mathcal{U}_i'$, because S_i is a cautiously optimistic strategy, $x_i^*(\tau, u_i) \cap X_{it}^E = \emptyset$. Also, because the PED mechanism is OSP, by the Single Continuation Property, $x_i^*(\tau, u_i) \subseteq a_{it}$. Therefore, $u_i \in \mathcal{U}_i(\tau_i)$ and therefore $\mathcal{U}_i' \subseteq \mathcal{U}_i(\tau_i)$.

Let $u_i \in \mathcal{U}_i(\tau_i)$. Let I_{it} denote the information set at period t. Then by the descending property of the PED mechanism, for all $x_i \in a_{it}$ and $a_{it'} \in \psi_i(I_i)$, there exists $x_i' \in a_{it'}$ such that $x_i' \geq x_i$. Also, by the strong Persistent of Exit property, $X_{it'} \subseteq X_{it}$ for all $I_{it'} \in \psi_i(I_{it})$. Therefore, the most preferred outcome at $I_{it'}$ should also be in $a_{it'}$. Thus, the cautiously optimistic strategy for u_i should choose $a_{it'}$ at all information set $I_{it'}$. Therefore, u_i reaches I_{it} and thus $u_i \in \mathcal{U}_i'$. Therefore, $\mathcal{U}_i(\tau_i) \subseteq \mathcal{U}_i'$ and the two sets are equal. ∎

Proof of Lemma 9. Suppose the claim is not true. Then there exists $u_i \in \mathcal{U}_i(\tau)$ such that $u_i \notin \mathcal{U}_i(\hat{\tau})$. Then there exists $x_i^e \in x_i^*(\hat{\tau}_i, u_i) \cap \hat{X}_i^E$. Also, because $u_i \in \mathcal{U}_i(\tau_i)$, there exists $x_i^c \in x_i^*(\tau, u_i) \cap X_i^C$ such that $u_i(x_i^c) > u_i(x_i^e)$. Then there exists $\hat{x}_i^c \in \hat{X}_i^C$ such that $\hat{x}_i^c \geq x_i^c$. Therefore, $u_i(\hat{x}_i^c) \geq u_i(x_i^c) > u_i(x_i^e)$. This contradicts that $u_i(x_i^e) \in x_i^*(\hat{\tau}_i, u_i)$ and completes the proof. ∎

Proof of Theorem 13. We first construct $\bar{X}_i(u_N)$ for all $i \in N$ and $U_N(x_N, u_N)$ for all $x_i \in \bar{X}_i(u_N)$.

For $i \in N$, let $Y_i^M = \alpha_i(\mathcal{U}_N)$ and for each $y_i \in Y_i^M$, define $\bar{p}_i(y_i) = \sup\{\rho_i(u_N) | u_N \in \mathcal{U}_N, \alpha_i(u_N) = y_i\}$. Let $X_i^0 = \{(y_i, \bar{p}(y_i)) | y_i \in Y_i^M\}$. Let $\tau_i^0 = (X_i^0, \{x_i^0\})$. Thus, X_i^0 is the set of best possible outcomes that i could achieve in mechanism M. τ_i^0 defines a initial threshold of bidder i.

Because M is PED-implementable, let $u_N \in \mathcal{U}_N$, then there exists cautiously optimistic strategy S_i for u_i for $i \in N$. With S_N, the path of play in D is fixed. Now suppose all bidders play according to S_N. Let $A(t)$ be the set of active bidders at each period t.

Let T_i be the last period i is called to action, which is also the period when he chooses to exit. Let $\bar{X}_i(u_N) = X^E_{iT_i}$. Given $i \in N$ and $x_i \in \bar{X}_i(u_N)$, let $t(x_i)$ be the time period when x_i is first added to E_{it}.

For all $j \in A(t(x_i))$, if j has never been called to action before $t(x_i)$, set $\tau_j(x_i) = \tau_j^0$; else, set t_j to be the last period that bidder $j \neq i$ is called to action before $t(x_i)$. Because j is active, S_j must have chosen a continuation action $a^c_j \in X^C_{jt_j}$ at t_j. Let $\tau_j(x_i) = (a^c_j, X^E_{jt_j})$.

Let $\mathcal{U}_N(x_N, u_N) = \mathcal{U}_N(u_N, (\tau_j(x_i))_{j \in A(t(x_i))})$. By construction, $\mathcal{U}_N(x_N, u_N)$ is a threshold space. Now we show the claimed results.

1. $\chi_i(u_N) \in \bar{X}_i(u_N)$: Because i chooses to exit at T_i, then $\chi_i(u_N) \in X^E_{iT_i} = \bar{X}_i(u_N)$.

2. We order $\cup_{i \in N} \bar{X}_i(u_N)$ by the time they are first added to the exit outcome. That is, $t(x_{(1)}) \leq t(x_{(2)}) \leq \cdots \leq t(x_{(k)})$. The evolution rule of the PED mechanism guarantees that if $t(x) \geq t(x')$, then $A(t(x)) \subseteq A(t(x'))$, and for all $j \in A(t(x))$, $\tau_j(x) \leq \tau_j(x')$. Therefore, $\mathcal{U}_N(u_N, x) \subseteq \mathcal{U}_N(u_N, x')$.

3. By Lemma 8, the set $\mathcal{U}_j(\tau_j(x_i))$ is the set of u_j that are still active when $x_i = (y_i, p_i)$ becomes an exit outcome. Because once x_i becomes the exit outcome, it will be persistent throughout the rest of the mechanism, therefore, any report in $\mathcal{U}_j(\tau_j(x_i))$ will not affect the pricing of y_i. Thus, the product

threshold space $\mathcal{U}_N(u_N, (\tau_j(x_i))_{j\in A(t(x_i))})$ is a constant pricing space for x_i.

4. For condition 3a: because $X^E_{jt_j}$ is the set of exit outcomes at period t_j, therefore, for each $x_j \in X^E_{jt_j}$, there exists $t(x_j) \leq t_j < t(x_i)$ when it becomes the exit outcome. That implies that $x_j \in \bar{X}_j(u_N)$ and x_j precedes x_i in the hierarchy. Conversely, if x_j precedes x_i in the hierarchy, then it means that in the last period t_j when j is called to play before $t(x_i)$, $x_j \in X^E_{jt_j}$. Therefore, $x_j \in X^E_j$.

5. For condition 3b: if $x_j \in X_j(u_{-j})\backslash X^E_j$, than it means x_j has not been announced as an exit outcome at t_j. Therefore, all the u_j that will eventually be assigned x_j will have to choose to continue at t_j. Thus, the continuation action u_j choses should contain some $x'_j \geq x_j$. Therefore, $x'_j \in a^c_j = X^C_j$.

6. For condition 3c, if $j \notin A(t(x_i))$, then it means that j has chosen some exit outcome. The outcome he chooses should be $x_j = \chi_j(u_N)$. Then x_j must have become an exit outcome before $t(x_i)$. Therefore, there exists a constant pricing space that precedes x_i in the hierarchy.

■

Proof of Theorem 14. Suppose that y_i has winner influence over y_j for all $j \in S$, $j \neq i$. As we did in the proof of Theorem 13, let $t(x_i)$ be the period when x_i becomes the exit outcome, and define $\tau_j(x_i)$ as the announcements for j chooses to continue in the last period before $t(x_i)$.

Let $u_N \in \mathcal{U}_N(y_S, u_{N\backslash S})$. Then for all $j \in S$, we have $\alpha_j(u_N) = y_j$. Because for all period $t' < t$, x_i is not available to exit, therefore, for all $u'_i \in \mathcal{U}_i(y_i, u_{-i})$, the cautiously optimistic strategy has to continue and therefore must be the same. That implies that if y_j becomes an exit allocation before $t(x_i)$, then it is not possible for the pricing of y_j to be influenced by the reports in $\mathcal{U}_i(y_i, u_{-i})$. Hence, y_j is a continuation action in the last period before $t(x_i)$. Therefore, $u_j \in \mathcal{U}_j(\tau_j(x_i))$. Therefore, $u_N \in \mathcal{U}_N(u_N, x_i)$ and $\mathcal{U}_S(y_S, u_{N\setminus S}) \subseteq \mathcal{U}_N(u_N, x_i)$.

■

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