ABSTRACT

Title of dissertation: APPLICATION OF DISPERSIVE PDE
TECHNIQUES TO THE STUDIES OF
THE TIME-DEPENDENT HARTREE-FOCK
-BOGOLIUBOV SYSTEM FOR BOSONS

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Doctor of Philosophy, 2019

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The thesis provides rigorous quantitative analyses for studying quantum fluctuations of a non-relativistic Bose gas about a Bose-Einstein condensate. In recent years, the dynamics of the condensate and the excitations of a Bose gas was shown to be well approximated by the quasifree dynamics with governing equations given by a system of coupled nonlinear dispersive PDEs called the time-dependent Hartree-Fock-Bogoliubov (HFB) system (c.f. [GM13a, GM17, BBC$^+$18, BSS18]). However, both the quantitative and qualitative analysis of the time-dependent HFB system are still in their early stages of development. Thus, the primary purposes of the thesis are to further the development of some analytic tools necessary for studying the time-dependent HFB system and use these effective equations to provide quantitative estimates for the true dynamics of the Bose gas at absolute zero temperature in Fock space norm.

The thesis comprises the entirety of the author's current and past projects on the time-dependent HFB system. Each project falls into one or more of two categories: studying the local or global well-posedness of the time-dependent HFB system, or obtaining global-in-time Fock space estimates for the error terms of the quasifree approximation to the dynamics of a system of interacting bosons. In the former category, the author employs techniques of dispersive PDE theory developed in the past three decades along with classical methods of harmonic analysis to study lower regularity solutions to the time-dependent HFB system. The lower regularity well-posedness of solutions for the time-dependent HFB system is necessary for studying norm approximation of the dynamics of a dilute Bose gas with strong interactions. In the latter category, we prove global a-priori estimates for the solutions to the HFB system and use them to obtain estimates for the error terms of the Fock space approximation.

APPLICATION OF DISPERSIVE PDE TECHNIQUES TO THE STUDIES OF THE TIME-DEPENDENT HARTREE-FOCK-BOGOLIUBOV SYSTEM FOR BOSONS

by

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Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2019

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Acknowledgments

Let me take this opportunity to express my deepest gratitude towards my two advisors, Manoussos Grillakis and Matei Machedon, for their guidance and support. I believe one could not wish for better advisors than these two fine gentlemen. They have taught me a great deal of mathematics. It was also a great learning experience for me to see how these two master craftsmen of mathematics work relentlessly together to chip away on a big problem one small piece at a time. In many ways, they have shaped my taste for mathematics and help me understand how to ask meaningful questions. I attribute much of my success in graduate school to their guidance and support if not all.

Several other professors have also played significant roles during my graduate school career. First, I would like to thank Professor David C. Levermore. In many ways, Professor Levermore has helped me improve both as a researcher and as an educator. In terms of research, Professor Levermore plays a big role in helping me hone my presentation skills. In the many talks that I have given at his Applied PDE RIT (Research Interaction Team) seminar, Professor Levermore was always very enthusiastic and encouraging, which gave me confidence when giving talks outside of UMD. As an educator, I have benefited a great deal from the many discussions I had with him about teaching Math 246 (elementary differential equations).

Next, I would like to thank Professor Dionisios Margetis and Professor Konstantina Trivisa. I thank Professor Margetis for the many insightful discussions we have had. I would also like to thank him for making time to serve as a committee member on both my preliminary oral exam and final oral defense. Moreover, I am truly grateful for his careful review of my thesis manuscript. I thank Professor Trivisa for her constant encouragement and warm welcoming personality which always brightens my day. Also, I am indebted to her for writing my letter of recommendation under short notice. She definitely helped reduce a lot of my stress during my postdoc application process.

I want to thank the Dean's representative Professor John Weeks for agreeing to serve on my final defense committee. I had the pleasure to meet Professor Weeks through auditing his graduate thermodynamics course. He is a fantastic educator with very deep insight on the subject. I attended all his lectures with great anticipation and he never fails to deliver.

Many thanks to my friends and colleagues who were with me since the very beginning of my graduate school career: Patrick Daniels, Danul Gunatilleka, Siming He, Ian Johnson, Minsung Kim, Weilin Li, Mark Magsino, Xuesen Na, Yousheng Shi, Weikun Wang, Zhenfu Wang, Dong Xin, Tao Zhang, Jing Zhou and many others which I have not named. I also had the good fortune to meet many wonderful junior graduate students along the way. I wish them the best of luck as they head toward the finish line. I would also like to thank Elif Kuz and Zhenfu Wang for their hospitalities during my visits to the University of Iowa and the University of Pennsylvania.

Finally, I am grateful to the Graduate School for awarding me the 2018-2019 Anne G. Wylie Fellowship to fund for my living expenses during the preparation of the dissertation.

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Chapter 1: Introduction

1.1 Background

The studies of dynamical behaviors of systems with many interacting bodies from first principles of quantum mechanics are of paramount interest in many branches of physics and chemistry. A prominent example is the studies of systems of interacting *bosons*. More precisely, the studies of *Bose-Einstein Condensate* (BEC), a state of matter of a dilute gas of bosons when cooled to near absolute zero temperature, has gained eminence in the world of experimental physics after its initial realization in atomic gases a little over two decades ago [AEM+95, BSTH95, DMA+95]. Notably, for the groundbreaking achievement of exhibiting condensation limits in dilute gases of alkali atoms, Eric Cornell and Carl Wieman of JILA/NIST and Wolfgang Ketterle of MIT were awarded the 2001 Nobel Prize in Physics[1]. The experimental success has and continues to garner substantial attention of scientists and mathematicians.

Due to the subsequent voluminous influx of research activities in the field of many-body boson systems, the demand for a firm mathematical foundation also grew. Moreover, a rigorous understanding of the dynamics of such systems is one

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[1] `https://www.nobelprize.org/prizes/physics/2001/press-release`

of the main challenges of modern mathematical physics and provides fundamental insights into quantum mechanical systems, as well as offering potential applications to the sciences. However, the immediate pressing difficulty one encounters when studying large particle systems is often the size of the system. In many applications of chemistry or physics, the size of the system of interest typically ranges between thousands to an Avogadro's number ($\sim 10^{23}$) of particles. Hence, even if one manages to exhibit a *many-body wave function* which solves the many-body Schrödinger equation analytically, the sheer number of particles of the system will render the usage of the wave function to analyze the dynamical behaviors of the system obsolete, or at the very least, not very effective with the current available tools.

It is no news that the size of the system presents a formidable obstacle for studying dynamical properties of the system. Indeed, it is rudimentary knowledge among researchers that the wave function contains more information than one could process since it encapsulates all the microscopic details of the many-body system. Even in practice, experiments are conducted and measured at a *macroscopic scale* where a lot of the quantum information are overlooked. Thus, to compare the experimental data against the full-theoretical description of the system from the wave function is impractical. As a matter of fact, based on heuristics and scaling arguments, many areas of chemistry and physics employ macroscopic equations to approximate the behaviors of the system. Therefore, it seems fair to study *effective descriptions* of the many-body system, which allows one to approximate the macroscopic dynamics of the large particle system but with much lesser variables, rather than the full quantum mechanical description. However, to understand the validity and qualities of any of these approximations, a rigorous derivation of effective macroscopic equations from quantum mechanical laws is essential, but this, in general, is a challenging task.

In this thesis, we study a *coupled nonlinear system* of effective macroscopic equations, parametrized by the particle-number N, called the *time-dependent Hartree-Fock-Bogoliubov (HFB) system (equations)*, for describing the quantum fluctuations about a BEC and use them to obtain global quantitative estimates for the true dynamics of the many-body system in *Fock space*. The main contribution of our work is the nonlinear analysis of the coupled system through the lens of dispersive PDE theory. We show that by employing dispersive PDE techniques to our analysis of the coupled system we could improve upon results which only uses "standard" mathematical physics techniques. More specifically, by applying dispersive PDE techniques, we were able to obtain nonlinear approximations to the dynamics of a N-body quantum system which are valid for a longer period of time and more singular interaction potentials without imposing further assumptions or restrictions on the approximation. In fact, we show the reader glimpses of the harmonious marriage between dispersive PDE theory and the studies of many-body interacting boson systems.

1.1.1 Mean-Field Model

We begin by considering a system of N interacting non-relativistic spinless bosons[2] in three dimensional space whose evolution is governed by the N-body linear Schrödinger equation

$$\left(\frac{1}{i}\frac{\partial}{\partial t} - \sum_{j=1}^{N} \Delta_{x_j} + \frac{1}{N}\sum_{i>j} v_N(x_i - x_j)\right)\Psi_N(t, x_1, \ldots, x_N) = 0 \tag{1.1}$$

where $x_i \in \mathbb{R}^3$ for $1 \leq i \leq N$ and $v_N(x) := N^{3\beta} v(N^\beta x)$ for $0 \leq \beta \leq 1$ where $v \in C_0^\infty(\mathbb{R})$. The reader should note that as $N \to \infty$ we have that $v_N(x) \to c\delta(x)$, in the sense of distribution, for some constant c. However, it should also be noted that in our work N is typically large but fixed. In the literature, it is common to refer

$$H_{N,\mathrm{mf}} := -\sum_{j=1}^{N} \Delta_{x_j} + \frac{1}{N}\sum_{i>j} v_N(x_i - x_j) \tag{1.2}$$

as the *mean-field Hamiltonian* and (1.1) the *mean-field model.*

There are many interpretations for for the coupling constant N^{-1} in front of the interaction potential. The simplest argument for having N^{-1} is based on the heuristic that the coupling constant allows for the balance between the kinetic energy

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[2] In relativistic quantum mechanics, bosons are classified, by the Spin-Statistic theorem, to be particles with integer intrinsic spin. However, in this thesis, we work in the realm of non-relativistic quantum physics where the bosonic property of a system of particles is captured by the symmetric structure of the wave function.

and the interaction potential energy. More precisely, since a general Hamiltonian

$$H_N := \sum_{i=1}^{N} \Delta_{x_i} - \lambda \sum_{1\le i<j\le N} v(x_i - x_j) \tag{1.3}$$

scales like $\mathcal{O}(N) + \lambda\mathcal{O}(N^2)$, then the energy of each particle is $\mathcal{O}(1)$ provided the coupling constant λ is $\mathcal{O}(N^{-1})$, which we called the *mean-field scaling* of (1.3). With this scaling, we define the *mean-field limit* to be the singular limit of (1.3) as the particle-number N tends to infinity. The coupling constant N^{-1} could also be understood with the law of large **numbers**. If the particles are independent and identically distributed according to an underlying density distribution ρ then we see that

$$\frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N} v_N(x_i - x_j) \simeq \int_{\mathbb{R}^3} v_N(x_i - y)\rho(y)\ dy = (v_N * \rho)(x_i) \simeq c\rho(x_i),$$

that is, the interaction potential has a **non-zero** limit with respect to the particle limit.

To physically motivate the mean-field model, let us consider N particles inside a fixed box[3] with volume $V = \ell^d$ subjected to either Robin or Neumann boundary conditions. Furthermore, assume the particles interact through a two-body repulsive potential v (with coupling constant λ set to 1). Then the particles will uniformly

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[3] The box model is used to simplify the exposition. Alternatively, we could have considered N particles in $\mathbb{R}^d$ subjected to some harmonic trapping potential, i.e.

$$H_N = \sum_{i=1}^{N}\{\Delta_{x_i} - V_{\text{ext}}(x_i)\} - \sum_{1\le i<j\le N} v(x_i - x_j)$$

where V_{ext} is small inside the box $[-L, L]$ and large otherwise.

spread themselves inside the box with an average *interparticle separation distance* of $N^{-1/d}\ell$ since the average volume occupied by a particle is $N^{-1}\ell^d$. In particular, we are interested in the *dilute gas model*, that is the case when $N^{-1/d}\ell \gg a$, where a is the scattering length. Following a scaling argument, one can show that the dynamics generated by the Hamiltonian (1.3) is equivalent to the dynamics generated by the rescaled Hamiltonian[4]

$$\frac{1}{N^{3-d}}\sum_{i=1}^{N}\Delta_{y_i} - \frac{1}{N}\sum_{1\leq i<j\leq N} v_N(y_i - y_j) \tag{1.4}$$

provided we set the length scale of y_i to order 1[5]. In the case $d = 3$, we see that (1.4) gives us a mean-field model for the particles in a unit box with interactions v_N. Finally, if we take the *dilute limit*, $N^{-1/d}\ell \to \infty$, in the box ℓ^3, we essentially recover the mean-field limit of N weakly interacting particles in the unit box. In particular, the 3D mean-field model in the unit box is equivalent to the strongly interacting dilute gas model in a box. The studies of weakly interacting bose gases can be trace back to the pioneering work of Bogoliubov in [Bog47], and later by Lee, Wang, and Yang in [LHY57] and as well as Dyson in [Dys57]. We refer the interested reader to [Lew15, LSSY05, Gol16] for more in-depth discussions.

1.1.2 Short-Range Scaling

The reader should take note of the two scaling processes that are involved in the interactions of this mean-field model. Aside from the obvious mean-field

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[4] To preserve the dynamics, we will need to rescale the time by a factor of N^{-2}.

[5] Here we are assuming x_i is on the length scale $\ell \sim N$.

scaling, we also have the *short-range scaling* of the interaction v given by v_N with the tuning parameter $\beta > 0$. Let us consider the dynamics generated by the mean-field Hamiltonian and let Ψ_N be the solution to

$$\frac{1}{i}\frac{\partial}{\partial t}\Psi_N = H_{N,\mathrm{mf}}\Psi_N \tag{1.5}$$

then by rescaling the solution, i.e. defining $\Phi(\tau, y) = \Psi_N(N^{-2\beta}\tau, N^{-\beta}y)$, we see the dynamics of the rescaled system is governed by the equation

$$\frac{1}{i}\frac{\partial}{\partial \tau}\Phi = \sum_{i=1}^{N}\Delta_{y_i}\Phi - N^{(d-2)\beta-1}\sum_{1\leq i<j\leq N} v(y_i - y_j)\Phi \tag{1.6}$$

In the instance of $d = 3$, we see, at least heuristically, the appearance of a critical scaling when $\beta = 1$, which we called the *Gross-Pitaveskii scaling*. With this consideration, we restrict ourselves to the case $0 \leq \beta \leq 1$, at least for the initial investigation. In some sense, the parameter β is a mathematical apparatus which was introduced to aid with the study of (1.1); in fact, the analysis of the dynamics of the system becomes more difficult as β approaches 1. Nevertheless, the are physical interpretations for β. Physically, for $0 < \beta < \frac{1}{3}$, we are in the regime of weakly interacting dense gas; since the scale size is N^β which means the size of the volume is effectively $V \simeq N^{3\beta}$, then we see that the interparticle separation distance is given by $\ell/N^{-1/3} = N^\beta/N^{-1/3}$, which tends to 0 as $N \to \infty$ provided $0 < \beta < \frac{1}{3}$. But once $\frac{1}{3} \leq \beta \leq 1$, we enter the *self-interacting regime* or sometimes called the strongly interacting diluted gas regime, depending on the modeling situation. In this regime, each particle is said to only feel the potential generated by itself; in some sense, this is a reminiscence of the Gross-Pitaevskii (GP) theory which proposed to model the many-body effects by a nonlinear strong on-site self interaction of a complex order parameter (the "condensation wave function") [Gro61, Gro63, Pit61]. But, strictly speaking, only $\beta = 1$ captures the **2 scale** structure postulated by the GP theory. See Chapter 6 of [LSSY05] for a survey of the GP theory for trapped bosons.

1.1.3 Initial Condition: the Ground State

The main interests of the thesis are to study the effective dynamics describing the evolution of the above many-body system, parametrized by N and β, and provide a quantitative method for tracking the evolution of the many-body quantum system in state space. Unfortunately, the problem of tracking the exact dynamics of bosonic systems in state space with arbitrary initial condition, at least to the author's knowledge, is still not tractable with the current available tools. Nevertheless, if we restrict ourselves to a special class of initial datum, then we are able to obtain some positive results in the direction of understanding the exact evolution in state space via studying some effective dynamics of the system.

In this thesis, we consider the evolution problem (1.1) with initial condition given by the *ground state* of the mean-field hamiltonian, i.e., the lowest energy state of (1.2). This choice of initial condition is natural for modeling Bose gases at lower temperature. In fact, by cooling a system of bosons to the absolute-zero temperature, we are essentially forcing the system to its ground state since the temperature of the system is directly proportional to its kinetic energy. However, solving the static (eigenvalue) problem for a many-body hamiltonian to determine the ground state remains a highly non-trivial open problem if not completely intractable. Nevertheless, based on the experimental observation of formation of BEC inside a trap potential at lower temperature, that is, the individual particles of the Bose gas coalesce into a single quantum entity, one could speculate that the ground state of a boson system has the form

$$\Psi_{N,\text{ ground}}(x_1,\ldots,x_N) = \phi_0^{\otimes N} + \text{important corrections} \tag{1.7}$$

where $\phi_0^{\otimes N} := \prod_{j=1}^N \phi_0(x_j)$ for some ϕ_0 which we called the *condensate wave function.* Furthermore, it is also speculated that in the particle limit the correction terms will tend to zero, i.e., $\Psi_{N,\text{ ground}} \simeq \ \phi_0^{\otimes N}$.

In fact, shortly after the discovery of atomic BEC in laboratory, Lieb, Seiringer, and Yngvason were able to rigorously verified the GP theory for the ground state of a dilute trapped Bose gas in [LSY00], i.e., they were able to show that the energy per particle of the ground state in the particle limit satisfies a variational principle, $\min\{\mathcal{E}_{\text{GP}}(\phi) \mid \|\,\phi\,\| = 1\}$, where

$$\mathcal{E}_{\text{GP}}(\phi) = \int_{\mathbb{R}^3} dx \ \ \left\{|\nabla\phi(x)|^2 + V_{\text{ext}}(x)|\phi(x)|^2 + 4\pi a|\phi(x)|^4\right\} \tag{1.8}$$

is the *Gross-Pitaevskii (GP) energy functional.* Here, a is the scattering length associated to the potential v. Subsequently, Lieb and Seiringer prove the existence of BEC in a dilute trapped gas at absolute zero temperature [LS02] by demonstrating that the *1-particle marginal density (operator)*, denoted by $\gamma_N^{(1)}$, with kernel

$$\gamma_N^{(1)}(x,x') := \int_{\mathbb{R}^{3(N-1)}} d\mathbf{x}_{N-1}\ \Psi_N(x,\mathbf{x}_{N-1})\overline{\Psi_N(x',\mathbf{x}_{N-1})} \to \phi_{\mathrm{GP}}(x)\overline{\phi_{\mathrm{GP}}(x')}$$

in trace norm where ϕ_{GP} is the minimizer of (1.8). This formulation of the definition of BEC in terms of the marginal density was first stated by Penrose and Onsager in [PO56].

Hence it strongly suggest that the ground state is well-approximated by tensor products of the form $\phi_0^{\otimes N}$ where ϕ_0 satisfies a variational principle.

1.1.4 Recent Advancements

In recent years, many have contributed to the studies of effective dynamics for many particle systems. In the case of $\beta = 0$ with repulsive Coulomb interactions, Erdös and Yau in [EY01] prove the qualitative result, via the method of BBGKY hierarchy, that the one-particle marginal density $\gamma_{N,t}^{(1)}$ associated to the wave function $\Psi_{N,t}$ with asymptotically factorized initial state, i.e. $\Psi_{N,0} \to \phi^{\otimes N}$ as $N \to \infty$, converges to $|\phi_t\rangle\langle\phi_t|$ in trace norm in the mean-field limit of $N \to \infty$ where ϕ_t satisfies the Hartree equation

$$\frac{1}{i}\frac{\partial}{\partial t}\phi(t,x) - \Delta_x\phi(t,x) + \left(\frac{1}{|\cdot|} * |\phi_t|^2\right)\phi(t,x) = 0. \tag{1.9}$$

In fact, the approach mentioned above is based on the earlier work of Spohn [Spo80] proving the statement for bounded potentials in the case $\beta = 0$. Using the Fock space method introduced by Hepp in [Hep74] and subsequently extended by Ginibre and Velo in [GV79a, GV79b], Rodnianski and Schlein in [RS09] provide a rate of convergence of the one-particle marginal associated to the many-body quantum system towards the Hartree dynamics in trace norm, that is[6]

$$\operatorname{Tr}\left|\gamma_{N,t}^{(1)} - |\phi_t\rangle\langle\phi_t|\right| \lesssim \frac{e^{Kt}}{\sqrt{N}}$$

for some constant $K > 0$ independent of N. The estimate was later improved to $e^{Kt}N^{-1}$ in [ES09, CLS11]. Using a second-order correction Fock space method introduced by Grillakis, Machedon, and Margetis in [GMM10, GMM11], Kuz in [Kuz15b] provides a rate of convergence of the many-body quantum system to the Hartree dynamics in the sense of Fock space marginal density[7]. Consequently, Kuz shows that

$$\operatorname{Tr}\left|\gamma_{N,t}^{(1)} - |\phi_t\rangle\langle\phi_t|\right| \lesssim \frac{\sqrt{1+t}}{N^{\frac{1}{4}}}$$

which in turn establishes the validity of the approximation for time t of the order $\sqrt{N}$. Similar results are derived in [FKS09, KP10] but the approaches are completely

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[6] We adopt the standard notation $A \lesssim B$ to mean there exists a constant, depending on some parameters, such that $A \leq CB$.

[7] One should note the main result in Rodnianski and Schlein's paper is their result on the rate of convergence of the one-particle Fock marginal towards the Hartree dynamics. Whereas, the significance of Kuz's paper is that she was able to show that the mean-field estimate is actually valid for a much longer period of time then most proceeding results had indicated.

different from the above methods.

For the case $0 < \beta \leq 1$, Erdös, Schlein, and Yau in a series of papers [ESY06, ESY07, ESY10, ESY09] show qualitatively that the many-body dynamics with asymptotically factorized initial data converges to the cubic nonlinear Schrödinger dynamics when $0 < \beta < 1$ or the Gross-Pitaevskii dynamics when $\beta = 1$. More precisely, they prove that $\gamma^{(1)}_{N,t} \to |\phi_t\rangle\langle\phi_t|$ in trace norm where ϕ_t satisfies

$$\frac{1}{i}\frac{\partial}{\partial t}\phi(t,x) - \Delta_x\phi(t,x) = \begin{cases} -\left(\int v\right)|\phi_t|^2\phi_t & \text{if } 0 < \beta < 1 \\ -8\pi a|\phi_t|^2\phi_t & \text{if } \beta = 1 \end{cases}$$

where a is the scattering length corresponding to the potential v. Results on the rate of convergence of Fock space marginals can be can be found in [KP10, BdOS15, Kuz15b].

Despite the success founded in the rigorous study of the mean-field behaviors of BEC, recent experiments suggest that mean-field dynamics may not account for the *depletion of the condensate*, the phenomenon where particles in the condensate escape to higher energy states [XLM$^+$06, LEN$^+$17]. Hence, this warrants the rigorous studies of quantum fluctuations about the mean-field dynamic of BEC. A natural setting to account for the fluctuation is in the *bosonic (Symmetric) Fock space*

$$\mathcal{F}_s(\mathfrak{h}) = \mathbb{C} \oplus \bigoplus_{n\geq 1} \mathrm{Sym}\left(\mathfrak{h}^{\otimes}\right)$$

where $\mathfrak{h} := L^2(\mathbb{R}^3)$. Introducing $\mathcal{F}_s$ allows us to deal with states with varying number of particles. Recent works on evolution of *coherent states* in Fock space with quantum fluctuations can be found in [RS09, GMM10, GMM11, Che12, GM13a, GM13b, Kuz15b, Kuz15a, BCS17, NN17, Cho16]. Hence, by accounting for some quantum fluctuation, one is able to estimate the evolution of the coherent state in Fock space norm, which in effect allows one to obtain L^2-norm approximation of the evolution of many-body quantum system with factorized initial data. This is the main setting of the thesis which we will elaborate more on in the next section.

We refer the reader to [LSSY05, Gol16, GMM17] for a complete survey of the subject.

1.2 Mathematical Framework

1.2.1 Fock Space Formalism

In this section, we provide the reader with a brief account of the main mathematical framework for the thesis. For a more comprehensive treatment of the *second quantization formalism*, we refer the reader to [Ber66].

Let us introduce the mathematical setting for our work. The one-particle base space, denoted by $\mathfrak{h} = L^2(\mathbb{R}^3, dx)$, is a complex separable Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{h}}$ which is linear in the second variable and conjugate linear (or anti-linear) in the first variable [8].

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[8] This is the physicists' inner product.

We define the *bosonic Fock space over* $\mathfrak{h}$ to be the closure of

$$\mathcal{F}_s(\mathfrak{h}) = \mathcal{F}_s := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \operatorname{Sym}(\mathfrak{h}^{\otimes n})$$

with respect to the norm induced by the *Fock inner product*

$$\langle \varphi, \psi \rangle_{\mathcal{F}} = \bar{\varphi}_0 \psi_0 + \sum_{n=1}^{\infty} \langle \varphi_n, \psi_n \rangle_{\mathfrak{h}^{\otimes n}}$$

where $\varphi = (\varphi_0, \varphi_1, \ldots), \psi = (\psi_0, \psi_1, \ldots) \in \mathcal{F}_s(\mathfrak{h})$. For convenience, we shall refer $\mathcal{F}_s$ simply as the Fock space henceforth. The *vacuum*, denoted by Ω, is define to be the Fock vector $(1, 0, 0, \ldots) \in \mathcal{F}_s$.

For every field $\phi \in \mathfrak{h}$ we can define the associated *creation* and *annihilation* operators on $\mathcal{F}_s$, denoted respectively by $a^\dagger(\phi)$ and $a(\bar{\phi})$, as follow

$$(a^\dagger(\phi)\psi)_n(x_1, \ldots, x_n) := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \phi(x_j) \psi_{n-1}(x_1, \ldots, \widehat{x}_j, \ldots, x_n) \tag{1.10a}$$

$$(a(\bar{\phi})\psi)_n(x_1, \ldots, x_n) := \sqrt{n+1} \int dx\ \bar{\phi}(x) \psi_{n+1}(x, x_1, \ldots, x_n). \tag{1.10b}$$

with the property that $a(\phi)\Omega = 0$. We can also define the corresponding creation and annihilation distribution-valued operators associated to (1.10a) and (1.10b), denoted by $a_x^\dagger$ and a_x, as follow

$$(a_x^\dagger \psi)_n := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \delta(x - x_j) \psi_{n-1}(x_1, \ldots, \widehat{x}_j, \ldots, x_n) \tag{1.11a}$$

$$(a_x \psi)_n := \sqrt{n+1} \psi_{n+1}(x, x_1, \ldots, x_n). \tag{1.11b}$$

In short, we have the relations

$$a^\dagger(\phi) = \int dx\ \{\phi(x)a_x^\dagger\} \quad \text{and} \quad a(\bar{\phi}) = \int dx\ \{\bar{\phi}(x)a_x\}.$$

Let us note that the creation and annihilation operators $a(\bar{\phi})$ and $a^\dagger(\phi)$ associated to the field ϕ are unbounded, densely defined, closed operators. Moreover, one can easily verify, formally, $(a_x^\dagger, a_x)$ satisfy the canonical commutation relation (CCR): $[a_x, a_y^\dagger] = \delta(x-y)$, $[a_x, a_y] = [a_x^\dagger, a_y^\dagger] = 0$[9], and the *number operator* defined by

$$\mathcal{N} := \int dx\ a_x^\dagger a_x \tag{1.12}$$

is a diagonal operator on $\mathcal{F}$ that counts the number of particles in each sector.

As mentioned in the introduction we are interested in studying the time evolution of the coherent state in Fock space. Before doing so, let us define the initial datum, the coherent state and the Fock Hamiltonian. For each $\phi \in \mathfrak{h}$, we associate the corresponding unique closure of the operator

$$\mathcal{A}(\phi) = a(\bar{\phi}) - a^\dagger(\phi) \tag{1.13}$$

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[9] The reader should note for any $f, g \in \mathfrak{h}$ the CCR for $a^\dagger(f)$ and $a(g)$ are not well defined since there are domain issues that need to be resolved for the given unbounded operators. For an exotic example of an ill-defined commutator of unbounded operators, we refer the reader to Chapter VIII.5 of [RS80].

then the *Weyl operator*[10] is defined to be

$$e^{-\sqrt{N}\mathcal{A}(\phi)}. \tag{1.14}$$

Let us note the operator $\mathcal{A}(\phi)$ is a skew-Hermitian unbounded operator which means the corresponding Weyl operator is unitary. The *coherent state* associated to ϕ is given by

$$\psi(\phi) := e^{-\sqrt{N}\mathcal{A}(\phi)}\Omega. \tag{1.15}$$

Using the Baker-Campbell Hausdorff formula, one can show

$$e^{-\sqrt{N}\mathcal{A}(\phi)}\Omega = \left(\ldots, c_n\phi^{\otimes n}, \ldots\right) \quad \text{where} \quad c_n = \left(e^{-N\|\,\phi\,\|_{\mathfrak{h}}^2} N^n/n!\right).$$

For a fixed $N \in \mathbb{N}$, we defined the *Fock Hamiltonian* associated to N, denoted by $\mathcal{H}_N$, to be the diagonal operator on the Fock space given by

$$(\mathcal{H}_N\psi)_n = \left(\sum_{j=1}^{n} \Delta_{x_j} - \frac{1}{N}\sum_{i<j}^{n} v_N(x_i - x_j)\right)\psi_n = H_{N,n}\psi_n$$

where $v_N(x) = N^{3\beta}v(N^\beta x)$. Rewrite $\mathcal{H}_N$ using creation and annihilation operators

---

[10] To avoid the unfavorable technicality associated with the unbounded natural of our creation and annihilation operators one often choose to work with the corresponding Weyl algebra, the C^*-algebra generated by the exponential of $\mathcal{A}(\phi)$ where $\phi \in \mathfrak{h}$ (cf. chapter 9 of [DG13] and chapter 5.2 of [BR03]).

we get

$$\mathcal{H}_N := \mathcal{H}_1 - \frac{1}{N}\mathcal{V} \tag{1.16a}$$

$$\mathcal{H}_1 := \int dxdy\ \{\Delta_x \delta(x-y) a_x^\dagger a_y\} \quad \text{and} \tag{1.16b}$$

$$\mathcal{V} := \frac{1}{2}\int dxdy\ \{v_N(x-y) a_x^\dagger a_y^\dagger a_x a_y\}. \tag{1.16c}$$

In light of (1.16a), we are interested in the solution to the following Cauchy problem in Fock space

$$\frac{1}{i}\frac{\partial}{\partial t}\psi = \mathcal{H}_N \psi \quad \text{with initial datum} \quad \psi_0 = e^{-\sqrt{N}\mathcal{A}(\phi_0)}\Omega \tag{1.17}$$

which we write

$$\psi_{\text{exact}} = e^{it\mathcal{H}_N} e^{-\sqrt{N}\mathcal{A}(\phi_0)}\Omega. \tag{1.18}$$

An important fact to note about the Fock Hamiltonian is its action on the Nth sector of the Fock space. There, the Fock Hamiltonian acts as a mean-field Hamiltonian for the N-particle system, that is

$$(\mathcal{H}_N \psi)_N = \left(\sum_{j=1}^{N} \Delta_{x_j} - \frac{1}{N}\sum_{i<j}^{N} v_N(x_i - x_j)\right)\psi_N = H_{N,\text{mf}}\psi_N. \tag{1.19}$$

Since the Nth coefficient c_N could be approximated, using Stirling's formula, by $N^{-\frac{1}{4}}$ and the coherent state is a simple N-tensor of ϕ in the N sector, then heuristically we see how by understanding the evolution of the coherent state we would also understand the mean-field evolution of the N-particle factorized state.

Based on the earlier works of Hepp and Ginibre & Velo in [Hep74, GV79a, GV79b], Rodnianski and Schlein in [RS09] studies the one-particle Fock marginal, which is defined as follows: for every $\psi \in \mathcal{F}_s$ the *one-particle Fock marginal of* ψ, denote by $\Gamma^{(1)}_{\psi}$, is a positive trace class integral operator on $\mathfrak{h}$ with kernel given by

$$\Gamma^{(1)}_{\psi}(x,y) = \frac{\langle \psi, a^\dagger_x a_y \psi\rangle_{\mathcal{F}}}{\langle \psi, \mathcal{N}\psi\rangle_{\mathcal{F}}} = \frac{1}{N}\langle \psi, a^\dagger_x a_y \psi\rangle_{\mathcal{F}}. \tag{1.20}$$

They were about to show that the one-particle Fock marginal with an initial coherent state converges to the Hartree dynamics in trace norm for the case $\beta = 0$. Furthermore, they were also able to obtain a rate of convergence

$$\operatorname{Tr}\left|\Gamma^{(1)}_{N,t} - |\phi_t\rangle\langle\phi_t|\right| \lesssim \frac{e^{Kt}}{N} \tag{1.21a}$$

where $\Gamma^{(1)}_{N,t}$ denotes the one-particle Fock marginal for ψ_{exact} and ϕ_t satisfies the Hartree equation. Later, Kuz in [Kuz15b] improved the estimate substantially in time and obtain the estimate

$$\operatorname{Tr}\left|\Gamma^{(1)}_{N,t} - |\phi_t\rangle\langle\phi_t|\right| \lesssim \frac{t}{N}. \tag{1.21b}$$

Unlike the approach of Rodnianski and Schlein which uses the mean-field approximation of the form

$$\psi_{\text{mf}} = e^{-\sqrt{N}\mathcal{A}(\phi_t)}\Omega = e^{-\sqrt{N}\mathcal{A}(t)}\Omega, \tag{1.22}$$

Kuz uses the method of second-order correction introduced in the works of Grillakis, Machedon, and Margetis in [GMM10, GMM11, GM13a] to establish (1.21b), which relies on tracking the exact dynamics of the evolution of the coherent state in Fock space.

To track the exact dynamics in Fock space, we need to introduce the *pair excitation function*, $k(x, y) = k(y, x)$, and its corresponding quadratic operator $\mathcal{B}(k)$ with kernel

$$\mathcal{B}(k_t)(x, y) = \mathcal{B}(t)(x, y) = \int dxdy\ \{\bar{k}(t, x, y)a_x a_y - k(t, x, y)a_x^\dagger a_y^\dagger\}. \tag{1.23}$$

From the pair excitation, we concoct a new approximation scheme, which is a second order correction to the mean field (1.22), given by

$$\psi_{\text{approx}} = e^{iN\chi(t)}e^{-\sqrt{N}\mathcal{A}(t)}e^{-\mathcal{B}(t)}\Omega \tag{1.24}$$

where $\chi(t)$ is some phase factor to be determined. Approximation (1.24) is inspired by the earlier work of Wu in [Wu61] on weakly interacting bose gas in non-periodic settings. In the literature, the unitary operator $e^{\mathcal{B}(k_t)}$ is called the *Segal-Shale-Weil metaplectic representation* or *Bogoliubov transformation* by physicists (c.f. [Sha62], chapter 4 of [Fol89], and chapter 11 of [DG13]). With some appropriate choice of evolution equations for ϕ and k we will later see that (1.24) will indeed allow use to track the exact dynamics of the evolution of coherent state (or *quasifree state*) in Fock space.

1.2.2 Uncoupled Time-Dependent HFB System

Incidentally, one could show via a Lie algebra isomorphism argument established in [GM13a,GM13b,GM17] that the evolution of k could be described by some nonlinear evolution equations of

$$\operatorname{sh}(k) := k + \frac{1}{3!}k \circ \bar{k} \circ k + \frac{1}{5!}k \circ \bar{k} \circ k \circ \bar{k} \circ k + \dots \tag{1.25a}$$

$$\operatorname{ch}(k) := \delta + \frac{1}{2!}\bar{k} \circ k + \frac{1}{4!}\bar{k} \circ k \circ \bar{k} \circ k + \dots \tag{1.25b}$$

where $\circ$ denotes the composition of operators. Moreover, in [GM13b], Grillakis and Machedon show, by using a specific coordinate, that the nonlinear equation of the pair excitation could be express as a system of coupled linear equations in $\operatorname{sh}(2k)$ and $\operatorname{ch}(2k)$.

Let us introduce some notation to help us compactly write out the evolution equations for ϕ and k

$$\begin{aligned} g_N(t,x,y) :=& -\Delta_x \delta(x-y) + (v_N * |\phi|^2)(t,x)\delta(x-y) \\ & + v_N(x-y)\bar{\phi}(t,x)\phi(t,y) \\ m_N(t,x,y) :=& -v_N(x-y)\phi(t,x)\phi(t,y) \\ \mathbf{S}_{\text{old}}(s) :=& \frac{1}{i}\partial_t s + g_N^T \circ s + s \circ g_N \quad \text{(Schrödinger-type operator)} \\ \mathbf{W}_{\text{old}}(p) :=& \frac{1}{i}\partial_t p + [g_N^T, p] \qquad\qquad \text{(Wigner-type operator)} \end{aligned}$$

then the desired evolution equations of ϕ and k are given by

$$\frac{1}{i}\partial_t \phi - \Delta_x \phi + (v_N * |\phi|^2)\phi = 0 \qquad \text{(Hartree-type equation)} \tag{1.26a}$$

$$\mathbf{S}_{\text{old}}(\text{sh}(2k)) = m_N \circ \text{ch}(2k) + \overline{\text{ch}(2k)} \circ m_N \tag{1.26b}$$

$$\mathbf{W}_{\text{old}}(\text{ch}(2k)) = m_N \circ \overline{\text{sh}(2k)} - \text{sh}(2k) \circ \overline{m}_N. \tag{1.26c}$$

The system of equations (1.26) is referred to as the *uncoupled time-dependent Hartree-Fock-Bogoliubov system* in contrast to the coupled system introduced in [GM13a, GM17] where the equation for ϕ and the pair excitation equations are coupled.

Now, let us summarize the results in [GM13b, Kuz15b], which built on earlier works by Grillakis, Machedon, and Margetis in [GMM10, GMM11].

Theorem 1.1 (Grillakis & Machedon '13, Kuz '16). *Let* $v \in C_c^1(\mathbb{R}^3)$ *and* $v \geq 0$. *Assume* ϕ *and* k *satisfy* (1.26) *with initial conditions* $\phi(0,\cdot) = \phi_0 \in L^2(\mathbb{R}^2) \cap W^{m,1}(\mathbb{R}^3)$ *for some sufficiently large* m *and* $k(0,\cdot) = 0$. *If* ψ_{exact} *and* ψ_{approx} *are* *defined by* (1.18) *and* (1.24) *respectively, then we have the following estimate*

$$\| \psi_{exact(t)} - \psi_{approx}(t) \|_{\mathcal{F}} \lesssim \frac{(1+t)\log^4(1+t)}{N^{(1-3\beta)/2}} \tag{1.27}$$

provided $0 < \beta < \frac{1}{3}$. *Moreover, if* $(\partial_t \operatorname{sh}(2k))(0,\cdot)$ *is sufficiently regular, then for any* $\epsilon > 0$ *and* j *a positive integer, we have*

$$\begin{aligned} &\| \psi_{exact}(t) - \psi_{approx}(t) \|_{\mathcal{F}} \\ &\lesssim t^{\frac{j+3}{2}} \log^6(1+t) \cdot \begin{cases} N^{-1/2+\beta(1+\epsilon)} & \frac{1}{3} \leq \beta < \frac{2j}{(1-2\epsilon+4j)}, \\ N^{\frac{-3+7\beta}{2}+(j-1)(-1+2\beta)} & \frac{2j}{(1-2\epsilon+4j)} \leq \beta < \frac{1+2j}{3+4j}. \end{cases} \end{aligned} \tag{1.28}$$

Remark 1.2. It should be noted that the assumption $(\partial_t \operatorname{sh}(2k))(0,\cdot)$ must be sufficiently regular imposes a restriction on the form of the initial condition; in particular, $k(0,\cdot)$ cannot be zero. Due to the restriction, we could not choose the coherent state as our initial condition since $e^{-\sqrt{N}\mathcal{A}_0}e^{-\mathcal{B}_0}\Omega$ is a coherent state if and only if $k(0,\cdot) = 0$.

Remark 1.3. In §2 of [Kuz15a], Kuz provides a heuristic argument showing that the system (1.26) has limitations. In fact, Kuz argued that (1.26) will not be able to provide any Fock space estimate for $\beta \geq \frac{1}{2}$ which indicates a revision to (1.26) is necessary in order to study the case of large β.

1.2.3 (Coupled) Time-Dependent HFB System

Let $\mathcal{M} = e^{-\sqrt{N}\mathcal{A}}e^{-\mathcal{B}}$. Following [GM13a, GM17], we work with the reduced dynamic. More specifically, since $\mathcal{M}$ is unitary then it follows

$$\left\| \psi_{\text{exact}}(t) - e^{iN\int_0^t \chi_0(s)\ ds}\psi_{\text{approx}}(t) \right\|_{\mathcal{F}} = \left\| e^{-iN\int_0^t \chi_0(s)\ ds}\psi_{\text{red}}(t) - \Omega \right\|_{\mathcal{F}}$$

where

$$\psi_{\text{red}}(t) = e^{\mathcal{B}(t)}e^{\sqrt{N}\mathcal{A}(t)}e^{it\mathcal{H}}e^{-\sqrt{N}\mathcal{A}_0}e^{-\mathcal{B}_0}\Omega. \tag{1.29}$$

Then by considering the evolution equation of ψ_{red} given by

$$\frac{1}{i}\frac{\partial}{\partial t}\psi_{\text{red}} = \mathcal{H}_{\text{red}}\psi_{\text{red}} \quad \text{where} \quad \mathcal{H}_{\text{red}} = \frac{1}{i}(\partial_t\mathcal{M}^*)\mathcal{M} + \mathcal{M}^*\mathcal{H}\mathcal{M} \tag{1.30}$$

we see that

$$\left(\frac{1}{i}\frac{\partial}{\partial t} - \mathcal{H}_{\text{red}} + X_0\right)\left(e^{-iN\int_0^t \chi_0(s)\ ds}\psi_{\text{red}} - \Omega\right) = \mathcal{H}_{\text{red}}\Omega - X_0\Omega$$

with

$$\mathcal{H}_{\text{red}}\Omega = (X_0, X_1, X_2, X_3, X_4, 0, 0, \ldots). \tag{1.31}$$

Thus, to estimate the Fock space error, we need to be able to control $\mathcal{H}_{\text{red}}\Omega$. A direct calculation reveals that X_3 and X_4 are heuristically small since they are proportional to $N^{-1/2}$ and N^{-1}, respectively. On the other hand, X_1 and X_2 are proportional to $N^{1/2}$ and constant, respectively. Hence, $X_1 = X_2 = 0$ are natural conditions to impose on ϕ_t and k_t.

Following [GM17], we define the monomial $\mathcal{P}_{n,m} := a^\dagger_{x_1} \cdots a^\dagger_{x_n} a_{y_1} \cdots a_{y_m}$ and consider the $\mathcal{L}$-matrices whose kernels are defined by

$$\mathcal{L}_{n,m}(t, x_1, \dots, x_n; y_1, \dots, y_m) = \frac{1}{N^{(n+m)/2}} \langle \mathcal{M}\Omega, P_{n,m}\mathcal{M}\Omega \rangle. \tag{1.32}$$

In particular, let us focus on the matrices $\mathcal{L}_{0,1}, \mathcal{L}_{1,1}$ and $\mathcal{L}_{0,2}$, which we will denote by ϕ, Γ and Λ respectively. It is shown in [GM17] that the conditions $X_1 = X_2 = 0$ is equivalent to the fact that (ϕ, Γ, Λ) forms a closed system of coupled nonlinear equations

$$\left\{\frac{1}{i}\frac{\partial}{\partial t} - \Delta_{x_1}\right\}\phi(x_1) = -\int dy\ \{v_N(x_1 - y)\,\mathrm{diag}\,\Gamma(t, y)\} \cdot \phi(x_1) \tag{1.33a}$$

$$-\int dy\ \{v_N(x_1 - y)(\Gamma(y, x_1) - \bar{\phi}(y)\phi(x_1))\phi(y)\}$$

$$-\int dy\ \{v_N(x_1 - y)(\Lambda(x_1, y) - \phi(y)\phi(x_1))\bar{\phi}(y)\}$$

$$\left\{\frac{1}{i}\frac{\partial}{\partial t} - \Delta_{x_1} + \Delta_{x_2}\right\}\Gamma(x_1, x_2) \tag{1.33b}$$

$$= -\int dy\ \{(v_N(x_1 - y) - v_N(x_2 - y))\overline{\Lambda}(x_1, y)\Lambda(y, x_2)\}$$

$$-\int dy\ \{(v_N(x_1 - y) - v_N(x_2 - y))\Gamma(x_1, y)\Gamma(y, x_2)\}$$

$$-\int dy\ \{(v_N(x_1 - y) - v_N(x_2 - y))\,\mathrm{diag}\,\Gamma(t, y)\Gamma(x_1, x_2)\}$$

$$+2\int dy\ \{(v_N(x_1 - y) - v_N(x_2 - y))|\phi(y)|^2\bar{\phi}(x_1)\phi(x_2)\}$$

$$
\begin{aligned}
&\left\{\frac{1}{i}\frac{\partial}{\partial t} - \Delta_{x_1} - \Delta_{x_2} + \frac{1}{N}v_N(x_1 - x_2)\right\}\Lambda(x_1, x_2) \qquad (1.33c)\\
&= -\int dy\ \{(v_N(x_1 - y) + v_N(x_2 - y))\,\mathrm{diag}\,\Gamma(t, y)\Lambda(x_1, x_2)\}\\
&-\int dy\ \{(v_N(x_1 - y) + v_N(x_2 - y))\Lambda(x_1, y)\Gamma(y, x_2)\}\\
&-\int dy\ \{(v_N(x_1 - y) + v_N(x_2 - y))\bar{\Gamma}(x_1, y)\Lambda(y, x_2)\}\\
&+2\int dy\ \{(v_N(x_1 - y) + v_N(x_2 - y))|\phi(y)|^2\phi(x_1)\phi(x_2)\}
\end{aligned}
$$

where $\mathrm{diag}\,F(t, x) = F(t, x, x)$[11]. Note, we have suppressed the time dependence to compactify the notation. We refer (1.33) as the *time-dependent Hartree-Fock-Bogoliubov (HFB) system*. It is also instructive to consider $v(x) = g\delta(x)$ which yields the following system

$$
\begin{aligned}
&\left\{\frac{1}{i}\frac{\partial}{\partial t} - \Delta_x\right\}\phi(t, x) = -g\,\mathrm{diag}\,\Lambda(t, x)\bar{\phi}(t, x) \qquad (1.34a)\\
&\quad - 2g\,\mathrm{diag}\,\Gamma(t, x)\phi(t, x) + 2g|\phi(t, x)|^2\phi(t, x)\\
&\left\{\frac{1}{i}\frac{\partial}{\partial t} - \Delta_x + \Delta_y\right\}\Gamma(t, x, y) \qquad (1.34b)\\
&= -g\,\mathrm{diag}\,\overline{\Lambda}(t, x)\Lambda(x, y) + g\,\mathrm{diag}\,\Lambda(t, y)\overline{\Lambda}(t, x, y)\\
&\quad - 2g\{\mathrm{diag}\,\Gamma(t, x) - \mathrm{diag}\,\Gamma(t, y)\}\Gamma(t, x, y)\\
&\quad + 2g\left\{|\phi(t, x)|^2 - |\phi(t, y)|^2\right\}\bar{\phi}(t, x)\phi(t, y)
\end{aligned}
$$

---

[11] In the literature, it is common to denote $(\mathrm{diag}\,\Gamma)(t, x)$ by $\rho(t, x)$, which we will also use. In general, we called the restricted kernels *Schrödinger-type densities*.

$$\left\{\frac{1}{i}\frac{\partial}{\partial t} - \Delta_x - \Delta_y\right\}\Lambda(t,x,y) = -\frac{g}{N}\operatorname{diag}\Lambda(t,x) \tag{1.34c}$$

$$- g\operatorname{diag}\Lambda(t,x)\Gamma(t,x,y) - g\operatorname{diag}\Lambda(t,y)\overline{\Gamma}(t,x,y)$$

$$- 2g\{\operatorname{diag}\Gamma(t,x) + \operatorname{diag}\Gamma(t,y)\}\Lambda(t,x,y)$$

$$+ 2g\{|\phi(t,x)|^2 + |\phi(t,y)|^2\}\phi(t,x)\phi(t,y).$$

Remark 1.4. The physical interpretation of $(\phi(t), \Gamma(t), \Lambda(t))$ is as follows: The function $\phi(t)$ is the one-particle wave function called the condensate wave function which describes the BEC. Following [BBC⁺18], $\gamma(t,x,y) := N(\overline{\Gamma}(t,x,y) - \phi(t,x)\bar{\phi}(t,y)) = \overline{\operatorname{sh}(k)}\circ\operatorname{sh}(k)$ and $\sigma(t,x,y) := N(\Lambda(t,x,y) - \phi(t,x)\phi(t,y)) = \operatorname{sh}(k)\circ\operatorname{ch}(k)$ describe the dynamics of sound waves in the quasifree approximation; in particular, $\operatorname{diag}\gamma(t,x)$ determines the density of the "thermal cloud" of atoms, i.e. the excitation density of the Bose gas. (In the physics literature, $n = \operatorname{diag}\gamma$ and $m = \operatorname{diag}\sigma$ are called the non-condensate density and anomalous density, respectively.)

By direct calculation, it is shown in [GM17] that

$$\Gamma(t,x,y) = \bar{\phi}(t,x)\phi(t,y) + \frac{1}{N}\left(\overline{\operatorname{sh}(k)}\circ\operatorname{sh}(k)\right)(t,x,y) \tag{1.35a}$$

$$\Lambda(t,x,y) = \phi(t,x)\phi(t,y) + \frac{1}{2N}\operatorname{sh}(2k)(t,x,y). \tag{1.35b}$$

The local well-posedness of (1.33) were established in [GM17] using techniques from dispersive PDEs. Consequently, the authors were able to obtain a Fock space estimate for small time. The following theorem summarizes the main result of [GM17]

Theorem 1.5 (Grillakis & Machedon '17). *Let $\frac{1}{3} \leq \beta < \frac{2}{3}$ and $v \in \mathcal{S}$ a nonnegative* *interaction potential satisfying the condition that* $|\hat{v}| \leq \hat{w}$ *for some* $w \in \mathcal{S}$. *Suppose* $(\phi_t, \Gamma_t, \Lambda_t)$ *are solutions to the time-dependent HFB system with some smooth initial conditions* $(\phi_0, \Gamma_0, \Lambda_0)$ *satisfying the following regularity condition uniformly in* N*: for some* $\varepsilon > 0$ *and* $0 \leq i \leq 1, 0 \leq j \leq 2$

$$\begin{aligned}
&\left\| \left\langle \nabla_x \right\rangle^{1/2+\varepsilon} \partial_t^i \nabla_x^j \phi(t,\cdot)\big|_{t=0} \right\|_{L^2(dx)} \lesssim 1 \\
&\left\| \left\langle \nabla_x \right\rangle^{1/2+\varepsilon} \left\langle \nabla_y \right\rangle^{1/2+\varepsilon} \partial_t^i \nabla_{x+y}^j \Gamma(t,\cdot)\big|_{t=0} \right\|_{L^2(dxdy)} \lesssim 1 \\
&\left\| \left\langle \nabla_x \right\rangle^{1/2+\varepsilon} \left\langle \nabla_y \right\rangle^{1/2+\varepsilon} \partial_t^i \nabla_{x+y}^j \Lambda(t,\cdot)\big|_{t=0} \right\|_{L^2(dxdy)} \lesssim 1 \\
&\left\| \nabla_{x+y}^j \operatorname{sh}(2k)(0,x,y) \right\|_{L^2(dxdy)} \lesssim 1.
\end{aligned}$$

Then there exists constants $\delta = \delta(\varepsilon), \kappa = \kappa(\varepsilon), C = C(\varepsilon, \beta)$, *a phase function* $\chi(t)$, *depending on* N, *and* T_0 $(T_0 \sim 1)$ *independent of* N *such that we have the Fock space estimate*

$$\left\| e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}(\phi_0)} e^{-\mathcal{B}(k_0)}\Omega - e^{i\chi(t)} e^{-\sqrt{N}(\phi_t)} e^{-\mathcal{B}(k_t)}\Omega \right\|_{\mathcal{F}} \leq \frac{C}{N^{1/6}}$$

for all $0 \leq t \leq T_0$.

These estimates were later extended by the author to a global-in-time result in [Cho17], which is also the main focus of Chapter 4 of the thesis. More recently, Grillakis and Machedon extended the local well-posedness of the time-dependent Hartree-Fock-Bogoliubov system to the case $\frac{2}{3} < \beta < 1$ in [GM18].

Independently and in a different frame work, Bach, Breteaux, Chen, Fröhlich, and Sigal derived equations closely related to the above equations in [BBC+18]. In particular, the two sets of equations are equivalent in the case of pure states.

More recently, Benedikter, Sok, and Solovej use the reformulated Dirac-Frenkel variational principle in the space of reduced density matrices to geometrically approximate[12] the dynamics of both the bosonic and fermionic many-body systems in [BSS18]. Using the variational principle, they provide a rigorous derivation of both the time-dependent HFB equations and the Bogoliubov-de-Gennes equations, also known as the fermionic time-dependent HFB equations, and show that the equations are optimal approximations of the many-body dynamics when restricted to the manifold of quasifree states[13]. We also refer the interested reader to [HLLS10] for a study of the pseudo-relativistic version of the Bogoliubov-de-Gennes equations.

1.3 Outline and Main Results of the Thesis

In chapter 2, we study the uncoupled HFB equations in the case of attractive interaction potentials. The main results of the chapter is Theorem 2.1 and Theorem 2.5. Theorem 2.1 generalizes the Fock space estimate in [GM13a,Kuz15a] to the case of attractive boson systems. In fact, using Theorem 2.1, we provide two derivations of the focusing NLS from a quantum many-body system with attractive interactions for $0 < \beta < 1/6$, which is the result of Theorem 2.5. This chapter is based on the author's paper [Cho16].

In chapter 3, we study the uniform in N global well-posedness of the time-dependent HFB system in 1D. The main result of this chapter is Theorem 3.3. More

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[12] They were able to show that the Dirac-Frenkel variational principle implies the quasifree reduction principle which was used in [BBC$^+$18].

[13] See §10 in [Sol14] for a definition of quasifree states.

precisely, we show for any $\beta > 0$ the corresponding time-dependent HFB system is uniform in N globally well-posed. It should also be noted that the main tools used in the proving Theorem 3.3 are the linear estimates in §3.2, 3.3, 3.5, and 3.6. This chapter is based on the author's paper [Cho18]

In chapter 4, we extend the local-in-time [GM17] Fock space estimate to a global-in-time for a system of bosons in $\mathbb{R}^3$ for $0 < \beta < \frac{2}{3}$. The main result of the chapter is Theorem 4.1. This chapter is based on the author's paper [Cho17]

In chapter 5, we study some global estimates for the time-dependent HFB system. The main results of the chapter are Theorem 5.3 and Theorem 5.5 which are natural generalization of Morawetz identity and interaction Morawetz estimates for the cubic NLS in $\mathbb{R}^3$.

In chapter 6, we study collapsing estimates for Schrödinger-type densities on closed Riemannian manifolds, which are crucial to proving local well-posedness of the time-dependent HFB system closed manifolds. The main result of the chapter is Theorem 6.1.

Chapter 2: Uncoupled Time-Dependent HFB systems

2.1 Main Results

One of the main purposes of this chapter is to extend the results in [GM13b, Kuz15b, Kuz15a] to the case of arbitrary $v \in C_0^\infty$ with sufficiently small L^1-norm allowing non-positive v. Let us state the first main statement.

Theorem 2.1. *Let $v \in C_0^\infty(\mathbb{R}^3)$. Assume ϕ and k satisfy (1.26) with initial conditions $\phi_0 \in L^2(\mathbb{R}^2) \cap W^{m,1}(\mathbb{R}^3)$ for some sufficiently large m and sufficiently small $\dot{H}_x^{1/2}$-norm, depending on v, and $k(0,\cdot) = 0$. If ψ_{exact} and ψ_{approx} are defined by (1.18) and (1.24) respectively, then we have the following estimate*

$$\| \psi_{exact}(t) - \psi_{approx}(t) \|_{\mathcal{F}} \lesssim \frac{t}{N^{(1-3\beta)/2}} \tag{2.1}$$

provided $0 < \beta < \frac{1}{3}$. Moreover, if $(\partial_t \operatorname{sh}(2k))(0,\cdot)$ is sufficiently regular, then for *any* $\epsilon > 0$ *and* j *a positive integer, we have*

$$\begin{aligned} &\| \psi_{exact}(t) - \psi_{approx}(t) \|_{\mathcal{F}} \\ &\lesssim t^{\frac{j+3}{2}} \log^6(1+t) \cdot \begin{cases} N^{-1/2+\beta(1+\epsilon)} & 0 < \beta < \frac{2j}{(1-2\epsilon+4j)}, \\ N^{\frac{-3+7\beta}{2}+(j-1)(-1+2\beta)} & \frac{2j}{(1-2\epsilon+4j)} \leq \beta < \frac{1+2j}{3+4j}. \end{cases} \end{aligned} \tag{2.2}$$

Remark 2.2. Let us note that there is a tradeoff between the size of the data ϕ_0 and the size of the interaction potential v (c.f. Remark 2.9). Due to the nature of our proof, if we want to assume ϕ_0 is large, i.e. $\| \phi_0 \|_{L^2} = 1$ and $\| \nabla^{1/2}\phi_0 \|_{L^2}$ large , then we need to restrict the L^1-norm of the potential v, and vice versa.

Remark 2.3. A similar result was obtained in [NN17] for the case of repulsive interaction. As stated in Remark 4 in [NN17], their method also extends to the case of attractive interaction provided the uniform in N well-posedness and decay estimates for the corresponding Hartree equations hold, which we will show in the next section.

Remark 2.4. The second estimate in Theorem 2.1 could be improved. In particular, we can get rid of the logarithmic terms. However, to keep the organization of the chapter simple, we decided to keep the logarithmic terms. Nevertheless, we have included a proof of how to remove the logarithmic terms in §2.4.

The second purpose of the chapter is to derive the focusing cubic NLS in $\mathbb{R}^3$ from a many-body boson system as in [CH16a, CH17, CH16b]. For this purpose, we assume $v \leq 0$, i.e. the interaction is attractive. In this case, we have the following statement.

Theorem 2.5. *(Factorized Initial Condition) Assume* $v \in C_c^\infty(\mathbb{R}^3)$ *and* $v \leq 0$*. Suppose* $\Psi_N(t,\mathbf{x})$ *solves the initial value problem*

$$\frac{1}{i}\partial_t \Psi_N(t,\mathbf{x}) = H_{N,mf}\Psi_N(t,\mathbf{x}), \quad \Psi_N(0,\cdot) = \phi_0^{\otimes N}$$

where ϕ_0 *satisties the same conditions as in Theorem 2.1 and* $\| \phi_0 \|_{L^2(dx)} = 1$*. Denote the one-particle density associated to* $\Psi_N(t,x)$ *by* $\gamma_{N,t}^{(1)}$*. Then we have the estimate*

$$\operatorname{Tr}\left|\gamma_N^{(1)}(t,\cdot) - |\phi_t\rangle\langle\phi_t|\right| \lesssim N^\delta$$

for some $\delta < 0$ *provided* $0 < \beta < \frac{1}{6}$*.*

Remark 2.6. The reader should note that Theorem 2.5 only addresses the derivation of the focusing NLS for a system of weakly-interacting dense bose gas since $\beta \in (0, \frac{1}{6})$.

Remark 2.7. As pointed out by the referee, the case at hand deals with the situation where (1.26a) does not exhibit soliton solutions. C.f. Remark 2.11 and Remark 2.21.

2.2 Estimates for the Solution to the Hartree-Type Equations

Let us consider the following family of Hartree-type PDE

$$\begin{aligned} &\frac{1}{i}\partial_t\phi - \Delta_x\phi + (v_N * |\phi|^2)\phi = 0 \\ &\phi(0,\cdot) = \phi_0 \qquad \phi_0 \in H^s(\mathbb{R}^3) \end{aligned} \tag{2.3}$$

where $v_N(x) = N^{3\beta}v(N^\beta x)$ for $0 \leq \beta \leq 1$ and $v \in C_0^\infty(\mathbb{R}^3)$ is not necessary nonnegative. In this section we prove the uniform in N well-posedness of the Hartree-type equation for small data and the corresponding decay estimates.

2.2.1 Uniform in N Global Well-posedness of the Hartree Equations

In this subsection we prove the uniform in N global well-posedness of (2.3) assuming small data. Let us recall the Strichartz norm. We said a pair of numbers (q,r) is *admissible* provided $q, r \geq 2$ and

$$\frac{2}{q} + \frac{3}{r} = \frac{3}{2}.$$

Then the Strichartz norm is defined by

$$\| \phi \|_{S^0} := \sup_{(q,r) \text{ admissible}} \| \phi \|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^3)}.$$

Proposition 2.8 (a-priori estimates)**.** *Let ϕ be a solution to (2.3), then we have* *the estimate*

$$\| \, |\nabla_x|^{\frac{1}{2}} \phi \, \|_{S^0} \lesssim \| \, \phi_0 \, \|_{\dot{H}_x^{1/2}} + \| \, v \, \|_{L^1(dx)} \| \, |\nabla_x|^{\frac{1}{2}} \phi \, \|_{S^0}^3 \tag{2.4}$$

which is independent of N*. Moreover, if* $\| \, v \, \|_{L^1}$ *is sufficiently small then we obtain the estimate*

$$\| \, |\nabla_x|^{\frac{1}{2}} \phi \, \|_{S^0} \lesssim 1 \tag{2.5}$$

which depends only on $\| \, \phi_0 \, \|_{\dot{H}_x^{\frac{1}{2}}}$ *and independent of* N*. Similar estimates holds for time and higher spatial derivatives, that is*

$$\| \, \partial_t^m |\nabla_x|^s \phi \, \|_{S^0} \lesssim 1 \tag{2.6}$$

where the estimate only depends on m, s *and the initial datum.*

Remark 2.9. Observe (2.5) is a consequence of the following elementary observation: if F is continuous on $[0, \infty)$ with $F(0) = A$ and $F(x) \leq A + xF(x)^3$ then there exists $\varepsilon = \varepsilon(A) > 0$ such that $F(x) \leq 2A$ whenever $x \leq \varepsilon$.

Proof. Similar to the local estimate, we begin by differentiating (2.3)

$$\frac{1}{i}\partial_t |\nabla_x|^{\frac{1}{2}} \phi - \Delta_x |\nabla_x|^{\frac{1}{2}} \phi + |\nabla_x|^{\frac{1}{2}} ((v_N * |\phi|^2) \cdot \phi) = 0$$

where

$$|\nabla_x|^{\frac{1}{2}}((v_N * |\phi|^2)\cdot\phi) = (v_N * |\phi|^2)\cdot|\nabla_x|^{\frac{1}{2}}\phi + (v_N * |\nabla_x|^{\frac{1}{2}}|\phi|^2)\cdot\phi$$

$$+ \text{``lower order" terms.}$$

Applying the $L^2L^{6/5}$- endpoint Strichartz estimate of [KT98] and the fractional Leibniz rule, we obtain the following estimate

$$\| \, |\nabla_x|^{\frac{1}{2}}\phi \, \|_{S^0} \lesssim \| \, \phi_0 \, \|_{\dot{H}_x^{1/2}} + \| \, v_N * |\phi|^2 \, \|_{L^2(dt)L^3(dx)} \| \, |\nabla_x|^{\frac{1}{2}}\phi \, \|_{L^\infty(dt)L^2(dx)}$$

$$+ \| \, v_N * |\nabla_x|^{\frac{1}{2}}|\phi|^2 \, \|_{L^2(dtdx)} \| \, \phi \, \|_{L_t^\infty L^3(dx)}.$$

For the first forcing term we have the estimate

$$\begin{aligned}
&\| \, v_N * |\phi|^2 \, \|_{L^2(dt)L^3(dx)} \| \, |\nabla_x|^{\frac{1}{2}}\phi \, \|_{L^\infty(dt)L^2(dx)} \\
\lesssim \;& \| \, v \, \|_{L^1(dx)} \| \, \phi \, \|^2_{L^4(dt)L^6(dx)} \| \, |\nabla_x|^{\frac{1}{2}}\phi \, \|_{L^\infty(dt)L^2(dx)} \\
\lesssim \;& \| \, v \, \|_{L^1(dx)} \| \, |\nabla_x|^{\frac{1}{2}}\phi \, \|^2_{L^4(dt)L^3(dx)} \| \, |\nabla_x|^{\frac{1}{2}}\phi \, \|_{L^\infty(dt)L^2(dx)} \\
\lesssim \;& \| \, v \, \|_{L^1(dx)} \| \, |\nabla_x|^{\frac{1}{2}}\phi \, \|^3_{S^0}.
\end{aligned}$$

The other term can be estimated in a similar fashion. Moreover, estimate (2.6) follows from the observation

$$\| \, \partial_t^m|\nabla_x|^s\phi \, \|_{S^0} \lesssim \| \, \partial_t^m|\nabla_x|^s\phi\big|_{t=0} \, \|_{L^2(dx)} + \| \, v \, \|_{L^1(dx)} \| \, \partial_t^m|\nabla_x|^s\phi \, \|_{S^0} \| \, |\nabla_x|^{\frac{1}{2}}\phi \, \|^2_{S^0}.$$

□

As an immediate corollary of Proposition 2.8, we have

Corollary 2.10 (Uniform in N global well-posedness)**.** *For any* $R > 0$ *there exists* $\varepsilon = \varepsilon(R) > 0$*, independent of* N*, such that when* $\| v \|_{L^1} < \varepsilon$ *then the family of Hartree equations is uniform in* N *globally well-posed. More precisely, we have for any* $\varphi_0 \in \{\varphi \in \dot{H}_x^{1/2} \mid \| \varphi \|_{\dot{H}_x^{1/2}} < R\}$ *there exists a unique solution to* (2.3) *with initial data* φ_0 *satisfying* $\varphi_t \in C([0,\infty) \to \dot{H}_x^{1/2}) \cap S^0$.

Remark 2.11*.* In this paper, we always consider sufficiently smooth initial data. In particular, we could take any $\phi_0 \in H_x^1$ and obtain a uniform in N local well-posedness of solutions to the family of Hartree-type equations. Of course, the tradeoff is that we can only have uniform in N local well-posedness for short time. C.f. chapter 3.3 Proposition 3.19 in [Tao06].

2.2.2 Decay Estimates

In this subsection we prove the uniform in N decay estimates for ϕ_t following the approach in [GM13b], which is in the spirit of [LS78]. Before we begin let us make a note on the notation used in this section. The notation $\alpha\pm$ means $\alpha \pm \varepsilon$ for some fixed $0 < \varepsilon \ll 1$.

Proposition 2.12. *Suppose* $\phi_0 \in W_x^{k,1}$ *for some sufficiently large* k*. Let* ϕ *be a solution to* (2.3) *with sufficiently small potential* v*, depending on the size of data.*

Then we have the decay estimate

$$\| \phi(t,\cdot) \|_{L^\infty(dx)} \lesssim \frac{1}{1+t^{3/2}}$$

which only depends on $\| \phi_0 \|_{W_x^{k,1}}$ *and independent of* N.

Let us first prove the following lemmas.

Lemma 2.13. *Assuming the same conditions as in Proposition 2.12. Then* $\| \phi(t,\cdot) \|_{L^\infty(dx)} \to 0$ *as* $t \to \infty$.

Proof of lemma 2.13. By Proposition 2.8 and Sobolev embedding, we have the estimates

$$\| \phi \|_{L^{10/3}_{t,x}} \le C \quad \text{and} \quad \| \phi \|_{L^\infty_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \lesssim \| \partial_t \nabla_x \phi \|_{L^2(dt)L^6(dx)} \le C.$$

Hence by interpolation we have

$$\begin{aligned} \| \phi \|_{L^p([n,n+1])L^p_x} &\le \| \phi \|^{1-\theta}_{L_t^{10/3}([n,n+1])L^{10/3}(dx)} \| \phi \|^{\theta}_{L_t^\infty([n,n+1])L^\infty(dx)} \\ &\le \| \phi \|^{1-\theta}_{L_t^{10/3}([n,n+1])L^{10/3}(dx)} \| \partial_t \nabla_x \phi \|^{\theta}_{L_t^2([n,n+1])L^6(dx)} \to 0 \end{aligned}$$

as $n \to \infty$ for all $10/3 < p < \infty$. Letting $p \to \infty$ yields the desired result. □

Remark 2.14. The slight analytic gymnastic is a consequence of the fact that we do not have the endpoint Sobolev estimate.

Lemma 2.15. *Assuming the same conditions as in Proposition 2.12. There exists*

$k \in L^1([0,\infty))$ *and* $\delta > 0$ *such that*

$$\| \, e^{i(t-s)\Delta}((v_N * |\phi|^2) \cdot \phi(s)) \, \|_{L^\infty(dx)} \le k(t-s) \| \, \phi(s,\cdot) \, \|^{1+\delta}_{L^\infty(dx)}. \tag{2.7}$$

Proof of Lemma 2.15. Using the $L^\infty L^1$ -decay and conservation of probability, we have

$$\begin{aligned} \| \, e^{i(t-s)\Delta}((v_N * |\phi|^2) \cdot \phi(s)) \, \|_{L^\infty(dx)} &\lesssim \frac{1}{|t-s|^{3/2}} \| \, (v_N * |\phi|^2) \cdot \phi(s) \, \|_{L^1(dx)} \\ &\lesssim \frac{1}{|t-s|^{3/2}} \| \, \phi(s,\cdot) \, \|_{L^\infty(dx)} \end{aligned} \tag{2.8}$$

On the other hand, applying Sobolev embedding, $L^{3+}L^{3/2-}$- decay estimate and interpolation yields

$$\begin{aligned} \| \, e^{i(t-s)\Delta}((v_N * |\phi|^2) \cdot \phi(s) \, \|_{L^\infty(dx)} &\lesssim \; \| \, \nabla_x e^{i(t-s)\Delta}((v_N * |\phi|^2) \cdot \phi(s)) \, \|_{L^{3+}(dx)} \\ &\lesssim \; \frac{1}{|t-s|^{1/2+}} \| \, \nabla_x \phi \, \|_{L^2(dx)} \| \, \phi \, \|^2_{L^{12-}(dx)} \\ &\lesssim \; \frac{1}{|t-s|^{1/2+}} \| \, \phi \, \|^{13/9-}_{L^\infty(dx)}. \end{aligned} \tag{2.9}$$

In the case $|t-s| < 1$, we could simply take $k(t-s) = |t-s|^{1/2+}$. In the case $|t-s| \ge 1$ we interpolate estimates (2.8) and (2.9). $\square$

Proof of Proposition 2.12. Let ϕ_0 be a test function and write (for $t > 0$)

$$\phi(t) = e^{it\Delta}\phi_0 - i \left[\int_0^{t/2} + \int_{t/2}^{t} \right] e^{i(t-\tau)\Delta}(v_N * |\phi(\tau)|^2)\phi(\tau) \, d\tau.$$

Taking the L^∞ norm yields

$$\| \phi(t) \|_{L^\infty(dx)} \lesssim \frac{\| \phi_0 \|_{L^1(dx)}}{t^{3/2}} + \left[\int_0^{t/2} + \int_{t/2}^{t} \right] \| e^{i(t-\tau)\Delta}(v_N * |\phi|^2)\phi(\tau) \|_{L^\infty(dx)} \, d\tau$$

where the first term is a consequence of the $L^\infty L^1$-decay estimate for the free evolution. For the second term, we apply the $L^\infty L^1$-decay estimate and Young's convolution estimate to get

$$\begin{aligned} \int_0^{t/2} \| e^{i(t-\tau)\Delta}(v_N * |\phi|^2)\phi(\tau) \|_{L^\infty(dx)} \, d\tau &\leq \int_0^{t/2} \frac{\| (v_N * |\phi|^2)\phi(\tau) \|_{L^1(dx)}}{|t-\tau|^{3/2}} \, d\tau \\ &\lesssim \frac{1}{t^{3/2}} \int_0^{t/2} \| \phi(\tau) \|_{L^\infty(dx)} \, d\tau. \end{aligned}$$

Lastly, by Lemma 2.15 there exists $k \in L^1([0, \infty])$ and $\delta > 0$ such that

$$\int_{t/2}^{t} \| e^{i(t-\tau)\Delta}(v_N * |\phi|^2)\phi(\tau) \|_{L^\infty(dx)} \, d\tau \lesssim \int_{t/2}^{t} k(t-\tau) \| \phi(\tau) \|_{L^\infty(dx)}^{1+\delta} \, d\tau.$$

Combining all the estimates, we have

$$\| \phi(t) \|_{L^\infty(dx)} \lesssim \frac{\| \phi_0 \|_{L^1(dx)}}{t^{3/2}} + \int_0^{t/2} \frac{\| \phi(\tau) \|_{L^\infty(dx)}}{t^{3/2}} \, d\tau + \int_{t/2}^{t} k(t-\tau) \| \phi(\tau) \|_{L^\infty(dx)}^{1+\delta} \, d\tau$$

which holds for all $t > 0$.

Since we care about large time behavior we may assume $t \geq 1$. In particular, we get the equivalent estimate

$$\| \phi(t) \|_{L^\infty(dx)} \lesssim \frac{\| \phi_0 \|}{1+t^{3/2}} + \int_0^{t/2} \frac{\| \phi(\tau) \|_{L^\infty(dx)}}{1+t^{3/2}} \, d\tau + \int_{t/2}^t k(t-\tau) \| \phi(\tau) \|_{L^\infty(dx)}^{1+\delta} \, d\tau. \tag{2.10}$$

Multiplying estimate (2.10) by $1+t^{3/2}$ yields

$$\begin{aligned}(1+t^{3/2}) \| \phi(t) \|_{L^\infty(dx)} &\lesssim \| \phi_0 \|_{L^1(dx)} + \int_0^{t/2} \| \phi(\tau) \|_{L^\infty(dx)} \, d\tau \\ &\quad + (1+t^{3/2}) \int_{t/2}^t k(t-\tau) \| \phi(\tau) \|_{L^\infty(dx)}^{1+\delta} \, d\tau \\ &\lesssim \| \phi_0 \|_{L^1(dx)} + \int_0^{t/2} \| \phi(\tau) \|_{L^\infty(dx)} \, d\tau \\ &\quad + \sup_{t/2 \le s \le t} (1+s^{3/2}) \| \phi(s) \|_{L^\infty(dx)}^{1+\delta}\end{aligned}$$

since $k \in L^1([0,\infty))$. Next, by Lemma 2.13, there exists $T > 0$ such that

$$\begin{aligned}(1+t^{3/2}) \| \phi(t) \|_{L^\infty(dx)} &\le c \| \phi_0 \|_{L^1(dx)} + c \int_0^{t/2} \| \phi(\tau) \|_{L^\infty(dx)} \, d\tau \\ &\quad + \frac{1}{2} \sup_{t/2 \le s \le t} (1+s^{3/2}) \| \phi(s) \|_{L^\infty(dx)}\end{aligned}$$

whenever $t \ge 2T$ for some constant $c > 0$.

Let $M(t) := \sup_{T \le s \le t} (1+s^{3/2}) \| \phi(s) \|_{L^\infty(dx)}$ and $C := \sup_{0 \le s \le 2T} (1+s^{3/2}) \| \phi(s) \|_{L^\infty(dx)}$ then for all $t \ge T$ we have either

$$(1+t^{3/2}) \| \phi(t) \|_{L^\infty(dx)} \le c \| \phi_0 \|_{L^1(dx)} + c \int_0^{t/2} \frac{M(\tau)}{1+\tau^{3/2}} \, d\tau + \frac{1}{2} M(t)$$

or $M(t) \leq C$. Note, for all $T < s < t$ we also have the following estimate

$$(1 + s^{3/2})\| \phi(s) \|_{L^\infty(dx)} \leq \max\left(c\| \phi_0 \|_{L^1(dx)} + c\int_0^{t/2} \frac{M(\tau)}{1+\tau^{3/2}}\ d\tau + \frac{1}{2}M(t), C \right).$$

Hence it follows

$$M(t) \leq \max\left(c\| \phi_0 \|_{L^1(dx)} + c\int_0^{t/2} \frac{M(\tau)}{1+\tau^{3/2}}\ d\tau + \frac{1}{2}M(t), C \right)$$

for all $t \geq T$. Then by Gronwall's inequality, we have the estimate

$$M(t) \lesssim \max\left(\| \phi_0 \|_{L^1(dx)} \exp\left(\int_0^t \frac{d\tau}{1+\tau^{3/2}} \right), C \right) \lesssim 1.$$

Thus, we have proved

$$\sup_{0 \leq s \leq t} (1 + s^{3/2})\| \phi(s) \|_{L^\infty(dx)} \lesssim \max(M(t), C) \lesssim 1.$$

□

Corollary 2.16. *Assume the same conditions as Proposition 2.12 then there exists a constant C depending only on $\| \phi_0 \|_{W^{k,1}}$ and $\| \partial_t \phi_0 \|_{W^{k,1}}$ such that*

$$\| \partial_t \phi(t, \cdot) \|_{L^\infty(dx)} \lesssim \frac{1}{1+t^{3/2}}. \tag{2.11}$$

Proof. Begin by taking the time derivative of (2.3)

$$\frac{1}{i}\frac{\partial}{\partial t}\partial_t\phi - \Delta_x\partial_t\phi + \partial_t(v_N * |\phi|^2)\phi = 0.$$

Then applying the $L^\infty L^1$ decay estimate yields ($t \geq 1$)

$$\begin{aligned}\| \partial_t\phi(t) \|_{L^\infty(dx)} \lesssim \frac{\| \partial_t\phi(t)\big|_{t=0} \|_{L^1(dx)}}{t^{3/2}} &+ \int_0^{t/2} \| e^{i(t-\tau)\Delta}\partial_\tau(v_N * |\phi|^2)\phi(\tau) \|_{L^\infty(dx)}\, d\tau \\ &+ \int_{t/2}^{t} \| e^{i(t-\tau)\Delta}\partial_\tau(v_N * |\phi|^2)\phi(\tau) \|_{L^\infty(dx)}\, d\tau.\end{aligned}$$

For the first integral, we shall apply the $L^\infty L^1$-decay estimate and Proposition 2.12 to get

$$\begin{aligned}\int_0^{t/2} \| e^{i(t-\tau)\Delta}\partial_\tau(v_N * |\phi|^2)\phi(\tau) \|_{L^\infty(dx)}\, d\tau &\lesssim \int_0^{t/2} \frac{\| \partial_\tau(v_N * |\phi|^2)\phi(\tau) \|_{L^1(dx)}}{|t-\tau|^{3/2}}\, d\tau \\ &\lesssim \frac{1}{1+t^{3/2}} \int_0^{t/2} \| \phi(\tau) \|_{L^\infty(dx)} \| \phi(\tau) \|_{L^2(dx)} \| \partial_\tau\phi(\tau) \|_{L^2(dx)}\, d\tau \\ &\lesssim \frac{1}{1+t^{3/2}} \int_0^{t/2} \frac{d\tau}{1+\tau^{3/2}} \lesssim \frac{1}{1+t^{3/2}}.\end{aligned}$$

Note we have used the fact $\| \partial_t^m \nabla_x^s \phi \|_{S^0} \lesssim_{m,s} 1$.

For the second integral, we use Sobolev embedding and $L^{3+}L^{3/2-}$ decay estimate to obtain the bound

$$\begin{aligned}
&\int_{t/2}^{t} \| e^{i(t-\tau)\Delta}\partial_\tau[(v_N * |\phi|^2)\phi(\tau)] \|_{L^\infty(dx)}\, d\tau \\
&\lesssim \int_{t/2}^{t} \frac{1}{|t-\tau|^{1/2+}} \| \nabla_x \partial_t \phi \|_{L^2(dx)} \| \phi \|_{L^\infty(dx)}^{5/3-}\, d\tau \\
&\quad + \int_{t/2}^{t} \frac{1}{|t-\tau|^{1/2+}} \| \partial_t \phi \|_{L^2(dx)}^{1/3+} \| \partial_t \phi \|_{L^\infty(dx)}^{2/3-} \| \nabla_x \phi \|_{L^2(dx)} \| \phi \|_{L^\infty(dx)}\, d\tau.
\end{aligned}$$

Note the last inequality is a consequence of Hölder inequalities and space interpolation. Since $\partial_t \phi$ is bounded by Proposition 2.8, then by Proposition 2.12 it follows

$$\begin{aligned}
\int_{t/2}^{t} \| e^{i(t-\tau)\Delta}\partial_\tau(v_N * |\phi|^2)\phi(\tau) \|_{L^\infty(dx)}\, d\tau &\lesssim \int_{t/2}^{t} \frac{1}{|t-\tau|^{1/2+}} \| \phi(\tau) \|_{L^\infty(dx)}\, d\tau \\
&\lesssim \frac{1}{1+t^{3/2}} \int_{t/2}^{t} \frac{1}{|t-\tau|^{1/2+}}\, d\tau \lesssim \frac{1}{1+t^{3/2}}.
\end{aligned}$$

□

2.3 Estimates for the Pair Excitations

There are two goals in this section. The first goal is to extend the estimates for $\mathrm{sh}(2k)$ to the case of non-positive interacting potential, which we will see only depends on the decay estimate of ϕ. The other goal is to provide a way to improve the estimate in Theorem 2.1 which we have mentioned in Remark 2.4. However, for the sake of simplicity, we will not propagate the improvement to the rest of the paper and happily leave it as an exercise(s) for the interested reader.

Let us define the shorthand notation $\mathrm{ch}(k) := \delta + p_1, \mathrm{sh}(k) := s_1$, and also

$\mathrm{ch}(2k) := \delta + p_2, \mathrm{sh}(2k) := s_2.$

Proposition 2.17. *Assume $\phi_0 \in W^{k,1}$ for k sufficiently large. The following estimates hold:*

$$\| s_2(t,\cdot) \|_{L^2(dxdy)} + \| p_2(t,\cdot) \|_{L^2(dxdy)} \lesssim 1$$

where the estimate only depends on $\| \phi_0 \|_{W^{k,1}}$ for some k.

To prove the above proposition, we begin by proving a few preliminary lemmas.

Lemma 2.18. *Let $m_N(t,x,y) := -v_N(x-y)\phi(t,x)\phi(t,y)$. Then we have the following estimates*

$$\int \frac{|\widehat{m_N}(t,\xi,\eta)|^2}{(|\xi|^2+|\eta|^2)^2}\, d\xi d\eta \lesssim \| \phi(t,\cdot) \|^4_{L^3(dx)} \tag{2.12}$$

and

$$\int_{|\xi-\eta|>1} \frac{|\partial_t \widehat{m_N}(t,\xi,\eta)|^2}{(|\xi|^2+|\eta|^2)^2}\, d\xi d\eta \lesssim \| \partial_t \phi(t,\cdot) \|^2_{L^4(dx)} \| \phi(t,\cdot) \|^2_{L^4(dx)}. \tag{2.13}$$

Proof. The proof of the first estimate can be found in [GM13b]. We shall focus on the proof of the second estimate where the proof is a slight modification of the first. First, observe

$$v_N(x-y)\phi(x)\phi(y) = \int \delta(x-y-z) v_N(z)\phi(x)\phi(y)\, dz$$

then the Fourier transform of $\delta(x-y-z)\phi(x)\phi(y)$ is given by

$$
\begin{aligned}
&\int e^{-i(x\cdot\eta+y\cdot\xi)}\delta(x-y-z)\phi(x)\phi(y)\ dxdy \\
&\quad = \int e^{-i((x\cdot\eta+(y-z)\cdot\xi)}\delta(x-y)\phi(x)\phi(y-z)\ dxdy \\
&\quad = e^{iz\cdot\xi}\int e^{-ix\cdot(\eta+\xi)}\phi(x)\phi(x-z)\ dx = e^{iz\cdot\xi}\widehat{\phi\phi_z}(t,\eta+\xi)
\end{aligned}
$$

which means

$$
\begin{aligned}
|\partial_t\widehat{m_N}(t,\eta,\xi)|^2 &= \left|\int e^{iz\cdot\xi}v_N(z)\widehat{\partial_t(\phi\phi_z)}(t,\xi+\eta)\ dz\right|^2 \\
&\lesssim \|\, v\,\|_{L^1(dx)}\int |v_N(z)||\widehat{\partial_t(\phi\phi_z)}(t,\eta+\xi)|^2\ dz.
\end{aligned}
$$

Then it follows

$$
\begin{aligned}
\int_{|\xi-\eta|>1}\frac{|\partial_t\widehat{m_N}(t,\eta,\xi)|^2}{(|\eta|^2+|\xi|^2)^2}\ d\eta d\xi &\lesssim \int |v_N(z)|\int_{|\xi-\eta|>1}\frac{|\widehat{\partial_t(\phi\phi_z)}(t,\eta+\xi)|^2}{(|\eta|^2+|\xi|^2)^2}\ d\eta d\xi dz \\
&\lesssim \int |v_N(z)|\int_{|\eta'|>1}\frac{|\widehat{\partial_t(\phi\phi_z)}(t,\xi')|^2}{(|\eta'|^2+|\xi'|^2)^2}\ d\eta' d\xi' dz \\
&\lesssim \int |v_N(z)||\widehat{\partial_t(\phi\phi_z)}(t,\xi')|^2\ d\xi' dz \\
&\lesssim \|\,\partial_t\phi(t,\cdot)\,\|^2_{L^4(dx)}\|\,\phi(t,\cdot)\,\|^2_{L^4(dx)}.
\end{aligned}
$$

$\square$

Lemma 2.19. *Let s_a^0 be the solution to*

$$
\left(\frac{1}{i}\frac{\partial}{\partial t}-\Delta_x-\Delta_y\right)s_a^0(t,x,y) = 2m_N(t,x,y), \quad s_a^0(0,x,y)=0.
$$

Then it follows

$$\| \, s_a^0(t,\cdot) \, \|_{L^2(dxdy)} \lesssim 1 \tag{2.14}$$

where the estimate only depends on $\| \, \phi_0 \, \|_{W^{k,1}}$.

Proof. Using Duhamel's principle, we have

$$\begin{aligned}
\| \, s_a^0(t,\cdot) \, \|_{L^2(dxdy)} &= 2 \left\| \int_0^t e^{i(t-s)\Delta} m_N(s,\cdot) \; ds \right\|_{L^2(dxdy)} \\
&\lesssim \left\| P_{|\xi-\eta|\le 1} \int_0^t e^{i(t-s)\Delta} m_N(s,\cdot) \; ds \right\|_{L^2(dxdy)} \\
&\quad + \left\| P_{|\xi-\eta|>1} \int_0^t e^{i(t-s)\Delta} m_N(s,\cdot) \; ds \right\|_{L^2(dxdy)} .
\end{aligned}$$

For the first term we shall directly apply Minkowski's inequality to get

$$\begin{aligned}
&\left\| P_{|\xi-\eta|\le 1} \int_0^t e^{i(t-s)\Delta} m_N(s,\cdot) \; ds \right\|_{L^2(dxdy)} \\
&\qquad \lesssim \int_0^t \left[\int_{|\xi-\eta|\le 1} |\widehat{m}(s,\xi,\eta)|^2 \; d\xi d\eta \right]^{1/2} \; ds \\
&\qquad \lesssim \int_0^t \left[\int |v_N(z)| \int_{|\eta'|\le 1} |\widehat{\phi\phi_z}(s,\xi')|^2 \; d\xi' d\eta' dz \right]^{1/2} \; ds \\
&\qquad \lesssim \int_0^t \| \, \phi(s,\cdot) \, \|_{L^4(dx)}^2 \; ds.
\end{aligned}$$

Using the decay estimate, we have that the first term is bounded. For the second term we have

$$
\begin{aligned}
&\left\| P_{|\xi-\eta|>1} \int_0^t e^{i(t-s)\Delta} m_N(s,\cdot)\ ds \right\|_{L^2(dxdy)} \\
&\quad = \left\| \chi_{|\xi-\eta|>1} \int_0^t \partial_s e^{i(t-s)(|\eta|^2+|\xi|^2)} \frac{\widehat{m}(s,\xi,\eta)}{|\eta|^2+|\xi|^2}\ ds \right\|_{L^2(dxdy)} \\
&\quad \lesssim \left\| \frac{\widehat{m}(0,\xi,\eta)}{|\eta|^2+|\xi|^2} \right\|_{L^2(d\xi d\eta)} + \left\| \frac{\widehat{m}(t,\xi,\eta)}{|\eta|^2+|\xi|^2} \right\|_{L^2(d\xi d\eta)} \\
&\qquad + \left\| \chi_{|\xi-\eta|>1} \int_0^t e^{i(t-s)(|\eta|^2+|\xi|^2)} \frac{\partial_s \widehat{m}(s,\xi,\eta)}{|\eta|^2+|\xi|^2}\ ds \right\|_{L^2(d\xi d\eta)}.
\end{aligned}
$$

It's clear the first two terms are bounded by the previous lemma. For the last term, using Minkowski's and the previous lemma we have

$$
\begin{aligned}
&\left\| \chi_{|\xi-\eta|>1} \int_0^t e^{i(t-s)(|\eta|^2+|\xi|^2)} \frac{\partial_s \widehat{m}(s,\eta,\xi)}{|\eta|^2+|\xi|^2}\ ds \right\|_{L^2(d\xi d\eta)} \\
&\quad \lesssim \int_0^t \| \partial_t \phi(s,\cdot) \|_{L^4(dx)} \| \phi(s,\cdot) \|_{L^4(dx)}\ ds.
\end{aligned}
$$

Again by the decay estimate, the second term is also bounded. □

Lemma 2.20. *Let s_a be a solution to*

$$\mathbf{S}_{old}(s_a) = 2m_N(t,x,y), \quad s_a(t,\cdot) = 0.$$

Then

$$\| s_a(t,\cdot) \|_{L^2(dxdy)} \lesssim 1$$

where the estimate depends only on the $\| \phi_0 \|_{W^{k,1}}$.

Sketch of the Proof. The essential idea is to decompose the solution into a two parts

$$s_a = s_a^0 + s_a^1$$

where s_a^0 satisfies the equation in the previous lemma and s_a^1 solves

$$\mathbf{S}_{\text{old}}(s_a^1) = -V(s_a^0(t,\cdot)).$$

By the previous lemma, we know the L^2-norm of s_a^0 is uniformly bounded in time. Next, we shall cite [GM13b], Lemma 4.5, for the proof that the L^2-norm of s_a^1 is also uniformly bounded in time. □

Sketch of the Proof of Proposition 2.17. The proof is the same as the proof of Theorem 4.1 in [GM13b]. Again, the only difference comes from the replacement of the estimate to the solution of $\mathbf{S}_{\text{old}}(s_a) = 2m_N$ by the result of the previous lemma. □

Remark 2.21. Following Remark 2.11, if we consider the subcritical uniform in N well-posedness for H_x^1 data, we would obtain the estimate $\sup_{t\in[0,T]}(\| s_2(t,\cdot) \|_{L^2(dxdy)} + \| p_2(t,\cdot) \|_{L^2(dxdy)}) \lesssim 1$ for some small time T and independent of N.

2.4 Proof of Theorem 2.1

The proof of Theorem 2.1 is essentially the same as the proof given in [GM13b, Kuz15a] provided we have established the decay estimate for ϕ. For the sake of simplicity, we shall only provide a complete proof of the first part of Theorem 2.1 since the second part of the theorem is significantly lengthier to present. We shall refer the interested reader to [Kuz15a] for a complete proof of the second part of Theorem 2.1.

2.4.1 List of Error Terms

For convenience, we shall include the list of error terms which were explicitly computed in §5 of [GM13b].

Recall the error terms are defined to be

$$\mathcal{E}(t) = e^{\mathcal{B}}([\mathcal{A}, \mathcal{V}] + N^{-1/2}\mathcal{V})e^{-\mathcal{B}} \tag{2.15}$$

where $[\mathcal{A}, \mathcal{V}]$ and $N^{-1/2}\mathcal{V}$ are cubic and quartic polynomials in $(a_x, a_x^\dagger)$ respectively. Using the conjugation formulae

$$e^{\mathcal{B}}a_x e^{-\mathcal{B}} = \int dy\ \{\mathrm{ch}(k)(y,x)a_y + \mathrm{sh}(k)(y,x)a_y^\dagger\} \tag{2.16a}$$

$$e^{\mathcal{B}}a_x^\dagger e^{-\mathcal{B}} = \int dy\ \{\overline{\mathrm{sh}(k)}(y,x)a_y + \overline{\mathrm{ch}(k)}(y,x)a_y^\dagger\} \tag{2.16b}$$

we could further expand the error terms into another fourth-order polynomial in

$(a_x^\dagger, a_x)$.

The following is the result of expanding $\mathcal{E}(t)$. First, let us list all the error terms of $N^{-1/2}e^{\mathcal{B}}\mathcal{V}e^{-\mathcal{B}}$ which is a fourth-order polynomial in $(a_x^\dagger, a_x)$ with no linear nor cubic terms. The quartic term is given by

$$\frac{1}{2N}\int dy_1 dy_2 dy_3 dy_4 \,\Big\{$$

$$v_N(y_1-y_2)\,\mathrm{sh}(k)(y_3,y_1)\,\mathrm{sh}(k)(y_2,y_4)\;+ \tag{2.17a}$$

$$\int dx\;\{\bar{p}(y_2,x)v_N(y_1-x)\,\mathrm{sh}(k)(x,y_4)\}\,\mathrm{sh}(k)(y_3,y_1)\;+ \tag{2.17b}$$

$$\int dx\;\{\bar{p}(y_1,x)v_N(x-y_2)\,\mathrm{sh}(k)(y_3,x)\}\,\mathrm{sh}(k)(y_2,y_4)\;+ \tag{2.17c}$$

$$\int dx_1 dx_2\{\bar{p}(y_1,x_1)p(x_2,y_2)v_N(x_1-x_2)\,\mathrm{sh}(k)(y_3,x_1)\,\mathrm{sh}(k)(x_2,x_4)\} \tag{2.17d}$$

$$\Big\}\; a_{y_1}^\dagger a_{y_2}^\dagger a_{y_3}^\dagger a_{y_4}^\dagger.$$

The quadratic term is given by

$$\frac{1}{2N}\int dy_1 dy_2 dx_1 dx_2 \,\Big\{$$

$$\overline{\mathrm{ch}(k)}(y_1,x_2)\,\mathrm{sh}(k)(x_2,y_2)(\overline{\mathrm{sh}(k)}\circ\mathrm{sh}(k))(x_1,x_1)v_N(x_1-x_2)\;+ \tag{2.18a}$$

$$\overline{\mathrm{ch}(k)}(y_1,x_2)\,\mathrm{sh}(k)(x_1,y_2)(\overline{\mathrm{sh}(k)}\circ\mathrm{sh}(k))(x_1,x_2)v_N(x_1-x_2)\;+ \tag{2.18b}$$

$$\overline{\mathrm{ch}(k)}(y_1,x_1)\,\mathrm{sh}(k)(x_2,y_2)(\mathrm{sh}(k)\circ\overline{\mathrm{sh}(k)})(x_1,x_2)v_N(x_1-x_2)\;+ \tag{2.18c}$$

$$\overline{\mathrm{ch}(k)}(y_1,x_1)\,\mathrm{sh}(k)(x_1,y_2)(\mathrm{sh}(k)\circ\overline{\mathrm{sh}(k)})(x_2,x_2)v_N(x_1-x_2)\;+ \tag{2.18d}$$

$$\mathrm{sh}(k)(y_1,x_1)\,\mathrm{sh}(k)(x_2,y_2)(\overline{\mathrm{sh}(k)}\circ\overline{\mathrm{sh}(k)})(x_1,x_2)v_N(x_1-x_2)\;+ \tag{2.18e}$$

$$\overline{\mathrm{ch}(k)}(y_1,x_1)\,\mathrm{ch}(k)(x_2,y_2)(\overline{\mathrm{ch}(k)}\circ\mathrm{sh}(k))(x_1,x_2)v_N(x_1-x_2)\Big\}\; a_{y_1}^\dagger a_{y_2}^\dagger \tag{2.18f}$$

The zeroth-order term is given by

$$\frac{1}{2N}\int dx_1 dx_2 \, \Big\{$$

$$(\mathrm{sh}(k)\circ\overline{\mathrm{sh}(k)})(x_1,x_2)v_N(x_1-x_2)(\overline{\mathrm{sh}(k)}\circ\mathrm{sh}(k))(x_1,x_2)\; + \tag{2.19a}$$

$$(\mathrm{sh}(k)\circ\overline{\mathrm{sh}(k)})(x_1,x_1)v_N(x_1-x_2)(\overline{\mathrm{sh}(k)}\circ\mathrm{sh}(k))(x_2,x_2)\; + \tag{2.19b}$$

$$(\overline{\mathrm{sh}(k)}\circ\overline{\mathrm{ch}}(k))(x_1,x_2)v_N(x_1-x_2)(\overline{\mathrm{ch}(k)}\circ\mathrm{sh}(k))(x_1,x_2)\Big\}. \tag{2.19c}$$

In the case of $e^{\mathcal{B}}[\mathcal{A},\mathcal{V}]e^{-\mathcal{B}}$ we have a cubic polynomial in $(a^\dagger_x, a_x)$ with no quadratic nor zeroth-order terms. The cubic term is given by

$$\frac{1}{\sqrt{N}}\int dy_1 dy_2 dy_3 \, \Big\{ v_N(y_1-y_2)\phi(y_2)\,\mathrm{sh}(k)(y_3,y_1)\; + \tag{2.20a}$$

$$\int dx\; \{v_N(y_1-x)\bar{\phi}(x)\,\mathrm{sh}(k)(x,y_3)\}\,\mathrm{sh}(k)(y_2,y_1)\; + \tag{2.20b}$$

$$\int dx\; \{\bar{p}(y_1,x)v_N(x-y_2)\,\mathrm{sh}(k)(y_3,x)\}\phi(y_2)\; + \tag{2.20c}$$

$$\int dx\; \{\bar{p}(y_2,x)v_N(y_1-x)\phi(x)\}\,\mathrm{sh}(k)(y_3,y_1)\; + \tag{2.20d}$$

$$\int dx_1 dx_2\; \{\bar{p}(y_1,x_1)v_N(x_1-x_2)\bar{\phi}(x_2)\,\mathrm{sh}(k)(y_2,x_1)\,\mathrm{sh}(k)(x_2,y_3)\}\; + \tag{2.20e}$$

$$\int dx_1 dx_2\; \{\bar{p}(y_1,x_1)p(x_2,y_2)v_N(x_1-x_2)\phi(x_2)\,\mathrm{sh}(k)(y_3,x_1)\}\Big\}\; a^\dagger_{y_1}a^\dagger_{y_2}a^\dagger_{y_3}. \tag{2.20f}$$

Lastly the linear term is given by

$$\frac{1}{\sqrt{N}}\int dy dx_1 dx_2 \Big\{$$

$$\mathrm{sh}(k)(y,x_2)(\overline{\mathrm{sh}(k)}\circ \mathrm{sh}(k))(x_1,x_1)\bar{\phi}(x_2)v_N(x_1-x_2)\ + \tag{2.21a}$$

$$\mathrm{sh}(k)(y,x_1)(\overline{\mathrm{sh}(k)}\circ \mathrm{sh}(k))(x_1,x_2)\bar{\phi}(x_2)v_N(x_1-x_2)\ + \tag{2.21b}$$

$$\overline{\mathrm{ch}(k)}(y,x_1)(\overline{\mathrm{ch}(k)}\circ \mathrm{sh}(k))(x_1,x_2)\bar{\phi}(x_2)v_N(x_1-x_2)\ + \tag{2.21c}$$

$$\overline{\mathrm{ch}(k)}(y,x_1)(\mathrm{sh}(k)\circ \overline{\mathrm{sh}(k)})(x_1,x_2)\phi(x_2)v_N(x_1-x_2)\ + \tag{2.21d}$$

$$\overline{\mathrm{ch}(k)}(y,x_2)(\mathrm{sh}(k)\circ \overline{\mathrm{sh}(k)})(x_1,x_1)\phi(x_2)v_N(x_1-x_2)\ + \tag{2.21e}$$

$$\mathrm{sh}(k)(y,x_1)(\overline{\mathrm{sh}(k)}\circ \overline{\mathrm{ch}(k)})(x_1,x_2)\phi(x_2)v_N(x_1-x_2)\ \Big\}\ a_y^{\dagger}. \tag{2.21f}$$

2.4.2 Estimates for the Error Terms

To prove theorem 2.1 it suffices to establish the following estimates on $\mathcal{E}(t)$.

Proposition 2.22. *For the two error terms we have the following estimates*

$$\frac{1}{\sqrt{N}}\|\, e^{\mathcal{B}}[\mathcal{A},\mathcal{V}]e^{-\mathcal{B}}\Omega\,\|_{\mathcal{F}} \lesssim \frac{N^{\frac{3\beta-1}{2}}}{1+t^{3/2}} \tag{2.22}$$

and

$$\frac{1}{N}\|\, e^{\mathcal{B}}\mathcal{V}e^{-\mathcal{B}}\Omega\,\|_{\mathcal{F}} \lesssim N^{3\beta-1}. \tag{2.23}$$

Proof. Since many of the terms are similar, without loss of generality, we shall pick representatives in each category and prove the bound holds for the representatives.

First, let us look at the quartic term. The two representatives are (2.17a) and (2.17d) since (2.17b) and (2.17c) could be handled similarly by the techniques in bounding (2.17d). In the case of (2.17a), we see that

$$\begin{aligned}&\frac{1}{N}\left\| v_N(y_1-y_2)\operatorname{sh}(k)(y_3,y_1)\operatorname{sh}(k)(y_2,y_4)\right\|_{L^2(dy_1dy_2dy_3dy_4)}\\&\lesssim\frac{1}{N}\| v_N\|_{L^\infty(dx)}\|\operatorname{sh}(k)\|^2_{L^2(dxdy)}\lesssim N^{3\beta-1}\end{aligned}$$

where we have used Proposition 2.17. For (2.17d), we have

$$\begin{aligned}&\frac{1}{N}\left\|\int dx_1dx_2\{\bar{p}(y_1,x_1)p(x_2,y_2)v_N(x_1-x_2)\operatorname{sh}(k)(y_3,x_1)\operatorname{sh}(k)(x_2,x_4)\right\|_{L^2(dy_1dy_2dy_3dy_4)}\\&\lesssim\ \frac{1}{N}\| v_N\|_{L^\infty(dx)}\| p(k)\|^2_{L^2(dxdy)}\|\operatorname{sh}(k)\|^2_{L^2(dxdy)}\lesssim N^{3\beta-1}.\end{aligned}$$

For the quadratic term, the worse term is given by (2.18f) due to the δ function contribution. Looking term with the most δ function contribution, we have

$$\frac{1}{N}\|\operatorname{sh}(k)(y_1,y_2)v_N(y_1-y_2)\|_{L^2(dy_1dy_2)}\lesssim\frac{1}{N}\| v_N\|_{L^\infty(dx)}\|\operatorname{sh}(k)\|_{L^2(dxdy)}\lesssim N^{3\beta-1}.$$

For the cubic term, we shall consider (2.20a) and (2.20f). In the case of (2.20a), we have

$$\begin{aligned}&\frac{1}{\sqrt{N}}\| v_N(y_1-y_2)\phi(y_2)\operatorname{sh}(k)(y_3,y_1)\|_{L^2(dy_1dy_2dy_3)}\\&\lesssim\frac{1}{\sqrt{N}}\|\phi\|_{L^\infty(dx)}\| v_N\|_{L^2(dx)}\|\operatorname{sh}(k)\|_{L^2(dxdy)}\lesssim\frac{N^{(3\beta-1)/2}}{1+t^{3/2}}.\end{aligned}$$

And for (2.20f), it follows

$$\frac{1}{\sqrt{N}}\left\| \int dx_1 dx_2 \ \{\bar{p}(y_1,x_1)p(x_2,y_2)v_N(x_1-x_2)\phi(x_2)\,\mathrm{sh}(k)(y_3,x_1)\} \right\|_{L^2(dy_1dy_2dy_3)}$$
$$\lesssim \ \frac{1}{\sqrt{N}}\|\,\phi\,\|_{L^\infty(dx)}\|\,p(k)\,\|^2_{L^2(dxdy)}\|\,\mathrm{sh}(k)\,\|_{L^2(dxdy)}\|\,v_N\,\|_{L^2(dx)} \lesssim \frac{N^{(3\beta-1)/2}}{1+t^{3/2}}.$$

Lastly, for the linear term, we shall consider (2.21c). Again, consider the term with the δ contribution, we have

$$\frac{1}{\sqrt{N}}\left\| \int dx_2\,\mathrm{sh}(k)(y,x_2)\bar{\phi}(x_2)v_N(y-x_2) \right\|_{L^2(dy)}$$
$$\lesssim \frac{1}{\sqrt{N}}\|\,\phi\,\|_{L^\infty(dx)}\|\,v_N\,\|_{L^2(dx)}\|\,\mathrm{sh}(k)\,\|_{L^2(dxdy)} \lesssim \frac{N^{(3\beta-1)/2}}{1+t^{3/2}}.$$

□

2.5 Application: Derivation of The Focusing NLS in $\mathbb{R}^3$

We provide two derivation of the focusing nonlinear Schrödinger equation. For the first derivation we will use the method of pair excitation developed in the previous sections and the second derivation will be via a method introduced by Pickl in [Pic11, Pic10].

2.5.1 Pair Excitation Method

In this section, we provide the Fock space method[1] for analyzing the rate of convergence of the one-particle marginal toward mean field. However, in the next

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[1] The pair excitation method is also referred to as the Fock space method.

subsection, we shall provide Pickl's method which offers an error bound which will be independent of time. Nevertheless, the purpose of this section is to show that one could still derive the focusing NLS from the pair excitation method developed thus far in the chapter.

Let us recall a couple results proven in [Kuz15b]:

Lemma 2.23. *Let* $k(x, y) \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ *symmetric in* (x, y)*. Then the following operator inequality holds*

$$e^{\mathcal{B}(k)}\mathcal{N}e^{-\mathcal{B}(k)} \lesssim \mathcal{N} + 1 \tag{2.24}$$

uniformly in time.

Lemma 2.24. *We define the reduced dynamics, denoted by* ψ_{red}*, to be*

$$\psi_{red}(t) := e^{\mathcal{B}(k_t)}e^{\sqrt{N}\mathcal{A}(\phi_t)}e^{it\mathcal{H}_N}e^{-\sqrt{N}\mathcal{A}(\phi_0)}\Omega. \tag{2.25}$$

Then we have the following estimates

$$\left\| \mathcal{N}e^{-\mathcal{B}}\psi_{red} \right\|_{\mathcal{F}} \lesssim \sqrt{N} \left\| (\mathcal{N}+1)^{1/2}\psi_{red} \right\|_{\mathcal{F}} \quad \textit{and} \tag{2.26}$$

$$\| \mathcal{N}^{1/2}\psi_{red} \|_{\mathcal{F}} \lesssim \sqrt{N} \| \psi_{exact} - \psi_{approx} \|_{\mathcal{F}}. \tag{2.27}$$

Following [RS09] and [Kuz15b], we rewrite the one-particle marginal density as follows

$$
\begin{aligned}
&\gamma^{(1)}_{N,t}(t,x,y)\\
&\quad= \frac{1}{c_N^2 N}\left(e^{it\mathcal{H}}P_N e^{-\sqrt{N}\mathcal{A}}\Omega, a_x^\dagger a_y e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}}\Omega\right)\\
&\quad= \frac{1}{c_N^2 N}\left(e^{it\mathcal{H}}P_N e^{-\sqrt{N}\mathcal{A}}\Omega, e^{-\sqrt{N}\mathcal{A}}\underbrace{e^{\sqrt{N}\mathcal{A}}a_x^\dagger e^{-\sqrt{N}\mathcal{A}}}_{a_x^\dagger+\sqrt{N}\bar{\phi}}\underbrace{e^{\sqrt{N}\mathcal{A}}a_y e^{-\sqrt{N}\mathcal{A}}}_{a_y+\sqrt{N}\phi} e^{\sqrt{N}\mathcal{A}} e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}}\Omega\right)\\
&\quad= \frac{1}{c_N^2 N}\left(e^{it\mathcal{H}}P_N e^{-\sqrt{N}\mathcal{A}}\Omega, e^{-\sqrt{N}\mathcal{A}}a_x^\dagger a_y e^{\sqrt{N}\mathcal{A}} e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}}\Omega\right)\\
&\qquad+\ \frac{\phi(t,x)}{c_N^2\sqrt{N}}\left(e^{it\mathcal{H}}P_N e^{-\sqrt{N}\mathcal{A}}\Omega, e^{-\sqrt{N}\mathcal{A}}a_y e^{\sqrt{N}\mathcal{A}} e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}}\Omega\right)\\
&\qquad+\ \frac{\bar{\phi}(t,y)}{c_N^2\sqrt{N}}\left(e^{it\mathcal{H}}P_N e^{-\sqrt{N}\mathcal{A}}\Omega, e^{-\sqrt{N}\mathcal{A}}a_x^\dagger e^{\sqrt{N}\mathcal{A}} e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}}\Omega\right) + \phi(t,x)\bar{\phi}(t,y).
\end{aligned}
$$

Here P_N is the projection operator onto the Nth sector of the Fock space. Moreover, the identities

$$e^{\sqrt{N}\mathcal{A}}a_x^\dagger e^{-\sqrt{N}\mathcal{A}} = a_x^\dagger + \sqrt{N}\bar{\phi} \tag{2.28a}$$

$$e^{\sqrt{N}\mathcal{A}}a_x e^{-\sqrt{N}\mathcal{A}} = a_x + \sqrt{N}\phi \tag{2.28b}$$

are direct consequences of the Lie-type identity used in [GM13b].

Using the above calculation, we have

$$
\begin{aligned}
\left|\gamma^{(1)}_N(t,x,y) - \phi_N(t,x)\bar{\phi}_N(t,y)\right| \le{}& \frac{1}{c_N N}\left\| a_x^\dagger a_y e^{\sqrt{N}\mathcal{A}} e^{it\mathcal{H}} P_N e^{-\sqrt{N}\mathcal{A}}\Omega\right\|_{\mathcal{F}}\\
&+ \frac{|\phi_N(t,y)|}{c_N\sqrt{N}}\left\| a_x e^{\sqrt{N}\mathcal{A}} e^{it\mathcal{H}} P_N e^{-\sqrt{N}\mathcal{A}}\Omega\right\|_{\mathcal{F}}\\
&+ \frac{|\phi_N(t,x)|}{c_N\sqrt{N}}\left\| a_y e^{\sqrt{N}\mathcal{A}} e^{it\mathcal{H}} P_N e^{-\sqrt{N}\mathcal{A}}\Omega\right\|_{\mathcal{F}}
\end{aligned}
$$

which means

$$\begin{aligned}
&\int dxdy\, \left|\gamma_N^{(1)}(t,x,y) - \phi_N(t,x)\bar{\phi}_N(t,y)\right|^2 \\
&\qquad \lesssim \frac{1}{c_N^2 N^2} \left\| \mathcal{N} e^{\sqrt{N}\mathcal{A}} e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}} \Omega \right\|_{\mathcal{F}}^2 + \frac{1}{c_N^2 N} \left\| \mathcal{N}^{1/2} e^{\sqrt{N}\mathcal{A}} e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}} \Omega \right\|_{\mathcal{F}}^2 \\
&\qquad = \frac{1}{c_N^2 N^2} \left\| \mathcal{N} e^{-\mathcal{B}} \psi_{\text{red}} \right\|_{\mathcal{F}}^2 + \frac{1}{c_N^2 N} \left\| \mathcal{N}^{1/2} e^{-\mathcal{B}} \psi_{\text{red}} \right\|_{\mathcal{F}}^2 .
\end{aligned}$$

Applying lemma (2.23) and (2.24), we get

$$\begin{aligned}
&\int dxdy\, \left|\gamma_N^{(1)}(t,x,y) - \phi_N(t,x)\bar{\phi}_N(t,y)\right|^2 \\
&\qquad \lesssim \frac{1}{N^{3/2}} \left\| \mathcal{N} e^{-\mathcal{B}} \psi_{\text{red}} \right\|_{\mathcal{F}}^2 + \frac{1}{\sqrt{N}} \left\| \mathcal{N}^{1/2} \psi_{\text{red}} \right\|_{\mathcal{F}}^2 \\
&\qquad \lesssim \sqrt{N} \| \psi_{\text{exact}} - \psi_{\text{approx}} \|_{\mathcal{F}}^2 .
\end{aligned}$$

Finally, by the appendix in [Kuz15b] and remark 1.4 in [RS09], we have provided both a derivation of the focusing Schrödinger equation and a rate of convergence of the N body interacting bosonic system toward mean field for β in the range $0 < \beta < \frac{1}{6}$.

Remark 2.25. One should note we could only use part one of Theorem 2.1 for our derivation of the focusing NLS since we are considering evolution of coherent states, i.e. $k(0, \cdot) = 0$.

2.5.2 Pickl's Method

Following closely the presentation in [Pic10], we consider the quantities

Definition 2.26. Let $\phi \in L^2(\mathbb{R}^3)$

(a) For each $1 \leq j \leq N$ we define the projectors $p_j^\phi : L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})$ and $q_j^\phi : L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})$ given by

$$p_j^\phi \Psi_N(x_1, \ldots, x_N) = \phi(x_j) \int \phi^*(x_j') \Psi_N(x_1, \ldots, x_j', \ldots, x_N) dx_j$$

and $q_j^\phi = 1 - p_j^\phi$ respectively.

(b) Furthermore, for any $1 \leq k \leq N$ we defined $P_k^\phi : L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})$ given by

$$P_k^\phi := \sum_{a \in \mathcal{A}_k} \prod_{\ell=1}^{N} (p_\ell^\phi)^{1-a_\ell} (q_\ell^\phi)^{a_\ell}$$

where

$$\mathcal{A}_k = \{(a_1, \ldots, a_N) \mid a_i \in \{0, 1\} \text{ and } \sum_{i=1}^{N} a_i = k\}$$

(c) Assume $0 < \lambda \leq 1$. Let us define the function $m^\lambda : \{1, \ldots, N\} \to \mathbb{R}_{\geq 0}$ given by

$$m^\lambda(k) := \begin{cases} k/N^\lambda, \text{ for } k \leq N^\lambda, \\ 1, \text{ otherwise} \end{cases}$$

and a corresponding functional $\alpha_N^\lambda : L^2(\mathbb{R}^{3N}) \times L^2(\mathbb{R}^3) \to \mathbb{R}_{\geq 0}$ given by

$$\begin{aligned}\alpha_N^\lambda(\Psi_N, \phi) &:= \langle \Psi_N, \sum_{k=1}^{N} m^\lambda(k) P_j^\phi \Psi_N \rangle \\ &= \langle \Psi_N, \widehat{m}^{\lambda,\phi} \Psi_N \rangle = \| \, (\widehat{m}^{\lambda,\phi})^{1/2} \Psi_N \, \|_{L^2_{\mathbf{x}}}^2.\end{aligned}$$

For convenience, we shall use the notation α_N instead of α_N^1.

As a direct consequence of the definitions, one could verify the following

$$\alpha_N(\Psi_N, \phi) = \| \, q_1^\phi \Psi_N \, \|_{L^2_{\mathbf{x}}}^2 \leq \alpha_N^\lambda(\Psi_N, \phi)$$

for $0 < \lambda < 1$. Again, by the definition, we could derive an error bound for the rate of convergence of the one particle density towards the mean field limit

$$\begin{aligned}\| \, \gamma_N^{(1)} - |\phi\rangle\langle\phi| \, \|_{\mathrm{op}} &\leq \Big| \| \, p_1^\phi \Psi_N \, \|_{L^2_{\mathbf{x}}}^2 - 1 \Big| \| \, |\phi\rangle\langle\phi| \, \|_{\mathrm{op}} \\ &\quad + 2 \| \, q_1^\phi \Psi_N \, \|_{L^2_{\mathbf{x}}} \| \, p_1^\phi \Psi_N \, \|_{L^2_{\mathbf{x}}} + \| \, q_1^\phi \Psi_N \, \|_{L^2_{\mathbf{x}}}^2 \\ &\leq \Big| \| \, p_1^\phi \Psi_N \, \|_{L^2_{\mathbf{x}}}^2 - 1 \Big| + 2 \| \, q_1^\phi \Psi_N \, \|_{L^2_{\mathbf{x}}} \| \, p_1^\phi \Psi_N \, \|_{L^2_{\mathbf{x}}} \\ &\quad + \| \, q_1^\phi \Psi_N \, \|_{L^2_{\mathbf{x}}}^2 \\ &\lesssim \| \, q_1^\phi \Psi_N \, \|_{L^2_{\mathbf{x}}}^2 + \| \, q_1^\phi \Psi_N \, \|_{L^2_{\mathbf{x}}}.\end{aligned}$$

Since $|\phi\rangle\langle\phi|$ is a rank one projection operator, by remark 1.4 in [RS09] the trace norm is two times the operator norm, i.e., $2 \| \, \gamma_N^{(1)} - |\phi\rangle\langle\phi| \, \|_{\mathrm{op}} = \mathrm{Tr} \Big| \gamma_N^{(1)} - |\phi\rangle\langle\phi| \Big|$.

Then it follows from the above estimates

$$\operatorname{Tr}\left|\gamma_{N,t}^{(1)} - |\phi_t\rangle\langle\phi_t|\right| \lesssim \alpha_N^\lambda(\Psi_N, \phi_t) + \sqrt{\alpha_N^\lambda(\Psi_N, \phi_t)}. \tag{2.29}$$

Thus, to obtain a rate of convergence for the error it suffices to prove an estimate for $\alpha_N^\lambda(\Psi_N, \phi)$. Let us now state the main theorem in [Pic10] which we will use to derive the focusing NLS:

Theorem 2.27. *Assume $0 < \lambda, \beta < 1$ and v_N satisfies the same conditions as before. Assume for every $N \in \mathbb{N}$ there exists a solution to the linear N-body Schrödinger equation $\Psi_N(t,x)$ and a L^∞ solution of the mean field equation ψ_t on some interval $[0,T)$ with $T \in \mathbb{R}_{>0} \cup \{\infty\}$. Then for any $t \in [0,T)$*

$$\begin{aligned}\alpha_N^\lambda(\Psi_{N,t}, \psi_t) \leq\ & \exp\left(\int_0^t C_v \| \phi_s \|_{L^\infty(dx)}^2\, ds\right) \alpha_N^\lambda(\Psi_{N,0}, \phi_0) \\ & + \left[\exp\left(C_v \int_0^t \| \phi_s \|_{L^\infty(dx)}^2\, ds\right) - 1\right] \sup_{0\leq s\leq t} K^{\phi_s} N^{\delta_\lambda}\end{aligned}$$

where $\delta_\lambda = \frac{1}{2}\max\{1-\lambda-4\beta, 3\beta-\lambda, -1+\lambda+3\beta\}$, C_v is some constant depending only on v and

$$K^\phi := C_v(\| \Delta|\phi|^2 \|_{L^2(dx)} + \| \phi \|_{L^\infty(dx)} + 1)\| \phi \|_{L^\infty(dx)}.$$

Proof of Theorem 2.5. Note if $\Psi_N(0,x) = \phi^{\otimes N}$ then $\alpha_N^\lambda(\phi^{\otimes N}, \phi) = 0$. Hence combining with our above decay result for ϕ satisfying the Hartree equation

$$\frac{1}{i}\partial_t\phi - \Delta\phi + (\int v)|\phi|^2\phi = 0$$

we have that

$$\alpha_N^\lambda(\Psi_{N,t}, \phi_t) \leq \left[\exp\left(C_v \int_0^t \| \phi_s \|_{L^\infty(dx)}^2 \, ds\right) - 1\right] \sup_{0\leq s\leq t} K^{\phi_s} N^{\delta_\lambda}$$

where

$$\begin{aligned}
K^{\phi_t} &= C_v(\| \Delta|\phi_t|^2 \|_{L^2(dx)} + \| \phi_t \|_{L^\infty(dx)} + 1)\| \phi_t \|_{L^\infty(dx)} \\
&\lesssim (\| |\nabla_x\phi_t|^2 \|_{L^2(dx)} + \| \phi_t\Delta\bar{\phi}_t \|_{L^2(dx)} + \| \phi_t \|_{L^\infty(dx)} + 1)\| \phi_t \|_{L^\infty(dx)} \\
&\lesssim (\| \nabla_x\phi_t \|_{L^\infty(dx)}\| \nabla_x\phi_t \|_{L^2(dx)} + \| \phi_t \|_{L^\infty(dx)}\| \nabla_x^2\phi_t \|_{L^2(dx)} + \| \phi_t \|_{L^\infty(dx)} + 1)\| \phi_t \|_{L^\infty(dx)} \\
&\lesssim \frac{1}{1+t^{3/2}}.
\end{aligned}$$

Thus, it follows

$$\mathrm{Tr}\left|\gamma_{N,t}^{(1)} - |\phi_t\rangle\langle\phi_t|\right| \lesssim \sqrt{\alpha_N^\lambda(\Psi_{N,t}, \varphi_t)} \lesssim N^{\delta_\lambda/2}.$$

By remark 1 in [Pic10], we see there is a choice of λ such that $\delta_\lambda < 0$ when $0 < \beta < \frac{1}{6}$. $\square$

Chapter 3: Uniform in N Global Well-posed of the Time-Dependent HFB system in $\mathbb{R}^{1+1}$

Based and the discussion in the introduction and (1.6), it is heuristically clear that there is no critical scaling when $d = 1, 2$. To be more specific, for $d \leq 2$, the coupling constant for the interaction of the rescaled system is inversely proportional to the number of particles which means the mean-field scaling is more prominent than the short-range scaling effect. Thus, we do not expect to see any short scale correlation effects. One of the purposes of this chapter is to offer a preliminary step to a rigorous demonstration of the fact that there is no development of short scale correlation structure when $d = 1$ for the effective description by showing the effective equations are well-posed for all $\beta > 0$. The case $d = 2$ for all $\beta > 0$ is still open.

Another reason to consider the entire range of β in $\mathbb{R}^{1+1}$ is inspired by the Lieb-Liniger model [LL63, Lie63] which is a 1D model for a system of ultradcold Bose particles inside the torus endowed with a pairwise interaction given by the repulsive δ-function, i.e. the Lieb-Liniger Hamiltonian for the N-particle Bose gas, in appropriate units, is

$$H_N = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{1 \le i < j \le N} \delta(x_i - x_j) \tag{3.1}$$

where $c \geq 0$ denotes the repulsion strength. More specifically, one can view the Lieb-Liniger model on $\mathbb{R}$ as a heuristic endpoint case of our analysis of the dynamics generated by (1.2) in the weak-coupling limit regime, $c \to 0$.

Our interest in the model is twofold. From a phsyics point of view, the model has an important feature of being exactly solvable in the ground state with computable spectrum. Moreover, the recent advancement in the techniques of trapping and cooling atoms has opened up a variety of possible experimental studies for ultracold Bose gases that are effectively one-dimensional; for a comprehensive survey on the subject, we refer the reader to [BDZ08]. Hence a firm mathematical understanding of the dynamics generated by the Hamiltonian (3.1) is an indispensable theoretical tool to suggest further experimental investigation of certain 1D properties for ultracold Bose gases. In particular, an effective description of the dynamics generated by the Lieb-Liniger model would provide a simplified way to analyze the dynamics of these effectively one-dimensional Bose gases. From a mathematical perspective, the Lieb-Liniger model on $\mathbb{R}$ is the simplest instance of a many-body quantum mechanical model with interaction given by the δ-potential. Up to date, there is no rigorous results on the effective description of the evolution of any quantum system with δ-interaction.

3.1 Notations and Main Statements

Let us indicate some of the notations adopted by this chapter.

Notations. Following [GM17], we use the notations

$$\mathbf{S}_{\pm} := \frac{1}{i}\frac{\partial}{\partial t} - \Delta_x + \Delta_y \quad \text{and} \quad \mathbf{S} := \frac{1}{i}\frac{\partial}{\partial t} - \Delta_x - \Delta_y$$

to denote the two Schrödinger-type differential operators. Moreover, unless specified, $x, y \in \mathbb{R}$, which means $\Delta_x = \partial_{xx}$ and, similarly, $\Delta_y = \partial_{yy}$. The two types of semilinear equations, corresponding to the above operators, considered are the inhomogeneous von-Neumann Schrödinger equation

$$\mathbf{S}_{\pm}\Gamma = F \tag{3.2}$$

and the inhomogeneous Schrödinger equation

$$\left(\mathbf{S} + \frac{1}{N}v_N(x-y)\right)\Lambda = F \tag{3.3}$$

where $v_N(x) = N^{\beta}v(N^{\beta}x)$ and $v \in L^1(\mathbb{R}) \cap C^{\infty}(\mathbb{R})$.

Remark 3.1. We assume v is non-negative and even in our presentation since our prime interest is in studying $v_N \to c\delta$. However, it should be noted that v can be asymmetric and negative when we study the local well-posedness of the time-dependent HFB; whether these facts have interesting physical consequences will not be explored in this chapter.

Next, let us define the space for the initial data. For every $s > 0$, we define the space

$$\mathcal{X}^s = \{(\varphi, \Gamma, \Lambda) \in H^s \times H^s_{\text{Herm}} \times H^s_{\text{sym}}\}$$

with H^s being the Sobolev space $H^s(\mathbb{R})$, H^s_{Herm} the Sobolev space $H^s(\mathbb{R}^2)$ restricted to functions Γ such that $\Gamma(x, y) = \overline{\Gamma(y, x)}$, and H^s_{sym} the Sobolev space space $H^s(\mathbb{R}^2)$ restricted to functions Λ such that $\Lambda(x, y) = \Lambda(y, x)$. More specifically, $\mathcal{X}^s$ is endowed with the norm

$$\begin{aligned}\| (\varphi, \Gamma, \Lambda) \|_{\mathcal{X}^s} &:= \| \langle \nabla_x \rangle^s \phi \|_{L^2(\mathbb{R})} + \| (\langle \nabla_x \rangle^2 \otimes 1 + 1 \otimes \langle \nabla_y \rangle^2)^{s/2} \Gamma \|_{L^2(\mathbb{R}^2)} \\ &\quad + \| (\langle \nabla_x \rangle^2 \otimes 1 + 1 \otimes \langle \nabla_y \rangle^2)^{s/2} \Lambda \|_{L^2(\mathbb{R}^2)}.\end{aligned}$$

When the context is clear, we use the symbol $\langle \nabla_{x,y} \rangle^s$ in place of $(\langle \nabla_x \rangle^2 \otimes 1 + 1 \otimes \langle \nabla_y \rangle^2)^{s/2}$. Furthermore, we study the local well-posedness of our equations in some Strichartz spaces, which are mixed L^p spaces endowed with the norm

$$\| u \|_{L^q[0,T] L^r(dx) L^s(dy)} := \left(\int_0^T dt \, \left(\int dx \, \| u(t, x, \cdot \|^r_{L^s(dy)} \right)^{q/r} \right)^{1/q}$$

where the triplet (q, r, s) satisfies some Strichartz admissible conditions, which will be made clear in the following sections. We also adopt the equivalent notation $L^q(dt) L^r(dx) L^s(dy)$, with the implicit assumption that it depends on T, in place of $L^q([0, T]) L^r(dx) L^s(dy)$.

The hyperbolic trigonometric integral operators introduced in §1 are defined as follows

$$\begin{aligned}
\operatorname{sh}(k) &:= k + \frac{1}{3!} k \circ \bar{k} \circ k + \frac{1}{5!} k \circ \bar{k} \circ k \circ \bar{k} \circ k + \dots \\
\operatorname{ch}(k) &:= \delta + p(k) := \delta + \frac{1}{2!} \bar{k} \circ k + \frac{1}{4!} \bar{k} \circ k \circ \bar{k} \circ k + \dots
\end{aligned}$$

where $\circ$ indicates composition of operators. The symmetric kernel of k, $k(t,x,y) = k(t,y,x)$, is called the pair excitation function. The following are some useful trigonometric identities

$$\operatorname{sh}(2k) = \ 2\operatorname{sh}(k) \circ \operatorname{ch}(k), \quad \operatorname{ch}(2k) = \ \delta + 2\overline{\operatorname{sh}(k)} \circ \operatorname{sh}(k) \tag{3.4a}$$

$$\operatorname{ch}(k) \circ \operatorname{ch}(k) - \overline{\operatorname{sh}(k)} \circ \operatorname{sh}(k) = \delta. \tag{3.4b}$$

Lastly, we use the usual conventional notation

$$\rho_\Gamma(t,x) := \Gamma(t,x,x)$$

to define the restriction of Γ to the diagonal of the plane.

Remark 3.2. We adopt the usual convention of identifying the collection of Hilbert-Schmidt integral operators on $L^2(\mathbb{R}^d)$, denoted by $\mathcal{L}^2$, with their integral kernels in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$.

Main Statement and Structure. Let us state the main results of the chapter

Theorem 3.3 (Uniform in N Local Well-Posedness of the time-dependent HFB in

$\mathbb{R}^{1+1}$). *Suppose* $\beta > 0$ *and* $R > 0$. *Then there exist* $T = T(\beta, R) > 0$, $\sigma = \sigma(\beta)$, *both independent of* N, *and a corresponding spacetime function space* X_T, *depending only on* T *and* σ, *such that for any given*

$$(\varphi_0, \Gamma_0, \Lambda_0) \in \{(\varphi, \Gamma, \Lambda) \in \mathcal{X}^\sigma \mid \|(\varphi, \Gamma, \Lambda)\|_{\mathcal{X}^\sigma} < R\},$$

there exists a unique solution to the time-dependent HFB equations (1.33) *with initial data* $(\varphi_0, \Gamma_0, \Lambda_0)$ *satisfying* $(\varphi_t, \Gamma_t, \Lambda_t) \in C([0, T] \to \mathcal{X}^\sigma) \cap X_T$.

Remark 3.4. The proof is based on Picard-Lindelöf theorem or sometimes known as the Banach fixed-point method. We refer the reader to §3.7 for the definition of the function space X_T and Theorem 3.36 for the a-priori estimates involved in the proof of Theorem 3.3.

Remark 3.5. Given $\beta > 0$, we will later see that the choice of σ must satisfy the conditions $1-\beta\sigma > 0$ and $0 < \sigma < \frac{1}{2}$; see Remark 3.15 and Remark 4.10. Informally, this means when β is large we can only have uniform control of low Sobolev norms. Ideally, we would like to choose $\sigma = 0$, but the nonlinearity requires us to choose $\sigma > 0$; see Remark 3.16. Hence an interesting point to observe is the competition between large $\beta > 0$, which requires low regularity of the initial condition, and the non-linearity, which requires some regularity.

Remark 3.6. The choice of the Banach space X_T is sufficient, maybe necessary, for our analysis of the time-dependent HFB equations. Heuristically, the space X_T is an intersection of Strichartz spaces, which capture evolution due to the Schrödinger-type operators, plus a trace-type space, which captures the interactions coming from the nonlinearity of the coupled equations.

Corollary 3.7 (Uniform in N Global Well-Posedness of the time-dependent HFB in $\mathbb{R}^{1+1}$)**.** *Suppose* $\beta > 0$ *and* $R > 0$*. Then for any*

$$(\varphi_0, \Gamma_0, \Lambda_0) \in \{(\varphi, \Gamma, \Lambda) \in \mathcal{X}^\sigma \mid \| (\varphi, \Gamma, \Lambda) \|_{\mathcal{X}^\sigma} + \| (\nabla_x \varphi, \nabla_{x,y} \Gamma, \nabla_{x,y} \Lambda) \|_{\mathcal{X}^\sigma} < R\},$$

the corresponding local solution to the time-dependent HFB equations (1.33) *given by Theorem 3.3 extends globally with* $(\varphi_t, \Gamma_t, \Lambda_t) \in C([0, \infty) \to \mathcal{X}^\sigma) \cap X_{\infty,loc}$ *(See §8 for definition of* $X_{\infty,loc}$*).*

Remark 3.8. To prove the global well-posedness it suffices to prove that the following estimates

$$\begin{aligned}
&\| \langle \nabla_x \rangle^\sigma \varphi(t, \cdot) \|_{L^2(dx)} \lesssim 1 \\
&\| \langle \nabla_{x,y} \rangle^\sigma \Gamma(t, \cdot) \|_{L^2(dxdy)} \lesssim 1 \\
&\| \langle \nabla_{x,y} \rangle^\sigma \Lambda(t, \cdot) \|_{L^2(dxdy)} \lesssim 1
\end{aligned}$$

hold uniformly in t and N, which is a consequence of the conservation laws proved in [GM13a]. See §3.7.3.

Remark 3.9. Our result does not require the condition $V^2 \leq C(I - \Delta)$ which is a standard assumption used to treat the multiplicative operator V as a perturbation of the non-interacting case. More precisely, since we are working with $V(x) =$

$N^{\beta-1}v(N^\beta x)$, then we see that

$$N^{\beta-2}\int dx\ |v(x)|^2|f(N^{-\beta}x)|^2 = \| Vf \|^2_{L^2(\mathbb{R})} \lesssim \| f' \|^2_{L^2(\mathbb{R})} + \| f \|^2_{L^2(\mathbb{R})}$$

can only be true uniformly in N provided $\beta < 2$. Nevertheless, in the one dimensional setting, V can still be considered as a perturbation even without the condition.

Now let us explain a bit the structure of the paper. In §3.2 and §3.3, we develop estimates that are essential for closing the iteration scheme of the Γ equation. The main results of those two sections necessary for the proof of Theorem 3.36 are Proposition 3.19, Proposition 3.20 and Propostion 3.21. Likewise, from §6 and §7, we will need Proposition 3.32, Corollary 3.33, Proposition 3.34 and Remark 4.10 to close the estimate for the Λ equation. Finally, in §3.7 we prove a-priori estimates that are necessary for us to establish the local well-posedness theory for the time-dependent HFB equations then extend the result to a global well-posedness result under further assumption on the initial data.

3.2 Estimates for the Homogeneous Γ Equation

The main purpose of this section is to prove (3.12) for the von-Neumann Schrödinger equation

$$\frac{1}{i}\frac{\partial}{\partial t}\Gamma + [-\Delta, \Gamma] = 0 \tag{3.5}$$

for arbitrarily smooth initial condition $\Gamma(0,x,y) = \Gamma_0(x,y)$. The two key ingredients involved in the proof of Corollary 3.14 are the collapsing estimate and the sharp trace theorem[1].

Let us adopt the following convention for our spacetime Fourier transform: the spacetime Fourier transform of a Schwartz function $f \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^d)$, denoted by $\widetilde{f}$, is defined to be

$$\widetilde{f}(\tau,\xi) = \int dtdx\; e^{-i(\tau t+\xi\cdot x)} f(t,x). \tag{3.6}$$

Likewise, the Fourier transform $\hat{f}$ of some function $f \in \mathcal{S}(\mathbb{R}^d)$ is defined by

$$\hat{f}(\xi) = \int dx\; e^{-i\xi\cdot x} f(x) \tag{3.7}$$

with corresponding inversion formula

$$f(x) = \frac{1}{(2\pi)^d}\int d\xi\; e^{i\xi\cdot x}\hat{f}(\xi). \tag{3.8}$$

Remark 3.10. The reader should be aware of our attempt to keep track of the values of the fractional derivatives in this section. Keeping a record of these values allows us to show that the mapping used when implementing the fixed-point argument is indeed a self map.

Now, using the spacetime Fourier transform, we can establish the following

---

[1] Here, sharp trace theorem refers to the statement: for any hyperplane $\Sigma \subset \mathbb{R}^d$ and $s > \frac{1}{2}$, the trace operator $T : H^s(\mathbb{R}^d) \to H^{s-\frac{1}{2}}(\Sigma)$ is bounded.

collapsing estimate for the solution to (3.5).

Proposition 3.11 (Collapsing Estimate)**.** *Suppose* Γ *is a solution to* $\mathbf{S}_{\pm}\Gamma = 0$*, then*

$$\| \nabla_x^{\frac{1}{2}} \rho_\Gamma(t,x) \|_{L^2(dtdx)} \lesssim \| \Gamma_0 \|_{L^2(dxdy)}. \tag{3.9}$$

Proof. Taking the spacetime Fourier transform of Γ yields

$$\begin{aligned}
\widetilde{\rho_\Gamma(t,x)} &= \int dtdx\; e^{-i\tau t - i\xi\cdot x}\rho_\Gamma(t,x) = \int dtdxdy\; e^{-i\tau t - i\xi\cdot x}\delta(x-y)\Gamma(t,x,y) \\
&= \frac{1}{(2\pi)^2}\int d\eta dt\; e^{-i\tau t}\widehat{\Gamma}(t,\xi-\eta,\eta) = \frac{1}{(2\pi)^2}\int d\eta dt\; e^{it(-\tau-|\xi-\eta|^2+|\eta|^2)}\widehat{\Gamma_0}(\xi-\eta,\eta) \\
&= \frac{1}{(2\pi)^2}\int d\eta\; \delta(\tau+|\xi-\eta|^2-|\eta|^2)\widehat{\Gamma_0}(\xi-\eta,\eta) \\
&= \frac{1}{8\pi^2|\xi|}\widehat{\Gamma_0}\left(\frac{\xi^2-\tau}{2\xi},\frac{\xi^2+\tau}{2\xi}\right).
\end{aligned}$$

Taking the $L^2_{\tau,\xi}(\mathbb{R}\times\mathbb{R})$ norm of $\widetilde{\nabla_x^{\frac{1}{2}}\rho_\Gamma}$ and applying Cauchy-Schwarz gives us the estimate

$$\int d\tau d\xi\; ||\nabla_x|^{\frac{1}{2}}\widetilde{\rho_\Gamma(t,x)}(\tau,\xi)|^2 \lesssim \| \Gamma_0 \|^2_{L^2(dxdy)}$$

since

$$\sup_{\tau,|\xi|}\int d\eta\; \delta(\tau+|\xi-\eta|^2-|\eta|^2)|\xi| \lesssim 1.$$

$\square$

Utilizing the above collapsing estimate, we prove a couple perturbed version of the collapsing estimate which will be crucial for the chapter.

Lemma 3.12. *Suppose* Γ *is a solution to* $\mathbf{S}_{\pm}\Gamma = 0$*. Then for any* $\varepsilon > 0$ *we have the estimate*

$$\| \nabla_x^{\varepsilon} \rho_\Gamma(t,x) \|_{L^\infty(dt)L^2(dx)} \lesssim \| \nabla_{x,y}^{\frac{1}{2}+\varepsilon} \Gamma_0 \|_{L^2(dxdy)}. \tag{3.10}$$

Proof. For any fixed t, it follows from the sharp trace theorem and the conservation of mass we have that

$$\| \nabla_x^{\varepsilon} \rho_\Gamma(t,x) \|_{L^2(dx)} \lesssim \| \nabla_{x,y}^{\frac{1}{2}+\varepsilon} \Gamma_0 \|_{L^2(dxdy)}.$$

□

Proposition 3.13. *Suppose* Γ *is a solution to* $\mathbf{S}_{\pm}\Gamma = 0$*. Then for any* $0 < \varepsilon < \frac{1}{2}$ *and* $0 < \varepsilon' < \frac{1}{2} - \varepsilon$ *there exist* $q = q(\varepsilon)$ *and* $\alpha = \alpha(\varepsilon)$ *such that the following estimate holds*

$$\| \nabla_x^{\frac{1}{2}-\varepsilon'} \rho_\Gamma(t,x) \|_{L^q(dt)L^2(dx)} \lesssim \| \nabla_{x,y}^{\alpha} \Gamma_0 \|_{L^2(dxdy)}. \tag{3.11}$$

Proof. Interpolating[2] estimates (3.9) and (3.10), we obtain the estimate

$$\| \nabla_x^{\frac{1}{2}-\varepsilon'} \rho_\Gamma \|_{L^q(dt)L^2(dx)} \lesssim \| \nabla_{x,y}^{\alpha} \Gamma_0 \|_{L^2(dxdy)}$$

---

[2] c.f. Chapter V §4 in [SW71].

with α given by

$$\alpha = \left(\frac{\frac{1}{2} + \varepsilon}{\frac{1}{2} - \varepsilon} \right) \varepsilon'.$$

Moreover, checking the arithmetic, we see that

$$q = \frac{1 - 2\varepsilon}{\frac{1}{2} - \varepsilon' - \varepsilon} \geq 2$$

since $\varepsilon' < \frac{1}{2} - \varepsilon$. $\square$

Corollary 3.14. *Suppose* Γ *is a solution to* $\mathbf{S}_{\pm}\Gamma = 0$. *Then for any* $0 < \varepsilon < \frac{1}{2}$ *there exists* $q = q(\varepsilon)$ *such that the following estimate holds*

$$\| \, |\nabla_x|^{\frac{1}{2} - \varepsilon} \rho_\Gamma(t,x) \, \|_{L^q(dt) L^2(dx)} \lesssim \| \, \nabla_{x,y}^{(\frac{3}{2} - \varepsilon)\varepsilon} \Gamma_0 \, \|_{L^2(dxdy)}. \tag{3.12}$$

Proof. Fix ε. Choose δ to be

$$\delta = \frac{1 - 2\varepsilon}{2(5 - 2\varepsilon)} \quad \Longrightarrow \quad \frac{\frac{1}{2} + \delta}{\frac{1}{2} - \delta} = \frac{3}{2} - \varepsilon.$$

To avoid confusion, the reader should note that ε and δ here correspond to ε' and ε in Proposition 3.4. Hence by the previous proposition, there exists $q(\varepsilon)$ given by

$$q = \frac{2}{(2 - \varepsilon)(\frac{1}{2} - \varepsilon)}$$

such that estimate (3.12) holds. $\square$

Remark 3.15. For convenience, we shall henceforth denote the quantity $(\frac{3}{2}-\varepsilon)\varepsilon$ by σ.

Remark 3.16. Heuristically, we want the estimate

$$\|\, \rho_\Gamma(t,x) \,\|_{L^2(dt)L^\infty(dx)} \lesssim \|\, \nabla_x^{\frac{1}{2}} \rho_\Gamma(t,x) \,\|_{L^2(dtdx)} \lesssim \|\, \Gamma_0 \,\|_{L^2(dxdy)}$$

but the estimate is a false endpoint of the Gagliardo-Nirenberg estimate. However, by using the above corollary and the fact that we are working on a finite interval $[0,T]$, we get that

$$\begin{aligned} \|\, \rho_\Gamma(t,x) \,\|_{L^2(dt)L^p(dx)} &\lesssim \|\, |\nabla_x|^{\frac{1}{2}-\varepsilon} \rho_\Gamma(t,x) \,\|_{L^2(dtdx)} \\ &\lesssim T^{\text{some power}} \|\, |\nabla_x|^{\frac{1}{2}-\varepsilon} \rho_\Gamma(t,x) \,\|_{L^q(dt)L^2(dx)} \\ &\lesssim T^{\text{some power}} \|\, |\nabla_{x,y}|^{\sigma} \Gamma_0 \,\|_{L^2(dxdy)}. \end{aligned}$$

We will elaborate more on this point in the next section.

Next, let us establish the homogeneous Strichartz estimate for the linear operator $\mathbf{S}_\pm$.

Proposition 3.17 (Non-Endpoint Strichartz)**.** *Suppose* Γ *is a solution to* $\mathbf{S}_\pm\Gamma = 0$ *with initial condition* Γ_0 *and* (k,ℓ) *is an admissible pair, i.e.*

$$\frac{2}{k} + \frac{1}{\ell} = \frac{1}{2} \tag{3.13}$$

where $(k, \ell) \in (2, \infty] \times [2, \infty]$. *Then it follows*

$$\| e^{it(\Delta_x - \Delta_y)} \Gamma_0 \|_{L^k(dt) L^\ell(dx) L^2(dy)} \lesssim \| \Gamma_0 \|_{L^2(dxdy)}. \tag{3.14}$$

Proof. The proof is essentially the same as the standard non-endpoint Strichartz estimate using both the TT^* principle and Christ-Kiselev lemma. See §2.3 in [Tao06]. □

3.3 Estimates for the Inhomogeneous Γ Equation

Let us now consider the inhomogeneous Γ equation

$$\mathbf{S}_{\pm}\Gamma = F \tag{3.15}$$

where F is smooth. The main purpose of this section is to obtain collapsing estimates similar to estimates proven in Proposition 6.1 and Corollary 3.14 but for that inhomogeneous equation. The main results of this section are Proposition 3.19 and Proposition 3.20.

Remark 3.18. For the purpose of obtaining estimates for (3.15), we do not need to assume F to have any symmetry. That being said, in order for our iteration scheme to preserve the symmetry $\Gamma(x, y) = \overline{\Gamma(y, x)}$, i.e. stay in the designated space which we have specified in Theorem 3.3, it is wise to assume F is skewed symmetric, i.e. $\overline{F(x, y)} = -F(y, x)$. Likewise, the forcing term with respect to the Λ equation should also satisfy $F(x, y) = F(y, x)$. Henceforth, we assume the forcing F for each of the three equations has the correct symmetry.

Observe the solution to the inhomogeneous equation can be written as

$$\Gamma(t,x,y) = e^{it(\Delta_x - \Delta_y)}\Gamma_0(x,y) + i\int_0^t e^{i(t-s)(\Delta_x-\Delta_y)}F(s,x,y)\ ds \tag{3.16}$$

which then yields

$$\rho_\Gamma(t,x) = [e^{it(\Delta_x-\Delta_y)}\Gamma_0](x,x) + i\int_0^t [e^{i(t-s)(\Delta_x-\Delta_y)}F](s,x,x)\ ds \tag{3.17}$$

Then it follows from the estimate (3.9) that

$$\begin{aligned}\|\,|\nabla_x|^{\frac{1}{2}}\rho_\Gamma\,\|_{L^2(dtdx)} &\lesssim \|\,\Gamma_0\,\|_{L^2(dxdy)} + \int_0^T \|\,|\nabla_x|^{\frac{1}{2}}[e^{i(t-s)(\Delta_x-\Delta_y)}F](s,x,x)\,\|_{L^2(dtdx)}\ ds \\ &\lesssim \|\,\Gamma_0\,\|_{L^2(dxdy)} + \|\,F\,\|_{L^1[0,T]L^2(dxdy)}.\end{aligned}$$

Hence we have obtained the following proposition

Proposition 3.19. *Suppose* Γ *solves* $\mathbf{S}_\pm\Gamma = F$*, then we have*

$$\|\,|\nabla_x|^{\frac{1}{2}}\rho_\Gamma\,\|_{L^2(dtdx)} \lesssim \|\,\Gamma_0\,\|_{L^2(dxdy)} + \|\,F\,\|_{L^1[0,T]L^2(dxdy)}. \tag{3.18}$$

The following is a perturbed version of the above proposition.

Proposition 3.20. *Suppose* Γ *solves* $\mathbf{S}_{\pm}\Gamma = F$*, then for every* $0 < \varepsilon < \frac{1}{2}$ *we have*

$$\| \, |\nabla_x|^{\frac{1}{2}-\varepsilon} \rho_\Gamma \, \|_{L^2(dtdx)} \lesssim T^{some\ power} \Big(\| \, |\nabla_{x,y}|^\sigma \Gamma(t,x,y) \, \|_{L^\infty[0,T] L^2(dxdy)} + \| \, |\nabla_{x,y}|^\sigma F \, \|_{L^1[0,T] L^2(dxdy)} \Big). \tag{3.19}$$

Proof. Applying Corollary 3.14 to (3.17) yields

$$\begin{aligned} &\| \, |\nabla_x|^{\frac{1}{2}-\varepsilon} \rho_\Gamma(t,x) \, \|_{L^2(dtdx)} \\ &\quad \lesssim T^{\text{some power}} \Big(\| \, |\nabla_{x,y}|^\sigma \Gamma(t,x,y) \, \|_{L^\infty[0,T] L^2(dxdy)} \\ &\qquad + \int_0^T ds \, \| \, |\nabla_x|^{\frac{1}{2}-\varepsilon} [e^{i(t-s)(\Delta_x - \Delta_y)} F](s,x,x) \, \|_{L^q[0,T] L^2(dx)} \Big) \\ &\quad \lesssim T^{\text{some power}} \left(\| \, |\nabla_{x,y}|^\sigma \Gamma(t,x,y) \, \|_{L^\infty[0,T] L^2(dxdy)} + \| \, \nabla_{x,y}^\sigma F \, \|_{L^1[0,T] L^2(dxdy)} \right) \end{aligned}$$

where $q = \frac{2}{(2-\varepsilon)(\frac{1}{2}-\varepsilon)}$. Then by Remark 3.16 we obtain the desired estimate. ☐

To conclude this section, let us state the inhomogeneous Strichartz estimate.

Proposition 3.21. *Suppose* Γ *is a solution to* $\mathbf{S}_{\pm}\Gamma = F$ *with initial condition* Γ_0 *and* (k,ℓ) *and* $(\tilde{k},\tilde{\ell})$ *are an admissible pairs (see* (3.13)*). Then it follows*

$$\| \, \Gamma(t,x,y) \, \|_{L^k(dt) L^\ell(dx) L^2(dy)} \lesssim \| \, \Gamma_0 \, \|_{L^2(dxdy)} + \| \, F \, \|_{L^{\tilde{k}'}(dt) L^{\tilde{\ell}'}(dx) L^2(dy)} \tag{3.20}$$

and

$$\| \, |\nabla_{x,y}|^\sigma \Gamma(t,x,y) \, \|_{L^k(dt)L^\ell(dx)L^2(dy)} \lesssim \| \, |\nabla_{x,y}|^\sigma \Gamma_0 \, \|_{L^2(dxdy)} + \| \, |\nabla_{x,y}|^\sigma F \, \|_{L^{\widetilde{k}'}(dt)L^{\widetilde{\ell}'}(dx)L^2(dy)} \tag{3.21}$$

where $(\widetilde{k}', \widetilde{\ell}')$ *denotes the Hölder conjugates of* $(\widetilde{k}, \widetilde{\ell})$.

3.4 Application of the Inhomogeneous Γ Estimates

The purpose of this section is to develop estimates which we will later use in the proof of our main theorem in §3.7. However, as an immediate application of the previous two sections, we are now ready to consider the uniform in N local well-posedness of the following Hartree-Fock equation

$$\frac{1}{i}\frac{\partial}{\partial t}\Gamma = [\Delta - v_N * \rho_\Gamma, \Gamma] \tag{3.22}$$

or equivalently

$$\mathbf{S}_\pm \Gamma(t,x,y) = [v_N * \rho_\Gamma(t,x) - v_N * \rho_\Gamma(t,y)]\Gamma(t,x,y) = F \tag{3.23}$$

in some Strichartz-type space X equipped with the norm

$$\begin{aligned} \| \, \Gamma \, \|_X := {} & \| \, |\nabla_x|^{\frac{1}{2}-\varepsilon} \rho_\Gamma(t,x) \, \|_{L^2[0,T]L^2(dx)} + \| \, |\nabla_x|^{\frac{1}{2}} \rho_\Gamma(t,x) \, \|_{L^2[0,T]L^2(dx)} \\ & + \| \, \langle \nabla_{x,y} \rangle^\sigma \Gamma(t,x,y) \, \|_{L^\infty[0,T]L^2(dxdy)} + \| \, \langle \nabla_{x,y} \rangle^\sigma \Gamma(t,x,y) \, \|_{L^4[0,T]L^\infty(dx)L^2(dy)} \end{aligned} \tag{3.24}$$

where ε is sufficiently small, say $\varepsilon < \frac{1}{5}$.

The uniform in N local well-posedness is proven using the standard Banach fixed-point argument. More precisely, we close the estimate for (3.22) in X. In particular, we close the estimate for each of the three norms indicated in (3.24). However, by Proposition 3.19 and Proposition 3.20, it suffices to consider estimates for the corresponding forcing terms.

First, let us estimate $\| F \|_{L^1[0,T]L^2(dxdy)}$. By Hölder's inequality, we see that

$$\begin{aligned}\| F \|_{L^1[0,T]L^2(dxdy)} &\lesssim \| v_N * \rho_\Gamma(t,x)\Gamma(t,x,y) \|_{L^1[0,T]L^2(dxdy)} \\ &\lesssim \| v_N * \rho_\Gamma(t,x) \|_{L^2(dt)L^p(dx)} \| \Gamma(t,x,y) \|_{L^2(dt)L^r(dx)L^2(dy)}\end{aligned}$$

where we made the choice $p = 1/\varepsilon$ and $r = 2(1-2\varepsilon)^{-1}$. Then by Gagliardo-Nirenberg-Sobolev inequality, Young's convolution inequality and Hölder inequality, in the time variable, we obtain the estimate

$$\begin{aligned}\| F \|_{L^1[0,T]L^2(dxdy)} &\lesssim \| v_N * \nabla_x^{\frac{1}{2}-\varepsilon} \rho_\Gamma(t,x) \|_{L^2(dtdx)} \| \Gamma(t,x,y) \|_{L^2(dt)L^r(dx)L^2(dy)} \\ &\lesssim T^{\text{some power}} \| \nabla_x^{\frac{1}{2}-\varepsilon} \rho_\Gamma(t,x) \|_{L^2(dtdx)} \| \Gamma(t,x,y) \|_{L^q(dt)L^r(dx)L^2(dy)}\end{aligned}$$

where $q = 2\varepsilon^{-1}$. Note that q is chosen so that (q, r) is a 1D Strichartz admissible pair. Hence by interpolation, we see that

$$\| F \|_{L^1[0,T]L^2(dxdy)} \lesssim T^{\text{some power}} \| \Gamma \|_X^2.$$

Likewise, we can show that $\| \, |\nabla_{x,y}|^{\frac{1}{2}} F \, \|_{L^1[0,T]L^2(dxdy)}$ also closes.

Next, let us estimate $\| \, |\nabla_{x,y}|^{\sigma} F \, \|_{L^1[0,T]L^2(dxdy)}$. Using the classical Kato-Ponce inequality, sometimes refers to as “fractional Leibniz rule", we see that

$$\begin{aligned}
\| \, |\nabla_{x,y}|^{\sigma} F \, \|_{L^1[0,T]L^2(dxdy)} &\lesssim \| \, |\nabla_{x,y}|^{\sigma}[v_N * \rho_\Gamma(t,x)\Gamma(t,x,y)] \, \|_{L^1[0,T]L^2(dxdy)} \\
&\lesssim \| \, v_N * \rho_\Gamma(t,x) \, \|_{L^2(dt)L^p(dx)} \| \, |\nabla_y|^{\sigma}\Gamma(t,x,y) \, \|_{L^2(dt)L^r(dx)L^2(dy)} \\
&\quad + \| \, v_N * |\nabla_x|^{\sigma}\rho_\Gamma(t,x) \, \|_{L^2(dt)L^{\widetilde{p}}(dx)} \| \, \Gamma(t,x,y) \, \|_{L^2(dt)L^{\widetilde{r}}(dx)L^2(dy)} \\
&\quad + \| \, v_N * \rho_\Gamma(t,x) \, \|_{L^2(dt)L^p(dx)} \| \, |\nabla_x|^{\sigma}\Gamma(t,x,y) \, \|_{L^2[0,T]L^r(dx)L^2(dy)}
\end{aligned}$$

where $\widetilde{p} = 2[(5-2\varepsilon)\varepsilon]^{-1}, \widetilde{r} = [\varepsilon^2 - \frac{5}{2}\varepsilon + \frac{1}{2}]^{-1}$ and p, r as defined above. Hence by the same argument as above with $\widetilde{q} = 2[(\frac{5}{2}-\varepsilon)\varepsilon]^{-1}$ we see that

$$\begin{aligned}
&\| \, |\nabla_{x,y}|^{\sigma}[v_N * \rho_\Gamma(t,x)\Gamma(t,x,y)] \, \|_{L^1[0,T]L^2(dxdy)} \\
&\quad \lesssim T^{\text{some power}} \| \, |\nabla_x|^{\frac{1}{2}-\varepsilon}\rho_\Gamma(t,x) \, \|_{L^2(dtdx)} \| \, |\nabla_y|^{\sigma}\Gamma(t,x,y) \, \|_{L^q(dt)L^r(dx)L^2(dy)} \\
&\qquad + T^{\text{some power}} \| \, |\nabla_x|^{\frac{1}{2}-\varepsilon}\rho_\Gamma(t,x) \, \|_{L^2(dtdx)} \| \, \Gamma(t,x,y) \, \|_{L^{\widetilde{q}}(dt)L^{\widetilde{r}}(dx)L^2(dy)} \\
&\qquad + T^{\text{some power}} \| \, |\nabla_x|^{\frac{1}{2}-\varepsilon}\rho_\Gamma(t,x) \, \|_{L^2(dtdx)} \| \, |\nabla_x|^{\sigma}\Gamma(t,x,y) \, \|_{L^q(dt)L^r(dx)L^2(dy)}
\end{aligned}$$

Again, note that $(\widetilde{q}, \widetilde{r})$ is an admissible pair which means the desired estimate holds by interpolation.

Remark 3.22. The belove estimate is included in this section purely for the author's own organizational purposes. Hence the reader may skip it for now and refer back to it in §8.

Lastly, observe we have

$$\begin{aligned}
\|\nabla_{x+y}^{\sigma} F\|_{L^1[0,T]L^2(dxdy)} &\lesssim \|\nabla_{x+y}^{\sigma}[v_N * \rho_\Gamma(t,x)\Gamma(t,x,y)]\|_{L^1[0,T]L^2(dxdy)} \\
&\lesssim \|v_N * |\nabla_x|^{\sigma}\rho_\Gamma(t,x)\|_{L^2(dt)L^{\widetilde{p}}(dx)}\|\Gamma(t,x,y)\|_{L^2(dt)L^{\widetilde{r}}(dx)L^2(dy)} \\
&\quad + \|v_N * \rho_\Gamma(t,x)\|_{L^2(dt)L^{p}(dx)}\||\nabla_{x+y}|^{\sigma}\Gamma(t,x,y)\|_{L^2[0,T]L^{r}(dx)L^2(dy)} \\
&\lesssim T^{\text{some power}}\||\nabla_x|^{\frac{1}{2}-\varepsilon}\rho_\Gamma(t,x)\|_{L^2(dtdx)}\|\Gamma(t,x,y)\|_{L^{\widetilde{q}}[0,T]L^{\widetilde{r}}(dx)L^2(dy)} \\
&\quad + T^{\text{some power}}\||\nabla_x|^{\frac{1}{2}-\varepsilon}\rho_\Gamma(t,x)\|_{L^2(dtdx)}\||\nabla_x|^{\sigma}\Gamma(t,x,y)\|_{L^{q}[0,T]L^{r}(dx)L^2(dy)}
\end{aligned}$$

where $\nabla_{x+y}F := \frac{1}{2}(\nabla_x F + \nabla_y F)$.

Remark 3.23. Since similar calculations will be performed in §8, then for convenience we shall fix the values of $p, q, r, \widetilde{p}, \widetilde{q}, \widetilde{r}$ as indicated above for a given ε in the remaining of this chapter.

As a result of the above calculation, we obtain the following proposition

Proposition 3.24. *Suppose* Γ *solves* (3.22) *with Schwartz initial condition* Γ_0 *and* $v \in L^1(\mathbb{R})$. *Then the following estimate holds*

$$\|\Gamma\|_X \lesssim \|\langle\nabla_{x,y}\rangle^{\sigma}\Gamma_0\|_{L^2(dxdy)} + T^{\textit{some power}}\|\Gamma\|_X^2.$$

Thus, there exists $T_0 > 0$ *such that for all* $0 < T \leq T_0$

$$\|\Gamma\|_X \lesssim \|\langle\nabla_{x,y}\rangle^{\sigma}\Gamma_0\|_{L^2(dxdy)}.$$

Similarly, we can show that

$$\| \partial_t \Gamma \|_X \lesssim \| \langle \nabla_{x,y} \rangle^{\sigma} \partial_t \Gamma_0 \|_{L^2(dxdy)} + T^{some\ power} \| \Gamma \|_X \| \partial_t \Gamma \|_X$$

which again means there exists $T_0 > 0$ such that

$$\| \partial_t \Gamma \|_X \lesssim \| \langle \nabla_{x,y} \rangle^{\sigma} \partial_t \Gamma_0 \|_{L^2(dxdy)}.$$

3.5 Homogeneous Λ Equation

In this section we prove collapsing estimates for the linear Schrödinger equations

$$\frac{1}{i}\frac{\partial}{\partial t}\Lambda - \Delta_x \Lambda - \Delta_y \Lambda = 0 \tag{3.25}$$

which we will need later. As mentioned in the introduction, one of the main difficulties in the analysis of equation (3.3) is that the L^p-norms of the potential $N^{-1}v_N(x-y)$ are not uniformly bounded in N when $p > 1$ and β arbitrarily large since $N^{-1}\| v_N(x-y) \|_p \sim N^{-1+\beta(1-\frac{1}{p})}$. More precisely, from Proposition 3.26, we see that the natural space to put the nonlinearity of equation (3.35) is in $L^1[0,T]L^2(dxdy)$. In particular, when handling the term $N^{-1}v_N(x-y)\Lambda(t,x,y)$ from equation (3.3) in $L^1([0,T] \times L^2(\mathbb{R}^2))$, we see there is no way (at least no simple way) to put the term $N^{-1}v_N(x-y)$ in $L^1(d(x-y))$. Thus, the purposes of §6 and §7 are to develop sufficient amount of tools to handle $N^{-1}v_N(x-y)\Lambda(t,x,y)$ and all the nonlinearity coming from the TDHBF equations.

One of the crucial tools for our analysis is the $X^{s,b}$ spaces (sometimes called the *Bourgain spaces* or *dispersive Sobolev spaces*) which is defined to be the closure of the Schwartz class, $\mathcal{S}_{t,x}(\mathbb{R}\times\mathbb{R}\times\mathbb{R})$ with respect to the norm

$$\| u \|_{X^{s,b}_{\mathbf{S}}} = \| (1+|\xi|^2+|\eta|^2)^s(1+|\tau+|\xi|^2+|\eta|^2|)^b\tilde{u}(\tau,\xi,\eta) \|_{L^2(d\tau)L^2(d\xi d\eta)}.$$

For this paper, s is always zero and we are only interested in defining the $X^{s,b}$ spaces for the operator $\mathbf{S}$. Hence we dropped both the s and $\mathbf{S}$ labels from the norm to simplify the notation. For instance, we have $\| u \|_{X^b} = \| u \|_{X^{0,b}_{\mathbf{S}}}$. We refer the interested reader to §2.6 in [Tao06] for a more complete introduction to these spaces.

Same as the von-Neumann Schrödinger equation, we first obtain a collapsing estimate for the above equation.

Proposition 3.25. *Suppose* $\mathbf{S}\Lambda = 0$ *with Schwartz initial condition* $\Lambda(0,x,y) = \Lambda_0(x,y)$ *then*

$$\| p\,(\partial_t,\nabla_x)\,\Lambda(t,x,x) \|_{L^2(dtdx)} \lesssim \| \Lambda_0 \|_{L^2(dxdy)}. \tag{3.26}$$

where $p(\partial_t,\nabla_x)$ *is a pseudodifferential operator with symbol* $\tilde{p}(\tau,\xi) = |\tau+|\xi|^2|^{1/4}$.

Proof. Let us begin by taking the spacetime Fourier transform of the trace of Λ to get

$$\begin{aligned}\widetilde{\Lambda(t,x,x)} &= \int dtdx\ e^{-i(\tau t+\xi\cdot x)}\Lambda(t,x,x) = \int dtdxdy\ e^{-i(\tau t+\xi\cdot x)}\delta(x-y)\Lambda(t,x,y)\\ &= \int d\eta dtdxdy\ e^{-i(\tau t+(\xi-\eta)\cdot x+\eta\cdot y)}\Lambda(t,x,y) = \int d\eta dt\ e^{-i\tau t}\widehat{\Lambda}(t,\xi-\eta,\eta)\\ &= \frac{1}{(2\pi)^2}\int d\eta\ \delta(\tau+|\xi-\eta|^2+|\eta|^2)\widehat{\Lambda_0}(\xi-\eta,\eta).\end{aligned}$$

Applying Cauchy-Schwarz inequality yields the following estimate

$$\int d\tau d\xi\ |(\tau+|\xi|^2)^{1/4}\widetilde{\Lambda(t,x,x)}(\tau,\xi)|^2 \lesssim \sup_{\tau,\xi}|I(\tau,\xi)| \| \Lambda_0 \|^2_{L^2(dxdy)}$$

where

$$I(\tau,\xi) := \sqrt{\tau+|\xi|^2}\int d\eta\ \delta(\tau+|\xi-\eta|^2+|\xi+\eta|^2).$$

Observe, we have the identity

$$\begin{aligned}\int d\eta\ \delta(\tau+|\xi-\eta|^2+|\xi+\eta|^2) &= \int_{\mathbb{R}} \frac{\delta(\eta-\sqrt{-\tau-|\xi|^2})+\delta(\eta+\sqrt{-\tau-|\xi|^2})}{4\sqrt{-\tau-|\xi|^2}}\ d\eta\\ &= \frac{1}{2\sqrt{-\tau-|\xi|^2}}.\end{aligned}$$

Thus, it follows

$$\int d\tau d\xi\ ||\tau+|\xi|^2|^{1/4}\widetilde{\Lambda(t,x,x)}(\tau,\xi)|^2 \lesssim \| \Lambda_0 \|^2_{L^2(dxdy)}.$$

$\square$

Unfortunately, the homogeneous derivative $p(\partial_t, \nabla_x)$ of the restriction of Λ to the diagonal is not of any immediate use to our studies of the nonlinear coupled equations. Since the nonlinearity in time-dependent HFB involves trace of Λ, we need estimates that will allow us to control the restricted Λ by the spacetime derivative $p(\partial_t, \nabla_x)$ of the restriction of $\Lambda(t, x, y)$ to the diagonal. One such estimate is given by the following proposition.

Proposition 3.26. *Suppose* $\mathbf{S}\Lambda = 0$*, then we have*

$$\| \Lambda(t, x, x) \|_{L^4(dt)L^2(dx)} \lesssim \| p(\partial_t, \nabla_x)\Lambda(t, x, x) \|_{L^2(dtdx)} \tag{3.27}$$

Proof. We prove the above estimate using a TT^* argument. Consider $T : L^2_{t,x} \to L^4_t L^2_x$ defined by

$$(TF)(t, x) = \left(\frac{\widetilde{F}}{|\tau + |\xi|^2|^{1/4}} \right)^{\vee} := \frac{1}{2\pi} \int d\xi \; e^{i(\xi \cdot x + \tau t)} \frac{\widetilde{F}(\tau, \xi)}{|\tau + |\xi|^2|^{1/4}}$$

then we see that $TT^* : L^{4/3}_t L^2_x \to L^4_t L^2_x$ is given by

$$TT^* F = \left(\frac{\widetilde{F}}{|\tau + |\xi|^2|^{\frac{1}{2}}} \right)^{\vee} = F * \left(\frac{1}{|\tau + |\xi|^2|^{\frac{1}{2}}} \right)^{\vee} =: F * K.$$

By triangle inequality and Plancherel, we obtain the estimate

$$\| K * F(t, \cdot) \|_{L^2(dx)} \leq \int ds \; \| \hat{K}(t - s, \xi)\hat{F}(s, \xi) \|_{L^2(d\xi)} \lesssim \int ds \; \frac{1}{|t - s|^{\frac{1}{2}}} \| \hat{F}(s, \cdot) \|_{L^2(d\xi)}$$

since we have

$$|\hat{K}(t-s,\xi)| = \left| \int_{-\infty}^{\infty} e^{i\tau(t-s)} \frac{e^{i|\xi|^2(t-s)}}{|\tau|^{\frac{1}{2}}} \, d\tau \right| \lesssim \frac{1}{|t-s|^{\frac{1}{2}}}$$

which is independent of ξ. Thus, it follows

$$\| TT^* F \|_{L^4(dt)L^2(dx)} \lesssim \left\| \int_{-\infty}^{\infty} ds \, \frac{\| \hat{F}(s,\cdot) \|_{L^2(d\xi)}}{|t-s|^{\frac{1}{2}}} \right\|_{L^4(dt)}.$$

Now, apply Hardy-Littlewood-Sobolev inequality $\frac{n}{p} = \frac{n}{q} - n + \alpha$, with $n = 1, p = 4/3$ and $q = 4$ we have that

$$\left\| \int_{-\infty}^{\infty} ds \, \frac{\| \hat{F}(s,\cdot) \|_{L^2(d\xi)}}{|t-s|^{\frac{1}{2}}} \right\|_{L^4(dt)} \lesssim \| F \|_{L^{4/3}(dt)L^2(dx)}$$

which means TT^* is a bounded operator. Hence it follows from the TT^* principle that T is also a bounded operator, i.e.

$$\| TF \|_{L^4(dt)L^2(dx)} \lesssim \| F \|_{L^2(dtdx)}$$

or equivalently

$$\| F \|_{L^4(dt)L^2(dx)} \lesssim \| \, |\tau + |\xi|^2|^{1/4} \tilde{F}(\tau,\xi) \, \|_{L^2(d\tau d\xi)}.$$

$\square$

As an immediate corollary of Proposition 3.26, we have that

Corollary 3.27. *Suppose* Λ *solves* $\mathbf{S}\Lambda = 0$*, then for every* $0 < \varepsilon < 1$ *we have*

$$\| \nabla_x^{\varepsilon} \Lambda(t,x,x) \|_{L^4(dt)L^2(dx)} \lesssim \| \nabla_{x+y}^{\varepsilon} \Lambda_0 \|_{L^2(dxdy)} \tag{3.28}$$

where $\nabla_{x+y}\Lambda := \frac{1}{2}(\nabla_x \Lambda + \nabla_y \Lambda)$.

Proof. If $\mathbf{S}\Lambda = 0$, then $\mathbf{S}\nabla_{x+y}\Lambda = 0$. Applying the previous estimate, we obtain the estimate

$$\begin{aligned} \| (\nabla_{x+y}\Lambda)(t,x,x) \|_{L^4(dt)L^2(dx)} &\lesssim \| p(\partial_t, \nabla_x)(\nabla_{x+y}\Lambda)(t,x,x) \|_{L^2(dtdx)} \\ &\lesssim \| \nabla_{x+y}\Lambda_0 \|_{L^2(dxdy)}. \end{aligned}$$

Noting the identity

$$(\nabla_{x+y}\Lambda)(t,x,x) = \frac{1}{2}\nabla_x \left(\Lambda(t,x,x) \right), \tag{3.29}$$

we get the estimate

$$\| \nabla_x \Lambda(t,x,x) \|_{L^4(dt)L^2(dx)} \lesssim \| \nabla_{x+y}\Lambda_0 \|_{L^2(dxdy)}.$$

Interpolating the above estimate with the estimate

$$\| \Lambda(t,x,x) \|_{L^4(dt)L^2(dx)} \lesssim \| \Lambda_0 \|_{L^2(dxdy)}$$

yields the desired result. □

Let us also record the following non-endpoint Strichartz estimate for the homogeneous Λ equation:

Proposition 3.28 (Non-endpoint Strichartz)**.** *Suppose Λ is a solution to $\mathbf{S}\Lambda = 0$ with initial condition Λ_0 and (k, ℓ) is an admissible pair as defined in Proposition 3.17. Then it follows*

$$\| e^{it\Delta_x}\Lambda_0 e^{it\Delta_y} \|_{L^k(dt)L^\ell(dx)L^2(dy)} \lesssim \| \Lambda_0 \|_{L^2(dxdy)}. \tag{3.30}$$

Proposition 3.29. *For any number $1^+ > 1$ and arbitrarily close to 1 there exists $\delta > 0$ such that the following estimate holds*

$$\| F \|_{X^{-\frac{1}{2}+\delta}} \lesssim T^{some\ power} \| F \|_{L^2[0,T]L^{1^+}(dx)L^2(dy)}. \tag{3.31}$$

Proof. By Proposition 3.28 and Lemma 2.9 in [Tao06], we have the estimate

$$\| F \|_{L^4[0,T]L^\infty(dx)L^2(dy)} \lesssim \| F \|_{X^{\frac{1}{2}+\delta}} \tag{3.32}$$

for all $\delta > 0$. Moreover, from (3.32) we also get the dual estimate

$$\| F \|_{X^{-\frac{1}{2}-\delta}} \lesssim \| F \|_{L^{4/3}[0,T]L^1(dx)L^2(dy)} \lesssim T^{1/4} \| F \|_{L^2[0,T]L^1(dx)L^2(dy)}. \tag{3.33}$$

By linearly interpolating (3.33) with

$$\| F \|_{X^{-\frac{1}{2}+\frac{1}{2}}} = \| F \|_{L^2[0,T]L^2(dx)L^2(dy)}$$

yields

$$\| F \|_{X^{-\frac{1}{2}+\lambda}} \lesssim T^{\text{some power}} \| F \|_{L^2[0,T] L^{1^+}(dx) L^2(dy)}$$

for $-\delta < \lambda < \frac{1}{2}$ and some number 1^+ depending on λ. In particular, for any number 1^+ arbitrarily close to 1 we can choose δ sufficiently small such that (3.31) holds. □

Remark 3.30. Let us make the observation: since

$$|\xi + \eta|^2 + |\xi - \eta|^2 = 2|\xi|^2 + 2|\eta|^2 \tag{3.34}$$

then we also have the estimate

$$\| F \|_{X^{-\frac{1}{2}+\delta}} \lesssim T^{\text{some power}} \| F \|_{L^2[0,T] L^{1^+}(d(x-y)) L^2(d(x+y))}.$$

3.6 Inhomogeneous Λ Equation

The main result in this section is Corollary 3.33 which allows us to obtain a collapsing-type estimate for equation (3.3) and essentially show that $N^{-1} v_N \Lambda$, mentioned in the previous section, can be viewed as a uniformly in N perturbation of equation (3.35).

Consider the inhomogeneous equation

$$\mathbf{S}\Lambda = F \tag{3.35}$$

then it follows from the $X^{s,b}$ energy estimate[3] and Proposition 3.29 that we have

$$\begin{aligned}\| \Lambda(t,x,x) \|_{L^4(dt)L^2(dx)} &\lesssim \| \Lambda_0 \|_{L^2(dxdy)} + \| F \|_{X^{-\frac{1}{2}+\delta}} \\ &\lesssim \| \Lambda_0 \|_{L^2(dxdy)} + T^{\text{some power}} \| F \|_{L^2(dt)L^{1+}(d(x-y))L^2(d(x+y))}.\end{aligned}$$

Summarizing the above result we obtain the following proposition:

Proposition 3.31. *Suppose* Λ *solves* $\mathbf{S}\Lambda = F$*, then we have*

$$\| \Lambda(t,x,x) \|_{L^4(dt)L^2(dx)} \lesssim \| \Lambda_0 \|_{L^2(dxdy)} + T^{\textit{some power}} \| F \|_{L^2(dt)L^{1+}(d(x-y))L^2(d(x+y))}. \tag{3.36}$$

Using the above proposition, we establish the following proposition:

Proposition 3.32. *Suppose* Λ *solves* (3.3) *with initial condition* Λ_0*. Then we have*

$$\| \Lambda(t,x,x) \|_{L^4(dt)L^2(dx)} \lesssim \| \Lambda_0 \|_{L^2(dxdy)} + \| F \|_{X^{-\frac{1}{2}+\delta}}. \tag{3.37}$$

Proof. Since by Proposition 3.26 we have

$$\left\| e^{it\Delta_x} \Lambda_0 e^{it\Delta_y} \big|_{x=y} \right\|_{L^2(dtdx)} \lesssim \| \Lambda_0 \|_{L^2(dxdy)},$$

then it follows from Lemma 2.9 in [Tao06]

$$\| F \|_{L^4(dt)L^2(dx)} \lesssim \| F \|_{X^{\frac{1}{2}+\delta}}$$

---

[3] cf. [Tao06] section 2.6

for any $\delta > 0$. In particular, applying the $X^{s,b}$ energy estimate we get that

$$\| \chi(t)\Lambda \|_{X^{\frac{1}{2}+\delta}} \lesssim \| \frac{1}{N} v_N \chi(t)\Lambda \|_{X^{-\frac{1}{2}+\delta}} + \| F \|_{X^{-\frac{1}{2}+\delta}} + \| \Lambda_0 \|_{L^2(dxdy)}$$

where $\chi(t)$ is a time localization bump function. Applying Proposition 3.29, we see that

$$\begin{aligned}
\frac{1}{N} \| v_N(x-y)\Lambda \|_{X^{-\frac{1}{2}+\delta}} &\lesssim \frac{1}{N} T^{\text{some power}} \| v_N \chi(t)\Lambda \|_{L^2(dt)L^{1^+}(d(x-y))L^2(d(x+y))} \\
&\lesssim \frac{1}{N} T^{\text{some power}} \| v_N \|_{L^{1^+}(d(x-y))} \| \chi(t)\Lambda \|_{L^2(dt)L^\infty(d(x-y))L^2(d(x+y))} \\
&\lesssim \frac{1}{N^{1-\beta+\beta/(1^+)}} T^{\text{some power}} \| \chi(t)\Lambda \|_{L^4(dt)L^\infty(d(x-y))L^2(d(x+y))} \\
&\lesssim \frac{1}{N^{1-\beta+\beta/(1^+)}} T^{\text{some power}} \| \chi(t)\Lambda \|_{X^{\frac{1}{2}+\delta}}.
\end{aligned}$$

Hence for 1^+ sufficiently close to 1 we are in the perturbative regime. This allows us to absorb the contribution from the potential term $\frac{1}{N} v_N(x-y)\Lambda$ when N is sufficiently large. □

Using the above proposition we could show that

Corollary 3.33. *Suppose Λ solves (3.3) with initial condition Λ_0. Then for every $0 < \sigma < \frac{1}{2}$ we have*

$$\| |\nabla_x|^\sigma \Lambda(t,x,x) \|_{L^4(dt)L^2(dx)} \lesssim \| |\nabla_{x+y}|^\sigma \Lambda_0 \|_{L^2(dxdy)} + \| |\nabla_{x+y}|^\sigma F \|_{X^{-\frac{1}{2}+\delta}}. \quad (3.38)$$

Proof. Taking the spatial derivative ∇_{x+y} of (3.3) yields

$$\left(\mathbf{S} + \frac{1}{N} v_N(x-y)\right) \nabla_{x+y}\Lambda = \nabla_{x+y}F \tag{3.39}$$

since $[\nabla_{x+y}, N^{-1}v_N(x-y)] = 0$. Hence by Proposition 3.32, we obtain the estimate

$$\|\, (\nabla_{x+y}\Lambda)(t,x,x) \,\|_{L^4(dt)L^2(dx)} \lesssim \|\, \nabla_{x+y}\Lambda_0 \,\|_{L^2(dxdy)} + \|\, \nabla_{x+y}F \,\|_{X^{-\frac{1}{2}+\delta}}.$$

Again, noting the identity (3.29), we obtain the estimate

$$\|\, \nabla_x\Lambda(t,x,x) \,\|_{L^4(dt)L^2(dx)} \lesssim \|\, \nabla_{x+y}\Lambda_0 \,\|_{L^2(dxdy)} + \|\, \nabla_{x+y}F \,\|_{X^{-\frac{1}{2}+\delta}}. \tag{3.40}$$

Interpolating (3.37) with (3.40) yields the desired result. □

Now, let us record some Strichartz estimates:

Proposition 3.34. *Suppose Λ is a solution to $\mathbf{S}\Lambda = F$ with initial condition Λ_0 and $(k,\ell), (\widetilde{k}, \widetilde{\ell})$ are Strichartz admissible pairs. Then it follows*

$$\|\, \Lambda(t,x,y) \,\|_{L^k(dt)L^\ell(dx)L^2(dy)} \lesssim \|\, \Lambda_0 \,\|_{L^2(dxdy)} + \|\, F \,\|_{L^{\widetilde{k}'}(dt)L^{\widetilde{\ell}'}(dx)L^2(dy)}. \tag{3.41}$$

In particular, it follows

$$\|\, |\nabla_{x,y}|^\sigma\Lambda(t,x,y) \,\|_{L^k(dt)L^\ell(dx)L^2(dy)} \lesssim \|\, |\nabla_{x,y}|^\sigma\Lambda_0 \,\|_{L^2(dxdy)} + \|\, |\nabla_{x,y}|^\sigma F \,\|_{L^{\widetilde{k}'}(dt)L^{\widetilde{\ell}'}(dx)L^2(dy)}. \tag{3.42}$$

Remark 3.35. Let us note that Proposition 3.34 also holds for solution to (3.3) when N is sufficiently large. More specifically, by interpolation, we can show

$$\frac{1}{N}\| \, |\nabla_x|^{\sigma}[v_N(x-y)]\Lambda \, \|_{L^{4/3}[0,T]L^1(d(x-y))L^2(d(x+y))} \lesssim \frac{T^{\frac{1}{2}}}{N^{1-\sigma\beta}}\| \, \Lambda \, \|_{L^4[0,T]L^\infty(d(x-y))L^2(d(x+y))}. \tag{3.43}$$

Thus, for any $\beta > 0$, we can choose $\sigma = \sigma(\beta)$ so that $1 - \sigma\beta > 0$.

3.7 The time-dependent HFB System in 1D

In this section we prove the local well-posedness of our system of nonlinear equations addressed in the introduction. First, let us recall the time-dependent HFB equations in 1D

$$\begin{aligned}\left\{\frac{1}{i}\frac{\partial}{\partial t} - \Delta_{x_1}\right\}\varphi_t(x_1) = &- \int dy \, \{v_N(x_1-y)\rho_\Gamma(t,y)\} \cdot \varphi_t(x_1) \qquad (3.44\text{a})\\ &- \int dy \, \{v_N(x_1-y)(\Gamma_t(y,x_1) - \overline{\varphi}_t(y)\varphi_t(x_1))\varphi_t(y)\}\\ &- \int dy \, \{v_N(x_1-y)(\Lambda_t(x_1,y) - \varphi_t(y)\varphi_t(x_1))\overline{\varphi}_t(y)\}\end{aligned}$$

$$\begin{aligned}&\left\{\frac{1}{i}\frac{\partial}{\partial t} - \Delta_{x_1} + \Delta_{x_2}\right\}\Gamma_t(x_1,x_2) \qquad (3.44\text{b})\\ &= - \int dy \, \{(v_N(x_1-y) - v_N(x_2-y))\overline{\Lambda}_t(x_1,y)\Lambda_t(y,x_2)\}\\ &\quad - \int dy \, \{(v_N(x_1-y) - v_N(x_2-y))\Gamma_t(x_1,y)\Gamma_t(y,x_2)\}\\ &\quad - \int dy \, \{(v_N(x_1-y) - v_N(x_2-y))\rho_\Gamma(t,y)\Gamma_t(x_1,x_2)\}\\ &\quad + 2\int dy \, \{(v_N(x_1-y) - v_N(x_2-y))|\varphi_t(y)|^2\overline{\varphi}_t(x_1)\varphi_t(x_2)\}\end{aligned}$$

$$\left\{\frac{1}{i}\frac{\partial}{\partial t} - \Delta_{x_1} - \Delta_{x_2} + \frac{1}{N} v_N(x_1 - x_2)\right\}\Lambda_t(x_1, x_2) \tag{3.44c}$$

$$\begin{aligned}
= & -\int dy\ \{(v_N(x_1 - y) + v_N(x_2 - y))\rho_\Gamma(t, y)\Lambda_t(x_1, x_2)\} \\
& -\int dy\ \{(v_N(x_1 - y) + v_N(x_2 y))\Lambda_t(x_1, y)\Gamma_t(y, x_2)\} \\
& -\int dy\ \{(v_N(x_1 - y) + v_N(x_2 - y))\overline{\Gamma}_t(x_1, y)\Lambda_t(y, x_2)\} \\
& + 2\int dy\ \{(v_N(x_1 - y) + v_N(x_2 - y))|\varphi_t(y)|^2\varphi_t(x_1)\varphi_t(x_2)\}
\end{aligned}$$

The space X_T is a Strichartz-type space equipped with a norm which is the sum of the following norms

$$\mathbf{N}_T(\varphi) := \|\, \langle \nabla_x \rangle^\sigma \varphi(t, x) \,\|_{L^4[0,T]L^\infty(dx)} + \|\, \langle \nabla_x \rangle^\sigma \varphi(t, x) \,\|_{L^\infty[0,T]L^2(dx)} \tag{3.45a}$$

$$\begin{aligned}
\mathbf{N}_T(\Gamma) := & \|\, \langle \nabla_{x,y} \rangle^\sigma \Gamma(t, x, y) \,\|_{L^4[0,T]L^\infty(dx)L^2(dy)} \\
& + \|\, \langle \nabla_{x,y} \rangle^\sigma \Gamma(t, x, y) \,\|_{L^\infty[0,T]L^2(dxdy)} \\
& + \sup_z \|\, |\nabla_x|^{\frac{1}{2}} \Gamma(t, x + z, x) \,\|_{L^2(dtdx)} + \sup_z \|\, |\nabla_x|^{\frac{1}{2} - \varepsilon} \Gamma(t, x + z, x) \,\|_{L^2(dtdx)}
\end{aligned} \tag{3.45b}$$

$$\begin{aligned}
\mathbf{N}_T(\Lambda) := & \|\, \langle \nabla_{x,y} \rangle^\sigma \Lambda(t, x, y) \,\|_{L^4[0,T]L^\infty(dx)L^2(dy)} \\
& + \|\, \langle \nabla_{x,y} \rangle^\sigma \Lambda(t, x, y) \,\|_{L^\infty[0,T]L^2(dxdy)} + \sup_z \|\, \langle \nabla_x \rangle^\sigma \Lambda(t, x + z, x) \,\|_{L^4[0,T]L^2(dx)}.
\end{aligned} \tag{3.45c}$$

Moreover, let us denote the space of functions $(\varphi_t, \Gamma_t, \Lambda_t)$ where the above norms are finite for any $0 \leq T < \infty$ by $X_{\infty,\text{loc}}$.

Let us present the main a-priori estimates of the chapter

Theorem 3.36. *Suppose* φ, Γ *and* Λ *solve* (3.44a), (3.44b), *and* (3.44c) *respectively* *with Schwartz initial condition* $(\varphi_0, \Gamma_0, \Lambda_0)$. *Then we have the following estimates*

$$\mathbf{N}_T(\varphi) \lesssim \| \langle \nabla_x \rangle^\sigma \varphi_0 \|_{L^2(dx)} + T^{some\ power} \left(\mathbf{N}_T(\varphi)^2 + \mathbf{N}_T(\Gamma) + \mathbf{N}_T(\Lambda) \right) \mathbf{N}_T(\varphi) \tag{3.46a}$$

$$\mathbf{N}_T(\Gamma) \lesssim \| \langle \nabla_{x,y} \rangle^\sigma \Gamma_0 \|_{L^2(dxdy)} + T^{some\ power} \left(\mathbf{N}_T(\Gamma)^2 + \mathbf{N}_T(\Lambda)^2 + \mathbf{N}_T(\varphi)^4 \right) \tag{3.46b}$$

$$\mathbf{N}_T(\Lambda) \lesssim \| \langle \nabla_{x,y} \rangle^\sigma \Lambda_0 \|_{L^2(dxdy)} + T^{some\ power} \left(\mathbf{N}_T(\Gamma)\mathbf{N}_T(\Lambda) + \mathbf{N}_T(\varphi)^4 \right). \tag{3.46c}$$

In particular, there exists T_0 *such that for all* $T \leq T_0$ *we have that*

$$\mathbf{N}_T(X) := \mathbf{N}_T(\varphi)^2 + \mathbf{N}_T(\Gamma) + \mathbf{N}_T(\Lambda) \lesssim 1.$$

Similarly, the following estimates hold for the time derivative of φ, Γ, Λ, *i.e.*

$$\begin{aligned} \mathbf{N}_T(\partial_t \varphi) \lesssim & \left\| \langle \nabla_x \rangle^\sigma \partial_t \varphi \Big|_{t=0} \right\|_{L^2(dx)} && (3.47a) \\ & + T^{some\ power} \left(\mathbf{N}_T(X)\mathbf{N}_T(\partial_t \varphi) + \mathbf{N}_T(\partial_t X)\mathbf{N}_T(\varphi) \right) \\ \mathbf{N}_T(\partial_t \Gamma) \lesssim & \left\| \langle \nabla_{x,y} \rangle^\sigma \partial_t \Gamma \Big|_{t=0} \right\|_{L^2(dxdy)} && (3.47b) \\ & + T^{some\ power} \mathbf{N}_T(X) \left(\mathbf{N}_T(\partial_t X) + \mathbf{N}_T(\varphi)\mathbf{N}_T(\partial_t \varphi) \right) \\ \mathbf{N}_T(\partial_t \Lambda) \lesssim & \left\| \langle \nabla_{x,y} \rangle^\sigma \partial_t \Lambda \Big|_{t=0} \right\|_{L^2(dxdy)} && (3.47c) \\ & + T^{some\ power} \mathbf{N}_T(X) \left(\mathbf{N}_T(\partial_t X) + \mathbf{N}_T(\varphi)\mathbf{N}_T(\partial_t \varphi) \right) \end{aligned}$$

which again means there exists T_0 such that for all $T < T_0$ we have

$$\mathbf{N}_T\left(\partial_t X\right) := \mathbf{N}_T\left(\partial_t \varphi\right)^2 + \mathbf{N}_T\left(\partial_t \Gamma\right) + \mathbf{N}_T\left(\partial_t \Lambda\right) \lesssim 1.$$

Indeed, for any $i, j \in \mathbb{N}$ we have the estimates

$$\mathbf{N}_T\left(\partial_t^i \nabla_{x+y}^j X\right) \lesssim 1. \tag{3.48}$$

Remark 3.37. The reader should note that the solution obtained from the Banach fixed-point theorem is smooth if the initial data $(\varphi_0, \Gamma_0, \Lambda_0)$ is sufficiently smooth. Indeed, for each fixed N, one can show

$$\mathbf{N}_T\left(\partial_t^i \nabla_x^j \nabla_y^k X\right) \lesssim N^{\text{some non-negative power}}.$$

Despite the fact that the higher Sobolev norms are not uniformly bounded in N, each of the solutions has sufficient smoothness for us to apply the conservation laws which we will state later in the section.

We split the presentation of the proof of the theorem into two subsections.

3.7.1 Proofs of Estimates (3.46b) and (3.46c)

Let us first consider equation (3.44b). Since the term $(v_N * \rho_\Gamma) \cdot \Gamma$ has already been handled in §5, it suffices to consider only the terms $(v_N \bar{\Lambda}) \circ \Lambda$ and $(v_N \Gamma) \circ \Gamma$. In particular, it suffices to consider just the derivative of the terms since any computation for the derivatives will encompass the computation for the non-derivative terms.

Let us first handle the term $(v_N * \rho_\Gamma) \cdot \Gamma$. By a direct change of variables, we can rewrite the kernel composition as follows

$$(v_N\Gamma \circ \Gamma)(x,y) = \int dw\ v_N(x-w)\Gamma(x,w)\Gamma(w,y) = \int dz\ v_N(z)\Gamma(x,x-z)\Gamma(x-z,y).$$

Then by Kato-Ponce inequality we obtain the following

$$\begin{aligned}
&\|\, |\nabla_{x,y}|^\sigma [(v_N\Gamma)\circ\Gamma]\, \|_{L^1[0,T]L^2(dxdy)} \\
&\qquad \le \int dz\ |v_N(z)| \|\, |\nabla_{x,y}|^\sigma [\Gamma(x,x-z)\Gamma(x-z,y)]\, \|_{L^1[0,T]L^2(dxdy)} \\
&\qquad \lesssim \int dz\ |v_N(z)| \|\, |\nabla_x|^\sigma \Gamma(x,x-z)\, \|_{L^2[0,T]L^{\widetilde{p}}(dx)} \|\, \Gamma(x,y)\, \|_{L^2[0,T]L^{\widetilde{r}}(dx)L^2(dy)} \\
&\qquad\quad + \int dz\ |v_N(z)| \|\, \Gamma(x,x-z)\, \|_{L^2[0,T]L^p(dx)} \|\, |\nabla_x|^\sigma \Gamma(x,y)\, \|_{L^2[0,T]L^r(dx)L^2(dy)}
\end{aligned}$$

where $p, r, \widetilde{p}, \widetilde{r}$ are the values stated in Remark 3.23. Applying Cauchy-Schwarz inequality in the time variable gives us

$$\begin{aligned}
&\|\, |\nabla_{x,y}|^\sigma [(v_N\Gamma)\circ\Gamma]\, \|_{L^1[0,T]L^2(dxdy)} \\
&\qquad \lesssim T^{\text{some power}} \int dz\ |v_N(z)| \|\, |\nabla_x|^{\frac{1}{2}-\varepsilon} \Gamma(x,x-z)\, \|_{L^2(dtdx)} \|\, \Gamma(x,y)\, \|_{L^{\widetilde{q}}(dt)L^{\widetilde{r}}(dx)L^2(dy)} \\
&\qquad\quad + T^{\text{some power}} \int dz\ |v_N(z)| \|\, |\nabla_x|^{\frac{1}{2}-\varepsilon} \Gamma(x,x-z)\, \|_{L^2(dtdx)} \|\, |\nabla_{x,y}|^\sigma \Gamma(x,y)\, \|_{L^q(dt)L^r(dx)L^2(dy)} \\
&\qquad \lesssim T^{\text{some power}} \mathbf{N}_T(\Gamma)^2
\end{aligned}$$

where (q,r) and $(\widetilde{q},\widetilde{r})$ are admissible pairs. Likewise, we have

$$
\begin{aligned}
&\|\, |\nabla_{x,y}|^{\sigma}[(v_N\bar{\Lambda})\circ\Lambda]\,\|_{L^1[0,T]L^2(dxdy)}\\
&\qquad\le \int dz\ |v_N(z)|\|\,|\nabla_{x,y}|^{\sigma}[\bar{\Lambda}(x,x-z)\Lambda(x-z,y)]\,\|_{L^1[0,T]L^2(dxdy)}\\
&\qquad\lesssim \int dz\ |v_N(z)|\|\,|\nabla_x|^{\sigma}\Lambda(x,x-z)\,\|_{L^4[0,T]L^2(dx)}\|\,\Lambda(x,y)\,\|_{L^{4/3}[0,T]L^{\infty}(dx)L^2(dy)}\\
&\qquad\quad+\int dz\ |v_N(z)|\|\,\Lambda(x,x-z)\,\|_{L^4[0,T]L^2(dx)}\|\,|\nabla_x|^{\sigma}\Lambda(x,y)\,\|_{L^{4/3}[0,T]L^{\infty}(dx)L^2(dy)}\\
&\qquad\lesssim T^{\frac{1}{2}}\mathbf{N}_T(\Lambda)^2.
\end{aligned}
$$

As for equation (3.44c), there are essentially three terms we need to estimate, namely $(v_N*\rho_\Gamma)\Lambda$, $(v_N\Gamma)\circ\Lambda$ and $(v_N\Lambda)\circ\Gamma$. Similar to the handling of the nonlinear terms for the Γ equation, it suffices to look at just the derivatives of the nonlinear terms.

For the first term, observe we have

$$
\begin{aligned}
&\|\, |\nabla_{x+y}|^{\sigma}[(v_N*\rho_\Gamma)\Lambda]\,\|_{L^2[0,T]L^{1+}(d(x-y))L^2(d(x+y))}\\
&\qquad\lesssim \|\, v_N*\rho_\Gamma\,\|_{L^2[0,T]L^{2+}(dx)}\|\,|\nabla_{x+y}|^{\sigma}\Lambda\,\|_{L^{\infty}[0,T]L^2(d(x-y))L^2(d(x+y))}\\
&\qquad\quad+\ \|\, v_N*|\nabla_x|^{\sigma}\rho_\Gamma\,\|_{L^2[0,T]L^{2+}(dx)}\|\,\Lambda\,\|_{L^{\infty}[0,T]L^2(d(x-y))L^2(d(x+y))}\\
&\qquad\lesssim \|\,|\nabla_x|^{\frac{1}{2}-\varepsilon}\rho_\Gamma\,\|_{L^2(dtdx)}\|\,|\nabla_{x+y}|^{\sigma}\Lambda\,\|_{L^{\infty}[0,T]L^2(d(x-y))L^2(d(x+y))}\\
&\qquad\quad+\|\,|\nabla_x|^{\frac{1}{2}-\varepsilon}\rho_\Gamma\,\|_{L^2(dtdx)}\|\,\Lambda\,\|_{L^{\infty}[0,T]L^2(d(x-y))L^2(d(x+y))}\\
&\qquad\lesssim \mathbf{N}_T(\Gamma)\mathbf{N}_T(\Lambda)
\end{aligned}
$$

The terms $F=(v_N\Lambda)\circ\Gamma$ and $\Lambda\circ(v_N\Gamma)$ are handled similarly.

3.7.2 Proof of Estimate (3.46a)

Let us begin by stating the following Strichartz estimate

Proposition 3.38. *Suppose φ is a solution to $\mathbf{S}\varphi = F$ with initial condition φ_0 and let (k, ℓ) be an admissible pair. Then it follows for all $\alpha > 0$ we have*

$$\| \, |\nabla_x|^\alpha \varphi \, \|_{L^k[0,T]L^\ell(dx)} \lesssim \| \, |\nabla_x|^\alpha \varphi_0 \, \|_{L^2(dx)} + \| \, |\nabla_x|^\alpha F \, \|_{L^{4/3}[0,T]L^1(dx)}. \tag{3.49}$$

It suffice to consider only $(v_N * \rho_\Gamma) \cdot \varphi$ and $(v_N \Lambda) \circ \varphi$ since the method applies word-for-word to the remaining nonlinear terms.

For the first nonlinearity, we apply Kato-Ponce inequality to get the estimate

$$\begin{aligned} \| \, |\nabla_x|^\sigma [(v_N * \rho_\Gamma) \cdot \varphi] \, \|_{L^{4/3}[0,T]L^1(dx)} &\lesssim \| \, v_N * |\nabla_x|^\sigma \rho_\Gamma \, \|_{L^{4/3}[0,T]L^2(dx)} \| \, \varphi \, \|_{L^\infty[0,T]L^2(dx)} \\ &\quad + \| \, v_N * \rho_\Gamma \, \|_{L^{4/3}[0,T]L^2(dx)} \| \, |\nabla_x|^\sigma \varphi \, \|_{L^\infty[0,T]L^2(dx)} \\ &\lesssim T^{\text{some power}} \mathbf{N}_T(\Gamma) \mathbf{N}_T(\varphi). \end{aligned}$$

For the second nonlinear term, we have

$$\begin{aligned} &\| \, |\nabla_x|^\sigma [(v_N \Lambda) \circ \varphi] \, \|_{L^{4/3}[0,T]L^1(dx)} \\ &\quad \lesssim \int dz \; |v_N(z)| \| \, |\nabla_x|^\sigma \Lambda(x, x-z) \, \|_{L^{4/3}[0,T]L^2(dx)} \| \, \varphi(x-z) \, \|_{L^\infty[0,T]L^2(dx)} \\ &\qquad + \int dz \; |v_N(z)| \| \, \Lambda(x, x-z) \, \|_{L^{4/3}[0,T]L^2(dx)} \| \, |\nabla_x|^\sigma \varphi(x-z) \, \|_{L^\infty[0,T]L^2(dx)} \\ &\quad \lesssim T^{\text{some power}} \mathbf{N}_T(\Lambda) \mathbf{N}_T(\varphi). \end{aligned}$$

3.7.3 Global Well-Posedness of the Time-Dependent HFB Equations

In this subsection, we prove the global well-posedness of the time-dependent HFB equations. Let us begin by recalling the number and energy conservation laws derived in §9 of [GM13a][4]. Recall the total particle number is given by

$$\mathcal{N} := N \int dx \; \rho_\Gamma(t,x) \tag{3.50}$$

and the energy is defined by

$$\begin{aligned}\mathcal{E} :=\ & N\Big\{ \int dx \; |\nabla_x \varphi(x)|^2 + \frac{1}{2N}\int dxdy \; |\nabla_{x,y} \operatorname{sh}(k)(x,y)|^2 \\ &+ \frac{1}{2N}\int dxdydz \; v_N(x-y)|\varphi(x)\operatorname{sh}(k)(y,z) + \varphi(y)\operatorname{sh}(k)(x,z)|^2 \\ &+ \frac{1}{4}\int dxdy \; v_N(x-y)\left\{2|\Lambda(x,y)|^2 + |\Gamma(x,y)|^2 + \Gamma(x,x)\Gamma(y,y)\right\}\Big\}\end{aligned} \tag{3.51}$$

Note that we have suppress the dependence on t in $\mathcal{E}$ for the sake of compactness of notation.

Theorem 3.39 (Conservation Laws)**.** *Suppose* $(\varphi_t, \Gamma_t, \Lambda_t)$ *solves the time-dependent HFB equations and* $v \in L^1(\mathbb{R}) \cap C^\infty(\mathbb{R})$*. Then the total particle number and energy is conserved.*

Proof. See §8 in [GM13a]. □

As an immediate corollary of Theorem 3.39, we have

---

[4] cf. Corollary 2.7. and Theorem 2.8 in [BBC+18]

Corollary 3.40. *Let $(\varphi_t, \Gamma_t, \Lambda_t)$ be a solution to the time-dependent HFB equations. Then there exists a constant $C > 0$ such that for any $T > 0$ and $0 < s < 1$ we have that*

$$\sup_{t\in[0,T]} \| (\varphi_t, \Gamma_t, \Lambda_t) \|_{\mathcal{X}^s} \le C, \tag{3.52}$$

independent of N.

Proof. The estimate for φ_t follows immedately by interpolating between the conservation of total particle number and conservation of energy. Next, applying Cauchy-Schwarz and the conservation of total particle number, we obtain the estimate

$$\| \Gamma(t,\cdot) \|_{L^2(dxdy)} \le \| \varphi_t \|^2_{L^2(dxdy)} + \frac{1}{N} \| \operatorname{sh}(k_t) \|^2_{L^2(dx)} \lesssim 1. \tag{3.53}$$

Similarly, using Cauchy-Schwarz and the conservation of energy, we obtain

$$\begin{aligned} &\| \nabla_x \Gamma(t,\cdot) \|_{L^2(dxdy)} \\ &\le \| \varphi_t \|_{L^2(dx)} \| \nabla_x \varphi_t \|_{L^2(dx)} + \frac{1}{N} \| \operatorname{sh}(k_t) \|_{L^2(dxdy)} \| \nabla_x \operatorname{sh}(k_t) \|_{L^2(dxdy)} \lesssim 1. \end{aligned} \tag{3.54}$$

Interpolating (3.53) and (3.54) yields a desired bound for Γ_t.

To uniformly bound Λ_t, we use the trig identity (3.4a) to get the estimate

$$\begin{aligned} \| \Lambda(t,\cdot) \|_{L^2(dxdy)} \le \| \varphi_t \|^2_{L^2(dx)} &+ \frac{1}{N} \| \operatorname{sh}(k_t) \|_{L^2(dxdy)} \\ &+ \frac{1}{N} \| \operatorname{sh}(k_t) \|_{L^2(dxdy)} \| p(k_t) \|_{L^2(dxdy)}. \end{aligned} \tag{3.55}$$

By identity (3.4b), we see that $p \circ p + 2p = \overline{\mathrm{sh}} \circ \mathrm{sh}$ which means

$$\| \, p(k) \, \|_{L^2(dx)}^2 \leq \| \, p \circ p + 2p \, \|_{\mathrm{Tr}} = \| \, \mathrm{sh}(k) \, \|_{L^2(dxdy)}^2$$

since $p(k)(x,x) \geq 0$. Hence by the conservation of total particle number we have that

$$\| \, \Lambda(t,\cdot) \, \|_{L^2(dxdy)} \lesssim 1.$$

Similarly, we can show that $\| \, \nabla_x \Lambda(t,\cdot) \, \|_{L^2(dxdy)} \lesssim 1.$ □

Chapter 4: Global Well-posed of the Time-Dependent HFB system in $\mathbb{R}^{1+3}$ and Fock Space Estimate

4.1 Main Result

The main goal of this chapter is to extend Theorem 1.5 to obtain a global in time result. Let us state the main result of this chapter

Theorem 4.1. *Let* $\frac{1}{3} \leq \beta < \frac{2}{3}$ *and* $v \in \mathcal{S}$ *a nonnegative interaction potential satisfying the condition that* $|\hat{v}| \leq \hat{w}$ *for some* $w \in \mathcal{S}$. *Suppose* $(\phi_t, \Gamma_t, \Lambda_t)$ *are solutions to the time-dependent HFB equations with some smooth initial conditions* $(\phi_0, \Gamma_0, \Lambda_0)$ *satisfying the following regularity condition uniformly in* N*: for some* $\varepsilon > 0$ *and* $0 \leq i \leq 1, 0 \leq j \leq 2$

$$
\begin{aligned}
&\left\| \langle \nabla_x \rangle^{1/2+\varepsilon} \partial_t^i \nabla_x^j \phi(t, \cdot)\big|_{t=0} \right\|_{L^2(dx)} \lesssim 1\\
&\left\| \langle \nabla_x \rangle^{1/2+\varepsilon} \langle \nabla_y \rangle^{1/2+\varepsilon} \partial_t^i \nabla_{x+y}^j \Gamma(t, \cdot)\big|_{t=0} \right\|_{L^2(dxdy)} \lesssim 1\\
&\left\| \langle \nabla_x \rangle^{1/2+\varepsilon} \langle \nabla_y \rangle^{1/2+\varepsilon} \partial_t^i \nabla_{x+y}^j \Lambda(t, \cdot)\big|_{t=0} \right\|_{L^2(dxdy)} \lesssim 1\\
&\left\| \nabla_{x+y}^j \operatorname{sh}(2k)(0, x, y) \right\|_{L^2(dxdy)} \lesssim 1.
\end{aligned}
$$

Then there exists constants $\delta = \delta(\varepsilon), \kappa = \kappa(\varepsilon), C = C(\varepsilon, \beta)$ *and a phase function*

$\chi(t)$, depending on N, such that we have the Fock space estimate

$$\left\| e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}(\phi_0)} e^{-\mathcal{B}(k_0)}\Omega - e^{i\chi(t)} e^{-\sqrt{N}(\phi_t)} e^{-\mathcal{B}(k_t)}\Omega \right\|_{\mathcal{F}} \leq \frac{C\exp\left(\kappa T^{5+\varepsilon}\right)}{N^{1/6}}$$

for all $0 \leq t \leq T$.

As remarked in §2 of [GM17], we need to first prove the following a-priori estimates

$$\| \langle\nabla_x\rangle^{\frac{1}{2}+\varepsilon} \langle\nabla_y\rangle^{\frac{1}{2}+\varepsilon} \Lambda(t) \|_{L^2(dxdy)} \leq C(t)$$
$$\| \langle\nabla_x\rangle^{\frac{1}{2}+\varepsilon} \langle\nabla_y\rangle^{\frac{1}{2}+\varepsilon} \Gamma(t) \|_{L^2(dxdy)} \leq C$$
$$\| \langle\nabla_x\rangle^{\frac{1}{2}+\varepsilon} \phi(t) \|_{L^2(dx)} \leq C$$

and use them to obtain appropriate norm bounds on the solutions of the time-dependent HFB equations; see Proposition 4.14, 4.16, 4.19, 4.20. Afterward, by replicating the proof of Theorem 1.5 in §9 and 10 of [GM17], one can obtain the desired Fock space estimate.

Remark 4.2. Recently, Grillakis and Machedon extended the local well-posedness result of the time-dependent HFB system to the range $0 < \beta < 1$ for more general initial data in [GM18]. By a similar argument as in the proof of Theorem 4.1, we could also extend result of Theorem 4.1 to the range $0 < \beta < 1$.

4.2 Global Estimates for the Time-Dependent HFB Equations

In this section we prove, for a sufficiently small $\varepsilon > 0$, the following estimates

$$\| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \langle \nabla_y \rangle^{\frac{1}{2}+\varepsilon} \Lambda(t) \|_{L^2(dxdy)} \leq C(t) \tag{4.1a}$$

$$\| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \langle \nabla_y \rangle^{\frac{1}{2}+\varepsilon} \Gamma(t) \|_{L^2(dxdy)} \leq C \tag{4.1b}$$

$$\| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \phi(t) \|_{L^2(dx)} \leq C \tag{4.1c}$$

hold uniformly in N for any fixed time t. The proof of estimates (4.1a)-(4.1c) relies on the conservation laws established in [GM13a]. For the reader's convenience, we restate the conservation laws for the time-dependent HFB system in the following proposition. Let us recall the total particle number and energy, which we denote by $\mathcal{N}$ and $\mathcal{E}$ respectively, can be evaluated explicitly as follows:

$$\mathcal{N} = N \left\{ \int dx \; |\phi(x)|^2 + \frac{1}{N} \int dxdy \; |\operatorname{sh}(k)(x,y)|^2 \right\} \tag{4.2a}$$

and

$$\begin{aligned} \mathcal{E} = N \Bigg\{ & \int dx \; |\nabla \phi(x)|^2 + \frac{1}{2N} \int dxdy \; |\nabla_{x,y} \operatorname{sh}(k)(x,y)|^2 \\ & + \frac{1}{2N} \int dxdydz \; v_N(x-y) |\phi(x) \operatorname{sh}(k)(y,z) + \phi(y) \operatorname{sh}(k)(x,z)|^2 \\ & + \frac{1}{4} \int dxdy \; v_N(x-y) \left\{ 2|\Lambda(x,y)|^2 + |\Gamma(x,y)|^2 + \Gamma(x,x)\Gamma(y,y) \right\} \Bigg\}. \end{aligned} \tag{4.2b}$$

For the sake of compactness of notation, we have suppressed the dependence on the time variable since it only plays a passive role in our studies of the equations.

Proposition 4.3 (Conservation Quantities). *Suppose* $(\phi(t), \Gamma(t), \Lambda(t))$ *is a smooth solution to the time-dependent HFB system with* $v \in L^1(\mathbb{R}) \cap C^\infty(\mathbb{R})$*. Then the total particle number and energy for the system are conserved.*

Remark 4.4. The reader should be aware of the fact that we are assuming that the energy per particle is constant and independent of N. More precisely, we make the assumption that $\mathcal{N}$ and $\mathcal{E}$ are proportional to N for some fixed N. In fact, we have that $\mathcal{N} = N$ and $\mathcal{E} \sim N$ or, equivalently, $N^{-1}\mathcal{E} \sim 1$.

As an immediate corollary of the conservation quantities, we prove estimate (4.1b) and (4.1c).

Corollary 4.5. *Let* $\phi(t)$ *and* $\Gamma(t)$ *be smooth solutions to the time-dependent HFB equations. Then, for any* $0 < \varepsilon \leq \frac{1}{2}$*, we have the estimates*

$$\| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \langle \nabla_y \rangle^{\frac{1}{2}+\varepsilon} \Gamma(t) \|_{L^2(dxdy)} \lesssim 1$$

$$\| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \phi(t) \|_{L^2(dx)} \lesssim 1$$

which hold uniformly in N *and independent of* t*.*

Proof. It suffices to prove estimate (4.1b) since the prove of (4.1c) is similar. By

Proposition 4.3 and Cauchy-Schwarz inequality, we obtain the estimate[1]

$$\begin{aligned} \| \Gamma(t) \|_{L^2(dxdy)} &\le \| \phi(t) \|^2_{L^2(dx)} + N^{-1} \| \overline{\mathrm{sh}(k_t)} \circ \mathrm{sh}(k_t) \|_{L^2(dxdy)} \\ &\le \| \phi(t) \|^2_{L^2(dx)} + N^{-1} \| \mathrm{sh}(k_t) \|^2_{L^2(dxdy)} = 1 \end{aligned} \tag{4.3a}$$

independent of t and N. Likewise, we see that

$$\| \nabla_x \nabla_y \Gamma(t) \|_{L^2(dxdy)} \le \| \nabla_x \phi(t) \|^2_{L^2} + N^{-1} \| \nabla_x \,\mathrm{sh}(k_t) \|^2_{L^2} \lesssim 1. \tag{4.3b}$$

Hence interpolating (4.3a) and (4.3b) yields the desired result. □

In the remainder of the section, we shall prove estimate (4.1a) holds for some sub-linear function $C(t)$. To this end, let us begin by making the observation that proving estimate (4.1a) is equivalent to establishing the estimate

$$N^{-1} \| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \langle \nabla_y \rangle^{\frac{1}{2}+\varepsilon} \mathrm{sh}(2k_t) \|_{L^2(dxdy)} \lesssim C(t) \tag{4.4}$$

for some sufficiently small $\varepsilon > 0$. Furthermore, to aid us in proving estimate (4.4), we apply the operator identity

$$\mathrm{sh}(2k) = 2\,\mathrm{sh}(k) \circ \mathrm{ch}(k) = 2\,\mathrm{sh}(k) + 2\,\mathrm{sh}(k) \circ p \tag{4.5}$$

---

[1] Here, we abused the notation by identifying the composition operator $T_k = T_f \circ T_g$, where T_f and T_g are integral operators with kernel $f(x,y)$ and $g(x,y)$, with its kernel

$$k(x,y) = \int dz\ f(x,z) g(z,y).$$

and the triangle inequality to obtain a preliminary estimate

$$\begin{aligned}
&\|\, \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \langle \nabla_y \rangle^{\frac{1}{2}+\varepsilon} \operatorname{sh}(2k_t) \,\|_{L^2(dxdy)} \\
&\lesssim \|\, \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \langle \nabla_y \rangle^{\frac{1}{2}+\varepsilon} \operatorname{sh}(k_t) \,\|_{L^2} + \|\, \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \langle \nabla_y \rangle^{\frac{1}{2}+\varepsilon} \operatorname{sh}(k_t) \circ p_t \,\|_{L^2} \\
&=: I_1(t) + I_2(t).
\end{aligned}$$

Hence it remains to show $N^{-1} I_i(t) \lesssim C(t)$ for $i = 1, 2$.

To estimate $I_2(t)$, we use the following lemma

Lemma 4.6. *We have the following estimates*

$$N^{-1} \left\| \, |\nabla_x|^{\frac{1}{2}} |\nabla_y|^{\frac{1}{2}} \operatorname{sh}(k_t) \circ p_t \right\|_{L^2(dxdy)} \lesssim 1 \tag{4.6a}$$

and

$$N^{-1} \|\, \nabla_x \operatorname{sh}(k_t) \circ \nabla_y p_t \,\|_{L^2(dxdy)} \lesssim 1 \tag{4.6b}$$

where both are independent of time t. In particular, by interpolating estimates (4.6a) and (4.6b), we obtain the estimate

$$N^{-1} \left\| \, |\nabla_x|^{\frac{1}{2}+\varepsilon} |\nabla_y|^{\frac{1}{2}+\varepsilon} \operatorname{sh}(k_t) \circ p_t \right\|_{L^2(dxdy)} \lesssim 1 \tag{4.7}$$

for any $0 \leq \varepsilon \leq \frac{1}{2}$.

Proof. Using Plancherel identity and Cauchy-Schwarz inequality, we establish the estimate

$$\begin{aligned}&\left\| \, |\nabla_x|^{\frac{1}{2}} |\nabla_y|^{\frac{1}{2}} \operatorname{sh}(k_t) \circ p_t \, \right\|_{L^2(dxdy)} \\ &\lesssim \; \| \, \nabla_x \operatorname{sh}(k_t) \, \|_{L^2(dxdy)} \, \| \, p_t \, \|_{L^2(dxdy)} + \| \operatorname{sh}(k_t) \, \|_{L^2(dxdy)} \, \| \, \nabla_y p_t \, \|_{L^2(dxdy)} \, .\end{aligned} \tag{4.8}$$

Next, taking derivatives of the kernel of the operator identity

$$\overline{\operatorname{sh}(k)} \circ \operatorname{sh}(k) = p \circ p + 2p$$

yields the operator identity

$$\overline{\nabla_x \operatorname{sh}(k)} \circ \nabla_y \operatorname{sh}(k) = \nabla_x p \circ \nabla_y p + 2 \nabla_x \nabla_y p.$$

In particular, we have that

$$\| \, \nabla_x \operatorname{sh}(k) \, \|^2_{L^2(dxdy)} = \| \, \nabla_x p \circ \nabla_y p + 2 \nabla_x \nabla_y p \, \|_{\mathrm{tr}} \geq \| \, \nabla_x p \, \|^2_{L^2(dxdy)}$$

since both $\nabla_x \nabla_y (p \circ p + 2p)$ and $2 \nabla_x \nabla_y p$ are positive trace class operators. Hence combining estimate (4.8) with the conservation laws, we obtain the estimate

$$\begin{aligned}&N^{-1} \left\| \, |\nabla_x|^{\frac{1}{2}} |\nabla_y|^{\frac{1}{2}} \operatorname{sh}(k_t) \circ p_t \, \right\|_{L^2(dxdy)} \\ &\lesssim N^{-1} \, \| \, \nabla_x \operatorname{sh}(k_t) \, \|_{L^2(dxdy)} \, \| \operatorname{sh}(k_t) \, \|_{L^2(dxdy)} \lesssim 1.\end{aligned}$$

Likewise, we have shown

$$N^{-1} \| \nabla_x \operatorname{sh}(k_t) \circ \nabla_y p_t \|_{L^2(dxdy)} \lesssim N^{-1} \| \nabla_x \operatorname{sh}(k_t) \|_{L^2}^2 \lesssim 1.$$

□

Next, to estimate $I_1(t)$, we first prove a couple preliminary lemmas.

Lemma 4.7. *Let $\Gamma(t)$ and $\Lambda(t)$ be solutions the time-dependent HFB equations. Then it follows we have the estimates*

$$\| \nabla_{x,y} \Lambda(t) \|_{L^2(dxdy)} \lesssim 1 \tag{4.9a}$$

and

$$\| \nabla_{x,y} \Gamma(t) \|_{L^2(dxdy)} \lesssim 1 \tag{4.9b}$$

which holds uniformly in N and independent of time t.

Proof. This is an immediate corollary of Lemma 4.6. □

Lemma 4.8. *Let $\Lambda(t)$ be a solution to the time-dependent HFB equations. Then we have the following energy estimate*

$$\| \nabla_x \nabla_y \Lambda(t) \|_{L^2(dxdy)} \lesssim \| \nabla_x \nabla_y \Lambda_0 \|_{L^2(dxdy)} + N^{3\beta} t. \tag{4.10}$$

Proof. For convenience, let us restate the equation for $\Lambda(t)$ which is

$$\begin{aligned}(\mathbf{S}+\mathbf{V})\,\Lambda = &-(v_N\Lambda)\circ\Gamma - \bar{\Gamma}\circ(v_N\Lambda) - (v_N\bar{\Gamma})\circ\Lambda - \Lambda\circ(v_N\Gamma) \\ &+ 2(v_N * |\phi|^2)(x)\phi(x)\phi(y) + 2(v_N * |\phi|^2)(y)\phi(y)\phi(x) =: F\end{aligned} \tag{4.11}$$

where $v_N\Lambda = v_N(x-y)\Lambda(x,y)$ and

$$\mathbf{V} = \frac{1}{N}v_N + (v_N * \operatorname{diag}\Gamma)(x) + (v_N * \operatorname{diag}\Gamma)(y).$$

Differentiating equation (4.11) by $\nabla_x\nabla_y$ gives us the following equation

$$(\mathbf{S}+\mathbf{V})(\nabla_x\nabla_y\Lambda) = \left[\mathbf{S}+\mathbf{V},\nabla_x\nabla_y\right]\Lambda + \nabla_x\nabla_y F$$

and

$$\begin{aligned}[\mathbf{S}+\mathbf{V},\nabla_x\nabla_y] = &N^{-1}(\nabla_x\nabla_y v_N)\Lambda + N^{-1}\nabla_y v_N\nabla_x\Lambda + N^{-1}\nabla_x v_N\nabla_y\Lambda \\ &+ [(\nabla_y v_N) * \operatorname{diag}\Gamma(y)]\nabla_x\Lambda + [(\nabla_x v_N) * \operatorname{diag}\Gamma(x)]\nabla_y\Lambda.\end{aligned}$$

Using the energy method, we obtain the estimate

$$
\begin{aligned}
&\frac{d}{dt}\| \nabla_x \nabla_y \Lambda(t) \|^2_{L^2(dxdy)} \\
&= \ 2\,\mathrm{Re}\langle \partial_t \nabla_x \nabla_y \Lambda(t), \nabla_x \nabla_y \Lambda(t)\rangle \\
&= \ 2\,\mathrm{Re}\langle (\mathbf{S}+\mathbf{V})(\nabla_x \nabla_y \Lambda(t)), \nabla_x \nabla_y \Lambda(t)\rangle \\
&\leq \ 2\,\| \, [\mathbf{S}+\mathbf{V}, \nabla_x \nabla_y]\, \Lambda(t) + \nabla_x \nabla_y F \, \|_{L^2(dxdy)} \, \| \nabla_x \nabla_y \Lambda(t) \|_{L^2(dxdy)}
\end{aligned}
$$

which leads to the energy estimate

$$
\begin{aligned}
&\| \nabla_x \nabla_y \Lambda(t) \|_{L^2(dxdy)} \leq \ \| \nabla_x \nabla_y \Lambda_0 \|_{L^2(dxdy)} \\
&+ \int_0^t ds \ \, \| \, [\mathbf{S}+\mathbf{V}, \nabla_x \nabla_y]\, \Lambda(s) + \nabla_x \nabla_y F(s) \, \|_{L^2(dxdy)} \, .
\end{aligned}
\tag{4.12}
$$

We are now ready to estimate the forcing terms. First, for the commutator term we have the estimate

$$
\begin{aligned}
&\| \, [\mathbf{S}+\mathbf{V}, \nabla_x \nabla_y] \Lambda(t) \, \|_{L^2(dxdy)} \\
&\leq \ N^{-1} \| \, (\nabla_x \nabla_y v_N) \Lambda(t) \, \|_{L^2(dxdy)} + 2N^{-1} \, \| \nabla_y v_N \nabla_x \Lambda(t) \|_{L^2(dxdy)} \\
&\quad + 2 \, \| \, [(\nabla_y v_N) * \mathrm{diag}\, \Gamma(y)] \nabla_x \Lambda(t) \, \|_{L^2(dxdy)} \\
&\lesssim \ N^{4\beta-1} \Big(\| \nabla_x \Lambda(t) \|_{L^2(dxdy)} + \| \, \mathrm{diag}\, \Gamma(t) \, \|_{L^1(dx)} \, \| \nabla_x \Lambda(t) \|_{L^2(dxdy)} \Big) \lesssim N^{4\beta-1}.
\end{aligned}
$$

The other forcing term in estimate (4.12) can be handled in a similar fashion. We shall estimate only one of the terms since the proof is exactly the same for the other terms. Observe, for the $(v_N\Lambda \circ \Gamma)$ we have that

$$\begin{aligned}
&\| \nabla_y (v_N\Lambda) \circ \nabla_y \Gamma \|_{L^2(dxdy)} \\
\leq\ & \| \nabla_x (v_N\Lambda(t)) \|_{L^2(dxdy)} \| \nabla_y \Gamma(t) \|_{L^2(dxdy)} \lesssim N^{3\beta}.
\end{aligned}$$

Hence combining all the estimates yields the desired estimate. □

Lemma 4.9. *There exists* $\varepsilon_0 > 0$ *such that*

$$N^{-1} \| \, |\nabla_x|^{\frac{1}{2}+\varepsilon} |\nabla_y|^{\frac{1}{2}+\varepsilon} \operatorname{sh}(k_t) \|_{L^2(dxdy)} \lesssim C(t) \tag{4.13}$$

for $0 < \varepsilon \leq \varepsilon_0$ *where* $C(t)$ *is a sub-linear function independent of* N*.*

Proof. Applying Lemma 4.8 and (4.5) give us the estimate

$$\begin{aligned}
&N^{-1} \| \nabla_x \nabla_y \operatorname{sh}(k_t) \|_{L^2(dxdy)} \\
\lesssim\ & N^{-1} \| \nabla_x \nabla_y \operatorname{sh}(2k_t) \|_{L^2(dxdy)} + N^{-1} \| \nabla_x \operatorname{sh}(k_t) \circ \nabla_y p_t \|_{L^2(dxdy)} \\
\lesssim\ & \| \nabla_x \phi(t) \|^2_{L^2(dx)} + \| \nabla_x \nabla_y \Lambda(t) \|_{L^2(dxdy)} + N^{-1} \| \nabla_x \operatorname{sh}(k_t) \circ \nabla_y p_t \|_{L^2(dxdy)} \\
\lesssim\ & 1 + N^{3\beta} t.
\end{aligned}$$

Interpolating the above inequality with the estimate

$$N^{-1} \| \, |\nabla_x|^{\frac{1}{2}} |\nabla_y|^{\frac{1}{2}} \operatorname{sh}(k_t) \|_{L^2(dxdy)} \lesssim N^{-1} \| \nabla_x \operatorname{sh}(k_t) \|_{L^2(dxdy)} \lesssim \frac{1}{\sqrt{N}}$$

we have shown that there exists $\varepsilon_0 > 0$ such that

$$N^{-1} \| \, |\nabla_x|^{\frac{1}{2}+\varepsilon_0} |\nabla_y|^{\frac{1}{2}+\varepsilon_0} \operatorname{sh}(k_t) \, \|_{L^2(dxdy)} \lesssim C(t)$$

where $C(t)$ is a sub-linear function independent of N. In fact, we see that for all $0 < \varepsilon \leq \varepsilon_0 = \frac{1}{2(6\beta+1)}$ we have the estimate

$$N^{-1} \| \, |\nabla_x|^{\frac{1}{2}+\varepsilon} |\nabla_y|^{\frac{1}{2}+\varepsilon} \operatorname{sh}(k_t) \, \|_{L^2(dxdy)} \lesssim N^{-\frac{1}{2}+(6\beta+1)\varepsilon}(1+t)^{2\varepsilon}. \tag{4.14}$$

□

Remark 4.10. With an eye for $0 < \beta < 1$, we need $2\beta(\frac{1}{2} + \varepsilon) < 1$, as assumed in Theorem 3.4 of [GM18], then it follows we need the assumption that $0 < \varepsilon \leq \varepsilon_0 = \min(\frac{1-\beta}{2\beta}, \frac{1}{2(6\beta+1)})$ in our proof of the global well-posedness of the time-dependent HFB system.

Let us summarize our findings.

Proposition 4.11. *Let $\Lambda(t)$ be a solution of the time-dependent HFB equation for $0 < \beta < \frac{2}{3}$. Then for any $0 < \varepsilon \leq \varepsilon_0 = \frac{1}{2(6\beta+1)}$ we have the estimate*

$$\| \, \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \langle \nabla_y \rangle^{\frac{1}{2}+\varepsilon} \Lambda(t) \, \|_{L^2(dxdy)} \lesssim 1 + N^{-\frac{1}{2}+(6\beta+1)\varepsilon}(1+t)^{2\varepsilon}.$$

4.3 Global Well-posedness of the Time-Dependent HFB System

Let us define the norms which we shall use in the proof of the uniform in N global well-posedness of the time-dependent HFB equations for $0 < \beta < \frac{2}{3}$, c.f. [GM17].

Fix $\varepsilon > 0$ as in Remark 4.10 and define the norms

$$\begin{aligned}\mathbf{N}_{[T_0,T_1]}(\Lambda) :=& \sup_z \| \langle\nabla_x\rangle^{\frac{1}{2}+\varepsilon}\Lambda(t,x,x+z) \|_{L^2([T_0,T_1])L^2(dx)} \\ &+ \| \langle\nabla_x\rangle^{\frac{1}{2}+\varepsilon}\langle\nabla_y\rangle^{\frac{1}{2}+\varepsilon}\Lambda \|_{L^\infty([T_0,T_1])L^2(dxdy)} \end{aligned} \tag{4.15a}$$

$$\begin{aligned}\dot{\mathbf{N}}_{[T_0,T_1]}(\Gamma) :=& \sup_z \| |\nabla_x|^{\frac{1}{2}}\langle\nabla_x\rangle^{\varepsilon}\Gamma(t,x,x+z) \|_{L^2([T_0,T_1])L^2(dx)} \\ &+ \| \langle\nabla_x\rangle^{\frac{1}{2}+\varepsilon}\langle\nabla_y\rangle^{\frac{1}{2}+\varepsilon}\Gamma \|_{L^\infty([T_0,T_1])L^2(dxdy)} \end{aligned} \tag{4.15b}$$

$$\mathbf{N}_{[T_0,T_1]}(\phi) := \| \langle\nabla_x\rangle^{\frac{1}{2}+\varepsilon}\phi \|_{L^\infty([T_0,T_1])L^2(dx)} + \| \langle\nabla_x\rangle^{\frac{1}{2}+\varepsilon}\phi \|_{L^2([T_0,T_1])L^6(dx)}. \tag{4.15c}$$

For convenience, we denote the sum of the three norms by

$$\mathbf{N}_{[T_0,T_1]}(X) := \mathbf{N}_{[T_0,T_1]}(\phi) + \dot{\mathbf{N}}_{[T_0,T_1]}(\Gamma) + \mathbf{N}_{[T_0,T_1]}(\Lambda).$$

If $[T_0, T_1] = [0, T]$ then we denote $N_T(X) := N_{[0,T]}(X)$ (similarly for the other norms). Moreover, we adopt the notation

$$\mathbf{N}_{[T_0,T_1]}(DX) := \mathbf{N}_{[T_0,T_1]}(D\phi) + \mathbf{N}_{[T_0,T_1]}(D\Gamma) + \mathbf{N}_{[T_0,T_1]}(D\Lambda)$$

where D is some differential operator.

The goal of this section is to prove the uniform in N global well-posedness of solutions for the time-dependent HFB equations. However, it suffices to prove an a-priori estimate of the form

$$\mathbf{N}_T(DX) \lesssim F(T) \tag{4.16}$$

for some positive real-valued function F defined on all of $[0,\infty)$.

We begin by proving a couple lemmas to aid us in establishing (4.16).

Lemma 4.12. *Let* $(\phi(t), \Gamma(t), \Lambda(t))$ *be a solution to the time-dependent HFB system. Then there exists* $\alpha > 0$ *such that we have the following estimates*

$$\mathbf{N}_{[T_0,T_1]}(X) \lesssim C_0(T_0) + (T_1 - T_0)^\alpha C_0(T_1)\mathbf{N}_{[T_0,T_1]}(X) \tag{4.17}$$

where

$$\begin{aligned} C_0(T) := & \, \| \left\langle \nabla_x \right\rangle^{\frac{1}{2}+\varepsilon} \phi(T,\cdot) \|_{L^2(dx)} \\ & + \| \left\langle \nabla_x \right\rangle^{\frac{1}{2}+\varepsilon} \left\langle \nabla_y \right\rangle^{\frac{1}{2}+\varepsilon} \Gamma(T,\cdot) \|_{L^2(dxdy)} \\ & + \| \left\langle \nabla_x \right\rangle^{\frac{1}{2}+\varepsilon} \left\langle \nabla_y \right\rangle^{\frac{1}{2}+\varepsilon} \Lambda(T,\cdot) \|_{L^2(dxdy)}. \end{aligned}$$

Proof. It suffices to consider the proof of estimate (4.17) for Γ and Λ since the proof for ϕ is similar. Recall the equation for Γ is given by

$$\begin{aligned}\mathbf{S}_{\pm}\bar{\Gamma} = &-(v_N\Lambda)\circ\bar{\Lambda}+\Lambda\circ(v_N\bar{\Lambda})-(v_N\bar{\Gamma})\circ\bar{\Gamma}+\bar{\Gamma}\circ(v_N\bar{\Gamma})\\ &-(v_N * \operatorname{diag}\Gamma)\cdot\bar{\Gamma}+\bar{\Gamma}\cdot(v_N * \operatorname{diag}\Gamma)\\ &+2(v_N * |\phi|^2)\cdot|\phi\rangle\langle\phi|-2|\phi\rangle\langle\phi|\cdot(v_N * |\phi|^2)=: F\end{aligned}\tag{4.18}$$

Then, by Proposition 5.8 in [GM17] and Lemma 4.2 in [GM18], we have the estimate

$$\begin{aligned}\dot{\mathbf{N}}_{[T_0,T_1]}(\Gamma)\lesssim &\,\|\,\langle\nabla_x\rangle^{\frac{1}{2}+\varepsilon}\langle\nabla_y\rangle^{\frac{1}{2}+\varepsilon}\Gamma(T_0,\cdot)\,\|_{L^2(dxdy)}\\ &+\left\|\,\langle\nabla_x\rangle^{\frac{1}{2}+\varepsilon}\langle\nabla_y\rangle^{\frac{1}{2}+\varepsilon}F\,\right\|_{X^{-\frac{1}{2}+\delta}}\\ &\|\,\langle\nabla_x\rangle^{\frac{1}{2}+\varepsilon}\langle\nabla_y\rangle^{\frac{1}{2}+\varepsilon}\Gamma(T_0,\cdot)\,\|_{L^2(dxdy)}\\ &+(T_1-T_0)^{\frac{\varepsilon-2\delta}{2}}\|\,\langle\nabla_x\rangle^{\frac{1}{2}+\varepsilon}\langle\nabla_y\rangle^{\frac{1}{2}+\varepsilon}F\,\|_{L^2([T_0,T_1])L^{\frac{6}{5}+}(dx)L^2(dy)}.\end{aligned}$$

where we choose $0<\delta<\frac{\varepsilon}{2}$. Here, the symbol $\frac{6}{5}+$ denotes a fixed number slightly bigger than $\frac{6}{5}$ with dependence on ε, in fact, $\frac{6}{5}+=\frac{6}{5-2\varepsilon}$.

For the forcing term $\Gamma\cdot(v_N * \operatorname{diag}\Gamma)$, we apply Young's convolution inequality, Sobolev inequality, and Corollary 4.5 to obtain the estimate

$$\begin{aligned}&\|\,\langle\nabla_x\rangle^{\frac{1}{2}+\varepsilon}\langle\nabla_y\rangle^{\frac{1}{2}+\varepsilon}[\bar{\Gamma}\cdot(v_N * \operatorname{diag}\Gamma)]\,\|_{L^2([T_0,T_1])L^{\frac{6}{5}+}(dx)L^2(dy)}\\ \lesssim &\;\|\,\langle\nabla_y\rangle^{\frac{1}{2}+\varepsilon}\Gamma\,\|_{L^2([T_0,T_1])L^{3+}(dx)L^2(dy)}\|\,\langle\nabla_x\rangle^{\frac{1}{2}+\varepsilon}\operatorname{diag}\Gamma\,\|_{L^2([T_0,T_1])L^2(dx)}\\ &\;+\|\,\langle\nabla_x\rangle^{\frac{1}{2}+\varepsilon}\langle\nabla_y\rangle^{\frac{1}{2}+\varepsilon}\Gamma\,\|_{L^2([T_0,T_1])L^2(dxdy)}\|\,\operatorname{diag}\Gamma\,\|_{L^2([T_0,T_1])L^{3+}(dx)}\\ \lesssim &\;\dot{\mathbf{N}}_{[T_0,T_1]}(\Gamma).\end{aligned}$$

where $3+ = \frac{3}{1-\varepsilon}$. Next, for the forcing term $\Lambda \circ (v_N \bar{\Lambda})$, we apply Kato-Ponce, Sobolev, and estimate (4.1c) from the previous section to obtain the estimate

$$
\begin{aligned}
&\| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \langle \nabla_y \rangle^{\frac{1}{2}+\varepsilon} [\Lambda \circ (v_N \bar{\Lambda})] \|_{L^2([T_0,T_1]) L^{\frac{6}{5}+}(dx) L^2(dy)} \\
&\lesssim \int dz\ v_N(z) \| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \bar{\Lambda}(x, x-z) \langle \nabla_y \rangle^{\frac{1}{2}+\varepsilon} \Lambda(x-z, y) \|_{L^2([T_0,T_1]) L^{\frac{6}{5}+}(dx) L^2(dy)} \\
&\quad + \int dz\ v_N(z) \| \bar{\Lambda}(x, x-z) \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \langle \nabla_y \rangle^{\frac{1}{2}+\varepsilon} \Lambda(x-z, y) \|_{L^2([T_0,T_1]) L^{\frac{6}{5}+}(dx) L^2(dy)} \\
&\lesssim \int dz\ v_N(z) \| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \Lambda(x, x-z) \|_{L^2(dt dx)} \| \langle \nabla_y \rangle^{\frac{1}{2}+\varepsilon} \Lambda \|_{L^\infty([T_0,T_1]) L^{3+}(dx) L^2(dy)} \\
&\quad + \int dz\ v_N(z) \| \Lambda(x, x-z) \|_{L^2(dt) L^{3+}(dx)} \| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \langle \nabla_y \rangle^{\frac{1}{2}+\varepsilon} \Lambda \|_{L^\infty([T_0,T_1]) L^2(dx dy)} \\
&\lesssim C_0(T_1) \mathbf{N}_{[T_0,T_1]}(\Lambda)
\end{aligned}
$$

The remaining nonlinear terms $(v_N \Gamma) \circ \Gamma$ and $(v_N * |\phi|^2) \cdot |\phi\rangle\langle\phi|$ can be handled in a similar manner. Thus, we have shown

$$
\dot{\mathbf{N}}_{[T_0,T_1]}(\Gamma) \lesssim C_0(T_0) + (T_1 - T_0)^{\frac{\varepsilon - 2\delta}{2}} C_0(T_1) \mathbf{N}_{[T_0,T_1]}(X).
$$

Next, let us recall the equation for Λ given by

$$
\begin{aligned}
\mathbf{S}\Lambda = &- \frac{1}{N} v_N \Lambda - (v_N * \operatorname{diag} \Gamma) \cdot \Lambda - \Lambda \cdot (v_N * \operatorname{diag} \Gamma) \\
&- (v_N \Lambda) \circ \Gamma - \bar{\Gamma} \circ (v_N \Lambda) - (v_N \bar{\Gamma}) \circ \Lambda - \Lambda \circ (v_N \Gamma) \\
&+ 2(v_N * |\phi|^2) \cdot \phi \otimes \phi - 2\phi \otimes \phi \cdot (v_N * |\phi|^2) =: G
\end{aligned}
\tag{4.19}
$$

To estimate Λ, we employ Proposition 5.9 of [GM17] and Lemma 4.2 of [GM18] to get the estimate

$$\begin{aligned}\mathbf{N}_{[T_0,T_1]}(\Lambda) \lesssim &\, \| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \langle \nabla_y \rangle^{\frac{1}{2}+\varepsilon} \Lambda(T_0,\cdot) \|_{L^2(dxdy)} \\ &+ (T_1-T_0)^{\frac{\varepsilon-2\delta}{2}} \| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \langle \nabla_y \rangle^{\frac{1}{2}+\varepsilon} G \|_{L^2([T_0,T_1])L^{\frac{6}{5}+}(dx)L^2(dy)}\end{aligned}$$

Following the same argument as for the Γ equation, we arrive at the desired result.

□

To obtain a-priori estimates of the form (4.16) we need to employ the following elementary lemma.

Lemma 4.13. *Let* $\delta_1, \delta_2 > 0$ *and* $C > 0$*. Then there exists a monotone sequence of positive real numbers* T_k *such that*

$$\lim_{k\to\infty} T_k = \infty \quad \textit{and} \quad (T_{k+1}-T_k)^{\delta_1} T_{k+1}^{\delta_2} \leq \frac{1}{C} \quad \forall\ k \in \mathbb{N}.$$

Proof. Consider the sequence T_k defined by

$$T_k := \frac{(1-\alpha)^{\alpha}}{C^{\alpha/\delta_2}} \left(1 + \frac{1}{2^{\alpha}} + \ldots + \frac{1}{k^{\alpha}}\right) \leq \frac{1}{(1-\alpha)^{1-\alpha} C^{\alpha/\delta_2}} k^{1-\alpha} \tag{4.20}$$

where $\alpha = \frac{\delta_2}{\delta_1+\delta_2}$. It is clear that $\{T_k\}$ is monotone increasing and tends to infinity as $k \to \infty$. Moreover, by estimate (4.20) we immediately see that

$$(T_{k+1}-T_k)^{\delta_1} T_{k+1}^{\delta_2} \leq \frac{1}{C}$$

which completes the proof. $\square$

Proposition 4.14. *Let $T > 0$. Assume $(\phi(t), \Gamma(t), \Lambda(t))$ is a solution to the time-dependent HFB system, then we have the following a-priori estimate*

$$\mathbf{N}_T(X) \lesssim 1 + T^{5+3\varepsilon}. \tag{4.21}$$

Proof. Define T_k as in the previous lemma with $\delta_1 = \frac{\varepsilon - 2\delta}{2}, \delta_2 = 2\varepsilon, \delta = \frac{\varepsilon^2}{8+2\varepsilon}$, and $C > 0$ sufficiently large. Using estimate (4.17), we obtain the estimate

$$\mathbf{N}_{[T_k, T_{k+1}]}(X) \lesssim C_0(T_k) \lesssim T_k^{2\varepsilon} \tag{4.22}$$

which means

$$\begin{aligned} \mathbf{N}_{T_{k+1}}(X) \leq \mathbf{N}_{T_k}(X) + \mathbf{N}_{[T_k, T_{k+1}]}(X) \leq\ & \mathbf{N}_{T_k}(X) + CT_k^{2\varepsilon} \\ \lesssim\ & (\text{data}) + T_1^{2\varepsilon} + \ldots + T_k^{2\varepsilon}. \end{aligned}$$

Switching to continuous T-variable yields the desired estimate

$$\mathbf{N}_T(X) \lesssim (\text{data}) + \int_0^{T^{1/(1-\alpha)}} x^{2\varepsilon(1-\alpha)}\, dx \lesssim_\varepsilon (\text{data}) + T^{\frac{4\varepsilon}{\varepsilon - 2\delta} + 1 + 2\varepsilon}.$$

$\square$

Lemma 4.15. *Let $(\phi(t), \Gamma(t), \Lambda(t))$ be a solution to the time-dependent HFB system.*

Then there exists $\alpha > 0$ such that we have the following estimates

$$\mathbf{N}_{[T_0,T_1]}(\partial_t X) \lesssim C_1(T_0) + (T_1 - T_0)^{\alpha}\mathbf{N}_{[T_0,T_1]}(X)\mathbf{N}_{[T_0,T_1]}(\partial_t X), \tag{4.23a}$$

$$\mathbf{N}_{[T_0,T_1]}(\nabla_{x+y} X) \lesssim C_2(T_0) + (T_1 - T_0)^{\alpha}\mathbf{N}_{[T_0,T_1]}(X)\mathbf{N}_{[T_0,T_1]}(\nabla_{x+y} X) \tag{4.23b}$$

where

$$\begin{aligned} C_1(T) := {} & \| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \partial_t \phi(T, \cdot) \|_{L^2(dx)} \\ & + \| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \langle \nabla_y \rangle^{\frac{1}{2}+\varepsilon} \partial_t \Gamma(T, \cdot) \|_{L^2(dxdy)} \\ & + \| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \langle \nabla_y \rangle^{\frac{1}{2}+\varepsilon} \partial_t \Lambda(T, \cdot) \|_{L^2(dxdy)} \end{aligned}$$

and

$$\begin{aligned} C_2(T) := {} & \| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \nabla_x \phi(T, \cdot) \|_{L^2(dx)} \\ & + \| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \langle \nabla_y \rangle^{\frac{1}{2}+\varepsilon} (\nabla_{x+y} \Gamma)(T, \cdot) \|_{L^2(dxdy)} \\ & + \| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \langle \nabla_y \rangle^{\frac{1}{2}+\varepsilon} (\nabla_{x+y} \Lambda)(T, \cdot) \|_{L^2(dxdy)}. \end{aligned}$$

Proof. Taking the time derivative of (4.11) yields

$$\begin{aligned} \left(\mathbf{S} + N^{-1} v_N\right) \partial_t \Lambda = {} & - (v_N \Lambda) \circ \partial_t \Gamma - (v_N * \operatorname{diag} \Gamma) \cdot \partial_t \Lambda \\ & + \text{similar terms} =: F \end{aligned}$$

By the same argument as in Lemma 4.12, we have the estimate

$$\begin{aligned}
&\mathbf{N}_{[T_0,T_1]}(\partial_t\Lambda)\\
&\lesssim C_1(T_0) + (T_1-T_0)^{\frac{\varepsilon-2\delta}{2}}\|\, \langle\nabla_x\rangle^{\frac{1}{2}+\varepsilon}\langle\nabla_y\rangle^{\frac{1}{2}+\varepsilon}F\,\|_{L^2([T_0,T_1])L^{\frac{6}{5}+}(dx)L^2(dy)}.
\end{aligned}$$

We shall look at two generic cases, as stated above, to deduce (4.23). In the first case, we estimate the term $(v_N\Lambda)\circ\partial_t\Gamma$, which goes as follow

$$\begin{aligned}
&\|\, \langle\nabla_x\rangle^{\frac{1}{2}+\varepsilon}\langle\nabla_y\rangle^{\frac{1}{2}+\varepsilon}(v_N\Lambda)\circ\partial_t\Gamma\,\|_{L^2([T_0,T_1])L^{\frac{6}{5}+}(dx)L^2(dy)}\\
&\lesssim \int dz\ v_N(z)\|\, \langle\nabla_x\rangle^{\frac{1}{2}+\varepsilon}\Lambda(x,x-z)\langle\nabla_y\rangle^{\frac{1}{2}+\varepsilon}\partial_t\Gamma(x-z,y)\,\|_{L^2([T_0,T_1])L^{\frac{6}{5}+}(dx)L^2(dy)}\\
&\quad+\int dz\ v_N(z)\|\,\Lambda(x,x-z)\langle\nabla_x\rangle^{\frac{1}{2}+\varepsilon}\langle\nabla_y\rangle^{\frac{1}{2}+\varepsilon}\partial_t\Gamma(x-z,y)\,\|_{L^2([T_0,T_1])L^{\frac{6}{5}+}(dx)L^2(dy)}\\
&\lesssim \int dz\ v_N(z)\|\, \langle\nabla_x\rangle^{\frac{1}{2}+\varepsilon}\Lambda(x,x-z)\,\|_{L^2(dtdx)}\|\, \langle\nabla_x\rangle^{\frac{1}{2}+\varepsilon}\langle\nabla_y\rangle^{\frac{1}{2}+\varepsilon}\partial_t\Gamma\,\|_{L^\infty([T_0,T_1])L^2(dxdy)}\\
&\quad+\int dz\ v_N(z)\|\,\Lambda(x,x-z)\,\|_{L^2(dt)L^{3+}(dx)}\|\, \langle\nabla_x\rangle^{\frac{1}{2}+\varepsilon}\langle\nabla_y\rangle^{\frac{1}{2}+\varepsilon}\partial_t\Gamma\,\|_{L^\infty([T_0,T_1])L^2(dxdy)}\\
&\lesssim \mathbf{N}_{[T_0,T_1]}(\Lambda)\dot{\mathbf{N}}_{[T_0,T_1]}(\partial_t\Gamma).
\end{aligned}$$

In the second case, we estimate the term $(v_N * \operatorname{diag}\Gamma)\cdot\partial_t\Lambda$ as follows

$$\begin{aligned}
&\|\, \langle\nabla_x\rangle^{\frac{1}{2}+\varepsilon}\langle\nabla_y\rangle^{\frac{1}{2}+\varepsilon}[(v_N * \operatorname{diag}\Gamma)\cdot\partial_t\Lambda]\,\|_{L^2([T_0,T_1])L^{\frac{6}{5}+}(dx)L^2(dy)}\\
&\lesssim\ \|\,(v_N * \langle\nabla_x\rangle^{\frac{1}{2}+\varepsilon}\operatorname{diag}\Gamma)\cdot\langle\nabla_y\rangle^{\frac{1}{2}+\varepsilon}\partial_t\Lambda\,\|_{L^2([T_0,T_1])L^{\frac{6}{5}+}(dx)L^2(dy)}\\
&\quad+\|\,(v_N * \langle\nabla_x\rangle^{\frac{1}{2}+\varepsilon}\operatorname{diag}\Gamma)\cdot\langle\nabla_y\rangle^{\frac{1}{2}+\varepsilon}\partial_t\Lambda\,\|_{L^2([T_0,T_1])L^{\frac{6}{5}+}(dx)L^2(dy)}\\
&\lesssim\ \dot{\mathbf{N}}_{[T_0,T_1]}(\Gamma)\mathbf{N}_{[T_0,T_1]}(\partial_t\Lambda).
\end{aligned}$$

Hence combining the above estimates yields

$$\mathbf{N}_{[T_0,T_1]}(\partial_t\Lambda) \lesssim C_1(T_0) + (T_1 - T_0)^{\frac{\varepsilon-2\delta}{2}}\{\mathbf{N}_{[T_0,T_1]}(\Lambda)\dot{\mathbf{N}}_{[T_0,T_1]}(\partial_t\Gamma)$$

$$+ \mathbf{N}_{[T_0,T_1]}(\partial_t\Lambda)\dot{\mathbf{N}}_{[T_0,T_1]}(\Gamma) + \mathbf{N}_{[T_0,T_1]}(\partial_t\phi)\mathbf{N}_{[T_0,T_1]}(\phi)\}$$

$$\lesssim C_1(T_0) + (T_1 - T_0)^{\frac{\varepsilon-2\delta}{2}}\mathbf{N}_{[T_0,T_1]}(X)\mathbf{N}_{[T_0,T_1]}(\partial_t X).$$

Similarly, we can show

$$\dot{\mathbf{N}}_{[T_0,T_1]}(\partial_t\Gamma) \lesssim C_1(T_0) + (T_1 - T_0)^{\frac{\varepsilon-2\delta}{2}}\mathbf{N}_{[T_0,T_1]}(X)\mathbf{N}_{[T_0,T_1]}(\partial_t X)$$

and

$$\mathbf{N}_{[T_0,T_1]}(\partial_t\phi) \lesssim C_1(T_0) + (T_1 - T_0)^{\frac{\varepsilon-2\delta}{2}}\mathbf{N}_{[T_0,T_1]}(X)\mathbf{N}_{[T_0,T_1]}(\partial_t X).$$

Therefore, summing up the three inequalities yields (4.23a). Moreover, the prove of (4.23b) is exactly the same since ∇_{x+y} commutes with $N^{-1}v_N(x-y)$, i.e. $[\nabla_{x+y}, N^{-1}v_N(x-y)] = 0$. □

Using the above lemma we could again prove some a-priori estimates for both the norm of $\partial_t X$ and $\nabla_{x+y}X$.

Proposition 4.16. *Let $T > 0$. Suppose $(\phi(t), \Gamma(t), \Lambda(t))$ is a solution to the time-dependent HFB system, then we have the following uniform in N a-priori estimates*

$$\mathbf{N}_T(\partial_t X) \lesssim \exp\left(\alpha T^{5+\varepsilon}\right), \tag{4.24a}$$

$$\mathbf{N}_T(\nabla_{x+y} X) \lesssim \exp\left(\alpha' T^{5+\varepsilon}\right) \tag{4.24b}$$

for some $\alpha, \alpha' > 0$, which are independent of T.

Proof. Again we choose the sequence T_k defined by (4.20) for some sufficiently large $C > 0$. Applying Lemma 4.15 and estimate (4.22) yield the estimate

$$\begin{aligned}\mathbf{N}_{[T_k,T_{k+1}]}(\partial_t X) &\lesssim C_1(T_k) + (T_{k+1}-T_k)^{\frac{\varepsilon-2\delta}{2}}\mathbf{N}_{[T_k,T_{k+1}]}(X)\mathbf{N}_{[T_k,T_{k+1}]}(\partial_t X)\\ &\lesssim C_1(T_k) + (T_{k+1}-T_k)^{\frac{\varepsilon-2\delta}{2}} T_{k+1}^{2\varepsilon}\mathbf{N}_{[T_k,T_{k+1}]}(\partial_t X).\end{aligned}$$

Then it follows

$$\mathbf{N}_{[T_k,T_{k+1}]}(\partial_t X) \lesssim C_1(T_k).$$

In particular, we have the estimate

$$\begin{aligned}\mathbf{N}_{T_{k+1}}(\partial_t X) &\le \mathbf{N}_{T_k}(\partial_t X) + \mathbf{N}_{[T_k,T_{k+1}]}(\partial_t X)\\ &\lesssim \mathbf{N}_{T_k}(\partial_t X) + C_1(T_k) \lesssim (\text{data}) + \sum_{j=1}^{k} C_1(T_j)\end{aligned}$$

By switching to the continuous T-variable and set $\delta = \frac{\varepsilon^2}{8+2\varepsilon}$, we obtain the estimate

$$\begin{aligned}\mathbf{N}_T(\partial_t X) &\lesssim C_1(T_0) + \int_0^T \, d\tau \; C_1(\tau)\tau^{4+\varepsilon} \\ &\lesssim C_1(T_0) + \int_0^T d\tau \; \mathbf{N}_\tau(\partial_t X)\tau^{4+\varepsilon}.\end{aligned}$$

Finally, applying Gronwall's inequality yields

$$\mathbf{N}_T(\partial_t X) \lesssim C_1(T_0) \exp\left(\alpha T^{5+\varepsilon}\right).$$

The proof for (4.24b) is exactly the same. $\square$

Remark 4.17. Note, from the a-priori estimate (4.24a), we could deduce

$$\begin{aligned}&\| \, \partial_t \phi \, \|_{L^1([0,T]\times L^2(\mathbb{R}^3))} \leq T \| \, \partial_t \phi \, \|_{L^\infty([0,T]\times L^2(\mathbb{R}^3))} \lesssim T \exp\left(\alpha T^{5+\varepsilon}\right) \\ &\| \, \partial_t \Gamma \, \|_{L^1([0,T]\times L^2(\mathbb{R}^6))} \leq T \| \, \partial_t \Gamma \, \|_{L^\infty([0,T]\times L^2(\mathbb{R}^6))} \lesssim T \exp\left(\alpha T^{5+\varepsilon}\right) \\ &\| \, \partial_t \Lambda \, \|_{L^1([0,T]\times L^2(\mathbb{R}^6))} \leq T \| \, \partial_t \Lambda \, \|_{L^\infty([0,T]\times L^2(\mathbb{R}^6))} \lesssim T \exp\left(\alpha T^{5+\varepsilon}\right).\end{aligned}$$

Then by the 1D Sobolev inequality we have that $\phi \in C([0,T] \times L^2(\mathbb{R}^3))$ and $\Gamma, \Lambda \in C([0,T] \times L^2(\mathbb{R}^6))$, that is, ϕ, Γ, and Λ are strong solutions to the nonlinear equations.

Let us conclude this section with some a-priori estimates for the higher order derivatives of (ϕ, Γ, Λ) which we will later use to estimate $\mathrm{sh}(2k)$.

Lemma 4.18. *Suppose $\phi(t), \Gamma(t)$, and $\Lambda(t)$ are solutions to the time-dependent HFB* *equations, then we have the following estimates*

$$\begin{aligned}
&\mathbf{N}_{[T_0,T_1]}(\partial_t \nabla_{x+y} X) \qquad (4.25a)\\
&\lesssim C_3(T_0) + (T_1 - T_0)^{\frac{\varepsilon-2\delta}{2}} \mathbf{N}_{[T_0,T_1]}(X) \mathbf{N}_{[T_0,T_1]}(\partial_t \nabla_{x+y} X)\\
&\quad + (T_1 - T_0)^{\frac{\varepsilon-2\delta}{2}} \mathbf{N}_{[T_0,T_1]}(\partial_t X) \mathbf{N}_{[T_0,T_1]}(\nabla_{x+y} X)\\
&\mathbf{N}_{[T_0,T_1]}(\nabla^2_{x+y} X) \qquad (4.25b)\\
&\lesssim C_4(T_0) + (T_1 - T_0)^{\frac{\varepsilon-2\delta}{2}} \mathbf{N}_{[T_0,T_1]}(X) \mathbf{N}_{[T_0,T_1]}(\nabla^2_{x+y} X)\\
&\quad + (T_1 - T_0)^{\frac{\varepsilon-2\delta}{2}} \mathbf{N}^2_{[T_0,T_1]}(\nabla_{x+y} X)\\
&\mathbf{N}_{[T_0,T_1]}(\partial_t \nabla^2_{x+y} X) \qquad (4.25c)\\
&\lesssim C_5(T_0) + (T_1 - T_0)^{\frac{\varepsilon-2\delta}{2}} \mathbf{N}_{[T_0,T_1]}(X) \mathbf{N}_{[T_0,T_1]}(\partial_t \nabla^2_{x+y} X)\\
&\quad + (T_1 - T_0)^{\frac{\varepsilon-2\delta}{2}} \mathbf{N}_{[T_0,T_1]}(\nabla_{x+y} X) \mathbf{N}_{[T_0,T_1]}(\partial_t \nabla_{x+y} X)\\
&\quad + (T_1 - T_0)^{\frac{\varepsilon-2\delta}{2}} \mathbf{N}_{[T_0,T_1]}(\nabla^2_{x+y} X) \mathbf{N}_{[T_0,T_1]}(\partial_t X)
\end{aligned}$$

where

$$
\begin{aligned}
C_3(T) &= \| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \partial_t \nabla_x \phi(T,\cdot) \|_{L^2(dx)} \\
&\quad + \| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \langle \nabla_y \rangle^{\frac{1}{2}+\varepsilon} (\partial_t \nabla_{x+y} \Gamma)(T,\cdot) \|_{L^2(dxdy)} \\
&\quad + \| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \langle \nabla_y \rangle^{\frac{1}{2}+\varepsilon} (\partial_t \nabla_{x+y} \Lambda)(T,\cdot) \|_{L^2(dxdy)} \\
C_4(T) &= \| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \nabla_x^2 \phi(T,\cdot) \|_{L^2(dx)} \\
&\quad + \| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \langle \nabla_y \rangle^{\frac{1}{2}+\varepsilon} (\nabla_{x+y}^2 \Gamma)(T,\cdot) \|_{L^2(dxdy)} \\
&\quad + \| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \langle \nabla_y \rangle^{\frac{1}{2}+\varepsilon} (\nabla_{x+y}^2 \Lambda)(T,\cdot) \|_{L^2(dxdy)} \\
C_5(T) &= \| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \partial_t \nabla_x^2 \phi(T,\cdot) \|_{L^2(dx)} \\
&\quad + \| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \langle \nabla_y \rangle^{\frac{1}{2}+\varepsilon} (\partial_t \nabla_{x+y}^2 \Gamma)(T,\cdot) \|_{L^2(dxdy)} \\
&\quad + \| \langle \nabla_x \rangle^{\frac{1}{2}+\varepsilon} \langle \nabla_y \rangle^{\frac{1}{2}+\varepsilon} (\partial_t \nabla_{x+y}^2 \Lambda)(T,\cdot) \|_{L^2(dxdy)}.
\end{aligned}
$$

Proof. The proof is similar to the proof of Lemma 4.15. □

Proposition 4.19. *Let $T > 0$. Suppose $(\phi(t), \Gamma(t), \Lambda(t))$ is a solution to the time-dependent HFB system, then we have the following uniform in N a-priori estimates*

$$\mathbf{N}_T(\partial_t \nabla_{x+y} X) \lesssim \exp\left(\kappa_1 T^{5+\varepsilon}\right), \tag{4.26a}$$

$$\mathbf{N}_T(\nabla_{x+y}^2 X) \lesssim \exp\left(\kappa_2 T^{5+\varepsilon}\right), \tag{4.26b}$$

$$\mathbf{N}_T(\partial_t \nabla_{x+y}^2 X) \lesssim \exp\left(\kappa_3 T^{5+\varepsilon}\right) \tag{4.26c}$$

for some constants $\kappa_1, \kappa_2, \kappa_3 > 0$, which are independent of T.

Proof. Let us begin by choosing the same sequence T_k defined by (4.20) for some sufficiently large $C > 0$. By Lemma 4.18 and (4.22), we obtain the estimate

$$\begin{aligned}
&\mathbf{N}_{[T_k,T_{k+1}]}(\partial_t \nabla_{x+y} X) \\
&\lesssim \; C_3(T_k) + (T_{k+1} - T_k)^{\frac{\varepsilon-2\delta}{2}} \mathbf{N}_{[T_k,T_{k+1}]}(\partial_t X) \mathbf{N}_{[T_k,T_{k+1}]}(\nabla_{x+y} X).
\end{aligned}$$

In particular, if we set $\delta = \frac{\varepsilon^2}{8+2\varepsilon}$ then we have the estimate

$$\mathbf{N}_{[T_k,T_{k+1}]}(\partial_t \nabla_{x+y} X) \lesssim C_3(T_k) + \frac{C_1(T_k) C_2(T_k)}{T_{k+1}^{2\varepsilon}}.$$

Hence it follows

$$\begin{aligned}
\mathbf{N}_{T_{k+1}}(\partial_t \nabla_{x+y} X) &\leq \mathbf{N}_{T_k}(\partial_t \nabla_{x+y} X) + \mathbf{N}_{[T_k,T_{k+1}]}(\partial_t \nabla_{x+y} X) \\
&\lesssim \mathbf{N}_{T_k}(\partial_t \nabla_{x+y} X) + C_3(T_k) + \frac{C_1(T_k) C_2(T_k)}{T_{k+1}^{2\varepsilon}} \\
&\lesssim C_3(T_0) + \sum_{j=1}^{k} \left[C_3(T_j) + \frac{C_1(T_j) C_2(T_j)}{T_{j+1}^{2\varepsilon}} \right].
\end{aligned}$$

Switching to continuous T-variable yields the estimate

$$\begin{aligned}
\mathbf{N}_T(\partial_t \nabla_{x+y} X) &\lesssim C_3(T_0) + \int_0^T d\tau \; \mathbf{N}_\tau(\nabla_{x+y} X) \mathbf{N}_\tau(\partial_t X) \tau^{4-\varepsilon} \\
&\quad + \int_0^T d\tau \; \mathbf{N}_\tau(\partial_t \nabla_{x+y} X) \tau^{4+\varepsilon} \\
&\lesssim C_3(T_0) + T^{5-\varepsilon} \exp\left(k T^{5+\varepsilon}\right) \\
&\quad + \int_0^T d\tau \; \mathbf{N}_\tau(\partial_t \nabla_{x+y} X) \tau^{4+\varepsilon}.
\end{aligned}$$

Using Gronwall's inequality, we obtain the estimate

$$\begin{aligned}\mathbf{N}_T(\partial_t \nabla_{x+y} X) &\lesssim (C_3(T_0) + T^{5-\varepsilon} \exp\left(kT^{5+\varepsilon}\right)) \exp\left(cT^{5+\varepsilon}\right) \\ &\lesssim \exp\left(\kappa T^{5+\varepsilon}\right)\end{aligned}$$

for some $\kappa > 0$. The proofs for the other two estimates are similar. □

4.4 Estimates for $\mathrm{sh}(2k)$

The purpose of this section is to obtain estimates for $\mathrm{sh}(2k)$, which will be used to obtain Fock space estimates. Recall the equation for $\mathrm{sh}(2k)$ is given by

$$\begin{aligned}\mathbf{S}(\mathrm{sh}(2k)) = &- 2v_N\Lambda - (v_N\Lambda) \circ p_2 - \bar{p}_2 \circ (v_N\Lambda) \\ &- ((v_N * \operatorname{diag}\Gamma)(x) + (v_N * \operatorname{diag}\Gamma)(y))\,\mathrm{sh}(2k) \\ &- (v_N\Gamma) \circ \mathrm{sh}(2k) - \mathrm{sh}(2k) \circ (v_N\Gamma)\end{aligned} \tag{4.27}$$

where $\mathbf{S} = \frac{1}{i}\partial_t - \Delta_{\mathbb{R}^6}$.

Proposition 4.20. *Let* $\mathrm{sh}(2k)$ *satisfy* (4.27) *with some initial conditions. Then for any fixed* $T > 0$ *and* $0 \leq j \leq 2$ *we have that*

$$\| \nabla^j_{x+y} \,\mathrm{sh}(2k)(t,\cdot) \|_{L^2(dxdy)} \lesssim \exp\left(\alpha_j T^{5+3\varepsilon}\right) \tag{4.28a}$$

$$\sup_x \| \,\mathrm{sh}(2k)(t,x,\cdot) \|_{L^2(dy)} \lesssim \exp\left(\alpha T^{5+3\varepsilon}\right) \; . \tag{4.28b}$$

for some $\alpha_j, \alpha > 0$.

To prove the above proposition we will need a couple lemmas

Lemma 4.21. *Let* s_a^0 *be the solution to*

$$\begin{aligned} \mathbf{S}s_a^0 =& -2v_N(x-y)\Lambda \\ s_a^0(0,x,y) =& \operatorname{sh}(2k)(0,x,y) \end{aligned}$$

on the interval $[0,T]$*. Then there exists* $\kappa_j > 0$ *for* $0 \leq j \leq 2$ *such that*

$$\| \nabla_{x+y}^j s_a^0(t,\cdot,\cdot) \|_{L^2(dxdy)} \lesssim \exp\left(\kappa_j T^{5+\varepsilon}\right) \tag{4.29}$$

for all $t \in [0,T]$*.*

Proof. Observe we could write the solution as

$$s_a^0(t,x,y) = e^{it\Delta_{x,y}} \operatorname{sh}(2k)(0,x,y) + i\int_0^t e^{i(t-s)\Delta_{x,y}} v_N(x-y)\Lambda(s)\; ds$$

then it follows

$$\begin{aligned} &\| s_a^0(t,\cdot) \|_{L^2(dxdy)} \\ &\leq \| \operatorname{sh}(2k_0) \|_{L^2(dxdy)} + \left\| \int_0^t e^{i(t-s)\Delta_{x,y}} v_N(x-y)\Lambda(s)\; ds \right\|_{L^2(dxdy)}. \end{aligned}$$

Let us focus on the nonlinear term. By a change of variables, we get

$$\begin{aligned}
&\left\| \int_0^t e^{i(t-s)\Delta_{x,y}} v_N(x-y)\Lambda(s)\ ds \right\|_{L^2(dxdy)} \\
&\lesssim \left\| P_{|\xi|>1} \int_0^t e^{i(t-s)\Delta_{x,y}} v_N(y)\Lambda\left(s, \frac{x+y}{2}, \frac{x-y}{2}\right)\ ds \right\|_{L^2(dxdy)} \\
&\quad + \left\| P_{|\xi|\le 1} \int_0^t e^{i(t-s)\Delta_{x,y}} v_N(y)\Lambda\left(s, \frac{x+y}{2}, \frac{x-y}{2}\right)\ ds \right\|_{L^2(dxdy)}.
\end{aligned}$$

Let us denote $\Lambda\left(s, \frac{x+y}{2}, \frac{x-y}{2}\right)$ by $\tilde{\Lambda}(s,x,y)$. For the first term we shall rewrite the integral using integration by parts, i.e.

$$\begin{aligned}
\int_0^t e^{i(t-s)\Delta_{x,y}} v_N(y)\tilde{\Lambda}(s)\ ds &\sim -\int_0^t ds\ \frac{\partial}{\partial s} e^{i(t-s)\Delta_{x,y}} \Delta_{x,y}^{-1}(v_N(y)\tilde{\Lambda}(s)) \\
&= \int_0^t e^{i(t-s)\Delta_{x,y}} \Delta_{x,y}^{-1}(v_N(y)\frac{\partial}{\partial s}\tilde{\Lambda}(s)) \\
&\quad + e^{it\Delta}\Delta_{x,y}^{-1}(v_N(y)\tilde{\Lambda}(0)) - \Delta_{x,y}^{-1}(v_N(y)\tilde{\Lambda}(t))
\end{aligned}$$

then it follows

$$\begin{aligned}
&\left\| P_{|\xi|\ge 1} \int_0^t e^{i(t-s)\Delta_{x,y}} v_N(y)\tilde{\Lambda}(s,\cdot)\ ds \right\|_{L^2(dxdy)} \\
&\lesssim \int_0^t ds\ \left\| P_{|\xi|\ge 1}\Delta_{x,y}^{-1}(v_N(y)\frac{\partial}{\partial s}\tilde{\Lambda}(s,\cdot)) \right\|_{L^2(dxdy)} \\
&+ \| P_{|\xi|\ge 1}\Delta_{x,y}^{-1}(v_N(y)\tilde{\Lambda}(0,\cdot)) \|_{L^2(dxdy)} + \| P_{|\xi|\ge 1}\Delta_{x,y}^{-1}(v_N(y)\tilde{\Lambda}(t,\cdot)) \|_{L^2(dxdy)}.
\end{aligned}$$

Next, by Plancherel, we obtain the estimate

$$
\begin{aligned}
&\int_0^t ds \, \left\| P_{|\xi|\geq 1}\Delta_{x,y}^{-1}(v_N(y)\frac{\partial}{\partial s}\tilde{\Lambda}(s,\cdot)) \right\|_{L^2(dxdy)} \\
&\lesssim \int_0^t ds \, \left\| \frac{\widehat{v_N \partial_s \tilde{\Lambda}}(s,\xi,\eta)}{|\xi|^2+|\eta|^2} \right\|_{L^2(|\xi|\geq 1, d\xi)L^2(d\eta)} \\
&\lesssim \int_0^t ds \, \| v_N(y)\partial_s\tilde{\Lambda}(s,x,y) \|_{L^1(dy)L^2(dx)} \\
&\sim \int_0^t ds \, \| \partial_s\Lambda(s,x+y,x) \|_{L^\infty(dy)L^2(dx)} \\
&\lesssim \sqrt{t}\| \partial_s\Lambda \|_{L^2([0,t])L^\infty(dy)L^2(dx)} \lesssim \exp\left(\kappa_0 t^{5+\varepsilon}\right).
\end{aligned}
$$

For the second and third terms, we have the estimate

$$
\begin{aligned}
&\| P_{|\xi|\geq 1}\Delta_{x,y}^{-1}(v_N(y)\tilde{\Lambda}(t,x,y)) \|_{L^2(dxdy)} \\
&\lesssim \ \| \Lambda(t,x,x-y) \|_{L^\infty(dy)L^2(dx)} \\
&\lesssim \ \| \Lambda(s,x,x-y) \|_{L^\infty([0,t])L^\infty(dy)L^2(dx)} \\
&\lesssim \ \| \partial_s\Lambda(s,x,x-y) \|_{L^2([0,t])L^\infty(dy)L^2(dx)} \lesssim \exp\left(\kappa_0 t^{5+\varepsilon}\right).
\end{aligned}
$$

Thus, we have shown

$$
\| s_a^0 \|_{L^\infty([0,T])L^2(dxdy)} \lesssim \exp\left(\kappa_0 T^{5+\varepsilon}\right).
$$

In particular, it is easy to check

$$\begin{aligned}\| \nabla^j_{x+y} s^0_a \|_{L^\infty([0,T])L^2(dxdy)} &\lesssim \| \partial_t \nabla^j_{x+y} \Lambda \|_{L^2([0,T])L^\infty(dx)L^2(dy)} \\ &\lesssim \mathbf{N}_T(\partial_t \nabla^j_{x+y} X) \lesssim \exp\left(\kappa_j T^{5+\varepsilon}\right).\end{aligned}$$

□

Lemma 4.22. *Let s_a be the solution to*

$$\begin{aligned}\tilde{\mathbf{S}} s_a =& \ -2 v_N \Lambda \\ s_a(0,x,y) =& \ \mathrm{sh}(2k)(0,x,y)\end{aligned}$$

on the interval $[0,T]$. Then there exists $\kappa_j > 0$ for $0 \leq j \leq 2$ such that

$$\| \nabla^j_{x+y} s_a(t,\cdot,\cdot) \|_{L^2(dxdy)} \lesssim \exp\left(\kappa_j T^{5+\varepsilon}\right)$$

for all $t \in [0,T]$.

Proof. Recall $\tilde{\mathbf{S}} = \mathbf{S} + V$ where

$$V(u) = ((v_N * \operatorname{diag}\Gamma)(x) + (v_N * \operatorname{diag}\Gamma)(y))u + (v_N \Gamma) \circ u + u \circ (v_N \Gamma).$$

Using the previous result, we see that

$$\tilde{\mathbf{S}} s^1_a = -V(s^0_a) \quad s^1_a(0,x,y) = 0$$

where $s_a = s_a^1 + s_a^0$. It's not hard to see

$$\begin{aligned}(\mathbf{S}+V)(\nabla_{x+y}^j s_a^1) &= [\mathbf{S}+V, \nabla_{x+y}^j] s_a^1 - \nabla_{x+y}^j V(s_a^0) \\ &= [V, \nabla_{x+y}^j] s_a - \nabla_{x+y}^j V(s_a^0).\end{aligned}$$

Using energy estimate, we have

$$\| \nabla_{x+y}^j s_a^1 \|_{L^2} \le \int_0^t ds \ \| [V, \nabla_{x+y}^j] s_a^1 \|_{L^2} + \| \nabla_{x+y}^j V(s_a^0) \|_{L^2}.$$

Let us consider the case when $j = 0$. Observe we have

$$\| s_a^1 \|_{L^2} \le \int_0^t ds \ \| V(s_a^0) \|_{L^2}.$$

Observe

$$\begin{aligned}&\int_0^t ds \ \| (v_N * \operatorname{diag}\Gamma)(x) \cdot s_a^0 \|_{L^2(dxdy)} \\ \le \ &\int_0^t ds \ \| v_N * \operatorname{diag}\Gamma(x) \|_{L^2(dx)} \| s_a^0 \|_{L^\infty(dx)L^2(dy)} \\ \le \ &\int_0^t ds \ \| v_N * \operatorname{diag}\Gamma(x) \|_{L^2(dx)} \| \, \| s_a^0(x, x+y) \|_{L^\infty(dx)} \|_{L^2(dy)} \\ \lesssim \ & t^{\frac{1}{2}} \sup_{0<s<t} \| \, \| (\nabla_{x+y}^j s_a^0)(s,x,y) \|_{L^2(dy)} \|_{L^2(dx)} \\ \lesssim \ & \exp\left(\kappa_0 T^{5+\varepsilon}\right)\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t ds \,\| \, (v_N \Gamma) \circ s_a^0 \, \|_{L^2(dxdy)} \\
&\leq \int_0^t ds \int dz v_N(z) \| \, \Gamma(x, x-z) s_a^0(x-z, y) \, \|_{L^2(dxdy)} \\
&\lesssim \int_0^t ds \int dz v_N(z) \| \, \Gamma(x, x-z) \, \|_{L^\infty(dx)} \| \, s_a^0 \, \|_{L^2(dxdy)} \\
&\lesssim \exp\left(\kappa_0 T^{5+\varepsilon}\right).
\end{aligned}$$

The proof for $j = 1, 2$ is the same. $\square$

Proof of Proposition 4.20. The proof is exactly the same as the one given for Theorem 7.1 in [GM17]. $\square$

Chapter 5: Morawetz Estimates of the time-dependent HFB System

Consider the Bogoliubov state (quasifree state)

$$\Psi_{\mathrm{HFB}}(t) = \mathcal{M}\Omega = e^{-\sqrt{N}\mathcal{A}(\phi)}e^{-\mathcal{B}(k)}\Omega \tag{5.1}$$

where $\Omega = (1, 0, 0, \ldots)$ denote the vacuum vector in Fock space. Following [GM17], we define the kernel of the $\mathcal{L}$-matrices (operators)

$$\mathcal{L}_{n,m}(t, x_1, \ldots, x_n; y_1, \ldots, y_m) := \frac{1}{N^{(m+n)/2}}\langle \mathcal{M}\Omega, \mathcal{P}_{n,m}\mathcal{M}\Omega\rangle \tag{5.2}$$

where $\mathcal{P}_{n,m} = a^\dagger_{x_1} \cdots a^\dagger_{x_n} a_{y_1} \cdots a_{y_m}$.

5.1 Main Results

Let us recall the evolution equation for the $\mathcal{L}_{1,1}$-density matrix derived in [GM17]

$$\frac{1}{i}\frac{\partial}{\partial t}\mathcal{L}_{1,1}(t, x_1; x_1') = (-\Delta_{x_1} + \Delta_{x_1'})\mathcal{L}_{1,1}(t, x_1; x_1') + B_v(\mathcal{L}_{2,2})(t, x_1; x_1') \tag{5.3}$$

where

$$
\begin{aligned}
B_v(\mathcal{L}_{2,2})(t,x_1;x_1') = &\int dx_2\ v(x_1-x_2)\mathcal{L}_{2,2}(t,x_1,x_2;x_1',x_2) \\
&- \int dx_2'\ v(x_1'-x_2')\mathcal{L}_{2,2}(t,x_1,x_2';x_1',x_2').
\end{aligned}
$$

To effectively describe the conservation law associated with (5.3), we introduce the *pseudo-stress-energy tensor* $T_{\mu\nu}$ for $\mu,\nu=0,1,2,3$ of the one-particle Fock marginal density matrix defined by

$$T_{00} = \rho :=\ \mathcal{L}_{1,1}(t,x;x), \tag{5.4a}$$

$$T_{j0} = T_{0j} = p_j :=\ \frac{1}{2i}\left(\partial_{x_j}\mathcal{L}_{1,1} - \partial_{x_j'}\mathcal{L}_{1,1}\right)(t,x;x), \tag{5.4b}$$

$$
\begin{aligned}
T_{jk} = \sigma_{jk} := &\int dx'\ \delta(x-x')(\partial_{x_k}\partial_{x_j'} + \partial_{x_k'}\partial_{x_j})\mathcal{L}_{1,1}(t,x;x') \\
&+ \frac{1}{2}\delta_{jk}\int dy\ v(x-y)\mathcal{L}_{2,2,}(t,x,y,x,y) - \frac{1}{2}\delta_{jk}\Delta_x[\mathcal{L}_{1,1}(t,x;x)]
\end{aligned}
\tag{5.4c}
$$

where δ_{jk} is the Kronecker delta. The quantity T_{00} is called the *total-particle-number density or mass density*, the quantity $T_{0j}=T_{j0}$ is called *total-particle-number current or, equivalently, momentum density*, and the quantity T_{jk} is the *stress tensor*[1]. For

---

[1] Since the Fock marginal density matrix is defined with a normalizing factor, then $N\cdot T_{00}$ is the true total-particle-number density. We lack a better name for T_{jk}.

a sufficiently smooth solution to (5.3), a short computation shows that

$$\begin{aligned}\partial_t T_{00} &= \frac{\partial}{\partial t}\int dudu' e^{i(u-u')\cdot x}\widehat{\mathcal{L}}_{1,1}(t,u;u') \\ &= i\int dudu'\ e^{i(u-u')\cdot x}(|u|^2-|u'|^2)\widehat{\mathcal{L}}_{1,1}(t,u;u') \\ &\quad + i\int dudu'\ e^{i(u-u')\cdot x}\widehat{B_v(\mathcal{L}_{2,2})}(t,u;u') \\ &= \nabla_x\cdot\int dudu'\ e^{i(u-u')\cdot x}(u+u')\widehat{\mathcal{L}}_{1,1}(t,u;u') = -2\partial_{x_j}T_{0j}\end{aligned}$$

where the last equality follows from $B_v(\mathcal{L}_{2,2})(x;x)=0$. Let us summarize the above result and introduce some useful notation in the next proposition.

Proposition 5.1 (local conservation of total particle number)**.** *Let $\mathcal{L}_{1,1}(t)$ be a sufficiently smooth solution to* (5.3)*. Then we have the following conservation law*

$$\frac{\partial\rho}{\partial t}+2\nabla_x\cdot\mathbf{p}=0 \tag{5.5}$$

where the total-particle-number current is given by

$$\mathbf{p}(t,x) = -\int dudu'\ e^{i(u-u')\cdot x}\frac{(u+u')}{2}\widehat{\mathcal{L}}_{1,1}(t,u;u'). \tag{5.6}$$

In fact, with a bit more work, we also obtain the following continuity equation for the total-particle-number current.

Proposition 5.2. *Let $\mathcal{L}_{1,1}(t)$ be a sufficiently smooth solution to* (5.3)*. Then we* *have the following continuity equation*

$$\partial_t p_j + \sum_k \partial_k \sigma_{kj} + l_j = 0 \tag{5.7}$$

where the source term l_j is given by

$$l_j = \frac{1}{2}\int dy\ v(x-y)\{\nabla_y \mathcal{L}_{2,2}(x,y;x,y) - \nabla_x \mathcal{L}_{2,2}(x,y;x,y)\}.$$

Since we will prove Proposition 5.2 directly from the Morawetz identity, let us postponed the proof till a later section.

We define the *Morawetz action associated to the observable* $a(x) \in C^\infty(\mathbb{R}^3)$ of the one-particle Fock marginal density to be[2]

$$M_a(t) := \int dx\ \nabla a(x) \cdot \mathbf{p}(t,x). \tag{5.8}$$

Then we have the following Morawetz identity for the one-particle Fock marginal density matrix.

---

[2] The definition comes from looking at the evolution of the expectation value of an observable a against our mass density. More precisely, using (5.5) and integration by parts, we get that

$$\partial_t \int dx\ a(x)\mathcal{L}_{1,1}(t,x;x)\ dx = \int dx\ \nabla a(x) \cdot \mathbf{p}_t(x).$$

Theorem 5.3 (Morawetz identity). *The following identity holds*

$$\begin{aligned}\partial_t M_a = & -\frac{1}{2}\int dx\ (\Delta_x\Delta_x a(x))\mathcal{L}_{1,1}(x;x) \\ & +\frac{1}{2}\int dxdy\ \Delta_x a(x)v(x-y)\mathcal{L}_{2,2}(x,y;x,y) \\ & +2\Re\int dxdx'\ \delta(x-x')\sum_{j,\ell}(\partial_{x_j}\partial_{x_\ell}a(x))\partial_{x_\ell}\partial_{x'_j}\mathcal{L}_{1,1}(x;x') \\ & +\frac{1}{2}\int dx\ \nabla_x a(x)\cdot\int dy\ v(x-y)\{\nabla_x\mathcal{L}_{2,2}(x,y;x,y)-\nabla_y\mathcal{L}_{2,2}(x,y;x,y)\}\end{aligned} \tag{5.9}$$

for the one-particle Fock marginal density.

Remark 5.4. The one-particle Morawetz identity for the Gross-Pitaevskii hierarchy has been considered in Theorem 3.1 of [CPT12]. In fact our result coincides with Theorem 3.1 when $v=\delta$ even though our contexts are different.

For the convenience of notations, we have suppressed all the t variable in our calculation since it only plays a passive role in the proof of Theorem 5.3. The main result of this chapter is to establish the following interaction Morawetz estimate for the one-particle Fock density matrix.

Theorem 5.5 (Interaction Morawetz-Type Estimate). *Let $\Gamma(t)$ be a sufficiently smooth global solution to (5.3). Then we have the following estimate*

$$\int dtdx\ |\Gamma(t,x;x)|^2 \lesssim 1 \tag{5.10}$$

which is uniform in N and depends only on the initial data.

5.2 Proof of Theorem 5.3

By direct calculation, we see that

$$\begin{aligned}\partial_t \mathbf{p}(x) &= \frac{1}{i}\int dudu' \; e^{i(u-u')\cdot x}\frac{(u+u')}{2}(u^2-(u')^2)\widehat{\mathcal{L}}_{1,1}(u,u') \\ &\quad + \frac{1}{i}\int dudu' \; e^{i(u-u')\cdot x}\frac{(u+u')}{2}\widehat{B_v(\mathcal{L}_{2,2})}(u,u') \\ &= \; -\frac{1}{2}\nabla_x \cdot \int dudu' \; e^{i(u-u')\cdot x}(u+u')\otimes(u+u')\widehat{\mathcal{L}}_{1,1}(u,u') \\ &\quad + \frac{1}{i}\int dudu' \; e^{i(u-u')\cdot x}\frac{(u+u')}{2}\widehat{B_v(\mathcal{L}_{2,2})}(u,u') \\ &=: J_1 + J_2.\end{aligned}$$

For the J_1 term, applyinf integration by parts yields

$$J_1 = \frac{1}{2}\int dxdudu' \; e^{i(u-u')\cdot x}\{\nabla^2 a(x)(u+u',u+u')\}\widehat{\mathcal{L}}_{1,1}(u;u').$$

Next, expand the expression into components and apply Fourier inversion, we see that

$$
\begin{aligned}
J_1 &= \frac{1}{2}\int dxdudu'\ e^{i(u-u')\cdot x}\left\{\sum_{i,j}(\partial_{x_i}\partial_{x_j}a(x))(u_i+u'_i)(u_j+u'_j)\right\}\widehat{\mathcal{L}}_{1,1}(u;u') \\
&= -\frac{1}{2}\int dxdx'\ \delta(x-x')\sum_{i,j}(\partial_{x_i}\partial_{x_j}a(x))(\partial_{x_i}-\partial_{x'_i})(\partial_{x_j}-\partial_{x'_j})\mathcal{L}_{1,1}(x;x') \\
&= -\frac{1}{2}\int dxdx'\ \delta(x-x')\sum_{i,j}(\partial_{x_i}\partial_{x_j}a(x))(\partial_{x_i}\partial_{x_j}+\partial_{x'_i}\partial_{x'_j})\mathcal{L}_{1,1}(x;x') \\
&\quad + \frac{1}{2}\int dxdx'\ \delta(x-x')\sum_{i,j}(\partial_{x_i}\partial_{x_j}a(x))(\partial_{x_i}\partial_{x'_j}+\partial_{x'_i}\partial_{x_j})\mathcal{L}_{1,1}(x;x') \\
&= -\frac{1}{2}\int dxdx'\ \delta(x-x')\sum_{i,j}(\partial_{x_i}\partial_{x_j}a(x))(\partial_{x_i}\partial_{x_j}+\partial_{x_i}\partial_{x'_j}+\partial_{x'_i}\partial_{x_j}+\partial_{x'_i}\partial_{x'_j})\mathcal{L}_{1,1}(x;x') \\
&\quad + \int dxdx'\ \delta(x-x')\sum_{i,j}(\partial_{x_i}\partial_{x_j}a(x))(\partial_{x_i}\partial_{x'_j}+\partial_{x'_i}\partial_{x_j})\mathcal{L}_{1,1}(x;x') \\
&= -\frac{1}{2}\int dx\ \sum_{i,j}(\partial_{x_i}\partial_{x_j}a(x))\partial_{x_i}\partial_{x_j}[\mathcal{L}_{1,1}(x;x)] \\
&\quad + 2\Re\int dxdx'\ \delta(x-x')\sum_{i,j}(\partial_{x_i}\partial_{x_j}a(x))\partial_{x_i}\partial_{x'_j}\mathcal{L}_{1,1}(x;x').
\end{aligned}
$$

Now, let us handle J_2 which contains $B_v(\mathcal{L}_{2,2})$. Assume v is radially symmetric. Then consider the Fourier transform of $B_v(\mathcal{L}_{2,2})$. For convenience, we decompose the operator into two parts

$$B_v = B_v^+ - B_v^- \tag{5.11}$$

where

$$B_v^+(\mathcal{L}_{2,2})(x_1;x_1') = \int dx_2 dx_2'\ v(x_1 - x_2)\delta(x_2 - x_2')\mathcal{L}_{2,2}(x_1, x_2, x_1', x_2') \tag{5.12a}$$

and

$$B_v^-(\mathcal{L}_{2,2})(x_1;x_1') = \int dx_2 dx_2'\ v(x_1' - x_2')\delta(x_2 - x_2')\mathcal{L}_{2,2}(x_1, x_2, x_1', x_2'). \tag{5.12b}$$

By direct calculations, we see that

$$\begin{aligned}
\widehat{B_v^+(\mathcal{L}_{2,2})}(\xi;\xi') &= \int dx_1 dx_1' dx_2 dx_2'\ e^{-i(x_1\cdot\xi - x_1'\cdot\xi')}v(x_1 - x_2)\delta(x_2 - x_2')\mathcal{L}_{2,2}(\mathbf{x}, \mathbf{x}') \\
&= \int dx_1 dx_1' dx_2 dx_2'\ e^{-i(x_1\cdot\xi - x_1'\cdot\xi')}\mathcal{L}_{2,2}(\mathbf{x}, \mathbf{x}') \\
&\quad \times \int d\xi_2'\ e^{-i(x_2 - x_2')\xi_2'} \int d\xi_2\ e^{i(x_1 - x_2)\xi_2}\widehat{v}(\xi_2) \\
&= \int d\xi_2 d\xi_2'\ \widehat{v}(\xi_2)\widehat{\mathcal{L}}_{2,2}(\xi - \xi_2, \xi_2 + \xi_2', \xi', \xi_2') \\
&= \int d\xi_2 d\xi_2'\ \widehat{v}(\xi_2 - \xi_2')\widehat{\mathcal{L}}_{2,2}(\xi - \xi_2 + \xi_2', \xi_2, \xi', \xi_2')
\end{aligned}$$

and, similarly, we also have

$$\widehat{B_v^-(\mathcal{L}_{2,2})}(\xi, \xi') = \int d\xi_2 d\xi_2'\ \widehat{v}(\xi_2' - \xi_2)\widehat{\mathcal{L}}_{2,2}(\xi, \xi_2, \xi' - \xi_2' + \xi_2, \xi_2').$$

Then it follows

$$
\begin{aligned}
&\frac{1}{i}\int dudu'\ e^{i(u-u')\cdot x}\frac{(u+u')}{2}\widehat{B_v(\mathcal{L}_{2,2})}(u;u') \qquad (5.13)\\
&= \frac{1}{i}\int dudu'd\xi_2 d\xi_2'\ e^{i(u-u')x}\frac{(u+u')}{2}\hat{v}(\xi_2-\xi_2')\hat{\mathcal{L}}_{2,2}(u-\xi_2+\xi_2',\xi_2;u',\xi_2')\\
&\quad -\frac{1}{i}\int dudu'd\xi_2 d\xi_2'\ e^{i(u-u')x}\frac{(u+u')}{2}\hat{v}(\xi_2'-\xi_2)\hat{\mathcal{L}}_{2,2}(u,\xi_2;u'-\xi_2'+\xi_2,\xi_2')
\end{aligned}
$$

Now, apply the change of variables $u \mapsto u-\xi_2+\xi_2'$ and $u' \mapsto u'-\xi_2+\xi_2'$ to the second term on the RHS of (5.13) yields

$$
\begin{aligned}
&\text{RHS of (5.13)}\\
&= \frac{1}{i}\int dudu'd\xi_2 d\xi_2'\ e^{i(u-u')\cdot x}(\xi_2-\xi_2')\widehat{v}(\xi_2-\xi_2')\widehat{\mathcal{L}}_{2,2}(u-\xi_2+\xi_2',\xi_2;u',\xi_2').
\end{aligned}
$$

Hence, let us focus on the term

$$
\begin{aligned}
&\frac{1}{i}\int dx\ \nabla a(x)\cdot\int dudu'd\xi_2 d\xi_2'\ e^{i(u-u')\cdot x}(\xi_2-\xi_2')\\
&\times\ \widehat{v}(\xi_2-\xi_2')\widehat{\mathcal{L}}_{2,2}(u-\xi_2+\xi_2',\xi_2;u',\xi_2').
\end{aligned} \qquad (5.14)
$$

We begin by rewriting (5.14) in physical coordinates

$$
\begin{aligned}
(5.14) = &\frac{1}{i}\int dx\ \nabla a(x)\cdot\int du du' d\xi_2 d\xi_2' \int dX dY dX' dY' e^{i(u-u')\cdot x}(\xi_2-\xi_2')\\
&\times \widehat{v}(\xi_2-\xi_2')e^{-i(u-\xi_2+\xi_2')\cdot X - i\xi_2 Y + iu'X' + i\xi_2' Y'}\mathcal{L}_{2,2}(X,Y;X',Y')\\
=&\frac{1}{i}\int dX dY dX' dY' \mathcal{L}_{2,2}(X,Y;X',Y')\int d\xi_2 d\xi_2'\\
&\times \delta(X-X')\nabla_X a(X)\cdot(\xi_2-\xi_2')\widehat{v}(\xi_2-\xi_2')e^{i(X-Y)\xi_2 - i(X-Y')\xi_2'}\\
=&-\int dX dY dY' \mathcal{L}_{2,2}(X,Y;X,Y')\\
&\times \nabla_X a(X)\cdot\nabla_X\int d\xi_2 d\xi_2'\ e^{i(X-Y)\xi_2 - i(X-Y')\xi_2'}\widehat{v}(\xi_2-\xi_2')\\
=&-\int dX dY dY' \mathcal{L}_{2,2}(X,Y;X,Y')\\
&\times \nabla_X a(X)\cdot\nabla_X\left[\delta(Y-Y')v\left(X-\frac{Y+Y'}{2}\right)\right]\\
=&-\int dX dY \mathcal{L}_{2,2}(X,Y;X,Y)\nabla_X a(X)\cdot\nabla_X v\,(X-Y)\\
=&\int dX dY\ v(X-Y)\{\Delta_X a(X)+\nabla_X a(X)\cdot\nabla_X\}\mathcal{L}_{2,2}(X,Y;X,Y).
\end{aligned}
$$

Combining the calculation for J_1 and J_2, we obtain the Morawetz identity

$$
\begin{aligned}
\partial_t M_a = &-\frac{1}{2}\int dx\ (\Delta_x\Delta_x a(x))\mathcal{L}_{1,1}(x;x)\\
&+\int dx dy\ v(x-y)\ \{\Delta_x a(x)+\nabla_x a(x)\cdot\nabla_x\}\mathcal{L}_{2,2}(x,y;x,y) \qquad (5.15)\\
&+2\Re\int dx dx'\ \delta(x-x')\sum_{j,\ell}(\partial_{x_j}\partial_{x_\ell}a(x))\partial_{x_\ell}\partial_{x_j'}\mathcal{L}_{1,1}(x;x')
\end{aligned}
$$

which completes the proof of Theorem 5.3. Moreover, if we define the pseudo stress-energy tensor

$$\begin{aligned} T_{jk} = & \int dx' \ \delta(x-x')(\partial_{x_k}\partial_{x'_j} + \partial_{x'_k}\partial_{x_j})\mathcal{L}_{1,1}(x;x') \\ & + \delta_{jk}\frac{1}{2}\{\int dy \ v(x-y)\mathcal{L}_{2,2,}(x,y;x,y) - \Delta[\mathcal{L}_{1,1}(x;x)]\}. \end{aligned}$$

then, by (5.9), we have the continuity equation

$$\partial_t T_{0j} + \sum_k \partial_k T_{kj} + l_j = 0$$

where

$$l_j = \frac{1}{2}\int dy \ v(x-y)\{\nabla_y \mathcal{L}_{2,2}(x,y;x,y) - \nabla_x \mathcal{L}_{2,2}(x,y;x,y)\}.$$

This completes the proof of Proposition 5.2.

It is instructive to consider the special case $v(x) = \delta(x)$, c.f. equations (19)-(20) in [BBC$^+$18]. If $v = \delta$, then we have the following one-particle Morawetz identity for the $\mathcal{L}_{1,1}$ density matrix

$$\begin{aligned} \partial_t M_a = & -\frac{1}{2}\int dx \ (\Delta_x \Delta_x a(x))\mathcal{L}_{1,1}(x;x) \\ & + \frac{1}{2}\int dx \ (\Delta_x a(x))\mathcal{L}_{2,2}(x,x;x,x) \\ & + 2\Re \int dx dx' \ \delta(x-x') \sum_{j,\ell}(\partial_{x_j}\partial_{x_\ell} a(x))\partial_{x_\ell}\partial_{x'_j}\mathcal{L}_{1,1}(x;x'). \end{aligned} \tag{5.16}$$

Again, from the Morawetz identity we can read off the T_{jk} components of the pseudo stress-energy tensor for $j, k = 1, 2, 3$, which are given by

$$\begin{aligned}\partial_t \mathbf{p}(x) = \frac{1}{2}\nabla\Delta[\mathcal{L}_{1,1}(x;x)] - \frac{1}{2}\nabla[\mathcal{L}_{2,2}(x,x;x,x)] \\ - 2\Re\int dx'\ \delta(x-x')(\nabla_{x'}\Delta_x + \nabla_x\Delta_{x'})\mathcal{L}_{1,1}(x;x')\end{aligned} \tag{5.17a}$$

or

$$\begin{aligned}\partial_t p_k(x) = \frac{1}{2}\partial_{x_k}\Delta_x[\mathcal{L}_{1,1}(x;x)] - \frac{1}{2}\partial_{x_k}[\mathcal{L}_{2,2}(x,x;x,x)] \\ - \sum_j \partial_{x_j}\int dxdx'\ \delta(x-x')(\partial_{x_k}\partial_{x'_j} + \partial_{x'_k}\partial_{x_j})\mathcal{L}_{1,1}(x;x').\end{aligned} \tag{5.17b}$$

Thus, we see that

$$\begin{aligned}T_{jk} = \int dx'\ \delta(x-x')(\partial_{x_k}\partial_{x'_j} + \partial_{x'_k}\partial_{x_j})\mathcal{L}_{1,1}(x;x') \\ + \delta_{jk}\frac{1}{2}\{\mathcal{L}_{2,2}(x,x;x,x) - \Delta[\mathcal{L}_{1,1}(x;x)]\}\end{aligned}$$

and

$$\partial_t T_{0j} + \sum_k \partial_k T_{kj} = 0.$$

Remark 5.6. The reader should note that when $v(x) = \delta(x)$ we have that $l_j \equiv 0$ since $\mathcal{L}_{2,2}(x,y;x,y) = \mathcal{L}_{2,2}(y,x;y,x)$. Hence, this shows that the T_{jk} are conserved for every fixed k.

5.3 Morawetz Estimates

In this section we use (5.9) to prove the classical Morawetz estimates for the time-dependent HFB system in 3D. Here, we consider the cases $v = \delta$ and v positive radially symmetric. In particular, the main results of this section are Proposition 5.7 and Proposition 5.8.

Case 1: $v(x) = \delta(x)$

Multiplying (5.5) by $a(x)$ and integrate in space yields

$$\begin{aligned}&\partial_t \int dx\ a(x)\mathcal{L}_{1,1}(x;x)\\&+\frac{1}{2}\int dxdx'\ \delta(x-x')a(x)(\nabla_x+\nabla_{x'})\cdot(\nabla_{x'}-\nabla_x)\mathcal{L}_{1,1}(x;x')=0\end{aligned} \tag{5.18}$$

Next, apply integration by parts, we get that

$$\partial_t \int dx\ a(x)\mathcal{L}_{1,1}(x;x)=\frac{1}{2}\int dx\ \nabla a(x)\cdot[(\nabla_{x'}-\nabla_x)\mathcal{L}_{1,1}](x;x).$$

Then, by (5.16), it follows we have that

$$\begin{aligned}\partial_t^2\int dx\ a(x)\mathcal{L}_{1,1}(x;x)=&-\frac{1}{2}\int dx\ (\Delta_x\Delta_x a(x))\mathcal{L}_{1,1}(x;x)\\&+\frac{1}{2}\int dx\ (\Delta_x a(x))\mathcal{L}_{2,2}(x,x;x,x)\\&+2\Re\int dxdx'\ \delta(x-x')\sum_{j,\ell}(\partial_{x_j}\partial_{x_\ell}a(x))\partial_{x_\ell}\partial_{x'_j}\mathcal{L}_{1,1}(x;x').\end{aligned} \tag{5.19}$$

If $a(x) = |x|^2$, then we have obtained the following viriel identity

$$
\begin{aligned}
&\partial_t^2 \int dx\ |x|^2 \mathcal{L}_{1,1}(x;x) \\
&= \int dx\ \mathcal{L}_{2,2}(x,x;x,x) + 4 \sum_{j=1} \int dx\ [\partial_{x_j} \partial_{x'_j} \mathcal{L}_{1,1}](x;x).
\end{aligned}
\tag{5.20}
$$

If $a(x) = |x|$, then by simple calculations using the Green's identities we can show that the relevant distributional derivatives of $a(x)$ in $\mathbb{R}^3$ are given by

$$
\Delta |x| = \frac{2}{|x|} \quad \text{and} \quad \Delta^2 |x| = -8\pi \delta(x).
$$

Hence we have

$$
\begin{aligned}
&\partial_t^2 \int dx\ |x| \mathcal{L}_{1,1}(x;x) \\
&= 4\pi \mathcal{L}_{1,1}(t,0,0) + \int dx\ \frac{\mathcal{L}_{2,2}(x,x;x,x)}{|x|} \\
&\quad + \int dx\ \sum_{j,\ell} \left(\frac{\delta_{j\ell}}{|x|} - \frac{x_j x_\ell}{|x|^3} \right) \int dx'\ \delta(x-x') \{ \partial_{x_j} \partial_{x'_\ell} + \partial_{x'_j} \partial_{x_\ell} \} \mathcal{L}_{1,1}(x;x').
\end{aligned}
\tag{5.21a}
$$

Define the matrix

$$
A_{ij} = \int dx'\ \delta(x-x') \{ \partial_{x_j} \partial_{x'_\ell} + \partial_{x'_j} \partial_{x_\ell} \} \mathcal{L}_{1,1}(x;x')
$$

then we can rewrite (5.21a) as follows

$$\begin{aligned}&\partial_t \int dx\ \frac{x}{|x|}\cdot[(\nabla_{x'}-\nabla_x)\mathcal{L}_{1,1}](x;x)\\ =\ & 4\pi\mathcal{L}_{1,1}(t,0,0)+\int dx\ \frac{\mathcal{L}_{2,2}(x,x;x,x)}{|x|}+2\int dx\ \frac{\operatorname{Tr}A-u^TAu}{|x|}\end{aligned} \tag{5.21b}$$

where $u=\frac{x}{|x|}$. Integrate (5.21b) with respect to t yields

$$\begin{aligned}&4\pi\int_{-T}^{T}\mathcal{L}_{1,1}(t,0,0)\ dt+\int_{-T}^{T}\int\frac{\mathcal{L}_{2,2}(t,x,x,x,x)}{|x|}+2\frac{\operatorname{Tr}A-u^TAu}{|x|}\ dxdt\\ =\ &\int dx\ \frac{x}{|x|}\cdot[(\nabla_{x'}-\nabla_x)\mathcal{L}_{1,1}](T,x,x)-\int dx\ \frac{x}{|x|}\cdot[(\nabla_{x'}-\nabla_x)\mathcal{L}_{1,1}](-T,x,x).\end{aligned}$$

Recall, by Lemma 4.4 in [GM17], we have

$$\mathcal{L}_{1,1}(x;x')=\frac{1}{N}\overline{\operatorname{sh}}\circ\operatorname{sh}(x;x')+\bar{\phi}(x)\phi(x')=\Gamma(x;x') \tag{5.22a}$$

and

$$\begin{aligned}\mathcal{L}_{2,2}(t,x_1,x_2;x_1',x_2')=\ &\frac{1}{2}\Gamma(x_1;x_1')\Gamma(x_2;x_2')+\frac{1}{2}\Gamma(x_1;x_2')\Gamma(x_2;x_1')\\ &+\frac{1}{2}\Gamma(x_2;x_1')\Gamma(x_1;x_2')+\frac{1}{2}\Gamma(x_2;x_2')\Gamma(x_1;x_1')\\ &+\overline{\Lambda}(x_1,x_2)\Lambda(x_1',x_2')-2\bar{\phi}(x_1)\bar{\phi}(x_2)\phi(x_1')\phi(x_2').\end{aligned} \tag{5.22b}$$

In particular, it follows that

$$\begin{aligned}&\mathcal{L}_{2,2}(t,x,y;x,y)\\ &=|\Lambda(x,y)|^2+\Gamma(x;x)\Gamma(y;y)+|\Gamma(x;y)|^2-2|\phi(x)|^2|\phi(y)|^2\end{aligned} \tag{5.23a}$$

and

$$\mathcal{L}_{2,2}(t,x,x;x,x) = |\Lambda(x,x)|^2 + 2|\Gamma(x;x)|^2 - 2|\phi(x)|^4. \tag{5.23b}$$

Denote $u := \mathrm{sh}(k)$ and write

$$T_{00} = |\phi(t,x)|^2 + \frac{1}{N}\int dy\ |u(x,y)|^2 \tag{5.24a}$$

$$T_{0j} = \mathrm{Im}\left(\bar{\phi}(x)\partial_j\phi(x)\right) + \frac{1}{N}\int dy\ \ \mathrm{Im}\left(\bar{u}(x,y)\partial_j u(x,y)\right). \tag{5.24b}$$

Then observe we have

$$\begin{aligned} T_{jk} = &\frac{1}{N}\int dy\ \{\partial_{x_j}\bar{u}(x,y)\partial_{x_k}u(x,y) + \partial_{x_k}\bar{u}(x,y)\partial_{x_j}u(x,y)\} \\ &+\ \bar{\phi}_j(x)\phi_k(x) + \bar{\phi}_k(x)\phi_j(x) \\ &+\ \frac{1}{2}\delta_{jk}\{\mathcal{L}_{2,2}(x,x;x,x) - \Delta_x\Gamma(x;x)\}. \end{aligned} \tag{5.24c}$$

By (5.24b) we have

$$\begin{aligned} &\int dx\ \frac{x}{|x|}\cdot[(\nabla_{x'} - \nabla_x)\Gamma](t,x;x) \\ &= \int dxdz\,\mathrm{Im}\left(u(x)\frac{x_j}{|x|}\partial_{x_j}\bar{u}(x)\right) + \int dx\,\mathrm{Im}\left(\phi(x)\frac{x_j}{|x|}\partial_{x_j}\phi(x)\right), \end{aligned}$$

then, by Proposition A.10 in [Tao06] and the conservation law in [GM13a], we establish the estimate

$$\left|\int dx\ \frac{x}{|x|}\cdot[(\nabla_{x'} - \nabla_x)\Gamma](t,x;x)\right| \lesssim\ \|\,|\nabla_{x,y}|^{\frac{1}{2}}\Gamma_0\,\|^2_{L^2(dxdy)}$$

which means

$$\int dtdx \, \frac{\mathcal{L}_{2,2}(t,x,x;x,x)}{|x|} \lesssim 1 \tag{5.25}$$

since $\operatorname{Tr} A - u^T A u \geq 0$. Finally, by (5.23b) and the fact that $\Gamma(x;x) \geq |\phi(x)|^2$, we obtain the estimate

$$\int dtdx \, \frac{|\Lambda(t,x;x)|^2}{|x|} \lesssim 1 \tag{5.26a}$$

and

$$\frac{1}{N^2} \int dtdx \, \frac{|(\bar{u} \circ u)(t,x;x)|^2}{|x|} \lesssim 1 \tag{5.26b}$$

Hence we have proven the desired Morawetz estimate when $v = \delta$.

Similarly, if we consider the observable $a(x) = |x - z|$, then by (5.19) we have

$$\begin{aligned} &\partial_t^2 \int dx \, |x-z| \mathcal{L}_{1,1}(x;x) \qquad (5.27) \\ &= 4\pi \mathcal{L}_{1,1}(t,z,z) + \int dx \, \frac{\mathcal{L}_{2,2}(x,x;x,x)}{|x-z|} \\ &\quad + \int dx \, \sum_{j,\ell} \left(\frac{\delta_{j\ell}}{|x-z|} - \frac{x_j x_\ell}{|x-z|^3} \right) \int dx' \, \delta(x-x') \{ \partial_{x_j} \partial_{x'_\ell} + \partial_{x'_j} \partial_{x_\ell} \} \mathcal{L}_{1,1}(x;x'). \end{aligned}$$

which again leads to

$$\sup_z \int dtdx \, \frac{|\Lambda(t,x;x)|^2}{|x-z|} \lesssim 1 \tag{5.28a}$$

and

$$\sup_z \frac{1}{N^2} \int dtdx\ \frac{|(\bar{u} \circ u)(t,x;x)|^2}{|x-z|} \lesssim 1. \tag{5.28b}$$

Note, we also have

$$\sup_z \int dt\ \Gamma(t,z,z) \lesssim 1. \tag{5.29}$$

Let us summarize our result in the following proposition.

Proposition 5.7. *Let $(\phi(t), \Gamma(t), \Lambda(t))$ be a smooth solution to the time-dependent HFB system corresponding to the case $v(x) = \delta(x)$. Then we have the global estimates*

$$\sup_z \int dtdx\ \frac{|\Lambda(t,x;x)|^2 + 2|\Gamma(t,x;x)|^2 - 2|\phi(t,x)|^4}{|x-z|} \lesssim 1$$

and

$$\sup_z \int dt\ \Gamma(t,z,z) \lesssim 1.$$

Case 2: $v(x)$ positive radial

Let us assume $v(x) = v(|x|)$ and v is monotone decreasing away from the origin.

Take $a(x) = |x|$ in (5.15), we obtain the estimate

$$
\begin{aligned}
\partial_t M_a \geq 4\pi \mathcal{L}_{1,1}(0,0) + \int dxdy \ \frac{v(x-y)\mathcal{L}_{2,2}(x,y;x,y)}{|x|} \\
+ \frac{1}{2} \int dx \ \frac{x}{|x|} \cdot \int dy \ v(x-y)\{\nabla_x \mathcal{L}_{2,2}(x,y,x,y) - \nabla_y \mathcal{L}_{2,2}(x,y,x,y)\}
\end{aligned}
$$

Next, integrating by parts yields

$$
\begin{aligned}
& \frac{1}{2} \int dxdy \ v(x-y) \frac{x}{|x|} \cdot \{\nabla_x \mathcal{L}_{2,2}(x,y,x,y) - \nabla_y \mathcal{L}_{2,2}(x,y,x,y)\} \\
& = -\frac{1}{2} \int dxdy \ \left(\nabla_x \cdot \left(v(x-y) \frac{x}{|x|} \right) + \nabla_x (v(x-y)) \frac{x}{|x|} \right) \mathcal{L}_{2,2}(x,y,x,y) \\
& = \ -\int dxdy \ \left[v'(|x-y|) \frac{(x-y) \cdot x}{|x-y||x|} + v(x-y) \frac{1}{|x|} \right] \mathcal{L}_{2,2}(x,y,x,y)
\end{aligned}
$$

since v is radial, that is, $v(x-y) = v(|x-y|)$. By direct calculation, we see that

$$
\begin{aligned}
& \int dxdy \ \frac{v(x-y)\mathcal{L}_{2,2}(x,y;x,y)}{|x|} \\
& + \frac{1}{2} \int dx \ \frac{x}{|x|} \cdot \int dy \ v(x-y)\{\nabla_x \mathcal{L}_{2,2}(x,y,x,y) - \nabla_y \mathcal{L}_{2,2}(x,y,x,y)\} \\
& = \ -\int dxdy \ v'(|x-y|) \frac{(x-y) \cdot x}{|x-y||x|} \mathcal{L}_{2,2}(x,y,x,y) \\
& = \ -\left[\int_{x<y} + \int_{x>y} \right] dxdy \ v'(|x-y|) \frac{(x-y) \cdot x}{|x-y||x|} \mathcal{L}_{2,2}(x,y,x,y) \\
& = \ -\int_{x>y} dxdy \ v'(|x-y|) \left(\frac{(x-y) \cdot x}{|x-y||x|} + \frac{(y-x) \cdot y}{|x-y||y|} \right) \mathcal{L}_{2,2}(x,y,x,y) \\
& = \ -\int_{x>y} dxdy \ v'(|x-y|) \left(\frac{|x| - |y| \cos\theta}{|x-y|} + \frac{|y| - |x| \cos\theta}{|x-y|} \right) \mathcal{L}_{2,2}(x,y,x,y) \geq 0
\end{aligned}
$$

Thus, by conservation of energy, we obtain the estimate

$$\int_{-\infty}^{\infty} d\tau \ \Gamma(\tau, 0, 0) \lesssim |M(T)| + |M(-T)| \lesssim 1.$$

Taking $a(x) = |x - z|$, one can deduce

$$\sup_z \int dt \ \Gamma(t, z, z) \lesssim 1. \tag{5.30}$$

Let us summarize our findings

Proposition 5.8. *Let $(\phi(t), \Gamma(t), \Lambda(t))$ be a smooth solution to the time-dependent HFB system corresponding to a positive radially symmetric interaction potential v. Then we have the global estimate*

$$\sup_z \int dt \ \Gamma(t, z, z) \lesssim 1.$$

Unlike the case $v = \delta$, we were not able to obtain Morawetz-type estimate for the Λ equation. The lack of localization of v introduces the source term l_j in (5.7) which make the the previous approach with of the δ infeasible in the current setting.

5.4 Proof of the Interaction Morawetz Estimate for Γ

Case 1: $v(x) = \delta(x)$

Let us write down the Virial interaction potential V^a corresponding to $a(x)$ as

$$V^a(t) := \int dxdy\ T_{00}(t,x)T_{00}(t,y)a(x-y) \tag{5.31}$$

and let us define the Morawetz interaction potential $M^a := \partial_t V^a$ or

$$\begin{aligned}
&M^a(t) := \partial_t V^a(t)\\
&= \int dxdy\ [\partial_t T_{00}(t,x)T_{00}(t,y)a(x-y) + T_{00}(t,x)\partial_t T_{00}(t,y)]a(x-y)\\
&= -2\int dxdy\ [\partial_j T_{0j}(t,x)T_{00}(t,y)a(x-y) + T_{00}(t,x)\partial_j T_{0j}(t,y)]a(x-y)\\
&= 2\int dxdy\ [T_{00}(t,y)T_{0j}(t,x) - T_{0j}(t,y)T_{00}(t,x)]a_j(x-y)
\end{aligned}$$

Thus it follows

$$\partial_t M^a(t) = 2\int dxdy\ (T_{00}(x)T_{jk}(y) - 4T_{0j}(x)T_{0k}(y) + T_{jk}(x)T_{00}(y))a_{jk}(x-y) \tag{5.32}$$

provided $a(x)$ is even. First, let us consider the integrand of the off-diagonal terms

$$\begin{aligned}
2\Bigg(&\Gamma(x,x)\left[\frac{2}{N}\int dz\,\mathrm{Re}(\partial_{y_j}\bar{u}(y,z)\partial_{y_k}u(y,z)) + 2\,\mathrm{Re}(\bar{\phi}_j(y)\phi_k(y))\right]\\
&-4\left[\frac{1}{N}\int dz\;\;\mathrm{Im}(\bar{u}(x,z)\partial_{x_j}u(x,z)) + \mathrm{Im}(\bar{\phi}(x)\phi_j(x))\right]\\
&\;\times\left[\frac{1}{N}\int dz\;\;\mathrm{Im}(\bar{u}(y,z)\partial_{y_k}u(y,z)) + \mathrm{Im}(\bar{\phi}(y)\phi_k(y))\right]\\
&+\Gamma(y,y)\left[\frac{2}{N}\int dz\,\mathrm{Re}(\partial_{x_j}\bar{u}(x,z)\partial_{x_k}u(x,z)) + 2\,\mathrm{Re}(\bar{\phi}_j(x)\phi_k(x))\right]\Bigg)a_{jk}(x-y)
\end{aligned}$$

which have good terms given by

$$\phi(x)\bar{\phi}(x)(\bar{\phi}_j(y)\phi_k(y) + \phi_j(y)\bar{\phi}_k(y)) \tag{5.33a}$$

$$+\,(\bar{\phi}(x)\phi_j(x) - \phi(x)\bar{\phi}_j(x))(\bar{\phi}(y)\phi_k(y) - \phi(x)\bar{\phi}_k(y)) \tag{5.33b}$$

$$+\;\;\phi(y)\bar{\phi}(y)(\bar{\phi}_j(x)\phi_k(x) + \phi_j(x)\bar{\phi}_k(x)) \tag{5.33c}$$

$$+\;\;\frac{1}{N^2}\int dzdz'\;\{\partial_{y_j}\bar{u}(y,z)\partial_{y_k}u(y,z) + \partial_{y_j}u(y,z)\partial_{y_k}\bar{u}(y,z)\}u(x,z')\bar{u}(x,z') \tag{5.33d}$$

$$\begin{aligned}
+\;\;&\frac{1}{N^2}\int dzdz'\;\{\bar{u}(x,z')\partial_{x_j}u(x,z') - u(x,z')\partial_{x_j}\bar{u}(x,z')\}\\
&\quad\times\{\bar{u}(y,z)\partial_{y_k}u(y,z) - u(y,z)\partial_{y_k}\bar{u}(y,z)\}
\end{aligned} \tag{5.33e}$$

$$+\;\;\frac{1}{N^2}\int dzdz'\;\{\partial_{x_j}\bar{u}(x,z')\partial_{x_k}u(x,z') + \partial_{x_j}u(x,z')\partial_{x_k}\bar{u}(x,z')\}u(y,z)\bar{u}(y,z) \tag{5.33f}$$

and troublesome terms given by

$$(\bar{\phi}_j(y)\phi_k(y)+\phi_j(y)\bar{\phi}_k(y))\frac{1}{N}\int dz\ \bar{u}(x,z)u(x,z) \tag{5.34a}$$

$$+\,\phi(x)\bar{\phi}(x)\frac{1}{N}\int dz\ \{\partial_{y_j}\bar{u}(y,z)\partial_{y_k}u(y,z)+\partial_{y_j}u(y,z)\partial_{y_k}\bar{u}(y,z)\} \tag{5.34b}$$

$$+\,(\bar{\phi}(x)\phi_j(x)-\phi(x)\bar{\phi}_j(x))\frac{1}{N}\int dz\ \{\bar{u}(y,z)\partial_{y_k}u(y,z)-u(y,z)\partial_{y_k}\bar{u}(y,z)\} \tag{5.34c}$$

$$+\,(\bar{\phi}(y)\phi_k(y)-\phi(y)\bar{\phi}_k(y))\frac{1}{N}\int dz\ \{\bar{u}(x,z)\partial_{x_j}u(x,z)-u(x,z)\partial_{x_j}\bar{u}(x,z)\} \tag{5.34d}$$

$$+\,(\bar{\phi}_j(x)\phi_k(x)+\phi_j(x)\bar{\phi}_k(x))\frac{1}{N}\int dz\ \bar{u}(y,z)u(y,z) \tag{5.34e}$$

$$+\,\phi(y)\bar{\phi}(y)\frac{1}{N}\int dz\ \{\partial_{x_j}\bar{u}(x,z)\partial_{x_k}u(x,z)+\partial_{x_j}u(x,z)\partial_{x_k}\bar{u}(x,z)\}. \tag{5.34f}$$

For the good terms, the ϕ terms, (5.33a)-(5.33c), could be rewritten as follows

$$(5.33a)+(5.33b)+(5.33c)=P_j(x,y)\overline{P_k(x,y)}+Q_j(x,y)\overline{Q_k(x,y)} \tag{5.35a}$$

where

$$P_j(x,y):=\phi(x)\bar{\phi}_j(y)+\phi_j(x)\bar{\phi}(y)$$

and

$$Q_j(x,y):=\phi(x)\phi_j(y)-\phi_j(x)\phi(y).$$

Likewise, the u terms, (5.33d)-(5.33f), could also be rewritten as

$$
\begin{aligned}
&(5.33\text{d}) + (5.33\text{e}) + (5.33\text{f}) \\
&= \frac{1}{N^2} \int dz dz' \; \{R_j(z,z',x,y)\overline{R_k(z,z',x,y)} + S_j(z,z',x,y)\overline{S_k(z,z',x,y)}\}
\end{aligned}
\tag{5.35b}
$$

where

$$R_j(z,z',x,y) := u(x,z')\partial_{y_j}\bar{u}(y,z) + \partial_{x_j}u(x,z')\bar{u}(y,z)$$

and

$$S_j(z,z',x,y) := u(x,z')\partial_{y_j}u(y,z) - \partial_{x_j}u(x,z')u(y,z).$$

For the troublesome (mixed) terms, let us consider their integrand, that is

$$(\bar{\phi}_j(y)\phi_k(y) + \phi_j(y)\bar{\phi}_k(y))\bar{u}(x,z)u(x,z) \tag{5.36a}$$

$$+ \phi(x)\bar{\phi}(x)\{\partial_{y_j}\bar{u}(y,z)\partial_{y_k}u(y,z) + \partial_{y_j}u(y,z)\partial_{y_k}\bar{u}(y,z)\} \tag{5.36b}$$

$$+ \frac{1}{2}(\bar{\phi}(x)\phi_j(x) - \phi(x)\bar{\phi}_j(x))\{\bar{u}(y,z)\partial_{y_k}u(y,z) - u(y,z)\partial_{y_k}\bar{u}(y,z)\} \tag{5.36c}$$

$$+ \frac{1}{2}(\bar{\phi}(y)\phi_k(y) - \phi(y)\bar{\phi}_k(y))\{\bar{u}(x,z)\partial_{x_j}u(x,z) - u(x,z)\partial_{x_j}\bar{u}(x,z)\} \tag{5.36d}$$

$$+ \frac{1}{2}(\bar{\phi}(y)\phi_j(y) - \phi(y)\bar{\phi}_j(y))\{\bar{u}(x,z)\partial_{x_k}u(x,z) - u(x,z)\partial_{x_k}\bar{u}(x,z)\} \tag{5.36e}$$

$$+ \frac{1}{2}(\bar{\phi}(x)\phi_k(x) - \phi(x)\bar{\phi}_k(x))\{\bar{u}(y,z)\partial_{y_j}u(y,z) - u(y,z)\partial_{y_j}\bar{u}(y,z)\} \tag{5.36f}$$

$$+ (\bar{\phi}_j(x)\phi_k(x) + \phi_j(x)\bar{\phi}_k(x))\bar{u}(y,z)u(y,z) \tag{5.36g}$$

$$+ \phi(y)\bar{\phi}(y)\{\partial_{x_j}\bar{u}(x,z)\partial_{x_k}u(x,z) + \partial_{x_j}u(x,z)\partial_{x_k}\bar{u}(x,z)\}. \tag{5.36h}$$

Define the terms

$$A_j^{\pm}(x,y) := u(x)\bar{\phi}_j(y) \pm \bar{u}(x)\phi_j(y) \tag{5.37a}$$

$$B_j^{\pm}(x,y) := A_j^{\pm}(y,x) \tag{5.37b}$$

$$C_j^{\pm}(x,y) := \bar{\phi}(x)\partial_{y_j}u(y) \pm \phi(x)\partial_{y_j}\bar{u}(y) \tag{5.37c}$$

$$D_j^{\pm}(x,y) := C_j^{\pm}(y,x) \tag{5.37d}$$

$$E_j^{\pm}(x,y) := \bar{u}(x)\bar{\phi}_j(y) \pm u(x)\phi_j(y) \tag{5.37e}$$

$$F_j^{\pm}(x,y) := E_j^{\pm}(y,x) \tag{5.37f}$$

$$G_j^{\pm}(x,y) := \partial_{x_j}\bar{u}(x)\bar{\phi}(y) \pm \partial_{x_j}u(x)\phi(y) \tag{5.37g}$$

$$H_j^{\pm}(x,y) := G_j^{\pm}(y,x). \tag{5.37h}$$

By direct computation, one can check that (5.36) can be written as follows

$$(5.36) = \frac{1}{4}(A_j^+ + D_j^+)\overline{(A_k^+ + D_k^+)} + \frac{1}{4}(A_j^- + D_j^-)\overline{(A_k^- + D_k^-)} \tag{5.38a}$$

$$+ \frac{1}{4}(B_j^+ + C_j^+)\overline{(B_k^+ + C_k^+)} + \frac{1}{4}(B_j^- + C_j^-)\overline{(B_k^- + C_k^-)} \tag{5.38b}$$

$$+ \frac{1}{4}(E_j^+ - G_j^+)\overline{(E_k^+ - G_k^+)} + \frac{1}{4}(E_j^- - G_j^-)\overline{(E_k^- - G_k^-)} \tag{5.38c}$$

$$+ \frac{1}{4}(F_j^+ - H_j^+)\overline{(F_k^+ - H_k^+)} + \frac{1}{4}(F_j^- - H_j^-)\overline{(F_k^- - H_k^-)}. \tag{5.38d}$$

For the strictly-diagonal terms, we have that the $\mathcal{L}_{2,2}$ term is given by

$$\Delta a(x-y)\{\Gamma(x,x)\mathcal{L}_{2,2}(y,y,y,y) + \Gamma(y,y)\mathcal{L}_{2,2}(x,x,x,x)\} \tag{5.39a}$$

and the Laplacian term

$$-\Delta a(x-y)\{\Gamma(x,x)\Delta_y\Gamma(y,y)+\Gamma(y,y)\Delta_x\Gamma(x,x)\}. \tag{5.39b}$$

Hence combining (5.35), (5.38), and(5.39) yields the following Morawetz Identity

$$\begin{aligned}
\partial_t M^a(t) = {}& 2\int dxdy\ \Gamma(t,x,x)\Gamma(t,y,y)(-\Delta\Delta a(x-y)) \qquad (5.40)\\
&+\int dxdy\ \{\Gamma(t,x,x)\mathcal{L}_{2,2}(t,y,y,y,y)+\Gamma(t,y,y)\mathcal{L}_{2,2}(t,x,x,x,x)\}\Delta a(x-y)\\
&+2\int dxdy\ \{P_j(t,x,y)\overline{P_k(t,x,y)}+Q_j(t,x,y)\overline{Q_k(t,x,y)}\}a_{jk}(x-y)\\
&+\frac{2}{N^2}\int dxdydzdz'\ \{R_j(z,z',x,y)\overline{R_k(z,z',x,y)}\\
&+S_j(z,z',x,y)\overline{S_k(z,z',x,y)}\}a_{jk}(x-y)\\
&+\frac{1}{2N}\int dxdydz\ \{(A_j^+ + D_j^+)\overline{(A_k^+ + D_k^+)}+(A_j^- + D_j^-)\overline{(A_k^- + D_k^-)}\\
&+(B_j^+ + C_j^+)\overline{(B_k^+ + C_k^+)}+(B_j^- + C_j^-)\overline{(B_k^- + C_k^-)}\\
&+(E_j^+ - G_j^+)\overline{(E_k^+ - G_k^+)}+(E_j^- - G_j^-)\overline{(E_k^- - G_k^-)}\\
&+(F_j^+ - H_j^+)\overline{(F_k^+ - H_k^+)}+(F_j^- - H_j^-)\overline{(F_k^- - H_k^-)}\}a_{jk}(x-y).
\end{aligned}$$

Take $a(x)=|x|$. Then we obtain the estimate

$$\partial_t M(t) \gtrsim \int dx\ |\Gamma(t,x,x)|^2+\int dxdy\ \frac{\Gamma(t,x,x)\mathcal{L}_{2,2}(t,y,y,y,y)}{|x-y|} \tag{5.41}$$

In particular, we have that

$$\int_{-T}^{T} dt \int dx\ |\Gamma(t,x,x)|^2 \lesssim M(T) - M(-T)$$

and

$$\int_{-T}^{T} dt \int dxdy\ \frac{\Gamma(t,x,x)\mathcal{L}_{2,2}(t,y,y,y,y)}{|x-y|} \lesssim M(T) - M(-T).$$

Finally, observe by Proposition A.10 in [Tao06] we have

$$\begin{aligned} |M(t)| \leq \frac{4}{N}&\left|\int dxdydz\ \Gamma(y,y)\operatorname{Im}\left(\bar{u}(x,z)\frac{x_j-y_j}{|x-y|}\partial_j u(x,z)\right)\right| \\ &+ 4\left|\int dxdy\ \Gamma(y,y)\operatorname{Im}\left(\bar{\phi}(x)\frac{x_j-y_j}{|x-y|}\partial_j\phi(x)\right)\right| \\ &\lesssim \int dy\ \Gamma(t,y,y)\left\{\frac{1}{N}\int dz\ \|\, u(t,\cdot,z)\,\|^2_{\dot{H}^{1/2}_x} + \|\,\phi(t)\,\|^2_{\dot{H}^{1/2}_x}\right\}\end{aligned}$$

which means

$$\int_{-T}^{T} dt \int dx\ |\Gamma(t,x,x)|^2 \lesssim \sup_{t=T,-T}\|\,\Gamma(t)\,\|_{L^1_x}\left\{\frac{1}{N}\int dz\ \|\,u(t,\cdot,z)\,\|^2_{\dot{H}^{1/2}_x} + \|\,\phi(t)\,\|^2_{\dot{H}^{1/2}_x}\right\}$$

By the conservation laws in [GM13a], we see that indeed

$$\|\,\Gamma(t,x,x)\,\|^2_{L^2(dtdx)} \lesssim 1. \tag{5.42}$$

Likewise, we also have

$$\int dtdxdy\ \frac{\Gamma(t,x,x)\mathcal{L}_{2,2}(t,y,y,y,y)}{|x-y|} \lesssim 1, \tag{5.43a}$$

equivalently,

$$\int dtdxdy\ \frac{\Gamma(t,x,x)\{|\Lambda(t,y,y)|^2 + 2|\Gamma(t,y,y)|^2 - 2|\phi(t,y)|^4\}}{|x-y|} \lesssim 1. \tag{5.43b}$$

Case 2: $v(x)$ positive radial

As in the previous case, we write down the Morawetz interaction potential

$$M^a(t) = 2\int dxdy\ (T_{00}(t,y)T_{0j}(t,x) - T_{0j}(t,y)T_{00}(t,x))a_j(x-y)$$

then observe

$$\begin{aligned}\partial_t M^a(t) = 2\int dxdy\ &(\partial_t T_{00}(t,y)T_{0,j}(t,x) + T_{00}(t,y)\partial_t T_{0j}(t,x)\\ &- \partial_t T_{0j}(t,y)T_{00}(t,x) - T_{0j}(t,x)\partial_t T_{00}(t,x))a_j(x-y)\\ = 2\int dxdy\ &(-2\partial_k T_{0k}(t,y)T_{0j}(t,x) - T_{00}(t,y)(\partial_k T_{kj}(t,x) + l_j(t,x))\\ &+ (\partial_k T_{kj}(t,y) + l_j(t,y))T_{00}(t,x) + 2T_{0j}(t,x)\partial_k T_{0k}(t,x))a_j(x-y).\end{aligned}$$

Hence we obtain the following identity

$$\begin{aligned}\partial_t M^a(t) = 2\int dxdy\ &\{T_{00}(x)T_{jk}(y) - 2T_{0j}(x)T_{0k}(y)\\ &- 2T_{0j}(y)T_{0k}(x) + T_{jk}(x)T_{00}(y)\}a_{jk}(x-y)\ \ \text{(main term)}\\ &+\ 2\int dxdy\ \{l_j(t,y)T_{00}(t,x) - l_j(t,x)T_{00}(t,y)\}a_j(x-y)\ \ \text{(error term)}\end{aligned} \tag{5.44}$$

Applying a similar calculation as in the case when $v(x) = \delta(x)$, we see that

$$\begin{aligned}\text{main term }\ = 2\int &dxdy\ \Gamma(t,x,x)\Gamma(t,y,y)(-\Delta\Delta a(x-y))\\ &+ \int dxdy\ \{\Gamma(t,x,x)\int dz\ v(y-z)\mathcal{L}_{2,2}(t,y,z,y,z)\\ &+ \Gamma(t,y,y)\int dz\ v(x-z)\mathcal{L}_{2,2}(t,x,z,x,z)\}\Delta a(x-y)\\ &+ 2\int dxdy\ \{P_j(t,x,y)\overline{P_k(t,x,y)} + Q_j(t,x,y)\overline{Q_k(t,x,y)}\}a_{jk}(x-y)\\ &+ \frac{2}{N^2}\int dxdydzdz'\ \{R_j(z,z',x,y)\overline{R_k(z,z',x,y)}\\ &+ S_j(z,z',x,y)\overline{S_k(z,z',x,y)}\}a_{jk}(x-y)\\ &+ \frac{1}{2N}\int dxdydz\ \{(A_j^+ + D_j^+)\overline{(A_k^+ + D_k^+)} + (A_j^- + D_j^-)\overline{(A_k^- + D_k^-)}\\ &+ (B_j^+ + C_j^+)\overline{(B_k^+ + C_k^+)} + (B_j^- + C_j^-)\overline{(B_k^- + C_k^-)}\\ &+ (E_j^+ - G_j^+)\overline{(E_k^+ - G_k^+)} + (E_j^- - G_j^-)\overline{(E_k^- - G_k^-)}\\ &+ (F_j^+ - H_j^+)\overline{(F_k^+ - H_k^+)} + (F_j^- - H_j^-)\overline{(F_k^- - H_k^-)}\}a_{jk}(x-y).\end{aligned} \tag{5.45}$$

Therefore, it suffices to focus on the error term for the remaining of the section. The treatment of the error term will follow that of [GM13b].

Observe

$$\begin{aligned}
\text{error term} &= -4\int dxdy\ l_j(t,x)T_{00}(t,y)a_j(x-y)\\
&= -2\int dxdydz\ v(|x-z|)\{\partial_{z_j}\mathcal{L}_{2,2}(x,z;x,z)-\partial_{x_j}\mathcal{L}_{2,2}(x,z;x,z)\}\\
&\quad \times T_{00}(y)a_j(x-y)\\
&= -4\int dxdydz\ v'(|x-z|)\frac{x_j-z_j}{|x-z|}\mathcal{L}_{2,2}(x,z;x,z)T_{00}(y)a_j(x-y)\\
&\quad -2\int dxdydz\ v(x-z)\mathcal{L}_{2,2}(x,z;x,z)\Gamma(t,y;y)\partial_{x_j}a_j(x-y)
\end{aligned}$$

To further the computation let us take $a(x)=|x|$, then it follows

$$\begin{aligned}
\text{error term} &= -4\int dxdydz\ v'(|x-z|)\frac{x_j-z_j}{|x-z|}\frac{x_j-y_j}{|x-y|}\mathcal{L}_{2,2}(x,z,x,z)T_{00}(y)\\
&\quad -2\int dxdydz\ v(y-z)\mathcal{L}_{2,2}(x,z,x,z)T_{00}(y)\Delta|x-y|\\
&= -2\int dxdydz\ v'(|x-z|)\left(\frac{x_j-z_j}{|x-z|}\frac{x_j-y_j}{|x-y|}+\frac{z_j-x_j}{|z-x|}\frac{z_j-y_j}{|z-y|}\right)\\
&\quad \times \mathcal{L}_{2,2}(x,z,x,z)T_{00}(y)\\
&\quad -2\int dxdydz\ v(y-z)\mathcal{L}_{2,2}(x,z,x,z)T_{00}(y)\Delta|x-y|.
\end{aligned}$$

Thus, we have the following Morawetz estimate

$$\begin{aligned}
\partial_t M^a(t) &= (\text{main term})+(\text{error term})\\
&\geq 16\pi\int dx\ |\Gamma(t,x,x)|^2
\end{aligned}\tag{5.46}$$

which again means

$$\| \Gamma(t,x,x) \|^2_{L^2(dtdx)} \lesssim \| \Gamma_0 \|_{L^1_x} \left\{ \frac{1}{N} \int dz \, \| \operatorname{sh}(k_0)(\cdot, z) \|^2_{\dot{H}^{\frac{1}{2}}(dx)} + \| \phi_0 \|^2_{\dot{H}^{\frac{1}{2}}(dx)} \right\}.$$

In particular, we also have

$$\| \phi \|_{L^4(dxdy)} \leq C \tag{5.47a}$$

and

$$\frac{1}{N^2} \int dxdt \, \left| \int dz \, |\operatorname{sh}(k)(x,z)|^2 \right|^2 \leq C. \tag{5.47b}$$

Also we have that

$$|\Gamma(x,y)| \leq \Gamma(x,x)^{1/2} \Gamma(y,y)^{1/2}$$

which means

$$\int dt \left(\int dxdy \, |\Gamma(t,x,y)|^4 \right)^{1/2} \leq \int dtdx \, \Gamma(t,x,x)^2 \lesssim 1. \tag{5.48}$$

Chapter 6: Collapsing Estimates on Closed Manifolds

6.1 Main Results

Let (M, g) be a closed Riemannian manifold with dimension $d \geq 3$ and denote Δ_g the corresponding Laplace-Beltrami operator associated to the metric g. Since we only consider closed manifold with a fixed metric g, it is convenient to write Δ in place of Δ_g when the context is clear. We begin by considering both the linear Schrödinger equation

$$\begin{aligned} \mathbf{S}_g\Lambda(t, x_1, x_2) = &\ (i\partial_t + \Delta_1 + \Delta_2)\,\Lambda(t, x_1, x_2) = 0, \\ \Lambda\big|_{t=0} &= \Lambda_0 \in C^\infty(M \times M) \end{aligned} \tag{6.1}$$

and the von-Neumann Schrödinger equation

$$\begin{aligned} \mathbf{S}_g^{\pm}\Gamma(t, x_1, x_2) = &\ (i\partial_t + \Delta_1 - \Delta_2)\,\Gamma(t, x_1, x_2) = 0, \\ \Gamma\big|_{t=0} &= \Gamma_0 \in C^\infty(M \times M) \end{aligned} \tag{6.2}$$

defined on the product manifold $(M \times M, G = g \oplus g)$.

The first goal of the chapter is to establish collapsing estimates, which are natural generalization of bilinear Strichartz-type estimates to the case of arbitrary tensor products, for both (6.1) and (6.2) on general closed manifolds.

Theorem 6.1. *Assume* $d \geq 3$*. Suppose* $\Lambda(t), \Gamma(t)$ *are solutions to* (6.1) *and* (6.2)*, respectively. Then there exists* $\alpha > 0$ *such that for all* $\sigma > \sigma_* = \frac{d-1}{2}$ *we have the estimates*

$$\begin{aligned} &\| \operatorname{diag} \Lambda \|_{L^2([-\alpha,\alpha]\times \dot{H}^1(M))} \\ &\lesssim \min\left(\left\| (-\Delta_1)^{\frac{1}{2}} (-\Delta_2)^{\frac{\sigma}{2}} \Lambda_0 \right\|_{L^2(M\times M)}, \left\| (-\Delta_1)^{\frac{\sigma}{2}} (-\Delta_2)^{\frac{1}{2}} \Lambda_0 \right\|_{L^2(M\times M)} \right) \end{aligned} \tag{6.3a}$$

and

$$\begin{aligned} &\| \operatorname{diag} \Gamma \|_{L^2([-\alpha,\alpha]\times \dot{H}^1(M))} \\ &\lesssim \min\left(\left\| (-\Delta_1)^{\frac{1}{2}} (-\Delta_2)^{\frac{\sigma}{2}} \Gamma_0 \right\|_{L^2(M\times M)}, \left\| (-\Delta_1)^{\frac{\sigma}{2}} (-\Delta_2)^{\frac{1}{2}} \Gamma_0 \right\|_{L^2(M\times M)} \right). \end{aligned} \tag{6.3b}$$

Here, $\operatorname{diag} F$ *denotes the restriction of* F *to the diagonal subset* $\{(x, y) \in M \times M \mid x = y\}$*.*

The proof of Theorem 6.1 is based on ideas introduced in [Sog93b] to prove the local well-posedness of the nonlinear wave equations with variable coefficients and the semiclassical techniques used in [BGT04] to prove the Strichartz estimates for the Schrödinger equation on closed manifold. Moreover, the collapsing estimates for (6.1) can be viewed as a generalization of the bilinear Strichartz estimates on closed manifold as proved in [Han12].

6.2 Collapsing Estimates

This section is devoted to the proof of Theorem 6.1. To begin, we follow the ideas used in [BGT04] by first describing the effect of spectral localization relative to the elliptic operator Δ_g on local coordinate patches of the product manifold. Then we prove the spectral localized versions of Theorem 6.1. Finally, we use spectral dyadic techniques to sum up the different range of the spectrum to obtain the results of Theorem 6.1. Moreover, since the underlying geometries of (6.1) and (6.2) are different, we will treat the equations separately.

The author would like to begin by apologizing to the reader for the fact that this section will not be self-contained. In fact, we borrow many tools from [BGT04, Han12]. Nevertheless, we will provide the reader with detailed reference to the relevant section or statement of these papers when necessary.

We adopt the Kohn-Nirenberg pseudodifferential quantization rule, that is

$$a(x,D)u = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x,\xi)\widehat{u}(\xi)\, d\xi$$

for every smooth symbol a and $u \in C_0^\infty(\mathbb{R}^d)$. See [Sog93a, Hör94].

Let us state the following proposition which says the spectral localization operator on product manifold is well-approximated by pseudodifferential operators.

Lemma 6.2. *Let* $\varphi \in C_c^\infty(\mathbb{R})$ *and* $\operatorname{supp}\varphi \subset [\frac{1}{2}, 1]$, $\kappa_{\alpha\beta} := \kappa_\alpha \times \kappa_\beta : U_\alpha \times U_\beta \subset \mathbb{R}^d \to V_\alpha \times V_\beta \subset M \times M$ *a coordinate patch, and* $\chi_1, \chi_2 \in C_0^\infty(V_1 \times V_2)$ *such that* $\chi_2 = 1$ *near the support of* χ_1. *Then there exist sequences of symbols* $(\psi_j^k)_{j\geq 0}$ *of*

$C_c^\infty(U_k \times \mathbb{R}^d)$, $k = 1, 2$, *such that for every* $N \in \mathbb{N}$, *for every* $h, m \in (0, 1]$, *for every* $s \in [0, N]$, *and for every* $f \in C^\infty(M \times M)$, *we have the estimate*

$$\left\| \kappa^*_{\alpha\beta}(\chi_1 P^1_{\frac{1}{h}} P^2_{\frac{1}{m}} f) - \sum_{j,\ell=0}^{N-1} h^j m^\ell \psi^1_j(x, hD_x)\psi^2_\ell(y, mD_y)\kappa^*_{\alpha\beta}(\chi_2 f) \right\|_{\dot{H}^s(\mathbb{R}^{2d})}$$
$$\lesssim_N \sum_{s_1+s_2=s} h^{N-s_1} m^{N-s_2} \| f \|_{L^2(M\times M)}$$

where κ^* *denotes the standard pullback,* $P^j_{\frac{1}{h}} = \varphi(h^2\Delta_j)$, *and* $D = -i\nabla$. *Applying the sharp trace theorem, we immediately get the estimate*

$$\left\| \kappa^*_{\alpha\beta}(\chi_1 P^1_{\frac{1}{h}} P^2_{\frac{1}{m}} f) - \sum_{i,j=0}^{N-1} h^i m^j \psi^1_i(x, hD_x)\psi^2_j(y, mD_y)\kappa^*_{\alpha\beta}(\chi_2 f) \Big|_{x=y} \right\|_{L^2(\mathbb{R}^d)}$$
$$\lesssim_N \sum_{s_1+s_2=s} h^{N-s_1} m^{N-s_2} \| f \|_{L^2(M\times M)}.$$

Proof. This follows immediately from Proposition 2.1 in [BGT04]. □

6.2.1 Estimates for (6.1)

For brevity of notation, let us adopt the convention $D_i = (-\Delta_i)^{\frac{1}{2}}$. We are ready to prove the following proposition

Proposition 6.3. *Let* $h \in (0, 1]$, $\varphi \in C_c^\infty(\mathbb{R})$ *with* $\operatorname{supp}\varphi \subset [\frac{1}{2}, 1]$ *and define* $P_{\frac{1}{h}} = \varphi(h^2\Delta)$. *Assume* $P^1_{\frac{1}{h}} P^2_{\frac{1}{h}} \Lambda_0 = \Lambda_0$. *Then we have the estimate*

$$\| \operatorname{diag}\Lambda \|_{L^2([-\alpha h, \alpha h], \dot{H}^1(M))} \lesssim \left\| D_1^{\frac{1}{2}} D_2^{\sigma_*} \Lambda_0 \right\|_{L^2(M\times M)} \tag{6.4a}$$

of some $\alpha > 0$, independent of h, or equivalently

$$\| \operatorname{diag} \Lambda \|_{L^2([-\alpha,\alpha],\dot{H}^1(M))} \lesssim \| D_1 D_2^{\sigma_*} \Lambda_0 \|_{L^2(M\times M)} \tag{6.4b}$$

where σ_* is defined in Theorem 6.1.

Remark 6.4. The reader should note that the choice of derivatives in Proposition 6.3 is superficial. In fact, the spectral localization allows us to rewrite (6.4a) as

$$\| \operatorname{diag} \Lambda \|_{L^2([-\alpha h,\alpha h],\dot{H}^1(M))} \lesssim h^{-\frac{d}{2}} \| \Lambda_0 \|_{L^2(M\times M)} .$$

However, our choice of derivatives will be more transparent later in Section 6.2.3 when we discuss the Bourgain refinement estimates. Moreover, following our proof of Proposition 6.3, one could also prove

$$\| \operatorname{diag} \Lambda \|_{L^2([-\alpha h,\alpha h]\times M)} \lesssim h^{-\frac{d-2}{2}} \| \Lambda_0 \|_{L^2(M\times M)} .$$

Following closely the presentation of [BGT04], we start by proving Proposition 6.3 on local coordinate charts.

Proposition 6.5 (Local Collapsing Estimate for (6.1))**.** *Let $U_1 \times V_1 \subset \mathbb{R}^{2d}$ be a product of open balls $U_1, V_1 \subset \mathbb{R}^d$, endowed with Riemannian metrics g_α, g_β, respectively, such that $U_1 \cap V_1 \neq \emptyset$ and $g_\alpha\big|_{U_1\cap V_1} = g_\beta\big|_{U_1\cap V_1}$. Let $U_2 \Subset U_1$ and $V_2 \Subset V_1$ be open balls, again with $U_2 \cap V_2 \neq \emptyset$, $\chi_0 \in C_0^\infty(U_2 \times V_2), \psi \in C_0^\infty(\mathbb{R}^{2d}\backslash\{|\xi|, |\eta| < \frac{1}{2}\})$. Then there exists $\alpha > 0$ such that for every $\delta > 0$, $h \in (0, 1]$, and $w_0 \in C_0^\infty(U_1 \times V_1)$,* *we can find a* $\widetilde{w} \in C^{\infty}([-\alpha, \alpha] \times U_2 \times V_2)$*, compactly supported, satisfying*

$$ih\partial_s \widetilde{w} + h^2 \Delta_G \widetilde{w} = r, \quad \widetilde{w}(0, x, y) = \chi_0(x, y)\psi(hD_x, hD_y)w_0(x, y),$$

such that we have the estimate

$$\| \nabla_x \widetilde{w}(s, x, x) \|_{L^2([-\alpha,\alpha]\times U_2 \cap V_2)} \lesssim h^{-\frac{1}{2}} \left\| |\nabla_x|^{\frac{1}{2}} |\nabla_y|^{\sigma_*} w_0 \right\|_{L^2(U_1 \times V_1)}.$$

Moreover, if w *is a solution to* (6.1) *written in the above local coordinate with the microlocalized initial data* $\widetilde{w}(0)$ *then we have* $w(s, x, y) = \widetilde{w}(s, x, y) + R(s, x, y)$ *where*

$$\| R(s, x, x) \|_{L^2([-\alpha,\alpha]\times U_2 \cap V_2)} \lesssim h^{\textit{some positive power}} \| w_0 \|_{L^2(U_1 \times V_1)}.$$

Proof of Proposition 6.5. The proof employs the WKB approximation to the solution of (6.1). More precisely, we seek an approximation $\widetilde{w}$ given by the oscillatory integral[1]

$$\widetilde{w}(s, x, y) = \int_{\mathbb{R}^{2d}} e^{\frac{i}{h}\phi(s,x,y,\xi,\eta)} a(s, x, y, \xi, \eta, h) \widehat{w}_0 \left(\frac{\xi}{h}, \frac{\eta}{h}\right) \frac{d\xi d\eta}{(2\pi h)^{2d}} \tag{6.5}$$

where

$$a(s, x, y, \xi, \eta, h) = \sum_{j=0}^{N} h^j a_j(s, x, y, \xi, \eta).$$

---

[1] Since w_0 is compactly supported then we could trivially extend w_0 to all of $\mathbb{R}^{2d}$. Also, $\widehat{w}_0$ is the Fourier transform of the microlocalized data $\widetilde{w}_0$ not w_0. Moreover, the variables ξ and η are scaled to have length on the order of 1, that is $\frac{1}{2} \leq |\xi|, |\eta| \leq 2$.

Here N is chosen to be sufficiently large, $a_j \in C_0^\infty([-\alpha,\alpha] \times U_2 \times V_2 \times \mathbb{R}^{2d})$ are solutions to some transport equations satisfying the initial data

$$a_0(0,x,y,\xi,\eta) = \chi_0(x,y)\psi(\xi,\eta) \quad \text{and} \quad a_j(0,x,y,\xi,\eta) = 0 \text{ for all } j \geq 1,$$

and the phase function $\phi \in C^\infty([-\alpha,\alpha] \times U_2 \times V_2 \times A)$, where the annulus $A = \{\frac{1}{4} \leq |\xi|^2 + |\eta|^2 \leq 4\}$ contains the support of ψ, is a real-valued smooth function on the support of a satisfying the eikonal equation[2]

$$\partial_s \phi + \sum_{1 \leq j,k \leq 2d} G^{jk} \partial_j \phi \partial_k \phi = 0 \tag{6.6}$$

where G^{jk} is the dual metric with initial condition $\phi(0,x,y,\xi,\eta) = x \cdot \xi + y \cdot \eta$. Since the Riemannian metric is given by $G = g \oplus g$, then the eikonal equation reduces to

$$\partial_s \phi + g(\nabla_x \phi, \nabla_x \phi) + g(\nabla_y \phi, \nabla_y \phi) = 0, \ \phi(0) = x \cdot \xi + y \cdot \eta. \tag{6.7}$$

One could make a further observation, if φ is a solution to

$$\partial_s \varphi + g(\nabla_x \varphi, \nabla_x \varphi) = 0, \quad \varphi(0,x,\xi) = x \cdot \xi \tag{6.8}$$

---

[2] By the standard Hamilton-Jacobi theory, we see that the eikonal equation is well-posed on some small time interval $[-\alpha,\alpha]$ (c.f. Chapter 9 of [Arn97]). Moreover, It is also convenient to write (6.8) in the form $\partial_s \varphi + |\nabla_x \varphi|_g^2 = 0$ where $|\xi|_g^2 := g^{ij}(x)\xi_i\xi_j$.

then $\varphi(s,x,\xi)+\varphi(s,y,\eta)$ uniquely solves (6.7). Rewrite (6.5), we get

$$\widetilde{w}(s,x,x)=\int e^{\frac{i}{h}\{\varphi(s,x,\xi)+\varphi(s,x,\eta)\}}\tilde{a}(s,x,\xi,\eta,h)\widehat{w}_0\left(\frac{\xi}{h},\frac{\eta}{h}\right)\frac{d\xi d\eta}{(2\pi h)^{2d}}$$

where $\tilde{a}$ is the collapsed function of a. Then it follows

$$\begin{aligned}\frac{1}{i}\partial_j\widetilde{w}(s,x,x)\sim\ &\int e^{\frac{i}{h}\{\varphi(s,x,\xi)+\varphi(s,x,\eta)\}}(\partial_j\varphi(s,x,\xi)+\partial_j\varphi(s,x,\eta))\\ &\qquad\times\tilde{a}(s,x,\xi,\eta,h)\widehat{w}_0\left(\frac{\xi}{h},\frac{\eta}{h}\right)\frac{d\xi d\eta}{(2\pi h)^{2d+1}}\\ &+\text{lower order term}\end{aligned}$$

for $j=1,\dots,d$.

To complete the proof of Proposition 6.5 it suffices to prove the following proposition.

Proposition 6.6. *Consider $T:L^2(\mathbb{R}^{2d})\to L^2(\mathbb{R}^{1+d})$ defined by*

$$(TF)(s,x)=\int e^{\frac{i}{h}\phi(s,x,\xi,\eta)}q_h(s,x,\xi,\eta)F(\xi,\eta)\frac{d\xi d\eta}{(2\pi h)^{2d+1}}\tag{6.9}$$

where ϕ is defined as above and q_h vanishes either on the complement of $\frac{1}{4}<|\xi|^2+|\eta|^2<4$ or $(s,x)\notin[-\varepsilon,\varepsilon]\times X$, X compact, and $|\partial^\alpha_{s,x}q_h|\le C_\alpha|\xi+\eta|$. If ε is sufficiently small then there exists a constant C, depending on finitely many derivatives of q_h, so that the following holds

$$\|\,TF\,\|_{L^2(\mathbb{R}^{1+d})}\le Ch^{-\frac{3d+1}{2}}\left\|\,|\xi|^{\frac{1}{2}}|\eta|^{\sigma_*}F\,\right\|_{L^2(\mathbb{R}^{2d})}.\tag{6.10}$$

As a preliminary to the proof of Proposition 6.6, let us recall some properties of the phase function which will also be useful for the proof and later on in the section.

Taylor expanding the solution of (6.8) about $t = 0$ yields

$$\begin{aligned} \varphi(t, x, \xi) &= \varphi(0, x, \xi) + t(\partial_t \varphi)(0, x, \xi) + \mathcal{O}(t^2) \\ &= x \cdot \xi - t|\nabla_x \varphi(0, x, \xi)|_g^2 + \mathcal{O}(t^2) \end{aligned} \tag{6.11}$$

for $|t| < \varepsilon$. In many of the proofs of the collapsing estimates, we will need to handle the phase function

$$\Phi^{\pm}(t, x, \xi, \eta, \xi', \eta') = \varphi(t, x, \xi) \pm \varphi(t, x, \eta) - \varphi(t, x, \xi') \mp \varphi(t, x, \eta')$$

which by (6.11) has the form

$$x \cdot \{(\xi - \xi') + (\eta - \eta')\} - t\left(|\xi|_g^2 \pm |\eta|_g^2 - |\xi'|_g^2 \mp |\eta'|_g^2\right) + \mathcal{O}(t^2)$$

when $|t| < \varepsilon$. Making the change of variables $(\xi, \eta, \xi', \eta') \longmapsto \frac{1}{2}(\xi + \eta, \xi - \eta, \xi' + \eta', \xi' - \eta')$ yields

$$x \cdot (\xi - \xi') - tp^{\pm}(x, \xi, \eta, \xi', \eta') + R(t, x, \xi, \eta, \xi', \eta') \tag{6.12}$$

where $p^+(x, \xi, \eta, \xi', \eta') = \frac{1}{2}(|\xi|_g^2 + |\eta|_g^2 - |\xi'|_g^2 - |\eta'|_g^2)$ and $p^-(x, \xi, \eta, \xi', \eta') = g(\xi, \eta) -$

$g(\xi', \eta')$ and the remainder R satisfies

$$R(0, \xi, \eta, \xi', \eta') = (\partial_t R)(0, \xi, \eta, \xi', \eta') = 0.$$

Then we have the following lemmas.

Lemma 6.7. *Let $(t, x) \in [-\varepsilon, \varepsilon] \times X$, $X \subset \mathbb{R}^d$ compact. If $\varepsilon > 0$ is sufficiently small then we have the estimate*

$$|\nabla_{t,x} \Phi(t, x)| \geq \frac{1}{2} \left(|\xi - \xi'| + |p(x, \xi, \eta, \xi', \eta')| \right). \tag{6.13}$$

Proof. By the Taylor expansion (6.12), we see that

$$\nabla_{t,x} \Phi = \begin{pmatrix} -p(\xi, \eta, \xi', \eta') \\ \\ \xi - \xi' \end{pmatrix} - t \begin{pmatrix} 0 \\ \\ \nabla_x p(\xi, \eta, \xi', \eta') \end{pmatrix} + \begin{pmatrix} \partial_t R \\ \\ \nabla_x R \end{pmatrix}.$$

Thus, we have that

$$|\nabla_{t,x} \Phi| = |\xi - \xi'| + |p(x, \xi, \eta, \xi', \eta')| + \mathcal{O}(t).$$

Hence when t is sufficiently small the $\mathcal{O}(t)$ error term will be dominated by $\frac{1}{2}|\xi - \xi'|$ which yield the desired result. □

Lemma 6.8. *Let $N > 0$ and $(t, x) \in [-\varepsilon, \varepsilon] \times X$, $X \subset \mathbb{R}^d$ compact. If $\varepsilon > 0$ is* *sufficiently small then there exists $C > 0$ such that*

$$|\partial_t^m \partial_x^\alpha \Phi(t,x)| \leq C \sup_{m,|\alpha|\leq N} \sup_{x\in X} |\partial_x^\alpha p(x,\xi,\eta,\xi',\eta')| \tag{6.14}$$

where $m \in \mathbb{N}_{>1}, \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_{>1}^d$.

Proof. The proof is essentially the same as the proof of Lemma 6.7. ☐

Let us continue with the proof of the proposition.

Proof of Proposition 6.6. Expanding the L^2 norm of TF and making the change of variables[3] $(\xi,\eta) \mapsto (\xi+\eta, \xi-\eta)$, we get

$$\| TF \|_2^2 \sim \int d\xi d\eta d\xi' d\eta' \; \frac{K(\xi,\eta,\xi',\eta')\widetilde{F}(\xi,\eta)\overline{\widetilde{F}(\xi',\eta')}}{|\xi+\eta|^{\frac{1}{2}}|\xi-\eta|^{\frac{d-1}{2}}|\xi'+\eta'|^{\frac{1}{2}}|\xi'-\eta'|^{\frac{d-1}{2}}}$$

where $\widetilde{F}(\xi,\eta) := |\xi+\eta|^{\frac{1}{2}}|\xi-\eta|^{\frac{d-1}{2}} F(\xi+\eta,\xi-\eta)$ and

$$K(\xi,\eta,\xi',\eta') := \frac{1}{(2\pi h)^{4d+2}} \int e^{\frac{i}{h}\Phi(s,x)} \tilde{q}_h(s,x,\xi,\eta,\xi',\eta') \; dxds$$

with phase function $\Phi(s,x) = \Phi(s,x,\xi,\eta,\xi',\eta')$ given by

$$\Phi(s,x,\xi,\eta,\xi',\eta') = \phi(s,x,\xi-\eta,\xi+\eta) - \phi(s,x,\xi'-\eta',\xi'+\eta')$$

and $|\tilde{q}_h(s,x,\xi,\eta,\xi',\eta')| \leq C|\xi||\xi'|$. Next, we employ the technique of non-stationary

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[3] Here, we abused notation. More accurately, we should have $(\tilde{\xi},\tilde{\eta})$ maps to $\xi = \tilde{\xi} + \tilde{\eta}$ and $\xi' = \tilde{\xi}' + \tilde{\eta}'$...etc.

phase to estimate the kernel K. Define the operator

$$\mathcal{L}(s,x,D_{s,x}) = i^{-1}\langle \nabla_{s,x}\Phi, \nabla_{s,x}\rangle |\nabla_{s,x}\Phi|^{-2}$$

then by applying integration by parts yields

$$K(\xi,\eta,\xi',\eta') = \frac{1}{(2\pi h)^{4d+2}} \int dxds\; e^{\frac{i}{h}\Phi(s,x)} (\mathcal{L}^*)^N \tilde{q}_h(s,x,\xi,\eta,\xi',\eta').$$

Here we note that there are essentially two types of terms which we need to handle, namely

$$\frac{\partial_{s,x}^{\alpha}\tilde{q}_h(\nabla_{s,x}\Phi)^{\gamma}}{|\nabla_{s,x}\Phi|^{2N}}, \quad |\gamma| = |\alpha| = N \tag{6.15}$$

and

$$\frac{(\partial_{s,x}^{\alpha_0}\tilde{q}_h \otimes \bigotimes_{k=1}^{m} \partial_{s,x}^{\alpha_k}\Phi)(\nabla_{s,x}\Phi)^{\gamma}}{|\nabla_{s,x}\Phi|^{2N+m}}, \quad |\gamma|, m \le N, \alpha_0 + \cdots + \alpha_m = N \tag{6.16}$$

since the general terms of $(\mathcal{L}^*)^N\tilde{q}_h$ are linear combination of (6.15) and (6.16). Applying Lemma 6.7 and 6.8 and the fact that $\tilde{q}_N$ vanishes on the complement of $\frac{1}{4} < |\xi|^2 + \rho^2 < 4$ then it follows

$$|(\mathcal{L}^*)^N\tilde{q}_h| \lesssim \frac{|\xi||\xi'|}{(h^{-1}|\xi-\xi'| + h^{-1}\big||\xi|_g^2 + |\eta|_g^2 - |\xi'|_g^2 - |\eta'|_g^2\big|)^N}.$$

Note, when $|\xi-\xi'| + \big||\xi|_g^2 + |\eta|_g^2 - |\xi'|_g^2 - |\eta'|_g^2\big| \le h$ we will not perform any integration by parts. Moreover, since $|\cdot|$ and $|\cdot|_g$ are comparable, then by change of variables, independent of h, we could estimate the kernel uniformly on $[-\varepsilon,\varepsilon]\times X$ as follows

$$|K(\xi,\eta,\xi',\eta')| \lesssim \frac{1}{h^{4d+2}} \frac{|\xi||\xi'|}{(1+h^{-1}|\xi-\xi'|+h^{-1}\big||\xi|^2+|\eta|^2-|\xi'|^2-|\eta'|^2\big|)^N}.$$

Next, using polar coordinates, $\eta=\rho\omega$ where $\rho>0, \omega\in\mathbb{S}^{d-1}$, we have

$$\begin{aligned}\|TF\|_2^2 \lesssim \frac{1}{h^{4d+2}} \int & d\xi \rho^{d-1} d\rho d\xi' (\rho')^{d-1} d\rho' d\omega d\omega' \ \widetilde{F}(\xi,\rho\omega)\overline{\widetilde{F}(\xi',\rho'\omega')} \\ & \times \frac{|\xi||\xi'||\xi+\rho\omega|^{-\frac{1}{2}}|\xi-\rho\omega|^{-\frac{d-1}{2}}|\xi'+\rho'\omega'|^{-\frac{1}{2}}|\xi'-\rho'\omega'|^{-\frac{d-1}{2}}}{(1+h^{-1}|\xi-\xi'|+h^{-1}\big||\xi|^2+\rho^2-|\xi'|^2-\rho'^2\big|)^N}\end{aligned} \tag{6.17}$$

To further estimate the RHS of (6.17), we begin by applying Cauchy-Schwarz inequality in the angular variables to get

$$\begin{aligned}&\left|\int_{\mathbb{S}^{d-1}} \frac{\widetilde{F}(\xi,\rho\omega)d\omega}{|\xi+\rho\omega|^{\frac{1}{2}}|\xi-\rho\omega|^{\frac{d-1}{2}}}\right|\left|\int_{\mathbb{S}^{d-1}} \frac{\widetilde{F}(\xi',\rho'\omega')d\omega'}{|\xi'+\rho'\omega'|^{\frac{1}{2}}|\xi'-\rho'\omega'|^{\frac{d-1}{2}}}\right| \\ &\lesssim \frac{h(\xi,\rho)h(\xi',\rho')}{(|\xi|^2+\rho^2)^{\frac{d}{4}}(|\xi'|^2+\rho'^2)^{\frac{d}{4}}}\end{aligned}$$

where $h(\xi,\rho) := \left(\int d\omega\ |\widetilde{F}(\xi,\rho\omega)|^2\right)^{\frac{1}{2}}$. Note we have used the fact that

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} \frac{d\omega}{|\xi+\rho\omega||\xi-\rho\omega|^{d-1}} &\sim \frac{1}{(|\xi|^2+\rho^2)^{\frac{d}{2}}} \int_{\delta}^{\pi-\delta} \frac{\sin^{d-2}\theta d\theta}{\sqrt{1-\kappa^2\cos^2\theta}(1-\kappa\cos\theta)^{\frac{d-2}{2}}} \\ &\lesssim \frac{1}{(|\xi|^2+\rho^2)^{\frac{d}{2}}},\end{aligned}$$

for some $0<\delta<\pi$ where $0<\kappa<1$ since $\frac{1}{2}\le|\xi+\eta|, |\xi-\eta|\le 2$.

Finally, let us estimate

$$\frac{1}{h^{4d+2}}\int \frac{d\xi d\rho d\xi' d\rho'}{(|\xi|^2+\rho^2)^{\frac{d}{4}}(|\xi'|^2+\rho'^2)^{\frac{d}{4}}}\frac{|\xi||\xi'|\rho^{d-1}(\rho')^{d-1}h(\xi,\rho)h(\xi',\rho')}{(1+h^{-2}|\xi-\xi'|^2+h^{-2}\mathcal{D}^2)^{\frac{N}{2}}} \tag{6.18}$$

where $\mathcal{D} = |\xi|^2+\rho^2-|\xi'|^2-\rho'^2$. We begin by making the change of variables $(\xi,\rho)\mapsto(\xi,\tau)=(\xi,|\xi|^2+\rho^2)$. Then (6.18) becomes

$$\frac{1}{h^{4d+2}}\int d\xi d\tau d\xi' d\tau' \frac{|\xi||\xi'|(\tau-|\xi|^2)^{\frac{d-2}{2}}(\tau'-|\xi'|^2)^{\frac{d-2}{2}}\tilde{h}(\xi,\tau)\tilde{h}(\xi',\tau')}{\tau^{\frac{d}{4}}\tau'^{\frac{d}{4}}(1+h^{-2}|\xi-\xi'|^2+h^{-2}(\tau-\tau')^2)^{\frac{N}{2}}} \tag{6.19}$$

where the integration take place over the region $\tau \geq |\xi|^2$ and $\tau' \geq |\xi'|^2$. By Young's inequality, we have that

$$\begin{aligned}
(6.19) &\lesssim \frac{1}{h^{3d+1}}\int d\xi d\tau \frac{|\xi|^2(\tau-|\xi|^2)^{d-2}}{\tau^{\frac{d}{2}}}|h(\xi,\rho)|^2 \\
&\sim \frac{1}{h^{3d+1}}\int d\xi \rho^{d-1}d\rho \frac{|\xi|^2\rho^{d-2}}{(|\xi|^2+\rho^2)^{\frac{d}{2}}}|h(\xi,\rho)|^2 \\
&\lesssim \frac{1}{h^{3d+1}}\int d\xi d\eta\ |\xi|^2|\eta|^{d-2}|F(\xi,\eta)|^2.
\end{aligned}$$

This completes the proof of Proposition 5. □

Now, let us conclude the proof of Proposition 6.5. First note, by rescaling ξ and η in estimate (6.31) and applying Plancherel and boundedness of projection operator, we get the desired estimate

$$\| \nabla_x \widetilde{w}(s,x,x) \|_{L^2([-\alpha,\alpha]\times U_2\cap V_2} \lesssim h^{-\frac{1}{2}} \left\| |\nabla_x|^{\frac{1}{2}} |\nabla_y|^{\frac{d-1}{2}} w_0 \right\|_{L^2(U_1\times V_1)}.$$

Finally, for the error term we see that

$$R(s,x,x) = \int_0^s [e^{i(s-\tau)\Delta_G} r](\tau,x,x)\ d\tau \sim \mathcal{O}(h^{N+1})$$

with

$$r(\tau,x,y) = h^{N+2} \int e^{\frac{i}{h}\phi(\tau,x,y)} b(\tau,x,y,\xi,\eta) \widehat{w}_0 \left(\frac{\xi}{h}, \frac{\eta}{h}\right) \frac{d\xi d\eta}{(2\pi h)^{2d}}.$$

where $b \in C_0^\infty([-\alpha,\alpha] \times U_2 \times V_2 \times B)$. By a straightforward application of the trace theorem, we see that

$$\begin{aligned} \| \nabla_x R(s,x,x) \|_{L^2(I \times L^2(U_2 \cap V_2))} &\lesssim_I \sup_s \int_I \| [e^{i(s-\tau)\Delta_G} r](s,x,x) \|_{H^1_x}\ d\tau \\ \text{by trace theorem on } \mathbb{R}^{2d} \quad &\lesssim_I \sup_s \int_I \| [e^{i(s-\tau)\Delta_G} r](s,x,y) \|_{\dot{H}^\mu_{x,y}}\ d\tau \\ \text{by Strichartz ineq.} \quad &\lesssim_I \| r \|_{L^\infty_s \dot{H}^\mu_{x,y}} \lesssim_\alpha\ h^{N+2-\mu} \| \widehat{w}_0 \|_{L^2_{\xi,\eta}} \end{aligned}$$

where $\mu > \frac{d}{2} + 1$. This completes the proof of Proposition 6.5. □

Remark 6.9. To get higher derivative Strichartz estimates on closed manifold, we used the fact $D_g = (-\Delta_g)^{\frac{1}{2}}$ commutes with the Schrödinger operator to get the Strichartz estimates

$$\| D_g^s \phi \|_{L^p(I \times L^q(M))} \lesssim \| D_g^s \phi_0 \|_{H^{\frac{1}{p}}(M)}.$$

Finally, using the fact that $\| D_g^s \phi \|_{L^q(M)} \sim \| \phi \|_{\dot{W}^{s,q}(M)}$ we get the desired result.

Proof of Proposition 6.3. Suppose $\{\chi_\nu\}$ is a partition of unity subordinate to a finite covering $\bigcup \Omega_\nu = M$, then we have that

$$\begin{aligned}\left\| \operatorname{diag}[P^1_{\frac{1}{h}} P^2_{\frac{1}{h}} \Lambda] \right\|^2_{L^2(I,H^1(M))} &= \int_I dt \, \| \operatorname{diag}[P^1_{\frac{1}{h}} P^2_{\frac{1}{h}} \Lambda] \|^2_{H^1(M)} \\ \lesssim \sum_\nu \int_I dt \, &\left\| \kappa^*_\nu (\chi_\nu \cdot \operatorname{diag}[P^1_{\frac{1}{h}} P^2_{\frac{1}{h}} \Lambda(t)]) \right\|^2_{H^1(\mathbb{R}^d)} \\ \sim \sum_\nu \int_I dt \, &\left\| \kappa^*_\nu (\chi^2_\nu \cdot \operatorname{diag}[P^1_{\frac{1}{h}} P^2_{\frac{1}{h}} \Lambda(t)]) \right\|^2_{H^1(\mathbb{R}^d)}\end{aligned}$$

where $\kappa : \Omega_\nu \subset M \to \mathbb{R}^d$ and $\kappa^*(f) := f \circ \kappa$ is the standard pullback. Hence it suffices to prove estimate (6.4) on a single coordinate chart.

Furthermore, as in [BGT04], we begin by making the observation

$$w(s, \cdot) = e^{ihs\Delta_G} P^1_{\frac{1}{h}} P^2_{\frac{1}{h}} \Lambda_0 \tag{6.20}$$

solves the semiclassical equation

$$ih\partial_s w + h^2 \Delta_G w = 0, \quad w(0) = w_0 = P^1_{\frac{1}{h}} P^2_{\frac{1}{h}} \Lambda_0.$$

Applying boundedness of $e^{it\Delta_G}$ on H^μ, Lemma 6.2 and the trace theorem, we see that

$$\begin{aligned}
&\left\| (\kappa_{\nu\nu})^*(\chi_\nu \otimes \chi_\nu \cdot P^1_{\frac{1}{h}} P^2_{\frac{1}{h}} \Lambda(t))\Big|_{x=y} \right\|_{H^1(\mathbb{R}^d)} \\
&\lesssim \left\| (1-\psi(hD_x, hD_y))(\kappa_{\nu\nu})^*(\chi_\nu \otimes \chi_\nu \cdot P^1_{\frac{1}{h}} P^2_{\frac{1}{h}} \Lambda(t))\Big|_{x=y} \right\|_{H^1(\mathbb{R}^d)} \\
&\quad + \left\| \psi(hD_x, hD_y)(\kappa_{\nu\nu})^*(\chi_\nu \otimes \chi_\nu \cdot P^1_{\frac{1}{h}} P^2_{\frac{1}{h}} \Lambda(t))\Big|_{x=y} \right\|_{H^1(\mathbb{R}^d)} \\
&\lesssim \left\| (1-\psi(hD_x, hD_y))(\kappa_{\nu\nu})^*(\chi_\nu \otimes \chi_\nu \cdot P^1_{\frac{1}{h}} P^2_{\frac{1}{h}} \Lambda_0) \right\|_{H^\mu(\mathbb{R}^{2d})} \\
&\quad + \left\| \psi(hD_x, hD_y)(\kappa_{\nu\nu})^*(\chi_\nu \otimes \chi_\nu \cdot P^1_{\frac{1}{h}} P^2_{\frac{1}{h}} \Lambda(t))\Big|_{x=y} \right\|_{H^1(\mathbb{R}^d)} \\
&\lesssim h^{N-\mu} \| \Lambda_0 \|_{L^2(M\times M)} \\
&\quad + \left\| (\kappa_{\nu\nu})^*(\chi_\nu \otimes \chi_\nu)\psi(hD_x, hD_y)(\kappa_{\nu\nu})^*(P^1_{\frac{1}{h}} P^2_{\frac{1}{h}} \Lambda(t))\Big|_{x=y} \right\|_{H^1(\mathbb{R}^d)} \\
&\quad + \text{ lower order terms.}
\end{aligned}$$

Note that $\psi(hD_x, hD_y)$ is a shorthand expression for the sum of product of pseudodifferential operators in Lemma 6.2. Finally, apply Proposition 6.5 and sum up the $\frac{1}{h}$ number of small-time intervals completes the proof of Proposition 6.3. □

6.2.2 Estimates for the Γ Equation

In this subsection, we prove some estimates for (6.2) similar to ones in Proposition 6.3.

Based on the Strichartz estimates for $\rho(t, x)$ in the Euclidean space setting established in Theorem 3.3 of [CHP17], we prove the following proposition.

Proposition 6.10. *Let* $h \in (0, 1]$, $\varphi \in C_c^\infty(\mathbb{R})$ *with* $\operatorname{supp} \varphi \subset [\frac{1}{2}, 1]$ *and define*

$P_{\frac{1}{h}} = \varphi(h^2\Delta)$. *Assume* $P^1_{\frac{1}{h}} P^2_{\frac{1}{h}} \Gamma_0 = \Gamma_0$. *Then we have the estimate*

$$\| \operatorname{diag} \Gamma \|_{L^2([-\alpha h, \alpha h], H^1(M))} \lesssim \left\| D_1^{\frac{1}{2}} D_2^{\sigma_*} \Gamma_0 \right\|_{L^2(M \times M)} \tag{6.21a}$$

of some $\alpha > 0$, *or equivalently*

$$\| \operatorname{diag} \Gamma \|_{L^2([-\alpha, \alpha], H^1(M))} \lesssim \| D_1 D_2^{\sigma_*} \Gamma_0 \|_{L^2(M \times M)} \tag{6.21b}$$

where σ_* *is as defined in Theorem 6.1.*

Following the same strategy as in the proof of Proposition 6.3, it suffices to prove the statement

Proposition 6.11 (Local Coordinate Collapsing Estimate for Γ)**.** *Let* $U_1 \times V_1 \subset \mathbb{R}^{2d}$ *be a product of open balls* $U_1, V_1 \subset \mathbb{R}^d$, *endowed with Riemannian metrics* g_α, g_β, *respectively, such that* $U_1 \cap V_1 \neq \emptyset$ *and* $g_\alpha\big|_{U_1 \cap V_1} = g_\beta\big|_{U_1 \cap V_1}$. *Let* $U_2 \Subset U_1$ *and* $V_2 \Subset V_1$ *be open balls, again with* $U_2 \cap V_2 \neq \emptyset$, $\chi_0 \in C_0^\infty(U_2 \times V_2)$, $\psi \in C_0^\infty(\mathbb{R}^{2d} \backslash \{|\xi|, |\eta| < \frac{1}{2}\})$. *Then there exists* $\alpha > 0$ *such that, for every* $h \in (0, 1], u_0 \in C_0^\infty(U_1 \times V_1)$, *we can find a* $\widetilde{u} \in C^\infty([-\alpha, \alpha] \times U_2 \times V_2)$, *compactly supported, satisfying*

$$ih\partial_s \widetilde{u} + h^2(\Delta_x - \Delta_y)\widetilde{u} = r, \quad \widetilde{u}(0, x, y) = \chi_0(x, y)\psi(hD_x, hD_y)u_0(x, y),$$

such that we have the estimate

$$\| \nabla_x \widetilde{u}(s, x, x) \|_{L^2([-\alpha, \alpha] \times L^2(U_2 \cap V_2))} \lesssim h^{-\frac{1}{2}} \left\| |\nabla_x|^{\frac{1}{2}} |\nabla_y|^{\sigma_*} u_0 \right\|_{L^2(U_1 \times V_1)}. \tag{6.22}$$

Moreover, if u is a solution to (6.2) written in the above local coordinate with initial data $\widetilde{u}(0)$ then we have $u(s,x,y) = \widetilde{u}(s,x,y) + R(s,x,y)$ where

$$\| R(s,x,x) \|_{L^2([-\alpha,\alpha]\times L^2(U_2\cap V_2))} \lesssim h^{\textit{positive power}} \| u_0 \|_{L^2(U_1\times V_1)}. \tag{6.23}$$

Again, the proof of Proposition 6.11 relies on the following proposition

Proposition 6.12. *Consider $T : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{1+d})$ defined by*

$$(TF)(s,x) = \int e^{\frac{i}{h}\psi(s,x,\xi,\eta)} q_h(s,x,\xi,\eta) F(\xi,\eta) \frac{d\xi d\eta}{(2\pi h)^{2d+1}} \tag{6.24}$$

where $\psi(s,x,\xi,\eta) = \varphi(s,x,\xi) - \varphi(s,x,\eta)$ and q_h vanishes either on the complement of $\frac{1}{4} < |\xi|^2 + |\eta|^2 < 4$ or $(s,x) \notin [-\varepsilon,\varepsilon] \times X$, X compact, and $|\partial^\alpha_{s,x} q_h| \le C_\alpha |\xi - \eta|$. If ε is sufficiently small then there exists a constant C, depending on finitely many derivatives of q_h, so that the following holds

$$\| TF \|_{L^2(\mathbb{R}^{1+d})} \le C h^{-\frac{3d+1}{2}} \left\| |\xi|^{\frac{1}{2}} |\eta|^{\sigma_*} F \right\|_{L^2(\mathbb{R}^{2d})}. \tag{6.25}$$

Proof of Proposition 6.12. Expanding the L^2 norm of TF and making the change of variables $(\xi,\eta) \mapsto (\xi-\eta, \xi+\eta)$, we get

$$\| TF \|_2^2 \sim \int d\xi d\eta d\xi' d\eta' \, \frac{K(\xi,\eta,\xi',\eta') \widetilde{F}(\xi,\eta) \overline{\widetilde{F}(\xi',\eta')}}{|\xi+\eta|^{\frac{1}{2}} |\xi-\eta|^{\frac{d-1}{2}} |\xi'+\eta'|^{\frac{1}{2}} |\xi'-\eta'|^{\frac{d-1}{2}}}$$

where $\widetilde{F}(\xi,\eta) := |\xi+\eta|^{\frac{1}{2}}|\xi-\eta|^{\frac{d-1}{2}}F(\xi-\eta,\xi+\eta)$ and

$$K(\xi,\eta,\xi',\eta') = \frac{1}{(2\pi h)^{4d+2}}\int e^{\frac{i}{h}\Psi(s,x)}\widetilde{q}_h(s,x,\xi,\eta,\xi',\eta')\ dxds$$

with the phase function $\Psi(s,x) = \Psi(s,x,\xi,\eta,\xi',\eta')$ given by

$$\Psi(s,x) = \psi(s,x,\xi-\eta,\xi+\eta) - \psi(s,x,\xi'-\eta',\xi'+\eta')$$

and

$$|\widetilde{q}_h(s,x,\xi,\eta,\xi',\eta')| \le C|\xi||\xi'|.$$

Applying the method of non-stationary phase along with Lemma 6.7 and 6.8, we obtain the estimate

$$\|\,TF\,\|_2^2 \lesssim \frac{1}{h^{4d+2}}\int_{|\xi+\eta|\sim|\xi-\eta|\sim 1}\frac{|\xi||\xi'|\widetilde{F}(\xi,\eta)\overline{\widetilde{F}(\xi',\eta')}\ d\xi d\eta d\xi' d\eta'}{(1+h^{-1}|\xi-\xi'|+h^{-1}\big|\xi\cdot\eta-\xi'\cdot\eta'\big|)^N}. \tag{6.26}$$

Note that we have again used the fact that $g(\xi,\eta)\sim\xi\cdot\eta$ since $g\sim \mathrm{Id}$. Next, write $\eta=(p,\widetilde{\eta})$ then consider the integration with respect to η. Without loss of generality, take $\xi=(|\xi|,0,\ldots,0)$, then we have the integral

$$\begin{aligned}&\int_{|\xi|^2+p^2+|\tilde{\eta}|^2\sim 1} dpd\tilde{\eta}dp'd\tilde{\eta}'\frac{\widetilde{F}(\xi,p,\tilde{\eta})\overline{\widetilde{F}(\xi',p',\tilde{\eta}')}}{(1+h^{-1}|\xi-\xi'|+h^{-1}\big||\xi|p-|\xi'|p'\big|)^N}\\ &\lesssim\int_{|\xi|^2+p^2\lesssim 1} dpdp'\frac{G(\xi,p)G(\xi',p')}{(1+h^{-1}|\xi-\xi'|+h^{-1}\big||\xi|p-|\xi'|p'\big|)^N}\end{aligned}$$

where $G(\xi,p) = \left(\int d\widetilde{\eta}\ |\widetilde{F}(\xi,p,\widetilde{\eta})|^2\right)^{\frac{1}{2}}$. Finally, we see that

$$\begin{aligned}
(6.26) &\lesssim \frac{1}{h^{4d+2}}\int d\xi dp d\xi' dp' \frac{|\xi||\xi'|G(\xi,p)G(\xi',p')}{(1+h^{-1}|\xi-\xi'|+h^{-1}\big||\xi|p-|\xi'|p'\big|)^N}\\
&\lesssim \frac{1}{h^{4d+2}}\int d\xi d\tau d\xi' d\tau' \frac{G(\xi,\tau/|\xi|)G(\xi',\tau'/|\xi'|)}{(1+h^{-1}|\xi-\xi'|+h^{-1}|\tau-\tau'|)^N}\\
\text{Young's ineq.}\quad &\lesssim \frac{1}{h^{3d+1}}\int d\xi d\tau\ |G(\xi,\tau/|\xi|)|^2\\
&\lesssim \frac{1}{h^{3d+1}}\int d\xi dp d\widetilde{\eta}\ |\widetilde{F}(\xi,p,\widetilde{\eta})|^2.
\end{aligned}$$

Hence we arrive at the desired inequality. □

6.2.3 Bourgain Refinement of the Collapsing Estimates

In this subsection we study the collapsing estimates where the spectral variables corresponding to the two spatial variables of $M \times M$ are localized to different ranges, that is, we choose $\varphi \in C_0^\infty(\mathbb{R}\backslash\{0\})$ and

$$P^1_{\frac{1}{h}}P^2_{\frac{1}{m}}F_0 = F_0 \tag{6.27}$$

for any $h, m \in (0,1]$ where $F_0 = \Gamma_0$ or Λ_0. The proof of the estimates is based on Hani's work on the bilinear Strichartz estimates on closed manifold [Han12] .

Proposition 6.13. *For every* $0 < h < m \leq 1, \lambda = \frac{h}{m}$ *and* $\varphi \in C_0^\infty(\mathbb{R})$ *with* $\operatorname{supp}\varphi \subset [\frac{1}{2},1]$. *Assume* $P^1_{\frac{1}{h}}P^2_{\frac{1}{m}}F_0 = F_0$. *Suppose* $F(t)$ *is a solution to either* (6.1) *or* (6.2)*, then we have the estimate*

$$\| \operatorname{diag} F \|_{L^2([-\alpha h,\alpha h],\dot{H}^1(M))} \lesssim \left\| D_1^{\frac{1}{2}} D_2^{\sigma_*} F_0 \right\|_{L^2(M\times M)} \tag{6.28a}$$

of some $\alpha > 0$*, or equivalently*

$$\| \operatorname{diag} F \|_{L^2([-\alpha,\alpha],\dot{H}^1(M))} \lesssim \| D_1 D_2^{\sigma_*} F_0 \|_{L^2(M\times M)} \,. \tag{6.28b}$$

Let us state the local version of Proposition 6.13.

Proposition 6.14. *Let* $U_1 \times V_1 \subset \mathbb{R}^{2d}$ *be a product of open balls* $U_1, V_1 \subset \mathbb{R}^d$*, endowed with Riemannian metrics* g_α, g_β*, respectively, such that* $U_1 \cap V_1 \neq \emptyset$ *and* $g_\alpha\big|_{U_1\cap V_1} = g_\beta\big|_{U_1\cap V_1}$*. Let* $U_2 \Subset U_1$ *and* $V_2 \Subset V_1$ *be open balls, with* $U_2 \cap V_2 \neq \emptyset$*,* $\chi_0 \in C_0^\infty(U_2 \times V_2), \psi \in C_0^\infty(\mathbb{R}^{2d}\backslash\{|\xi|, |\eta| < \frac{1}{2}\})$*. Then there exists* $\alpha > 0$ *such that, for every* $h, m \in (0, 1]$ *with* $h < m$ *and* $\lambda = \frac{h}{m}$*,* $w_0 \in C_0^\infty(U_1 \times V_1)$*, we can find a* $\widetilde{w} \in C^\infty([-\alpha, \alpha] \times U_2 \times V_2)$*, compactly supported, satisfying the problem*

$$\begin{aligned} &ih\partial_s \widetilde{w} + h^2 \Delta_x \widetilde{w} \pm h^2 \Delta_y \widetilde{w} = r, \\ &\widetilde{w}(0, x, y) = \chi_0(x, y)\psi(hD_x, mD_y)w_0(x, y) \end{aligned}$$

with estimates

$$\| \nabla_x \widetilde{w}(s, x, x) \|_{L^2([-\alpha,\alpha]\times H^1(U_2\cap V_2))} \lesssim h^{-\frac{1}{2}} \left\| |\nabla_x|^{\frac{1}{2}} |\nabla_y|^{\sigma_*} w_0 \right\|_{L^2(U_1\times V_1)} . \tag{6.29}$$

Moreover, if w *is a solution to either* (6.1) *or* (6.2)*, satisfying* (6.27)*, written in local* *coordinates with initial data* $\widetilde{w}(0)$ *then we have* $w(s,x,y) = \widetilde{w}(s,x,y) + R(s,x,y)$ *where*

$$\| R(s,x,x) \|_{L^2([-\alpha,\alpha]\times L^2(U_2\cap V_2))} \lesssim h^{positive\ power} \| w_0 \|_{L^2(U_1\times V_1)}.$$

Proof of Proposition 6.14. Similar to the proof of Proposition 6.5, we consider the WKB approximation (6.5) where a_h is supported in $[-\alpha,\alpha]\times K\times[1/2,1]\times[\lambda/2,\lambda]$. Making the rescaling $\eta \mapsto \lambda\eta$, it follows

$$\partial_j \widetilde{w}(s,x,x) \sim \int e^{\frac{i}{h}\{\varphi(s,x,\xi)\pm\varphi(s,x,\lambda\eta)\}}(\partial_j\varphi(s,x,\xi) \pm \partial_j\varphi(s,x,\lambda\eta)) \\ \times a_h(s,x,\xi,\lambda\eta)\widehat{w}_0\left(\frac{\xi}{h},\frac{\eta}{m}\right)\frac{d\xi d\eta}{h^{d+1}m^d}.$$

for $j = 1,\ldots,d$. Similar to the proof of Proposition 6.5, we first prove the following proposition.

Proposition 6.15. *Consider* $T : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{1+d})$ *defined by*

$$(TF)(s,x) = \int e^{\frac{i}{h}\phi(s,x,\xi,\eta)} q_{h,m}(s,x,\xi,\eta)F(\xi,\eta)\frac{d\xi d\eta}{m^d h^{d+k}} \tag{6.30}$$

where $\phi(s,x,\xi,\eta) = \varphi(s,x,\xi) \pm \varphi(s,x,\lambda\eta)$ *and* $q_{h,m}$ *vanishes either when* $|\xi| \not\sim 1$ *or* $|\eta| \not\sim 1$, *or* $(s,x) \notin [-\varepsilon,\varepsilon]\times X$, *X compact, and* $|\partial^\alpha_{s,x} q_{h,m}| \leq C_\alpha |\xi \pm \lambda\eta|^k$. *If* ε *is sufficiently small then there exists a constant* C, *depending only on finitely many* *derivatives of $q_{h,m}$, so that the following holds*

$$\| TF \|_{L^2(\mathbb{R}^{1+d})} \le Ch^{-\frac{d}{2}-k} m^{-d+\frac{1}{2}} \left\| |\xi|^{k-\frac{1}{2}} |\eta|^{\sigma_*} F \right\|_{L^2(\mathbb{R}^{2d})}. \tag{6.31}$$

Remark 6.16. The proof of Proposition 6.15 is similar to the proof of Theorem 1.1 given in [Han12]. However, for completeness, we have included a sketch of the proof of Proposition 6.15 and refer the reader to [Han12] for the details. The key ingredient in the argument is the uniform transversality condition satisfied by the two surfaces $\nabla_{t,x}\varphi(t,x,\xi)$ and $\nabla_{t,x}[\pm\lambda^{-1}\varphi(t,x,\lambda\eta)]$ in $T_{(t,x)}\mathbb{R}^{d+1}$ whenever $|\xi|_g \sim |\eta|_g \sim 1$. More precisely, let n_1 and n_2 be unit normal vectors to the two surfaces, respectively, then for every $\delta \in (0,1]$ there exists λ_0 such that for all $\lambda \le \lambda_0$ we have

$$|\langle n_1(\xi), n_2(\eta)\rangle_g| \le 1-\delta$$

whenever $|\xi|_g \sim 1$ and $|\eta|_g \sim 1$.

Sketch of the Proof of Proposition 6.15. Consider the change of variables $(\xi,\eta) \mapsto (\xi-\lambda\eta,\eta)$ which gives

$$(TF)(t,x) = \int e^{\frac{i}{h}\phi(s,x,\xi-\lambda\eta,\eta)} q_{h,m}(s,x,\xi-\lambda\eta,\eta) F(\xi-\lambda\eta,\eta) \frac{d\xi d\eta}{m^d h^{d+1}}.$$

Let us note that for t sufficiently small (6.11) gives

$$\frac{1}{\lambda}\frac{\partial^2}{\partial\eta\partial(t,x)}\varphi(t,x,\lambda\eta) = \pm\begin{pmatrix} -2\lambda\eta^T g^{-1}(x) \\ \\ I_{d\times d}\end{pmatrix} + \mathcal{O}(t),$$

which is clearly maximal rank when t is sufficiently small, and the unit normal vector to $\varphi(t,x,\xi)$ is given by

$$n_1(\xi) = \frac{1}{\sqrt{1+4|\xi|_g^2}}\begin{pmatrix} 1 \\ \\ 2g^{-1}(x)\xi\end{pmatrix} + \mathcal{O}(t).$$

Hence we have

$$\frac{1}{\lambda}n_1(\xi-\lambda\eta)^T\frac{\partial^2}{\partial\eta\partial(t,x)}\varphi(t,x,\lambda\eta) = \pm\frac{2(\xi-\lambda\eta)^T g^{-1}(x)}{\sqrt{1+4|\xi-\lambda\eta|_g^2}} + \mathcal{O}(t) \neq \mathbf{0}$$

on the region $|\xi|_g \sim 1$ and $|\eta|_g \sim 1$ whenever t is sufficiently small. In particular, if we consider the unit vector v_1 in the direction of $g(x)^{-1}(\xi-\lambda\eta)$, it is clear that

$$\left|\frac{\langle g^{-1}(x)(\xi-\lambda\eta), v_1\rangle_g}{\sqrt{1+4|\xi-\lambda\eta|_g^2}}\right| \sim \frac{|\xi-\lambda\eta|_g}{\sqrt{1+4|\xi-\lambda\eta|_g^2}} \gtrsim_\delta 1$$

whenever λ is sufficiently small. Hence the transversality condition holds.

Now let us rewrite η in terms of a new basis $\eta = pv_1 + \widetilde{\eta}$ or, simply, $\eta = (p, \widetilde{\eta})$.

Then we have that

$$\begin{aligned}
&\| \, TF \, \|_{L^2} \\
&\quad = \left\| \int \frac{d\widetilde{\eta} dp d\xi}{h^{d+1} m^d} e^{\frac{i}{h}\{\varphi(t,x,\xi-\lambda\eta) \pm \varphi(t,x,\lambda\eta)\}} \tilde{a}_{h,m}(t,x,\xi,\eta) F(\xi-\lambda\eta,\eta) \right\|_2 \\
&\quad \leq \int \frac{d\widetilde{\eta}}{h^{d+1} m^d} \left\| \int dp d\xi \; e^{\frac{i}{h}\{\varphi(t,x,\xi-\lambda\eta) \pm \varphi(t,x,\lambda\eta)\}} \tilde{a}_{h,m}(t,x,\xi,\eta) F(\xi-\lambda\eta,\eta) \right\|_2 .
\end{aligned}$$

Let us define the freezing operator $S_{\widetilde{\eta}} : L^2(\mathbb{R}^{d+1}) \to L^2(\mathbb{R}^{d+1})$.

$$S_{\widetilde{\eta}}(w) := \int dp d\xi \; e^{\frac{i}{h}\{\varphi(t,x,\xi-\lambda\eta) \pm \varphi(t,x,\lambda\eta)\}} \tilde{a}_{h,m}(t,x,\xi,p,\widetilde{\eta}) F(\xi-\lambda\eta,\eta)$$

where $\widetilde{\eta}$ is frozen. Thus, it suffices to prove that

$$\| \, S_{\widetilde{\eta}}(w) \, \|_{L^2([-\alpha,\alpha]\times\mathbb{R}^d)} \lesssim h^{\frac{d}{2}} m^{\frac{1}{2}} \| \, F \, \|_{L^2(\mathbb{R}^{2d})}$$

since $|\widetilde{\eta}| \lesssim 1$. Using a TT^*-argument, we see that

$$\| \, S_{\widetilde{\eta}}(w) \, \|_{L^2}^2 = \int d\xi dp d\xi' dp' \; K(\xi,p,\xi',p') G(\xi,p) G(\xi',p')$$

where $G(\xi,p) = G_{\widetilde{\eta}}(\xi,p) = |\xi - \lambda(p,\widetilde{\eta})|^a |(p,\widetilde{\eta})|^b F(\xi - \lambda(p,\widetilde{\eta}),(p,\widetilde{\eta}))$ and

$$K(\xi,p,\xi',p') = \int dt dx \; e^{\frac{i}{h}\{\phi(t,x,\xi,p) - \phi(t,x,\xi',p')\}} c(t,x,\xi,p,\xi',p').$$

for some smooth compactly supported function c. By applying Lemma 2.1 in

[Han12], we can estimate the kernel as follow

$$|K(\xi,p,\xi',p')| \lesssim (1+h^{-1}|\xi-\xi'|+m^{-1}|p-p'|)^{-N}.$$

Finally, by Young's inequality, we have that

$$\int d\xi dp d\xi' dp' \frac{G(\xi,p)G(\xi',p')}{(1+h^{-1}|\xi-\xi'|+m^{-1}|p-p'|)^{N}} \lesssim \| K \|_{L^1} \| G \|^2_{L^2(d\xi dp)}.$$

where

$$\| K \|_{L^1} = h^d m \int \frac{d(\xi/h)d(p/m)}{(1+h^{-1}|\xi|+m^{-1}p)^N} \lesssim h^d m.$$

Thus we arrive at the desired result. □

The remainder of the proof of Proposition 6.14 follows exactly the same line of arguments as in the proof of Proposition 6.5. □

6.2.4 Proof of Theorem 6.1 for (6.1)

In this section we prove Proposition 6.1 for (6.1). The key ingredients involved in establishing (6.3a) are Proposition 6.3, Proposition 6.13, and the following two lemmas.

Lemma 6.17. *Assume* $0 < h < m \leq 1$ *with* $\lambda = \frac{h}{m}$ *and* $\varphi \in C^\infty(\mathbb{R})$ *with* $\operatorname{supp}\varphi \subset$

$[\frac{1}{2}, 1]$. *Assume* $P^1_{\frac{1}{h}} P^2_{\frac{1}{h}} \Lambda_0 = \Lambda_0$. *Then we have the estimate*

$$\left\| P_{\frac{1}{m}} \operatorname{diag} \Lambda \right\|_{L^2([-\alpha,\alpha],\dot{H}^1(M))} \lesssim \lambda \, \| D_1 D_2^{\sigma_*} \Lambda_0 \|_{L^2(M\times M)}$$

for some $\alpha > 0$.

Proof of Lemma 6.17. It suffices to prove the statement in local coordinates. By Theorem 2.1 in [BGT04] and Proposition 6.3, we have that

$$\begin{aligned} &\left\| \kappa^*(\chi_1 \cdot P_{\frac{1}{m}} \operatorname{diag} \Lambda) - \sum_{j=0}^{N-1} m^j \psi_j(x, mD)\kappa^*(\chi_2 \cdot \operatorname{diag} \Lambda) \right\|_{L^2([-\alpha h,\alpha h],\dot{H}^1(\mathbb{R}^d))} \\ &\lesssim h^N \left\| D_1^{\frac{1}{2}} D_2^{\sigma_*} \Lambda_0 \right\|_{L^2(M\times M)} \end{aligned}$$

for any $N \geq 1$. Finally, following the proofs of Proposition 6.5 and Proposition 6.6, we see that

$$\begin{aligned} &\left\| \psi(x, mD)\kappa^*(\chi_2 \cdot \operatorname{diag}[P^1_{\frac{1}{h}} P^2_{\frac{1}{h}} \Lambda]) \right\|^2_{L^2([-\alpha h,\alpha h],\dot{H}^1(M))} \\ &\lesssim \int d\xi d\eta d\xi' d\eta' \, \frac{K(\xi,\eta,\xi',\eta')\widetilde{F}(\xi,\eta)\overline{\widetilde{F}(\xi',\eta')}}{|\xi+\eta|^{\frac{1}{2}}|\xi-\eta|^{\frac{d-1}{2}}|\xi'+\eta'|^{\frac{1}{2}}|\xi'-\eta'|^{\frac{d-1}{2}}} \\ &\qquad + \text{lower order terms} \\ &\lesssim \frac{1}{h^{3d+1}} \int d\xi \rho^{d-1} d\rho \frac{|\xi|^2 \rho^{d-2}}{(|\xi|^2+\rho^2)^{\frac{d}{2}}} |h(\xi,\rho)|^2 \\ &\lesssim \frac{\lambda^2}{h^{3d+1}} \int d\xi d\eta \; |\xi|^2 |\eta|^{d-2} |F(\xi,\eta)|^2 \end{aligned}$$

where the last inequality is a result of the facts that $|\xi| \sim \lambda$ and $\rho \sim 1$, which are consequences of rescaling and the restriction imposed by $\psi(x, mD)$. The remainder of the argument is similar to the proof of Proposition 6.5. $\square$

Lemma 6.18. *Assume $0 < h < h' < m \leq 1$. Then we have that*

$$\langle D_g P_{\frac{1}{m}} \operatorname{diag}[P^1_{\frac{1}{h}} P^2_{\frac{1}{h}} \Lambda], D_g P_{\frac{1}{m}} \operatorname{diag}[P^1_{\frac{1}{h'}} P^2_{\frac{1}{h'}} \Lambda] \rangle \lesssim_N h^N \| D_1^{\frac{1}{2}} D_2^{\sigma_*} \Lambda_0 \|^2_{L^2(M \times M)}$$

for any N and where the $\langle \cdot, \cdot \rangle$ is the standard inner product on $L^2([-\alpha h, \alpha h] \times M)$.

Proof of Lemma 6.18. It suffices to prove the statement in local coordinates. Applying Theorem 2.1 in [BGT04], Cauchy-Schwarz inequality and Proposition 6.3, we see that

$$\begin{aligned}
&\left| \langle \nabla_x \kappa^* (\chi_1 \cdot P_{\frac{1}{m}} \operatorname{diag}[P^1_{\frac{1}{h}} P^2_{\frac{1}{h}} \Lambda]), \nabla_x \kappa^* (\chi_1 \cdot P_{\frac{1}{m}} \operatorname{diag}[P^1_{\frac{1}{h'}} P^2_{\frac{1}{h'}} \Lambda]) \rangle \right| \\
&\lesssim \left| \langle \nabla_x [\psi(x, mD) \kappa^* (\chi_2 \operatorname{diag}[P^1_{\frac{1}{h}} P^2_{\frac{1}{h}} \Lambda])], \nabla_x [\psi(x, mD) \kappa^* (\chi_2 \operatorname{diag}[P^1_{\frac{1}{h'}} P^2_{\frac{1}{h'}} \Lambda])] \rangle \right| \\
&\quad + h^{\text{large positive power}} \left\| D_1^{\frac{1}{2}} D_2^{\sigma_*} \Lambda_0 \right\|^2_{L^2(M \times M)} = I + \text{ small term}
\end{aligned}$$

where the inner product is defined on $L^2([-\alpha h, \alpha h] \times \mathbb{R}^d)$. Finally, using WKB approximation, we have that

$$I \lesssim \int \frac{d\xi d\eta d\xi' d\eta'}{h^{2d+1} (h')^{2d+1}} K_{h', h}(\xi, \eta, \xi', \eta') \widetilde{F} \left(\frac{\xi}{h}, \frac{\eta}{h} \right) \overline{\widetilde{F} \left(\frac{\xi'}{h'}, \frac{\eta'}{h'} \right)}$$

where the kernel $K_{h,h'}$ can be estimated as follows

$$|K_{h,h'}(\xi, \eta, \xi', \eta')| \lesssim_N \frac{1}{(1 + h^{-1}|\xi - \xi'| + h^{-1}\mathcal{D})^N} \lesssim_N \frac{1}{(1 + h^{-1})^N}$$

since $|\xi| \sim 1$ and $|\xi'| \ll 1$. Hence by the same arguments as in the proof of Proposition 6.5 and Proposition 6.6, we arrive at the desired estimate. □

Proof of Estimate (6.3a). By the almost orthogonality property of the spectral localization of $\operatorname{diag}\Lambda$, we see that

$$\| D_g \operatorname{diag}\Lambda \|^2_{L^2([-\alpha,\alpha]\times M)} \lesssim \sum_{i=0}^{\infty} \| D_g P_{2^i} \operatorname{diag}\Lambda \|^2_{L^2([-\alpha,\alpha]\times M)} .$$

Next, employing the standard Littlewood-Paley product decomposition, we see that for each fixed i we have

$$\begin{aligned} P_{2^i} \operatorname{diag}\Lambda \approx & \left(\sum_{2^i \ll 2^j \sim 2^k} + \sum_{2^i \sim 2^j \sim 2^k} + \sum_{2^i \sim 2^j \gg 2^k} + \sum_{2^i \sim 2^k \gg 2^j} \right) P_{2^i} \operatorname{diag}(P^1_{2^j} P^2_{2^k} \Lambda) \\ =: & \; HH_{i<}\Lambda + HH_{i\sim}\Lambda + HL_{i>}\Lambda + LH_{i>}\Lambda. \end{aligned}$$

Then it follows

$$\begin{aligned} & \| D_g P_{2^i} \operatorname{diag}\Lambda \|^2_{L^2([-\alpha,\alpha]\times M)} \\ \lesssim & \; \| D_g HH_{i<}\Lambda \|^2_{L^2([-\alpha,\alpha]\times M)} + \| D_g HH_{i\sim}\Lambda \|^2_{L^2([-\alpha,\alpha]\times M)} \\ & + \| D_g HL_{i>}\Lambda \|^2_{L^2([-\alpha,\alpha]\times M)} + \| D_g LH_{i>}\Lambda \|^2_{L^2([-\alpha,\alpha]\times M)} . \end{aligned}$$

Next, let us estimate each term.

For the first term, observe by Lemma 6.17 and Lemma 6.18 we have that

$$\begin{aligned}
&\| D_g HH_{i<}\Lambda \|^2_{L^2([-\alpha,\alpha]\times M)} \\
&\lesssim \sum_{2^i \ll 2^j} \sum_{2^i \ll 2^{j'}} \langle D_g P_{2^i} \operatorname{diag}(P^1_{2^j} P^2_{2^j} \Lambda), D_g P_{2^i} \operatorname{diag}(P^1_{2^{j'}} P^2_{2^{j'}} \Lambda) \rangle \\
&\sim \sum_{2^i \ll 2^j} \| D_g P_{2^i} \operatorname{diag}(P^1_{2^j} P^2_{2^j} \Lambda) \|^2_{L^2([-\alpha,\alpha]\times M)} \\
&\lesssim \sum_{j=i+1}^{\infty} 2^{2(i-j)} \| D_1 D_2^{\sigma_*} P^1_{2^j} P^2_{2^j} \Lambda_0 \|^2_{L^2(M\times M)} .
\end{aligned}$$

Then taking the summation in i yields

$$\begin{aligned}
\sum_{i=1}^{\infty} \| D_g HH_{i<}\Lambda \|^2_{L^2([-\alpha,\alpha]\times M)} &\lesssim \sum_{j=2}^{\infty} \| D_1 D_2^{\sigma_*} P^1_{2^j} P^2_{2^j} \Lambda_0 \|^2_{L^2(M\times M)} \\
&\lesssim \| D_1 D_2^{\sigma_*} \Lambda_0 \|^2_{L^2(M\times M)} .
\end{aligned}$$

We can handle the $HH_{i\sim}\Lambda$ term in a similar manner.

Next, we consider the term $HL_{i>}\Lambda$. By Cauchy-Schwarz inequality and Proposition 6.13, we have that

$$\begin{aligned}
&\| D_g HL_{i>}\Lambda \|^2_{L^2([-\alpha,\alpha]\times M)} \\
&\sim \sum_{2^i \gg 2^k} \sum_{2^i \gg 2^{k'}} \langle D_g P_{2^i} \operatorname{diag}(P^1_{2^i} P^2_{2^k} \Lambda), D_g P_{2^i} \operatorname{diag}(P^1_{2^i} P^2_{2^{k'}} \Lambda) \rangle \\
&\lesssim \left(\sum_{k=1}^{i-1} \| D_g \operatorname{diag}(P^1_{2^i} P^2_{2^k} \Lambda) \|_{L^2([-\alpha,\alpha]\times M)} \right)^2 \\
&\lesssim \sum_{k=1}^{i-1} \| D_1 D_2^{\sigma} P^1_{2^i} P^2_{2^k} \Lambda_0 \|^2_{L^2(M\times M)} \lesssim \| D_1 D_2^{\sigma} P^1_{2^i} \Lambda_0 \|^2_{L^2(M\times M)} .
\end{aligned}$$

Finally, by almost orthogonality, we obtain the desired result. The proof is similar for the $LH_{i>}\Lambda$ term. ☐

6.2.5 Proof of Theorem 6.1 for (6.2)

Lemma 6.19. *Let* $0 < h < m \leq 1$ *with* $\lambda = \frac{h}{m}$ *and* $\varphi \in C^\infty(\mathbb{R})$ *with* $\operatorname{supp}\varphi \subset [\frac{1}{2}, 1]$. *Assume* $P^1_{\frac{1}{h}} P^2_{\frac{1}{h}} \Gamma_0 = \Gamma_0$. *Then we have the estimate*

$$\left\| P_{\frac{1}{m}} \operatorname{diag}\Gamma \right\|_{L^2([-\alpha,\alpha],\dot{H}^1(M))} \lesssim \lambda^{\frac{1}{2}} \left\| D_1 D_2^{\sigma_*} \Gamma_0 \right\|_{L^2(M\times M)}$$

for some $\alpha > 0$.

Sketch of Proof of Lemma 6.19. If suffices to consider the modification of the proof of Proposition 6.12. In the current case, we have $|\xi| \sim \lambda$ which means

$$\begin{aligned}
\text{RHS (6.26)} &\lesssim \frac{\lambda^2}{h^{4d+2}} \int_{|\eta|\sim 1, |\eta'|\sim 1} \frac{\widetilde{F}(\xi,\eta)\overline{\widetilde{F}(\xi',\eta')}\ d\xi d\eta d\xi' d\eta'}{(1 + h^{-1}|\xi - \xi'| + h^{-1}\left|\xi\cdot\eta - \xi'\cdot\eta'\right|)^N} \\
&\lesssim \frac{\lambda^2}{h^{4d+2}} \int \frac{G(\xi,p)G(\xi',p')\ d\xi dp d\xi' dp'}{(1 + h^{-1}|\xi - \xi'| + h^{-1}\left||\xi|p - |\xi'|p'\right|)^N} \\
&\lesssim \frac{\lambda}{h^{3d+1}} \int d\xi d\eta\ |\xi|^2 |\eta|^{d-2} |F(\xi,\eta)|^2.
\end{aligned}$$

The rest of the proof is standard. ☐

Let us now complete the proof of Theorem 6.1 for (6.2).

Proof of Estimate (6.3b). Fix i. As in the proof of estimate (6.3a), we immediately have that

$$\begin{aligned}
&\| D_g P_{2^i} \Gamma \|^2_{L^2([-\alpha,\alpha]\times M)} \\
&\lesssim \| D_g HH_{i<} \Gamma \|^2_{L^2([-\alpha,\alpha]\times M)} + \| D_g HH_{i\sim} \Gamma \|^2_{L^2([-\alpha,\alpha]\times M)} \\
&\quad + \| D_g HL_{i>} \Gamma \|^2_{L^2([-\alpha,\alpha]\times M)} + \| D_g LH_{i>} \Gamma \|^2_{L^2([-\alpha,\alpha]\times M)} .
\end{aligned}$$

For the first term, applying Lemma 6.19 and Cauchy-Schwarz inequality yields

$$\begin{aligned}
&\| D_g HH_{i<} \Gamma \|^2_{L^2([-\alpha,\alpha]\times M)} \\
&\lesssim \sum_{2^i \ll 2^{j'} \sim 2^j} \left\| D_g P_{2^i} \operatorname{diag}(P^1_{2^j} P^2_{2^j} \Gamma) \right\|^2_{L^2([-\alpha,\alpha]\times M)} \\
&\quad + \sum_{2^i \ll 2^{j'} \ll 2^j} \langle D_g P_{2^i} \operatorname{diag}(P^1_{2^j} P^2_{2^j} \Gamma), D_g P_{2^i} \operatorname{diag}(P^1_{2^{j'}} P^2_{2^{j'}} \Gamma) \rangle \\
&\lesssim \sum_{j=i+1}^{\infty} 2^{i-j} \left\| D_1 D_2^{\sigma_*} P^1_{2^j} P^2_{2^j} \Gamma_0 \right\|^2_{L^2(M\times M)} \\
&\qquad + \sum_{j'=i+1}^{\infty} 2^{i-j'} \left\| D_1 D_2^{\sigma_*} P^1_{2^{j'}} P^2_{2^{j'}} \Gamma_0 \right\|_{L^2(M\times M)} \\
&\qquad\qquad \times \left(\sum_{j=j'+1}^{\infty} 2^{\frac{j'-j}{2}} \left\| D_1 D_2^{\sigma_*} P^1_{2^j} P^2_{2^j} \Gamma_0 \right\|^2_{L^2(M\times M)} \right)^{\frac{1}{2}} .
\end{aligned}$$

Then it follows from Cauchy-Schwarz inequality that

$$\begin{aligned}
&\sum_{i=1}^{\infty} \| D_g H H_{i<} \Gamma \|^2_{L^2([-\alpha,\alpha]\times M)} \\
&\lesssim \sum_{j=1}^{\infty} \| D_1 D_2^{\sigma_*} P^1_{2^j} P^2_{2^j} \Gamma_0 \|^2_{L^2(M\times M)} \\
&\qquad + \sum_{j'=2}^{\infty} \| D_1 D_2^{\sigma_*} P^1_{2^{j'}} P^2_{2^{j'}} \Gamma_0 \|_{L^2(M\times M)} \\
&\qquad\qquad \times \left(\sum_{j=j'+1}^{\infty} 2^{\frac{j'-j}{2}} \| D_1 D_2^{\sigma_*} P^1_{2^j} P^2_{2^j} \Gamma_0 \|^2_{L^2(M\times M)} \right)^{\frac{1}{2}} \\
&\lesssim \ \sum_{j=1}^{\infty} \| D_1 D_2^{\sigma_*} P^1_{2^j} P^2_{2^j} \Gamma_0 \|^2_{L^2(M\times M)} \\
&\qquad + \left(\sum_{j'=1}^{\infty} \| D_1 D_2^{\sigma_*} P^1_{2^{j'}} P^2_{2^{j'}} \Gamma_0 \|^2_{L^2(M\times M)} \right)^{\frac{1}{2}} \\
&\qquad\qquad \times \left(\sum_{j'=1}^{\infty} \sum_{j=j'+1}^{\infty} 2^{\frac{j'-j}{2}} \| D_1 D_2^{\sigma_*} P^1_{2^j} P^2_{2^j} \Gamma_0 \|^2_{L^2(M\times M)} \right)^{\frac{1}{2}} \\
&\lesssim \ \| D_1 D_2^{\sigma_*} \Gamma_0 \|^2_{L^2(M\times M)}
\end{aligned}$$

The remainder of the proof is similar to case of (6.1). □

Chapter 7: Conclusion and Discussion

In many ways, the time-dependent HFB system for bosons offers a remarkable generalization of cubic NLS and Hartree equation. From a mathematical physics perspective, the system provides a nontrivial correction to the mean-field equation which allows for the studies of effective dynamics of the excitation of quantum gas. In fact, Theorem 4.1 states that the time-dependent HFB system is rich enough to capture details of the many-body system and provide Fock space estimates for the true dynamics of quasifree states.

To put the results of Chapter 4 in context, we compare Theorem 4.1 and Remark 4.2 to Theorem 1.1 in [BCS17]. Reader should note that the theorems are similar in spirit, but the nature of the results are different. In [BCS17], the authors imposed a condition on the structure of the pair excitation function k, which only depends dynamically on the evolution of the condensate, then use it to obtain Fock space approximation to the true dynamic when $0 < \beta < 1$. In fact, they were able to prove a global-in-time Fock space error estimate which is double-exponential in time.

On the other hand, Theorem 4.1 allows for the dynamical development of the pair excitation function, which in some sense is more general than the results in [BCS17]. But, the regularity assumption on the initial data in Theorem 4.1 imposes restriction on the form of k. In particular, one could show that the regularity assumption rules out the case of the coherent states, i.e. the case $k = 0$. However, the recent result of Grillakis and Machedon in [GM18] suggests that more general data is permissible. It is conjectured that we could even consider coherent states initial data and obtain Fock space estimate on the development of correlation structure in the long time dynamics of the initial state. Nevertheless, even with the restriction, the set of permissible initial conditions in Theorem 4.1 is still comparable to that of [BCS17].

Lastly, the error estimate of Theorem 4.1 is valid for a much longer period of time when compare to Theorem 1.1 in [BCS17]. The improvement in time of the Fock space estimate in Theorem 4.1 is a result of our analysis of Λ. More precisely, the improvement is the result of (4.1a), which ultimately helped us to avoid the usage of Grönwall estimate. In fact, we believe the Fock space estimate could be further improved if we improve our estimates for Λ. One conjecture we expect to be true is that there exists $\kappa > 0$ such that

$$\| \psi_{\text{exact}}(t) - \psi_{\text{approx}}(t) \|_{\mathcal{F}} \lesssim \frac{\exp(\kappa T)}{N^{\frac{1-\beta}{2}}} \tag{7.1}$$

for all $t \in [0, T]$ for all N. In fact, to establish (7.1) it suffices to show

$$\| \nabla_x \nabla_y \Lambda(t, x, y) \|_{L^\infty(dt) L^2(dxdy)} \lesssim N^{\text{some power}}. \tag{7.2}$$

Note that (7.2) is interesting in its own right. Problems of this flavor can be traced back to the works of Bourgain on the growth of Sobolev norms for linear Schrödinger equations (c.f. [Bou99a, Bou99b]).

A close examination of (7.2) shows that the current approach of estimating $\nabla_x \nabla_y \Lambda(t)$ via energy method will not be sufficient in establishing the estimate. In fact, to prove (7.2), we will need to employ global-in-time Strichartz estimates. Unfortunately, obtaining global-in-time Strichartz estimates for the time-dependent HFB equations is in general a formidable task. In chapter 5, we establish the interaction Morawetz-type estimate for Γ using the Virial interaction potential

$$V^a(t) = \int dxdy\ \Gamma(t, x, x)a(x - y)\Gamma(t, y, y).$$

The approach adopts the method used in [CPT12] to establish the interaction Morawetz estimates for the BBGKY (Gross-Pitaevskii) hierarchy. However, a major difference between the BBGKY (Gross-Pitaevskii) hierarchy and the time-dependent HFB system is the obvious fact that the time-dependent HFB system is nonlinear whereas the Gross-Pitaevskii hierarchy is linear. Hence instead of following [CPT12] and use the second marginal density, i.e. consider the Virial interaction potential

$$V^a(t) = \int dxdy\ \mathcal{L}_{2,2}(t, x, y, x, y)a(x - y),$$

we replaced $\mathcal{L}_{2,2}(t, x, y, x, y)$ by $\Gamma(t, x, x)\Gamma(t, y, y)$. In doing so, we were able to prove the interaction Morawetz estimate

$$\| \Gamma(t,x,x) \|_{L^2(dtdx)} \lesssim \| \Gamma_0 \|_{H^{1/2}(dx)} .$$

Of course, the immediate question that follows is whether a similar type of estimate holds for Λ. The calculation using $\mathcal{L}_{2,2}$ in the Virial interaction potential is highly involved and we have no guarantee whether the idea will bear fruit.

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