ABSTRACT

Title of Dissertation: BESOV WEll–POSEDNESS FOR HIGH DIMENSIONAL NON–LINEAR WAVE EQUATIONS

Jacob Sterbenz, Doctor of Philosophy, 2003

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Following work of Tataru, [13] and [11], we solve the division problem for wave equations with generic quadratic non-linearities in high dimensions. Specifically, we show that non–linear wave equations which can be written as systems involving equations of the form $\Box\phi = \phi\,\nabla\phi$ and $\Box\phi = |\nabla\phi|^2$ are well-posed with scattering in (6+1) and higher dimensions if the Cauchy data are small in the scale invariant ℓ^1 Besov space $\dot{B}^{s_c,1}$.

BESOV WEll–POSEDNESS FOR HIGH DIMENSIONAL NON–LINEAR WAVE EQUATIONS

by

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Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2003

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ACKNOWLEDGEMENTS

The author was supported by NSF grant DMS-0100406.

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Chapter 1

Introduction

In this paper, our aim is to give a more or less complete description of the global regularity properties of generic homogeneous quadratic non–linear wave equations on $(6+1)$ and higher dimensional Minkowski space. The equations we will consider are all of the form:

$$\Box\phi \ = \ \mathcal{N}(\phi, D\phi) \ , \tag{1.1}$$

where $\mathcal{N}$ is a smooth function of ϕ and its first partial derivatives, which we denote by $D\phi$. For all of the nonlinearities we study here, $\mathcal{N}$ will be assumed to be at least quadratic in nature, that is:

$$\mathcal{N}(X,Y) \ = \ O(|(X,Y)|^2) \ , \qquad\qquad (X,Y) \ \sim \ 0 \ .$$

The homogeneity condition we require $\mathcal{N}$ to satisfy is that there exist a (vector) σ such that:

$$\mathcal{N}(\lambda^\sigma\phi, \lambda^{\sigma+1}D\phi) \ = \ \lambda^{\sigma+2}\mathcal{N}(\phi, D\phi) \ , \tag{1.2}$$

where we use multiindex notation for vector $\mathcal{N}$. The condition (1.2) implies that solutions to the system (1.1) are invariant (again solutions) if one performs the scale transformations:

$$\phi(\cdot) \ \rightsquigarrow \ \lambda^\sigma\phi(\lambda\,\cdot) \ . \tag{1.3}$$

The general class of equations which falls under this description contains virtually all massless non–linear field theories on Minkowski space, including the Yang Mills equations (YM), the wave–maps equations (WM), and the Maxwell–Dirac equations (MD). We list the schematics for these systems respectively as:

$$\Box A \ = \ A\, DA + A^3 \ , \tag{YM}$$

$$\Box \phi \ = \ |D\phi|^2 \ , \tag{WM}$$

$$\begin{aligned} \Box u \ &= \ A\, Du \ , \\ \Box A \ &= \ |Du|^2 \ . \end{aligned} \tag{MD}$$

The various values of σ for these equations are (respectively) $\sigma = 1$, $\sigma = 0$, and $\sigma = (\frac{1}{2}, 1)$.

The central problem we will be concerned with is that of giving a precise description of the regularity assumptions needed in order to guarantee that the Cauchy problem for the system (1.1) is globally well posed with scattering (GWPS). That is, given initial data:

$$\phi(0) \ = \ f \ , \qquad\qquad \partial_t \phi\,(0) \ = \ g \ , \tag{1.4}$$

we wish to describe how much smoothness and decay (f, g) needs to possess in order for there to exist a unique global solution to the system (1.1) with this given initial data. We also wish to show that the solutions we construct depend continuously on the initial data, and are asymptotic to solutions of the linear part of (1.1). We will describe shortly in what sense we will require these notions to hold.

There are many reasons for the importance of discussing the low regularity properties of non–linear wave equations of the form (1.1) which go well beyond the simple desire to "count derivatives". In fact, our work here is not at all motivated by the desire to prove "local well posedness in the energy space" type results.[1] Rather, our main concern is to be able to prove global well posedness and scattering in a context where the initial data may be very smooth but does not possess enough decay at infinity to be in L^2. That is, we would like to consider data sets which are in homogeneous Sobolev spaces. From the point of view of homogeneity, it is not really possible to separate the smoothness and decay properties of initial data for equations like (1.1). Furthermore, from the point of view of constructing a Picard iteration for (1.1) which works globally in time, it is most natural to work with function spaces which are scale invariant with respect to (1.3). This leads directly to considerations of the low regularity properties of these equations as follows: By a simple scaling argument[2], one can see that the most efficient L^2 based regularity assumption possible on the initial data involves $s_c = \frac{n}{2} - \sigma$ derivatives. Again, by scale invariance and looking at unit frequency initial data, one can see that if we are to impose only an L^2 smallness condition

[1] In fact the type of function spaces we use here are not geared toward that kind of work as they involve an ℓ^1 structure.

[2] In conjunction with finite time blowup for large data. This phenomena is known to happen for higher dimensional equations with derivative non–linearities even in the presence of positive conserved quantities.

with no physical space weights, then $s_c = \frac{n}{2} - \sigma$ is in fact the *largest* amount of derivatives we may work with. This leads us to consider the question of GWPS for initial data in the homogeneous Besov spaces $\dot{B}^{s_c,p}$.

In recent years, there has been significant progress in our understanding of the low regularity local theory for general non–linear wave equations of the form (1.1). In the lower dimensional setting, i.e. when $n = 2, 3, 4$, it is known from counterexamples of Lindblad (see [8]) that there is ill posedness for initial data in the Sobolev space H^{s_0}, where $s_0 \leqslant s_c + \frac{5-n}{4}$. Intimately connected with this phenomena is the failure of certain space–time estimates for the linear wave equation known as *Strichartz estimates*. Specifically, one does not have anything close to an $L^2(L^4)$ estimate in these dimensions. Such an estimate obviously plays a crucial role (via Duhamel's principle) in the quadratic theory. Also, using the Strichartz estimates available in these dimensions along with Picard iteration in certain function spaces, one can show that the Lindblad counterexamples are sharp in that there is local well–posedness for initial data in the spaces H^s when $s_c + \frac{5-n}{4} < s$. Once in the higher dimensional setting, i.e. when the number of spatial dimensions is $n = 5$ or greater, one does have access to the $L^2(L^4)$ estimate (see [1]), and it is possible to push the local theory down to $H^{s_c+\epsilon}$, where $0 < \epsilon$ is arbitrary (see [13]).

In all dimensions, the single most important factor which determines the local theory as well as the range of validity for Strichartz estimates is the existence of free waves which are highly concentrated along null directions in Minkowski space. These waves, known as *Knapp counterexamples*, resemble a single beam of light, which remains coherent for a long period of time before dispersing. For a special class of non–linearities, known as "null structures", interactions between these coherent beams are effectively canceled, and one gains an improvement in the local theory of equations whose nonlinearities have this form (see for example [3], [7]).

In both high and low dimensional settings, the analysis of certain null structures, specifically non–linearities containing the Q_0 null from[3], has led to the proof that the wave–maps model equations[4] are well posed in the scale invariant ℓ^1 Besov space $\dot{B}^{\frac{n}{2},1}$ (see [11] and [12]). While the proof of this result is quite simple for high dimensions, it relies in an essential way on the structure of the Q_0 null form. In fact, there is no direct way to extend the proof of this result to include the less regular Q_{ij} null forms which show up in the equations of gauge field theory. However, the high dimensional non–linear interaction of coherent waves is quite weak (e.g. giving the desired range of validity for Strichartz estimates), and one would expect that it is possible to prove local well posedness for quadratic equations with initial data in the scale invariant ℓ^1 Besov space without resorting to any additional structure in the nonlinearity. For $n = 5$ dimensions, it may be that this is not quite possible, and we have no proof or conjecture here.[5] For $n = 6$ and higher dimensions, we will prove that in fact no null structure is

[3]This is defined by the equation $Q_0(\phi, \psi) = \partial_\alpha \phi \, \partial^\alpha \psi$.

[4]Not the rough schematic we have listed here, but rather equations of the form $\Box \phi = \Gamma(\phi) Q_0(\phi, \phi)$.

[5]If true, this is probably quite technical, being that the $L^2(L^4)$ Strichartz estimate is an endpoint estimate in this dimension ([1]). In some sense, the $n = 5$ case is the endpoint of an endpoint.

needed for there to be well posedness in $\dot{B}^{s_c,1}$. This leads to the statement of our main result which is as follows:

Theorem 1.0.1 (Global Well Posedness). *Let $6 \leqslant n$ be the number of spatial dimensions. For any of the generic equations listed above: YM, WM, or MD, let (f, g) be a (possibly vector valued) initial data set. Let $s_c = \frac{n}{2} - \sigma$ be the corresponding L^2 scaling exponent. Then there exists constants $0 < \epsilon_0, C$ such that if*

$$\| (f, g) \|_{\dot{B}^{s_c,1} \times \dot{B}^{s_c - 1,1}} \quad \leqslant \quad \epsilon_0 \ , \tag{1.5}$$

there exits a global solution ψ which satisfies the continuity condition:

$$\| \psi \|_{C(\dot{B}^{s_c,1}) \cap C^{(1)}(\dot{B}^{s_c-1,1})} \quad \leqslant \quad C \| (f, g) \|_{\dot{B}^{s_c,1} \times \dot{B}^{s_c-1,1}} \ . \tag{1.6}$$

The solution ψ is unique in the following sense: There exists a sequence of smooth functions (f_N, g_N) such that:

$$\lim_{N \to \infty} \| (f, g) - (f_N, g_N) \|_{\dot{B}^{s_c,1} \times \dot{B}^{s_c-1,1}} \ = \ 0 \ .$$

For this sequence of functions, there exists a sequence of unique smooth global solutions ψ_N of (1.1) *with this initial data. Furthermore, the ψ_N converge to ψ as follows:*

$$\lim_{N \to \infty} \| \psi - \psi_N \|_{C(\dot{B}^{s_c,1}) \cap C^{(1)}(\dot{B}^{s_c-1,1})} \ = \ 0 \ .$$

Also, ψ is the only solution which may be obtained as a limit (in the above sense) of solutions to (1.1) *with regularizations of (f, g) as initial data. Finally, ψ retains any extra smoothness inherent in the initial data. That is, if (f, g) also has finite $\dot{H}^s \times \dot{H}^{s-1}$ norm, for $s_c < s$, then so does ψ at fixed time and one has the following estimate:*

$$\| \psi \|_{C(\dot{H}^s) \cap C^{(1)}(\dot{H}^{s-1})} \quad \leqslant \quad C \| (f, g) \|_{\dot{H}^s \times \dot{H}^{s-1}} \ . \tag{1.7}$$

In a straightforward way, our estimates also prove the following scattering result:

Theorem 1.0.2. *Using the same notation as above we have that there exists data sets* $(f^{\pm}, g^{\pm})$*, such that if* $\psi^{\pm}$ *is the solution to the homogeneous wave equation with the corresponding initial data, the following asymptotics hold:*

$$\lim_{t\to\infty} \| \psi^{+} - \psi \|_{\dot{B}^{s_c,1}\cap\partial_t\dot{B}^{s_c-1,1}} = 0 \ , \tag{1.8}$$

$$\lim_{t\to-\infty} \| \psi^{-} - \psi \|_{\dot{B}^{s_c,1}\cap\partial_t\dot{B}^{s_c-1,1}} = 0 \ . \tag{1.9}$$

Furthermore, the scattering operator retains any additional regularity inherent in the initial data. That is, if (f, g) *has finite* $\dot{H}^s$ *norm, then so does* $(f^{\pm}, g^{\pm})$*, and the following asymptotics hold:*

$$\lim_{t\to\infty} \| \psi^{+} - \psi \|_{\dot{H}^{s}\cap\partial_t\dot{H}^{s-1}} = 0 \ , \tag{1.10}$$

$$\lim_{t\to-\infty} \| \psi^{-} - \psi \|_{\dot{H}^{s}\cap\partial_t\dot{H}^{s-1}} = 0 \ . \tag{1.11}$$

Remark 1.0.3. The dividends of the low regularity point of view are already apparent here in the high dimensional setting. For instance, they show that the results of [2] can be recovered in a context which is translation independent and where there is relatively little decay of the initial data and no reference to uniform decay of the solution. Furthermore, our method leads to a precise description of what needs to be "small" in order for there to global existence for (even smooth) initial data sets. Also, we construct a proof of scattering which shows exactly which frequency interactions in the non–linear part of (1.1) contribute to the scattering operator, and which frequency interactions "drop out" at infinity. However, we expect that the greatest payoffs of this work will come when investigating the small data global theory for $(3+1)$ and $(4+1)$ dimensional gauge field equations. Specifically, we hope that the methods here will[6] be able to resolve the GWPS problem for gauge field theories in the presence of charge (see [10] and [9]). This will be the subject of further work.

[6]With the help of some new technical devices in the $(3+1)$ dimensional regime.

Chapter 2

Preliminary Notation

For quantities A and B, we denote by $A \lesssim B$ to mean that $A \leqslant C \cdot B$ for some large constant C. The constant C may change from line to line, but will always remain fixed for any given instance where this notation appears. Likewise we use the notation $A \sim B$ to mean that $\frac{1}{C} \cdot B \leqslant A \leqslant C \cdot B$. We also use the notation $A \ll B$ to mean that $A \leqslant \frac{1}{C} \cdot B$ for some large constant C. This is the notation we will use throughout the paper to break down quantities into the standard cases: $A \sim B$, or $A \ll B$, or $B \ll A$; and $A \lesssim B$, or $B \ll A$, without ever discussing which constants we are using.

For a given function of two variables $(t, x) \in \mathbf{R} \times \mathbf{R^3}$ we write the spatial and space–time Fourier transform as:

$$\widehat{f}(t, \xi) = \int e^{-2\pi i \xi \cdot x} \, f(t, x) \; dx \ ,$$
$$\widetilde{f}(\tau, \xi) = \int e^{-2\pi i (\tau t + \xi \cdot x)} \; f(t, x) \; dt dx \ .$$

respectively. At times, we will also write $\mathcal{F}[f] = \widetilde{f}$.

For a given set of functions of the spatial variable only, we denote by $W(f, g)$

the solution of the homogeneous wave equation with Cauchy data (f, g). If F is a function on space–time, we will denote by $W(F)$ the function $W\left(F(0), \partial_t F\,(0)\right)$.

Let E denote any fundamental solution to the homogeneous wave equation. i.e., one has the formula $\Box E = \delta$. We define the standard Cauchy parametrix for the wave equation by the formula:

$$\frac{1}{\Box}F \ = \ E * F - W(E * F) \ .$$

Explicitly, one has the identity:

$$\widehat{\frac{1}{\Box}F}\,(t,\xi) \ = \ \int_0^t \frac{sin\,(2\pi|\xi|(t-s))}{2\pi|\xi|}\widehat{F}(s,\xi)\ ds \ . \tag{2.1}$$

For any function F which is supported away from the light cone in Fourier space, we shall use the following notation for division by the symbol of the wave equation:

$$\frac{1}{\varXi}F \ = \ E * F \ .$$

Of course, the definition of $\frac{1}{\varXi}$ does not depend on E so long as for F is supported away from the light cone; for us that will always be the case when we use this notation. Explicitly, one has the formula:

$$\mathcal{F}\left[\frac{1}{\varXi}F\right](\tau,\xi) \ = \ \frac{1}{4\pi^2(\tau^2 - |\xi|^2)}\widetilde{F}(\tau,\xi) \ .$$

Chapter 3

Multipliers and Function Spaces

Let φ be a smooth bump function (i.e. supported on the set $|s| \leqslant 2$ such that $\varphi = 1$ for $|s| \leqslant 1$). In what follows, it will be a great convenience for us to assume that φ may change its exact form for two separate instances of the symbol φ (even if they occur on the same line). In this way, we may assume without loss of generality that in addition to being smooth, we also have the idempotence identity $\varphi^2 = \varphi$. We shall use this convention for all the cutoff functions we introduce in the sequel.

For $\lambda \in 2^{\mathbb{Z}}$, we denote the dyadic scaling of φ by $\varphi_\lambda(s) = \varphi(\frac{s}{\lambda})$. The most basic Fourier localizations we shall use here are with respect to the space-time variable and the distance from the cone. Accordingly, we form the Littlewood-Paley type cutoff functions:

$$s_\lambda(\tau, \xi) = \varphi_{2\lambda}(|(\tau, \xi)|) - \varphi_{\frac{1}{2}\lambda}(|(\tau, \xi)|) \ , \tag{3.1}$$

$$c_d(\tau, \xi) = \varphi_{2d}(|\tau| - |\xi|) - \varphi_{\frac{1}{2}d}(|\tau| - |\xi|) \ . \tag{3.2}$$

We now denote the corresponding Fourier multiplier operator via the formulas $\widetilde{S_\lambda u} = s_\lambda \widetilde{u}$ and $\widetilde{C_d u} = c_d \widetilde{u}$ respectively. We also use a multi-subscript notation to denote products of the above operators, e.g. $S_{\lambda,d} = S_\lambda C_d$. We shall use the notation:

$$S_{\lambda,\bullet\leqslant d} = \sum_{\delta\leqslant d} S_{\lambda,\delta} \ , \tag{3.3}$$

to denote cutoff in an $O(d)$ neighborhood of the light cone in Fourier space. At times it will also be convenient to write $S_{\lambda,d\leqslant\bullet} = S_\lambda - S_{\lambda,\bullet<d}$. We shall also use the notation $S^{\pm}_{\lambda,d}$ etc. to denote the multiplier $S_{\lambda,d}$ cutoff in the half space $\pm\tau > 0$.

The other type of Fourier localization which will be central to our analysis is the decomposition of the spatial variable into radially directed blocks of various sizes. To begin with, we denote the spatial frequency cutoff by:

$$p_\lambda(\xi) = \varphi_{2\lambda}(|\xi|) - \varphi_{\frac{1}{2}\lambda}(|\xi|) \ , \tag{3.4}$$

with P_λ the corresponding operator. For a given parameter $\delta \leqslant \lambda$, we now decompose P_λ radially as follows. First decompose the the unit sphere $S^{n-1} \subset \mathbb{R}^n$ into angular sectors of size $\frac{\delta}{\lambda}\times\ldots\times\frac{\delta}{\lambda}$ with bounded overlap (independent of δ). These angular sectors are then projected out to frequency λ via rays through the origin. The result is a decomposition of $supp\{p_\lambda\}$ into radially directed blocks of size $\lambda \times \delta \times \ldots \times \delta$ with bounded overlap. We enumerate these blocks and label the corresponding partition of unity by $b^\omega_{\lambda,\delta}$. It is clear that things may be arranged so that upon rotation onto the ξ_1–axis, each $b^\omega_{\lambda,\delta}$ satisfies the bound:

$$|\partial_1^N b^\omega_{\lambda,\delta}| \leqslant C_N \lambda^{-N} \ , \qquad\qquad |\partial_i^N b^\omega_{\lambda,\delta}| \leqslant C_N \delta^{-N} \ . \tag{3.5}$$

In particular, each $B^\omega_{\lambda,\delta}$ is given by convolution with an L^1 kernel. We shall also denote:

$$S^\omega_{\lambda,d} = B^\omega_{\lambda,(\lambda d)^{\frac{1}{2}}} S_{\lambda,d} \ , \qquad\qquad S^\omega_{\lambda,\bullet\leqslant d} = B^\omega_{\lambda,(\lambda d)^{\frac{1}{2}}} S_{\lambda,\bullet\leqslant d} \ .$$

Note that the operators $S^{\omega}_{\lambda,d}$ and $S^{\omega}_{\lambda,\bullet\leqslant d}$ are only supported in the region where $|\tau| \lesssim |\xi|$.

We now use these multipliers to define the following dyadic norms, which will be the building blocks for the function spaces we will use here.

$$\| u \|^p_{X^{\frac{1}{2}}_{\lambda,p}} = \sum_{d\in 2^{\mathbb{Z}}} d^{\frac{p}{2}} \| S_{\lambda,d} u \|^p_{L^2} \ , \qquad (\text{“classical” } H^{s,\delta}) \tag{3.6}$$

$$\| u \|_{Y_\lambda} = \lambda^{-1} \| \Box S_\lambda u \|_{L^1(L^2)} \ , \qquad (\text{Duhamel}) \tag{3.7}$$

$$\| u \|_{Z_\lambda} = \lambda^{\frac{2-n}{2}} \sum_d \left(\sum_\omega \| S^{\omega}_{\lambda,d} u \|^2_{L^1(L^\infty)} \right)^{\frac{1}{2}} \ . \qquad (\text{outer block}) \tag{3.8}$$

We define the spaces $X^{\frac{1}{2}}_{\lambda,p}$ and Y_λ to be the completion of test functions under the respective (semi) norms. It is no too difficult to see that the resulting space of distributions contains more than Fourier transforms of L^p functions with finite weighted norms, and includes L^2 measures supported on the light cone in Fourier space.[1] Because of this it will be convenient for us to include an extra $L^\infty(L^2)$ norm in the definition of our function spaces. However, it should be noted that this extra norm is implicit in the completion of test functions under (3.6)–(3.7). These considerations lead us to define, at fixed frequency, the (semi) norms:

$$\| u \|_{F_\lambda} = \left(X^{\frac{1}{2}}_{\lambda,1} + Y_\lambda \right) \cap S_\lambda \left(L^\infty(L^2) \right) \ . \tag{3.9}$$

Unfortunately, the above norm is still not strong enough for us to be able to iterate equations of the form (1.1) which contain derivatives. This is due to

[1] See the next section for details.

a very specific $Low \times High$ frequency interaction in quadratic non–linearities. Fortunately, this problem has been effectively handled by Tataru in [13], based on ideas from [5] and [6]. What is necessary is to add some extra $L^1(L^\infty)$ norms on "outer block" regions of Fourier space. This is the essence of the norm (3.8) above, which is a slight variant of that which appeared in [13]. This leads to our second main dyadic norm:

$$\| u \|_{G_\lambda} = \left(X^{\frac{1}{2}}_{\lambda,1} + Y_\lambda \right) \cap S_\lambda \left(L^\infty(L^2) \right) \cap Z_\lambda \ . \tag{3.10}$$

Finally, the spaces we will iterate in are produced by adding the appropriate number of derivatives combined with the necessary Besov structures:

$$\| u \|^2_{F^s} \ = \ \sum_\lambda \lambda^{2s} \| u \|^2_{F_\lambda} \ , \tag{3.11}$$

$$\| u \|_{G^s} \ = \ \sum_\lambda \lambda^{s} \| u \|_{G_\lambda} \ . \tag{3.12}$$

Due to the need for precise microlocal decompositions, of crucial importance to us will be the boundedness of certain multipliers on the components (3.6)–(3.7) of our function spaces as well as mixed Lebesgue spaces. We state these as follows:

Lemma 3.0.4 (Multiplier boundedness).

1. *The following multipliers are given by L^1 kernels:* $\lambda^{-1}\nabla S_\lambda$, $S^\omega_{\lambda,d}$, $S^\omega_{\lambda,\bullet\leqslant d}$, *and* $(\lambda d)\Xi^{-1}S^\omega_{\lambda,d}$ *. In particular, all of these are bounded on every mixed Lebesgue space* $L^q(L^r)$.

2. *The following multipliers are bounded on the spaces* $L^q(L^2)$*, for* $1 \leqslant q \leqslant \infty$*:* $S_{\lambda,d}$ *and* $S_{\lambda,\bullet\leqslant d}$.

Proof of Lemma 3.0.4 (1). First, notice that after a rescaling, the symbol for the multiplier $\lambda^{-1}\nabla S_\lambda$ is a C^∞ bump function with $O(1)$ support. Thus, its kernel is in L^1 with norm independent of λ.

For the remainder of the operators listed in (1) above, it suffices to work with $(\lambda d)\Xi^{-1}S^\omega_{\lambda,d}$. The boundedness of the others follows from a similar argument. We let $\chi^\pm$ denote the symbol of this operator cut off in the upper resp. lower half plane. After a rotation in the spatial domain, we may assume that the spatial projection of $\chi^\pm$ is directed along the positive ξ_1 axis. Now look at $\chi^+(s,\eta)$ with coordinates:

$$\begin{aligned} s \ &= \ \frac{1}{\sqrt{2}}(\tau - \xi_1) \ , \\ \eta_1 \ &= \ \frac{1}{\sqrt{2}}(\tau + \xi_1) \ , \\ \eta' \ &= \ \xi' \ . \end{aligned}$$

It is apparent that $\chi^+(s,\eta)$ has support in a box of dimension $\sim \lambda \times \sqrt{\lambda d} \times \ldots \times \sqrt{\lambda d} \times d$ with sides parallel to the coordinate axis and longest side in the η_1 direction and shortest side in the s direction. Furthermore, a direction calculation shows that one has the bounds:

$$|\partial^N_{\eta_1}\chi^+| \leqslant C_N\lambda^{-N} \ , \qquad |\partial^N_{\eta'}\chi^+| \leqslant C_N(\lambda d)^{-N/2} \ , \qquad |\partial^N_s\chi^+| \leqslant C_N d^{-N} \ .$$

Therefore, we have that χ^+ yields an L^1 kernel. A similar argument works for the cutoff function χ^-, using the rotation:

$$\begin{aligned} s \ &= \ \frac{1}{\sqrt{2}}(\tau + \xi_1) \ , \\ \eta_1 \ &= \ \frac{1}{\sqrt{2}}(-\tau + \xi_1) \ , \\ \eta' \ &= \ \xi' \ . \end{aligned}$$

$\square$

Proof of Lemma 3.0.4 (2). We will argue here for $S_{\lambda,d}$. The estimates for the others follow similarly. If we denote by $K^{\pm}(t,x)$ the convolution kernel associated with $S^{\pm}_{\lambda,d}$, then a simple calculation shows that:

$$e^{\mp 2\pi i t|\xi|}\widehat{K^{\pm}}(t,\xi) \ = \ \int e^{2\pi i t\tau}\psi(\tau,\xi)\,d\tau \ ,$$

where $supp\{\psi\}$ is contained in a box of dimension $\sim \quad \lambda\times\ldots\times\lambda\times d$ with sides along the coordinate axis and short side in the τ direction. Furthermore, one has the estimate:

$$|\partial_\tau^N\psi| \ \leqslant \ C_N\,d^{-N} \ .$$

This shows that we have the bound:

$$\|\,\widehat{K^{\pm}}\,\|_{L^1_\tau(L^\infty_\xi)} \ \lesssim \ 1 \ ,$$

independent of λ and d. Thus, we get the desired bounds for the convolution kernels. $\square$

As an immediate application of the above lemma, we show that the extra Z_λ intersection in the G_λ norm above only effects the $X^{\frac{1}{2}}_{\lambda,1}$ portion of things.

Lemma 3.0.5 (Outer block estimate on Y_λ). *For $5 < n$, one has the following uniform inclusion:*

$$Y_\lambda \ \subseteq \ Z_\lambda \ . \tag{3.13}$$

proof of (3.13). It is enough to show that:

$$\left(\sum_{\omega} \| \Xi^{-1} S^{\omega}_{\lambda,d} u \|^2_{L^1(L^\infty)}\right)^{\frac{1}{2}} \lesssim \lambda^{\frac{n-4}{2}} \left(\frac{d}{\lambda}\right)^{\frac{n-5}{4}} \| S_\lambda u \|_{L^1(L^2)} \ .$$

First, using a local Sobolev embedding, we see that:

$$\| B^{\omega}_{\lambda,(\lambda d)^{\frac{1}{2}}} \Xi^{-1} S^{\omega}_{\lambda,d} u \|_{L^1(L^\infty)} \lesssim \lambda^{\frac{n+1}{4}} d^{\frac{n-1}{4}} \| \Xi^{-1} S^{\omega}_{\lambda,d} u \|_{L^1(L^2)} \ .$$

Therefore, using the boundedness Lemma 3.0.4, it suffices to note that by Minkowski's inequality we can bound:

$$\begin{aligned} \left(\sum_{\omega} \Big(\int \| S^{\omega}_{\lambda,d} u \|_{L^2_x}\Big)^2\right)^{\frac{1}{2}} &\lesssim \int \left(\sum_{\omega} \| S^{\omega}_{\lambda,d} u \|^2_{L^2_x}\right)^{\frac{1}{2}} \ , \\ &\lesssim \| S_{\lambda,d} u \|_{L^1(L^2)} \ . \end{aligned}$$

□

The last line of the above proof showed that it is possible to bound a square sum over an angular decomposition of a given function in $L^1(L^2)$. It is also clear that this same procedure works for the $X^{\frac{1}{2}}_{\lambda,1}$ spaces because one can use Minkowski's inequality for the ℓ^1 sum with respect to the cone variable d. This fact will be of great importance in what follows and we record it here as:

Lemma 3.0.6 (Angular reconstruction of norms). *Given a test function u and parameter $\delta \leqslant \lambda$, one can bound:*

$$\left(\sum_{\omega} \| B^{\omega}_{\lambda,\delta} u \|^2_{X^{\frac{1}{2}}_{\lambda,1}, Y_\lambda}\right)^{\frac{1}{2}} \lesssim \| u \|_{X^{\frac{1}{2}}_{\lambda,1}, Y_\lambda} \ . \tag{3.14}$$

Chapter 4

Structure of the F_λ spaces

The purpose of this section is to clarify some remarks of the previous section and write down two integral formulas for functions in the F_λ space. This material is all more or less standard in the literature and we include it here primarily because the notation will be useful for our scattering result. Our first order of business is to write down a decomposition for functions in the F_λ space:

Lemma 4.0.7 (F_λ decomposition). *For any $u_\lambda \in F_\lambda$, one can write:*

$$u_\lambda \ = \ u_{\mathring{X}_\lambda} + u_{X^{1/2}_{\lambda,1}} + u_{Y_\lambda} \ , \tag{4.1}$$

where $u_{\mathring{X}_\lambda}$ is a solution to the homogeneous wave equation, $u_{X^{1/2}_{\lambda,1}}$ is the Fourier transform of a function, and u_{Y_λ} satisfies:

$$u_{Y_\lambda}(0) = \partial_t u_{Y_\lambda}(0) = 0 \ .$$

Furthermore, one has the norm bounds:

$$\frac{1}{C}\| \, u_\lambda \, \|_{F_\lambda} \ \leqslant \ \left(\| \, u_{\mathring{X}_\lambda} \, \|_{L^\infty(L^2)} + \| \, u_{X^{1/2}_{\lambda,1}} \, \|_{X^{\frac{1}{2}}_{\lambda,1}} + \| \, u_{Y_\lambda} \, \|_{Y_\lambda} \right) \ \leqslant \ C \| \, u_\lambda \, \|_{F_\lambda} \ . \tag{4.2}$$

We now show that the two inhomogeneous terms on the right hand side of (4.1) can be written as integrals over solutions to the wave equation with L^2 data. This fact will be of crucial importance to us in the sequel. The first formula is simply a restatement of (2.1):

Lemma 4.0.8 (Duhamel's principle). *Using the same notation as above, for any u_{Y_λ}, one can write:*

$$u_{Y_\lambda}(t) \ = \ \int_0^t |D_x|^{-1} sin\big((t-s)|D_x|\big)\, \Box u_{Y_\lambda}(s)\ ds \ . \tag{4.3}$$

Likewise, one can write the $u_{X^{1/2}_{\lambda,1}}$ portion of the sum (4.1) as an integral over modulated solutions to the wave equation be foliating Fourier space by forward and backward facing light–cones:

Lemma 4.0.9 ($X^{\frac{1}{2}}_{\lambda,1}$ Trace lemma). *For any $u_{X^{1/2}_{\lambda,1}}$, let $u^\pm_{X^{1/2}_{\lambda,1}}$ denote its restriction to the frequency half space $0 < \pm\tau$. Then one can write:*

$$u^\pm_{X^{1/2}_{\lambda,1}}(t) \ = \ \int e^{\pm 2\pi i t(s+|D_x|)}\, u^\pm_{\lambda,s}\ ds \ , \tag{4.4}$$

where $u^\pm_{\lambda,s}$ is the spatial Fourier transform of $\widetilde{u^\pm}$ to the forward or backward light–cone, i.e.:

$$\widehat{u^\pm}_{\lambda,s}(\xi) \ = \ \int \delta(\tau \mp s \mp |\xi|)\, \widetilde{u^\pm}(\tau,\xi)\ d\tau \ .$$

In particular. one has the formula:

$$\int \| u^\pm_{\lambda,s} \|_{L^2}\, ds \ \lesssim \ \| u^\pm_{X^{1/2}_{\lambda,1}} \|_{X^{\frac{1}{2}}_{\lambda,1}} \ . \tag{4.5}$$

Chapter 5

Strichartz estimates

Our inductive estimates will be based on a method of bilinear decompositions and local Strichartz estimates as in the work [13]. We first state the standard Strichartz from which the local estimates follow.

Lemma 5.0.10 (Homogeneous Strichartz estimates (see [1])). *Let $5 < n$, $\sigma = \frac{n-1}{2}$, and suppose u is a given function of the spatial variable only. Then if $\frac{1}{q} + \frac{\sigma}{r} \leqslant \frac{\sigma}{2}$ and $\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - \gamma$, the following estimate holds:*

$$\| e^{\pm 2\pi i t |D_x|} P_{\bullet \leqslant \lambda} u \|_{L^q_t(L^r_x)} \lesssim \lambda^{\gamma} \| P_{\bullet \leqslant \lambda} u \|_{L^2} . \tag{5.1}$$

Combining the $L^2(L^{\frac{2(n-1)}{n-3}})$ endpoint of the above estimate with a local Sobolev in the spatial domain, we arrive at the following local version of (5.1):

Lemma 5.0.11 (Local Strichartz estimate). *Let $5 < n$, then the following estimate holds:*

$$\| e^{\pm 2\pi i t |D_x|} B^{\omega}_{\lambda, (\lambda d)^{\frac{1}{2}}} u \|_{L^2_t(L^\infty_x)} \lesssim \lambda^{\frac{n+1}{4}} d^{\frac{n-3}{4}} \| B^{\omega}_{\lambda, (\lambda d)^{\frac{1}{2}}} u \|_{L^2} . \tag{5.2}$$

Using the integral formulas (4.3) and (4.4), we can transfer the above estimates to the F_λ spaces:

Lemma 5.0.12 (**F_λ Strichartz estimates**). *Let $5 < n$ and set $\sigma = \frac{n-1}{2}$. Then if $\frac{1}{q} + \frac{\sigma}{r} \leqslant \frac{\sigma}{2}$ and $\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - \gamma$, the following estimates hold:*

$$\| S_\lambda u \|_{L^q(L^r)} \ \lesssim \ \lambda^\gamma \| u \|_{F_\lambda} \ , \tag{5.3}$$

$$\left(\sum_\omega \| S^\alpha_{\lambda, \bullet \leqslant d} u \|^2_{L^2(L^\infty)} \right)^{\frac{1}{2}} \ \lesssim \ \lambda^{\frac{n+1}{4}} d^{\frac{n-3}{4}} \| u \|_{F_\lambda} \ . \tag{5.4}$$

Proof of 5.0.12. Using the decomposition (4.0.7) and the angular reconstruction formula (3.14), it is enough to prove (5.3) for functions $u_{X^{1/2}_{\lambda,1}}$ and u_{Y_λ}. Using the integral formula (4.4), we see immediately that:

$$\begin{aligned} \| u_{X^{1/2}_{\lambda,1}} \|_{L^q(L^r)} \ &\leqslant \sum_\pm \int \| e^{\pm 2\pi i t |D_x|} \, u^\pm_{\lambda,s} \|_{L^q(L^r)} \ ds \ , \\ &\lesssim \lambda^\gamma \sum_\pm \int \| u^\pm_{\lambda,s} \|_{L^2} \ ds \ , \\ &\lesssim \lambda^\gamma \| u_{X^{1/2}_{\lambda,1}} \|_{X^{\frac{1}{2}}_{\lambda,1}} \ . \end{aligned}$$

For the u_{Y_λ} portion of things, we can chop the function up into a fixed number of space–time angular sectors using L^1 convolution kernels. Doing this and using R_α to denote an operator from the set $\{I, \partial_i |D_x|^{-1}\}$, we estimate:

$$\begin{aligned} \| u_{Y_\lambda} \|_{L^q(L^r)} \ &\leqslant \lambda^{-1} \sum_\alpha \ \| \partial_\alpha u_{Y_\lambda} \|_{L^q(L^r)} \ , \\ &\lesssim \lambda^{-1} \sum_{\pm\, , \, \alpha} \int \ \| e^{\pm 2\pi i t |D_x|} \left(e^{\mp 2\pi i s |D_x|} R_\alpha \, \Box u_{Y_\lambda} \, (s,x) \right) \|_{L^q_t(L^r_x)} \ ds \ , \\ &\lesssim \lambda^\gamma \lambda^{-1} \sum_\alpha \int \ \| e^{\mp 2\pi i s |D_x|} R_\alpha \, \Box u_{Y_\lambda} \, (s,x) \|_{L^2_x} \ ds \ , \\ &\leqslant \lambda^\gamma \ \| u_{Y_\lambda} \|_{Y_\lambda} \ . \end{aligned}$$

$\square$

A consequence of (5.3) is that we have the embedding:

$$X^{\frac{1}{2}}_{\lambda,1} \ \subseteq \ L^\infty(L^2) \ .$$

Using a simple approximation argument along with uniform convergence, we arrive at the following energy estimate for the F^s and G^s spaces:

Lemma 5.0.13 (Energy estimates). *For space–time functions u, one has the following estimates:*

$$\| u \|_{C(\dot{H}^s)\cap C^{(1)}(\dot{H}^{s-1})} \ \lesssim \ \| u \|_{F^s} \ , \tag{5.5}$$

$$\| u \|_{C(\dot{B}^s)\cap C^{(1)}(\dot{B}^{s-1})} \ \lesssim \ \| u \|_{G^s} \ . \tag{5.6}$$

Also, by duality and the estimate (5.5), we have that:

$$\lambda \varXi^{-1} L^1(L^2) \ \subseteq \ \lambda \varXi^{-1} X^{-\frac{1}{2}}_{\lambda,\infty} \ \subseteq \ X^{\frac{1}{2}}_{\lambda,\infty} \ . \tag{5.7}$$

This proves shows:

Lemma 5.0.14 (L^2 estimate for Y_λ). *The following inclusion holds uniformly:*

$$d^{\frac{1}{2}} S_{\lambda,d}(Y_\lambda) \ \subseteq \ L^2(L^2) \ , \tag{5.8}$$

in particular, by dyadic summing one has:

$$d^{\frac{1}{2}} S_{\lambda,d\leqslant\bullet}(F_\lambda) \ \subseteq \ L^2(L^2) \ .$$

Chapter 6

Scattering

It turns out that our scattering result, Theorem 1.0.2, is implicitly contained in the function spaces F^s and G^s. That is, there is scattering in these spaces independently of any specific equation being considered. Therefore, to prove Theorem 1.0.2, it will only be necessary to show that our solution to (1.1) belongs to these spaces.

Using a simple approximation argument, it suffices to deal with things at fixed frequency. Because the estimates in Theorem 1.0.2 deal with more than one derivative, we will show that:

Lemma 6.0.15 (F_λ scattering). *For any function $u_\lambda \in F_\lambda$, there exists a set of initial data $(f^\pm_\lambda, g^\pm_\lambda) \in P_\lambda(L^2) \times \lambda P_\lambda(L^2)$ such that the following asymptotic holds:*

$$\lim_{t\to\infty} \| u_\lambda(t) - W(f^+_\lambda, g^+_\lambda)(t) \|_{\dot{H}^1 \cap \partial_t(L^2)} = 0 , \tag{6.1}$$

$$\lim_{t\to -\infty} \| u_\lambda(t) - W(f^-_\lambda, g^-_\lambda)(t) \|_{\dot{H}^1 \cap \partial_t(L^2)} = 0 . \tag{6.2}$$

Proof of Lemma 6.0.15. Using the notation of Section 4, we may write:

$$u_\lambda \ = \ u_{\mathring{X}_\lambda} + u^+_{X^{1/2}_{\lambda,1}} + u^-_{X^{1/2}_{\lambda,1}} + u_{Y_\lambda} \ ,$$

We now define the scattering data implicitly by the relations:

$$\begin{aligned} W(f^+_\lambda, g^+_\lambda)(t) \ &= \ u_{\mathring{X}_\lambda} + \int_0^\infty |D_x|^{-1} sin\left(|D_x|(t-s)\right) \Box u_{Y_\lambda}(s)\, ds \ , \\ W(f^-_\lambda, g^-_\lambda)(t) \ &= \ u_{\mathring{X}_\lambda} + \int_{-\infty}^0 |D_x|^{-1} sin\left(|D_x|(t-s)\right) \Box u_{Y_\lambda}(s)\, ds \ . \end{aligned}$$

Using the fact that $\Box u_{Y_\lambda}$ has finite $L^1(L^2)$ norm, it suffices to show that one has the limits:

$$\lim_{t\to\pm\infty} \| \, u^+_{X^{1/2}_{\lambda,1}}(t) + u^-_{X^{1/2}_{\lambda,1}}(t) \, \|_{\dot{H}^1 \cap \partial_t(L^2)} \ = \ 0 \ .$$

Squaring this, we see that we must show the limits:

$$\lim_{t\to\pm\infty} \int |D_x| u^+_{X^{1/2}_{\lambda,1}}(t) \ \overline{|D_x| u^\pm_{X^{1/2}_{\lambda,1}}}(t) \ = \ 0 \ , \tag{6.3}$$

$$\lim_{t\to\pm\infty} \int \partial_t u^+_{X^{1/2}_{\lambda,1}}(t) \ \overline{\partial_t u^\pm_{X^{1/2}_{\lambda,1}}}(t) \ = \ 0 \ . \tag{6.4}$$

We'll only deal here with the limit (6.3), as the limit (6.4) follows from a virtually identical argument. Using the trace formula (4.4) along with the Plancherel theorem, we compute:

$$\begin{aligned} &(L.H.S.)(6.3) \\ &\qquad = \ \lim_{t\to\pm\infty} \int e^{2\pi i t(|\xi| \mp |\xi|)} \, |\xi|^2 \ \int e^{2\pi i t s_1} \ \widehat{u^+_{\lambda, s_1 \pm s_2}}(\xi) \ \overline{\widehat{u^\pm_{\lambda, s_2}}(\xi)} \ \ ds_1\, ds_2\, d\xi. \end{aligned}$$

By (4.5) we have the bounds:

$$\left\| \, |\xi|^2 \ \int e^{2\pi i t s_1} \widehat{u^+_{\lambda, s_1 \pm s_2}}(\xi) \ \overline{\widehat{u^\pm_{\lambda, s_2}}(\xi)} \ \ ds_1\, ds_2 \right\|_{L^1_\xi} \ \lesssim \ \lambda^2 \, \| \, u^-_{X^{1/2}_{\lambda,1}} \, \|_{X^{\frac12}_{\lambda,1}} \, \| \, u^\pm_{X^{1/2}_{\lambda,1}} \, \|_{X^{\frac12}_{\lambda,1}} \ .$$

Furthermore, by Fubini's theorem and the Riemann–Lebesgue Lemma, the following pointwise limit holds for almost every fixed ξ:

$$\lim_{t\to\pm\infty} \ |\xi|^2 \ \int e^{2\pi i t s_1} \widehat{u^+_{\lambda, s_1 \pm s_2}}(\xi) \ \overline{\widehat{u^\pm_{\lambda, s_2}}(\xi)} \ \ ds_1\, ds_2 \ \ = \ \ 0 \ .$$

The desired result now follows from the Dominated Convergence Theorem. This completes the proof of estimates (6.1)–(6.2). $\square$

Chapter 7

Inductive Estimates I

Our solution to (1.1) will be produced through the usual procedure of Picard iteration. Because the initial data and our function spaces are both invariant with respect to the scaling (1.3), any iteration procedure must effectively be global in time. Therefore, we shall have no need of an auxiliary time cutoff system as in the works [4]–[13]. Instead, we write (1.1) directly as an integral equation:

$$\phi \ = \ W(f,g) + \Box^{-1}\mathcal{N}(\phi, D\phi) \ . \tag{7.1}$$

By the contraction mapping principle and the quadratic nature of the nonlinearity, to produce a solution to (7.1) which satisfies the regularity assumptions of our main theorem, it suffices to prove the following two sets of estimates:

Theorem 7.0.16 (Solution of the division problem). *Let $5 < n$, then the F and G spaces solve the division problem for quadratic wave equations in the sense that for any of the model systems we have written above: YM, WM, or MD, one* *has the following estimates:*

$$\| \Box^{-1}\mathcal{N}(u, Dv) \|_{G^{s_c}} \lesssim \| u \|_{G^{s_c}} \| v \|_{G^{s_c}} \ , \tag{7.2}$$

$$\| \Box^{-1}\mathcal{N}(u, Dv) \|_{F^s} \lesssim \| u \|_{G^{s_c}} \| v \|_{F^s} + \| u \|_{F^s} \| v \|_{G^{s_c}} \ . \tag{7.3}$$

The remainder of the paper is devoted to the proof of Theorem 7.0.16. In what follows, we will work exclusively with the equation:

$$\phi = W(f, g) + \Box^{-1}(\phi \nabla \phi) \ . \tag{7.4}$$

In this case, we set $s_c = \frac{n-2}{2}$. The proof of Theorem 7.0.16 for the other model equations can be achieved through a straightforward adaptation of the estimates we give here. In fact, after the various derivatives and values of s_c are taken into account, the proof in these cases follows verbatim from estimates (7.6) and (7.7) below.

Our first step is to take a Littlewood-Paley decomposition of $\Box^{-1}(u \nabla v)$ with respect to space–time frequencies:

$$\Box^{-1}(u \nabla v) = \sum_{\mu_i} \Box^{-1}(S_{\mu_1} u \nabla S_{\mu_2} v) \ . \tag{7.5}$$

We now follow the standard procedure of splitting the sum (7.5) into three pieces depending on the cases $\mu_1 \ll \mu_2$, $\mu_2 \ll \mu_1$, and $\mu_2 \sim \mu_1$. Therefore, due to the ℓ^1 Besov structure in the F spaces, in order to prove both (7.2) and (7.3), it suffices to show the two estimates:

$$\| \Box^{-1}(S_{\mu_1} u \nabla S_{\mu_2} v) \|_{G_\lambda} \ \lesssim \ \lambda^{-1} \mu_1^{\frac{n}{2}} \| u \|_{F_{\mu_1}} \| v \|_{F_{\mu_2}} \ \ , \ \mu_1 \sim \mu_2 \ , \tag{7.6}$$

$$\| \Box^{-1}(S_{\mu} u \nabla S_{\lambda} v) \|_{G_\lambda} \ \lesssim \ \mu^{\frac{n-2}{2}} \| u \|_{G_{\mu}} \| v \|_{F_{\lambda}} \ \ , \ \mu \ll \lambda \ . \tag{7.7}$$

Notice that after some weight trading, the estimates (7.2) and (7.3) follow from (7.7) in the case where $\mu_2 \ll \mu_1$.

proof of (7.6). It is enough if we show the following two estimates:

$$\| S_\lambda(S_{\mu_1} u \nabla S_{\mu_2} v) \|_{L^1(L^2)} \ \lesssim \ \mu_1^{\frac{n}{2}} \| u \|_{F_{\mu_1}} \| v \|_{F_{\mu_2}} \ \ , \ \mu_1 \sim \mu_2 \ , \tag{7.8}$$

$$\| S_\lambda \Box^{-1}(S_{\mu_1} u \nabla S_{\mu_2} v) \|_{L^\infty(L^2)} \ \lesssim \ \lambda^{-1} \mu_1^{\frac{n}{2}} \| u \|_{F_{\mu_1}} \| v \|_{F_{\mu_2}} \ \ , \ \mu_1 \sim \mu_2 \ . \tag{7.9}$$

In fact, it (essentially) suffices to prove (7.8). To see this, notice that one has the formula:

$$\left[S_\lambda, \Box^{-1} \right] G \ = \ W(E * S_\lambda F) - S_\lambda W(E * F) \ .$$

Thus, after multiplying by S_λ, we see that:

$$\begin{aligned} S_\lambda \left[S_\lambda, \Box^{-1} \right] G \ &= \ P_\lambda \big(W(E * S_\lambda F) - S_\lambda W(E * F) \big) \ , \\ &= \ W(E * S_\lambda P_\lambda F) - S_\lambda W(E * P_\lambda F) \ , \\ &= \ S_\lambda \left[S_\lambda, \Box^{-1} \right] P_\lambda G \ . \end{aligned}$$

Therefore, by the (approximate) idempotence of S_λ one has:

$$\begin{aligned} S_\lambda \Box^{-1} G \ &= \ S_\lambda \Box^{-1} S_\lambda G + S_\lambda \left[S_\lambda , \Box^{-1} \right] G \ , \\ &= \ S_\lambda \Box^{-1} S_\lambda G + S_\lambda \left[S_\lambda , \Box^{-1} \right] P_\lambda G \ . \end{aligned}$$

Thus, by the boundedness of S_λ on the spaces $L^\infty(L^2)$ and$L^1(L^2)$ and the energy estimate, one can bound:

$$\| \, S_\lambda \Box^{-1} G \, \|_{L^\infty(L^2)} \ \lesssim \ \lambda^{-1} \left(\| \, S_\lambda G \, \|_{L^1(L^2)} + \| \, P_\lambda G \, \|_{L^1(L^2)} \right) \ .$$

To proceed, we now estimate:

$$\begin{aligned} \| \, S_{\mu_1} u \, \nabla S_{\mu_2} v \, \|_{L^1(L^2)} \ &\lesssim \ \| \, S_{\mu_1} u \, \|_{L^2(L^4)} \| \, \nabla S_{\mu_2} v \, \|_{L^2(L^4)} \ , \\ & \mu_1^{\frac{n-2}{4}} \mu_2^{\frac{n+2}{4}} \, \| \, u \, \|_{F_{\mu_1}} \| \, v \, \|_{F_{\mu_2}} \ . \end{aligned}$$

Taking into account the the bound $\mu_1 \sim \mu_2$, the claim now follows. $\Box$

Next, we'll deal with the estimate (7.7). For the remainder of the paper we shall fix both λ and μ and assume they such that $\mu \ll \lambda$ for a fixed constant. We now decompose the product $S_\lambda (S_\mu u \, \nabla S_\lambda v)$ into a sum of three pieces:

$$S_\lambda (S_\mu u \, \nabla S_\lambda v) \ = \ A + B + C \ ,$$

where

$$A \ = \ S_\lambda (S_\mu u \, \nabla S_{\lambda, c\mu \leqslant \bullet} v) \ ,$$

$$B \ = \ S_{\lambda, c\mu \leqslant \bullet} (S_\mu u \, \nabla S_{\lambda, \bullet < c\mu} v) \ ,$$

$$C \ = \ S_{\lambda, \bullet < c\mu} (S_\mu u \, \nabla S_{\lambda, \bullet < c\mu} v) \ .$$

Here c is a suitably small constant which will be chosen later. It will be needed to make explicit a dependency between some of the constants which arise in a specific frequency localization in the sequel. We now work to recover the estimate (7.7) for each of the three above terms separately.

proof of (7.7) *for the term A.* Following the remarks at the beginning of the proof of (7.6), it suffices to compute:

$$\begin{aligned}
\| S_\lambda (S_\mu u \, \nabla S_{\lambda, c\mu \leqslant \bullet} v) \|_{L^1(L^2)} &\lesssim \lambda \, \| S_\mu u \|_{L^2(L^\infty)} \| S_{\lambda, c\mu \leqslant \bullet} v) \|_{L^2(L^2)} \ , \\
&\lesssim \lambda \mu^{\frac{n-1}{2}} \| u \|_{F_\mu} \ (c\mu)^{-\frac{1}{2}} \| v \|_{F_\lambda} \ , \\
&\lesssim c^{-1} \lambda \mu^{\frac{n-2}{2}} \| u \|_{F_\mu} \| v \|_{F_\lambda} \ .
\end{aligned}$$

For a fixed c, we obtain the desired result. □

We now move on to showing the inclusion (7.7) for the B term above. In this range, we are forced to work outside the context of $L^1(L^2)$ estimates. This is the reason we have included the $L^2(L^2)$ based $X^{\frac{1}{2}}_{\lambda,1}$ spaces. This also means that we will need to recover Z_λ norms by hand (because they are only covered by the Y_λ spaces). However, because this last task will require a somewhat finer analysis than what we will do in this section, we contend ourselves here with showing:

proof of the $X^{\frac{1}{2}}_{\lambda,1} \cap S_\lambda(L^\infty(L^2))$ estimates for the term B. Our first task will be deal with the energy estimate which we write as:

$$\| S_\lambda \Box^{-1} S_{\lambda, c\mu \leqslant \bullet} (S_\mu u \, \nabla S_{\lambda, \bullet < c\mu} v) \|_{L^\infty(L^2)} \ \lesssim \ \mu^{\frac{n-2}{2}} \| u \|_{F_\mu} \| v \|_{F_\lambda} \ .$$

For G supported away from the light–cone in Fourier space, we have the identity:

$$S_\lambda \Box^{-1} S_\lambda G \ = \ \Xi^{-1} S_\lambda G - W(\Xi^{-1} P_\lambda S_\lambda G) \ .$$

By energy, this allows us to estimate:

$$\| S_\lambda \Box^{-1} S_\lambda G \|_{L^\infty(L^2)} \ \lesssim\ \| \Xi^{-1} S_\lambda G \|_{L^\infty(L^2)} \ \lesssim\ \| \Xi^{-1} S_\lambda G \|_{X^{\frac{1}{2}}_{\lambda,1}} \ .$$

Therefore, we are left with estimating the term B in the $X^{\frac{1}{2}}_{\lambda,1}$ space. For a fixed distance d from the cone, we compute that:

$$\begin{aligned} \| \Xi^{-1} S_{\lambda,d} S_{\lambda, c\mu \leqslant \bullet} (S_\mu u \, \nabla S_{\lambda, \bullet < c\mu} v) \|_{L^2(L^2)} \ &\lesssim\ d^{-1} \ \| S_\mu u \|_{L^2(L^\infty)} \| S_\lambda v \|_{L^\infty(L^2)} \ , \\ &\lesssim\ d^{-1} \mu^{\frac{n-2}{2}} \| v \|_{F_\mu} \| u \|_{F_\lambda} \ . \end{aligned}$$

Summing $d^{\frac{1}{2}}$ times this last expression over all $c\mu \leqslant d$ yields:

$$\begin{aligned} &\sum_{c\mu \leqslant d} d^{\frac{1}{2}} \| \Xi^{-1} S_{\lambda,d} S_{\lambda, c\mu \leqslant \bullet} (S_\mu u \, \nabla S_{\lambda, \bullet < c\mu} v) \|_{L^2(L^2)} \\ &\qquad \lesssim\ \sum_{c\mu \leqslant d} \left(\frac{\mu}{d}\right)^{\frac{1}{2}} \mu^{\frac{n-2}{2}} \| v \|_{F_\mu} \| u \|_{F_\lambda} \ . \end{aligned}$$

For a fixed c we obtain the desired result.

$\Box$

Chapter 8

Interlude: Some bilinear decompositions

To proceed further, it will be necessary for us to take a closer look at the expression:

$$S^{\omega}_{\lambda,d}(S_{\mu}u\,\nabla S_{\lambda,\bullet<c\mu}v)\ , \qquad\qquad c\mu\leqslant d\ , \tag{8.1}$$

as well as the sum:

$$C\ =\ S_{\lambda,\bullet<c\mu}(S_{\mu}u\,\nabla S_{\lambda,\bullet<c\mu}v)\ =\ C_I+C_{II}+C_{III}\ ,$$

where

$$C_I=\sum_{d<c\mu}S_{\lambda,d}(S_{\mu,\bullet\leqslant d}u\ \nabla S_{\lambda,\bullet\leqslant d}v)\ ,$$

$$C_{II}=\sum_{d<c\mu}S_{\lambda,\bullet\leqslant d}(S_{\mu,\bullet\leqslant d}u\ \nabla S_{\lambda,d}v)\ ,$$

$$C_{III}=\sum_{d\leqslant\mu}S_{\lambda,\bullet<min\{c\mu,d\}}(S_{\mu,d}u\ \nabla S_{\lambda,\bullet<min\{c\mu,d\}}v)\ .$$

We'll begin with a decomposition of C_I and C_{II}. The C_{III} term is basically the same but requires a slightly more delicate analysis. All of the decompositions we compute here will be for a fixed d. The full decomposition will then be given by summing over the relevant values of d. Because our decompositions will be with respect to Fourier supports, it suffices to look at the convolution product of the corresponding cutoff functions in Fourier space. In what follows, we'll only deal with the C_I term. It will become apparent that the same idea works for C_{II}. Therefore, without loss of generality, we shall decompose the product:

$$s^+_{\lambda,d}(s^\pm_{\mu,\bullet\leqslant d} * s^+_{\lambda,\bullet\leqslant d}) \ . \tag{8.2}$$

To do this, we use the standard device of restricting the angle of interaction in the above product. It will be crucial for us to be able to make these restrictions based only on the spatial Fourier variables, because we will need to reconstruct our decompositions through square–summing. For $(\tau', \xi') \in supp\{s^\pm_{\mu,\bullet\leqslant d}\}$ and $(\tau, \xi) \in supp\{s^+_{\lambda,\bullet\leqslant d}\}$ we compute that:

$$\begin{aligned} O(d) &= \big||\tau' + \tau| - |\xi' + \xi|\big| \ , \\ &= \Big|\big| \pm |\xi'| + |\xi| + O(d)\big| - |\xi' + \xi|\Big| \ , \\ &= \Big|O(d) + \big| \pm |\xi'| + |\xi|\big| - |\xi' + \xi|\Big| \ . \end{aligned}$$

Using now the fact that $d < c\mu$ and $\mu < c\lambda$ to conclude that $|\xi'| \sim \mu$ and $|\xi| \sim \lambda$, we see that one has the angular restriction:

$$\mu\Theta^2_{\pm\xi',\xi} \ \lesssim \ \Big| \pm |\xi'| + |\xi| - |\xi' + \xi|\Big| \ = \ O(d) \ .$$

In particular we have that $\Theta_{\pm\xi',\xi} \lesssim \sqrt{\frac{d}{\mu}}$. This allows us to decompose the product (8.2) into a sum over angular regions with $O(\sqrt{\frac{d}{\mu}})$ spread. The result is:

Lemma 8.0.17 (Wide angle decomposition). *In the ranges stated for the C_I term above, one can write:*

$$s^{+}_{\lambda,d}(s^{\pm}_{\mu,\bullet\leqslant d} * s^{+}_{\lambda,\bullet\leqslant d}) = \sum_{\substack{\omega_1,\omega_2,\omega_3 \ : \\ |\omega_1\mp\omega_2|\sim(d/\mu)^{\frac{1}{2}} \\ |\omega_1-\omega_3|\sim(d/\mu)^{\frac{1}{2}}}} b^{\omega_1}_{\lambda,\lambda(\frac{d}{\mu})^{\frac{1}{2}}} s^{+}_{\lambda,d} \left(s^{\omega_2}_{\mu,\bullet\leqslant d}{}^{\pm} * b^{\omega_3}_{\lambda,\lambda(\frac{d}{\mu})^{\frac{1}{2}}} s^{+}_{\lambda,\bullet\leqslant d} \right) . \tag{8.3}$$

for the convolution of the associated cutoff functions in Fourier space.

We note here that the key feature in the decomposition (8.3) is that the sum is (essentially) diagonal in all three angles which appear there $(\omega_1, \omega_2, \omega_3)$. It is useful here to keep in mind the following diagram:

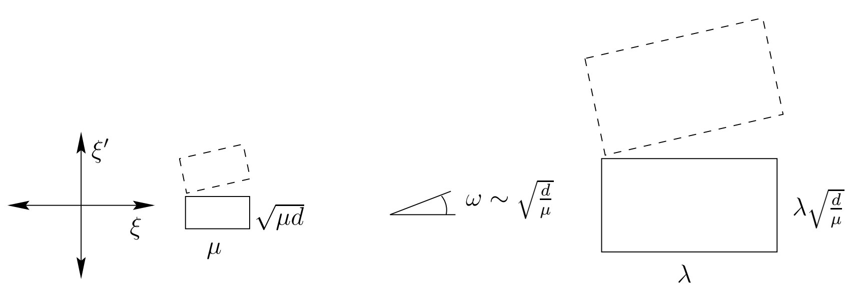

Figure 8.1: Spatial supports in the wide angle decomposition.

We now focus our attention on decomposing the convolution:

$$s^{+}_{\lambda,\bullet\leqslant min\{c\mu,d\}}(s^{\pm}_{\mu,d} * s^{+}_{\lambda,\bullet\leqslant min\{c\mu,d\}}) . \tag{8.4}$$

If it is the case that $d \ll \mu$, then the same calculation which was used to produce (8.3) works and we end up with the same type of sum. However, if we are in the case where $d \sim \mu$, we need to compute things a bit more carefully. We will now assume that things are set up so that $c\mu \ll d$. It is clear that all the previous decompositions can be made so that we can reduce things to this consideration. If we now take $(\tau', \xi') \in supp\{s^{\pm}_{\mu,d}\}$ and $(\tau, \xi) \in supp\{s^{+}_{\lambda,\bullet \leqslant min\{c\mu,d\}}\}$, we can use the facts that $\tau' = O^{\mp}(d) \pm |\xi'|$, $\tau = O(c\mu) + |\xi|$, and $|\xi| \gg \mu$ to compute that:

$$\begin{aligned} O(c\mu) &= \Big| \big|\tau' + \tau\big| - |\xi' + \xi| \Big| \ , \\ &= \Big| O^{\mp}(d) \pm |\xi'| + O(c\mu) + |\xi| - |\xi' + \xi| \Big| \ , \end{aligned} \tag{8.5}$$

where the term $O^{\mp}(d)$ in the above expression is such that $|O^{\pm}(d)| \sim d$. In fact, one can see that the equality (8.5) forces $\pm O^{\mp}(d) > 0$ on account of the fact that $\pm(\pm|\xi'| + |\xi| - |\xi' + \xi|) > 0$ and the assumption $|O(c\mu) + O^{\mp}(d)| \sim d$. In particular, this means that we can multiply $s^{\pm}_{\mu,d}$ in the product (8.4) by the cutoff $s_{|\tau|<|\xi|}$ without effecting things. This in turn shows that we may decompose the product (8.4) based solely on restriction of the spatial Fourier variables, just as we did to get the sum in Lemma 8.0.17.

We now return to the C_I term. For the sequel, we will need to know what the contribution of the factor $S_{\lambda,\bullet\leqslant d}v$ to the following localized product is:

$$S^{\omega_1}_{\lambda,d}(S_{\mu,\bullet\leqslant d}u\ \nabla S_{\lambda,\bullet\leqslant d}v)\ .$$

Using Lemma 8.0.17, we see that we may write:

$$s^{\omega_1}_{\lambda,d}(s_{\mu,\bullet\leqslant d} * s_{\lambda,\bullet\leqslant d}) \ = \ s^{\omega_1}_{\lambda,d}\left(s^{\omega_2}_{\mu,\bullet\leqslant d} \ * \ b^{\omega_3}_{\lambda,\lambda(\frac{d}{\mu})^{\frac{1}{2}}} s_{\lambda,\bullet\leqslant d}\right) \ , \tag{8.6}$$

where $|\omega_1 - \pm\omega_2| \sim |\omega_3 - \pm\omega_2| \sim \sqrt{\frac{d}{\mu}}$. However, this can be refined significantly. To see this, assume that the spatial support of $s^{\omega_1}_{\lambda,d}$ lies along the positive ξ_1 axis. We'll label this block by $b^{\omega_1}_{\lambda,d}$. Because we are in the range where $\sqrt{\mu d} \ll \sqrt{\lambda d}$, we see that any $\xi \in supp\{b^{\omega_3}_{\lambda,\lambda(\frac{d}{\mu})^{\frac{1}{2}}}\}$ and $\xi' \in supp_{\xi'}\{s^{\omega_2}_{\mu,\bullet\leqslant d}\}$ that the sum $\xi + \xi'$ must in fact belong to (a slight thickening of) $supp\{b^{\omega_1}_{\lambda,d}\}$. This allows us to write:

Lemma 8.0.18 (Small angle decomposition). *In the ranges stated for the C_I term above, we can write:*

$$s^{\omega_1}_{\lambda,d}(s_{\mu,\bullet\leqslant d} * s_{\lambda,\bullet\leqslant d}) \ = \ s^{\omega_1}_{\lambda,d}(s^{\omega_2}_{\mu,\bullet\leqslant d} \ * \ s^{\omega_3}_{\lambda,\bullet\leqslant d})\ . \tag{8.7}$$

where $|\omega_1 - \omega_3| \sim \sqrt{\frac{d}{\lambda}}$ *, and* $|\omega_1 - \pm\omega_2| \sim \sqrt{\frac{d}{\mu}}$.

It is important to note here that if one were to sum the expression (8.7) over ω_1, the resulting sum would be (essentially) diagonal in ω_3, but there would be *many* ω_1 which would contribute to a single ω_2. This means that the resulting would *not* be diagonal in ω_2 as was the case for the sum (8.3). It is helpful to visualize things through the following figure:

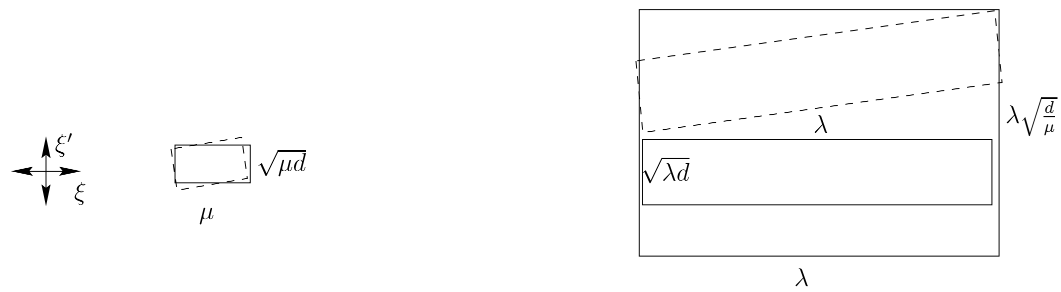

Figure 8.2: Spatial supports in the small angular decomposition.

Our final task here is to mention an analog of Lemma 8.0.18 for the term (8.1). Here we can frequency localize the factor $S_{\lambda,\bullet<c\mu}$ in the product using the fact that one has $\mu \ll c^{-\frac{1}{2}}\sqrt{\lambda d}$. The result is:

Lemma 8.0.19 (Small angle decomposition for the term B). *In the ranges stated for the B term above, we can write:*

$$s^{\omega_1}_{\lambda,d}(s_\mu * s_{\lambda,\bullet\leqslant c\mu}) \ = \ s^{\omega_1}_{\lambda,d}(s_\mu \ * \ b^{\omega_3}_{\lambda,(\lambda d)^{\frac{1}{2}}} s_{\lambda,\bullet\leqslant c\mu}) \ . \tag{8.8}$$

where $|\omega_1 - \omega_3| \sim \sqrt{\frac{d}{\lambda}}$.

Finally, we note here the important fact that in the decomposition (8.8) above, the range of interaction in the product forces $d \lesssim \mu$. This completes our list of bilinear decompositions.

Chapter 9

Inductive Estimates II: Remainder of the $High \times Low \Rightarrow High$ frequency interaction

It remains for us is to bound the term B from line (8.1) in the Z_λ space, as well as show the inclusion (7.7) for the terms $C_I - C_{III}$ from line (8.2). We do this now, proceeding in reverse order.

proof of estimate (7.7) *for the* C_{III} *term.* To begin with we fix d. Using the remarks at the beginning of the proof of (7.6), we see that it is enough to show that:

$$\| S_{\lambda,\bullet<min\{c\mu,d\}}(S_{\mu,d}u \ \nabla S_{\lambda,\bullet<min\{c\mu,d\}}v) \|_{L^1(L^2)} \lesssim \lambda \left(\sum_{\omega} \| S^{\omega}_{\mu,d}u \|^2_{L^1(L^\infty)} \right)^{\frac{1}{2}} \| v \|_{F_\lambda} \ . \quad (9.1)$$

To accomplish this, we first use the wide angle decomposition, (8.3), on the left hand side of (9.1). This allows us to compute:

$$\begin{aligned}
& \| S_{\lambda,\bullet<min\{c\mu,d\}}(S_{\mu,d}u\ \nabla S_{\lambda,\bullet<min\{c\mu,d\}}v) \|_{L^1(L^2)} \ , \\
\lesssim & \sum_{\substack{\omega_2,\omega_3\ : \\ |\omega_3\pm\omega_2|\sim(d/\mu)^{\frac{1}{2}}}} \| S^{\omega_2}_{\mu,d}u \|_{L^1(L^\infty)} \cdot \| \nabla B^{\omega_3}_{\lambda,\lambda(\frac{d}{\mu})^{\frac{1}{2}}} S_{\lambda,\bullet<min\{c\mu,d\}}v) \|_{L^\infty(L^2)} \ , \\
\lesssim & \ \lambda \left(\sum_{\omega} \| S^{\omega}_{\mu,d}u \|^2_{L^1(L^\infty)} \right)^{\frac{1}{2}} \left(\sum_{\omega} \| B^{\omega}_{\lambda,\lambda(\frac{d}{\mu})^{\frac{1}{2}}} S_{\lambda,\bullet<min\{c\mu,d\}}v) \|^2_{L^\infty(L^2_x)} \right)^{\frac{1}{2}} \ , \\
\lesssim & \ \lambda \left(\sum_{\omega} \| S^{\omega}_{\mu,d}u \|^2_{L^1(L^\infty)} \right)^{\frac{1}{2}} \| v \|_{F_\lambda} \ .
\end{aligned}$$

Summing over d yields the desired result. □

proof of (7.7) *for the C_{II} term.* Again, fixing d, and using the angular decomposition lemma 8.0.17, we compute that:

$$\begin{aligned}
& \| S_{\lambda,\bullet\leqslant d}(S_{\mu,\bullet\leqslant d}u\ \nabla S_{\lambda,d}v) \|_{L^1(L^2)} \ , \\
\lesssim & \ \lambda \sum_{\substack{\omega_2,\omega_3\ : \\ |\omega_3\pm\omega_2|\sim(d/\mu)^{\frac{1}{2}}}} \| S^{\omega_2}_{\mu,\bullet\leqslant d}u \|_{L^2(L^\infty)} \cdot \| B^{\omega_3}_{\lambda,\lambda(\frac{d}{\mu})^{\frac{1}{2}}} S_{\lambda,d}v \|_{L^2(L^2)} \ , \\
\lesssim & \ \lambda \left(\sum_{\omega} \| S^{\omega}_{\mu,\bullet\leqslant d}u \|^2_{L^2(L^\infty)} \right)^{\frac{1}{2}} \| S_{\lambda,d}v \|_{L^2(L^2)} \ , \\
\lesssim & \ \lambda\mu^{\frac{n-2}{2}} \left(\frac{d}{\mu} \right)^{\frac{n-5}{4}} \| u \|_{F_\mu} \| v \|_{F_\lambda} \ .
\end{aligned}$$

This last expression can now be summed over d, using the condition $d < c\mu$, to obtain the desired result. □

proof of (7.7) *for the C_I term.* This is the other instance where we will have to rely on the $X^{\frac{1}{2}}_{\lambda,1}$ space. Following the same reasoning used previously, we first bound:

$$
\begin{aligned}
& \| \varXi^{-1} S_{\lambda,d}(S_{\mu,\bullet\leqslant d} u \ \nabla S_{\lambda,\bullet\leqslant d} v) \|_{L^2(L^2)} \ , \\
\lesssim \ & d^{-1} \sum_{\substack{\omega_2,\omega_3 \ : \\ |\omega_3 \pm \omega_2| \sim (d/\mu)^{\frac{1}{2}}}} \| S^{\omega_2}_{\mu,\bullet\leqslant d} u \|_{L^2(L^\infty)} \cdot \| B^{\omega_3}_{\lambda,\lambda(\frac{d}{\mu})^{\frac{1}{2}}} S_{\lambda,\bullet\leqslant d} v) \|_{L^\infty(L^2)} \ , \\
\lesssim \ & d^{-1} \left(\sum_{\omega} \| S^{\omega}_{\mu,\bullet\leqslant d} u \|^2_{L^2(L^\infty)} \right)^{\frac{1}{2}} \cdot \left(\sum_{\omega} \| B^{\omega}_{\lambda,\lambda(\frac{d}{\mu})^{\frac{1}{2}}} S_{\lambda,\bullet\leqslant d} v) \|^2_{L^\infty(L^2_x)} \right)^{\frac{1}{2}} \ , \\
\lesssim \ & d^{-\frac{1}{2}} \mu^{\frac{n-2}{2}} \left(\frac{d}{\mu} \right)^{\frac{n-5}{4}} \| u \|_{F_\mu} \| v \|_{F_\lambda} \ .
\end{aligned}
$$

Multiplying this last expression by $d^{\frac{1}{2}}$ and then using the condition $d < c\mu$ to sum over d yields the desired result for the $X^{\frac{1}{2}}_{\lambda,1}$ space part of estimate (7.7). It remains to prove the Z_λ estimate. Here we use the second angular decomposition lemma 8.0.18 to compute that for fixed d:

$$
\begin{aligned}
& \left(\sum_{\omega_1} \| \varXi^{-1} S^{\omega_1}_{\lambda,d}(S_{\mu,\bullet\leqslant d} u \ \nabla S_{\lambda,\bullet\leqslant d} v) \|^2_{L^1(L^\infty)} \right)^{\frac{1}{2}} \ , \\
\lesssim \ & (\lambda d)^{-1} \left(\sum_{\substack{\omega_1,\omega_2,\omega_3 \ : \\ \omega_1 - \omega_3 \sim (d/\lambda)^{\frac{1}{2}} \\ \omega_1 \pm \omega_2 \sim (d/\mu)^{\frac{1}{2}}}} \| S^{\omega_1}_{\lambda,d}(S^{\omega_2}_{\mu,\bullet\leqslant d} u \ \nabla S^{\omega_3}_{\lambda,\bullet\leqslant d} v) \|^2_{L^1(L^\infty)} \right)^{\frac{1}{2}} \ , \\
\lesssim \ & d^{-1} \sup_{\omega} \| S^{\omega}_{\mu,\bullet\leqslant d} u \|_{L^2(L^\infty)} \cdot \left(\sum_{\omega} \| S^{\omega}_{\lambda,\bullet\leqslant d} v \|^2_{L^2(L^\infty)} \right)^{\frac{1}{2}} \ , \\
\lesssim \ & \left(\frac{d}{\mu} \right)^{\frac{n-5}{4}} \left(\frac{d}{\lambda} \right)^{\frac{n-5}{4}} \mu^{\frac{n-2}{2}} \lambda^{\frac{n-2}{2}} \| u \|_{F_\mu} \| v \|_{F_\lambda} \ .
\end{aligned}
$$

Multiplying this last expression by $\lambda^{\frac{2-n}{2}}$ and summing over d using the condition $d < \lambda, \mu$ yields the desired result. $\square$

proof of the Z_λ embedding for the B term. The pattern here follows that of the last few lines of the previous proof. Fixing d, we use the decomposition Lemma

8.0.19 to compute that:

$$
\begin{aligned}
&\Big(\sum_{\omega} \| \Xi^{-1} S^{\omega}_{\lambda,d}(S_\mu u \ \nabla S_{\lambda,\bullet\leqslant c\mu} v) \|^2_{L^1(L^\infty)}\Big)^{\frac{1}{2}} \ , \\
\lesssim \ & (\lambda d)^{-1} \Bigg(\sum_{\substack{\omega_1,\omega_3 \ : \\ |\omega_1-\omega_3|\sim (d/\lambda)^{\frac{1}{2}}}} \| S^{\omega_1}_{\lambda,d}(S_\mu u \ \nabla B^{\omega_3}_{\lambda,(\lambda d)^{\frac{1}{2}}} S_{\lambda,\bullet\leqslant c\mu} v) \|^2_{L^1(L^\infty)} \Bigg)^{\frac{1}{2}} \ , \\
\lesssim \ & d^{-1} \| S_\mu u \|_{L^2(L^\infty)} \cdot \Big(\sum_{\omega} \| B^{\omega}_{\lambda,(\lambda d)^{\frac{1}{2}}} S_{\lambda,\bullet\leqslant c\mu} v \|^2_{L^2(L^\infty)} \Big)^{\frac{1}{2}} \ , \\
\lesssim \ & \Big(\frac{\mu}{d}\Big)^{\frac{1}{2}} \Big(\frac{d}{\lambda}\Big)^{\frac{n-5}{4}} \mu^{\frac{n-2}{2}} \lambda^{\frac{n-2}{2}} \| u \|_{F_\mu} \| v \|_{F_\lambda} \ .
\end{aligned}
$$

Multiplying the last line above by a factor of $\lambda^{\frac{2-n}{2}}$ and using the conditions $d < \lambda$ and $c\mu < d \lesssim \mu$, we may sum over d to yield the desired result. □

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