ABSTRACT

Title of dissertation: Generalized Frame Potential and Problems Related to SIC-POVMs

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Frame theory generalizes the idea of bases in Hilbert space, and the frame potential is an important tool when studying frame theory. In this thesis, we first explore the minimization problem of a generalized definition of frame potential, namely the p-frame potential, and show there exists a universal optimizer under certain conditions by applying a method involving ultraspherical polynomials and spherical designs.

Next, we further discuss the topic on Grassmannian frames, which are special cases of minimizers of p-frame potentials. We present the construction of equiangular lines in lower dimensions since numerical result showed their connections with Grassmannian frames. We also derive properties of the $(6, 4)$-Grassmannian frame.

Then, we obtain lower bounds for the generalized frame potentials in the complex setting. The frame potentials may provide a different approach to determine the existence of Gabor frames that are equiangular. This relates the potential minimization problem to the unsolved Zauner conjecture. In addition, we study the

properties of Gramian matrices of Gabor frames in an attempt to search for Gabor frames with a small number of different inner products. We also calculate the number of different inner products in Gabor frames generated by Alltop sequences and Björck sequences.

In addition, we also present examples related to a generalized support uncertainty inequality and shift-invariant spaces on LCA groups.

Generalized Frame Potential and Problems Related to SIC-POVMs

by

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Chapter 1: Background and Introduction

1.1 Definitions and Notation

Definition 1.1.1. A finite frame for a Hilbert space $\mathbb{H}^d$ is a set of vectors $\{x_i\}_{i=1}^N \subset \mathbb{H}^d$ such that for any $x \in \mathbb{H}^d$,

$$A\|x\|^2 \leq \sum_{i=1}^{N} |\langle x, x_i \rangle|^2 \leq B\|x\|^2$$

for $0 < A \leq B < \infty$.

If, in addition, each x_k is unit-norm, we say that X is a *unit-norm frame.* X is called *tight* if $A = B$. A tight unit-norm frame is called a *finite unit-norm tight frame (FUNTF).* One attractive feature of FUNTFs is the fact that they can be used to decompose and reconstruct any vector x via the following formula:

$$x = \frac{d}{N} \sum_{k=1}^{N} \langle x, x_k \rangle x_k. \tag{1.1}$$

A set of lines in Euclidean space is called *equiangular* if the angles between each pair of lines are the same. A frame X is said to be *equiangular* if there exists

$c > 0$ such that

$$\left|\left\langle \frac{x_k}{\|x_k\|}, \frac{x_l}{\|x_l\|} \right\rangle\right| = c \quad \text{for all } k \neq l.$$

If in addition X is tight, then X is called an *equiangular tight frame (ETF)*. It follows from [14, Proposition 1.2] that the vectors of an ETF have necessarily equal norm. Consequently, and without loss of generality, all ETFs considered in the sequel will be unit-norm frames, i.e., FUNTFs.

Denote $S(N, d)$ to be the collection of all sets of N unit norm vectors in $\mathbb{R}^d$. For $p \in [0, \infty]$, we can define the p-frame potential of a finite unit norm frame.

Definition 1.1.2. Let $X = \{x_i\}_{i=1}^N \subset \mathbb{H}^d$ be a finite unit norm frame. Then its *p-frame potential* is defined as

$$\mathrm{FP}_{p,N,d}(X) := \begin{cases} \displaystyle\sum_{k=1}^{N} \sum_{l \neq k}^{N} |\langle x_k, x_\ell \rangle|^p, & \text{when } p < \infty \\ \displaystyle\max_{k \neq \ell} |\langle x_k, x_\ell \rangle|, & \text{when } p = \infty. \end{cases} \tag{1.2}$$

Definition 1.1.3.

$$\mathcal{F}_{p,N,d} = \inf_{X \in S(N,d)} \mathrm{FP}_{p,N,d}(X).$$

We say X is an optimal configuration of $FP_{p,N,d}$ if

$$\mathrm{FP}_{p,N,d}(X) = \mathcal{F}_{p,N,d}.$$

Definition 1.1.4. Let $\mathbb{H} \in \{\mathbb{R}, \mathbb{C}\}$. A unit sphere in $\mathbb{H}^{d-1}$ is the set of points of distance 1 from a fixed point, i.e.,

$$\mathbb{S}^{d-1} = \{x \in \mathbb{H}^d \mid \|x\| = 1\}.$$

Definition 1.1.5. A set of unit norm vectors $X = \{v_i\}_{i=1}^N \subset \mathbb{S}^{d-1}$ is called a (N,d)-Grassmannian frame if

$$\mathcal{M}_\infty(X) = \inf_{U \in S(N,d)} \mathrm{FP}_{\infty,N,d}(U). \tag{1.3}$$

Definition 1.1.6. A *spherical t-design* is a finite subset X of the sphere such that every polynomial on $\mathbb{R}^d$ of total degree at most t has the same average over the subset as over the entire sphere. i.e.,

$$\int_{S^{d-1}} p(t)dt = \frac{1}{|X|} \sum_{t \in X} p(t),$$

where p is any polynomial of degree at most t.

SIC-POVM, short for symmetric informationally complete positive operator-valued measure, is an important object in quantum measurement theory. The definition of POVM is as follow [17, II.1.2]:

Definition 1.1.7. Let Ω be a nonempty set and $\mathcal{F}$ a σ-algebra of subsets of Ω so that $(\Omega, \mathcal{F})$ is a measurable space. Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{L}(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. A POVM $E : \mathcal{F} \to \mathcal{L}(\mathcal{H})$ on $(\Omega, \mathcal{F})$ is defined by the following properties:

1. $E(X) \geq 0$ for all $X \in \mathcal{F}$;

2. $E(\Omega) = I$;

3. $E(\cup X_i) = \sum E(X_i)$ for all disjoint sequences $\{X_i\} \subset \mathcal{F}$,

where the series converges in the weak operator topology of $\mathcal{L}(\mathcal{H})$.

Definition 1.1.8. A *Gabor frame* for $\mathbb{C}^d$ with $g \in \mathbb{C}^d$ is the set of all vectors of the form $M^{\alpha k} T^{\beta l} g$ where $(k, l) \in (\mathbb{Z}/N\mathbb{Z})^2$, $\alpha, \beta > 0$. The operators M and T are defined as

$$T = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & \cdots & & 0 & 1 \end{bmatrix} \qquad M = \begin{bmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{d-1} \end{bmatrix},$$

where $\omega = e^{2\pi i/d}$.

1.2 History and Background

Frames are extensions of bases. In finite dimensional Hilbert spaces $\mathbb{H}^d$, frames are precisely spanning sets and can be used to reconstruct any vector in the space. Frames, and FUNTFs in particular, have significant applications in image processing [19], speech processing and $\Sigma\Delta$ quantization. More detailed discussions on finite frames can be found in [22, 49, 67].

The optimization of frame potential can be considered as an energy optimization problem, where the *energy* of a set $X = \{x_i\}_{i=1}^N \subset \mathbb{S}^{d-1} \subset \mathbb{R}^d$ with respect to a function F is

$$E_F(X) = \sum_{i \neq j} F(\langle x_i, x_j \rangle). \tag{1.4}$$

The frame potential of a frame X is then $\mathrm{FP}_{2,N,d}(X) = E_{|t|^2}(X)$. There are other problems posed in this category. For example, part of the "Distribution of Points on the 2-Sphere" problem [57] is to optimize the energy when $F(t) = \frac{1}{(2-2t)^p}$ for $0 < p < 2$, $d = 3$. When $F(t) = \arccos(|t|)$, it becomes the Fejes-Tóth problem and remains open for $d \geq 2$, see [12].

Another function to consider is $F(t) = |t|^p$, which makes a ℓ^p-type norm comparing it to $\mathrm{FP}_{2,N,d}$ as ℓ^2-type norm. This leads to one of our main focus, the *p-frame potential optimization problem*. Our goal is to find $\mathcal{F}_{p,N,d}$ and the corresponding optimal configuration. The 2-frame potential and maximum correlation defined in 1.2 are special cases of the p-frame potential.

If equiangular tight frames exist for a given pair $\{N, d\}$, they minimize $\mathrm{FP}_{p,N,d}$ for all $p \in [2, \infty)$(e.g. [32]). It is then natural to ask whether universal optimizers exist for any fixed $\{N, d\}$.

Problem 1.2.1. *For what pair $\{N, d\}$, can we find a universal sequence $U = \{u_i\}_{i=1}^N \subset \mathbb{S}^{d-1}$ such that $\mathrm{FP}_{p,N,d}(U) = \mu_{p,d,N} \equiv \min_X \mathrm{FP}_{p,N,d}(X)$ for all $p \in [p_{N,d}, \infty]$, where $p_{N,d}$ depends on N, d?*

In fact, the optimal configuration of energy function and existence of universal optimizer is a fundamental problem in extremal geometry. In [26], Cohn and Kumar used linear programming method and characterized the optimal configurations of the f-energy $\sum_{i\neq j} f(|x_i - x_j|^2)$ for a class of functions in terms of spherical t-designs.

When $p = 2$, the minimizers of $\mathrm{FP}_{p,N,d}$ are called the tight frames [8]. The search for minimizers of $\mathrm{FP}_{\infty,N,d}$ (or Grassmannian frames) is a part of the packing problem in Grassmannian spaces. (N, d)-Grassmannian frames are also known as the optimal N packings in RP^{d-1}.

The packing problem in Grassmannian spaces is described by Conway, Hardin and Sloane [28]. The Grassmannian space $G(d, m)$ is a set of all m-dimensional subspaces of the real Euclidean d-dimensional space $\mathbb{R}^d$. The packing problem is given N, d, m, find a set of n dimensional planes $\{P_i\}_{i=1}^N \subset G(d, m)$ such that $\min_{i\neq j} dist(P_i, P_j)$ is as large as possible. Possible distances that can be used include geodesic and chordal distance. When $m = 1$, let v_i be a unit vector on P_i, we can define

$$dist(v_i, v_j) = \arccos |\langle v_i, v_j\rangle|.$$

We can now see that the problem of constructing the (N, d)-Grassmannian frames is equivalent to the Grassmannian packing problem when $m = 1$. Papers [28, 31] give an extensive set of numerical results. These papers also compared the minimizer of maximum correlation and the known "equiangular lines" in [46]. When equiangular tight frames for certain (N, d) exist, they are the (N, d)-Grassmannian frames. When an ETF does not exist, there are no general methods to construct a Grassmannian frame. It is proved that there exist Grassmannian frames that are equiangular but not tight, e.g. $(5, 3)$-Grassmannian frames [10]. Some of the earlier works [39, 46, 65] set the foundation of constructing equiangular lines. In [68], Welch gave a lower bound of ∞-frame potential, which can be achieved on some equiangular frames.

We can also ask what is the optimal configuration of p-frame potential when $\mathbb{H} = \mathbb{C}$. It is proved in [32], when equiangular tight frames exist, they are the optimizers of the p-frame potentials. Then we face the same question as in $\mathbb{R}$, that is the existence of the equiangular tight frames. For $N = d^2$, this problem links to a topic in quantum physics, that is the existence of SIC-POVM.

In mathematics, a POVM on a finite nonempty set Ω corresponds to a finite tight frame $\{\phi_k\}_{k=1}^n \subset \mathbb{C}^d$. [51]. Suppose $\Omega = \{x_1, x_2, ..., x_n\}$. Let $E(x_k) = \frac{d}{n}\phi_k \otimes \phi_k$ and define E on other sets in $\mathcal{F}$ by the 3rd condition in Definition 1.1.7. E satisfies 1st condition by definition, also the 2nd condition since $\sum_k E(x_k) = \frac{d}{n}S = \frac{d}{n} \cdot \frac{n}{d} I = I$ where S is the frame operator of $\{\phi_k\}_{k=1}^n$.

A POVM on a finite set is said to be informationally complete if for a tight frame $\{\phi_k\}_{k=1}^n$, the operators $\Pi_k = \phi_k \otimes \phi_k$ span the space of operators from $\mathbb{C}^{d^2}$ to $\mathbb{C}^{d^2}$ [51]. So if $\{\phi_k\}_{k=1}^n$ is informationally complete, n is at least d^2. Symmetric means all the inner products between pairs of the vectors in the tight frame are equal, that is, the frame is equiangular. The upper bound of cardinality of equiangular tight frames in $\mathbb{C}^d$ is d^2. So a SIC-POVM is equivalent to an equiangular tight frame with d^2 vectors in $\mathbb{C}^d$.

Then the natural question is whether SIC-POVMs exist in every dimension d. Zauner proposed a conjecture in his thesis.

Conjecture 1.2.2. *[71] In any $\mathbb{C}^d$,$d > 2$, there exist SIC-POVMs generated by* *a single unit norm vector g under the orbit of Heisenberg group. For any $k, l \in \mathbb{Z}/d\mathbb{Z} \backslash \{(0,0)\}$, $|\langle g, M^k T^l g\rangle| = \frac{1}{\sqrt{d+1}}$, where M, T are the modulation and translation operators.*

This conjecture remains open. Several papers have considered searching for the generating vector (or sometimes called a fiducial vector) and got results in dimension 2-16, 19, 24, 28, 35, 48, 124, 323 (see [3, 37, 71]) and numerical result in dimensions up to 67. Many known SIC-POVMs are Gabor frames. Finding vectors that generate SIC-POVMs proved difficult.

1.3 Results

Chapter 2 is joint work with Xuemei Chen, Victor Gonzalez, Eric Goodman and Kasso Okoudjou [24], and mainly focuses on the p-frame potential in $\mathbb{R}^d$. We first show some basic properties of the minimal p-frame potential in Section 2.2. We then present the minimum of p-frames potential $\mathcal{F}_{p,N,2}$ for large p, and the corresponding optimal configuration in Theorem 2.3.1. Finally, we give a partial result on optimal configuration for $\mathcal{F}_{p,d+1,d}$. The complete characterzation for $\mathcal{F}_{p,d+1,d}$ is proved independently in [69].

Chapter 3 is devoted for the construction of equiangular lines and deriving properties of Grassmannian frames. We show the construction for the Gramian matrices of equiangular lines in lower dimensions, which was first done in [39], and fill in more details. The method utilizes graph theory. We prove a necessary condition for being $(6, 4)$-Grassmannian frames in Lemma 3.4.4. The full construction for

$(6,4)$-Grassmannian frames is given in [48].

In Chapter 4 we explore the connection between p-frame potential in $\mathbb{C}^d$ and problems related to Zauner's conjecture. We first show properties of p-frame potential for Gabor frames. We prove a property of the Gramian matrices of Gabor frames in Proposition 4.2.3 and calculate its spectrum in Corollary 4.2.6. We also discuss the possibility of connecting the optimal configuration of p-potential with the (t,t)-design. Lastly, we consider the question of whether it is possible to find frames with a small number of different inner products, and examine the inner products of Björck sequences and Alltop sequences.

In Chapter 5 and Chapter 6, we discuss two different topics and provide more detailed proof for several previously known results.

In Chapter 5 we discuss the refined version of classic Support Uncertainty Inequality, which is derived in [52], and give an example of the mutually unbiased bases.

In Chapter 6 we characterize the shift invariant spaces in a locally compact abelian group. We then provide a method to determine whether a set that consist of translation of a vector is a frame on its closed span. The results in this chapter are in [18].

Chapter 2: Optimal Configuration of p-Frame Potential on $\mathbb{R}^d$

2.1 Introduction

Let $S(N,d)$ be the collection of all sets of N unit-norm vectors. We are interested in finding the infimum of the p-frame potential among all N-point configurations in $S(N,d)$. It is a standard argument to show that this infimum can be achieved due to the compactness of the sphere and the continuity of the function, so we can replace infimum by minimum and define

$$\mathcal{F}_{p,N,d} := \min_{X \in S(N,d)} \mathrm{FP}_{p,N,d}(X). \tag{2.1}$$

Any minimizer of (2.1) will be called an *optimal configuration* of the p-frame potential. We observe that if $X^* = \{x_1^*, \cdots, x_N^*\}$ is optimal, then with any orthogonal matrix U, any permutation π, and any $s_i \in \{1, -1\}$,

$$\{s_1 U x_{\pi_1}^*, \cdots, s_N U x_{\pi_N}^*\}$$

is optimal too. In other words, the optimal configuration is an equivalence class with respect to orthogonal transformations, permutations or sign switches. So when we say an optimal configuration is unique, we mean that it is unique up to this equivalence relation.

Note that in the definition of the frame potential, X does not necessarily need to be a frame of $\mathbb{R}^d$, but we will show in Proposition 2.2.1 that the minimizers of the p-frame potential must be a frame, as expected. Therefore problem (2.1) remains the same if we had restricted X to be a unit-norm frame with N frame vectors.

The name "frame potential" originates from the special case $p = 2$,

$$\mathrm{FP}_{2,N,d}(X) = \sum_{k=1}^{N} \sum_{l \neq k}^{N} |\langle x_k, x_l \rangle|^2 \tag{2.2}$$

which was studied by Benedetto and Fickus [8]. They proved that X^* is an optimal configuration of $\mathrm{FP}_{2,N,d}(X)$ if and only if $X^* = \{x_k^*\}_{k=1}^N$ is a FUNTF.

Another important special case is $p = \infty$. In this case, the quantity

$$c(X) := \mathrm{FP}_{\infty,N,d}(X) = \max_{k \neq \ell} |\langle x_k, x_\ell \rangle| \tag{2.3}$$

is also called *coherence* of $X = \{x_k\}_{k=1}^N \in S(N,d)$, and its minimizers are called *Grassmannian frames* [9, 16, 58, 68]. The following Welch bound [68] is well known:

$$\mathrm{FP}_{\infty,N,d}(X) \geq \sqrt{\frac{N-d}{d(N-1)}}, \tag{2.4}$$

and the equality in (2.4) holds if and only if $X = \{x_k\}_{k=1}^N$ is an ETF, which is only possible if $N \leq \dfrac{d(d+1)}{2}$. The coherence minimization problem corresponds to $p = \infty$ because it is the limiting case when p grows to infinity; see Proposition

2.2.2. It is known that ETFs, when exist, are minimizers of (1.2) for $p > 2$ [13, 32].

When p is an even integer, the minimizers of $\mathrm{FP}_{p,N,d}$ have long been investigated in the setting of spherical designs, see [32, 55, 66]. A set of N points $X \subset \mathbb{S}^{d-1}$ (the unit sphere in $\mathbb{R}^d$) is called a spherical t-design if for every homogeneous polynomial h of degree t or less,

$$\int_{\mathbb{S}^{d-1}} h(\xi) d\sigma(\xi) = \frac{1}{N} \sum_{x \in X} h(x),$$

where σ is the normalized surface measure on $\mathbb{S}^{d-1}$. For example, a spherical 1-design is a set of points whose center of mass is at the origin. More generally, as shown in [56, Corollary 1], if p is an even integer and $X \in S(N,d)$ is symmetric, that is $X = -X$, then

$$\mathrm{FP}_{p,N,d}(X) \geq N^2 \frac{1 \cdot 3 \cdot 5 \cdots (p-1)}{d(d+2) \cdots (d+p-2)} - N, \tag{2.5}$$

and equality holds if and only if X is a spherical p-design.

Optimal configurations of (2.1) are often not symmetric since x_i and $-x_i$ are considered the same points as far as frame potential is concerned. However, we can still use (2.5) by symmetrizing a frame. Given $X = \{x_i\}_{i=1}^N$ such that its coherence $c(X) < 1$ (i.e. no repeated vectors or opposite vectors), we let

$$X^{sym} := \{x_i\}_{i=1}^N \cup \{-x_i\}_{i=1}^N \in S(2N, d).$$

Some straightforward computations result in

$$\mathrm{FP}_{p,2N,d}(X^{sym}) = 4\,\mathrm{FP}_{p,N,d}(X) + 2N \tag{2.6}$$

which combined with (2.5), can be used to prove

Proposition 2.1.1. *Let p be an even integer, then*

$$\mathrm{FP}_{p,N,d}(X) = \frac{1}{4}(\mathrm{FP}_{p,2N,d}(X^{sym}) - 2N) \geq N^2 \frac{1 \cdot 3 \cdot 5 \cdots (p-1)}{d(d+2)\cdots(d+p-2)} - N,$$

and equality holds if and only if X^{sym} is a spherical p-design.

Not only is Proposition 2.1.1 limited to even p's, but it is also not trivial to find spherical t-designs for large t. More generally, and to the best of our knowledge, little is known about the complete solutions to (2.1) even in the simplest case $d = 2$. When $N = 3$, a solution is given in [32] for all positive p. See also [12,50] for related results. For any N and $p = \infty$, it is shown in [10] that the Grassmannian frame is

$$X_N^{(h)} = \left\{ \begin{bmatrix} \cos 0 \\ \\ \sin 0 \end{bmatrix}, \begin{bmatrix} \cos \frac{\pi}{N} \\ \\ \sin \frac{\pi}{N} \end{bmatrix}, \begin{bmatrix} \cos \frac{2\pi}{N} \\ \\ \sin \frac{2\pi}{N} \end{bmatrix}, \cdots, \begin{bmatrix} \cos \frac{(N-1)\pi}{N} \\ \\ \sin \frac{(N-1)\pi}{N} \end{bmatrix} \right\}, \tag{2.7}$$

which can be viewed as N equally spaced points on the half circle. The main result of this chapter establishes that the unique optimal configuration when $d = 2$, $N \geq 4$, and $p > 4\lfloor \frac{N}{2} \rfloor - 2$ is $X_N^{(h)}$, where $\lfloor c \rfloor$ is the largest integer that does not exceed c. Moreover for $N = 4$, our result is sharper as we prove this is the case for $p > 2$. Such a result is expected since optimal configurations for large p are approaching the Grassmannian frame. Moreover, we are able to show that $X_N^{(h)}$ is the optimal configuration for a large class of kernel functions. See Theorem 2.3.11. The phenomenon that a given configuration is the optimal configuration for a large range of functions is what we call universal. Such a name stems from the work [26]. In addition to these results, we present numerical results for all other values of p and N when $d = 2$. Finally, we also consider the special case of $N = d + 1$ and $d \geq 3$ and state a conjecture regarding the function $\mathcal{F}_{p,N,d}$ for $p \in (0, 2]$. Based on the results of the present paper, Table 2.1 gives the state of affairs concerning the solutions of (2.1) and is an invitation to initiate a broader discussion on the problem. We would like to remark that the case $N = d + 1$ has been solved during the revision of this manuscript; see Section 2.4 for more details.

The rest of this chapter is organized as follows. Section 2.2 states some basic results of the p-frame potential including some asymptotic results as $N \to \infty$. Section 2.3 presents the results for $d = 2$. Section 2.4 presents conjectures (now proved in [69]) and partial results for the case $N = d + 1$. Section 2.5 raises some questions that we would like to answer in the future. We will use $[m : n]$ for the index set $\{m, m + 1, \cdots, n\}$.

	$\mathbb{R}^2$	$\mathbb{R}^d$
$p \in (0, \frac{\ln 3}{\ln 2})$	$N = 3$: ONB+ [32]	$N = d + 1$: ONB+ [35]
$p \in (\frac{\ln 3}{\ln 2}, 2)$	$N = 3$: ETF [32]	$N = d + 1$: see Conjecture 2.4.5
$p \in (0, 2)$	$N = 2k$: k copies of ONB [32]	$N = kd$: k copies of ONB [32]
$p = 2$	FUNTF [8]	
$p \in (4\lfloor \frac{N}{2} \rfloor - 2, \infty)$	$N \geq 5 : X_N^{(h)}$ (Theorem 2.3.1)	ETF if exists [32, 58]
$p = \infty$	Any N: $X_N^{(h)}$ [10]	
$p \in (2, \infty)$	$N = 4$: $X_4^{(h)}$ (Theorem 2.3.1)	
$p \in (n - 4, n - 2)$	$N = kn$, $n \geq 6$ even: k copies of n-gon (Corollary 2.2.12)	

ONB+ refers to an orthonormal basis with a repeated vector. See Definition 2.4.1(a).

Table 2.1: Optimal configurations for the p-frame potential

2.2 Some basic results

Intuitively, minimizing the frame potential amounts to promoting large angles among vectors. Consequently, it is expected that optimal configurations will not be subsets of lower dimension subspaces. If X is a subset of a lower dimension subspace, then one can always find a vector e that is orthogonal to X, and replacing any vector in X by e won't increase the frame potential. In other words, it is trivial to show that problem (2.1) might as well be restricted to frames. The following result shows something stronger, that is, it excludes the possibility that a minimizer doesn't span $\mathbb{R}^d$.

Proposition 2.2.1. *For $p \in (0, \infty]$, any optimal configuration of (2.1) is a frame of $\mathbb{R}^d$.*

Proof. We first consider the case $p \in (0, \infty)$. Suppose not, and say $X^* = \{x_k^*\}_{k=1}^N \subset$

S^{d-1} is a minimizer so that $\operatorname{span} X^*$ is a strict subset of $\mathbb{R}^d$. Because there are $N \geq d$ vectors, it is possible to select two indices k_1 and k_2 such that $|\langle x_{k_1}^*, x_{k_2}^* \rangle| > 0$. Finally, select any unit-norm vector $x_0 \in (\operatorname{span} X^*)^{\perp}$ and replace $x_{k_1}^*$ with x_0; i.e., define $Y = \{x_k^*\}_{k \neq k_1} \cup \{x_0\}$. A direct computation shows that $\mathrm{FP}_{p,M,N}(Y) < \mathrm{FP}_{p,M,N}(X^*)$.

Now consider the case $p = \infty$ and let $X^* = \{x_k^*\}_{k=1}^N \subset S^{d-1}$ be a minimizer of $\mathrm{FP}_{\infty,N,d}$. Suppose that the dimension of $\operatorname{span}(X^*) \leq d-1$. Choose a unit vector $e \in (\operatorname{span} X^*)^{\perp}$. There could be multiple pairs of vectors that achieve the maximal inner product $\mathcal{F}_\infty = \mathrm{FP}_{\infty,N,d}(X^*)$. Without loss of generality, we assume these vectors are among the first K vectors, that is,

$$|\langle x_i^*, x_j^* \rangle| < \mathcal{F}_\infty, \qquad \text{if either } i \text{ or } j \text{ does not belong to } [1:K], i \neq j. \tag{2.8}$$

We will construct $Y = \{y_k\}_{k=1}^K \cup \{x_k^*\}_{k=K+1}^N$ that will have smaller coherence.

For $i = 1, 2, \cdots, K$, let $y_i = \sqrt{1-\epsilon_i}\, x_i^* + \sqrt{\epsilon_i}\, e$, where $0 < \epsilon_i < 1$. Define

$$f(a,b) := \frac{\sqrt{a}\sqrt{b}}{1 - \sqrt{1-a}\sqrt{1-b}} \text{ on } (0,1] \times (0,1].$$

If we choose ϵ_i, ϵ_j such that

$$f(\epsilon_i, \epsilon_j) = \frac{\sqrt{\epsilon_i}\sqrt{\epsilon_j}}{1 - \sqrt{1-\epsilon_i}\sqrt{1-\epsilon_j}} < \mathcal{F}_\infty, \tag{2.9}$$

then

$$|\langle y_i, y_j\rangle| = |\sqrt{1-\epsilon_i}\sqrt{1-\epsilon_j}\langle x_i^*, x_j^*\rangle + \sqrt{\epsilon_i}\sqrt{\epsilon_j}| \leq \sqrt{1-\epsilon_i}\sqrt{1-\epsilon_j}\mathcal{F}_\infty + \sqrt{\epsilon_i}\sqrt{\epsilon_j} < \mathcal{F}_\infty. \tag{2.10}$$

We will pick ϵ_i iteratively to satisfy (2.9):

Step 1: pick $0 < \epsilon_1 < 1$ arbitrarily.

Step i: given $\epsilon_1, \cdots, \epsilon_{i-1}$, pick $\epsilon_i > 0$ such that $f(\epsilon_j, \epsilon_i) < \mathcal{F}_\infty$, for all $j = 1, \cdots, i-1$. This is possible because $\lim_{\epsilon\to 0} f(\epsilon_j, \epsilon) = 0$ for all $j \leq i-1$.

For convenience, let $y_k = x_k^*$ for $k = K+1, \cdots, N$. The new frame $Y = \{y_k\}_{k=1}^K$ has a smaller coherence because for any pair i, j, if $i, j \in [1 : K]$, then $|\langle y_i, y_j\rangle| < \mathcal{F}_\infty$ by (2.10); if $i, j \in [K+1 : N]$, then $|\langle y_i, y_j\rangle| = |\langle x_i^*, x_j^*\rangle| < \mathcal{F}_\infty$ by (2.8); if $i \in [1 : K], j \in [K+1 : N]$, then $|\langle y_i, y_j\rangle| = |\langle \sqrt{1-\epsilon_i}x_i^* + \sqrt{\epsilon_i}e, x_j^*\rangle| = \sqrt{1-\epsilon_i}|\langle x_i^*, x_j^*\rangle| < \mathcal{F}_\infty$.

This is a contradiction, so the optimal configuration must be a frame. □

Now we establish the relationship between large p and $p = \infty$. We denote $X^{(p)}$ an optimal configuration for (2.1) when $p < \infty$.

Proposition 2.2.2. $\lim_{p\to\infty} \mathcal{F}_{p,N,d}^{1/p} = \mathcal{F}_{\infty,N,d}$. *Moreover, if X is a cluster point of the set $\{X^{(p)}\}_{p>0}$, then X optimizes the coherence as* $X = \arg\min_{Y\in S(N,d)} c(Y)$.

Proof. On one hand, we have

$$\mathcal{F}_{p,N,d}^{1/p} = \left(\sum_{i\neq j} |\langle x_i^{(p)}, x_j^{(p)}\rangle|^p\right)^{1/p} \geq c(X^{(p)}) \geq \mathcal{F}_{\infty,N,d}. \tag{2.11}$$

On the other hand,

$$\mathcal{F}_{p,N,d}^{1/p} \leq \left(\sum_{i\neq j} |\langle x_i^{(\infty)}, x_j^{(\infty)}\rangle|^p\right)^{1/p} \leq \left(\sum_{i\neq j} \mathcal{F}_{\infty,N,d}^p\right)^{1/p} = \mathcal{F}_{\infty,N,d}[N(N-1)]^{1/p}. \tag{2.12}$$

Taking the limit of both inequalities gives us the desired limit.

For the second part of the proposition, let $X = \lim_{k\to\infty} X^{(p_k)}$ where $p_k \to \infty$ as $k \to \infty$. Then by (2.11) and (2.12),

$$c(X^{(p_k)}) \leq \mathcal{F}_{p_k,N,d}^{1/p_k} \leq \mathcal{F}_{\infty,N,d}[N(N-1)]^{1/p_k}.$$

Letting $k \to \infty$, by continuity of the coherence, we get $c(X) \leq \mathcal{F}_{\infty,N,d}$ which forces $c(X) = \mathcal{F}_{\infty,N,d}$. □

Next, we establish a continuity result of $\mathcal{F}_{p,N,d}$.

Proposition 2.2.3. *The minimal frame potential $\mathcal{F}_{p,N,d}$ is a continuous and non-increasing function of $p \in (0,\infty)$.*

Proof. We first prove that the function is non-increasing. Letting $p > q$, for any $X \in S(N,d)$,

$$\mathrm{FP}_{q,N,d}(X) \geq \mathrm{FP}_{p,N,d}(X) \geq \mathcal{F}_{p,N,d},$$

so $\mathcal{F}_{q,N,d} = \mathrm{FP}_{q,N,d}(X^{(p)}) \geq \mathcal{F}_{p,N,d}$.

For continuity, we have

$$\sum_{i\neq j, |\langle x_i,x_j\rangle|\neq 0} |\langle x_i, x_j\rangle|^q \ln|\langle x_i, x_j\rangle| \leq \frac{\mathrm{FP}_{p,N,d}(X) - \mathrm{FP}_{q,N,d}(X)}{p-q},$$

which comes from applying the inequality $a^q \ln a \leq \dfrac{a^p - a^q}{p-q}$ for $0 < q < p, a > 0$ to every nonzero term in the frame potential.

So

$$\begin{aligned}
0 &\leq \frac{\mathcal{F}_{q,N,d} - \mathcal{F}_{p,N,d}}{p-q} \\
&= \frac{\mathcal{F}_{q,N,d} - \mathrm{FP}_{p,N,d}(X^{(p)})}{p-q} \\
&\leq \frac{\mathrm{FP}_{q,N,d}(X^{(p)}) - \mathrm{FP}_{p,N,d}(X^{(p)})}{p-q} \\
&\leq \sum_{i \neq j, |\langle x_i^{(p)}, x_j^{(p)}\rangle| \neq 0} |\langle x_i^{(p)}, x_j^{(p)}\rangle|^q \ln \frac{1}{|\langle x_i^{(p)}, x_j^{(p)}\rangle|} \\
&\leq \sum_{i \neq j, |\langle x_i^{(p)}, x_j^{(p)}\rangle| \neq 0} \ln \frac{1}{|\langle x_i^{(p)}, x_j^{(p)}\rangle|} := C_p.
\end{aligned}$$

Therefore $0 \leq \mathcal{F}_{q,N,d} - \mathcal{F}_{p,N,d} \leq (p-q)C_p$, which implies the continuity of $\mathcal{F}$. □

Next, for fixed p, d, we consider the asymptotics of $\mathcal{F}_{p,N,d}$ as the number of points N grows. In particular, we show that $\mathcal{F}_{p,N,d} \sim N^2$, see Proposition 2.2.6. We note that this behavior was numerically observed in [5]. We begin by establishing some preliminary results.

Lemma 2.2.4. *Given $d \geq 2$, and $p \in (0, \infty)$, the sequence $\left\{ \frac{\mathcal{F}_{p,N,d}}{N(N-1)} \right\}_{N \geq d+1}$ is a non-decreasing sequence.*

Proof. Let $X^{(N)} = \{x_i^{(N)}\}_{i=1}^N$ be an optimal configuration for $\mathrm{FP}_{p,N,d}$. For each

$k \in [1:N]$,

$$\mathcal{F}_{p,N,d} = \mathrm{FP}_{p,N,d}(X^{(N)}) = \mathrm{FP}_{p,N,d}\left(X^{(N)} \backslash \{x_k^{(N)}\}\right) + 2\sum_{j\neq k} |\langle x_k^{(N)}, x_j^{(N)}\rangle|^p \tag{2.13}$$

$$\geq \mathcal{F}_{p,N-1,d} + 2\sum_{j\neq k} |\langle x_k^{(N)}, x_j^{(N)}\rangle|^p. \tag{2.14}$$

Summing (2.13) over k, we obtain

$$N\mathcal{F}_{p,N,d} \geq N\mathcal{F}_{p,N-1,d} + 2\mathcal{F}_{p,N,d} \Longrightarrow (N-2)\mathcal{F}_{p,N,d} \geq N\mathcal{F}_{p,N-1,d}$$
$$\Longrightarrow \frac{\mathcal{F}_{p,N,d}}{N(N-1)} \geq \frac{\mathcal{F}_{p,N-1,d}}{(N-1)(N-2)}.$$

□

It follows that $\tau := \lim_{N\to\infty} \frac{\mathcal{F}_{p,N,d}}{N^2}$ exists. In fact, in the minimal energy literature, τ is called the transfinite diameter due to Fekete. Furthermore, τ is related to the continuous version of the frame potential, which is introduced in [32]. More specifically, given a probability measure μ on the sphere, the *probabilistic p-frame potential* is defined as

$$\mathrm{PFP}_{p,d}(\mu) := \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} |\langle x, y\rangle|^p d\mu(x)d\mu(y). \tag{2.15}$$

Let $\mathcal{M}(\mathbb{S}^{d-1})$ be the collection of all probability measures on the sphere. Simple compactness and continuity arguments show that

$$\mathcal{P}_{p,d} := \min_{\mu\in\mathcal{M}(\mathbb{S}^{d-1})} \mathrm{PFP}_{p,d}(\mu) \tag{2.16}$$

exists.

Given any N point configuration X, its normalized counting measure is defined as

$$\nu_X := \frac{1}{N}\sum_{x\in X}\delta_x.$$

We have

$$\mathrm{PFP}_{p,d}(\nu_X) = \int\int |\langle x,y\rangle|^p d\nu_X(x)d\nu_X(y) = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}|\langle x_i,x_j\rangle|^p = \frac{\mathrm{FP}_{p,N,d}(X)+N}{N^2}. \tag{2.17}$$

Consequently, if X is an optimal configuration, i.e., $\mathcal{F}_{p,N} = \mathrm{FP}_{p,N,d}(X)$, then by (2.17), it is plausible that $\tau = \mathcal{P}_{p,d}$. This is indeed the case, and it was proved in a more general setting by Farkas and Nagy [33]. For the sake of completeness, we reproduce their proof below.

Lemma 2.2.5. *Given $d \geq 2$ and $p \in (0,\infty)$, $\tau = \lim_{N\to\infty}\frac{\mathcal{F}_{p,N,d}}{N^2} \leq \mathcal{P}_{p,d}$.*

Proof. Let μ^* be the optimal probability measure, that is,

$$\int\int |\langle x,y\rangle|^p d\mu^*(x)d\mu^*(y) = \mathcal{P}_{p,d} = \mathrm{PFP}_{p,d}(\mu^*).$$

Consequently,

$$\begin{aligned}
\mathcal{F}_{p,N,d} &= \int\cdots\int\left[\min_X \mathrm{FP}(X)\right] d\mu^*(x_1)\cdots d\mu^*(x_N)\\
&\leq \int\cdots\int \mathrm{FP}(X)d\mu^*(x_1)\cdots d\mu^*(x_N)\\
&= \sum_{i\neq j}\int\cdots\int |\langle x_i,x_j\rangle|^p d\mu^*(x_1)\cdots d\mu^*(x_N) = \sum_{i\neq j}\mathcal{P}_{p,d} = N(N-1)\mathcal{P}_{p,d}.
\end{aligned}$$

The result follows by dividing N^2 on both sides and taking the limit. $\square$

We can now state and prove that $\mathcal{F}_{p,N,d} \sim N^2$ as $N \to \infty$.

Proposition 2.2.6. *Given $d \geq 2$ and $p \in (0, \infty)$, we have $\tau = \lim\limits_{N\to\infty} \dfrac{\mathcal{F}_{p,N,d}}{N^2} = \mathcal{P}_{p,N,d}$. Moreover, if $\{X_N\}_{N \geq d+1}$ is a sequence of N-point configurations such that $\lim_{N\to\infty} \frac{\mathrm{FP}_N(X_N)}{N^2} = \tau$, then every weak star cluster point ν^* of the normalized counting measure $\nu_{X_N} = \frac{1}{N}\sum_{x\in X_N} \delta_x$ solves (2.16), that is $\mathrm{PFP}_{p,d}(\nu^*) = \mathcal{P}_{p,d}$. In particular, this holds for any sequence of the optimal configurations of $\mathrm{FP}_{p,N,d}$.*

Proof. By weak star convergence and (2.17)

$$\mathcal{P}_{p,d} \leq \mathrm{PFP}_{p,d}(\nu^*) = \lim_{N\to\infty} \mathrm{PFP}_{p,d}(\nu_{X_N}) = \lim_{N\to\infty} \frac{\mathrm{FP}_{p,N,d}(X_N) + N}{N^2} = \tau.$$

In view of Lemma 2.2.5, we have $\tau = \mathcal{P}_{p,d}$ and ν^* is an optimal probability measure.

$\square$

The exact value of τ can be found in many cases. We list two examples in the following corollary.

Corollary 2.2.7. *(a) When $d \geq 2$ and $p \in (0, 2]$, we have $\lim\limits_{N\to\infty} \dfrac{\mathcal{F}_{p,N,d}}{N^2} = \mathcal{P}_{p,d} = \dfrac{1}{d}$.*

(b) When $d = 2$ and p is an even integer, we have $\lim\limits_{N\to\infty} \dfrac{\mathcal{F}_{p,N,2}}{N^2} = \mathcal{P}_{p,2} = \dfrac{1 \cdot 3 \cdot 5 \cdots (p-1)}{2 \cdot 4 \cdot 6 \cdots p}$.

Proof. (a) By [32, Theorem 3.5] we know that when $N = kd$, the frame potential is minimized by k copies of orthonormal basis. So $\lim_{N\to\infty} \frac{\mathcal{F}_{p,N,d}}{N^2} = \lim_{k\to\infty} \frac{\mathcal{F}_{p,kd,d}}{(kd)^2} = \lim_{k\to\infty} \frac{(k-1)kd}{(kd)^2} = \frac{1}{d}$. Note that this recovers [32, Theorem 4.9], which states that $\mathcal{P}_{p,d} = \frac{1}{d}$.

(b) In dimension $d = 2$, it is known that $2N$ equally spaced points on the unit circle forms a spherical $(2N-1)$-design ([66, Section 4]), so Proposition 2.1.1 implies that $X_N^{(h)}$ is an optimal configuration if $p \leq 2N-2$ is an even integer. In other words, with fixed even integer p, when N is large enough, $\left(X_N^{(h)}\right)^{sym}$ is going to be a $(2N-1)$-design (hence p-design), so the equality in Proposition 2.1.1 holds and we get the desired result. $\square$

Paper [11] provides a more detailed discussion on the minimizers of the probabilistic p-frame potential. It is proved that for certain p, the minimizers are discrete measures. The description of the result involves tight spherical designs, which are spherical designs with smallest possible cardinality.

Definition 2.2.8. A discrete set $\mathcal{C} \subset \mathbb{S}^{d-1}$ is a tight t-design if one of the following conditions is satisfied.

1. $\mathcal{C}$ is a spherical deisgn of degree $t = 2m$ and $|\mathcal{C}| = \binom{d+m-1}{d-1} + \binom{d+m-2}{d-1}$.

2. $\mathcal{C}$ is a spherical deisgn of degree $t = 2m+1$ and $|\mathcal{C}| = 2\binom{d+m-1}{d-1}$.

Equivalently, it can be also defined in term of number of distances between distinct elements in the set.

Definition 2.2.9. A discrete set $\mathcal{C} \subset \Omega$ is a *tight t-design* if one of the following conditions is satisfied.

1. $\mathcal{C}$ is a design of degree $t = 2m-1$ and there are m distances between its distinct elements, including at least one pair diameter apart;

2. $\mathcal{C}$ is a design of degree $t = 2m$ and there are m distances between its distinct elements.

The main result in [11] shows that the spherical tight designs are the minimizers of the probabilistic p-frame potential for certain p.

Theorem 2.2.10 ([11]). *If there exists a tight spherical $(2t+1)$-design $\mathcal{C} \subset \mathbb{S}^{d-1}$, then the measure*

$$\nu_{\mathcal{C}} = \frac{1}{|\mathcal{C}|} \sum_{x \in \mathcal{C}} \delta_x$$

is a minimizer of the probabilistic p-frame potential $\mathrm{PFP}_{p,d}$ with $2t - 2 \leq p \leq 2t$ over $\mu \in \mathcal{M}(\mathbb{S}^{d-1})$.

Let $N = k|\mathcal{C}|$, where $\mathcal{C}$ be the same as in Theorem 2.2.10 and $k \in \mathbb{N}$. Denote $\mathcal{C}'$ to be the configuration $\mathcal{C}$ repeated k times. Then

$$\nu_{\mathcal{C}} = \frac{1}{|\mathcal{C}|} \sum_{x \in \mathcal{C}} \delta_x = \frac{1}{|\mathcal{C}'|} \sum_{x \in \mathcal{C}'} \delta_x = \nu_{\mathcal{C}'}.$$

We can then obtain the following corollary:

Corollary 2.2.11 ([11]). *The N-point discrete p frame potential $\mathrm{FP}_{p,N,d}$ is minimized by the configuration $\mathcal{C}$ repeated k times, i.e.*

$$\frac{1}{N} \mathcal{F}_{p,N,d} = PFP_{p,d}(\nu_{\mathcal{C}})$$

Since the tight spherical designs in $\mathbb{R}^2$ are characterized, we also have the following result regarding $\mathcal{F}_{p,N,2}$.

Corollary 2.2.12. *Let $N \geq 6$ be a even integer, then*

1. *the optimal configuration of $FP_{p,N,2}$ is an N-gon with $N-4 \leq p \leq N-2$;*

2. *for $k \in \mathbb{N}$, the optimal configuration of $FP_{p,kN,2}$ is k exact copies of N-gon with $N-4 \leq p \leq N-2$.*

Proof. When N is even, then the N-gon is a tight $(N-1)$-design. Thus $t = \frac{N-2}{2}$ is an integer.

1. Denote

$$S'_N = \{\mu \mid \mu = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}, x_i \in \mathbb{S}^1\}.$$

By Theorem 2.2.10, if $\mathcal{C}$ is a N-gon, then $\mu = \frac{1}{|\mathcal{C}|}\sum_{x\in\mathcal{C}} \delta_x \in S'_N$ is optimal configuration of $\min_{\mu\in\mathcal{P}(\mathbb{S}^{d-1})} I_f(\mu)$. So N-gon is minimizer for $FP_{p,N,2}$.

2. This follows from Corollary 2.2.11.

□

2.3 Optimal configurations in dimension 2

This section focuses on the case $d = 2$, when the points are on the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$. Our main result is the following. It shows that for each p, N, there exist a p_N such that the optimal configuration of $FP_{p,N,2}$ is universal for all $p > p_N$. In Section 2.3.1 we present intermediate results that are necessary to prove Theorem 2.3.1. In Section 2.3.2, we will finish the proof of Theorem 2.3.1.

Theorem 2.3.1. *Let $X_N^{(h)}$ be the equally spaced points on half of the circle $\mathbb{S}^1$ as in (2.7). The following statements hold.*

(a) If $N = 4$ and $p > 2$, then $X_4^{(h)}$ is the unique optimal configuration of (2.1).

(b) If $N \geq 5$ and $p > \begin{cases} 2N-2, & N \text{ is even} \\ 2N-4, & N \text{ is odd} \end{cases}$, then $X_N^{(h)}$ is the unique optimal configuration of (2.1).

(c) If $N \geq 5$, and $2 < p \leq \begin{cases} 2N-2, & N \text{ is even} \\ 2N-4, & N \text{ is odd} \end{cases}$ is an even integer, then $X_N^{(h)}$ is an optimal configuration of (2.1), but it is unclear whether there are other optimal configurations.

2.3.1 A class of minimal energy problems

We recall that when $N = 2k$ is even and $0 < p < 2$, the solution to (2.1) was given in [32, Theorem 3.5], where it was established that the minimizers are k copies of any orthonormal basis of $\mathbb{R}^2$. The case $p = 2$ was settled by Benedetto and Fickus [8]. In order to address other values of p, we will consider a more general problem

$$\min_{X \subset C_r, |X| = N} \sum_{i \neq j} f(\|x_i - x_j\|^2), \tag{2.18}$$

where $f : (0, 4r^2] \to \mathbb{R}$ is a nonnegative and decreasing function, and C_r is a 1−dimensional circle with radius r. This circle C_r does not need to be centered at 0 and could be in any dimension. It will become clear later why we require points on a general circle instead of the usual $\mathbb{S}^1$.

The proof of Theorem 2.3.1 involves two results.

The first result only requires f to be convex, but it only works for up to 4 points.

Theorem 2.3.2. *Given $r > 0$, let $f : (0, 4r^2] \to \mathbb{R}$ be a decreasing convex function. Any configuration X_4^* of 4 equally spaced points on C_r is an optimal configuration of (2.18) with $N = 4$. If in addition, f is strictly convex, then no other 4-point configuration is optimal.*

Proof. Let $X_4 = \{x_i\}_{i=1}^4$ be an arbitrary configuration with x_i ordered counter clockwise. Let $\alpha_{ik} \in [0, 2\pi)$ be the angle between x_i and x_{i+k} for any $k \in [1 : 3]$. The index of the vectors is cyclic as $x_i = x_{i-4}$. Then $\|x_i - x_{i+k}\|^2 = 2r^2 - 2r^2 \cos \alpha_{ik} = 4r^2 \sin^2 \frac{\alpha_{ik}}{2}$. It is evident that $\sum_{i=1}^4 \alpha_{ik} = 2\pi k$. Using the convexity of f,

$$\begin{aligned}\sum_{i \neq j} f\left(\|x_i - x_j\|^2\right) &= \sum_{k=1}^{3} \sum_{i=1}^{4} f\left(\|x_i - x_{i+k}\|^2\right) = 4 \sum_{k=1}^{3} \frac{1}{4} \sum_{i=1}^{4} f\left(\|x_i - x_{i+k}\|^2\right) \\ &\geq 4 \sum_{k=1}^{3} f\left(\frac{1}{4} \sum_{i=1}^{4} \|x_i - x_{i+k}\|^2\right) = 4 \sum_{k=1}^{3} f\left(\frac{4r^2}{4} \sum_{i=1}^{4} \sin^2 \frac{\alpha_{ik}}{2}\right).\end{aligned} \tag{2.19}$$

Next, let $\beta_{ik} = \alpha_{ik}/2$. In order to minimize the right hand side of (2.19), we solve

$$\max \sum_{i=1}^{4} \sin^2 \beta_{ik} \quad \text{subject to} \quad \beta_{ik} \geq 0,\ \sum_{i=1}^{4} \beta_{ik} = \pi k.$$

When $k = 1$, we let $\beta_i = \beta_{i1}$ for short. Using Lagrange multipliers, we have $0 = \frac{\partial}{\partial \beta_j} [\sum_{i=1}^4 \sin^2 \beta_i + \lambda(\sum_{i=1}^4 \beta_i - \pi)] = \sin 2\beta_j + \lambda$, which implies that

$$\sin 2\beta_i = \sin 2\beta_j \Longrightarrow 2\beta_i = 2\beta_j, \text{ or } 2\beta_i + 2\beta_j = \pi,$$

since $\sum_{i=1}^{4} \beta_i = \pi$.

If we are in the case that $\beta_1 + \beta_2 = \pi/2$ (or any pair $i \neq j$ with $\beta_i + \beta_j = \pi/2$), then $\sum_{i=1}^{4} \sin^2 \beta_i = \sin^2(\beta_1) + \sin^2(\pi/2 - \beta_1) + \sin^2(\beta_3) + \sin^2(\pi/2 - \beta_3) = 2$. If we are in the other case that $\beta_1 = \beta_2 = \beta_3 = \beta_4$, then $\sum_{i=1}^{4} \sin^2 \beta_i = 4\sin^2 \frac{\pi}{4} = 2$. So for $k = 1$,

$$\sum_{i=1}^{4} \sin^2 \beta_{i1} \leq 4\sin^2 \frac{\pi}{4},$$

and the equality holds when $\beta_{i1} + \beta_{j1} = \pi/2$ for some $i \neq j$.

When $k = 2$, it is obvious that

$$\sum_{i=1}^{4} \sin^2 \beta_{i2} \leq 4 = 4\sin^2 \frac{\pi}{2}$$

with equality at $\beta_{i2} = \pi/2$, for all $i \in [1:4]$. This implies that $\beta_{i1} + \beta_{i+1,1} = \pi/2$ for some i.

When $k = 3$, $\sum_{i=1}^{4} \sin^2 \beta_{i3} = \sum_{i=1}^{4} \sin^2(\pi - \beta_{i1}) = \sum_{i=1}^{4} \sin^2 \beta_{i1}$ which reduces to the $k = 1$ case.

In summary, for any $k = 1, 2, 3$,

$$\sum_{i=1}^{4} \sin^2 \frac{\alpha_{ik}}{2} \leq 4\sin^2 \frac{\pi k}{4},$$

and the equality holds simultaneously when $\alpha_{i1} + \alpha_{i+1,1} = \pi$, or equivalently $x_1 + x_3 = 0, x_2 + x_4 = 0$.

Following (2.19), we have

$$\sum_{i\neq j} f\left(\|x_i - x_j\|^2\right) \geq 4\sum_{k=1}^{3} f\Big(\frac{4r^2}{4}\sum_{i=1}^{4}\sin^2\frac{\alpha_{ik}}{2}\Big) \geq 4\sum_{k=1}^{3} f\Big(4r^2\sin^2\frac{\pi k}{4}\Big) = 8f(2r^2) + 4f(4r^2). \tag{2.20}$$

It is easy to check that four equally spaced points on C_r achieve this minimum.

If f is strictly convex, then the inequality of (2.19) becomes equality if $\|x_i - x_{i+k}\| = \|x_j - x_{j+k}\|$ for every $i \neq j$, which only holds for equally spaced points. $\square$

Remark 2.3.3. The proof of Theorem 2.3.2 breaks down for $N \geq 5$ because $\sum_{i=1}^{N}\sin^2\beta_{i1}$ is not maximized at equally spaced points.

The second result regarding (2.18) is a variation of the main result of the work by Cohn and Kumar [26, Theorem 1.2]. Let m be a positive integer. An m-*sharp configuration* $X \subset \mathbb{S}^{d-1}$ is a spherical $(2m-1)$-design with m inner products between its distinct points. A list of known sharp configurations was given in [26]. For example, N equally spaced points on $\mathbb{S}^1$ is an $\lfloor N/2 \rfloor$-sharp configuration. A C^∞ function $f : I \to \mathbb{R}$ is called K-*completely monotonic* if $(-1)^k f^{(k)}(x) \geq 0$ for all $x \in I$ and all $k \leq K$, and *strictly* K-*completely monotonic* if strict inequality always holds in the interior of I. The notion ∞-completely monotonic is simply called *completely monotonic* as traditionally defined, which means $(-1)^k f^{(k)}(x) \geq 0$ for all $x \in I$ and all $k \geq 0$. It was proven in [26] that sharp configurations are the unique universal optimal configurations of the problem

$$\min_{X \in S(N,d)} \sum_{i \neq j} f(\|x_i - x_j\|^2), \tag{2.21}$$

for completely monotonic functions f.

Another notion that we will need is that of absolutely monotonic functions. A C^∞ function $f : I \to \mathbb{R}$ is called K-*absolutely monotonic* if $f^{(k)}(x) \geq 0$ for all $x \in I$ and all $k \leq K$. Similarly, ∞-absolutely monotonic means the inequality is true for all nonnegative integers k, and will be simply referred to as *absolutely monotonic*. It is straightforward that $f(t)$ being completely monotonic is equivalent to $f(-t)$ being absolutely monotonic.

As remarked by [26], the complete monotonicity assumption of f can be weakened slightly. The proof of the next result is a variation of [26, Theorem 1.2]. It is also proved in [11] after we submitted our work.

Theorem 2.3.4. *Fix a positive integer m and let $f : (0, 4] \to \mathbb{R}$ be a function such that $(-1)^k f^{(k)}(t) \geq 0$ for all $t \in (0, 4], k \leq 2m$. Then an m-sharp configuration is an optimal configuration of* (2.21)*. Furthermore, if $(-1)^k f^{(k)}(t) > 0$ for all $t \in (0, 4), k \leq 2m$, then the m-sharp configuration is the unique optimal configuration of* (2.21).

Similar to the main result [26, Theorem 1.2], Theorem 2.3.4 involves the idea of linear programming. We will need the following proposition by Yudin.

Proposition 2.3.5 ([70]). *Let $f : (0, 4] \to \mathbb{R}$ be any function. Suppose $h : [-1, 1] \to$*

$\mathbb{R}$ *is a polynomial such that*

$$h(t) \leq f(2-2t)$$

for all $t \in [-1,1)$*, and suppose there are nonnegative coefficients* $\alpha_0, \cdots, \alpha_d$ *such that* h *has the expansion*

$$h(t) = \sum_{i=0}^{d} \alpha_i C_i^{n/2-1}(t)$$

in terms of ultraspherical polynomials. Then every set of N *points on* $\mathbb{S}^{n-1}$ *has potential energy at least*

$$N^2\alpha_0 - Nh(1)$$

with respect to the potential function f*.*

Let $-1 \leq t_1 < t_2 < \cdots < t_m < 1$ be the m distinct inner products of the m-sharp configuration. Let $a(t) = f(2-2t)$ be defined on $[-1,1)$ and $h(t)$ be the Hermite interpolating polynomial that agrees with $a(t)$ to order 2 at each t_i (i.e. $h(t_i) = a(t_i)$ and $h'(t_i) = a'(t_i)$).

We claim that the Hermite interpolating polynomial h satisfies all the conditions described in Proposition 2.3.5. In order to prove our claim, we need several lemmas.

Let f be a smooth function. Given a polynomial g with $\deg(g) \geq 1$, let $H(f,g)$ denote the Hermite interpolating polynomial of degree less than $\deg(g)$ that agrees with f at each root of g to the order of that root. The following fact is proven in the proof of [26, Proposition 2.2].

Lemma 2.3.6. *Let* a *be differentiable up to* K *on a subset of* $[-1,1)$*, and* g_1, g_2

be two polynomials such that $\deg(g_1) + \deg(g_2) \le K$*, then* $H(a, g_1g_2) = H(a, g_1) + g_1H(Q(a, g_1), g_2)$ *where*

$$Q(a, g) := \frac{a - H(a, g)}{g}.$$

We provide a variation of [26, Proposition 2.2] below. The proof is also similar.

Proposition 2.3.7. *Let* $c, d \in \mathbb{R}$*. If* a *is (strictly)* K*-absolutely monotonic on* (c, d)*, then given any nonconstant polynomial* g*,* $Q(a, g) = \frac{a-H(a,g)}{g}$ *is (strictly) absolutely monotone up to* $K - \deg g$ *on* (c, d)*.*

Proof. By [26, Lemma 2.1],

$$Q(a, g)(t) = \frac{a(t) - H(a, g)(t)}{g(t)} = \frac{a^{(\deg g)}(\xi)}{\deg g!} \tag{2.22}$$

for some $\xi \in (c, d)$.

A direct consequence of Lemma 2.3.6 is that $Q(a, g_1g_2) = Q(Q(a, g_1), g_2)$. For $n \in [1 : K - \deg(g)]$, $s_0 \in (c, d)$, there exists $\xi' \in (c, d)$ such that

$$\frac{Q(a, g)^{(n)}(s_0)}{n!} = Q\left(Q(a, g), (t - s_0)^n\right)(s_0) = Q(a, (t - s_0)^n g)(s_0) = \frac{a^{(n+\deg g)}(\xi')}{(n + \deg g)!}. \tag{2.23}$$

The right hand side of (2.23) is nonnegative due to the absolute monotonicity of a. □

In order to prove that h is positive definite, Cohn and Kumar introduced the term conductive (see [26, Definition 5.2]). Since we want to show that the absolutely monotonic requirement of a can be loosened, we alter the definition slightly to keep track of the requirement.

Definition 2.3.8. A nonconstant polynomial g with all its roots in $[-1,1)$ is K-*conductive* if for any K-absolutely monotone function a on $[-1,1)$, $H(a,g)$ is positive definite.

The following Lemma is a variation of [26, Lemma 5.3].

Lemma 2.3.9. *If g_1 and g_2 are K-conductive and g_1 is positive definite, then g_1g_2 is $(K+\deg g_1)$-conductive.*

Proof. Let a be $(K+\deg g_1)$-absolutely monotone, then $Q(a,g_1)$ is K-absolutely monotone according to Proposition 2.3.7. Consequently, $H(Q(a,g_1),g_2)$ is positive definite due to the conductivity of g_2. Finally, $H(a,g_1g_2) = H(a,g_1)+g_1H(Q(a,g_1),g_2)$ is positive definite because all three functions are positive definite and positive definite functions are closed under taking products. $\square$

Proof of Theorem 2.3.4. Using our notation, $h = H(a,F^2)$ where $F = \prod_{i=1}^{m}(t-t_i)$.

For $r \in [-1,1)$, $l(t) = t-r$ is K-conductive for any $K \geq 0$ since $H(a,l)$ is the nonconstant polynomial $a(r)$. It is also proven in [26, Section 5] that $\prod_{i=1}^{j}(t-t_i)$ is strictly positive definite for all $j \leq m$.

For any $K \geq 0$, $g_1 = t-t_1, g_2 = t-t_2$ are both K-conductive and g_1 is positive definite, then Lemma 2.3.9 implies that g_1g_2 is $(K+1)$-conductive. Using Lemma 2.3.9 repeatedly on $g_1 = t-t_j, g_2 = \prod_{i=1}^{j-1}(t-t_i)$, we get that F^2 is K-conductive for any $K \geq 2m$. In particular F^2 is $2m$-conductive and it follows that $h = H(a,F^2)$ is positive definite.

It is also clear that $h(t) \leq a(t)$ by applying (2.22) with $g = F^2$. By 2.3.5, the energy has a lower bound that is achieved by the m-sharp configuration.

If further f is strictly $2m$-completely monotone, the uniqueness is the same as in [26, Section 6] where only $a^{(\deg h+1)}(t) > 0$ is needed in [26, Lemma 6.4]. This is true since $\deg h + 1 \leq 2m$. $\square$

A direct consequence of Theorem 2.3.4 for dimension $d = 2$ is that equally spaced points are optimal configurations if the energy kernel function f is completely monotonic up to certain order. But notice that $\sum_{i \neq j} f(\|x_i - x_j\|^2)$ only depends on the relative distances between x_i's so the result should be true for any circle C_r (whose radius is r) if we rescale f properly.

Corollary 2.3.10. *For $N \geq 4$, let $m = \lfloor N/2 \rfloor$. For $r > 0$, suppose that $f : (0, 4r^2] \to \mathbb{R}$ is completely monotonic up to $2m$. Then N equally spaced points on C_r is an optimal configuration of* (2.18)*. Moreover, if f is strictly completely monotonic up to $2m$, then the equally spaced points is the unique optimal configuration of* (2.18).

2.3.2 Proof of the main theorem: A lifting trick

How do Theorem 2.3.2 and Corollary 2.3.10 help to prove our main theorem on $\mathrm{FP}_{p,N,2}$? On the unit circle, we have $|\langle x_i, x_j \rangle|^p = \left| \frac{2 - \|x_i - x_j\|^2}{2} \right|^p = h(\|x_i - x_j\|^2)$, where $h(t) = \left|\frac{2-t}{2}\right|^p$. Unfortunately neither result can be applied because the function $h(t)$ is not differentiable at $t = 2$ unless p is an even integer; worse, it is not even decreasing on [0,4]. This should not come as a surprise since the frame potential does not distinguish between antipodal points. Consequently, rather than analyzing the frame potential in terms of the distance between vectors, we should consider it in terms of the distance between lines, as was done in [25].

Define $P : \mathbb{S}^{d-1} \to M(d,d)$ as $P(x) = xx^*$, where $M(d,d)$ is the space of $d \times d$ symmetric matrices endowed with the Frobenius norm. $P(\mathbb{S}^{d-1})$ identifies antipodal points, and is the projective space embedded in $M(d,d)$. We write $P(x)$ as P_x and list some of the properties.

$$\begin{cases} \langle P_x, P_y \rangle &= |\langle x, y \rangle|^2 \\ \|P_x - P_y\|^2 &= 2 - 2|\langle x, y\rangle|^2. \end{cases} \tag{2.24}$$

When $d = 2$, we can explicitly write the embedding as $P : \mathbb{S}^1 \to M(2,2)(= \mathbb{R}^3)$,

$$P(x) = P_x = xx^* = \begin{bmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{bmatrix} \longleftrightarrow (x_1^2, \sqrt{2}x_1x_2, x_2^2).$$

It is not hard to see that $P(\mathbb{S}^1)$ is a circle in $\mathbb{R}^3$ centered at $(\frac{1}{2}, 0, \frac{1}{2})$ with radius $r = \frac{1}{\sqrt{2}}$, and this is where we can apply Theorem 2.3.2 or Corollary 2.3.10. One can verify that equally spaced points on the circle $P(\mathbb{S}^1)$ are precisely $X_N^{(h)}$, equally spaced points on the half circle, so we have the following theorem.

Theorem 2.3.11. *Let $g : [0,1) \to \mathbb{R}$ and consider*

$$\min_{X \in S(N,2)} \sum_{i \neq j} g(|\langle x_i, x_j \rangle|^2), \tag{2.25}$$

Then the following statements hold.

(a) *If g is convex and increasing, then $X_4^{(h)}$ is an optimal configuration of* (2.25) *when $N = 4$. Moreover if g is strictly convex, then $X_4^{(h)}$ is the unique optimal configuration.*

(b) *If g is absolutely monotone up to $2\lfloor N/2 \rfloor$, then $X_N^{(h)}$ is an optimal configuration of* (2.25)*. Moreover if g is strictly absolutely monotone up to $2\lfloor N/2 \rfloor$, then $X_N^{(h)}$ is the unique optimal configuration.*

Proof. As defined, $P_{x_i} = x_i x_i^*$. Denote P_{x_i} by P_i for simplicity. By (2.24),

$$g(|\langle x_i, x_j \rangle|^2) = g(1 - \|P_i - P_j\|^2/2) =: f(\|P_i - P_j\|^2),$$

where $f(t) = g(1 - t/2)$ is defined on $(0, 2]$. As discussed earlier, view the points P_i on a circle in $\mathbb{R}^3$ with radius $1/\sqrt{2}$, so solving (2.25) is equivalent to solving (2.18) with $r = 1/\sqrt{2}$.

If g is convex and increasing, then f is convex and decreasing. Applying Theorem 2.3.2 gives equally spaced P_i, which is equally spaced points on the half circle. This is part (a).

If g is absolutely monotone up to $2\lfloor N/2 \rfloor$, then f is completely monotone up to $2\lfloor N/2 \rfloor$. Applying Corollary 2.3.10 gives part (b).

□

Remark 2.3.12. Observe that in Theorem 2.3.11, the assumption of (b) is much stronger than (a). If g is twice differentiable, then g being convex and decreasing is equivalent to g being absolutely monotone up to 2. Furthermore, Theorem 2.3.11 is a very general result that goes beyond frame potentials. Indeed, it cover the cases where the energy can be expressed as a function of squares of the inner products. We expect to pursue this line of investigations elsewhere, with the goal of analyzing other energy kernels suitable for finding certain well conditioned frames.

We are now ready to prove Theorem 2.3.1 as a special case of Theorem 2.3.11.

Proof of Theorem 2.3.1. The p-frame potential kernel $|\langle x_i, x_j\rangle|^p = g_p(|\langle x_i, x_j\rangle|^2)$ with $g_p(t) = t^{p/2}$. The function g_p is strictly convex and increasing on [0,1) if $p > 2$.

(a) This part is due to Theorem 2.3.11(a).

(b) We notice that g_p is strictly absolutely monotone up to $\lceil p/2\rceil$, where $\lceil c\rceil$ is the smallest integer that is no less than c. In order to apply Theorem 2.3.11(b), we require $\lceil p/2\rceil \geq 2\lfloor N/2\rfloor$, which is equivalent to $p > 2N - 2$ if N is even and $p > 2N - 4$ if N is odd.

(c) Finally, this part is true because g_p is absolutely monotone when p is an even integer.

□

Remark 2.3.13. By Proposition 2.2.6, we can let p go to infinity in Theorem 2.3.1 and get that $X_N^{(h)}$ is the Grassmannian frame, as was shown in [10].

As seen, the 1-dimensional projective space is isomorphic to a circle. It is well known that higher a dimensional projective space is not a higher dimensional sphere. This is why the main result Theorem 2.3.11 is limited to $d = 2$.

At this point, we summarize the p-frame potential results in $\mathbb{S}^1$ as the following remark.

Remark 2.3.14. Let $d = 2$.

(a) When $N = 4$ we have completed the characterization of $\mathcal{F}_{p,4,2}$.

(b) When $N \geq 6$ is even, then [32, Theorem 3.5], Corollary 2.2.12, and parts (b) and (c) of Theorem 2.3.1 give the value of $\mathcal{F}_{p,N,2}$ when $p \in (0,2] \cup \bigcup_{m|N, m\geq 6 \text{ even}} (m-4, m-2) \cup \{4, 6, \cdots, 2N-2\} \cup (2N-2, \infty)$. We further know that the minimizer is unique for $p \in (0,2) \cup (2N-2, \infty)$. The numerical result is displayed in Figure 2.1 for $N = 6$. Figure 2.2 show an example of Corollary 2.2.12 when $N = 6, 10, 30$.

(c) When $N \geq 5$ is odd, we know $\mathcal{F}_{p,N,2}$ for $p \in \{2, 4, \cdots, 2N-4\} \cup (2N-4, \infty)$. We suspect that for $p \in (2, 2N-4]$, $X_N^{(h)}$ will still be the minimizer. The case $p \in (0,2)$ seems rather intriguing as demonstrated in Figure 2.1 for $N = 5$.

Figure 2.1 displays the numerical experiment for $d = 2$ and $N = 5, 6$, as well as the known result for $d = 2$ and $N = 4$. It appears that for p from 0 to about 1.78, the optimal configuration for $\mathrm{FP}_{p,5,2}$ is two copies of ONB plus a repeated vector; for $p \in (1.78, 2)$, the optimal configuration has the structure $\{x, x, y, y, z\}$ whose angles vary as p changes; for $p \in (2, 6)$, the optimal configuration is $X_5^{(h)}$.

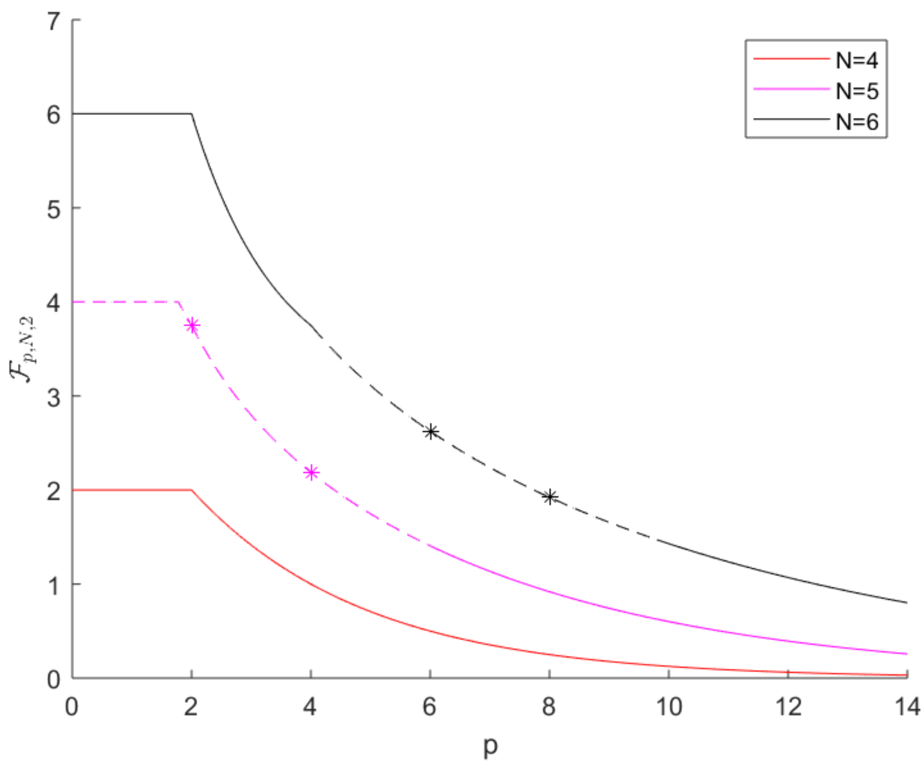

Figure 2.1: $\mathcal{F}_{p,N,2}$ for $N = 4, 5, 6$. The solid portion indicates proven cases as commented in Remark 2.3.14.

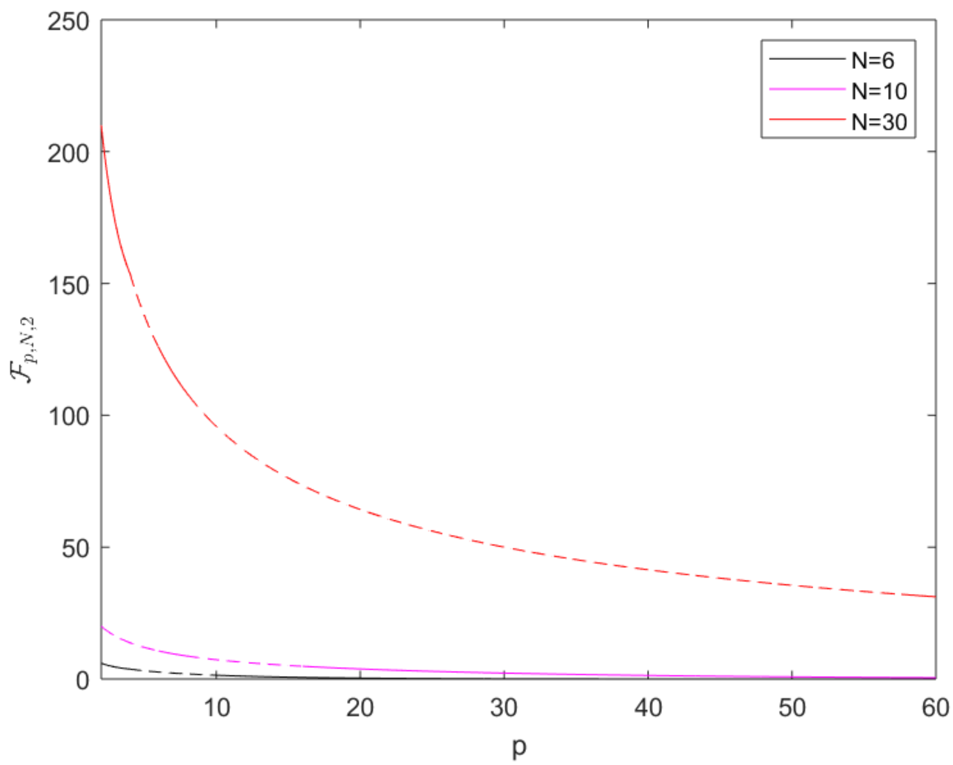

Figure 2.2: $\mathcal{F}_{p,N,2}$ for $N = 6, 10, 30$. The solid portion indicates proven cases.

2.4 Special case of $N = d + 1$ points in dimension d.

In this last section, we report on some numerical experiments and the resulting conjectures when minimizing the p-frame potential with $N = d + 1$ vectors in $\mathbb{R}^d$, and $p \in (0, \infty)$ and some partial results. Observe that the case $p = 2$ is a special case of the work by Benedetto and Fickus [8]. Additionally, the case $p > 2$ is handled by Ehler and Okoudjou [32, Proposition 3.1], for which the simplex is the optimal configuration. To be specific, the simplex is an ETF of $d+1$ vectors for $\mathbb{R}^d$. Therefore, the focus in this section are values $p < 2$. The following definition will be used through the rest of this section.

Definition 2.4.1.

(a) X is an ONB+ if X is formed by an orthonormal basis of $\mathbb{R}^d$ with one of the vectors repeated.

(b) Given $n \geq 2$, the simplex of $\mathbb{R}^n$ is denoted by ETF_n. An explicit construction of ETF_n is to project $e_1, e_2, \cdots, e_n, e_{n+1}$, the canonical basis of $\mathbb{R}^{n+1}$, onto the orthogonal complement of $\sum_{i=1}^{n+1} e_i$.

2.4.1 Embedded ETFs

From numerical tests, we have noticed that minimizers for $\mathcal{F}_{p,d+1,d}$ take forms similar to ETFs. In particular, they take the form of ETFs that have been embedded to higher dimensions.

Definition 2.4.2. For $1 \leq k \leq d$, the frame

$$\mathrm{L}_k^d = \begin{bmatrix} \mathrm{ETF}_k & 0 \\ 0 & I_{d-k} \end{bmatrix} = \begin{bmatrix} \mathrm{ETF}_k & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in S(d+1, d)$$

is called an *embedded ETF*.

Remark 2.4.3.

(a) In Definition 2.4.2, the entry ETF_k is the synthesis operator for the ETF_k configuration, and I_{d-k} is the $(d-k) \times (d-k)$ identity matrix. These frames are lifted in the sense that unit vectors for the remaining dimensions ($e_{k+1}, e_{k+2}, \ldots,$ and e_d) have been added such that the ETF_k frame is moved from $\mathbb{R}^k$ to $\mathbb{R}^d$. We refer to [63, 64] for more on constructions of these classes of ETFs.

(b) The L_k^d frames are not tight, except for the case $k = d$, and we have L_d^d is ETF_d.

(c) In addition to considering ETF_d as an L_d^d configuration, ONB+ is the L_1^d frame.

Example 2.4.4. The ETF in $\mathbb{R}^2$ can be embedded to $\mathbb{R}^3$ as

$$\mathrm{L}_2^3 = \begin{bmatrix} 1 & -1/2 & -1/2 & 0 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We see that this frame is neither tight nor equiangular by computing the frame operator and Grammian,

$$S = \begin{bmatrix} 3/2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad G = \begin{bmatrix} 1 & -1/2 & -1/2 & 0 \\ -1/2 & 1 & -1/2 & 0 \\ -1/2 & -1/2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

More generally, the Grammian of the L_k^d frame is

$$\begin{bmatrix} 1 & -1/k & -1/k & \cdots & -1/k & 0 & 0 & \cdots & 0 \\ -1/k & 1 & -1/k & \cdots & -1/k & 0 & 0 & \cdots & 0 \\ -1/k & -1/k & 1 & \cdots & -1/k & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1/k & -1/k & -1/k & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{2.26}$$

indicating that each L_k^d frame is a two-distance set (see, [4, 30]) with inner products $-1/k$ and 0; note, however, that L_d^d, or the ETF_d configuration, will have only one inner product, $-1/d$.

2.4.2 Embedded ETFs as the conjectured minimizers and partial results

Numerical computations suggest that the L_k^d frames are minimizers of $\mathrm{FP}_{p,d+1,d}$.

Conjecture 2.4.5. *Suppose $d \geq 2$ and for every natural number $1 \leq k \leq d-1$, let*

$$p_k = \frac{\log(k+2) - \log k}{\log(k+1) - \log k}.$$

We also define $p_0 = 0$. The following configurations minimize the p-frame potential $FP_{p,d+1,d}$:

- *when $p \in (p_{k-1}, p_k]$, the L_k^d configuration, $k = 1, 2, \cdots, d-1$;*
- *when $p \in (p_{d-1}, \infty]$, the ETF_d, or L_d^d configuration.*

Certain cases have been known for some time. The case $d = 2$ is completely established in [32]. For $d \geq 3$, the statement that ETF_d is the minimizer follows from [8] when $p = 2$, from [32, Proposition 3.1] when $p > 2$, and from [58] for $p = \infty$. A. Glazyrin [35] recently established that the ONB+, or L_1^d is the optimal configuration for $p \in (0, 2(\frac{\ln 3}{\ln 2} - 1)]$, leading to the fact that $\mathcal{F}_{p,d+1,d} = 2$ for all p in this range and all $d \geq 2$. The number $2(\frac{\ln 3}{\ln 2} - 1)$ is approximately 1.17 and is less than p_1.

The values p_k may be found by using the p-frame potentials of the L_k^d frames. By (2.26),

$$\mathrm{FP}_{p,d+1,d}(\mathrm{L}_k^d) = \left((k+1)^2 - (k+1)\right)\left(\frac{1}{k}\right)^p = (k+1)k\left(\frac{1}{k}\right)^p.$$

We find p_k so that the p-frame potentials of L_k^d and L_{k+1}^d are equal at the value p_k, so

$$(k+1)k\left(\frac{1}{k}\right)^{p_k} = (k+2)(k+1)\left(\frac{1}{k+1}\right)^{p_k}$$

leads to $p_k = \frac{\log(k+2) - \log k}{\log(k+1) - \log k}$.

Remark 2.4.6.

(a) The value p_k, where the p-frame potential of the L^d_{k+1} frame drops below the p-frame potential of the L^d_k frame, does not depend on d, the overall dimension.

(b) Following Conjecture 2.4.5, we will call the values p_k are the *switching points* as these are the values of p where the minimizing configuration seems to change. The final switching point is approaching to 2:

$$\lim_{d\to\infty} p_{d-1} = \lim_{d\to\infty} \frac{\log\left(\frac{d+1}{d-1}\right)}{\log\left(\frac{d}{d-1}\right)} = 2.$$

In [34], partial result of Conjecture 2.4.5 is obtained through applying the Lagrange multiplier method. It is proved that the critical configurations of $FP_{p,d+1,d}$ are all members of a set $L^d_{k,m}(\alpha,\beta)$, which is defined below.

Definition 2.4.7. Let $1 \le k \le m \le d+1$ and $0 \le \alpha, \beta \le 1$. Then $L^d_{k,m}(\alpha,\beta)$ is the set of unit vectors $\{x_i\}_{i=1}^{d+1}$ such that the following conditions hold:

1. $x_{m+1}, \cdots, x_{d+1}$ are pairwise orthogonal and orthogonal to all other vectors;

2. $\langle x_i, x_j\rangle = -\alpha^2$ when $1 \le i < j \le k$;

3. $\langle x_i, x_j\rangle = -\beta^2$ when $k+1 \le i < j \le m$;

4. $\langle x_i, x_j\rangle = -\alpha\beta$ when $1 \le i \le k$ and $k+1 \le j \le m$.

Theorem 2.4.8 ([34], Theorem 1)**.** *For any full-dimensional configuration X critical for $FP_{p,d+1,d}$, $p > 1$, one can change signs of vectors in X so that the resulting* *set is* $L^d_{k,m}(\alpha,\beta)$ *for some* $1 \le k < m \le d+1$ *and*

$$\alpha^p + \alpha^{p-2} = \beta^p + \beta^{p-2} = k\alpha^p + (m-k)\beta^p.$$

By reorganizing the vectors in $L^d_{k,m}(\alpha,\beta)$, we will be able to obtain a $L^d_{k',m'}(\alpha',\beta')$ such that $L^d_{k,m}(\alpha,\beta) = L^d_{k',m'}(\alpha',\beta')$ and $\alpha',\beta' \neq 0$. Suppose $d \ge 4$ and $m < d+1$, then $E_p(L^d_{k,m}(\alpha,\beta)) = E_p(L^{d-1}_{k,m}(\alpha,\beta))$. Since Conjecture 2.4.5 is true for dimension 2 and 3, without loss of generality, we only consider critical points of the form $L^d_{k,d+1}(\alpha,\beta)$ where $\alpha,\beta \neq 0$. In this case $L^d_{k,d+1}(\alpha,\beta)$ is an example of full spark frames in $\mathbb{R}^d$, which are frames $X \in \mathbb{R}^d$ such that any d vectors in X form a basis for $\mathbb{R}^d$. (Definition in [1]).

Then if we can prove the conjecture below, Conjecture 2.4.5 will follow.

Conjecture 2.4.9. *Let* $p_k = \frac{\log(k+2)-\log(k)}{\log(k+1)-\log(k)}$. *Then for given* $d \ge 4$, p *and* $N = d+1$, *the following statements hold:*

1. *when* $p \in [1, p_{d-1})$, *the minimizers of* $p-$*frame potential are not full spark frames;*

2. *when* $p \in [p_{d-1}, \infty)$, *the* L^d_d *configuration minimize the* $p-$*frame potential.*

To prove the first statement in the conjecture, we would like to further examine the frames $L^d_{k,d+1}(\alpha,\beta)$, $\alpha,\beta \neq 0$. Consider $f_{p_1,k,d}(\beta) \equiv E_p(L_{k,d+1}(\alpha,\beta))$ as function of $\beta \in (0, \sqrt{\frac{1}{d-k}})$. Since we only need ton consider critical points, it is sufficient to show $f_{p_1,k,d} > 2$ for all $d \ge 4$ and $1 \le k \le \lfloor \frac{d+1}{2} \rfloor$.

We have the following quick corollaries of [34, Thm 1].

Corollary 2.4.10. *For $d > 51$,*

$$E_p(L^d_{k,d+1}(\alpha,\beta)) > 2$$

for any $p \in [1, p_1]$,$1 \le k < d+1$ and $\alpha, \beta \in (0,1]$.

Proof. Suppose $L^d_{k,d+1}(\alpha,\beta)$ is a critical configuration of E_p where $\alpha, \beta \neq 0$. By (3) in [34],

$$\max(0, \frac{2-k}{2d-k}) \le \beta^2 < \frac{1}{d-k}.$$

α, β are symmetric, it is sufficient to consider $1 \le k \le \lfloor \frac{d+1}{2} \rfloor$. So for fixed d,

$$\beta < \frac{1}{\sqrt{d-k}} \le \sqrt{\frac{2}{d-1}}.$$

By [34, Theorem 1] , $\alpha^{p_1} + \alpha^{p_1-2} = \beta^{p_1} + \beta^{p_1-2}$. $g(y) = y^{p_1-2} + y^p$ changes from decreasing to increasing, so we consider two cases:

1. $\alpha = \beta$

   By (3) in [34],

$$d\beta^4 + (d-1)\beta^2 = 1,$$

$$\alpha^2 = \beta^2 = 1/d.$$

$$E_p(L^d_{k,d+1}(\alpha,\beta)) = (\frac{1}{d})^{p_1} d(d+1) \ge 2 \text{ } for \text{ } d \ge 2$$

2. $\alpha \neq \beta$

   Denote x_0 as the solution of $g(y) = g(1) = 2$ in interval $(0,1)$. Then $\alpha, \beta > x_0$.

By Intermediate Value Theorem, $x_0 > \frac{1}{5}$. Since $d > 51$,

$$\beta < \sqrt{\frac{2}{d-1}} < \frac{1}{5}.$$

This leads to a contradiction. So there is no possible critical point in this case.

$\square$

Corollary 2.4.11. *For fixed $d \geq 9$, $2 \leq k < \lfloor \frac{d+1}{2} \rfloor$, $f_{p_1,k,d}(\beta) > 2$ for any $\beta \in (0, \sqrt{\frac{1}{d-k}})$.*

Proof. Here we only need to consider the critical point of $f_{p_1,k,d}$. Since $0 < \beta < \sqrt{\frac{1}{d-k}} \leq \sqrt{\frac{2}{d-1}} \leq \sqrt{\frac{2-p_1}{p_1}}$, the function $g(\beta) = \beta^{p_1} + \beta^{p_1-2}$ is decreasing on the domain $\beta \in (0, \sqrt{\frac{1}{d-k}})$.

Now consider

$$g(\alpha) = \Big(\frac{1-(d-k)\beta^2}{d\beta^2+k-1}\Big)^{p_1/2} + \Big(\frac{1-(d-k)\beta^2}{d\beta^2+k-1}\Big)^{p_1/2-1}.$$

Since

$$\frac{d\alpha^2}{d\beta} = \frac{-\beta^2 k(d-k+1)}{(d\beta^2+k-1)^2} < 0,$$

we have $\frac{d\alpha}{d\beta} < 0$. Denote $b_0 = max(0, \frac{1+c-ck}{2d-k})$ and $\sqrt{\frac{2-p_1}{p_1}} \equiv c$. So $\frac{dg(\alpha)}{d\beta} \geq 0$ when $\beta \geq \sqrt{b_0}$; and $\frac{dg(\alpha)}{d\beta} \leq 0$ when $\beta \leq \sqrt{b_0}$. Since $\sqrt{b_0} < 1/5$ and $g(\beta) > 2 < g(\alpha)$ when $\beta < \sqrt{b_0}$. No α, β exist such that $g(\alpha) = g(\beta)$ and $\beta^2 \in (0, \sqrt{b_0})$. So there is only one critical point in $(0, \sqrt{\frac{1}{d-k}})$.

We know that $\alpha = \beta$ is one solution of $g(\alpha) = g(\beta)$, Thus $\alpha = \beta = \frac{1}{\sqrt{d}}$ is the only critical point of $f_{p_1,k,d}$. By checking the the critical point and endpoints we can conclude that $f_{p_1,k,d}(\beta) > 2$ for $\beta \in (0, \frac{1}{\sqrt{d-k}})$.

$\square$

Conjecture 2.4.5 can be proved for $\mathrm{FP}_{p,5,4}$.

Theorem 2.4.12. *Let $p_0 = 0$ and $p_k = \frac{\log(k+2)-\log(k)}{\log(k+1)-\log(k)}$ for $k = 1, 2, 3$. Then the p-frame potential for a frame of 5 vectors in $\mathbb{R}^4$ when $p \in [p_k, p_{k+1}]$ is L^d_{k+1} for $k = 1, 2, 3, 4$.*

Proof. The p-frame potential for $L^d_{k,m}(\alpha,\beta)$ is the same with $L^{d+1}_{k,m}(\alpha,\beta)$. Since the minimizing configuration is proved for $\mathbb{R}^3$, we only need to consider $m = 5$ for $d = 4$. $L^4_{1,5}$ and $L^4_{4,5}$ are the same . $L^4_{2,5}$ and $L^4_{3,5}$ are the same.

1. $p \in (1, p_1]$, $L^4_{1,5}$ case

$$\mathrm{FP}_{p,N,d}(L^4_{1,5}(\alpha,\beta)) = 8(\alpha\beta)^p + 12\beta^{2p},$$

with $4\alpha^2\beta^2 + 3\beta^2 = 1$. Take $y = \beta^2 \in [1/7, 1/3]$, we have

$$\mathrm{FP}_{p_1,N,d} = 8\big(\frac{1-3y}{4}\big)^{p_1/2} + 12y^{p_1} > 2.$$

Since $FP_{p,N,d}(X) \geq FP_{p_1}$ for any frame X with 5 unit vectors in $\mathbb{R}^4$,

$$\mathrm{FP}_p(L^4_{1,5}(\alpha,\beta)) > 2$$

for any α, β.

2. $p \in (1, p_1]$, $L^4_{2,5}$ case

$$\mathrm{FP}_{p,N,d}(L^4_{2,5}(\alpha,\beta)) = 2\alpha^{2p} + 12(\alpha\beta)^p + 6\beta^{2p},$$

with $4\alpha^2\beta^2 + \alpha^2 + 2\beta^2 = 1$. Take $y = \beta^2 \in [0, 1/2]$, we have

$$\mathrm{FP}_{p_2,N,d}(L^4_{2,5}(\alpha,\beta)) = 2\big(\frac{1-2y}{1+4y}\big)^{p_1} + 12\big(\frac{y-2y^2}{1+4y}\big)^{p_1/2} + 6y^{p_1} \geq 2.$$

with equality at $\beta = 0, 1/2$.

3. $p \in [p_1, p_2]$, $L^4_{1,5}(\alpha,\beta)$ case

We need

$$\mathrm{FP}_{p,N,d}(L^4_{1,5}(\alpha,\beta)) = 8(\alpha\beta)^p + 12\beta^{2p} = 8\big(\frac{1-3y}{4}\big)^{p/2} + 12y^p \geq 6(1/2)^p,$$

with $y \in [1/7, 1/3]$. The inequality is equivalent to

$$8(1-3y)^{p/2} + 12(2y)^p \geq 6.$$

If we fix y, and view $8(1-3y)^{p/2} + 12(2y)^p$ as a function of p, then it is decreasing by checking the derivative with respect to p. So we only need to consider the inequality at p_2.

$$E_{p_2}(L^4_{1,5}(\alpha,\beta)) \geq (1/2)^{p_2}$$

4. $p \in [p_1, p_2], L^4_{2,5}(\alpha, \beta)$ case We need

$$2(\frac{2-4y}{1+4y})^p + 12(\frac{4y-8y^2}{1+4y})^{p/2} + 6(2y)^p \geq 6.$$

In this case we only need to check $p = p_2$. And we have

$$2(\frac{2-4y}{1+4y})^{p_2} + 12(\frac{4y-8y^2}{1+4y})^{p_2/2} + 6(2y)^{p_2} \geq 6.$$

5. $p \in [p_2, p_3], L^4_{1,5}(\alpha, \beta)$ case

We need

$$\mathrm{FP}_{p,N,d}(L^4_{1,5}(\alpha, \beta)) = 8(\alpha\beta)^p + 12\beta^{2p} = 8(\frac{1-3y}{4})^{p/2} + 12y^p \geq 12(1/3)^p,$$

which is equivalent to

$$8(\frac{9(1-3y)}{4})^{p/2} + 12(3y)^p \geq 12.$$

If we fix y, $8(1-3y)^{p/2} + 12(2y)^p$ decreases with increasing p. So we only need to consider the inequality at p_3. We can check

$$8(\frac{9(1-3y)}{4})^{p_3/2} + 12(3y)^{p_3} \geq 12.$$

6. $p \in [p_2, p_3], L^4_{2,5}(\alpha, \beta)$ case We need

$$2(\frac{3-6y}{1+4y})^p + 12(\frac{9y-18y^2}{1+4y})^{p/2} + 6(3y)^p \geq 12.$$

LHS is decreasing function w.r.t. p, so we only need to check $p = p_3$. And the inequality holds at p_3.

7. $p \in [p_3, 2], L^4_{1,5}(\alpha, \beta)$ case We need

$$8(4-12y)^{p/2} + 12(4y)^p \geq 20.$$

Since $8(4-12y)^{p/2} + 12(4y)^p$ is increasing w.r.t. p except when $y = 1/4$.So again we only need to check the endpoints of p. And the inequality does hold.

8. $p \in [p_3, 2], L^4_{2,5}(\alpha, \beta)$ case We need

$$2(\frac{4-8y}{1+4y})^p + 12(\frac{16y-32y^2}{1+4y})^{p/2} + 6(4y)^p \geq 20.$$

LHS is increasing function w.r.t p when $y \in [1/4, 1/2]$ and decreasing when $y \in [0, 1/4)$. So we need to check the endpoints of p. And the inequality holds.

□

Remark 2.4.13. In the work by Zhiqiang Xu and Zili Xu [69], they prove the rest of the conjecture 2.4.5.

2.4.3 Description of the Numerical Computations

Numerical computations in Sage [62] were used to test Conjecture 2.4.5 numerically for $d+1$ vectors in $\mathbb{R}^d$. For each $d = 3, 4, 5, 6, 7$ and each $k = 1, 2, \ldots, d$, the program checked numerically whether L_k^d is the minimizer on the regions $[p_{k-1}, p_k]$. For $p = p_{k-1}$ and for $p = p_k$ specifically, along with some random values p in $[p_{k-1}, p_k]$, it used a basic gradient descent to numerically minimize the p-frame potential of several randomly chosen frames and then it compared these to the appropriate L_k^d frame. The only lower frame potential found seemed within the realm of numerical error (`<1e-15`). The number of comparisons was not selected rigorously; rather we only use the program as a guide. More details and the code may be found online at `https://www.math.umd.edu/~okoudjou/`.

2.5 Future research on optimal configurations in $\mathbb{R}^2$

In this section, we discuss two questions related to the unsolved cases in $\mathbb{R}^2$.

Question 1. What is the optimal configuration for $\mathrm{FP}_{p,5,2}$ when $p \in (0, \log 3/\log 2)$?

For any $X = \{x_i\}_{i=1}^5$, $p \in (0, \log 3/\log 2)$, we have

$$\begin{aligned} \mathrm{FP}_{p,5,2}(X) &= \mathrm{FP}_{p,3,2}(\{x_1, x_2, x_3\}) + \mathrm{FP}_{p,3,2}(\{x_1, x_4, x_5\}) + \mathrm{FP}_{p,3,2}(\{x_2, x_4, x_5\}) + \cdots \\ &\quad \mathrm{FP}_{p,3,2}(\{x_3, x_4, x_5\}) - 2|\langle x_4, x_5\rangle|^p \\ &\geq 4 - 2|\langle x_4, x_5\rangle|^p. \end{aligned} \tag{2.27}$$

Since permutation on the vectors in a set X does not change the p-frame potential,

we have

$$\mathrm{FP}_{p,5,2}(X) \geq 4 - 2 \min_{x_i, x_j \in X} |\langle x_i, x_j \rangle|^p.$$

If we can prove that the optimal configuration for $\mathrm{FP}_{p,5,2}$ with $p \in (0, \log 3/\log 2)$ contains at least one pair of orthogonal vectors, then we can conclude that $\mathcal{F}_{p,5,2} = 4$ and two copies of ONB plus a repeated vector is an optimal configuration for $\mathrm{FP}_{p,5,2}$. One of our future goals is to prove this statement.

It is then natural to ask whether this method could be generalized to any N, d. Consider the complete graph K_N with N vertices $x_1, \cdots, x_N$ and assign $|\langle x_i, x_j \rangle|$ to the edge connecting x_i and x_j. Then the process in equation (2.27) is covering the edges of K_5 with K_3 while having as few repeated edges as possible. This is similar to the goal of the edge clique covering problem (cf. [43]). In the future, we would like to further explore the possibility of utilizing the results in edge clique covering problem to construct the optimal configurations for $\mathrm{FP}_{p,N,d}$.

Question 2. What is the optimal configuration of $\mathrm{FP}_{p,N,2}$ for the p not listed in Remark 2.3.14 part(b)?

By Corollary 2.2.12, if k is an integer and $N \geq 6$ is even, the optimal configuration of $\mathrm{FP}_{p,kN,2}$ is known when $p \in (N-4, N-2)$. We would ask whether statement still true for any integer $N \geq 4$? The answer is no. We consider the following example.

Example 2.5.1. Let $N = 24$. Then the optimal configuration of $\mathrm{FP}_{p,N,2}$ is 4 copies of 6-gons when $p \in (2, 4)$, which is not equivalent to 6 copies of $X_4^{(h)}$.

It is still unknown whether it is possible to loosen the restriction on N in

Corollary 2.2.12 and if yes, to what extent. Settling this problem could give us partial results of Question 2.

Chapter 3: Equiangular Lines and Grassmannian Frames

3.1 Preliminaries

In this section we describes the method applied to construct equiangular lines, which is developed in [39]. Define a (N, d) equiangular line sets to be a set of N equiangular lines in $\mathbb{R}^d$.

Define the Gram matrix G of a set of vectors $\{v_i\}_{i=1}^N$ as $G_{i,j} = \langle v_i, v_j \rangle$. The following correspondence between equiangular line sets and matrices with certain properties is well known.

Theorem 3.1.1. *A set of N equiangular lines in $\mathbb{R}^d$ exists if and only if there exists a $N \times N$ Hermitian matrix G with the following properties:*

1. $G_{ii} = 1$, $|G_{ij}| = a \in [0, 1)$ *for* $i \neq j \in \{1, \cdots, N\}$.
2. *rank*$(G) \leq d$.
3. *All principal minors of G are non-negative.*

To prove the theorem we need the following results on Gram matrices and positive semidefinite matrices.

Theorem 3.1.2 ([44], Theorem 7.2.10)**.** *Suppose G is a $N \times N$ Gram matrix of a set of vectors $\{v_i\}_{i=1}^N$, then it is Hermitian and positive-semidefinite. Furthermore, $rank(G) = dim\, span\{v_i\}_{i=1}^N$.*

Theorem 3.1.3 (Cholesky factorization, [44], Corollary 7.2.9)**.** *Suppose A is a $N \times N$ Hermitian matrix, then A is positive semidefinite if and only if there is a lower triangular matrix $L \in M_n$ with nonnegative diagonal entries such that $A = LL^*$. If A is real, L may be taken to be real.*

We reproduce the proof here.

Proof. Suppose A is positive semidefinite, then there exist a unique square root $A^{1/2}$. Let $A^{1/2} = QR$ be a QR factorization and $L = R^*$. Then L is a lower triangle matrix. We have

$$A = (A^{1/2})^* A^{1/2} = R^* Q^* Q R = R^* R = LL^*.$$

Suppose $A = LL^* = (L^*)^* L^*$. Then A is the Gram for the set of columns vectors of L^*. By Theorem 3.1.2, A is positive semi-definite. □

Theorem 3.1.4 (Sylvester's criterion, [44],Observation 7.1.2 & Theorem 7.2.5)**.** *A $N \times N$ Hermitian matrix A is positive semidefinite*

1. *if and only if every principal minor of A is nonnegative.*

2. *if the first $N-1$ leading principal minors of A are positive and $det(A) \geq 0$.*

Proof of Theorem 3.1.1. Suppose first that there exist an (N, d) equiangular line set with unit vectors $\{v_i\}_{i=1}^N$ on each of the lines. Then by definition, its Gram matrix satisfies condition 1. By Theorem 3.1.2 and 3.1.4, the Gram matrix satisfies condition 2 and 3 correspondingly.

Suppose then there exist a Hermitian matrix G that satisfies condition 1-3. By Theorem 3.1.4 and condition 3, G is positive semidefinite. By Theorem 3.1.3 and condition 2, there exist a $N \times N$ matrix L such that $G = L^T L$. Let $L = [v_1' \cdots v_N']$, then G is Gram matrix of $\{v_i'\}_{i=1}^N$. By Theorem 3.1.2, dim $span\{v_i'\}_{i=1}^N \leq d$. So $\{v_i'\}_{i=1}^N$ can be embedded in $\mathbb{R}^d$. Therefore there exist a corresponding (N, d) equiangular line set. □

Remark 3.1.5. Given a Hermitian matrix A that satisfies the three conditions in Theorem 3.1.1, we could construct a set of equiangular lines concretely by Theorem 3.1.3. We give two examples.

Example 1

$$A = \begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{bmatrix}.$$

is a positive semidefinite matrix of rank 2, thus should be the Gram matrix for a set of 3 equiangular lines in $\mathbb{R}^2$. The eigenvalue of A are $0, 3/2, 3/2$.

$$A^{1/2} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ -1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

So

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \\ -1/2 & -\sqrt{3}/2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

A is then the Gram matrix of the set of 3 equiangular lines represented by the vectors $(1,0)$,$(-1/2, \sqrt{3}/2)$, $(-1/2, -\sqrt{3}/2)$.

Exmaple 2

$$A = \begin{bmatrix} 1 & 1/3 & 1/3 & -1/3 \\ 1/3 & 1 & -1/3 & 1/3 \\ 1/3 & -1/3 & 1 & 1/3 \\ -1/3 & 1/3 & 1/3 & 1 \end{bmatrix}$$

is a positive semidefinite matrix of rank 3. Using matlab, we have

$$A^{1/2} = \frac{\sqrt{3}}{6} \begin{bmatrix} 3 & 1 & 1 & -1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ -1 & 1 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3}/2 & 0 & 0 & 1/2 \\ \sqrt{3}/6 & \sqrt{6}/3 & 0 & -1/2 \\ \sqrt{3}/6 & -\sqrt{6}/6 & \sqrt{2}/2 & -1/2 \\ -\sqrt{3}/6 & \sqrt{6}/6 & \sqrt{2}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1/3 & 1/3 & -1/3 \\ 0 & 2\sqrt{2}/3 & -2\sqrt{2}/3 & 2\sqrt{2}/3 \\ 0 & 0 & \sqrt{6}/3 & \sqrt{6}/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/3 & 2\sqrt{2}/3 & 0 & 0 \\ 1/3 & -2\sqrt{2}/3 & \sqrt{6}/3 & 0 \\ -1/3 & 2\sqrt{2}/3 & \sqrt{6}/3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1/3 & 1/3 & -1/3 \\ 0 & 2\sqrt{2}/3 & -2\sqrt{2}/3 & 2\sqrt{2}/3 \\ 0 & 0 & \sqrt{6}/3 & \sqrt{6}/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then one configuration of equiangular lines that has Gram matrix A is $(1, 0, 0)$, $(1/3, 2\sqrt{2}/3, 0)$, $(1/3, -2\sqrt{2}/3, \sqrt{6/3})$, $(-1/3, 2\sqrt{2}/3, \sqrt{6}/3)$.

Given Theorem 3.1.1, equiangular line sets can be obtained by constructing Hermitian matrices that satisfy condition 1-3 in Theorem 3.1.1. We will start the construction with $(d+1, d)$ equiangular line sets, and construct their Gram matrices recursively.

3.1.1 Constructing $(d + 1, d)$ equiangular line sets

For any d and $N = d + 1$, condition 2 in Theorem 3.1.1 is equivalent to $det(G) = 0$. Paper [26] uses the following method to construct matrix G that satisfies conditions 1-3:

Step 1. List all the possible form of G that satisfy condition 1.

Step 2. Find $a \in [0, 1)$ such that $det(G) = 0$.

Step 3. Replace all the a with the solution in step 2. Since the 1×1 principal minor is always 1, by Theorem 3.1.4 we need to check whether G_k are positive for $k = 2, \cdots , N - 1$. Where G_k is the $k \times k$ leading principal minors of G .

Remark 3.1.6. Without loss of generality, orthogonal transformation and permutation of vectors in $\{v_i\}_{i=1}^N$ or replacing any v_i with $-v_i$ does not change the corresponding set of lines. In step 1, we say two Hermitian matrices A and B are equivalent if it is possible to obtain B by

- multiplying rows and corresponding columns of A by -1;
- exchanging rows and corresponding columns of A.

In step 1, to enumerate the possible forms of G, we consider the graph with the vectors $\{v_i\}_{i=1}^N$ as vertices, and connect v_i, v_j with an edge if $\langle v_i, v_j \rangle = -a$. Without loss of generality, we assume the non-diagonal entries of first row and column are a. So v_1 is disconnected in the graph. We can then examine the matrices corresponding to simple graphs with $N-1$ vertices.

3.1.2 Constructing (N, d) equiangular line sets with $N > d+1$

Suppose we have all the possible Gram matrices of (k, d) equiangular line sets for $k = d+1, \cdots, N$. If $(N+1, d)$ equiangular line sets exist, all its subsets are also equiangular line sets. So the Gram matrix of $(N+1, d)$ equiangular line sets can be constructed by adding an extra row and column to the Gram matrix of a (N, d) equiangular line set, such that the extended matrix still satisfies condition 1-3 in Theorem 3.1.1. In Section 3.2 and Section 3.3, we will give detailed construction of equiangular lines in $\mathbb{R}^3$ and $\mathbb{R}^4$ respectively. The results were first derived in [39]. In Section 3.4 we derive properties of $(6, 4)$- Grassmannian frames. In Section 3.5 we draw attention to some unsolved problems related to Equiangular lines and

Grassmannian frames.

3.2 Equiangular line sets in $\mathbb{R}^3$

3.2.1 $N = 4$

In this section we apply the method in Section 3.1.1 to construct $(4, 3)$ equiangular lines.

There are 3 cases to consider in step 1. For the matrix G in each case, we solve for $det(G) = 0$ and compute the principal minors G_2 and G_3. There are 2 configurations that satisfy all the conditions in Theorem 3.1.1.

3.2.1.1 Case 1

$$G = \begin{bmatrix} 1 & a & a & a \\ a & 1 & a & a \\ a & a & 1 & a \\ a & a & a & 1 \end{bmatrix}$$

Since $det(G) = (1-a)^3(1+3a) > 0$ for all $a \in [0, 1)$, $rank(G) = 4$. G does not satisfy condition 3. There exist no equiangular line set with $N = 4, d = 3$ that correspond to G in this case.

3.2.1.2 Case 2

$$G = \begin{bmatrix} 1 & a & a & -a \\ a & 1 & -a & a \\ a & -a & 1 & a \\ -a & a & a & 1 \end{bmatrix}$$

$det(G) = (1+a)^3(1-3a)$. Solution for $det(G) = 0$ is $a = \frac{1}{3}$. The leading principal minors are

$$G_2 = 1 - a^2 > 0;$$
$$G_3 = (1+a)^2(1-2a) > 0 \; when \; a = \frac{1}{3}.$$

So case 2 corresponds to a set of 4 equiangular lines in $\mathbb{R}^3$, and the angle between any two lines is $\cos^{-1}(\frac{1}{3})$.

3.2.1.3 Case 3

$$G = \begin{bmatrix} 1 & a & a & a \\ a & 1 & a & a \\ a & a & 1 & -a \\ a & a & -a & 1 \end{bmatrix}$$

$det(G) = (1-a^2)(1-5a^2)$. Solution for $det(G) = 0$ in $[0,1)$ is $a = \frac{1}{\sqrt{5}}$. The leading principal minors are

$$G_2 = 1 - a^2 > 0;$$

$$G_3 = (a-1)^2(2a+1) > 0.$$

Case 3 corresponds to a set of 4 equiangular lines in $\mathbb{R}^3$, and the angle between any two lines is $\cos^{-1}(\frac{1}{\sqrt{5}})$.

Therefore there are only 2 possible angles in a $(4,3)$ equiangular line sets and we have the following.

Theorem 3.2.1. $(4,3)$ *equiangular line sets exist, and* $a = \frac{1}{3}$ *or* $a = \frac{1}{\sqrt{5}}$.

3.2.2 $N = 5$

In this section and section 3.2.3, we build on the result in Section 3.2.1. A candidate G for the Gram matrix can be obtained by attaching a row and column to one of the possible Gram matrices of $(4,3)$ equiangular line set S. Then every $k \times k$ principal submatrix of G is the Gram of a subset of S, thus is the Gram matrix of a (k,d) equiangular line set with the same angle a. For 5 lines in $\mathbb{R}^3$, there exists one such extension that satisfies conditions 1-3 in Theorem 3.1.1.

Theorem 3.2.2. $(5,3)$ *equiangular line set exists, and the angle* $a = \frac{1}{\sqrt{5}}$.

Proof. Since the a value in different cases are all different, any 4×4 principal submatrix in the Gram matrix of $(5,3)$ equiangular line set belong to the same case in section 3.2.1.

Suppose there exists G', a matrix extension of case 2. Then the submatrices obtained by removing 1st or 4th row and the corresponding column are equivalent with case 2. The only G' for which that holds is

$$\begin{bmatrix} 1 & a & a & -a & a \\ a & 1 & -a & a & -a \\ a & -a & 1 & a & -a \\ -a & a & a & 1 & a \\ a & -a & -a & a & 1 \end{bmatrix}.$$

However if we remove 2nd row and column, the remaining submatrix is not equivalent to the Gram matrix in case 2. No such G can be obtained from case 2.

There exist one extension of case 3 that satisfies condition 1-3 in Theorem 3.1.1.

$$G = \begin{bmatrix} 1 & a & a & a & a \\ a & 1 & a & a & -a \\ a & a & 1 & -a & -a \\ a & a & -a & 1 & a \\ a & -a & -a & a & 1 \end{bmatrix}$$

$\square$

In this case $a = \frac{1}{\sqrt{5}}$. $det(G) = 25a^4 - 10a^2 + 1 = 0$. All the 4×4 principal submatrices are equivalent to case 3, so all principal minors of G are non-negative. By Theorem 3.1.4, G is a positive semidefinite matrix. Since all 4×4 submatrices are equivalent to case 3, their determinant are all 0. The leading principal minor is positive, so $rank(G) = 3$. By Theorem 3.1.1, G is the Gram matrix of a $(5,3)$ equiangular line set.

3.2.3 $N = 6$

The Gram matrix of $(6,3)$ equiangular line set can be obtained by further extending the Gram matrix of the $(5,3)$ equiangular line set. Applying the same procedure as in section 3.2.2, we obtain one extension of G in 3.2.2.

$$G = \begin{bmatrix} 1 & a & a & a & a & a \\ a & 1 & a & a & -a & -a \\ a & a & 1 & -a & -a & a \\ a & a & -a & 1 & a & -a \\ a & -a & -a & a & 1 & a \\ a & -a & a & -a & a & 1 \end{bmatrix}.$$

The angle $a = \frac{1}{\sqrt{5}}$ is the same with the $(5,3)$ equiangular line set. $det(G) = -125a^6 + 75a^4 - 15a^2 + 1 = 0$. G is a positive semidefinite matrix of rank 3.

3.3 Equiangular line sets in $\mathbb{R}^4$

3.3.1 N=5

In this section we construct the Gram matrix of $(5,4)$ equiangular line set.

To enumerate the possible forms of G, we consider the graph with the vectors

$\{v_i\}_{i=1}^5$ as vertices, and connect v_i, v_j with an edge if $\langle v_i, v_j \rangle = -a$. Without loss of generality, we assume the non-diagonal entries of the first row and column are a. So v_1 is disconnected in the graph. We will then examine the matrices corresponding to simple graphs with 4 vertices. [29] listed all simple graphs with 4 vertices, see also figure 3.1. The 11 different graphs correspond to 7 different cases. Using the same notation as in figure 3.1,we present the matrix that corresponds to each graph and check whether there exists a corresponding equiangular line set.

3.3.1.1 Case 1: $\bar{K_4}$

There is no edge in this graph, so the non-diagonal entries of corresponding G are all a. We have

$$G = \begin{bmatrix} 1 & a & a & a & a \\ a & 1 & a & a & a \\ a & a & 1 & a & a \\ a & a & a & 1 & a \\ a & a & a & a & 1 \end{bmatrix}$$

$det(G) = (1-a)^3(1+4a) > 0$ for all $a \in [0, 1)$, $rank(G) = 5$. There exist no corresponding equiangular line set.

3.3.1.2 Case 2: co-diamond, $K_{1,3}$

There exist one edge in 'co-diamond'. So there exist distinct $j, k \in \{1, \cdots, 4\}$ such that $\langle v_j, v_k \rangle = -a$. In $K_{1,3}$, there are three edges and there exist one vertex that is an endpoint of all three edges. Without loss of generality, we have $\langle v_2, v_3 \rangle =$

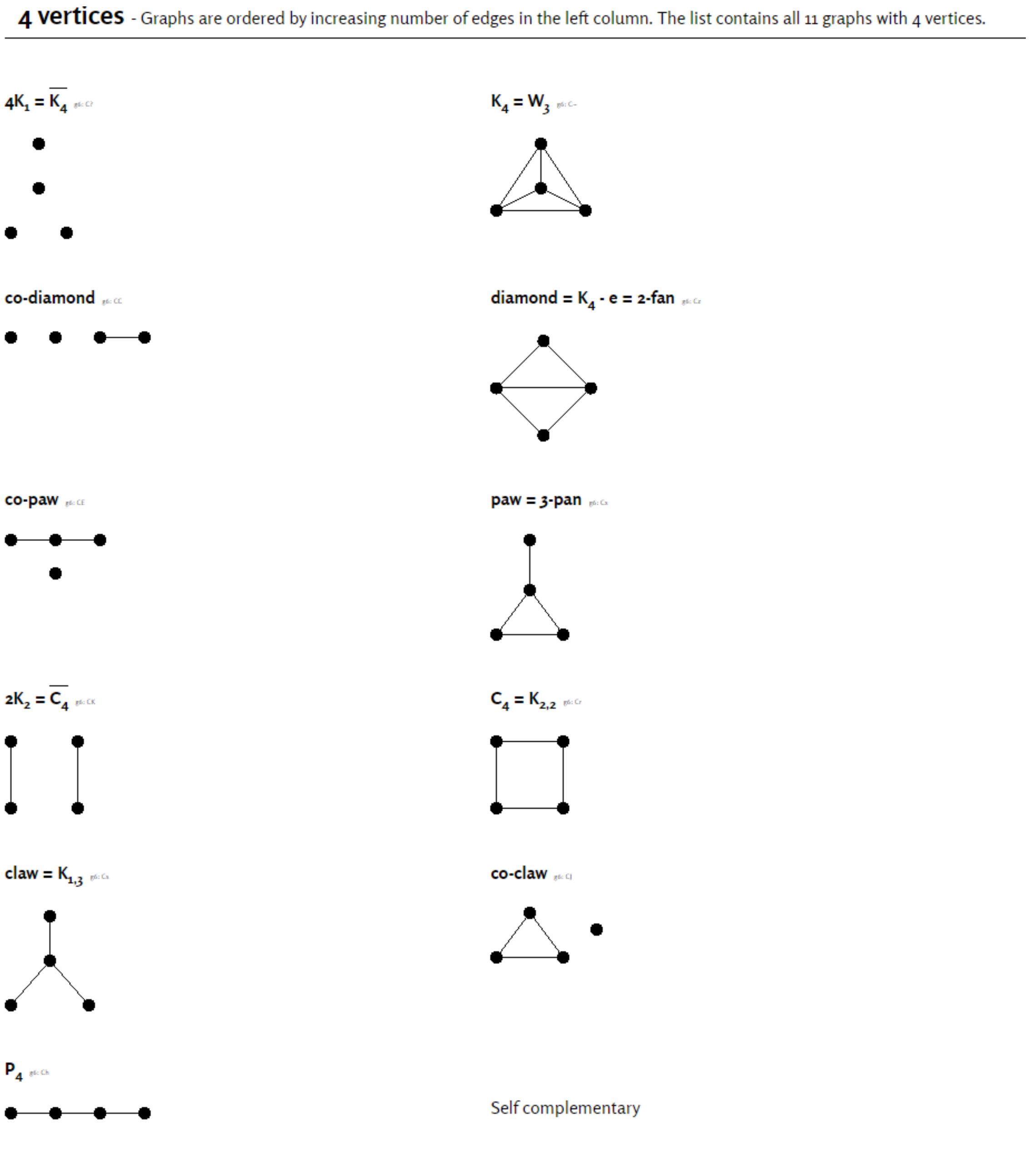

Figure 3.1: [29]List of all simple graphs with 4 vertices

$\langle v_2, v_4\rangle = \langle v_2, v_5\rangle = -a.$

$$G = \begin{bmatrix} 1 & a & a & a & a \\ a & 1 & a & a & a \\ a & a & 1 & a & a \\ a & a & a & 1 & -a \\ a & a & a & -a & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & a & a & a & a \\ a & 1 & -a & -a & -a \\ a & -a & 1 & a & a \\ a & -a & a & 1 & a \\ a & -a & a & a & 1 \end{bmatrix} = G'.$$

G and G' are matrices that correspond to 'co-diamond' and $K_{1,3}$ respectively. By multiplying -1 to 2nd row and column of G', we can see that G and G' are equivalent. We only need to check whether G satisfies all the conditions in Theorem 3.1.1.

$det(G) = (1-a)^2(1+a)(1+a-8a^2)$. Solution for $det(G) = 0$ in $[0, 1)$ is $a = \frac{1+\sqrt{33}}{16}$. The leading principal minors are

$$G_2 = 1 - a^2 > 0;$$

$$G_3 = (a-1)^2(2a+1) > 0;$$

$$G_4 = (1-a)^3(3a+1) > 0.$$

So G is a positive semidefinite matrix of rank 4 when $a = \frac{1+\sqrt{33}}{16}$. Case 2 corresponds to a set of 5 equiangular lines in $\mathbb{R}^4$, and the angle between any two lines is $\cos^{-1}(\frac{1+\sqrt{33}}{16})$.

3.3.1.3 Case 3: co-paw,C_4

There are two edges in 'co-paw' and one vertice that is the end point of both edges. So there exist distinct $j, k_1, k_2 \in \{1, \cdots, 4\}$ such that $\langle v_j, v_{k_1} \rangle = \langle v_j, v_{k_2} \rangle = -a$. In C_4, there are 4 edges and $\langle v_2, v_3 \rangle = \langle v_2, v_5 \rangle = \langle v_3, v_4 \rangle = \langle v_4, v_5 \rangle = -a$.

$$G = \begin{bmatrix} 1 & a & a & a & a \\ a & 1 & a & a & a \\ a & a & 1 & a & -a \\ a & a & a & 1 & -a \\ a & a & -a & -a & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & a & a & a & a \\ a & 1 & -a & a & -a \\ a & -a & 1 & -a & a \\ a & a & -a & 1 & -a \\ a & -a & a & -a & 1 \end{bmatrix} = G'.$$

G and G' are matrices that correspond to 'co-paw' and C_4 respectively. By multiplying -1 to 2nd and 4th rows and columns of G', we can see that G and G' are equivalent. We only need to check whether G satisfies all the conditions in Theorem 3.1.1.

$det(G) = (1-a)^2(1+3a)(1-a-4a^2)$. Solution for $det(G) = 0$ in $[0, 1)$ is $a = \frac{\sqrt{17}-1}{8}$. The leading principal minors are

$$G_2 = 1 - a^2 > 0;$$

$$G_3 = (a-1)^2(2a+1) > 0;$$

$$G_4 = (1-a)^3(3a+1) > 0.$$

G is a positive semidefinite matrix of rank 4 when $a = \frac{\sqrt{17}-1}{8}$. Case 3 correspond to a set of 5 equiangular lines in $\mathbb{R}^4$, and the angle between any two lines is $\cos^{-1}(\frac{\sqrt{17}-1}{8})$.

3.3.1.4 Case 4: $\overline{C_4}$, paw

For $\overline{C_4}$, $\langle v_2, v_5\rangle = \langle v_3, v_4\rangle = -a$. For 'paw', $\langle v_2, v_3\rangle = \langle v_2, v_4\rangle = \langle v_2, v_5\rangle = \langle v_3, v_4\rangle = -a$.

$$G = \begin{bmatrix} 1 & a & a & a & a \\ a & 1 & a & a & -a \\ a & a & 1 & -a & a \\ a & a & -a & 1 & a \\ a & -a & a & a & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & a & a & a & a \\ a & 1 & -a & -a & -a \\ a & -a & 1 & -a & a \\ a & -a & -a & 1 & a \\ a & -a & a & a & 1 \end{bmatrix} = G'.$$

G and G' are matrices that correspond to $\overline{C_4}$ and 'paw' respectively. By multiplying -1 to 2nd row and column of G', we can see that G and G' are equivalent. We only need to check whether G satisfies all the conditions in Theorem 3.1.1.

$det(G) = (1-a)^2(1-3a)(1+a-4a^2)$. Solutions for $det(G) = 0$ in $[0, 1)$ are $a = \frac{\sqrt{17}+1}{8}$ and $a = \frac{1}{3}$. The leading principal minors are

$$G_2 = 1 - a^2 > 0;$$

$$G_3 = (a-1)^2(2a+1) > 0$$

for both solutions. However $G_4 = (1-a)^2(1-5a^2)$. We have $G_4 < 0$ when $a = \frac{\sqrt{17}+1}{8}$ and $G_4 > 0$ when $a = \frac{1}{3}$. G is a positive semidefinite matrix of rank 4 when $a = \frac{1}{3}$. Case 4 correspond to a set of 5 equiangular lines in $\mathbb{R}^4$, and the angle between any two lines is $\cos^{-1}(\frac{1}{3})$.

3.3.1.5 Case 5: co-claw, diamond

There are 3 edges, and one isolated vertice in 'co-claw'. $\langle v_3, v_4\rangle = \langle v_3, v_5\rangle = \langle v_4, v_5\rangle = -a$. 'Diamond', or $K_4 - e$, is the complete graph minus one edge. Inner product between any two vectors in $\{v_i\}_{i=2}^5$ are $-a$ except one pair of vectors. Without loss of generality we let $\langle v_2, v_4\rangle = a$.

$$G = \begin{bmatrix} 1 & a & a & a & a \\ a & 1 & a & a & a \\ a & a & 1 & -a & -a \\ a & a & -a & 1 & -a \\ a & a & -a & -a & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & a & a & a & a \\ a & 1 & -a & a & -a \\ a & -a & 1 & -a & -a \\ a & a & -a & 1 & -a \\ a & -a & -a & -a & 1 \end{bmatrix}$$

G and G' are matrices that correspond to 'co-claw' and 'diamond' respectively. By multiplying -1 to 5nd and 3rd rows and columns of G', we can see that G and G' are equivalent. We only need to check whether G satisfies all the conditions in Theorem 3.1.1.

$det(G) = (1+a)^2(1-a)(1-a-8a^2)$. Solution for $det(G) = 0$ in $[0, 1)$ is

$a = \frac{\sqrt{33}-1}{16}$. The leading principal minors are

$$G_2 = 1 - a^2 > 0;$$

$$G_3 = (a-1)^2(2a+1) > 0;$$

$$G_4 = (1-a^2)(1-5a^2) > 0.$$

G is a positive semidefinite matrix of rank 4 when $a = \frac{\sqrt{33}-1}{16}$.Case 5 correspond to a set of 4 equiangular lines in $\mathbb{R}^3$, and the angle between any two lines is $\cos^{-1}(\frac{\sqrt{33}-1}{16})$.

3.3.1.6 Case 6: P_4

There are 3 edges in P_4. We have $\langle v_2, v_5\rangle = \langle v_3, v_4\rangle = \langle v_3, v_5\rangle = -a$.

$$G = \begin{bmatrix} 1 & a & a & a & a \\ a & 1 & a & a & -a \\ a & a & 1 & -a & -a \\ a & a & -a & 1 & a \\ a & -a & -a & a & 1 \end{bmatrix}$$

$det(G) = (1-5a^2)^2$. Solution for $det(G) = 0$ in $[0,1)$ is $a = \frac{1}{\sqrt{5}}$. This is the set of 5 equiangular lines in $\mathbb{R}^3$ embedded in $\mathbb{R}^4$.

3.3.1.7 Case 7: K_4

K_4 is the complete graph. So

$$G = \begin{bmatrix} 1 & a & a & a & a \\ a & 1 & -a & -a & -a \\ a & -a & 1 & -a & -a \\ a & -a & -a & 1 & -a \\ a & -a & -a & -a & 1 \end{bmatrix}$$

$det(G) = (1+a)^3(1-4a)$. Solution for $det(G) = 0$ in $[0, 1)$ is $a = \frac{1}{4}$. The leading principal minors are

$$G_2 = 1 - a^2 > 0;$$

$$G_3 = (a+1)^2(1-2a) > 0;$$

$$G_4 = (a+1)^3(1-3a) > 0.$$

Case 3 correspond to a set of 4 equiangular lines in $\mathbb{R}^3$, and the angle between any two lines is $\cos^{-1}(\frac{1}{4})$.

By the above construction, we have the following result.

Theorem 3.3.1. $(5, 4)$ *equiangular line sets exist.* $a \in \{\frac{1+\sqrt{33}}{16}, \frac{\sqrt{17}-1}{8}, \frac{1}{3}, \frac{\sqrt{33}-1}{16}, \frac{1}{\sqrt{5}}, \frac{1}{4}\}$.

3.3.2 N=6

If a equiangular line set with 6 lines exists, then any subset of 5 lines is also an equiangular line set with the same angle. Since the a value in different cases are all different, any 5×5 principal submatrix in the Gram matrix of $(6, 4)$ equiangular line set belong to one of the possible cases in section 3.3.1. Without loss of generality, suppose the number of $-a$ in the added row is less than 3.

Theorem 3.3.2 (Welch Bound)**.** *Suppose* $\{v_i\}_{i=1}^N \subset \mathbb{S}^{d-1}$ *with* $N > d$, *then*

$$\max_{i \neq j} |\langle v_i, v_j \rangle| \geq \sqrt{\frac{N-d}{d(N-1)}}.$$

The Welch Bound for $(6,4)$ equiangular line set is $\frac{1}{\sqrt{10}}$. So case 5 and 7 can be excluded.

If there exists a $(6,4)$ equiangular line set that has same a with any of the cases in 3.3.1, we can add a new vertex that is endpoint of at most 2 edges to the corresponding graph, such that any induced subgraph of the new graph with 5 vertices belong to the same case. This is not possible for case 2,3.

3.3.2.1 The extension of 3.3.1.4

Theorem 3.3.3.

$$G = \begin{bmatrix} 1 & a & a & a & a & -a \\ a & 1 & a & a & -a & a \\ a & a & 1 & -a & a & a \\ a & a & -a & 1 & a & a \\ a & -a & a & a & 1 & a \\ -a & a & a & a & a & 1 \end{bmatrix}$$

is the Gram matrix of a $(6,4)$ *equiangular line set.*

Proof. $det(G) = (1+a)^2(1-3a)^2(1+3a)$. The only solution for $det(G) = 0$ in $a \in [0,1)$ is $a = \frac{1}{3}$.

Since each column or row of G contains only one $-a$, by removing any row and corresponding column in G, the principal submatrix will be the Gram matrix of a $(5,4)$ line set $\{v_i\}_{i=1}^5$ with distinct $j_1, k_1, j_2, k_2 \in \{1,2,3,4\}$ such that

1. $\langle v_{j_1}, v_{k_1}\rangle = \langle v_{j_2}, v_{k_2}\rangle = -a$,

2. $\langle v_j, v_k\rangle = a$ for $(j,k) \notin \{(j_1,k_1),(j_2,k_2)\}$.

Changing the order of lines does not change the set, so all 5×5 principal submatrices are equivalent. By calculation in section 3.3.1, all the principal minors of G are non-negative. So G is positive semidefinite.

All 5×5 principal minors of G are 0. The leading 4×4 principal minor is positive. $rank(G) = 4$. G is the Gram matrix of a $(6,4)$ equiangular line set. $\square$

Remark 3.3.4. If a 4×4 principal minor of G is 0, then the corresponding subset of lines can be embedded into $\mathbb{R}^3$. Three 4×4 principal minors are 0. The corresponding subsets are $\{v_2, v_3, v_4, v_5\}$, $\{v_1, v_3, v_4, v_6\}$ and $\{v_2, v_2, v_5, v_6\}$.

3.3.2.2 The extension of 3.3.1.6

The extension of 3.3.1 is the same with the Gram matrix in 3.2.3. The corresponding line set can be embedded into $\mathbb{R}^3$.

3.4 Proposition of (6, 4)-Grassmannian Frame

In this section we will prove a proposition of $(6,4)$-Grassmannian frames that is similar to Lemma VI.6 in [10]. First we will need the following propositions that are proved in [10].

Proposition 3.4.1. *[10] Let $N \geq d$, $Y = \{y_1, \cdots, y_N\} \subset S^{d-1} \subset \mathbb{R}^d$, and assume $span(Y) = \mathbb{R}^d$. Let*

$$Q = \{v \in \mathbb{R}^d : |\langle v, y_k \rangle| \leq 1, \text{ for } k = 1, \cdots, N\}$$

and C be the set of extreme points of Q. Then

(a) Q is a bounded convex set,

(b) If $v_0 \in C$ then there are at least d distinct integers $k_1, \cdots, k_d \in \{1, \cdots, N\}$ such that $|\langle v_0, y_{k_i} \rangle| = 1$ for $1 = 1, \cdots, d$,

(c) $card(C) \leq \binom{N}{d} 2^d < \infty$.

Proposition 3.4.2. *[10] Let N, d, Y, Q and C be as in proposition 3.4.1, and $c \in C$ have the property that $\|c\| = max\{\|c'\| : c' \in C\}$. Then for any $v \in Q \setminus C$,*

$$\|v\| < \|c\|.$$

We can then prove a generalized version of [10, Lemma VI.5], which follows similar arguments in the proof of (5,3) version.

Lemma 3.4.3. *Let $U = \{b, y_1, \cdots, y_{d+1}\} \subset S^{d-1} \subset \mathbb{R}^d$, and $\alpha = \mathcal{M}_\infty(U)$. Then if $|\langle b, y_1 \rangle| < \alpha$,$|\langle b, y_2 \rangle| < \alpha$, there exist $c \in S^{d-1}$ such that*

$$|\langle c, y_k \rangle| < \alpha \text{ for } k = 1, \cdots, d+1.$$

Proof. We consider two cases:

Case 1. $dim(span\{y_1, \cdots, y_{d+1}\}) < d$.

Then choose $c \in (span\{y_1, \cdots, y_{d+1}\})^\perp$, we have

$$\langle \frac{c}{\|c\|}, y_k \rangle = 0 < \alpha \; for \; k = 1, \cdots, d+1.$$

Case 2. $span\{y_1, \cdots, y_{d+1}\} = \mathbb{R}^d$.

Let $Q = \{v \in \mathbb{R}^d : |\langle v, y_k \rangle| \leq 1, k = 1, \cdots, d+1\}$ and C be the extreme points of Q. Then by Proposition 3.4.1, Q is convex and C is finite. Then by assumption, there are at most $d-1$ distinct integers $k_1, \cdots, k_{d-1} \in \{1, \cdots, d+1\}$ such that $|\langle \frac{b}{\alpha}, y_{k_{d-1}} \rangle| = 1$. By Proposition 3.4.1, $\frac{b}{\alpha}$ is not a extreme point of Q.

Choose c with the property $\|c\| = max\{\|c'\| : c' \in C\}$. Then by Proposition 3.4.2, $\|c\| > \|\frac{b}{\alpha}\| = \frac{1}{\alpha}$. So

$$|\langle \frac{c}{\|c\|}, y_k \rangle| \leq \frac{1}{\|c\|} < \alpha \; for \; k = 1, \cdots, d+1.$$

□

Now we are ready to prove the following lemma on $(6,4)$-Grassmannian frames.

Lemma 3.4.4. *Let $U = \{u_i\}_{i=1}^6$ be a $(6,4)$-Grassmannian frame, and $\alpha = \mathcal{M}_\infty(U)$. Then for any j, there are distinct $j_1, j_2, j_3, j_4 \in \{1, \cdots, 6\} \setminus \{j\}$ such that*

$$|\langle u_j, u_{j_k} \rangle| = \alpha \; for \; k = 1, 2, 3, 4.$$

Proof. Assume the contrapositive. Without loss of generality, let $|\langle u_1, u_2 \rangle| < \alpha$ and

$|\langle u_1, u_3 \rangle| < \alpha$. We would like to show either U is not Grassmannian frame, or such U does not exist.

Under our assumption, by Lemma 3.4.3, there exist $c_1 \in S^3$ such that

$$|\langle c_1, u_k \rangle| < \alpha \text{ for } k = 2, \cdots, 6.$$

Let $\tilde{U} = \{u_2, \cdots, u_6\}$, we have two cases:

1. There exist $j_0, k_0 \in \{2, \cdots, 6\}$ such that $j_0 \neq k_0$, for which $|\langle u_{j_0}, u_{k_0} \rangle| < \alpha$.

   Following the same procedure as in [10] Lemma VI.6 case 1, we can construct a frame W such that $\mathcal{M}_\infty(W) = \mathcal{M}_\infty(U)$.

   Without loss of generality assume $|\langle u_2, u_3 \rangle| < \alpha$. Let $b = u_2$, and $\{y_1, \cdots, y_5\} = \{c_1, u_3, \cdots, u_6\}$. Then by Lemma 3.4.3, there exists $c_2 \in S^3$ such that $\langle c_2, y_k \rangle < \alpha$ for $k = 1, \cdots, 5$.

   By the construction above, let $b = u_3$, $\{y_1, \cdots, y_5\} = \{c_1, c_2, u_4, u_5, u_6\}$. We can apply Lemma 3.4.3 again and have $c_3 \in S^3$ such that $\langle c_3, y_k \rangle < \alpha$ for $k = 1, \cdots, 5$. Repeat this procedure and we have c_4,c_5 such that $|\langle c_i, c_j \rangle| < \alpha$ for $i \neq j$, and $|\langle c_i, u_6 \rangle| < \alpha$ for $i = 1, \cdots, 5$. Let $W = \{c_1, \cdots, c_5, u_6\}$. Then $\mathcal{M}_\infty(W) < \alpha = \mathcal{M}_\infty(U)$.

2. $\tilde{U}$ is equiangular

   $\tilde{U}$ has 5 vectors, so it has the same configuration of the 5 equiangular lines in $\mathbb{R}^4$. The calculation below is based on result in [39], where the possible angles between lines and the Gram matrices are constructed.

By [39], $\alpha \in \{\frac{\sqrt{33}\pm 1}{16}, \frac{\sqrt{17}-1}{8}, \frac{1}{3}, \frac{1}{\sqrt{5}}, \frac{1}{4}\}$. $\frac{1}{4}$ is less than the Welch bound of $N = 6, d = 4$, which is $\frac{1}{\sqrt{10}}$, so we can rule out the case $\alpha = 1/4$. [39] also shows there exists an equiangular frame $\{X\}_{i=1}^{6} \subset \mathbb{R}^4$ such that $\mathcal{M}_\infty(X) = 1/3$. So the only possibility we need to consider is $\alpha = \frac{1}{3}$.

Relabel $u_2, \cdots, u_6$ as $v_1, \cdots, v_5$. By [39], there exist a subset of four vectors in $\tilde{U}$ that can be embedded into $\mathbb{R}^3$. Without loss of generality, by we have

$$\begin{aligned}
v_1 &= (0, 0, 1, 0)^T, \\
v_2 &= (\sqrt{1-\alpha^2}, 0, \alpha, 0)^T, \\
v_3 &= (x_3, y_3, \alpha, 0)^T, \\
v_4 &= (x_4, y_4, \alpha, 0)^T, \\
v_5 &= (x_5, y_5, \alpha, z_5)^T,
\end{aligned}$$

and $\langle v_2, v_3\rangle = \langle v_2, v_4\rangle = \langle v_3, v_4\rangle = -\alpha$, $\langle v_2, v_5\rangle = \langle v_3, v_5\rangle = \langle v_4, v_5\rangle = \alpha$. So

$$x_3 = x_4 = -\alpha\sqrt{\frac{1+\alpha}{1-\alpha}} = -\frac{\sqrt{2}}{3}.$$

Also

$$\begin{aligned}
y_3 y_4 &= \frac{\alpha(\alpha+1)}{\alpha-1}, \\
y_3^2 = y_4^2 &= \frac{(2\alpha-1)(\alpha+1)}{\alpha-1},
\end{aligned}$$

so $(y_3, y_4) = (\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}})$ or $(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}})$.

By solving the equations, we have

$$
\begin{aligned}
v_1 &= (0,0,1,0)^T, \\
v_2 &= (\frac{2\sqrt{2}}{3}, 0, \frac{1}{3}, 0)^T, \\
v_3 &= (-\frac{\sqrt{2}}{3}, \sqrt{\frac{2}{3}}, \frac{1}{3}, 0)^T, \\
v_4 &= (\frac{\sqrt{2}}{3}, \sqrt{\frac{2}{3}}, -\frac{1}{3}, 0)^T, \\
v_5 &= (\frac{1}{3\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{3}, \pm\sqrt{\frac{2}{3}})^T,
\end{aligned}
$$

or

$$
\begin{aligned}
v_1 &= (0,0,1,0)^T, \\
v_2 &= (\frac{2\sqrt{2}}{3}, 0, \frac{1}{3}, 0)^T, \\
v_3 &= (-\frac{\sqrt{2}}{3}, -\sqrt{\frac{2}{3}}, \frac{1}{3}, 0)^T, \\
v_4 &= (\frac{\sqrt{2}}{3}, -\sqrt{\frac{2}{3}}, -\frac{1}{3}, 0)^T, \\
v_5 &= (\frac{1}{3\sqrt{2}}, -\frac{1}{\sqrt{6}}, \frac{1}{3}, \pm\sqrt{\frac{2}{3}})^T,
\end{aligned}
$$

We claim that such pair of $\tilde{U}$ and c_1 that satisfy our assumption does not exist. To prove that we first assume that it is possible to construct the c_1 such that $\|c_1\| = 1$ and $\langle c_1, v_i \rangle < \alpha$ for $v = 1, \cdots, 5$, and reach a contradiction. Denote $c_1 = (x_c, y_c, z_c, w_c)^T$.

(a) Suppose $v_1 = (0,0,1,0)^T, v_2 = (\frac{2\sqrt{2}}{3}, 0, \frac{1}{3}, 0)^T, v_3 = (-\frac{\sqrt{2}}{3}, \sqrt{\frac{2}{3}}, \frac{1}{3}, 0)^T, v_4 = (\frac{\sqrt{2}}{3}, \sqrt{\frac{2}{3}}, -\frac{1}{3}, 0)^T, v_5 = (\frac{1}{3\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{3}, \sqrt{\frac{2}{3}})^T, c = (x_c, y_c, z_c, \sqrt{1 - x_c^2 - y_c^2 - z_c^2})^T$.

Then the condition $\langle c_1, v_i\rangle < \alpha$ for $v = 1, \cdots, 5$ turns into the following system of inequalities:

$$-\frac{1}{3} < z_c < \frac{1}{3} \tag{3.1}$$

$$-\frac{1}{3} < \frac{2\sqrt{2}}{3}x_c + \frac{1}{3}z_c < \frac{1}{3} \tag{3.2}$$

$$-\frac{1}{3} < -\frac{\sqrt{2}}{3}x_c + \sqrt{\frac{2}{3}}y_c + \frac{1}{3}z_c < \frac{1}{3} \tag{3.3}$$

$$-\frac{1}{3} < \frac{\sqrt{2}}{3}x_c + \sqrt{\frac{2}{3}}y_c - \frac{1}{3}z_c < \frac{1}{3} \tag{3.4}$$

$$-\frac{1}{3} < \frac{1}{3\sqrt{2}}x_c + \frac{1}{\sqrt{6}}y_c + \frac{1}{3}z_c + \sqrt{\frac{2}{3}}\sqrt{1 - x_c^2 - y_c^2 - z_c^2} < \frac{1}{3} \tag{3.5}$$

Consider the function

$$f_1(x_c, y_c, z_c) = \frac{1}{3\sqrt{2}}x_c + \frac{1}{\sqrt{6}}y_c + \frac{1}{3}z_c + \sqrt{\frac{2}{3}}\sqrt{1 - x_c^2 - y_c^2 - z_c^2}.$$

We claim that the minimum of f_1 is equal to $\frac{1}{3}$ and is achieved at $(-\frac{1}{3\sqrt{2}}, -\frac{1}{\sqrt{6}}, -\frac{1}{3})$ if we include the boundary of (3.1)-(3.4). Since

$$df_1(-\frac{1}{3\sqrt{2}}, -\frac{1}{\sqrt{6}}, -\frac{1}{3}) = (\frac{\sqrt{2}}{3}, \sqrt{\frac{2}{3}}, \frac{2}{3}).$$

For any (x, y, z) that satisfies (3.1)-(3.4),the vector v' pointing from $(-\frac{1}{3\sqrt{2}}, -\frac{1}{\sqrt{6}}, -\frac{1}{3})$ to (x, y, z) has

$$D_{v'}f_1(-\frac{1}{3\sqrt{2}}, -\frac{1}{\sqrt{6}}, -\frac{1}{3}) = \frac{\sqrt{2}}{3}(x + \frac{1}{3\sqrt{2}}) + \sqrt{\frac{2}{3}}(y + \frac{1}{\sqrt{6}}) + \frac{2}{3}(z + \frac{1}{3}) > 0$$

by (3.1)(3.3), so its a local minimum. Let (x, y, z) be any point that satisfy (3.1)-(3.4),$w = \sqrt{1 - x^2 - y^2 - z^2}$. The Hessian of f_1 is

$$H = \begin{bmatrix} \frac{1}{w} + \frac{x^2}{w^3} & \frac{xy}{w^3} & \frac{xz}{w^3} \\ \frac{xy}{w^3} & \frac{1}{w} + \frac{y^2}{w^3} & \frac{yz}{w^3} \\ \frac{xz}{w^3} & \frac{yz}{w^3} & \frac{1}{w} + \frac{z^2}{w^3} \end{bmatrix}$$

The leading principal minors of H are $\frac{1}{w} + \frac{x^2}{w^3}$, $\frac{x^2y^2}{w^7}$, $\frac{1}{d^3} + \frac{x^2+y^2+z^2}{w^5} + \frac{x^2z^2+x^2y^2}{w^7}$, which are all positive. So f is convex on the domain defined by (3.1)-(3.3). The local minimum is then global minimum.

(b) $v_1 = (0, 0, 1, 0)^T$, $v_2 = (\frac{2\sqrt{2}}{3}, 0, \frac{1}{3}, 0)^T$, $v_3 = (-\frac{\sqrt{2}}{3}, -\sqrt{\frac{2}{3}}, \frac{1}{3}, 0)^T$, $v_4 = (\frac{\sqrt{2}}{3}, -\sqrt{\frac{2}{3}}, -\frac{1}{3}, 0)^T$, $v_5 = (\frac{1}{3\sqrt{2}}, -\frac{1}{\sqrt{6}}, \frac{1}{3}, \sqrt{\frac{2}{3}})^T$, $c = (x_c, y_c, z_c, \sqrt{1 - x_c^2 - y_c^2 - z_c^2})^T$. Then similarly

$$f_2(x_c, y_c, z_c) = \frac{1}{3\sqrt{2}}x_c - \frac{1}{\sqrt{6}}y_c + \frac{1}{3}z_c + \sqrt{\frac{2}{3}}\sqrt{1 - x_c^2 - y_c^2 - z_c^2}$$

is convex. So local minimum is the global minimum. The local minimum is achieved at $(-\frac{1}{3\sqrt{2}}, \frac{1}{\sqrt{6}}, -\frac{1}{3})$ and is equal to $\frac{1}{3}$. Since

$$D_{v'}f_2(-\frac{1}{3\sqrt{2}}, \frac{1}{\sqrt{6}}, -\frac{1}{3}) = \frac{\sqrt{2}}{3}(x + \frac{1}{3\sqrt{2}}) - \sqrt{\frac{2}{3}}(y + \frac{1}{\sqrt{6}}) + \frac{2}{3}(z + \frac{1}{3}) > 0$$

for any v' pointing from $(-\frac{1}{3\sqrt{2}}, \frac{1}{\sqrt{6}}, -\frac{1}{3})$ to any interior points.

(c) $v_1 = (0, 0, 1, 0)^T$, $v_2 = (\frac{2\sqrt{2}}{3}, 0, \frac{1}{3}, 0)^T$, $v_3 = (-\frac{\sqrt{2}}{3}, \sqrt{\frac{2}{3}}, \frac{1}{3}, 0)^T$, $v_4 = (\frac{\sqrt{2}}{3}, \sqrt{\frac{2}{3}}, -\frac{1}{3}, 0)^T$,

$v_5 = (\frac{1}{3\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{3}, \sqrt{\frac{2}{3}})^T$, $c = (x_c, y_c, z_c, -\sqrt{1 - x_c^2 - y_c^2 - z_c^2})^T$. Then

$$f_3(x_c, y_c, z_c) = -(\frac{1}{3\sqrt{2}}x_c + \frac{1}{\sqrt{6}}y_c + \frac{1}{3}z_c - \sqrt{\frac{2}{3}}\sqrt{1 - x_c^2 - y_c^2 - z_c^2})$$

is convex. Its local minimums is achieved at $(\frac{1}{3\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{3})$ and is equal to $\frac{1}{3}$. Since

$$D_{v'}f_3(\frac{1}{3\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{3}) = -(\frac{\sqrt{2}}{3}(x + \frac{1}{3\sqrt{2}}) + \sqrt{\frac{2}{3}}(y + \frac{1}{\sqrt{6}}) + \frac{2}{3}(z + \frac{1}{3})) > 0$$

for any v' pointing from $(-\frac{1}{3\sqrt{2}}, \frac{1}{\sqrt{6}}, -\frac{1}{3})$ to any interior points.

(d) $v_1 = (0, 0, 1, 0)^T$, $v_2 = (\frac{2\sqrt{2}}{3}, 0, \frac{1}{3}, 0)^T$, $v_3 = (-\frac{\sqrt{2}}{3}, -\sqrt{\frac{2}{3}}, \frac{1}{3}, 0)^T$, $v_4 = (\frac{\sqrt{2}}{3}, -\sqrt{\frac{2}{3}}, -\frac{1}{3}, 0)^T$, $v_5 = (\frac{1}{3\sqrt{2}}, -\frac{1}{\sqrt{6}}, \frac{1}{3}, \sqrt{\frac{2}{3}})^T$, $c = (x_c, y_c, z_c, -\sqrt{1 - x_c^2 - y_c^2 - z_c^2})^T$. Then

$$f_4(x_c, y_c, z_c) = -(\frac{1}{3\sqrt{2}}x_c - \frac{1}{\sqrt{6}}y_c + \frac{1}{3}z_c - \sqrt{\frac{2}{3}}\sqrt{1 - x_c^2 - y_c^2 - z_c^2})$$

is convex. Its local minimums is achieved at $(\frac{1}{3\sqrt{2}}, -\frac{1}{\sqrt{6}}, \frac{1}{3})$ and is equal to $\frac{1}{3}$. Since

$$D_{v'}f_4(\frac{1}{3\sqrt{2}}, -\frac{1}{\sqrt{6}}, +\frac{1}{3}) = -(\frac{\sqrt{2}}{3}(x + \frac{1}{3\sqrt{2}}) - \sqrt{\frac{2}{3}}(y + \frac{1}{\sqrt{6}}) + \frac{2}{3}(z + \frac{1}{3})) > 0$$

for any v' pointing from $(-\frac{1}{3\sqrt{2}}, \frac{1}{\sqrt{6}}, -\frac{1}{3})$ to any interior points.

So the system of inequalities has no solution. There exist no c_1 that satisfies the assumption. So $\tilde{U}$ is not equiangular.

$\square$

For frame $U = \{u_i\}_{i=1}^6$, we consider the graph with the vectors as vertices, and connect u_i, u_j with an edge when $|\langle u_i, u_j\rangle| < \alpha$. By Lemma 3.4.4, each vertex is connected to at least 4 edges. By [47], besides the equiangular frame, 3 possible simple graphs satisfy the condition in Lemma 3.4.4. Without loss of generality, we can put the corresponding Gram matrices into 3 cases. Denote $\alpha > 0$ the maximum correlation, and $x, y, z \in (-\alpha, \alpha)$. Then the possible Gram matrix of the frames is in one of the following forms:

$$G_1 = \begin{bmatrix} 1 & \alpha & \alpha & \alpha & \alpha & x \\ \alpha & 1 & y & \pm\alpha & \pm\alpha & \pm\alpha \\ \alpha & y & 1 & \pm\alpha & \pm\alpha & \pm\alpha \\ \alpha & \pm\alpha & \pm\alpha & 1 & z & \pm\alpha \\ \alpha & \pm\alpha & \pm\alpha & z & 1 & \pm\alpha \\ x & \pm\alpha & \pm\alpha & \pm\alpha & \pm\alpha & 1 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} 1 & \alpha & \alpha & \alpha & \alpha & \alpha \\ \alpha & 1 & \pm\alpha & \pm\alpha & \pm\alpha & \pm\alpha \\ \alpha & \pm\alpha & 1 & \pm\alpha & \pm\alpha & x \\ \alpha & \pm\alpha & \pm\alpha & 1 & y & \pm\alpha \\ \alpha & \pm\alpha & \pm\alpha & y & 1 & \pm\alpha \\ \alpha & \pm\alpha & x & \pm\alpha & \pm\alpha & 1 \end{bmatrix}$$

$$G_3 = \begin{bmatrix} 1 & \alpha & \alpha & \alpha & \alpha & \alpha \\ \alpha & 1 & \pm\alpha & \pm\alpha & \pm\alpha & \pm\alpha \\ \alpha & \pm\alpha & 1 & \pm\alpha & \pm\alpha & \pm\alpha \\ \alpha & \pm\alpha & \pm\alpha & 1 & \pm\alpha & \pm\alpha \\ \alpha & \pm\alpha & \pm\alpha & \pm\alpha & 1 & x \\ \alpha & \pm\alpha & \pm\alpha & \pm\alpha & x & 1 \end{bmatrix}$$

If the equiangular frame is not $(6,4)$-Grassmannian, then there exist a Gram matrix G such that $G \in \{G_1, G_2, G_3\}$ and

1. G is positive semi-definite;
2. $rank(G) \leq 4$;
3. $\alpha < \frac{1}{3}$.

Remark 3.4.5. In [48], the construction of $(6,4)$-Grassmannian frames is provided independently.

Theorem 3.4.6 ([48])**.** *The* $(6,4)$*-Grassmannian frames are the equiangular frames and is unique up to isometry.*

$$\mu_{6,4} = \frac{1}{3}.$$

3.5 Problems related to Grassmannian frames and equiangular lines

In the proof of Lemma 3.4.4, we used the information on the configurations of 6 equiangular lines in $\mathbb{R}^4$, and 5 equiangular lines in $\mathbb{R}^4$. In order to generalize the

Lemma to $(d+2,d)-$Grassmannian frames, we need the configurations of $(d+2)$ and $(d+1)$ equiangular lines in $\mathbb{R}^d$. This leads to some unsolved problems.

First, the d+1 equiangular tight frames are characterized as regular simplex. This can be shown applying the Naimark's Theorem.

Theorem 3.5.1 ([23] Naimark's Theorem). *A family of vectors $\{f_m\}_{m=1}^M$ is a Parseval frame for an $\mathbb{R}^N$ if and only if there is a an orthonormal projection P on $\mathbb{R}^M$ satisfying $Pe_m = f_m$ for all $m = 1, \cdots, M$ where $\{e_m\}_{m=1}^M$ is an orthonormal basis for $\mathbb{R}^N$.*

Furthermore, the complement preserves the "equiangular" property of the original frames.

Corollary 3.5.2 ([23]). *If $\{f_m\}_{m=1}^M$ is an equiangular tight frame for $\mathbb{R}^N$ with $Pe_m = \sqrt{\frac{N}{N}} f_m$, then $\{\sqrt{\frac{M}{M-N}}(I-P)e_m\}_{m=1}^M$ is an equiangular tight frame for $\mathbb{R}^{M-N}$. This is called the complementary equiangular tight frame.*

The ETFs with $d+1$ vectors in $\mathbb{R}^d$ are Naimark complements of ETF with $d+1$ vectors in $\mathbb{R}$, i.e. $\{1\}_{i=1}^{d+1}$. However if we remove the tight frame condition, there exist other possible configuration, as we can see in [39], there are 5 possible angles for set of 5 equiangular lines in $\mathbb{R}^4$. This leads to the first question.

Problem 3.5.3. *Is it possible to characterize the configurations of $d+1$ equiangular lines in $\mathbb{R}^d$.*

In [21], the concept of Naimark complement is extended to any frame in real space, using the fact that for any frame it is possible to construct a tight frame that contains it. It is then natural to ask whether the extended notion of Naimark complement would help with characterizing the equiangular lines. However the answer is not obvious. Since the complement of $\{f_n\}_{n=1}^{d+1} \subset \mathbb{R}^d$ is in a space of dimension $2d + 1 - K$. Denote F the synthesis operator of $\{f_n\}_{n=1}^{d+1}$, then K is the multiplicity of the largest eigenvalue of FF^*. To further explore this problem, we may start with the spectrum of FF^*, where F is the synthesis operator of a equiangular frame.

Our second problem considers $(d+2, d)$-Grassmannian frames and equiangular lines. By [23, Theorem 5.1] part (8), ETF with $d + 2$ vectors does not exist in $\mathbb{R}^d$. Since if it exists, then its Naimark complement is a equiangular tight frame of $d+2$ vectors in $\mathbb{R}^2$, which does not exist. Then we would like to ask what is the $(d+2, d)$-Grassmannian frames. One candidate is the $(d+2, d)$ equiangular lines. It is proved in [10] that $(5, 3)$-Grassmannian frames are equiangular frames. We would like to know whether that is true for any d.

Problem 3.5.4. *When does a $d + 2$ equiangular frame exist in $\mathbb{R}^d$? If they exist, are they the $(d + 2, d)$-Grassmannian frame?*

The third problem is how to determine whether (N, d) Grassmannian frames are tight frames. This question is also discussed in detail in [40].

Let $\overline{\Omega_{N,d}}(\mathbb{F})$ denote the space of unit-norm frames for $\mathbb{F}^d$ consisting of N vectors and let $\Omega_{N,d}(\mathbb{F})$ denote the space of unit-norm, tight frames for $\mathbb{F}^d$ consisting of N vectors. [40] gave the following definition.

Definition 3.5.5. The 1-Grassmannian constant is

$$\mu_{N,d}(\mathbb{F}) = \min_{\Phi \in \Omega_{N,d}(\mathbb{F})} \mu(\Phi).$$

And a frame $\Phi \in \Omega_{N,d}$ is a 1-Grassmannian frame if

$$\mu(\Phi) = \mu_{N,d}(\mathbb{F}).$$

Then Naimark's Theorem gives

Theorem 3.5.6 ([40]). *If a 1-Grassmannian frame $\Phi \in \Omega_{N,d}(\mathbb{F})$ has coherence $\mu_{N,d}(\mathbb{F})$, then a 1-Grassmannian frame $\Phi' \in \Omega_{N,N-d}(\mathbb{F})$ exists, and its coherence is $\frac{d}{N-d}\mu_{N,d}(\mathbb{F})$. More succinctly,*

$$\mu_{N,N-d}(\mathbb{F}) = \frac{d}{N-d}\mu_{N,d}(\mathbb{F}).$$

So we can determine whether (N, d) Grassmannian frames are tight without constructing the frame, given information on $(N, N-d)$-Grassmannian frames.

Example 3.5.7. We can determine whether the $(6, 4)$ Grassmannian frame is tight with Theorem 3.5.6. It is known that $\mu_{6,2}(\mathbb{R}) = \cos(\pi/6) = \sqrt{3}/2$. Then

$$\mu_{6,4} = \frac{2}{4}\mu_{6,2}(\mathbb{R}) = \frac{\sqrt{3}}{4} > \frac{1}{3}.$$

So $(6, 4)$-Grassmannian frames are not tight.

Chapter 4: p-Frame Potential of Finite Gabor Frames

4.1 Introduction and background

The Zauner Conjecture, which concerns ETFs with d^2 vectors in $\mathbb{C}^d$, is still open. Neither the construction nor the existence of SIC-POVMs in all $\mathbb{C}^d$ is established. We would like to approach the conjecture with alternate ways by asking three different questions. First, we would like to know if we can instead characterize POVMs that are informationally complete. Second, whether we can prove the existence by linking the SIC-POVMs to the minimizers of p-frame potential, which always exist. Third, whether it is possible to find frames that have a small number of different inner products. In this chapter we will further discuss topics related to these three questions.

In Section 4.2, we compute the spectrum of the Gram matrix of a finite Gabor frame. Section 4.3 discusses some known bounds for the p-frame potentials. Section 4.4 is focusd on the relation between spherical design and the optimizers of p-frame potentials. In section 4.5, we compute the inner products of vectors in Gabor frames that are generated by two type of special sequences. Section 4.3, 4.3, 4.5 deal with the three questions discussed in the previous paragraph correspondingly. We discuss some possible ways to approach the questions in Section 4.6.

4.2 Spectrum of Gram matrices

In this section we analyze the spectrum of the Gram matrix of a finite Gabor frame. The motivation is to characterize finite Gabor frames that are informationally complete. Recall that a set of operators $\{\Pi_{k,l}\}_{k,l=0}^{d-1}$ is informationally complete if it is linearly independent.

$\Phi = \{M^k T^l \phi\}_{(k,l)\in\mathbb{Z}^d\times\mathbb{Z}^d}$ is the orbit of ϕ under Weyl-Heisenberg group action. For convenience, we index the vectors in a finite Gabor frame Φ as $\phi_{dk+l} \equiv M^k T^l \phi$. Then the operator corresponding to ϕ_i is $E_i = \frac{1}{d}\phi_i \otimes \phi_i \equiv \frac{1}{d}\Pi_i$. In order to determine whether the operators $\{\Pi_{k,l}\}_{k,l=0}^{d-1}$ form a linear independent set, we can define its Gram matrix similarly as in $\mathbb{C}^d$ with the Frobenius inner product.

Definition 4.2.1 ([44]). The Frobenius inner product of two $m \times n$ matrices A and B is

$$\langle A, B\rangle_F = tr(B^*A) = \sum A_{i,j}\overline{B_{i,j}}$$

The Frobenius inner product is indeed an inner product in $M_{m\times n}(\mathbb{C})$, the space of $m \times n$ matrices. Suppose $a_i, b_i \in \mathbb{C}^m$ for $i = 1, \cdots, d$, and define the matrices $A = [a_1, \cdots, a_n]$, $B = [b_1, \cdots, b_n]$. Then $w_A = [a_1^T \cdots a_n^T]^T$, $w_B = [b_1^T \cdots b_n^T]^T$ are vectors in $\mathbb{C}^{mn}$, and

$$\langle w_A, w_B\rangle = \sum A_{i,j}\overline{B_{i,j}} = \langle A, B\rangle_F.$$

So the Frobenius inner product is equivalent to the inner product defined on $\mathbb{C}^{d^2}$ if we view the matrices $\Pi_{k,l}$ as vectors in $\mathbb{C}^{d^2}$.

Since a set of vectors is linear independent if and only if its Gram matrix is nonsingular, we can determine the linearly independence of $\{\Pi_{k,l}\}_{k,l=0}^{d-1}$ by examining the Gram matrix of the operators. First, we can compute the entries of the Gram matrix.

Proposition 4.2.2. *Let $\phi \in \mathbb{C}^d$ be a unit vector and $\Pi_{k,l} = M^kT^l\phi \otimes M^kT^l\phi$, G be the Gram matrix of $\{\Pi_{k,l}\}_{k,l=0}^{d-1}$. Then the $dk+l, dk'+l'$-th entry of G is*

$$G_{dk+l,dk'+l'} = |\langle \phi, M^{k'-k}T^{l'-l}\phi\rangle|^2, \tag{4.1}$$

where $k, k' \in \{0, \cdots, d-1\}$.

Proof.

$$\begin{aligned}
G_{dk+l,dk'+l'} &= \langle \Pi_{k,l}, \Pi_{k',l'}\rangle_F = \sum_{i,j} (\Pi_{k,l})_{i,j}\overline{(\Pi_{k',l'})_{i,j}} \\
&= \sum_{i,j} (\phi_{dk+l} \otimes \phi_{dk+l})_{i,j}\overline{(\phi_{dk'+l'} \otimes \phi_{dk'+l'})_{i,j}} \\
&= |\langle \phi_{dk+l}, \phi_{dk'+l'}\rangle|^2 = |\langle M^kT^l\phi, M^{k'}T^{l'}\phi\rangle|^2 = |\langle \phi, M^{k'-k}T^{l'-l}\phi\rangle|^2.
\end{aligned}$$

□

We observe from Proposition 4.2.2 that each row of G is a rearrangement of the first row of G. Entries of G follow a certain pattern.

Proposition 4.2.3. *G is a block circulant matrix with circulant blocks.*

Proof. The Gram matrix of $\{\Pi_{k,l}\}_{k,l=0}^{d-1}$ is a $d^2 \times d^2$ matrix. We separate G into d^2 blocks as following:

$$G = \begin{bmatrix} A_{0,0} & A_{0,1} & A_{0,2} & \cdots & A_{0,d-1} \\ A_{1,0} & A_{1,1} & A_{1,2} & \cdots & A_{1,d-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{d-1,0} & A_{d-1,1} & A_{d-1,2} & \cdots & A_{d-1,d-1} \end{bmatrix}$$

Each of the block is a $d \times d$ submatrix of G. The l, l'-th entry in block $A_{k,k'}$ is then $G_{dk+l,dk'+l'}$.

First we show that G is block circulant, that is, $A_{k,k'} = A_{k+1,k'+1}$ for any $k, k' \in \mathbb{Z}/d\mathbb{Z}$. For any $l, l' \in \mathbb{Z}/d\mathbb{Z}$ the l, l'-entry in $A_{k+1,k'+1}$ is

$$G_{d(k+1)+l,d(k'+1)+l} = |\langle M^{k+1}T^l\phi, M^{k'+1}T^{l'}\phi\rangle|^2 = |\langle \phi, M^{k'-k}T^{l'-l}\phi\rangle|^2.$$

Which is equal to the l, l'-entry in $A_{k,k'}$. So the Gram matrix is a block circulant matrix. And we have $A_{k,k'} = A_{0,k'-k}$. For simplicity, we denote $A_k \equiv A_{0,k}$. G can be then written as

$$G = \begin{bmatrix} A_0 & A_1 & A_2 & \dots & A_{d-1} \\ A_{d-1} & A_0 & A_1 & \dots & A_{d-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_1 & A_2 & A_3 & \dots & A_0 \end{bmatrix}.$$

Second we show each block itself is circulant. Without loss of generality, we only need to show that A_k is circulant when $k \in \mathbb{Z}/d\mathbb{Z}$. For any $k, l, l' \in \mathbb{Z}/d\mathbb{Z}$, the

$l+1, l'+1$-th entry of A_k is

$$A_{l+1,dk+l'+1} = |\langle T^l\phi, M^kT^{l'}\phi\rangle|^2 = |\langle \phi, M^kT^{l'-l}\phi\rangle|^2 = A_{l,dk+l'}.$$

Where $A_{l,dk+l'}$ is the l,l-th entry of A_k. Each block is circulant. So the block A_k is of the form

$$A_k = \begin{bmatrix} A_k^0 & A_k^1 & A_k^2 & \cdots & A_k^{d-1} \\ A_k^{d-1} & A_k^0 & A_k^1 & \cdots & A_k^{d-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_k^1 & A_k^2 & A_k^3 & \cdots & A_k^0 \end{bmatrix}$$

□

First we need the following well known result (see e.g. [38]).

Theorem 4.2.4. *Let*

$$B = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \\ b_{n-1} & b_0 & b_1 & \cdots & b_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_1 & b_2 & b_3 & \cdots & b_0 \end{bmatrix}$$

be a $n \times n$ circulant matrix. Then its n eigenvalues are

$$\lambda_j = \sum_{j=0}^{n-1} b_j \omega^j$$

for $j \in \mathbb{Z}/n\mathbb{Z}$.

Now we can calculate the eigenvalues of G. The following result is well known (see [27, 60]). It is an extension of the method used to calculate the eigenvalues of circulant matrices. Denote DFT_n the $n \times n$ DFT matrix. Where $DFT_n = (\frac{\omega^{jk}}{\sqrt{n}})_{j,k \in \mathbb{Z}/n\mathbb{Z}}$ and ω is the n-th root of unity. We can compute the spectrum of a matrix that is block circulant with circulant blocks.

Theorem 4.2.5. *Let G be a block circulant matrix with circulant blocks as in (4.2). G can be diagonalized by $DFT_d \otimes DFT_d$. The eigenvalues of G are*

$$\lambda_{a,b} = \sum_{l=0}^{d-1} \sum_{k=0}^{d-1} \omega^{ak+bl} A_k^l, \quad i, j \in \{0, 1, \cdots d-1\}.$$

Proof. For any $a \in \mathbb{Z}/d\mathbb{Z}$, consider the functions $h_a : \mathbb{C}^d \to \mathbb{C}^{d^2}$ defined as following

$$h_a(v) = (v^T, \rho_a v^T, \rho_a^2 v^T, \cdots, \rho_a^{d-1} v^T),$$

where $\rho_a = \omega^a$. We first prove the claim that the column vectors of $DFT_d \otimes DFT_d$ are linearly independent and are eigenvectors of G. As a result of this claim the matrix G can be diagonalized by $DFT_d \otimes DFT_d$.

Denote $H_a = A_0 + A_1\rho_a + A_2\rho_a^2 + \cdots + A_{d-1}\rho_a^{d-1}$. Each H_i is a circulant matrix, thus can be diagonalized by the $d \times d$ DFT matrix DFT_d. Suppose v is an eigenvector of H_a and $H_a v = \lambda v$. Then v is a column vector of DFT_d and

$$Gh_a(v) = \begin{bmatrix} A_0 & A_1 & A_2 & \dots & A_{d-1} \\ A_{d-1} & A_0 & A_1 & \dots & A_{d-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_1 & A_2 & A_3 & \dots & A_0 \end{bmatrix} \begin{bmatrix} v \\ \rho_a v \\ \vdots \\ \rho_a^{d-1} v \end{bmatrix} = \begin{bmatrix} H_a v \\ \rho_a H_a v \\ \dots \\ \rho_a^{d-1} H_a v \end{bmatrix} = \lambda h_a(v).$$

So the vectors $h_a(v)$, which by definition are exactly the columns of $DFT_d \otimes DFT_d$, are eigenvectors of G. Since $det(DFT_d \otimes DFT_d) = det(DFT_d)^{2d} \neq 0$, $DFT_d \otimes DFT_d$ is invertible. The matrix G can be diagonalized by $DFT_d \otimes DFT_d$.

By (4.2), the eigenvalues of G are the collections of eigenvalues of $\{H_a\}_{a=1}^{d}$. Denoting the 1,n-th entry in H_a as H_a^n, $n \in \{0, 1, \cdots, d-1\}$. The d eigenvalues of H_a are:

$$\begin{aligned} \lambda_{a,b} &= H_a^0 + \rho_b H_a^1 + \rho_b^2 H_a^2 + \cdots + \rho_b^{d-1} H_a^{d-1} \\ &= \sum_{l=0}^{d-1} \rho_b^l H_a^l = \sum_{l=0}^{d-1} \rho_b^l (\sum_{k=0}^{d-1} \rho_a^k A_k^l) = \sum_{l=0}^{d-1} \sum_{k=0}^{d-1} \rho_b^k \rho_a^l A_l^k \\ &= \sum_{l=0}^{d-1} \sum_{k=0}^{d-1} \omega^{ak+bl} A_k^l. \end{aligned}$$

The spectrum of G is then $\{\lambda_{a,b} = \sum_{l=0}^{d-1} \sum_{k=0}^{d-1} \omega^{ak+bl} A_k^l\}_{a,b=0}^{d-1}$. □

We can now compute the spectrum of the Gram matrix of $\{\Pi_{k,l}\}_{k,l=0}^{d-1}$ with the general result for block circulant matrices with circulant blocks.

Corollary 4.2.6. *Let $\phi \in \mathbb{C}^d$ be a unit vector and $\Pi_{k,l} = M^k T^l \phi \otimes M^k T^l \phi$, G be* *the Gram matrix of* $\{\Pi_{k,l}\}_{k,l=0}^{d-1}$. *Then*

$$\lambda_{a,b} = \begin{cases} \sum\limits_{k=0}^{d-1} \omega^{bl} |\langle g, T^l g\rangle|^2 + \sum\limits_{k=1}^{(d-1)/2} \sum\limits_{l=0}^{d-1} \cos\left(\frac{2\pi(ak+bl)}{d}\right) |\langle g, M^k T^l g\rangle|^2, & d \text{ odd} \\ \sum\limits_{l=0}^{d-1} \omega^{bl} |\langle g, T^l g\rangle|^2 + \sum\limits_{k=1}^{d/2-1} \sum\limits_{l=0}^{d-1} \cos\left(\frac{2\pi(ak+bl)}{d}\right) |\langle g, M^k T^l g\rangle|^2 + ... & \\ \sum\limits_{k=0}^{d-1} \omega^{ad/2+bl} |\langle g, M^{d/2} T^l g\rangle|^2, & d \text{ even} \end{cases}$$

Proof. 1. If d is odd, since $G = G^T$,

$A_0 = A_0^T$, $A_1 = A_{d-1}^T$, ... ,$A_{(d-1)/2} = A_{(d+1)/2}^T$. Then for $k = 1, \cdots, (d-1)/2$, $l = 0, \cdots, (d-1)/2$, we have $A_k^l = A_{d-k}^{d-l}$. For $l = 0, \cdots, (d-1)/2$, we have $A_0^l = A_0^{d-l}$. Applying Theorem 4.2.5,

$$\begin{aligned} \lambda_{a,b} &= \sum_{l=0}^{d-1} \omega^{bl} A_0^l + \sum_{k=1}^{(d-1)/2} \sum_{l=0}^{d-1} (\omega^{ak+bl} + \omega^{-(ak+bl)}) A_k^l \\ &= \sum_{k=0}^{d-1} \omega^{bl} |\langle g, T^l g\rangle|^2 + \sum_{k=1}^{(d-1)/2} \sum_{l=0}^{d-1} \cos\left(\frac{2\pi(ak+bl)}{d}\right) |\langle g, M^k T^l g\rangle|^2. \end{aligned}$$

2. If d is even, since $G = G^T$,

$A_0 = A_0^T$, $A_1 = A_{d-1}^T$, ... , $A_{d/2-1} = A_{d/2+1}^T$, $A_{d/2} = A_{d/2}^T$, since $G = G^T$.

Then for $k = 1, \cdots, d/2-1$, $l = 0, \cdots, d/2-1$, we have $A_k^l = A_{d-k}^{d-l}$. For

$k = 0, d/2, l = 0, \cdots, (d-1)/2$, we have $A_k^l = A_k^{d-l}$. Applying Theorem 4.2.5,

$$\begin{aligned}
\lambda_{a,b} &= \sum_{l=0}^{d-1} \omega^{bl} A_0^l + \sum_{k=1}^{d/2-1} \sum_{l=0}^{d-1} (\omega^{ak+bl} + \omega^{-(ak+bl)}) A_k^l + \sum_{l=0}^{d-1} \omega^{ad/2+bl} A_{d/2-1}^l \\
&= \sum_{l=0}^{d-1} \omega^{bl} |\langle g, T^l g\rangle|^2 + \sum_{k=1}^{d/2-1} \sum_{l=0}^{d-1} \cos\left(\frac{2\pi(ak+bl)}{d}\right) |\langle g, M^k T^l g\rangle|^2 + ... \\
&\sum_{k=0}^{d-1} \omega^{ad/2+bl} |\langle g, M^{d/2} T^l g\rangle|^2
\end{aligned}$$

□

4.2.1 Future problem: applying Inverse Function Theorem

One question we would like to consider is whether it is possible to construct a Gabor frame $\{M^k T^l g\}_{k,l=0}^{d}$ such that there is a ball $B \in \mathbb{C}^d$ with small radius and $\langle g, M^k T^l g\rangle \in B$ for all $k, l = 0, \cdots, d-1$. In this subsection we suggest applying the Inverse Function Theorem(IVT) and state the problem.

Define $F : \mathbb{C}^d \to \mathbb{C}^{d^2}$ as $F(g) = \{\langle g, M^k T^l g\rangle\}_{k,l=0}^{d-1}$, and let $F_{k,l}(g) = \langle g, M^k T^l g\rangle$. F has the following property.

Proposition 4.2.7. *F is real differentiable.*

Proof. Let $h \in \mathbb{C}^d$. Then

$$F(g+h) - F(g) = \{\langle g, M^k T^l h\rangle + \langle h, M^k T^l g\rangle + \langle h, M^k T^l h\rangle\}_{k,l=0}^{d-1}$$

Define $DF(g)(h) = \{\langle g, M^k T^l h\rangle + \langle h, M^k T^l g\rangle\}_{k,l=0}^{d-1}$. $DF(g) : \mathbb{C}^d \to \mathbb{C}^{d^2}$ is

$\mathbb{R}$-linear (but not $\mathbb{C}$-linear), since if $c \in \mathbb{R}$,

$$
\begin{aligned}
DF(g)(ch) &= \{\langle g, cM^kT^lh\rangle + \langle ch, M^kT^lg\rangle\}_{k,l=0}^{d-1} \\
&= \{c\langle g, M^kT^lh\rangle + c\langle h, M^kT^lg\rangle\}_{k,l=0}^{d-1}
\end{aligned}
$$

And $\lim_{\|h\|\to 0} \frac{\|F(g+h)-F(g)-DF(g)(h)\|}{\|h\|} = \lim_{\|h\|\to 0} \frac{\|\{\langle h, M^kT^lh\rangle\}_{k,l=0}^{d-1}\|}{\|h\|} = 0.$ □

For any $g \in \mathbb{C}^d$, $DF(g)$ is continuous. So we want to see whether we can get any conclusion by applying Inverse Function Theorem (for example guarantee F^{-1} exists on some set near or include the point $\frac{e^{i\theta_1}}{\sqrt{d+1}}, \frac{e^{i\theta_2}}{\sqrt{d+1}}, \dots)$).

Theorem 4.2.8 (Inverse Function Theorem)**.** *Suppose X and Y are Banach spaces, $U \subset X$ is open, $f \in C^1$, $x_0 \in U$ and $Df(x_0)$ is invertible. Then there is a ball $B = B(x_0, r)$ in U centered at x_0 such that*

1. *$V = f(B)$ is open,*

2. *$f|_B : B \to V$ is a homeomorphism,*

3. *$g = (f|_B)^{-1} \in C^k(V, B)$ and $g'(y) = [f'(g(y))]^{-1}$ for all $y \in V$.*

To apply this theorem, we need a $g \in \mathbb{C}^d$ such that $DF(g)$ is invertible. This is difficult because F is not complex differentiable and we want to separate it into real and imaginary part and consider it to be a $\mathbb{R}^{2d} \to \mathbb{R}^{2d^2}$ function. Furthermore, to achieve our goal, it's necessary to find the corresponding neighborhood U of g and the diameter of the corresponding $V = F(U)$, but the theorem and its proof does not provide much information on V besides it being an open set.

4.3 p-Frame potentials of Gabor frames

In this section we will investigate the lower bound of p-frame potential of Gabor frames in $\mathbb{C}^d$. We separate the problem into three cases: $0 < p < 2$, $p = 2$ and $p > 2$. For $p \in (0, \infty]$ and $d \geq 2$, define $Z_{p,d}$ on unit sphere by

$$Z_{p,d}(g) \equiv \sum_{k,l=0}^{d-1} |\langle g, M^k T^l g\rangle|^p = \frac{2\,\mathrm{FP}_{p,d^2,d}(\{M^k T^l g\}_{k,l=0}^{d-1}) + d^2}{d^2}$$

4.3.1 p=2

Under the special case $p = 2$. The minimizers of 2-frame potential among all frames are characterized in [8].

Theorem 4.3.1 ([8],Theorem 6.2)**.** *Given any d and N, let $\{x_i\}_{i=1}^N \subset \mathbb{C}^{d-1}$. Then*

$$\max\big(0, \frac{N^2 - dN}{2d}\big) \leq \mathrm{FP}_{2,N,d}(\{x_i\}_{i=1}^N) \leq N^2.$$

The lower bounds of N and N^2/d are achieved if and only if $\{x_i\}$ is an orthonormal set or a unit norm tight frame in $\mathbb{C}^d$, respectively.

For the Gabor Frames, we make the following observation.

Proposition 4.3.2. *Suppose $g \in S^{d-1}$, then $\{M^k T^l g\}_{k,l=0}^{d-1}$ is tight frame.*

Proof. To prove this we can show that the frame operator of $\{M^k T^l g\}_{k,l=0}^{d-1}$ is a constant times identity. It is sufficient to have

$$\sum_{k,l} \langle e_j, M^k T^l g\rangle M^k T^l g = C e_j, \quad \forall j \in \{0, 1, \cdots, d-1\}$$

for some constant C, where $\{e_j\}_{j=0}^{d-1}$ is the standard orthonormal basis of $\mathbb{C}^d$.

$$\begin{aligned}
\Big(\sum_{k,l} \langle e_j, M^k T^l g\rangle M^k T^l g\Big)_m &= \sum_{k,l} g_{j-l}\overline{g_{m-l}} e^{2\pi i(m-j)k/d} \\
&= \begin{cases} 0, & m \neq j, \\ d, & m = j. \end{cases}
\end{aligned}$$

Thus

$$\sum_{k,l} \langle e_j, M^k T^l g\rangle M^k T^l g = d e_j, \quad \forall j \in \{0, 1, \cdots, d-1\}.$$

$\square$

Remark 4.3.3. When $p = 2$, by Theorem 4.3.1 and Proposition 4.3.2 we conclude that for any $g \in \mathbb{S}^{d-1}$, $Z_{2,d}(g)$ is a constant and

$$Z_{2,d}(g) = \frac{1}{d}.$$

4.3.2 $p > 2$

Before we compute the lower bound for $Z_{p,d}$ for $p > 2$, we first show a general lower bound for the p-frame potential among any frames. The following lower bound is proved in [32].

Theorem 4.3.4. *Suppose $p > 2$ is any real number. For any frame $X = \{x_i\}_{i=1}^N \subset S^{d-1}$. Then*

$$\sum_{i<j} |\langle x_i, x_j\rangle|^p \geq \left(\frac{2N}{d(N-1)}\right)^{p/2} \binom{N}{2}.$$

To prove this theorem, we need the following well known result.

Proof of Theorem 4.3.4.

$$\begin{aligned}\left(\frac{\sum_{i<j}(|\langle x_i, x_j\rangle|^2)^{p/2}}{\binom{N}{2}}\right)^{2/p} &\geq \frac{\sum_{i<j}|\langle x_i, x_j\rangle|^2}{\binom{N}{2}} \\ &\geq \frac{(N^2/d - N)/2}{\binom{N}{2}} = \frac{N-d}{d(N-1)}. \end{aligned} \tag{4.2}$$

We have

$$\sum_{i<j} |\langle x_i, x_j\rangle|^p \geq \left(\frac{N-d}{d(N-1)}\right)^{p/2} \binom{N}{2}$$

The equality holds if and only if X is a ETF. $\square$

As a result we can get a lower bound for any Gabor frames generated by $g \in \mathbb{C}^d$.

Corollary 4.3.5. *Suppose $p > 2$ is any real number and $g \in \mathbb{C}^d$, then $Z_{p,d}(g) \geq \frac{d-1}{(d+1)^{p/2-1}} + 1$. The equality holds if and only if $X = \{M^k T^l g\}_{k,l=0}^{d-1}$ is ETF.*

Proof.

$$
\begin{aligned}
Z_{p,d}(g) &\geq \frac{2\left(\frac{d^2-d}{d(d^2-1)}\right)^{p/2}\binom{d^2}{2}+d^2}{d^2} \\
&= \frac{\frac{d^2(d^2-1)}{(d+1)^{p/2}}+d^2}{d^2} \\
&= \frac{d-1}{(d+1)^{p/2-1}}+1
\end{aligned}
$$

The condition equality hold is the same as in Theorem 4.3.4. □

Corollary 4.3.5 shows that the lower bound of $Z_{p,d}$ is achieved if and only if for a fixed d, there exist Gabor frames that are also ETFs. Zauner conjectured that Gabor frames that are ETFs exist for any d. The conjecture is still open, so it is not known whether the bound in Corollary 4.3.5 can be achieved for all d. However, the minimizers of $Z_{p,d}$ exist by a compactness argument. So a natural question to ask would be the following.

Problem 4.3.6. *Can the lower bound in Corollary 4.3.5 be achieved for all d? In another word, is it true that $g \in \mathbb{C}^d$ minimizes $Z_{p,d}$ if and only if the Gabor frame generated by g is ETF.*

This problem will be further discussed in Section 4.4.

4.3.2.1 Further questions

Theorem 4.3.5 solve the minimization problem for the dimensions which have known the exact construction of SIC-POVMs. In those dimensions, the minimizers of $Z_{p,d}$ when $2 < p < \infty$ also minimize $Z_{\infty,d}$. To better understand this minimization problem, it is then natural to ask whether the minimizers of $Z_{p,d}$ are the same for $2 < p < \infty$ and $p = \infty$ in the dimension when there is currently no known exact construction of SIC-POVM.

Another question is that, suppose g is a minimizer of $Z_{p,d}$, whether the operators $\{P_{k,l} = (M^k T^l g)(M^k T^l g)^*\}_{k,l=0}^{d-1}$ are linear independent. Denote the Gram matrix of $\{P_{k,l}\}_{k,l=0}^{d-1}$ as G. The $p/2$-th hadamard power of G, defined as $G^{(p/2)} = \{G_{i,j}^{p/2}\}_{i,j}$ is positive semidefinite, since G is positive semidefinite and all entries of G are non-negative. ([45])

Since $G^{(p/2)}$ is also a block circulant matrix with circulant blocks, by Theorem 4.2.2 the eigenvalues of $G^{(p/2)}$ are

$$\lambda_{a,b} = \sum_{k,l} \omega^{ak+bl} |\langle g, M^k T^l g\rangle|^p.$$

We had a few observations:

- Since $G^{(p/2)}$ is positive semidefinite, all $\lambda_{a,b}$ are non-negative.
- $Z_{p,d}(g)$ is the largest eigenvalue of $G^{(p/2)}$.

By the above observation, minimizing $Z_{p,d}$ can also be viewed as finding the lower bound for the largest eigenvalue of some symmetric positive semidefinite matrices that are block circulant with circulant blocks.

4.3.3 $0 < p < 2$

In this section we give a lower bound for $Z_{p,2}(g)$ when g is a unit vector in $\mathbb{R}^2$.

Theorem 4.3.7. *Suppose $0 < p < 2$ and $g \in \mathbb{R}^2$ is a unit vector, then $Z_{p,2}(g) \geq 2$. Equality holds when $g \in \{(1,0),(0,1),(-1,0),(0,-1)\}$.*

Proof. We prove this claim by applying method in calculus. Suppose $g = (x, y)$ with $x, y \in [-1, 1]$, then $|y| = (1 - x^2)^{1/2}$.

$$Z_{p,2} = 2^p |x(1-x^2)^{1/2}|^p + |2x^2 - 1|^p.$$

$Z_{p,2}$ is differentiable except at $x = 0$. For $x \neq 0$, let

$$\frac{dZ_{p,2}}{dx} = 2^p p |x(1-x^2)^{1/2}|^{p-1} sgn(x) \frac{1-2x^2}{(1-x^2)^{1/2}} + 4px|2x^2-1|^{p-1} sgn(2x^2-1) = 0 \tag{4.3}$$

Solving for (4.3), we have $x^2 = \frac{2 \pm \sqrt{2}}{4}$. So the critical points for $Z_{p,2}$ are $-1, 0, 1, \pm\sqrt{\frac{2\pm\sqrt{2}}{4}}$. Comparing $Z_{p,2}$ at these values, we get the minimum of $Z_{p,2}$ is 2 when $x = -1, 0, 1$.

□

4.3.3.1 Numerical result

Suppose $p = 1$ and $g \in \mathbb{C}^d$. Let $\{v_1, \cdots, v_d\}$ be the standard basis for $\mathbb{C}^d$. For $d = 2, 3, 4, 5$, matlab results show that $d \leq Z_{1,d} \leq \frac{d^2-1}{\sqrt{d+1}} + 1$. Which means $Z_{1,d}$ is minimized when $g \in \{e^{i\theta} v_i \mid \theta \in [0, 2\pi),\ and\ i = 1, \cdots, d\}$ and maximized when $\{M^k T^l g\}$ is equiangular tight frame.

Suppose $p = 3, 4$ and $g \in \mathbb{C}^d$. Numerical results show that for $d = 2, 3, 4, 5$, $Z_{p,d} \le d$ and is maximized when $g \in \{e^{i\theta} v_i \mid \theta \in [0, 2\pi),\ \textit{and } i = 1, \cdots, d\}$.

4.4 Optimization of $Z_{p,d}$ and spherical (t, t)-designs

In this section we give a more detailed discussion on the problem posed in Section 4.3.2: is it true that $g \in \mathbb{C}^d$ minimizes $Z_{p,d}$ if and only if $\{M^k T^l g\}_{k,l=0}^{d-1}$ is ETF. We will focus on one of the possible ways to connect the Zauner's conjecture and the minimization of $Z_{p,d}$ using the concept of spherical designs.

Denote g^* the minimizer for $Z_{p,d}$ on S^{d-1}. We first establish the existence of a minimizer g^* using the following proposition of $Z_{p,d}$.

Proposition 4.4.1. *$Z_{p,d}(g)$ is a continuous function of g under ℓ^1 norm.*

Proof. Let $g, g' \in \mathbb{S}^{d-1}$. Composition of continous functions is continuous, and $Z_{p,d} = \sum\limits_{k,l=0}^{d-1} f_1(\langle g, M^k T^l g \rangle)$ where $f_1 : \mathbb{C} \to \mathbb{R}$ is $f_1(t) = |t|^p$. Since f_1 is continuous function, we only need to prove that for any fixed k, l, $\langle g, M^k T^l g \rangle$ is continuous. Denote $g = (a_1, \cdots, a_d)$ and $g' = (a'_1, \cdots, a'_d)$.

$$\begin{aligned}
|\langle g, M^k T^l g \rangle - \langle g', M^k T^l g' \rangle| &= |\sum_{i=1}^{d} a_i a_{i+l} - a'_i a'_{i+l}| \\
&\le \sum_{i=1}^{d} |a_i a_{i+l} - a'_i a'_{i+l}| \\
&\le \sum (|a_i| \cdot |a_{i+l} - a'_{i+l}| + |a'_{i+l} - a_{i+l}| \cdot |a_i - a'_i| + |a_{i+l}| \cdot |a_i - a'_i|).
\end{aligned}$$

For any $\epsilon > 0$, let $\delta = \min\left(\frac{\epsilon}{3\|g\|_1}, \sqrt{\epsilon/3}\right)$. $|a_i| < \|g\|_1$ for any i. Then if $\|g - g'\|_1 < \delta$,

we have

$$|\langle g, M^kT^lg\rangle - \langle g', M^kT^lg'\rangle| \leq \|g-g'\|_1\|g\|_1 + \|g-g'\|_1\|g-g'\|_1 + \|g-g'\|_1\|g\|_1 < \epsilon.$$

$\langle g, M^kT^lg\rangle$ are continuous functions of g under ℓ^1 norm. So $Z_{p,d}$ are also continuous functions of g under ℓ^1 norm.

$\square$

In addition, we make the following observation.

Observation: For p an integer, $g_1, g_2 \in \mathbb{S}^{d-1}$, denote $h = g_1 - g_2$, then $Z_{p,d}$ has Lipschitz property.

Proof.

$$\begin{aligned}
|Z_{p,d}(g_1) - Z_{p,d}(g_2)| &= |\sum_{k,l} |\langle g_1, M^kT^lg_1\rangle|^p - \sum_{k,l} |\langle g_2, M^kT^lg_2\rangle|^p| \\
&\leq \sum_{k,l} |\langle g_1, M^kT^lg_1\rangle^p - \langle g_2, M^kT^lg_2\rangle^p| \\
&= \sum_{k,l} |\langle g_1, M^kT^lg_1\rangle - \langle g_2, M^kT^lg_2\rangle| \times |\sum_{q=0}^{p-1} \langle g_1, M^kT^lg_1\rangle^q \langle g_2, M^kT^lg_2\rangle^{p-1-q}| \\
&\leq p \sum_{k,l} |\langle g_1, M^kT^lg_1\rangle - \langle g_2, M^kT^lg_2\rangle| \\
&= p \sum_{k,l} |\langle h, M^kT^lg_2\rangle + \langle g_2, M^kT^lh\rangle + \langle h, M^kT^lh\rangle| \\
&\leq 3pd^2\|h\| = 3pd^2\|g_1 - g_2\|
\end{aligned}$$

$\square$

Remark 4.4.2. If we fix g and view $Z_{p,d}$ as a function of p, i.e. let $f_g(p) \equiv Z_{p,d}(g)$. Then for $p \in (0, \infty)$, $f_g(p)$ is a decreasing function. And we have $\lim_{p\to\infty} f_g(p) = (\sharp\ of\ (k,l)\ s.t.\ M^k T^l g = g)$. Since if $g \neq M^k T^l g$, then $|\langle g, M^k T^l g\rangle| < 1$ and $\lim_{p\to\infty} |\langle g, M^k T^l g\rangle|^p = 0$.

Since $Z_{p,d}$ is a continuous function and S^{d-1} is compact, minimizers of $Z_{p,d}$ always exist. We then would like to know whether the lower bound in Corollary 4.3.5 can always be achieved.

The relation between equiangular tight frames and spherical designs is established in [67]. In the rest of the section we will first describe the relation, then state our question that is equivalent to Problem 4.3.6 in terms of spherical designs and discuss our attempts of solving the problem.

Theorem 4.4.3 ([67], Thm 6.7)**.** *For any $\{f_i\}_{i=1}^N \subset \mathbb{C}^d$ and positive integer t,*

$$\sum_{j=1}^{N}\sum_{k=1}^{n} |\langle f_j, f_k\rangle|^{2t} \geq \frac{1}{\binom{d+t-1}{t}}\Big(\sum_{l=1}^{N} \|f_l\|^{2t}\Big)^2.$$

And if the equality hold, then $\{f_i\}_{i=1}^N$ is called a (t,t)-design for $\mathbb{C}^d$. Without loss of generality we only consider collection of vectors on unit sphere. $\{f_i\}_{i=1}^N \subset S^{d-1}$ is a spherical (t,t)-design if

$$\sum_{j=1}^{N}\sum_{k=1}^{N} |\langle f_j, f_k\rangle|^{2t} = \frac{N^2}{\binom{d+t-1}{t}}.$$

For any $t > 2$ we make the following observation:

Proposition 4.4.4. *If $t > 2$, there is no (t,t)-design for $\mathbb{C}^d$ with d^2 unit norm* *vectors.*

Proof. By theorem 4.3.4, for any $\{f_i\}_{i=1}^{d^2} \subset S^{d-1}$,

$$\sum_{j=1}^{N}\sum_{k=1}^{N} |\langle f_j, f_k\rangle|^{2t} \geq \frac{d^2(d-1)}{(d+1)^{t-1}} + d^2$$

Suppose $\{f_i\}_{i=1}^{d^2} \subset S^{d-1}$ is a (t,t)-design, then

$$\sum_{j=1}^{N}\sum_{k=1}^{N} |\langle f_j, f_k\rangle|^{2t} = \frac{d^4}{\binom{d+t-1}{t}}.$$

For $t > 2$, $\frac{d^4}{\binom{d+t-1}{t}} < d^2 \leq \frac{d^2(d-1)}{(d+1)^{t-1}} + d^2$, which is a contradiction with the bound in Theorem 4.3.4. So no (t,t)-design exist when $t > 2$.

□

For any $g \in S^{d-1}$ we have

$$Z_{4,d}(g) = 1/d^2 \sum_{k_1,l_1}\sum_{k_2,l_2} |\langle M^{k_1}T^{l_1}g, M^{k_2}T^{l_2}g\rangle|^4 \geq \frac{2d}{d+1} = \frac{d^2}{\binom{d+1}{2}}. \tag{4.4}$$

and our goal is to prove that the lower bound can be achieved. This is equivalent with showing that the minimizer of $Z_{4,d}(g)$ is $(2,2)$-design. Renes etc. has shown in [51] that SIC-POVMs are exactly $(2,2)$-designs with d^2 vectors.

By Theorem 6.7 in [67], the equality in (4.4) holds if and only if any of the following equivalent conditions hold.

1. Generalized Bessel identity

$$\frac{2d}{d+1}\|x\|^4 = \sum_{k,l=0}^{d-1} |\langle x, M^k T^l g\rangle|^4, \quad \forall x \in \mathbb{C}^d$$

2. Generalized Plancherel identity

$$\frac{2d}{d+1}\langle x, y\rangle^2 = \sum_{k,l=0}^{d-1} \langle x, M^k T^l g\rangle^2 \langle M^k T^l g, y\rangle^2, \quad \forall x, y \in \mathbb{C}^d$$

We can now restate Problem 4.3.6.

Problem 4.4.5. *Is it true that for all $g^* \in \mathbb{C}^d$ that are the minimizers of $Z_{p,d}$, the Generalized Bessel identity or the Generalized Plancherel identity holds?*

4.5 Sequences with small number of different inner products

This section concerns the first question mentioned in the beginning of the chapter. Instead of looking for frames with only one angle among vectors, characterizing frames with small number of different inner products may give us more insight into Zauner's conjecture. In this section we investigate the Gabor frames generated by two special sequences, Björck sequences and Alltop sequences.

4.5.1 Björck Sequences

The construction of Björck sequence is related to the study of ambiguity function and CAZAC sequences. In this section we will apply the result in [7] to compute the number of possible different values the inner products take in Gabor frames generated by Björck sequences.

Definition 4.5.1. Let $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$. The discrete narrow band ambiguity function $A_N(u) : \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ is defined as

$$A_N(u)[m,n] = \frac{1}{N}\sum_{k=0}^{N-1} u[m+k]\overline{u[k]}e^{-2\pi ikn/N}$$

for all $(m,n) \in \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$.

Consider the vector $g = (u(1), \cdots, u(N))/\sqrt{N} \in \mathbb{C}^N$, the value of ambiguity function $A_N(u)[m,n]$ can be viewed as a multiple of the inner product between vectors in $\{M^nT^mg\}_{m,n=0}^{N-1}$, the Gabor frame generated by g. Since

$$A_N(u)[m,n] = \frac{1}{N}\langle T^{-m}u, M^nu\rangle = \frac{1}{N}\langle u, M^nT^mu\rangle e^{2\pi imn/N}.$$

The Björck sequence is defined in term of Legendre symbol.

Definition 4.5.2. Let p be a prime number, k an integer. Denote $\chi[k] = (\frac{k}{p})$ the Legendre symbol of k modulo p, where

$$\chi[k] = \left(\frac{k}{p}\right) = \begin{cases} 1 & \textit{if } k \equiv m^2 \mod p \textit{ for some } m \in \mathbb{Z}/p\mathbb{Z}^\times; \\ 0 & \textit{if } k \equiv 0 \mod p; \\ -1 & \textit{if } k \not\equiv m^2 \mod p \textit{ for all } m \in \mathbb{Z}/p\mathbb{Z}^\times. \end{cases}$$

We say k is a quadratic residue modulo p if $\chi[k] = 1$ and denote $k \in \mathcal{Q}$; k is a quadratic nonresidue if $\chi[k] = -1$ and denote $k \in \mathcal{Q}^C$.

Definition 4.5.3. The Björck sequence of length N, where N is a prime and $N \equiv 1 \mod 4$ is defined by

$$u[k] = e^{i\theta\chi[k]}, \ \textit{where } \theta = \arccos\left(\frac{1}{1+\sqrt{N}}\right),$$

for all $k \in \mathbb{Z}/N\mathbb{Z}$.

The Björck sequence of length N, where p is a prime and $N \equiv 3 \mod 4$ is defined by

$$u[k] = \begin{cases} e^{i\phi} & \textit{if } k \in \mathcal{Q}^C \subseteq (\mathbb{Z}/N\mathbb{Z})^\times, \\ 1 & \textit{otherwise}, \end{cases}$$

for all $k \in \mathbb{Z}/N\mathbb{Z}$.

We will then prove the main result for this section.

Theorem 4.5.4. *Suppose d is prime and $d \equiv 3 \mod 4$, then $|A_N(U)[m,n]|$ take d different values. Furthermore, $|A_N(U)[m,n]| = |A_N(U)[m',n']|$ if $mn \equiv m'n' \mod d$.*

We need the following results to prove Theorem 4.5.4.

Lemma 4.5.5 ([7], Lemma 3.6)**.** *Suppose N is prime and $r, s, t \in \mathbb{C}$. Define a* *function* $U : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ *as*

$$U[k] = \begin{cases} r & \chi[k] = 1, \\ s & \chi[k] = -1, \\ t & k = 0. \end{cases}$$

Let $R = \frac{r+s}{2}$, $S = \frac{r-s}{2}$, $T = t - R$, $\zeta_d = e^{2\pi i/d}$. *Then*

$$A_N(U)[m,n] = |S|^2 A_N(\chi)[m,n] + \frac{1}{N}(E_1[m,n] + E_2[m,n])$$

for all $m, n \in \mathbb{Z}/N\mathbb{Z} \setminus \{0\}$, *where* $E_1[m,n] = R\bar{T} + \bar{R}T\zeta_p^{mn}$, *and*

$$E_2[m,n] = \begin{cases} (S\bar{T} + \bar{S}T\zeta_N^{mn})\chi[m] + (R\bar{S} + \bar{R}S\zeta_N^{mn})\chi[n]\sqrt{N} & if\, N \equiv 1 \mod 4, \\ (S\bar{T} - \bar{S}T\zeta_N^{mn})\chi[m] - (R\bar{S} + \bar{R}S\zeta_N^{mn})i\chi[n]\sqrt{N} & if\, N \equiv 3 \mod 4. \end{cases}$$

Suppose N is prime, for any integer a, b denote the quantity

$$K[a,b;N] = \sum_{x \in \mathbb{Z}/N\mathbb{Z}} exp(\frac{2\pi i(ax + bx^{-1})}{N}),$$

where x^{-1} is the multiplicative inverse of x in $\mathbb{Z}/N\mathbb{Z}$.

Lemma 4.5.6 ([7]). *Fix an odd prime* N, *then for all* $m, n \in \mathbb{Z}/N\mathbb{Z} \setminus \{0\}$,

$$e^{-\pi imn/N} A_N(\chi)[m,n] = \pm\frac{1}{N} K[1,a;N] \in \mathbb{R},$$

where $a = (mn)^2/16$ *in* $\mathbb{Z}/N\mathbb{Z}$.

Remark 4.5.7. By the proof of Lemma 4.5.6, denote $b = m/2$ in $\mathbb{Z}/N\mathbb{Z}$, we have

$$A_N(\chi)[m,n] = \frac{e^{2\pi ibn}}{N}K[1,a;N].$$

So by computation if mn is even in $\mathbb{R}$, then $A_N(\chi)[m,n] = \frac{e^{2\pi imn}}{N}K[1,a;N]$. Otherwise $A_N(\chi)[m,n] = -\frac{e^{2\pi imn}}{N}K[1,a;N]$.

Proof of Theorem 4.5.4. Using the same notation as in the definition of Björck sequence and in Lemma 4.5.5. Let U be the Björck sequence with length N, where $N \equiv 3 \mod 4$. We have $R = \frac{1+e^{i\phi}}{2}$, $S = T = \frac{1-e^{i\phi}}{2}$.

Then for any $m, n \in \mathbb{Z}/N\mathbb{Z} \setminus \{0\}$,

$$\begin{aligned}
E_1[m,n] + E_2[m,n] &= (R\bar{S} + \bar{R}S\zeta_N^{mn}) + (|S|^2 - |S|^2\zeta_N^{mn})\chi[m] - (R\bar{S} + \bar{R}S\zeta_N^{mn})i\chi[n]\sqrt{N} \\
&= (\frac{i\sqrt{N}}{1+N} - \frac{i\sqrt{N}}{1+N}\zeta_N^{mn}) + \frac{N}{1+N}(1-\zeta_N^{mn})\chi[m] + ... \\
&\quad (\frac{i\sqrt{N}}{1+N} - \frac{i\sqrt{N}}{1+N}\zeta_M^{mn})i\chi[n]\sqrt{N} \\
&= \frac{N(1-\zeta_N^{mn})}{1+N}\big(\frac{i}{\sqrt{N}} + \chi[m] + \chi[n]\big)
\end{aligned}$$

So again let $a = (mn)^2/16$ in $\mathbb{Z}/N\mathbb{Z}$,

$$B[m,n] \equiv (1+N)A_N(U)[m,n] = \pm K[1,a;N]e^{\pi imn/N} + (1-\zeta_N^{mn})(\frac{i}{\sqrt{N}} + \chi[m] + \chi[n]). \tag{4.5}$$

Note that in (4.5), all the terms other than $\chi[m] + \chi[n]$ depend on the product mn. Now suppose $mn \equiv m'n' \mod N$. Since $\chi[mn] = \chi[m]\chi[n]$, we consider the following subsets of $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$:

- $S_1 = \{(m,n)|\chi[m] = \chi[n] = 1, mn \textit{ is odd in } \mathbb{R}\}$
- $S_2 = \{(m,n)|\chi[m] = \chi[n] = 1, mn \textit{ is even in } \mathbb{R}\}$
- $S_3 = \{(m,n)|\chi[m] = \chi[n] = -1, mn \textit{ is odd in } \mathbb{R}\}$
- $S_4 = \{(m,n)|\chi[m] = \chi[n] = -1, mn \textit{ is even in } \mathbb{R}\}$
- $S_5 = \{(m,n)|\chi[mn] = -1, mn \textit{ is odd in } \mathbb{R}\}$,
- $S_6 = \{(m,n)|\chi[mn] = -1, mn \textit{ is even in } \mathbb{R}\}$,

If (m,n) and (m',n') are in the same set, then $B[m,n] = B[m',n']$. Else without loss of generality, we have the following possible outcomes:

1. $mn \in S_5, m'n' \in S_6$

$$\begin{aligned}
&|B[m,n]| - |B[m',n']| \\
=&K[1,a;N]^2 + 2Re(K[1,a;N]e^{\pi imn/N}(1-\zeta_N^{-mn})\frac{-i}{\sqrt{N}}) + \frac{\|1-\zeta_N^{mn}\|^2}{N} - \\
&K[1,a;N]^2 + 2Re(K[1,a;N]e^{\pi im'n'/N}(1-\zeta_N^{-m'n'})\frac{-i}{\sqrt{N}}) - \frac{\|1-\zeta_N^{m'n'}\|^2}{N} \\
&= 0
\end{aligned}$$

2. $mn \in S_2, m'n' \in S_4$

$$
\begin{aligned}
&|B[m,n]| - |B[m',n']| \\
=&|K[1,a;N]e^{\pi imn/N} + (1-\zeta_N^{mn}\frac{i}{\sqrt{N}})|^2 + 4\|1-\zeta_N^{mn}\|^2 \\
&+ 4Re\bigg((K[1,a;N]e^{\pi imn/N} + (1-\zeta_N^{mn})\frac{i}{\sqrt{N}})(1-\zeta_N^{-mn})\bigg) \\
&- |K[1,a;N]e^{\pi im'n'/N} + (1-\zeta_N^{mn}\frac{i}{\sqrt{N}})|^2 - 4\|1-\zeta_N^{m'n'}\|^2 \\
&+ 4Re\bigg((K[1,a;N]e^{\pi im'n'/N} + (1-\zeta_N^{m'n'})\frac{i}{\sqrt{N}})(1-\zeta_N^{-m'n'})\bigg)
\end{aligned}
$$

By computation

$$
\begin{aligned}
&Re(K[1,a;N]e^{\pi imn/N} + (1-\zeta_N^{mn})\frac{i}{\sqrt{N}})(1-\zeta_N^{-mn}) \\
=&Re\bigg(2iK[1,a;N]\sin(\pi mn/p) - 2[\cos(2\pi mn/N)+1]\frac{i}{\sqrt{N}}\bigg) = 0
\end{aligned}
$$

So $|B[m,n]| - |B[m',n']| = 0$.

3. $mn \in S_1, m'n' \in S_2$

$$
\begin{aligned}
&|B[m,n]| - |B[m',n']| \\
=&Re(1-\zeta_N^{mn})(\frac{i}{\sqrt{N}}+2)K[1,a;N]e^{-\pi imn/N} - Re(1-\zeta_N^{m'n'})(\frac{i}{\sqrt{N}}+2)K[1,a;N]e^{-\pi im'n'/N} \\
=&Re(1-\zeta_N^{mn})(\frac{i}{\sqrt{N}}+2)K[1,a;N](e^{-\pi imn/N} - e^{-\pi im'n'/N}) = 0
\end{aligned}
$$

The rest of the possible cases can be derived from case 2 and 3. So we have $|A_N(U)[m,n]| = |A_N(U)[m',n']|$ if $mn \equiv m'n' \mod d$. $\square$

4.5.2 Alltop Sequence

In this Section we show the inner products of Gabor frames generated by Alltop Sequence. The following results are proved in [2].

Let N be an odd integer greater than two, we can define a λ-th quadric phase sequence by

$$a_\lambda(k) \equiv N^{-1/2} e^{2\pi i \lambda k^2/N},$$

where $\lambda \in \mathbb{Z}/N\mathbb{Z}$.

Define the qubic phase sequence as

$$b_\lambda(k) \equiv N^{-1/2} e^{2\pi i (k^3+\lambda k)/N}.$$

Then $\{a_\lambda\}_{\lambda=0}^{N-1}$ is the same sequence as $\{M^\lambda v\}_{\lambda=0}^{N-1}$

Theorem 4.5.8 ([2]). *For odd $N \geq 3$, let p be the smallest prime divisor of N, then*

$$|\langle T^m a_\lambda, a_\mu \rangle| = \begin{cases} 1, & \textit{if } \lambda = \mu, m = 0, \\ 0, & \textit{if } \lambda = \mu, m \neq 0, \\ N^{-1/2}, & \textit{otherwise.} \end{cases}$$

Theorem 4.5.9 ([2]). *For every prime $p \geq 5$, then*

$$|\langle T^m b_\lambda, b_\mu \rangle| = \begin{cases} 1, & \text{if } \lambda = \mu, m = 0, \\ 0, & \text{if } \lambda = \mu, m \neq 0, \\ N^{-1/2}, & \text{otherwise.} \end{cases}$$

Numerical result shows for $1 \leq p < 2$,Gabor frame generated by Alltop sequence has higher potential; for $p > 2$, Gabor frame generated by Björck sequence has higher potential.

4.6 Future research

In this section, we discuss two possible approaches to solve the problems in this chapter.

4.6.1 The minimizer of $Z_{p,d}$ for $1 < p < 2$ and Hausdorff-Young Inequality

Not much is known about the minimizers of $Z_{p,d}$ when $0 < p < 2$. In this section we apply the Hausdorff-Young inequality to acquire an inequality considering $Z_{p,d}$ when $1 < p < 2$.

Suppose we fix a vector $g \in \mathbb{C}^d$. Let $G = \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$ and $f : \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$ be $f(k,l) = \langle g, M^k T^l g \rangle$ for any $(k,l) \in \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$. G is a finite abelian group.

We have

$$\Big(\frac{1}{|G|}\sum_{(k,l)\in G}|f(k,l)|^p\Big)^{1/p}=\Big(\frac{1}{d^2}Z_{p,d}(g)\Big)^{1/p}. \tag{4.6}$$

We can apply Hausdorff-Young Inequality to the right hand side of equation 4.6.

Theorem 4.6.1. *[59, Hausdorff-Young Inequality] Let H be a finite abelian group, and $f : H \to \mathbb{C}$ be a function. Let $\hat{H}$ be the group of characters $\chi : H \to \mathbb{S}^1$ of H, and define the Fourier transform $\hat{f} : \hat{H} \to \mathbb{C}$ by the formula*

$$\hat{f}(\xi) \equiv \frac{1}{|H|}\sum_{x\in H} f(x)\overline{\chi(x)}.$$

Then if $1 < p < 2$ and $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\left(\sum_{\xi\in\hat{H}}|\hat{f}(\xi)|^q\right)^{1/q} \leq \left(\frac{1}{|H|}\sum_{x\in H}|f(x)|^p\right)^{1/p}. \tag{4.7}$$

By [61, ch.10] , the characters of G can be defined as $\chi_{\alpha,\beta}(k,l)=e^{\frac{2\pi i(\alpha k+\beta l)}{d}}$, where $k,l,\alpha,\beta \in \mathbb{Z}/d\mathbb{Z}$. Denote $\hat{G}$ the group of characters of G. Then

$$\hat{f}(\alpha,\beta)=\frac{1}{|G|}\sum_{(k,l)\in G}\langle g, M^kT^lg\rangle e^{-\frac{2\pi i(\alpha k+\beta l)}{d}}.$$

So we can apply Hausdorff-Young inequality, and obtain the inequality

$$\Big(\frac{1}{d^2}Z_{p,d}(g)\Big)^{1/p} \geq \left(\sum_{(\alpha,\beta)\in\mathbb{Z}/d\mathbb{Z}\times\mathbb{Z}/d\mathbb{Z}}\Big|\frac{1}{|G|}\sum_{(k,l)\in G}\langle g, M^kT^lg\rangle e^{-\frac{2\pi i(\alpha k+\beta l)}{d}}\Big|^q\right)^{1/q}. \tag{4.8}$$

By Hewitt and Hirschman in [42], the equality (4.7) holds if and only if f is a subcharacter or translate of subcharacter. Where a subcharacter is defined as follow:

Definition 4.6.2. Let G be a locally compact abelian group, A be a compact and open subgroup of G. $c \in \mathbb{C}$, $\chi \in \hat{G}$. A function h defined on G such that

$$h(x) = c\chi(x)\delta_A(x)$$

where $x \in G$, is said to be a subcharacter of the group G.

Remark 4.6.3. When g is $(1, 0, \cdots, 0)$, f is a subcharacter of G (let $A = \mathbb{Z}/d\mathbb{Z} \times \{0\}$, $c = 1$ and $\chi(x) = 1$). Similar with when g is any translate of $(1, 0, \cdots, 0)$. However, the lower bound of right hand side of (4.8) does not occur with same g.

4.6.2 Finding the minimizer of $Z_{4,d}$ with the Lagrange multiplier method

In this section, we focus on the case $p = 4$. We will concentrate on the question whether the minimizers of $Z_{4,d}$ satisfy the Generalized Bessel identity or Generalized Plancherel identity. If they do, we will be able to establish the existence of $(2, 2)$-designs with d^2 vectors in $\mathbb{C}^d$. We use the Lagrange multiplier method to provide another angle to view Problem 4.3.6.

For any function $f : \mathbb{C}^d \to \mathbb{R}$ with $f(x_1 + iy_1, \cdots, x_d + iy_d)$ a differentiable function of real variables $x_1, y_1, \cdots, x_d, y_d \in \mathbb{R}$, define a gradient $\nabla f : \mathbb{C}^d \to \mathbb{C}^d$ by

$$\nabla f \equiv \big(\frac{\partial}{\partial x_j} f(x_1 + iy_1, \cdots, x_d + iy_d) + i\frac{\partial}{\partial y_j} f(x_1 + iy_1, \cdots, x_d + iy_d)\big)_{j=1}^{d}.$$

Then we can calculate the gradient of $Z_{4,d}(g)$ as following.

Proposition 4.6.4. *For $m \in \{1, \cdots, d\}$,*

$$\nabla Z_{4,d}(g) = 8 \sum_{k,l=0}^{d-1} |\langle g, M^k T^l g\rangle|^2 \langle g, M^k T^l g\rangle M^k T^l g \tag{4.9}$$

Proof. Denote $g = (g_1, \cdots, g_d) = (a_1 + ib_1, \cdots, a_d + ib_d)$ with $a_1, b_1, \cdots, a_d, b_d \in \mathbb{R}$

$$\begin{aligned} |\langle g, M^k T^l g\rangle|^2 &= |\sum_j g_j \overline{g_{j-l}} e^{-2\pi ijk/d}|^2 \\ &= |\sum_j \Bigg(\Big(a_j a_{j-l}\cos(2\pi jk/d) + b_j b_{j-l}\cos(2\pi jk/d) ... \\ &+ a_{j-l} b_j \sin(2\pi jk/d) + a_j b_{j-l}\sin(2\pi jk/d)\Big) ... \\ &+ i\Big(-a_j a_{j-l}\sin(2\pi jk/d) - b_j b_{j-l}\sin(2\pi jk/d) ... \\ &+ a_{j-l} b_j \cos(2\pi jk/d) - a_j b_{j-l}\cos(2\pi jk/d)\Big)\Bigg)|^2. \end{aligned}$$

Then

$$\frac{\partial |\langle g, M^k T^l g\rangle|^2}{\partial a_j} + i\frac{\partial |\langle g, M^k T^l g\rangle|^2}{\partial b_j} = 2\langle g, M^k T^l g\rangle g_{j-l} e^{2\pi ijk/d} + 2\overline{\langle g, M^k T^l g\rangle} g_{j+l} e^{-2\pi i(j+l)l/d}.$$

$$
\begin{aligned}
(\nabla Z_{4,d}(g))_j &= 4\sum_{k,l=0}^{d-1} |\langle g, M^kT^lg\rangle|^2 \left(\langle g, M^kT^lg\rangle g_{j-l}e^{2\pi ijk/d} + \overline{\langle g, M^kT^lg\rangle} g_{j+l}e^{-2\pi i(j+l)k/d} \right) \\
&= 4\sum_{k,l=0}^{d-1} |\langle g, M^kT^lg\rangle|^2 \left(\langle g, M^kT^lg\rangle (M^kT^lg)_j + \langle g, M^{-k}T^{-l}g\rangle (M^{-k}T^{-l}g)j \right)
\end{aligned}
$$

So

$$
\begin{aligned}
\nabla Z_{4,d}(g) &= 4\sum_{k,l=0}^{d-1} |\langle g, M^kT^lg\rangle|^2 \left(\langle g, M^kT^lg\rangle M^kT^lg + \langle g, M^{-k}T^{-l}g\rangle M^{-k}T^{-l}g \right) \\
&= 8\sum_{k,l=0}^{d-1} |\langle g, M^kT^lg\rangle|^2 \langle g, M^kT^lg\rangle M^kT^lg
\end{aligned}
$$

$\square$

The minimizer g^* should be a critical point of the minimization problem of $2d$ variables with restriction $\|g\|^2 = 1$. Applying the Lagrange multiplier method, at g^* we have

$$
4\sum_{k,l=0}^{d-1} |\langle g^*, M^kT^lg^*\rangle|^2 \langle g^*, M^kT^lg^*\rangle M^kT^lg^* = \lambda g^*, \quad \lambda \in \mathbb{R} \tag{4.10}
$$

and

$$
\sum_{k,l=0}^{d-1} |\langle g^*, M^kT^lg^*\rangle|^4 = \frac{\lambda}{4}|g^*|^2
$$

We can see that if $\{M^kT^lg\}_{k,l=0}^{d-1}$ is a $(2,2)$-design for $\mathbb{C}^d$, it is indeed a local minimum, and also global minimum in this case. But the inverse is not necessary true. By the Generalized Bessel identity, since $g^* \in \mathbb{C}^d$, $\lambda = \frac{8d}{d+1}$ being a solution of (4.10) is a necessary condition for $\{M^kT^lg^*\}_{k,l=0}^{d-1}$ to be a (2,2)-design.

As a result we can list some necessary conditions for $\{M^kT^lg^*\}_{k,l=0}^{d-1}$ to be a (2,2)-design.

1. By Proposition 4.3.2,

$$\begin{aligned}&\sum_{k_1,k_2,l_1,l_2=0}^{d-1}\langle x, M^{k_1}T^{l_1}g\rangle\langle x, M^{k_2}T^{l_2}g\rangle\langle M^{k_1}T^{l_1}g, y\rangle\langle M^{k_2}T^{l_2}g, y\rangle\\&=\Big(\sum_{k_1,l_1=0}^{d-1}\langle x, M^{k_1}T^{l_1}g\rangle\langle M^{k_1}T^{l_1}, y\rangle\Big)\Big(\sum_{k_2,l_2=0}^{d-1}\langle x, M^{k_2}T^{l_2}g\rangle\langle M^{k_2}T^{l_2}, y\rangle\Big)\\&=d^2\langle x,y\rangle^2\end{aligned}$$

   for any $x, y \in \mathbb{C}^d$. $\{M^kT^lg^*\}_{k,l=0}^{d-1}$ is a (2,2)-design if and only if Generalized Plancherel identity holds. So $\{M^kT^lg^*\}_{k,l=0}^{d-1}$ is a (2,2)-design if and only if

$$\sum_{k_1\neq k_2,l_1\neq l_2}^{d-1}\langle x, M^{k_1}T^{l_1}g^*\rangle\langle x, M^{k_2}T^{l_2}g^*\rangle\langle M^{k_1}T^{l_1}g^*, y\rangle\langle M^{k_2}T^{l_2}g^*, y\rangle=\frac{d^3+d^2-2d}{d+1}\langle x,y\rangle^2.$$

2. By (4.10), for any $a, b \in \mathbb{Z}_d$, if g^* is minimizer of $Z_{4,p}$, then it is also a critical point. So

$$M^aT^bg^*=\frac{4}{\lambda}\sum_{k,l=0}^{d-1}|\langle g^*, M^kT^lg^*\rangle|^2\langle g^*, M^kT^lg^*\rangle M^{a+k}T^{b+l}g^*e^{-2\pi ibk/d}.$$

   Then a necessary condition of $\{M^kT^lg^*\}_{k,l=0}^{d-1}$ being a (2,2)-design is

$$M^aT^bg^*=\frac{d+1}{2d}\sum_{k,l=0}^{d-1}|\langle g^*, M^kT^lg^*\rangle|^2\langle g^*, M^kT^lg^*\rangle M^{a+k}T^{b+l}g^*e^{-2\pi ibk/d}$$

Chapter 5: Generalization of Support Uncertainty Inequality

5.1 Introduction

The uncertainty principle originates in quantum physics and can be expressed mathematically. Its general idea is to show that different representations of a function can not be sharply concentrated.

The inequalities consist of three main components: a global setting, which is generally Hilbert spaces; an invertible linear transform mapping initial representation to the other one without information lost; and a concentration measure [53, p.630]. Based on the operators, there are different ways to define the concentration measure. In Section 5.2, we will give a few examples of classic uncertainty inequalities that are developed and stated in different settings. Then in Section 5.3, we will reproduce the proof of a recent generalization of the inequality, an extension into frame setting. Section 5.3.3 will give a specific example of mutually unbiased bases in finite dimensional Hilbert space.

5.2 Classical Uncertainty Inequalities

The Heisenberg inequality is the earliest version of the uncertainty inequalities, where variance (i.e. $\|(t-t_0)f(t)\|_2$ for $f \in L^2(\mathbb{R})$) is used as the concentration measure.

Theorem 5.2.1. *Let* $(t_0, \gamma_0) \in \mathbb{R} \times \hat{\mathbb{R}}$*. Then*

$$\forall f \in L^2(\mathbb{R}), \|f\|_2^2 \leq 4\pi \|(t-t_0)f(t)\|_2 \|(\gamma-\gamma_0)\hat{f}(\gamma)\|_2$$

This gives a specific case on $L^2(\mathbb{R})$, and the two ways of representing functions are the function itself and its Fourier transform. Later the inequality is generalized to Hilbert spaces, using the projection onto different orthonormal bases as the different representations of functions and define another way to measure variance.

Theorem 5.2.2. *Let* $f \in \mathbb{H}$ *with* $\|f\| = 1$*.* A *and* B *be self-adjoint operators on* $\mathcal{H}$ *with respective domains D(A) and D(B). Define the mean and variance of* A *in state* $f \in D(A)$ *by*

$$e_f(A) = \langle Af, f \rangle, \quad v_f(A) = e_f(A^2) - e_f(A)^2$$

Setting $[A, B] = AB - BA$ *and* $\{A, B\} = AB + BA$*, then* $\forall f \in D(AB) \cap D(BA)$*,*

$$v_f(A)v_f(B) \geq \frac{1}{4}[|e_f([A, B])|^2 + |e_f(\{A - e_f(A), B - e_f(B)\})|^2].$$

The inequality above is called Robertson-Schrödinger inequality. The variance of function f is given by its projection onto eigenspaces of the operators A and B. However Robertson-Schrödinger inequality has been criticized for several reasons. First, unlike Heisenburg inequality, which gives a uniform bound for all f, the bound here depends on the function itself. Also, the definition of variance gives trouble when applying to certain spaces [53, p.630].

The uncertainty inequality in discrete settings is developed in more recent years. We can start from projection onto orthonormal bases of finite dimensional spaces and generalized to infinite dimensional spaces. Here support is used to measure the variance of two representations. Elad and Bruckstein gave an inequality on the quasi-norm $\|\cdot\|_0$, which is defined for a sequence $\{a_n\}_{n\in\mathbb{Z}}$ to be $\|a\|_0 = \sum_n |sgn(a_n)|$.

Theorem 5.2.3. *Given two orthonormal bases in a finite-dimensional Hilbert space and any vector x with set of coefficients a and b with respect to the two bases, then*

$$\|a\|_0\|b\|_0 \geq \frac{1}{\mu^2}$$

where for orthonormal bases $\mathcal{U}$ and $\mathcal{V}$, the mutual coherence $\mu = \sup_{l,k}|\langle u_l, v_k\rangle|$

This inequality tells us the product of l^0 norm has a lower bound only depending on the two bases. Ricaud and Torrésani extended this result into a broader setting and showed we can obtain an uncertainty inequality for frames (possibly in infinite dimensional Hilbert spaces). We will discuss more about the refined inequality in the next section.

5.3 Refined Elad-Bruckstein ℓ^0 Inequalities

Since we have projection onto bases as representations of functions, it is natural to ask whether it is possible to generalize the uncertainty inequality to frames. This is possible because of the existence of dual frames. No information will be lost in the process of representing a function by different frames. In this section, I will first introduce the notation used and prove the generalization of Elad-Bruckstein inequality.

5.3.1 Notation

Given $\mathcal{U}$ a frame, let U be the analysis operator, given by $U : \mathbb{H} \to l^2(\mathbb{N})$

$$\forall x \in \mathbb{H} \quad Ux = \langle x, u_k \rangle_{k=1}^{\infty}$$

Let $\tilde{\mathcal{U}}$, $\tilde{\mathcal{V}}$ be the dual frames of $\mathcal{U}$ and $\mathcal{V}$. Each frame has at least one dual frame (which is the canonical dual frame). For some frames, there exist dual frame other than the canonical dual frame.

In the refined inequality, the order r coherence is introduced, for it is possible to give better bound than standard mutual coherence.

Definition 5.3.1. Let $r \in [1, 2]$ and r' be conjugate to r. The mutual coherence of order r of two frames $\mathcal{U}$ and $\mathcal{V}$ is defined by

$$\mu_r(\mathcal{U}, \mathcal{V}) = \sup_l (\sum_k |\langle u_k, v_l \rangle|^{r'})^{\frac{r}{r'}}$$

Remark 5.3.2.

(1) The case $r = 1$ correspond to standard definition of mutual coherence.

(2) For a finite dimensional Hilbert space, it is clear that order r mutual coherence is well defined. Suppose instead we have an infinite dimensional Hilbert space, then

$$\mu_2(\mathcal{U}, \mathcal{V}) = \sup_l \sum_k |\langle u_k, v_l \rangle|^2 \leq B_{\mathcal{U}} \sup_l \|v_l\|^2 \tag{5.1}$$

and

$$\begin{aligned} \mu_r^{r'/r}(\mathcal{U}, \mathcal{V}) &= \sup_l \sum_k |\langle u_k, v_l \rangle|^{r'} = \sup_l \sum_k |\langle u_k, v_l \rangle|^2 |\langle u_k, v_l \rangle|^{r'-2} \\ &\leq \sup_l \sum_k |\langle u_k, v_l \rangle|^2 \sup_{k,l} |\langle u_k, v_l \rangle|^{r'-2} = \mu_2(\mathcal{U}, \mathcal{V}) \sup_{k,l} |\langle u_k, v_l \rangle|^{r'-2} \end{aligned}$$

$\sup_l \|v_l\| < \infty$, since if $\sup_l \|v_l\| = \infty$ then exist $\|v_l\| > B_{\mathcal{V}}$. In this case $\sum_k |\langle u_k, v_l \rangle|^2 \geq |\langle u_l, v_l \rangle|^2 \geq B_{\mathcal{V}} \|v_l\|^2$, contradict to the definition of frame.

Also $\sup_{k,l} |\langle u_k, v_l \rangle|^2 \leq \max(B_{\mathcal{U}}, B_{\mathcal{V}}) \max(\sup \|v_l\|^2, \sup \|v_l\|^2) < \infty$. This implies $\mu_r(\mathcal{U}, \mathcal{V})$ for $r \in [1, 2]$ is finite. Thus the order r coherence is well-defined on infinite dimensional spaces.

Definition 5.3.3. Two orthonormal bases $\mathcal{U}$ and $\mathcal{V}$ in an N-dimensional Hilbert space $\mathbb{H}$ are mutually unbiased bases (MUB) if

$$|\langle u_k, v_l \rangle| = \frac{1}{\sqrt{N}}, \qquad \forall k, l = 0, ...N - 1$$

5.3.2 Refined Inequality

The following theorems are proved in [52, p. 4274]. The first theorem states the inequality and the second theorem gives the condition under which the inequality is sharp.

Theorem 5.3.4. *Let $\mathcal{U}$ and $\mathcal{V}$ be two frames of Hilbert space $\mathbb{H}$. For any $x \in \mathbb{H}, x \neq 0$, denote $a = Ux$ and $b = Vx$ the analysis coefficients of x with respect to the two frames.*

For all $r \in [1, 2]$, coefficient a and b satisfy the uncertainty inequality

$$\|a\|_0 \|b\|_0 \geq \frac{1}{\mu_r(\tilde{\mathcal{U}}, \mathcal{V})\mu_r(\tilde{\mathcal{V}}, \mathcal{U})} \tag{5.2}$$

Therefore, $\|a\|_0 \|b\|_0 \geq \frac{1}{\nu_*(\mathcal{U}, \tilde{\mathcal{U}}, \mathcal{V}, \tilde{\mathcal{V}})^2}$, where

$$\mu_*(\mathcal{U}, \tilde{\mathcal{U}}, \mathcal{V}, \tilde{\mathcal{V}}) = \inf_{r \in [1,2]} \sqrt{\mu_r(\tilde{\mathcal{U}}, \mathcal{V})\mu_r(\tilde{\mathcal{V}}, \mathcal{U})}$$

Proof.

$$\begin{aligned} \|a\|_\infty = \|Ux\|_\infty &= \sup_l |\langle x, u_l \rangle| \\ &= \sup_l |\langle \sum_k \langle x, v_k \rangle \tilde{v_k}, u_l \rangle| \\ &= \sup_l |\langle \sum_k b_k \tilde{v_k}, u_l \rangle| \\ &\leq \sup_l \sum_k |b_k| |\langle \tilde{v_k}, u_l \rangle|. \end{aligned} \tag{5.3}$$

By Holder's inequality, for any $r \in [1,2]$ and $l \in \mathbb{Z}^+$

$$\sum_k |b_k||\langle \tilde{v_k}, u_l \rangle| = \|b\langle \tilde{v}, u_l \rangle\|_1 \leq \|b\|_r \|\langle \tilde{v}, u_l \rangle\|_{r'}. \tag{5.4}$$

Thus by (5.3),

$$\|a\|_\infty \leq \|b\|_r \sup_l \|\langle \tilde{v}, u_l \rangle\|_{r'} = \|b\|_r \mu_r(\tilde{\mathcal{V}}, \mathcal{U})^{1/r}. \tag{5.5}$$

Similarly we get the same conclusion on b:

$$\|b\|_\infty \leq \|a\|_r \sup_l \|\langle \tilde{u}, v_l \rangle\|_{r'} = \|a\|_r \mu_r(\tilde{\mathcal{U}}, \mathcal{V})^{1/r}. \tag{5.6}$$

By definition and (5.5)(5.6),

$$\begin{aligned} \forall a, \quad \|a\|_r^r &= \sum_k |a_k|^r \leq \sum_k \sup_k |a_k|^r \leq \|a\|_0 \|a\|_\infty^r \leq \|a\|_0 \|a\|_r^r \mu_r(\tilde{\mathcal{U}}, \mathcal{V})^{1/r}, & (5.7) \\ \forall b, \quad \|b\|_r^r &\leq \|b\|_0 \|b\|_\infty^r \leq \|b\|_0 \|b\|_r^r \mu_r(\tilde{\mathcal{V}}, \mathcal{U})^{1/r}. & (5.8) \end{aligned}$$

Multiplying (5.7) and (5.8) yields $\|a\|_0 \|b\|_0 \geq \frac{1}{\mu_r(\tilde{\mathcal{U}}, \mathcal{V}) \mu_r(\tilde{\mathcal{V}}, \mathcal{U})}$. □

Theorem 5.3.5. $\forall r \in [1,2]$, *the inequality (5.2) is sharp if and only if the following is satisfied:*

i $|a|$ *and* $|b|$ *are constant on support of* a *and* b *resp;*

ii for all $k \in supp(a)$ *(resp.* $l \in supp(b)$*), if we fix* l*, then the sequence* $|\langle \tilde{u}_k, v_l \rangle|$

(resp. fix k, $|\langle\tilde{v_l}, u_k\rangle|$*)is constant on supp(b)(resp. on supp(a));*

iii for all $k \in supp(a)$, $l \in supp(b)$, $\arg(\langle\tilde{u_k}, v_l\rangle) = \arg(b_l) - \arg(a_k) = -\arg(\langle\tilde{v_l}, u_k\rangle)$.

Proof. If inequality (5.2) is sharp, then all the inequalities in the proof of theorem 4 have to be sharp. I will prove the conclusion for b and $r \neq 1$, similar argument will give same conclusion for a.

The three inequalities involved are in (5.4),(5.6) and (5.8)

Since

$$\|b\|_r^r = \sum_k |b_k|^r \leq \sum_k |sgn(b_k)| \|b\|_\infty^r, \tag{5.9}$$

equality in (5.8) holds when $|sgn(b_k)| = 1$, $b_k = \|b\|_\infty$ for all k. This implies the first condition has to hold.

For (5.4), if we keep l fixed, the condition for equality in Holder's inequality to hold is $\exists C$, such that $|\tilde{v_k}, u_l|^{r'} = C|b_k|^r$. Since by the first condition $|b_k|$ is constant, $|\tilde{v_k}, u_l|$ is also constant. This proves the second condition.

For equality in (5.6) to hold, $\|b\|_\infty = \sup_l \sum_k |a_k||\langle\tilde{u_k}, v_l\rangle|$

$$|b_l|e^{iArg(b_l)} = \sum_k |a_k||\langle\tilde{u_k}, v_l\rangle|e^{i(Arg(a_k) + Arg(\langle\tilde{u_k}, v_l\rangle))}. \tag{5.10}$$

Then

$$Arg(b_l) = Arg(a_k) + Arg(\langle\tilde{u_k}, v_l\rangle) \quad \forall k. \tag{5.11}$$

This proves the necessity for the third condition.

If $r = 1$, (5.6) and (5.8) give the same condition. For (5.4), the equality holds when $|b_k| = \|b\|_\infty \ \forall k$. $\square$

Remark 5.3.6.

(1) If both $\mathcal{U}$ and $\mathcal{V}$ are orthonormal bases of a finite dimensional Hilbert space, $\mathcal{U} = \tilde{\mathcal{U}}$ and $\mathcal{V} = \tilde{\mathcal{V}}$ we get a better bound since by definition $\mu_*(\mathcal{U}, \tilde{\mathcal{U}}, \mathcal{V}, \tilde{\mathcal{V}}) = \inf_{r\in[1,2]} \sqrt{\mu_r(\tilde{\mathcal{U}}, \mathcal{V})\mu_r(\tilde{\mathcal{V}}, \mathcal{U})} \leq \mu_1(\mathcal{U}, \mathcal{V})$.

(2) The inequality can be also used to generalize Maassen-Uffink uncertainty inequality on Renyi entropy to frame representation. See [52, p. 4276].

5.3.3 Example: mutually unbiased bases

The following example is mentioned in paper [52].

Corollary 5.3.7. *If $\mathcal{U}$ and $\mathcal{V}$ are mutually unbiased orthonormal basis, then the optimal bound of the refined inequality is attained when r=1.*

Proof. Since $\mathcal{U}$ and $\mathcal{V}$ are mutually unbiased

$$|\langle u_k, v_l\rangle| = \frac{1}{\sqrt{N}} \quad \forall l, k \tag{5.12}$$

where N is the dimension of the Hilbert space.

$$\begin{aligned} \mu_r(\mathcal{U}, \mathcal{V}) &= (\sum_{k=0}^{N-1} (N^{-1/2})^{r'})^{r/r'} && (5.13)\\ &= (N^{1-\frac{1}{2}r'})^{r/r'} && (5.14)\\ &= N^{\frac{r}{r'}-\frac{1}{2}r} = N^{\frac{1}{2}r-1} && (5.15)\end{aligned}$$

which is increasing on $[1, 2]$ thus has its minimal value at $r = 1$. $\square$

Remark 5.3.8. If $\mathcal{U}$ and $\mathcal{V}$ are general pairs of orthonormal bases which are not mutually unbiased, the refined inequality yield a strictly better bound (i.e. $\sup_r \frac{1}{\mu_r(\mathcal{U},\mathcal{V})\mu_r(\mathcal{V},\mathcal{U})} > \frac{1}{\mu_1(\mathcal{U},\mathcal{V})^2}$) [52, pp. 4275].

5.4 Conclusion

The uncertainty inequalities have been developed in different settings, both Euclidean space and more general Hilbert spaces. For finite dimensional spaces, we have already obtained optimal bounds for the inequalities. However for infinite dimensional spaces, the inequality can still be improved ([52, p. 4278]).

Chapter 6: Shift-invariant spaces on LCA groups

6.1 Introduction

In Euclidean spaces, a shift-invariant space is a closed subspace of $L^2(\mathbb{R}^d)$ that is invariant under integer lattice translations. This subject has been studied since the 1960s. Henry Helson developed the concept of range function and gave a characterization of shift-invariant spaces on $L^2(\mathbb{R})$ in [41].

The study of shift-invariant spaces also has recent development, cf. [15, 20]. Techniques from Fourier analysis as well as ideas such as fiberization and the concept of range functions are used to extend the theory into more general settings. The result was also used to give characterizations of frames in shift-invariant spaces, decompose the spaces into direct sum of smaller spaces, etc. It can be applied to various fields such as Gabor theory and wavelet theory.

Then it is reasonable to ask whether the theory fits into a setting that is more general than Euclidean space, and what are the properties of the groups $\mathbb{R}^d$ and $\mathbb{Z}^d$ that makes it possible to apply the method developed. In [18], the theory was extended to the setting of locally compact abelian group G. And the concept of uniform lattice, which plays a similar role with $\mathbb{Z}^d$ in $\mathbb{R}^d$, was introduced. With the existence of Haar measure on LCA groups, it is possible to define the Fourier transform of functions in $L^2(G)$, as well as to generalize related theorems from Euclidean Fourier analysis. Similarly, other techniques used to characterize the shift-invariant spaces in $L^2(\mathbb{R}^d)$ can also be generalized and applied in $L^2(G)$.

In this chapter, we briefly discuss the characterization of shift-invariant spaces on locally compact abelian groups, and frames of the shift-invariant spaces. The results in this chapter are originally derived in [18].

6.2 Background

The theory is developed under the following assumptions:

- G is a second countable LCA group with dual group Γ;
- H is a countable uniform lattice (i.e., a discrete subgoup of G such that the quotient group G/H is compact), and Δ is the annihilator of H (i.e., $\Delta = \{\gamma \in \Gamma : (h, \gamma) = 1, \forall h \in H\}$);
- The Haar measures m_H, m_Δ and $m_{\Gamma/\Delta}$ of H,Δ, and Γ/Δ are chosen as that $m_H(0) = m_\Delta(0) = m_{\Gamma/\Delta}(\Gamma/\Delta) = 1$, and the inversion formula for the Fourier transform holds.

If Ω is a Borel measurable secton of Γ/Δ (i.e., a set of representatives of the quotient group), we have $m_\Gamma(\Omega) = 1$. This will not cause problems since under the assumptions, Γ/Δ will be compact, then Ω is also compact. Further, we let $E_H(\mathcal{A}) = \{\tau_h\phi : h \in H, \phi \in \mathcal{A}\}$, where $\mathcal{A} \subseteq L^2(G)$ and $\tau_h\phi(x) = \phi(x - h)$ denotes translation of ϕ by h.

With the Haar measure on G, we will be able to define $L^p(G)$ in a similar way with $L^p(\mathbb{R}^d)$. Since the characters (x, γ) are extensions of the complex exponential functions, we define the Fourier transform on G in the following way.

Definition 6.2.1. Given a function $f \in L^1(G)$, the Fourier transform of f is defined as

$$\hat{f}(\gamma) = \int_G f(x)(x, -\gamma) dm_G(x), \gamma \in \Gamma,$$

and the inversion formula for the Fourier transform is

$$f(x) = \int_\Gamma \hat{f}(\gamma)(x, \gamma) dm_\Gamma(\gamma).$$

The Haar measures m_G and m_Γ are normalized such that the inversion formula holds, so the Fourier transform on $L^1(G) \cap L^2(G)$ can be extended uniquely to an isometry from $L^2(G)$ onto $L^2(\Gamma)$ [54, Theorem 1.6.1].

The following concepts are developed in the study of shift-invariant subspces of $L^2(\mathbb{R}^d)$, and modified for the setting of $L^2(G)$. They are essential in proving Theorme 6.3.1.

Proposition 6.2.2. *The mapping $\mathcal{T} : L^2(G) \to L^2(\Omega, \ell^2(\Delta))$, defined as*

$$\mathcal{T} f(\omega) = \{\hat{f}(\omega + \delta)\}_{\delta \in \Delta},$$

is an isomorphism that satisfies $\|\mathcal{T} f\|_2 = \|f\|_{L^2(G)}$,

where

$$\|\mathcal{T} f\|_2 = (\int_\Omega \|\mathcal{T} f(\omega)\|^2_{\ell^2(\Delta)} dm_\Gamma(\omega))^{1/2}.$$

Definition 6.2.3. A range function is a mapping,

$$J : \Omega \to \{\text{closed spaces of } \ell^2(\Delta)\}.$$

The subspace $J(\omega)$ is called the fiber space associated to ω.

For a given range function J, we associate to each $\omega \in \Omega$ the orthogonal projection onto $J(\omega)$, $P_\omega : \ell^2(\Delta) \to J(\omega)$. And denote $M_J = \{\Phi \in L^2(\Omega, \ell^2(\Delta)) : \Phi(\omega) \in J(\omega)\, a.e.\, \omega \in \Omega\}$. M_J is a closed subset of $L^2(\Omega, \ell^2(\Delta))$.

We say a range function J is measurable if for all $\Phi \in L^2(\Omega, \ell^2(\Delta))$ and all $b \in \ell^2(\Delta)$, the mapping $\omega \mapsto \langle P_\omega(\Psi(\omega)), b\rangle$ is measurable.

Proposition 6.2.4. *Let J be a measurable range function and P_ω the associated orthogonal projections. Denote by $\mathcal{P}$ the orthogonal projection onto M_J. Then,*

$$(\mathcal{P}\Phi)(\omega) = P_\omega(\Phi(\omega)),\ a.e.\, \omega \in \Omega,\ \forall \Phi \in L^2(\Omega, \ell^2(\Delta)).$$

Proof. Define the linear operator $\mathcal{Q} : L^2(\Omega, \ell^2(\Delta)) \to L^2(\Omega, \ell^2(\Delta))$ as

$$(\mathcal{Q}\Phi)(\omega) = P_\omega(\Phi(\omega)).$$

Then by definition of P_ω, $\mathcal{Q}$ is well-defined and is also an orthogonal projection.

Also, by assumption, $Ran(\mathcal{Q}) \subseteq M_J$. Suppose the inclusion is proper. Let $\Psi \perp Ran(\mathcal{Q})$ and $\Psi \in M_J$. Then for any $\Phi \in L^2(\Omega, \ell^2(\Delta))$, $0 = \langle \mathcal{Q}\Phi, \Psi\rangle = \langle \Phi, \mathcal{Q}\Psi\rangle$.

Since $\Psi \in M_J$, $\Psi(\omega) \in J(\omega)$, $(\mathcal{Q}\Psi)(\omega) = P_\omega(\Phi(\omega)) = \Psi(\omega)$ a.e. $\omega \in \Omega$. We have for any Φ, $\langle \Phi, \Psi\rangle = 0$. This implies $\Psi = 0$ a.e. $\omega \in \Omega$. We conclude that $M_J \subseteq Ran(\mathcal{Q})$. Thus $\mathcal{P} = \mathcal{Q}$. □

Proposition 6.2.4 is a generalization of a Lemma in [41, Chapter VI].

6.3 Characterizaton of shift-invariant spaces

Theorem 6.3.1. *[18] Let $V \subseteq L^2(G)$ be a closed subspace. Then V is H-invariant (i.e., if $f \in V$ then $t_h f \in V$ for any $h \in H$), if and only if there exist a measurable range function J such that*

$$V = \{f \in L^2(G) : \mathcal{T}f(\omega) \in J(\omega) \quad a.e.\, \omega \in \Omega\}.$$

If two range functions which are equal almost everywhere are identified, the correspondence is one-to-one and onto.

If $V = \overline{span}\{t_h\phi : h \in H, \phi \in \mathcal{A}\}$*, where* $\mathcal{A}$ *is a countable subset of* $L^2(G)$*, then*

$$J(\omega) = \overline{span}\{\mathcal{T}\phi(\omega) : \phi \in \mathcal{A}\} \tag{6.1}$$

Proof. ($\Rightarrow$) Assume $V \subseteq L^2(G)$ is H-invariant. Since $L^2(G)$ is separable, V is also separable. Therefore we claim that $\exists$ countable set $\mathcal{A}$ such that $V = S(\mathcal{A})$.

Let D be a countable dense subset of V. Since V is H-invariant, $span E_H(D) \subseteq V$. By assumption, V is a closed subspace, then $S(D) \subseteq V$. Also since D is dense, $\bar{D} = V$, $V \subseteq S(D) = \overline{span} E_H(D)$. Thus $V = S(D)$.

Define J as in (6.1). We will need to show the following:

1. $V = \{f \in L^2(G) : \mathcal{T}f(\omega) \in J(\omega) \quad a.e.\, \omega \in \Omega\}$.

2. J is measurable.

To prove statement 1, we need $\mathcal{T}V = M_J$.

- $\mathcal{T}V \subseteq M_J$:

Take $\Phi \in \mathcal{T}V$, then $\mathcal{T}^{-1}\Phi \in S(\mathcal{A})$. $\exists\{g_j\} \subset E_H(\mathcal{A})$ such that $g_j \to \mathcal{T}^{-1}\Phi$ in $L^2(G)$. $\mathcal{T}$ is an isometry, so $\Phi_j \equiv \mathcal{T}g_j \to \Phi$ in $L^2(\Omega, \ell^2(\Delta))$. We can find a subsequence Φ_{k_j} such that $\Phi_{k_j}(\omega) \to \Phi(\omega)$ a.e. $\omega \in \Omega$. By our definition of J in (6.1), clearly $\Phi_{k_j}(\omega) \in J(\omega)$. Since for any ω, $J(\omega)$ is closed subspace of $\ell^2(\Delta)$, $\Phi(\omega) \in J(\omega)$. Thus $\Phi \in M_J$.

- $M_J \subseteq \mathcal{T}V$:

It is enough to prove that if $\Psi \in L^2(\Omega, \ell^2(\Delta))$ satisfies $\Psi \perp \mathcal{T}V$ then $\Psi \perp M_J$. Let $\Psi \perp \mathcal{T}V$, then for all $\Phi \in \mathcal{T}\mathcal{A} \subset \mathcal{T}V$,

$$\langle \Phi, \Psi \rangle = \int_\Omega \langle \Phi(\omega), \Psi(\omega) \rangle dm_\Gamma(\omega) = 0.$$

V is H-invariant, for all h, $\mathcal{T}\tau_h\Phi = (h, -\cdot)\mathcal{T}\Phi \in \mathcal{T}V$. By the assumption,

$$\int_\Omega (h, -\omega) \langle \Phi(\omega), \Psi(\omega) \rangle dm_\Gamma(\omega) = 0.$$

The left hand side of the equation above is the Fourier transform of $\langle \Phi(\omega), \Psi(\omega) \rangle$. So we can conclude that $\langle \Phi(\omega), \Psi(\omega) \rangle = 0$ a.e. $\omega \in \Omega$. Since $J(\omega) = \overline{span}\{\Phi(\omega) = \mathcal{T}\phi(\omega) : \phi \in \mathcal{A}\}$, $\Psi(\omega) \perp J(\omega)$, we have $\Psi \perp M_J$.

To prove statement 2, we will show $P_\omega(\Psi(\omega)) = (\mathcal{P}\Psi)(\omega)$, where $\mathcal{P}$ is the orthogonal projection onto M_J. Note that Proposition 2 can not be directly applied here since the proposition require J to be measurable.

Denote $\mathcal{I}$ the identity operator. Let $\Psi \in L^2(\Omega, \ell^2(\Delta))$, then $(\mathcal{I} - \mathcal{P})\Psi \perp M_J$. By similar statement as proving $M_J \subseteq \mathcal{T}V$, for almost every $\omega \in \Omega$, $(\mathcal{I} - \mathcal{P})\Psi(\omega) \perp J(\omega)$. We have

$$0 = P_\omega((\mathcal{I} - \mathcal{P})\Psi(\omega)) = P_\omega(\Psi(\omega) - \mathcal{P}\Psi(\omega)) = P_\omega(\Psi(\omega)) - P_\omega(\mathcal{P}\Psi(\omega)).$$

By definition, $\mathcal{P}\Psi \in M_J$, $\mathcal{P}\Psi(\omega) \in J(\omega)$, then $P_\omega(\mathcal{P}\Psi(\omega)) = \mathcal{P}\Psi(\omega)$. We conclude that $P_\omega(\Psi(\omega)) = (\mathcal{P}\Psi)(\omega)$.

The mapping $\omega \mapsto (\mathcal{P}\Psi)(\omega)$ is measurable. Let $b \in \ell^2(\Delta)$, $\omega \mapsto \langle P_\omega(\Psi(\omega)), b \rangle = \langle (\mathcal{P}\Psi)(\omega), b \rangle$ is also measurable. J is a measurable range function.

($\Leftarrow$) Assuming there exist a measurable range function J such that $V = \mathcal{T}^{-1}M_J$, we need to show that V is H-invariant.

Let $f \in V$. $\forall h \in H$, $\mathcal{T}\tau_h f(\omega) = (h, -\omega)\mathcal{T}f(\omega) \in J(\omega)$ *a.e.* $\omega \in \Omega$ since $\mathcal{T}f(\omega) \in J(\omega)$ *a.e.* $\omega \in \Omega$ by assumption. Then $\mathcal{T}\tau_h f \in M_J$, $\tau_h f \in V$.

The uniqueness of the range function J will follow from Lemma 6.3.2, which is a consequence of Proposition 6.2.4.

Lemma 6.3.2. *If J and K are two measurable range functions such that $M_J = M_K$, then $J(\omega) = K(\omega)$ a.e. $\omega \in \Omega$.*

Suppose for a given H-invariant space V, there are two corresponding range function J and K. Since $M_J = \mathcal{T}V = M_K$, $J = K$ almost everywhere. $\square$

6.4 Frames for H-invariant spaces

Applying Theorem 6.3.1, we can determine whether a set $E_H(\mathcal{A})$ is a frame on its closed span by examining the fibers $\{\mathcal{T}\phi(\omega) : \phi \in \mathcal{A}\}$ when $\mathcal{A}$ is a countable subset of $L^2(G)$.

Theorem 6.4.1. *Let $\mathcal{A}$ be a countable subset of $L^2(G)$, J the measurable range function associated, and $A \leq B$ positive constants. Then the following are equivalent:*

(i) The set $\{\tau_h\phi : h \in H, \phi \in \mathcal{A}\}$ is a frame for its closed span with contants A and B.

(ii) For a.e. $\omega \in \Omega$, the set $\{\mathcal{T}\phi(\omega) : \phi \in \mathcal{A}\} \subseteq \ell^2(\Delta)$ is a frame for $J(\omega)$ with constants A and B.

Proof. Assuming either (i) or (ii) in the statement of Theorem 6.4.1, we have

$$\sum_{\phi\in\mathcal{A}}\sum_{h\in H}|\langle t_h\phi, f\rangle_{L^2(G)}|^2 = \sum_{\phi\in\mathcal{A}}\int_{\Omega}|\langle \mathcal{T}\phi(\omega), \mathcal{T}f(\omega)\rangle_{l^2(\Delta)}|^2 dm_\Gamma(\omega) \qquad (6.2)$$

by Parseval's identity.

$(i) \Rightarrow (ii)$: By Theorem 6.3.1, for any $f \in \overline{span}\{t_h\phi : h \in H, \phi \in \mathcal{A}\}$, we know $\mathcal{T}f \in J(\omega)$. Then $A\|\mathcal{T}f(\omega)\|^2 \le \sum_{\phi\in\mathcal{A}}|\langle \mathcal{T}\phi(\omega), \mathcal{T}f(\omega)\rangle|^2 \le B\|\mathcal{T}f(\omega)\|^2$. Since $\mathcal{T}$ is an isometry, by integrating the inequality over Ω, we get $(ii) \Rightarrow (i)$ from (6.2).

$(i) \Rightarrow (ii)$: It is sufficient to show for all $d \in D$

$$A\|P_\omega d\|^2 \le \sum_{\phi\in\mathcal{A}}|\langle \mathcal{T}\phi(\omega), P_\omega d\rangle|^2 \le B\|P_\omega d\|^2,\ a.e. \omega \in \Omega,$$

where D is a dense countable subset of $l^2(\Delta)$. If this is not true, then there exist $d_0 \in D$ such that either

$$\sum_{\phi\in\mathcal{A}}|\langle \mathcal{T}\phi(\omega), P_\omega d_0\rangle|^2 > (B+\epsilon)\|P_\omega d_0\|^2 \qquad (6.3)$$

or

$$\sum_{\phi\in\mathcal{A}}|\langle \mathcal{T}\phi(\omega), P_\omega d_0\rangle|^2 < (A-\epsilon)\|P_\omega d_0\|^2 \qquad (6.4)$$

on a measurable set $W \subseteq \Omega$ with positive measure.

Suppose (6.3) holds and take $f \in \overline{span}\{t_h\phi : h \in H, \phi \in \mathcal{A}\}$ such that $\mathcal{T}f(\omega) = \chi_W(\omega)P_\omega d_0$ (this is possible by Theorem 6.3.1). Then (i) and Proposition

1 give

$$A\|\mathcal{T}f\|^2 \leq \sum_{\phi\in\mathcal{A}} \int_\Omega |\langle \mathcal{T}\phi(\omega), \mathcal{T}f(\omega)\rangle_{l^2(\Delta)}|^2 dm_\Gamma(\omega) \leq B\|\mathcal{T}f\|^2,$$

which will lead to a contradiction with (6.3). This proves $(i) \Rightarrow (ii)$. □

Theorem 6.4.1 allows us to look at a fiber space $J(\omega)$, which is a smaller space compared to the larger space $\overline{span}(E_H(\mathcal{A}))$. For example, if $\mathcal{A}$ is a finite set, the corresponding $J(\omega)$ will be finite dimensional space, whereas $E_H(\mathcal{A})$ can be a infinite dimensional when H is infinite.

In [6, Chapter 3], the result was proved for principle shift-invariant space ($\mathcal{A}$ contains a single element) on $L^2(\mathbb{R})$, cf. [6, Theorem 3.56]. [36] also proved the theorem on LCA groups using similar method. The following corollary of Theorem 6.4.1 provides an alternate proof.

Corollary 6.4.2. *Let $\phi \in L^2(G)$ and $\Omega_\phi = \{\omega \in \Omega : \|\mathcal{T}\phi(\omega)\|^2 \neq 0\}$. Then the following are equivalent:*

1. *The set $E_H(\phi)$ is a frame for $S(\phi)$ with frame constants A and B.*

2. *For almost every $\omega \in \Omega_\phi$, $A \leq \|\mathcal{T}\phi(\omega)\|^2 \leq B$.*

Proof. By Theorem 6.4.1, statement 1 is equivalent with $\mathcal{T}\phi(\omega)$ is frame for its closed span. The elements in $\overline{span}\{\mathcal{T}\phi(\omega)\}$ is in the form of $c\mathcal{T}\phi(\omega)$, where c is any real number. Then the frame condition gives

$$A\|c\mathcal{T}\phi(\omega)\|^2 \leq |\langle c\mathcal{T}\phi(\omega), \mathcal{T}\phi(\omega)\rangle|^2 \leq B\|c\mathcal{T}\phi(\omega)\|^2.$$

Where

$$|\langle c\mathcal{T}\phi(\omega), \mathcal{T}\phi(\omega)\rangle|^2 = (|c|\|\mathcal{T}\phi(\omega)\|^2)^2 = |c|^2\|\mathcal{T}\phi(\omega)\|^4,$$

and

$$\|c\mathcal{T}\phi(\omega)\|^2 = |c|^2\|\mathcal{T}\phi(\omega)\|^2.$$

So when $\|\mathcal{T}\phi(\omega)\|^2 \neq 0$, that is, $\omega \in \Omega_\phi$,

$$A \leq \|\mathcal{T}\phi(\omega)\|^2 \leq B$$

□

A similar procedure yields an analogous result for Riesz bases. Given a countable subset $\mathcal{A} \subseteq L^2(G)$ and associated measurable range function J, the set $\{\tau_h\phi : h \in H, \phi \in \mathcal{A}\}$ is a Riesz basis for its closed span if and only if for almost every $\omega \in \Omega$, $\{\mathcal{T}\phi(\omega) : \phi \in \mathcal{A}\}$ is a Riesz basis for $J(\omega)$ with the same constants.

6.5 Example

Fix a function $g \in L^2(\mathbb{R})$, and α, β positive real numbers. Then the system $\{\tau_{n\beta}M_{m\alpha}g : m, n \in \mathbb{Z}\}$ is a Gabor system, where $M_x g(\gamma) = e^{2\pi i x\cdot\gamma}g(\gamma)$. Theorem 6.4.1 will provide another way to determine whether a Gabor system form a frame for its closed span.

As a simple example, take $f \in L^2(\mathbb{R})$ such that $\hat{f} = \chi_{[0,1]}$ and let $\beta = 1$. Then the Gabor system generated by f can be considered as $E_{\mathbb{Z}}(\mathcal{A})$, $\mathcal{A} \equiv \{M_{m\alpha}f : m \in \mathbb{Z}\}$. By Theorem 6.4.1, it is a frame if and only if for almost every $\omega \in \mathbb{T}$,

$\mathcal{T}(M_{m\alpha}f)(\omega)$ is a frame for its own closed span.

$$\begin{aligned}\mathcal{T}(M_{m\alpha})(\omega) &= \{\widehat{M_{m\alpha}f}(\omega+k)\}_{k\in\mathbb{Z}} \\ &= \{\hat{f}(\omega+k-m\alpha)\}_{k\in\mathbb{Z}} \\ &= \{\chi_{[m\alpha-k,m\alpha+1-k]}(\omega)\}_{k\in\mathbb{Z}}\end{aligned}$$

If we let $\alpha = 2$, then $\mathcal{T}(M_{m\alpha}f)(\omega)$ is a frame for its closed span. We can conclude that $\{\tau_{n\beta}M_{m\alpha}f : m,n \in \mathbb{Z}\}$ is a frame for its closed span.

Note: We can not conclude that whether $\{\tau_{n\beta}M_{m\alpha}g : m,n \in \mathbb{Z}\}$ is frame for $L^2(\mathbb{R})$ from Theorem 6.4.1. In fact, if $\alpha\beta > 1$, the Gabor system is not a frame for $L^2(\mathbb{R})$.

Aside from Gabor systems, the theory may also give us alternate method to analyze wavelet systems. For example, in an MRA, the set V_0 is a shift-invariant space. Theorem 6.3.1 may be helpful when constructing MRAs.

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