A FIELD THEORY OF EXTENDED PARTICLES BASED ON COVARIANT HARMONIC OSCILLATOR WAVEFUNCTIONS

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Title of Dissertation: A Field Theory of Extended Particles Based on Covariant Harmonic Oscillator Wavefunctions

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ABSTRACT

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We attempt to combine the covariant harmonic oscillator(CHO) quark model with second quantized field theory. We review the CHO formalism for a system of two quarks(meson). We introduce a mesonic field $\Phi(x_1,x_2)$ that depends on the position of both quarks, and then derive the field equations from a covariant lagrangian $L(x_1,x_2)$. The CHO equation allows a complete separation of the average meson coordinate X from the relative quark coordinate ξ. The CHO wavefunction in the field expresses the extended size and internal structure of the meson. Φ describes mesons in the ground state and any excited state, with angular momentum $\propto$ mass^2. From Φ we construct conserved tensors like $\hat{P}^\mu$, the meson momentum. We second quantize Φ in the X variable only and discuss the extended particle commutation relations.

We investigate a Φ^3-type meson interaction where the vertex function is an overlap integral of the wavefunctions entering the interaction region. We derive a nonlinear integrodifferential equation for the U matrix, linearize and solve it by perturbation theory. The result is simple diagrammatic rules for the S matrix. The S matrix is covariant and unitary. We do not find any contradiction between the principles of QFT and the CHO quark model. The

Φ field theory includes scalar meson (point particle) theory as a special case, while its greater generality illuminates the difference between point and extended particles.

PROLOGUE

We look for it and do not see it;
     Its name is The Invisible.

We listen to it and do not hear it;
     Its name is The Inaudible.

We touch it and do not find it;
     Its name is the Subtle(Formless).

These three cannot be further enquired into,
     And hence merge into one.

Going up high, it is not bright, and coming down low,
     it is not dark.

Infinite and boundless, it cannot be given any name;
     It reverts to nothingness.

This is called shape without shape,
     Form without object.

It is the Vague and Elusive.

Meet it and you will not see its head.

Follow it and you will not see its back.

Hold on to the Tao of old in order to master the
     things of the present.

From this one may know the primeval beginning[of
     the universe].

This is called the bond of Tao.

Lao Tzu
<u>Tao-Te Ching</u>
as translated in <u>A Source
Book in Chinese Philosophy</u>, compiled
by Wing-Tsit Chan

27

Myself when young did eagerly frequent
Doctor and Saint, and heard great Argument
About it and about; but evermore
Came out by the same Door as in I went.

28

With them the Seed of Wisdom did I sow,
And with my own hand labour'd it to grow;
And this was all the Harvest that I reap'd--
'I came like Water, and like Wind I go.'

Rubâiyât of Omar Khayyâm
as translated by Edward Fitzgerald

Mysterious even in open day,
Nature retains her veil, despite our clamors:
That which she does not willingly display
Cannot be wrenched from her with levers, screws,
and hammers.

Faust monologue, from
Goethe's Faust
Part I, Scene I
as translated by Anthony Scenna

DEDICATION

My life, like any other, is the encounter of many wandering souls. I dedicate this with love to them all,

Galo Aumada
Mary Gibson
Bill Wheatly
Susan Harvey
Dave Douglas
Phil Ochs
Valerie Blalock
Ted(Vasco)Miller
Duna Poteet
Mike Kozma
Anthony Holdsworth
Walt Whitman
Anne Henry
Tom Roberts
Charles Schwartz
Ho Chih Minh
Steve Slaby
Emma Goldman
Jules Henry
Lynn Fagot
Chuck Goldman
Herbert Marcuse
Pancho Otero
Maria Fletcher
Dennis(Mack)Greene
"Tucson" Salant

Salahuddin El Hilali
Halem
Anna & Tomas Gergerly
Maggie Aumada
Alan Ginzburg
Sue Ann
Lao-Tzu
Leo Tolstoy
Bertolt Brecht
Bob Dylan
Joseph & Valeria
Thure Meyer
Dave Dellinger
Ursula Leguin
Julia Vinograd
Clarence Didonato
Lina Wertmuller
Tiwiwas
Jaques Ellul
Sharon Fondiller
Max Planck
Ernest Rutherford
Gail Schatz
Stan Gooch
Dwight Wilkinson
Homa Touhidi

and all the others just a breath away...

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True education, if it is attained at all, happens by a process of fortuitous accident. Among these accidents we must count the circumstances of our early youth, the people we meet and the way they impress us, the interests they arrouse in us and our opportunity to develope those interests. I have made education the central activity of my life, so all the faculties of my self have been brought into play and been reciprocally affected. As a part of life, education is uncontrolled and unpredictable. The result becomes different from the sum of the parts. Recognizing this, I know there is no fair way to give appreciation to all who have helped me. I feel warmly towards those I have time to name, and warmly towards those I cannot name.

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CONTENTS

1. INTRODUCTION

Physicists have called a "point particle" any system whose structure was too small to be observable or relevant at the time. Early, striking experimental evidence that nucleons are not point particles came in the observation of nucleon form factors in 1956[1]. This was the first firm evidence that nucleons have an intrinsic size, which sharply distinguishes them from leptons. Beginning with πp scattering experiments and the discovery of the $\Delta(1232)$ resonance[2], high energy physics has unfolded a rich and complex spectrum of hadronic states[3]. All our previous experience with similar spectra in atoms and nuclei suggests that the behavior of each hadronic state is due to its specific internal structure or the interrelation of its parts. But our deepest view inside the hadron is in the contemporary version of the Rutherford and Frank-Hertz experiments, namely deep inelastic lepton-nucleon scattering[4]. Here we observe high momentum transfer events(which signal collision with a point constituent inside the nucleon)as well as asymptotic cross sections consistent with the parton picture[5]. Taking these developments together we see the pressing need for a dynamical theory which explains the evident structure and size of hadrons.

Relativistic dynamics is notoriously difficult and has confounded all attempts at such a theory. We are not even sure of what framework to pose the problem in. While we have a reliable nonrelativistic mechanics, classical and quantum, in which we can discuss and evaluate conjectured dynamical theories of nonrelativistic phenomena, we have no such ready-made framework for the relativistic phenomena that must exist inside hadrons. So particle physicists have tried to explain much of hadronic behavior by symmetry principles independent of detailed dynamics. SU(3) symmetry is the most famous and successful discovery of this program[6]; SU(3) classification has brought order to what would otherwise be a chaotic experimental situation[7]. The quark[8] was invented to explain SU(3) symmetry, although the inventors made no committment to its existence. The quark hypothesis became the first serious basis for deducing hadronic properties from the interrelation of constituents inside the hadron. Models that treat the quark as a real particle and a hadron as a multiquark bound system have produced many encouraging results[9]. But the validity of these models is uncertain(independent of their faithfulness to experiment) because their underlying dynamics is uncertain. For example, the "naive" or standard quark model is formulated in the language of nonrelativistic quantum mechanics: the hadron is represented by a multiparticle wavefunction, the quarks move in an effective or a mutual potential well corresponding to a postulated "force" between quarks, and the system is bound into a hadron because the wavefunction is confined to the potential well. But we expect all this is "wrong" because it is nonrelativistic, and we do not know how much of this model is deducible from a consistent relativistic theory. Another example is the SU(6) symmetry scheme, which is very plausible nonrelativistically and also phenomenologically successful but has been proven inconsistent with relativity[10].

Second quantized field theory is a successful framework in which to discuss the relativistic scattering of point particles. It seems plausible that we can construct a consistent relativistic dynamics of interacting quark fields. But such field theories have not told us much about real hadrons. For example, the early quark models put the quarks in an infinite potential well for calculational convenience[11], and this explained(in the context of the model)why isolated quarks are never observed. But in a quark field theory this simple fact is unexplained. We cannot obtain any bound state information from a finite order of perturbation theory and there are no practical nonperturbative methods. Also, it is not clear how the techniques that were successful for asymptotically free states(scattering)should be modified to apply to states that are never free. Finally, the field theory must replace the bound state wavefunction of the nonrelativistic model by an eigenstate of a hamiltonian operator built from quark(and other) fields. We do not even know in what Hilbert space to represent this state; finding such a state(or proving it exists)seems to require a nonperturbative solution to the field equations. Nevertheless there are many papers about local field mechanisms that might create permanently bound states[12].

In nonrelativistic quantum mechanics the dynamical origin of a bound state, its detailed properties and its spacial localization are all interdependent. Although the usual field theory has difficulty in reproducing any of these features, QFT is especially unsuccessful in incorporating the idea of spacial localization or size. To escape this problem some models modify the usual QFT, then make a semiclassical approximation to the new field equations and show that the approximate solution for the quark field(or products of fields)is "trapped" in a region of finite but nonzero size[13]. We do not know whether the solutions(if they exist)to the full quantum equations of these models have the same trapping feature, nor do we know how to relate the localized quantum field(if it exists)to the probabilistic wavefunction. Other models depart from local QFT more radically, by taking the spacial extension of the hadron to be a fundamental fact and attempting to describe the direct strong interaction of these extended objects. String models are examples of this approach[14,15]. These models bypass the question of how quarks dynamically bind thgether, since the extended object is fundamental and has no dynamical origin. But string models have not been able to incorporate a gauge invariant electromagnetic interaction or explain the observed hadronic form factors(the experimental basis for believing hadrons are extended particles). Some string models fail for more basic reasons--they require more than four spacetime dimensions, or they contain tachyon or ghost states[14].

Where does all this leave the standard quark model? The most serious fault of the early model was its nonrelativistic dynamics--wavefunction, potential, etc. Although the usual QFT has not become the relativistic dynamical foundation of the standard model, the standard model has developed its own relativistic form. A relativistic wavefunction in the harmonic oscillator quark model was first proposed by Feynman et al.[16], who stated that it would be difficult to expect dynamical regularities among resonances from the conventional field theory and that it is worth considering a new relativistic theory which is naive and obviously wrong in its simplicity but which is definite and enables us to calculate as many things as possible.

The nonrelativistic wavefunction has a clear interpretation as a probability amplitude, and probability has an important role in quark model calculations. A new theory clearly would be wrong or incomplete if its wavefunction did not have a probability interpretation.

The wavefunctions of Feynman et al. are not normalizable and do not produce correct form factors. Lipes[17] attempted to reformulate their work, but his wavefunctions do not covariantly satisfy the equation of motion. Kim and Noz[18] constructed the first normalized ghost-free wavefunctions that are fully covariant. They also defined an inner product that gives the wavefunction a Lorentz-contracted probability interpretation[18,19]. The asymptotic dipole behavior of nucleon form factors has been related to the Lorentz contraction of the wavefunction by Fujimura et al.[20]. Radial excitations of these wavefunctions produce linearly spaced energy levels, and Kim and Noz[21] have established the existence of radial-like modes for the nonstrange baryons(where there is just enough evidence to test the linearity hypothesis). By taking probability overlap integrals between wavefunctions, Kim and Noz[22] did a preliminary calculation of meson decay widths and polarizations and Ruiz[23] calculated the axial vector coupling constant in nuclear β-decay. Those calculations show that the covariant harmonic oscillator wavefunction with a probability interpretation combines mathematical simplicity with predictive accuracy.

But a wavefunction, even a covariant one, is only a first-quantized object. Second-quantized field theory seems the most natural language for describing the creation and annihilation of particles in high energy physics, while the first-quantized language of wavefunctions with a probability interpretation has been useful in the standard quark model for describing the internal structure of hadrons. What is the relation between these two languages[24]? Is there a quantum field theory that incorprates a probabilistic wavefunction description of a particle's internal structure, and that is also consistent with the accepted relativistic principles of covariance, causality, and unitarity? The purpose of this paper is to construct such a theory, using the wavefunctions of Kim and Noz.

We will construct an interacting quantum field theory of mesons in which each meson is a bound state of two quarks. The bound state is represented by a covariant harmonic oscillator wavefunction. Our main purpose is to show how quantum field theory ideas can be combined with wavefunction-probability ideas into a consistent and physically nontrivial theory. For simplicity, the quarks in this theory have no internal quantum numbers -- no charge, no SU(3) quantum numbers, no spin. The quantum field we will construct incorporates the orbital angular momentum of the quark wavefunction in a natural way, which gives the composite mesons intrinsic integral angular momentum.

In Section 2 we review and reformulate the formalism of the Kim-Noz oscillator wavefunctions and inner product. Our reformulation allows us to give an elegant and general discussion of the transformation properties of all the covariant harmonic oscillator(CHO)wavefunctions. In Section 3 we put the CHO theory in lagrangian form. This gives us a theory of a classical free meson field whose internal meson states are described by CHO wavefunctions. We show how Noether's theorem, appropriately modified for this lagrangian, gives the usual conservation laws. In the process we find several conditions which a wavefunction must satisfy for its lagrangian theory to make physical sense. In Section 4 we second quantize our lagrangian and discuss the Poincaré transformations of the quantized field. Our field does not have canonical commutation relations. The difference between our CR and the CCR is due to internal meson structure, expressed by the CHO wavefunction. In Section 5 we turn on interactions after first reviewing the interaction model of Kim-Noz[22]. In Section 6 we write a perturbation expansion for the S matrix of this field theory and derive covariant diagrammatic rules of calculation. We discuss some similiarites and differences with ordinary QFT. In Section 7 we prove that our perturbation series is unitary. Our analysis illuminates the different role of internal quark motion in the vertices and the propagators of our theory. In Section 8 we review the general features of our theory that make it work, and we discuss some possible extensions and applications.

2. COVARIANT HARMONIC OSCILLATOR WAVEFUNCTIONS

The notation in this paper is as follows. Four-vectors are written as x, X; their contravariant components are x^{μ} ($\mu = 0, 1, 2, 3$), with time component x^0 and space components x^i ($i = 1, 2, 3$) or $\vec{x}$. The metric tensor $g^{\mu\nu}$ is $g^{00} = -g^{ii} = +1$, all other components = 0. Four-dimensional volume elements are written as dx, three dimensional spacial volume elements as $d\vec{x}$. The summation convention is used throughout.

$x\cdot y \equiv x^{\mu}y_{\mu}$, $x^2 \equiv x\cdot x$, $\vec{x}\cdot\vec{y} \equiv x^i y^i$, $\frac{\partial}{\partial x^{\mu}} \equiv \partial_{x}\mu$, and $\partial_{x\mu}\partial_x^{\mu} \equiv \Box_x$. When Λ denotes a Lorentz transformation, $y = \Lambda x$ denotes the matrix equation $y^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu}$. When $(x-y)^2 < 0$, we write $x \sim y$. The complex conjugate of a c-number is denoted by *, and the hermitean adjoint of an operator is denoted by †. The normal ordering of creation/annihilation operators is denoted by double dots : : .

We consider a system of two quarks, with spacetime location x_1, x_2. The state of the system is specified by a two-point field $\Phi(x_1,x_2)$. We are interested in Φ which satisfy the harmonic oscillator equation[25]:

$$\left[\Box_{x_1} + \Box_{x_2} - \frac{\omega^2}{32}(x_1-x_2)^2 + \frac{m_0^2}{2}\right]\Phi(x_1,x_2) = 0 \quad . \tag{2.1}$$

We define new coordinates by

$$X \equiv \frac{1}{2}(x_1 + x_2) \qquad \xi \equiv \frac{1}{2}(x_1 - x_2) \; .$$

X is called the average or external coordinate, ξ is called the relative or internal coordinate. Then $\Phi(x_1,x_2) = \Phi(X,\xi)$, $\Box_{x_1} + \Box_{x_2} = \frac{1}{2}(\Box_X + \Box_{\xi})$ and Eq. (II.1) becomes

$$\left[\Box_X + \Box_{\xi} - \frac{\omega^2}{4}\xi^2 + m_0^2\right]\Phi(X,\xi) = 0 \quad . \tag{2.2}$$

Fourier analyze Φ into normal modes of ξ :

$$\Phi(X, \xi) = \sum_k \frac{1}{(2\pi)^{3/2}} \int dP\, e^{iPX}\, \Phi_k\,(P,\xi) \quad , \tag{2.3}$$

where

$$\left[\Box_\xi - \frac{\omega^2}{4}\, \xi^2 \right] \Phi_k(P,\xi) = E(k)\; \Phi_k\; (P,\xi) \tag{2.4}$$

with $E(k)$ a real number, and where

$$(-P^2 + m_0{}^2 + E(k))\Phi_k\; (P,\xi) = 0 \quad . \tag{2.5}$$

The general solution to Eqs. (2.4) and (2.5) is

$$\Phi_k(P,\xi) = \delta(P^2 - m_0{}^2 - E(k))\; a_k(P)\phi_k(P,\xi) \tag{2.6}$$

(no sum over k)

where $a_k(P)$ is an arbitrary function of P and $\phi_k(P,\xi)$ solves Eq. (2.4). k represents all the quantum numbers that identify ϕ_k.

The hyperbolic partial differential equation, Eq. (2.4), is a relativistic generalization of the Schrödinger equation for an isotropic harmonic oscillator. It has many solutions, depending on the boundary conditions imposed. The solutions used by Kim and Noz are constructed as follows. We define the 4-dimensional oscillator ladder operators in arbitrary coordinates ξ^μ as

$$b^\mu \equiv \frac{1}{\sqrt{\omega}}\; (-\partial_\xi^\mu + \frac{\omega}{2}\, \xi^\mu), \qquad b^{\mu\dagger} \equiv \frac{1}{\sqrt{\omega}}\; (\partial_\xi^\mu + \frac{\omega}{2}\, \xi^\mu) \tag{2.7}$$

so that

$$[b^\mu,\; b^{\nu\dagger}] = -g^{\mu\nu} \quad . \tag{2.8}$$

Then b^0, $b^{i\dagger}$ $(i = 1,2,3)$ are the creation operators. Define the number operators as

$$\hat{n}_0 \equiv b^0\; b^{0\dagger} = b^{0\dagger} b_0 - 1 \tag{2.9}$$

$$\hat{n}_i \equiv b^{i\dagger} b^i = -b^{i\dagger} b_i \;\; \text{(no sum over i)} \quad .$$

Then $b^{\mu\dagger}$, $\hat{n}_\mu$, and the differential operator of Eq. (2.4) are connected by the following hermitean relation,

$$b^{\mu\dagger}b_\mu = \hat{n}_0 - \hat{n}_1 - \hat{n}_2 - \hat{n}_3 + 1 = \frac{1}{\omega}\left[-\Box_\xi + \frac{\omega^2}{4}\xi^2 + 2\omega\right] . \tag{2.10}$$

Let k represent the set of non-negative integers (n_0, n_1, n_2, n_3), and define

$$N(k) \equiv n_1 + n_2 + n_3 - n_0 \tag{2.11}$$

$$E(k) \equiv \omega N(k) + \omega \tag{2.12}$$

$$= \omega(n_1 + n_2 + n_3 - n_0) + \omega$$

$$m_k^2 \equiv m_0^2 + E(k) \tag{2.13}$$

and, for $n_0 = 0$,

$$(P_o)^0 \equiv m_k, \ \vec{P}_o = 0 \quad . \tag{2.14}$$

Then we define the CHO wavefunction $\phi_k(P_o)$ as

$$\phi_k(P_o,\eta) \equiv A_k H_{n_1}(\sqrt{\tfrac{\omega}{2}}\,\eta_1) H_{n_2}(\sqrt{\tfrac{\omega}{2}}\,\eta_2) H_{n_3}(\sqrt{\tfrac{\omega}{2}}\,\eta_3) \exp\{-\tfrac{\omega}{4}[\eta_0^2+\eta_1^2+\eta_2^2+\eta_3^2]\} , \tag{2.15}$$

where H_n is the nth Hermite polynomial and A_k is a normalization factor (for the inner product defined below). The phase of A_k is adjusted so that $\phi_k(P_o,\eta)$ is real for all k,η. $\phi_k(P_o)$ clearly solves Eq. (2.4) in the η^μ coordinates. $\phi_k(P_o)$ is the *Kim-Noz wavefunction in the rest frame* of the two quark system.

Eq. (2.10) shows that N(k) and E(k) are the eigenvalues of Lorentz invariant operators, so we may define $\phi_k(P)$ for an arbitrary P with $P^2 = m_k^2$ by Lorentz transforming $\phi_k(P_o)$ to a new coordinate frame. Pick a set of Lorentz transformations $\{C(P)\}$ such that for each P with $P^0 > 0$ and $P^2 = m_k^2$,

$$P = C(P)P_o \tag{2.16}$$

and such that the matrix elements $C(P)^\mu_{\ \nu}$ are continuous functions of P^μ. Then define $\phi_k(P)$ as

$$\phi_k(P,\xi) \equiv \phi_k(P_o, C(P)^{-1}\xi) \quad . \tag{2.17}$$

$\phi_k(P)$ is the *wavefunction in a frame where the two quark system has momentum* P. It clearly solves Eq. (2.4). Eq. (2.17) is consistent with the definition used by Kim and Noz[18,19] for $\vec{P}$ parallel to the 3-axis and generalizes their definition to arbitrary $\vec{P}$.

Eq. (2.17) defines $\phi_k(P)$ for all P with $P^0 > 0$ and $P^2 = m_k^2$. The decomposition of $\Phi(X,\xi)$ in Eq. (2.3) includes a sum over momenta with $P^0 < 0$, so $\phi_k(P)$ must also be defined for negative energy. The simplest way to extend $\phi_k(P)$ from $P^0 > 0$ to $P^0 < 0$ is to apply Eqs. (2.15) and (2.17) for $(P_o)^0 = -m_k$. Then

$$\phi_k(P^0 < 0, \vec{P},\xi) = \phi_k(|P^0|,-\vec{P},\xi),$$

that is

$$\phi_k(P,\xi) = \phi_k(-P,\xi) \quad . \tag{2.18}$$

Eq. (2.18) should be compared with Dirac theory[28], where the negative energy spinor v(p,s) describes a positive energy antiparticle moving with momentum $-\vec{p}$.

For an arbitrary proper Lorentz transformation Λ, Eq. (2.17) implies that

$$\begin{aligned}\phi_k(\Lambda P, \Lambda\xi) &= \phi_k(P_o, C(\Lambda P)^{-1} \Lambda\xi) \\ &= \phi_k(P_o, C(\Lambda P)^{-1} \Lambda C(P)C(P)^{-1}\xi) \\ &= \phi_k(P_o, M(\Lambda,P)\ C(P)^{-1}\ \xi)\end{aligned} \tag{2.19}$$

where

$$M(\Lambda.P) \equiv C(\Lambda P)^{-1} \Lambda C(P) \tag{2.20}$$

is a rotation. It is usual[27] to call P_o the fundamental vector, {C(P)} the complementary set, and $M(\Lambda,P)$ the Wigner rotation. Different choices of complementary set give different (but simply related) definitions for $\phi_k(P)$. For any complementary set Eqs. (2.15) and (2.17) imply that the gaussian factor in $\phi_k(P_o)$ becomes

$$\exp\{-\frac{\omega}{4}[-\xi^2 + \frac{2(P\cdot\xi)^2}{m_k^2}]\} \tag{2.21}$$

in $\phi_k(P)$.

Since $n_0 = 0$ in $\phi_k(\overset{P}{o})$, which means there are no excitations of the time coordinate in the rest frame, it follows that $\phi_k(P)$ satisfies (in addition to Eq. (2.4)) the covariant subsidiary condition

$$P^\mu b_\mu^\dagger \phi_k(P) = P^\mu \left[\frac{1}{\sqrt{\omega}} \partial_{\xi\mu} + \frac{\sqrt{\omega}}{2} \xi_\mu \right] \phi_k(P,\xi) = 0 \quad . \qquad (2.22)$$

This condition distinguishes the wavefunctions used by Kim and Noz from other relativisitc quark wavefunctions[16,17]. Time coordinate excitations have caused trouble with the harmonic oscillator quark model in the past[16,17]. If n_0 is arbitrary, then there is an infinite degeneracy of wavefunctions $\phi_k(P)$ for fixed P and E(k) and there can exist wavefunctions with $m_k^2 < 0$. The subsidiary condition, Eq. (2.22), eliminates these undesirable features in a manifestly covariant way. Eq. (2.24) combined with Eqs. (2.3) and (2.6) gives a similar condition[28] on $\Phi(X,\xi)$:

$$\partial_X{}^\mu (\partial_\xi{}^\mu + \frac{\omega}{2} \xi_\mu) \Phi(X,\xi) = 0 \quad . \qquad (2.23)$$

Consider the rest frame wavefunction of Eq. (2.15). As a function of $\eta, \phi_k(\overset{P}{o},\eta)$ is an isotropic harmonic oscillator. The set of $\phi_k(\overset{P}{o})$ with E(k) (or N(k)) fixed is a basis for a linear space of functions that is invariant under rotations of $\vec{\eta}$. This set is called the "Hermite basis." Define the internal orbital angular momentum operators as

$$L_k \equiv i\varepsilon_{k\ell m} \eta^\ell \partial_\eta^m, \qquad L^2 = L_1^2 + L_2^2 + L_3^2 \qquad (2.24)$$

where $\varepsilon_{k\ell m}$ is the 3-dimensional antisymmetric tensor. By taking appropriate linear combinations of the $\phi_k(P)$ (with E(k) fixed) we can define[29] a new set of functions $\phi_\alpha(\overset{P}{o})$ that are eigenfunctions of L^2 and L_3, with eigenvalues L(L+1) and M respectively. α represents the quantum numbers (E, L, M) which completely identify $\phi_\alpha(P)$. We define $\phi_\alpha(P)$ for arbitrary P with $P^2 = m_k^2$ by Eq. (2.17) with $\phi_\alpha(P)$ replacing $\phi_k(P)$. Then

$$\phi_\alpha(P) = u_{\alpha k}\phi_k(P) \tag{2 .25}$$

where $u_{\alpha k}$ is a unitary matrix that depends on $E(\alpha)$ and $L(\alpha)$, and the sum runs over all k with $E(k)$ fixed. The set of $\phi_\alpha(P)$ is called the "orbital basis". Latin subscripts denote functions in the Hermite basis, $\phi_k(P)$, and Greek subscripts denote functions in the orbital basis, $\phi_\alpha(P)$. The subscript α on wavefunctions is not to be confused with the index on four-vector components like X^μ.

The space of CHO wavefunctions, invariant under rotations, labeled by $E(k)$ or $E(\alpha)$ is now split into irreducible subspaces (under rotations) labeled by $(E(\alpha), L(\alpha))$. Since the $\phi_\alpha(P_o,\eta)$ are just spherical harmonics (in their dependence on the direction $\vec{\eta}$), Eq. (2 .19) for $\phi_\alpha(P)$ is

$$\begin{aligned}\phi_\alpha(\Lambda P,\Lambda\xi) &= \phi_\alpha(P_o, M(\Lambda,P)C(P)^{-1}\xi)\\ &= Q_{\alpha\beta}[M(\Lambda,P)]\phi_\beta(P_o, C(P)^{-1}\xi)\\ &= Q_{\alpha\beta}[M(\Lambda,P)]\phi_\beta(P,\xi)\end{aligned} \tag{2 .26}$$

where $Q_{\alpha\beta}$ is a $(2L(\alpha)+1) \times (2L(\alpha)+1)$ unitary irreducible matrix representation of SU(2) and the sum runs over all β with $M(\beta)$ between $-L(\alpha)$ and $+L(\alpha)$. Compare Eq. (2 .26) with Dirac theory [26]: the free particle spinor $\psi_\alpha(P,x) = e^{\pm iP\cdot x}u_\alpha(P)$ under a Lorentz transformation Λ becomes $\phi_\alpha(\Lambda P,\Lambda x) = e^{\pm iP\cdot x}S_{\alpha\beta}(\Lambda)u_\beta(P)$ and the 4 x 4 matrix $S_{\alpha\beta}(\Lambda)$ is a (nonunitary) representation of the Lorentz group. We define the Lorentz transformation operator $u(\Lambda)$ on $\phi_\alpha(P)$ as

$$(u(\Lambda)\phi_\alpha(P))(\xi) \equiv \phi_\alpha(P,\Lambda^{-1}\xi) \tag{2 .27}$$

$$= Q^*_{\beta\alpha}[M(\Lambda,P)]\phi_\beta(\Lambda P,\xi)$$

$$= (Q^*_{\beta\alpha}[M(\Lambda,P)]\phi_\beta(\Lambda P))(\xi) \quad . \tag{2 .28}$$

Eq. (2 .28) says that $u(\Lambda)$ and the orbital basis functions form an irreducible unitary Wigner representation of the Poincaré group[27,30] for spin $L(\alpha)$ and mass m_α. The $Q_{\alpha\beta}[M(\Lambda,P)]$ matrix rotates the spin under the Lorentz transformation Λ.

Particle physicists do not usually use the Wigner representation of Eq. (2 .28) to express free-particle Lorentz transformation properties. But the Wigner representation is equivalent to more usual representations [31]. The Wigner representation has several advantages. There are no redundant degrees of freedom in it -- a vector field has only three components, etc. Also, it is a unified description of particles with all possible spins[32]. Since the CHO wavefunctions $\phi_\alpha(P)$ have arbitrarily high spin $L(\alpha)$, we use the Wigner representation of Lorentz transformations, Eq. (2 .26).

For definiteness and simplicity we will work below with only one complementary set. $C(P)$ denotes the "pure boost" that transforms P_o into P, viz. for $P = m_k(\cosh\theta, \vec{e}\sinh\theta)$, $\vec{e}$ a unit vector in the direction of $\vec{P}$ with components e_i :

$$C(P)^\mu{}_\nu = \begin{bmatrix} \cosh\theta & e_1\sinh\theta & e_2\sinh\theta & e_3\sinh\theta \\ e_1\sinh\theta & 1+(\cosh\theta-1)e_1^2 & (\cosh\theta-1)e_2e_1 & (\cosh\theta-1)e_1e_3 \\ e_2\sinh\theta & (\cosh\theta-1)e_2e_1 & 1+(\cosh\theta-1)e_2^2 & (\cosh\theta-1)e_3e_2 \\ e_3\sinh\theta & (\cosh\theta-1)e_1e_3 & (\cosh\theta-1)e_3e_2 & 1+(\cosh\theta-1)e_3^2 \end{bmatrix} . \tag{2 .29}$$

In Appendix A there is a discussion of another complementary set and the "helicity basis" that it generates.

Consider the behavior of $\phi_k(P,\xi)$ under the parity transformation $P = (P^0, \vec{P}) \to \mathcal{P}P = (P^0, -\vec{P})$ and $\xi = (\xi^0, \vec{\xi}) \to \mathcal{P}\xi = (\xi^0, -\vec{\xi})$. Inspection of $C(P)$ above shows that

$$
\begin{aligned}
C(P)^0{}_0 &= C(\mathcal{P}P)^0{}_0,\quad C(P)^i{}_j = C(\mathcal{P}P)^i{}_j \quad (i,\ j = 1,2,3) \\
-\ C(P)^0{}_i &= C(\mathcal{P}P)^0{}_i = C(\mathcal{P}P)^i{}_0 \quad (i = 1,2,3) \quad .
\end{aligned}
\tag{2.30}
$$

When $P \to \mathcal{P}P$, $\xi \to \mathcal{P}\xi$ the gaussian factor in $\phi_k(P)$, Eq. (2 .20), is unchanged. But Eq. (2.30) implies that

$$
C(\mathcal{P}P)^{-1}\ (\mathcal{P}\xi) \quad = \quad -\ C(P)^{-1}\xi \tag{2.31}
$$

which means that the argument of the Hermite polynomials in $\phi_k(\mathcal{P}P,\mathcal{P}\xi)$ has the opposite sign from the argument of the Hermite polynomials in $\phi_k(P,\xi)$. Since the Hermite polynomials are also parity eigenfunctions,

$$
H_n(z) \quad = \quad (-1)^n\ H_n(-z),
$$

it follows that

$$
\phi_k(\mathcal{P}P,\mathcal{P}\xi) \quad = \quad (-1)^{N(k)}\phi_k(P,\xi) \quad . \tag{2.32}
$$

Eq. (2 .31) directly implies that

$$
\phi_k(\mathcal{P}P,\mathcal{P}\xi) \quad = \quad \phi_k(P,-\xi) \quad . \tag{2.33}
$$

Therefore

$$
\phi_k(P,\xi) \quad = \quad (-1)^{N(k)}\ \phi_k(P,-\xi). \tag{ .34}
$$

Because of Eq. (2.25), these parity relations also hold for $\phi_\alpha(P)$ in place of $\phi_k(P)$.

The inner product of two CHO wavefunctions is defined [18] as

$$(\phi_k(P),\phi_\ell(Q)) \equiv \int d\xi \phi_k^*(P,\xi)\phi_\ell(Q,\xi) \quad . \tag{2.35}$$

It follows that for any Lorentz transformation Λ,

$$(\phi_k(\Lambda P),\phi_\ell(\Lambda Q)) = (\phi_k(P),\phi_\ell(Q)) \tag{2.36}$$

and therefore, the factor A_K in Eq. (2.15) can be adjusted so that $\phi_k(P)$ has a Lorentz invariant normalization,

$$(\phi_k(P),\phi_\ell(P)) = \delta_{k\ell} \quad . \tag{2.37}$$

From Eq.(2.10) we see that

$$(\phi_k(P),\phi_\ell(Q)) = 0 \quad \text{if } N(k) \neq N(\ell) \quad . \tag{2.38}$$

Ruiz[19] showed that for $\vec{P}$ along the 3-direction

$$(\phi_k(\underset{o}{P}),\phi_\ell(P)) = \left(\frac{m_k}{P^0}\right)^{n_3(k)+1} \delta_{k\ell} \quad . \tag{2.39}$$

Eqs. (2.37), (2.38) and (2.39) give the CHO wavefunction a covariant probability interpretation: $\phi_k(P,\xi)$ is the probability amplitude for observing the two quarks in the CHO bound state labeled by k and separated from each other in spacetime by ξ, when the two quark system has total momentum P. The inner product, Eq. (2.35), is a probability overlap integral. Eq. (2.39) says that this probability overlap becomes Lorentz contracted when one CHO state moves with respect to another. These properties are summarized in Figure 1.

3. Lagrangian Formulation

Our ultimate goal is a field theory for $\Phi(x_1,x_2)$. The only general formalisms available for constructing a physically sensible field theory are the lagrangian and hamiltonian formalisms. The lagrangian approach is (usually) more manifestly covariant. In this section the equation of motion for $\Phi(x_1,x_2)$ will be put into lagrangian form, and the momentum and other dynamical quantities will be explored.

Consider the lagrangian density

$$L_0(x_1,x_2) = \frac{1}{2}(\partial_{x_1}{}^{\mu}\Phi)(\partial_{x_1\mu}\Phi) + \frac{1}{2}(\partial_{x_2}{}^{\mu}\Phi)(\partial_{x_2\mu}\Phi) + \frac{\omega^2}{64}\Phi\ (x_1-x_2)^2\ \Phi - \frac{m_0^2}{4}\Phi^2 \qquad (3\ .1)$$

where $\Phi(x_1,x_2)$ is a real-valued field function of the two four-vectors x_1, x_2. We define the action J as

$$J \equiv \int_{x_{1A}^0}^{x_{1B}^0} dx_1^0 \int_{x_{2A}^0}^{x_{2B}^0} dx_2^0 \int_{\text{all } \vec{x}_1,\vec{x}_2} d\vec{x}_1\, d\vec{x}_2\, L_0(x_1,x_2) \quad .$$

If Φ is varied while $\delta\Phi = 0$ at $x_1^0 = x_{1A}^0, x_{1B}^0$ and $x_2^0 = x_{2A}^0, x_{2B}^0$, then $\delta J = 0$ implies the Euler-Lagrange equation

$$\frac{\partial L_0}{\partial\Phi} = \partial_{x_1}{}^{\mu}\left[\frac{\partial L_0}{\partial(\partial_{x_1}{}^{\mu}\Phi)}\right] + \partial_{x_2}{}^{\mu}\left[\frac{\partial L_0}{\partial(\partial_{x_2}{}^{\mu}\Phi)}\right] \qquad (3\ .2)$$

which is equivalent to Eq. (2 .1). If L_0 is considered as a function of (X,ξ) then

$$L_0(X,\xi) = \frac{1}{4}(\partial_X^{\mu}\Phi)(\partial_{X\mu}\Phi) + \frac{1}{4}(\partial_{\xi}{}^{\mu}\Phi)(\partial_{\xi\mu}\Phi) + \frac{\omega^2}{16}\Phi(\xi^2)\Phi - \frac{m_0^2}{4}\Phi^2 \qquad (3\ .3)$$

The action is

$$J = \int_{X_A^0}^{X_B^0} dx^0 \int_{\xi_A^0}^{\xi_B^0} d\xi^0 \int_{\text{all } \vec{X},\vec{\xi}} d\vec{X}\, d\vec{\xi}\; L_0(X,\xi) \; .$$

If Φ is varied while $\delta\Phi = 0$ at $X^0 = X_A^0,\ X_B^0$ and $\xi^0 = \xi_A^0,\ \xi_B^0$ then $\delta J = 0$ implies the Euler-Lagrange equation

$$\frac{\partial L_0}{\partial \Phi} = \partial_X^\mu \left[\frac{\partial L_0}{\partial(\partial_X^\mu \Phi)} \right] + \partial_\xi^\mu \left[\frac{\partial L_0}{\partial(\partial_\xi^\mu \Phi)} \right] \qquad (3.4)$$

which is equivalent to Eq. (2.2).

In ordinary field theory the lagrangian density $L(x)$ depends on only one space-time coordinate, and the straightforward application of Noether's theorem to the invariances of the lagrangian gives conserved quantities like the stress-energy tensor, angular momentum tensor, etc. For example, the invariance of the action under the translation $x \to x + a$ leads (via Noether's theorem) to the stress-energy tensor conservation law[33]

$$\partial_x^{\,\mu}\, T_{\mu\nu} = 0 \; . \qquad (3.5)$$

It follows that the total momentum vector P_μ, defined as

$$P_\mu(x^0) \equiv \int_{\text{all } \vec{x}} d\vec{x}\;\; T_{0\mu}(x^0,\vec{x}) \qquad (3.6)$$

is really independent of x^0, i.e. the total momentum carried by the fields in $L(x)$ is conserved. In the field theory defined by Eqs. (3.3) and (3.4) there are two space-time coordinates, interpreted as the positions of the two quarks (x_1,x_2) or the external and internal coordinates of the two quark system (X,ξ). For $L_0(X,\xi)$ the conservation laws analogous to Eq. (3.5) will contain the divergence of tensors with respect to both coordinates. Since there are two time coordinates, X^0 and ξ^0, the definition of conserved dynamical quantities in terms of divergenceless tensors must be modified. We work out consequences of the translation invariance of $L_0(X,\xi)$ in detail below to illustrate the modifications required by two coordinates.

Consider the simultaneous translation of both quarks: $x_1 \to x_1 + a$, $x_2 \to x_2 + a$. Then $X \to X + a$ while ξ is unchanged. $L_0(X,\xi)$ does not depend on X explicitly (but does depend on ξ explicitly). Then $L_0 \to L_0 + \delta L_0$ where

$$\delta L_0 = a^\mu(\partial_{X\mu} L_0) = \frac{\partial L_0}{\partial\Phi}\,\delta\Phi + \frac{\partial L_0}{\partial(\partial_X{}^\mu\Phi)}\,\delta(\partial_X{}^\mu\Phi) + \frac{\partial L_0}{\partial(\partial_\xi{}^\mu\Phi)}\,\delta(\partial_\xi{}^\mu\Phi) \quad . \qquad (3.7)$$

$\delta\Phi = a^\mu(\partial_{X\mu}\Phi)$, $\delta(\partial_X{}^\mu\Phi) = \partial_X{}^\mu(\delta\Phi)$, $\delta(\partial_\xi{}^\mu\Phi) = \partial_\xi{}^\mu(\delta\Phi)$ and $\frac{\partial L_0}{\partial\Phi}$ satisfies Eq. (3.4). Thus

$$a^\mu(\partial_{X\mu} L_0) = \partial_X^\mu\left[\frac{\partial L_0}{\partial(\partial_X^\mu\Phi)}\right] a^\nu(\partial_{X\nu}\Phi) + \partial_\xi^\mu\left[\frac{\partial L_0}{\partial(\partial_\xi{}^\mu\Phi)}\right] a^\nu(\partial_{X\nu}\Phi)$$
$$+ \frac{\partial L_0}{\partial(\partial_X{}^\mu\Phi)}\,\partial_X{}^\mu\,[a^\nu(\partial_{X\nu}\Phi)] \qquad (3.8)$$
$$+ \frac{\partial L_0}{\partial(\partial_\xi{}^\mu\Phi)}\,\partial_\xi{}^\mu\,[a^\nu(\partial_{X\nu}\Phi)] \quad .$$

We define the external or average stress-energy tensor $T_{\mu\nu}$ as

$$T_{\mu\nu} \equiv \frac{\partial L_0}{\partial(\partial_X{}^\mu\Phi)}\,\partial_{X\nu}\Phi - g_{\mu\nu}L_0 \qquad (3.9)$$

and define the internal or relative stress-energy tensor $t_{\mu\nu}$ as

$$t_{\mu\nu} \equiv \frac{\partial L_0}{\partial(\partial_\xi{}^\mu \Phi)} \partial_{X^\nu}\Phi \quad . \tag{3.10}$$

Eq. (3 .8) says that

$$\partial_X{}^\mu T_{\mu\nu} + \partial_\xi{}^\mu t_{\mu\nu} = 0 \quad . \tag{3.11}$$

Eq. (3 .11) is the two-coordinate analog of Eq. (3.5).

We define the external or average momentum $\hat{P}_\mu$ carried by the field Φ as

$$\hat{P}_\mu (X^0;\xi_A^0,\xi_B^0) \equiv \int_{\text{all } \vec{X}} d\vec{X} \int_{\xi_A^0}^{\xi_B^0} d\xi^0 \int_{\text{all } \vec{\xi}} d\vec{\xi} \; T_{0\mu}(X^0,\vec{X},\xi^0,\vec{\xi}) \tag{3.12}$$

and define the internal or relative momentum $\hat{p}_\mu$ as

$$\hat{p}_\mu(\xi^0;X_A^0,X_B^0) \equiv \int_{\text{all } \vec{\xi}} d\vec{\xi} \int_{X_A^0}^{X_B^0} dX^0 \int_{\text{all } \vec{X}} d\vec{X} \; t_{0\mu} (X^0,\vec{X},\xi^0,\vec{\xi}) \quad . \tag{3.13}$$

Then

$$\hat{P}_\mu (X_B^0;\xi_A^0,\xi_B^0) - \hat{P}_\mu (X_A^0;\xi_A^0,\xi_B^0)$$

$$= \int_{X_A^0}^{X_B^0} dX^0 d\vec{X} \int_{\xi_A^0}^{\xi_B^0} d\xi^0 d\vec{\xi} \; [\frac{\partial}{\partial X_0} T_{0\mu} (X,\xi)]$$

(after integrating by parts with respect to X^0)

$$= \int_{X_A^0}^{X_B^0} dX^0 d\vec{X} \int_{\xi_A^0}^{\xi_B^0} d\xi^0 d\vec{\xi} \; [- \frac{\partial}{\partial X_i} T_{i\mu} - \frac{\partial}{\partial \xi_\nu} t_{\nu\mu}]$$

(because of Eq. (3 .11))

$$= \hat{p}_\mu(\xi_A^0;X_A^0,X_B^0) - \hat{p}_\mu(\xi_B^0;X_A^0,X_B^0) - \int_{X_A^0}^{X_B^0} dX^0 d\vec{X} \int_{\xi_A^0}^{\xi_B^0} d\xi^0 d\vec{\xi} \left[\frac{\partial}{\partial X_i} T_{i\mu} + \frac{\partial}{\partial \xi_i} t_{i\mu}\right]. \quad (3.14)$$

The application of Gauss' theorem to the volume integral in Eq. (3.14) of the 3-divergences $\frac{\partial}{\partial X_i} T_{i\mu}$ and $\frac{\partial}{\partial \xi_i} t_{i\mu}$ give two surface integrals,

$$S_{\xi_\mu} = \int_{X_A^0}^{X_B^0} dX^0 d\vec{X} \int_{\xi_A^0}^{\xi_B^0} d\xi^0 \oint_{(|\vec{\xi}|\to\infty)} T_{i\mu} \; n_\xi^i \; dA_\xi$$

and

$$S_{X_\mu} = \int_{\xi_A^0}^{\xi_B^0} d\xi^0 d\vec{\xi} \int_{X_A^0}^{X_B^0} dX^0 \oint_{(|\vec{X}|\to\infty)} T_i \; n_X^i \; dA_X$$

where n_ξ is the vector normal to the ξ-surface and n_X is the vector normal to the X-surface. Writing $t_{i\mu}$ in terms of L_0, L_0 in terms of Φ, and Φ in terms of $\phi_k(P)$ according to Eqs. (2.3) and (2.5) shows that $S_{\xi\mu} = 0$ because of the gaussian factor in $\phi_k(P)$, Eq. (2.21). We make the assumption (customary in ordinary one-coordinate field theory) that $S_{X\mu} = 0$. Then

$$\hat{P}_\mu(X_A^0;\xi_A^0,\xi_B^0) + \hat{p}_\mu(\xi_A^0;X_A^0,X_B^0) = \hat{P}_\mu(X_B^0;\xi_A^0,\xi_B^0) + \hat{p}_\mu(\xi_B^0;X_A^0,X_B^0) \quad . \quad (3.15)$$

Eq. (3.15) is the general statement of momentum conservation for the two-coordinate field $\Phi(X,\xi)$.

Eq.(3.15) is derived from the invariance of $L_0(X,\xi)$ under a translation of the two quark system, and it certainly expresses the conservation of something. But the physical meaning (if any) of this 'something' is not clear from the form of Eq. (3.15) or the tensors $T_{\mu\nu}$, $t_{\mu\nu}$. However,

$$\lim_{\xi^0 \to \pm\infty} \hat{p}_\mu(\xi^0;\, X_A^0,X_B^0) = 0 \quad (3.16)$$

because the gaussian factor, Eq. (2 .21), in $\Phi(X,\xi)$ vanishes at ∞ faster than any polynomial. Therefore,

$$\hat{P}_\mu \equiv \int_{\text{all } \vec{X},\xi} d\vec{X}\, d\xi\; T_{0\mu}\;(X,\xi) \qquad (3\ .17)$$

is conserved. Then $\hat{P}_\mu$ may be physically interpreted as the external momentum (momentum of the two quark system as a whole) averaged over the relative motion of the two quarks. $\hat{P}_\mu$ thus deserves the name "total momentum of the field Φ". Eq. (3 .15) expresses the physically reasonable result that the total momentum is conserved.

$T_{00}(X,\xi)$ plays the same role as the hamiltonian plays in a one-coordinate field theory, because

$$T_{0\mu} = \frac{1}{2}\frac{\partial\Phi}{\partial X^0}\frac{\partial\Phi}{\partial X^\mu} - g_{0\mu}L_0(X,\xi) \quad . \qquad (3\ .18)$$

So T_{00} is just the Legendre transform of L_0 with the conjugate field Π, defined as

$$\Pi \equiv \frac{\partial L_0}{\partial(\partial_{X^0}\Phi)} = \frac{1}{2}\frac{\partial\Phi}{\partial X_0} \quad , \qquad (3\ .19)$$

which is formally the same as the relationship between the lagrangian and hamiltonian in ordinary field theory.

The physical interpretation of $\hat{P}_\mu$ can be clarified further by writing $T_{0\mu}$ explicitly in terms of the CHO wavefunctions. The general solution $\Phi(X,\xi)$ to the field equation, Eq. (3 .4), is given by Eqs. (2 .3) and (2 .6). We restrict the discussion to $\Phi(X,\xi)$ constructed by substituting the harmonic oscillator wavefunctions of Eqs. (2 .15), (2.17) and (2 .18) into Eq. (2 .6). We separate Φ into positive- and negative- frequency parts:

$$\Phi_k^{(\pm)}(X,\xi) \equiv \frac{1}{(2\pi)^{3/2}} \int dP\, e^{\mp iP\cdot X}\, \delta(P^2-m_0{}^2-E(k))a_k^{(\pm)}(P)\phi_k(P,\xi) \tag{3.20}$$

where $a_k^{(\pm)}(P) \equiv \theta(P^\circ)\ a_k(\mp P)$ (no sum over k) ,

$$\Phi_k(X,\xi) = \Phi_k^{(+)}(X,\xi) + \Phi_k^{(-)}(X,\xi) , \tag{3.21}$$

and

$$\Phi(X,\xi) = \sum_k \Phi_k(X,\xi) . \tag{3.22}$$

Then

$$\Phi_k^{(\pm)}(X,\xi) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{P}}{2P_0}\, e^{\mp iP\cdot X}\, a_k^{(\pm)}(P)\phi_k(P,\xi) \tag{3.23}$$

where

$$P^0 = (m_k{}^2 + \vec{P}^2)^{\frac{1}{2}} > 0 . \tag{3.24}$$

Sometimes P^0 will have a subscript (k) to emphasize the dependence of P^0 on k through Eq. (3 .24). We define

$$a_k^{(\pm)}(\vec{P}) \equiv \frac{a_k^{(\pm)}(P)}{(2P^0_{(k)})^{\frac{1}{2}}} \tag{3.25}$$

so that $\Phi\ (X,\xi)$ takes the form

$$\Phi_k^{(\pm)}(X,\xi) = \frac{1}{(2\pi)^{3/2}} \int \frac{dP}{(2P^0)^{\frac{1}{2}}}\, e^{\mp iP\cdot X}\, a_k^{(\pm)}(\vec{P})\phi_k(P,\xi) . \tag{3.26}$$

Since $\Phi(X,\xi)$ is real,

$$\left[a_k^{(+)}(\vec{P})\right]^* = a_k^{(-)}(\vec{P}) . \tag{.27}$$

We combine Eqs. (3 .17) and (3 .18) to put $\hat{P}_0$ in the form

$$\hat{P}_0 = \frac{1}{2} \int d\vec{X} \int d\xi \quad [(\partial_X{}^\mu\Phi)(\partial_X{}^\mu\Phi) + m_0{}^2\Phi^2 + \Phi\,\Box_\xi\Phi$$

$$- \partial_{\xi\mu}(\Phi\partial_\xi{}^\mu\Phi) - \Phi\frac{\omega^2}{4} \xi^2 \Phi] \quad . \qquad (3.28)$$

In Eq. (3 .28) we expand Φ into positive- and negative- frequency parts with Eq. (3 .23). Then a representative term from $\hat{P}_0$ is

$$\frac{1}{2}\sum_{k,\ell} \int d\vec{X} \int d\xi \; \{\frac{1}{(2\pi)^3} \int \frac{d\vec{P}d\vec{Q}}{(4P^0_{(k)}Q^0_{(\ell)})^{\frac{1}{2}}} e^{i(P-Q)\cdot X} a_k^{(-)}(\vec{P})a_\ell^{(+)}(\vec{Q})$$

$$\cdot\; [\phi_k(P,\xi)\;(m_0{}^2 + P^\mu Q^\mu + \Box_\xi - \frac{\omega^2}{4}\xi^2)\;\phi_\ell\;(Q,\xi) \qquad (3.29)$$

$$- \partial_{\xi\mu}(\phi_k(P,\xi)\partial_\xi{}^\mu\phi_\ell(Q,\xi))]\} \quad .$$

Application of Gauss' theorem turns the divergence term above into a surface integral on a surface with $|\xi_\mu| \to \infty$. This surface integral vanishes because of the gaussian factor in $\phi_k(P)$, Eq. (2.21). Then the integration over $\vec{X}$ may be performed to give a momentum delta function. The term in (3 .29) now is

$$\frac{1}{2}\sum_{k,\ell} \int d\xi \quad \frac{d\vec{P}}{(4P^0_{(k)}P^0_{(\ell)})^{\frac{1}{2}}} a_k^{(-)}(\vec{P})\; a_\ell^{(+)}(\vec{P}) \quad e^{i(P^0_k - P^0_\ell)\;X} \qquad (3.30)$$

$$\cdot\quad [\phi_K(P^0_{(k)},\vec{P},\xi)(m_0{}^2 + P^0_{(k)}P^0_{(\ell)} + \vec{P}^2 + \Box_\xi - \frac{\omega^2}{4}\xi^2)\phi_\ell(P^0_{(\ell)},\vec{P},\xi)] \quad .$$

The integration over ξ may be performed by applying Eqs. (2.37) and (2.38). Then (3 .30) equals

$$\frac{1}{2} \sum_k \int d\vec{P} \; P^0_{(k)} \; a_k^{(-)}(\vec{P}) \; a_k^{(+)}(\vec{P}) \; .$$

Evaluation of the other terms in $\hat{P}_0$ by similar methods shows that

$$\hat{P}_0 = \frac{1}{2} \sum_k \int d\vec{P} \; P^0_{(k)} \; [a_k^{(+)}(\vec{P}) a_k^{(-)}(\vec{P}) + a_k^{(-)}(\vec{P}) a_k^{(+)}(\vec{P})] \; .$$

Similar straightforward (but tedious) evaluation of $\hat{P}^\mu$ leads to the form

$$\hat{P}^\mu = \frac{1}{2} \sum_k \int d\vec{P} \; P^\mu_{(k)} \; [a_k^{(+)}(\vec{P}) \; a_k^{(-)}(\vec{P}) + a_k^{(-)}(\vec{P}) \; a_k^{(+)}(\vec{P})] \; . \qquad (3.31)$$

$\hat{P}^\mu$ is formally quite similar to the momentum of an ordinary one-coordinate free local field. $\hat{P}_0 \geq 0$ because of Eq. (3.27). $a_k^{(-)}(\vec{P}) \; a_K^{(+)}(\vec{P})$ may be interpreted as the number of two quark systems in harmonic oscillator state k with total momentum $P^\mu_{(k)}$. The mass of this two quark system is $P^2_{(k)} = m_k^{\;2} = m_0^{\;2} + E(k)$.

If the $\Phi(X,\xi)$ field is required to be a scalar under a simultaneous Lorentz transformation Λ of x_1, x_2 or X,ξ, i.e. $X \to \Lambda X$, $\xi \to \Lambda\xi$, then $L_0(X,\xi)$ is also a scalar and the action is Lorentz invariant. By carrying out an analysis similar to the one for translation invariance, it can be shown that

$$\begin{aligned} &\partial_X^{\;\alpha}(X_\gamma T_{\alpha\beta} - X_\beta T_{\alpha\gamma}) + \partial_\xi^{\;\alpha}(X_\gamma t_{\alpha\beta} - X_\beta t_{\alpha\gamma}) \\ &+ \partial_X^{\;\alpha}(\xi_\gamma t_{\alpha\beta} - \xi_\beta t_{\alpha\gamma}) + \partial_\xi^{\;\alpha}(\xi_\gamma \tau_{\alpha\beta} - \xi_\beta \tau_{\alpha\gamma}) = 0 \quad , \end{aligned} \qquad (3.32)$$

where

$$\tau_{\alpha\beta} \equiv \frac{\partial L_0}{\partial(\partial_\xi^{\;\alpha}\Phi)} \; \partial_{\xi\beta}\Phi \; .$$

From Eq. (3 .32) and the properties of $\phi_k(P)$ we can show that the conserved angular momentum tensor $M_{\alpha\beta}$ is

$$M_{\alpha\beta} = J_{\alpha\beta} + K_{\alpha\beta} , \qquad (3.33)$$

where

$$J_{\alpha\beta} \equiv \int d\vec{X}\, d\xi \, (\xi_\beta t_{0\alpha} - \xi_\alpha t_{0\beta})$$

and $\qquad (3.34)$

$$K_{\alpha\beta} \equiv \int d\vec{X}\, d\xi \, (X_\beta T_{0\alpha} - X_\alpha T_{0\beta})$$

($J_{\alpha\beta}$ and $K_{\alpha\beta}$ must be evaluated at the same value of X^0). $J_{\alpha\beta}$ may be interpreted as the internal angular momentum of the two quarks moving around each other in some oscillator state $\phi_k(P)$, and $K_{\alpha\beta}$ may be interpreted as the angular momentum of the motion of the two quark system as a whole. We do not work out the explicit dependence of $J_{\alpha\beta}$ and $K_{\alpha\beta}$ on $a_k^{(\pm)}(P)$ and $\phi_k(P)$ because we do not need it for the following sections.

Our analysis of $\Phi(X,\xi)$ may be summarized as follows. $\Phi(X,\xi)$ can be considered as a real field function of two space-time coordinates. Its equation of motion can be derived from a lagrangian action principle. The symmetries of the lagrangian imply natural conservation laws. When $\Phi(X,\xi)$ is a solution of the equation of motion constructed from the harmonic oscillator wavefunctions $\phi_k(P)$ in Sec. 2 , then the conservation laws can be greatly simplified. The resulting quantities can be given their usual physical interpretation (e.g. $\hat{P}^\mu$ is momentum) if $\Phi(X,\xi)$ is interpreted as the wavefunction of a system of two quarks bound together by the covariant harmonic oscillator potential of Eq. (II.4). $a_k^{(-)}(\vec{P})\, a_k^{(+)}(\vec{P})$ is the density (in momentum space) of two quark systems with momentum $\vec{P}$, quantum numbers k and mass m_k; $\phi_k(P,\xi)$ is the probability amplitude that the two quarks of this system are separated in space-time by ξ. The momentum $\hat{P}^\mu$ carried by the field $\Phi(X,\xi)$, i.e. carried by a two quark system, is conserved, so the systems move as free particles. Henceforth we call these systems mesons.

The purely formal aspects of the above analysis generalize to any lagrangian of two coordinates $L(X,\xi)$. For example, if $L(X,\xi)$ has a translation symmetry then Noether's theorem in the form of Eqs. (3 .7) - (3.11) implies a "momentum conservation" law like Eq. (3 .11). But for a general $L(X,\xi)$ this "conservation" law by itself is physically meaningless. To make even the formal step from Eq. (3 .11) to Eq. (3 .15) requires enough detailed knowledge of the wavefunctions $\phi_k(P)$, i.e. of the solutions to the equation of motion in ξ, to eliminate the surface integral S_ξ. Eq. (3 .15) contains four quantities, each dependent on three times. It is not clear that any of those quantities, or any combination of them, is physically interpretable as momentum. The first quantity that is a physical momentum is $\hat{P}^\mu$, Eq. (3 .17). $\hat{P}^\mu$ is an average over all the relative motion of the two quarks, which is what we intuitively expect the momentum of the two quark system as a whole to be. So a necessary condition for a physically meaningful momentum conservation law is $\hat{P}_\mu(+\infty;X_A^0,X_B^0) = \hat{P}(-\infty;X_A^0,X_B^0)$.

Wavefunctions that have the properties that allow us to deduce physical conservation laws we shall call physically " good " relativistic bound state wavefunctions. In contrast, "bad" bound state wavefunctions are those that diverge at $|\xi| \to \infty$ (the system is really unbound), or do not vanish fast enough as $|\xi| \to \infty$ to eliminate surface terms like S_ξ (the system leaks momentum out through the surface at $|\xi| \to \infty$). To have a manifestly covariant formalism $\Phi(X,\xi)$ and $\phi_k(P)$ must depend on the relative time ξ^0; then "bad" wavefunctions will not justify eliminating $\hat{p}^\mu$ in the limit $|\xi^0| \to \infty$.

The construction of "good" relativistic bound state wavefunctions is not trivial[18,24]. $L_0(X,\xi)$ generates a physically sensible theory of free mesons because the CHO wavefunctions of Sec. 2 are physically "good" wavefunctions in the above sense.

Our covariant harmonic oscillator wavefunctions also are "good" because they satisfy Lorentz covariant normalization and orthogonality relations, Eqs. (2 .37) and (2.38). This implies that $\hat{P}^\mu$ decomposes into a sum over normal modes, Eq. (3.31), i.e. the normal modes $\phi_k(P,\xi)$ of Eq. (2.3) really are normal. If $\hat{P}^\mu$ contained cross terms like $a_k^{(+)}(\vec{P})\ a_\ell^{(-)}(\vec{P})\ e^{i(P^0_{(k)}-P^0_{(\ell)})X^0}$ then $\Phi(X,\xi)$ could not be interpreted as the wavefunction of a freely moving meson made of two quarks, because such cross terms make the mesons with quantum numbers k,ℓ interact with each other. In a fixed Lorentz frame $\hat{P}^\mu$ can always be diagonalized. But to make $\hat{P}^\mu$ diagonal in every frame the wavefunctions must be covariantly normalized and orthogonal. The fact that $\phi_k(P)$ satisfies Eqs. (2 .37) and (2 .38) means that $\phi_k(P)$ may be interpreted as describing the internal structure of physically free mesons. Noncovariantly orthonormal wavefunctions clearly are "bad" wavefunctions for physically free systems. Examples of such "bad" wavefunctions are the wavefunctions of Feynman et al.[16] and of Lipes[17].

4. Second Quantization and the Free Meson Field

In Sec. 3 we discussed a new type of field theory. The lagrangian $L_0(X,\xi)$ of this theory depends on two space-time positions, and the classical field $\Phi(X,\xi)$ is interpreted as the first-quantized wavefunction of a freely moving meson that is really a system of two scalar quarks bound together by a relativistic harmonic oscillator potential. $\Phi(X,\xi)$ covariantly describes an infinite spectrum of meson excited states. There is no upper bound to the mass of the excited states, so the quarks are permanently bound. This should be compared with ordinary local field theory, where the lagrangian $L(x)$ depends on only one space-time position and the classical field $\Phi(x)$ is interpreted as the first-quantized wavefunction of a structureless point particle, either moving freely or under the influence of some other fields or external potentials.

There are two ways to second quantize an ordinary local field $\Phi(x)$. One way is to find a local lagrangian $L(x)$ for $\Phi(x)$, then define the conjugate field $\Pi(x)$ by

$$\Pi(x) \equiv \frac{\partial L(x)}{\partial\left(\frac{\partial\Phi}{\partial x_0}\right)}$$

and impose the canonical commutation relation[34]

$$[\Pi(x), \quad \Phi(y)]_{x_0=y_0} = -i\delta(\vec{x} - \vec{y}) \tag{4.1}$$

in addition to the Euler-Lagrange equation of motion for $\Phi(x)$. The other way is to find the hamiltonian H for $\Phi(x)$ and impose the Heisenberg (canonical) equation of motion[33]

$$i\,\frac{\partial\Phi(x)}{\partial x_0} = [\Phi(x), H] \quad . \tag{4.2}$$

For a local theory with a lagrangian L, H can be defined by a Legendre transformation

$$H \equiv \int d\vec{x} \quad [\Pi(x) \frac{\partial \Phi(x)}{\partial x_0} - L(x)] \quad . \qquad (4.3)$$

Then the two methods of field quantization are equivalent[35]. But (at least for free fields) Eq. (4 .2) is conceptually much closer to the physics that the field is describing than Eq. (4 .1). Also, the covariant generalization of Eq. (4 .2) can be derived[33] from the general principle of translation invariance,

$$i \frac{\partial \Phi(x)}{\partial x^\mu} = [\Phi(x), \hat{P}_\mu] \qquad (4.4)$$

where $\hat{P}_\mu$ is the generator of space-time translations. We take Eq. (4 .4) as the appropriate point to generalize second quantization from $\Phi(x)$ to $\Phi(X,\xi)$.

Consider the field $\Phi_k(X,\xi)$ of Sec. 3. As a classical field, $\Phi_k(X,\xi)$ describes a freely moving meson with quantum numbers k; the meson is really a two quark system in harmonic oscillator state $\phi_k(P)$. A translation of the meson by a^μ is a translation of both quarks by a^μ, and therefore is a translation of X^μ by a^μ while ξ^μ is unchanged. If $\Phi_k(X,\xi)$ is made an operator, with associated translation operator $U(a) = \exp[-i\tilde{P}^\mu_{(k)} a_\mu]$ then U(a) should only translate X, not ξ. Then $\Phi_k(X+a,\xi) = U^\dagger(a)\, \Phi_k(X,\xi)\, U(a)$, which implies

$$i \frac{\partial \Phi_k(X,\xi)}{\partial X_\mu} = [\Phi_k(X,\xi), \tilde{P}^\mu_{(k)}] \quad . \qquad (4.5)$$

Eq. (4 .5) is the two-coordinate generalization of the Heisenberg equation, Eq. (4 .4). It is a quantum equation of motion for the X coordinate. There is no similar argument to derive a quantum equation of motion for the ξ coordinate.

The operator $\tilde{P}^\mu_{(k)}$ should be constructed from $\Phi_k(X,\xi)$ and its derivatives. Since $P^\mu_{(k)}$ generates translations of the meson as a whole, $\tilde{P}^\mu_{(k)}$ should be the momentum of the meson, not of the individual quarks. The obvious candidate for momentum operator is the kth term of Eq. (3 .31) ,

$$\hat{P}^{\mu}_{(k)} \equiv \frac{1}{2} \int d\vec{P} \;\; P^{\mu}_{(k)} \; [a_k^{(+)}(\vec{P}) \; a_k^{(-)}(\vec{P}) + a_k^{(-)}(\vec{P}) a_k^{(+)}(\vec{P})] \text{ (no sum over k)} . \quad (4.6)$$

Recall that the construction of $\hat{P}^{\mu}_{(k)}$ from $\Phi_k(X,\xi)$, the proof that it is conserved and the interpretation of $\hat{P}^{\mu}_{(k)}$ as the momentum vector of mesons in state k all depend on $\phi_k(P)$ being a physically "good" bound state c-number wavefunction.

In the absence of cogent reasons for a new quantum equation of motion for the ξ coordinate, the above considerations suggest that $\Phi_k(X,\xi)$ should be second quantized by making $a_k^{(\pm)}(\vec{P})$ operators that satisfy Eq. (4.5) while leaving $\phi_k(P)$ unchanged as a covariant harmonic oscillator wavefunction. Loosely speaking, $\Phi_k(X,\xi)$ should be second quantized only in the external variable X; it should remain first quantized, i.e. a wavefunction, in the internal variable ξ[36]. Physically this means that $\phi_k(X,\xi)$ creates/destroys free meson quanta which are bound systems of two quarks; $\Phi_k(X,\xi)$ cannot create/destroy single quarks. The internal structure of the meson comes from the relative motion of the two quarks which is described by the wavefunction $\phi_k(P,\xi)$.

We have discussed the quantization of the field $\Phi_k(X,\xi)$ independently of other fields $\Phi_\ell(Y,\eta)$. The arguments above apply to any k. Since the classical $\Phi_k(X,\xi)$ is not coupled to any $\Phi_\ell(Y,\eta)$, $\ell \neq k$ then the second quantized $\Phi_k(X,\xi)$ and $\Phi_\ell(Y,\eta)$ should commute.

The mathematical expression of the above quantization procedures is as follows. $\Phi(X,\xi)$ satisfies the Euler-Lagrange equation of motion, Eq. (2.2), and the subsidiary condition, Eq. (2.23). We decompose $\Phi(X,\xi)$ by Eqs. (3.21)-(3.26); the $\phi_k(P,\xi)$ in Eq. (3.26) is the CHO wavefunction of Sec. 2. We define the operators

$$a_k(\vec{P}) \equiv a_k^{(+)}(\vec{P}), \qquad a_k^{\dagger}(\vec{P}) \equiv a_k^{(-)}(\vec{P}) \qquad (4.7)$$

and impose the commutation relation

$$[a_k(\vec{P}), \quad a_\ell^\dagger(\vec{Q})] = \delta_{k\ell} \, \delta(\vec{P} - \vec{Q}) \, , \tag{4.8}$$

setting all other commutators equal to zero. Now $\Phi(X,\xi)$ and each $\Phi_k(X,\xi)$ is a hermitean boson field.

Much of the formalism of free local fields is trivially extended to $\Phi(X,\xi)$. We assume there is a unique vacuum state $|0>$ such that

$$a_k(\vec{P})|0> \; = 0 \text{ for all } k \; . \tag{4 .9}$$

The Hilbert space that $\Phi(X,\xi)$ operates in is the Fock space spanned by state vectors

$$|\vec{P}_{(k)},k; \; \vec{P}_{(\ell)},\ell; \; \ldots; \; \vec{P}_{(m)},m> \; = a_k^\dagger(\vec{P}_{(k)})a_\ell^\dagger(\vec{P}_{(\ell)})\ldots a_m^\dagger(\vec{P}_{(m)})|0> \tag{4 .10}$$

with arbitrary k, ℓ, ...,m. With $\hat{P}^\mu$ defined in terms of $\Phi(X,\xi)$ by Eq. the momentum operator and space-time translation generator is

$$:\hat{P}^\mu: \; = \int d\vec{P} \;\; P^\mu_{(k)} \; a_k^\dagger(\vec{P}) \; a_k(\vec{P}) \quad . \tag{4 .11}$$

Then $\Phi(X+A,\xi) \; = \; e^{i:\hat{P}^\mu:A_\mu} \; \Phi(X,\xi) \; e^{-i:\hat{P}^\mu:A_\mu} \quad ,$ (4 .12)

and therefore $\Phi(X,\xi)$ and each $\Phi_k(X,\xi)$ satisfies Eq. (4 .5).

$\Phi(X,\xi)$ can also be expressed in terms of the orbital basis wavefunctions $\phi_\alpha(P)$. We define

$$a_\alpha(\vec{P}) \equiv u^*_{\alpha k} \; a_k(\vec{P}) \tag{4 .13}$$

where $u_{\alpha k}$ is the unitary matrix of Eq. (4.25) and the sum runs over all k with $E(k) = E(\alpha)$ fixed. From Eq. (4 .13) it follows that

$$a_k(\vec{P})\phi_k(P) \; = \; a_\alpha(\vec{P})\phi_\alpha(P) \tag{4.14}$$

where the sums run over all k, α with $E(k) = E(\alpha)$ fixed. We also define

$$\Phi_\alpha(X,\xi) \equiv \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{P}}{(2P^0_{(\alpha)})^{\frac{1}{2}}} \cdot [e^{-iP.X} a_\alpha(\vec{P})\phi_\alpha(P,\xi) + e^{iP.X} a^\dagger_\alpha(\vec{P})\phi^*_\alpha(P,\xi)] \quad (4.15)$$

where $P^0_{(\alpha)} = (m_\alpha{}^2 + \vec{P}^2)^{\frac{1}{2}} > 0$,

so that $\Phi(X,\xi) = \sum_\alpha \Phi_\alpha(X,\xi)$. (4.16)

The Fock space is spanned by

$$|\vec{P}_{(\alpha)},\alpha; \vec{P}_{(\beta)},\beta; \ldots; \vec{P}_{(\gamma)},\gamma> = a^\dagger_\alpha(P_{(\alpha)})\, a^\dagger_\beta(\vec{P}_{(\beta)}) \ldots a^\dagger_\gamma(\vec{P}_{(\gamma)})|0> \quad (4.17)$$

with arbitrary $\alpha,\beta,\ldots\gamma$, and the momentum operator is

$$:\hat{P}^\mu: = \int d\vec{P}\; P^\mu_{(\alpha)}(\vec{P}) a^\dagger_\alpha(\vec{P})\, a_\alpha(\vec{P}) \quad . \quad (4.18)$$

For an arbitrary proper Lorentz transformation Λ, define the unitary operator $U(\Lambda)$ representing Λ by

$$U(\Lambda)\; a^\dagger_\alpha(\vec{P})\; U^\dagger(\Lambda) = \left[\frac{(\Lambda P_{(\alpha)})^0}{P^0_{(\alpha)}}\right]^{\frac{1}{2}} \cdot a^\dagger_\beta(\Lambda P)\; Q^*_{\beta\alpha}[M(\Lambda,P)] , \quad (4.19)$$

where $Q_{\alpha\beta}$ is the $(2L(\alpha) + 1) \times (2L(\alpha) + 1)$ unitary irreducible matrix representation of SU(2) introduced in Eq. (2.26) and the sum runs over all β with $E(\beta) = E(\alpha)$ and $L(\beta) = L(\alpha)$. Then the single meson states $|\vec{P}, \alpha>$ belong to a unitary representation of the Poincaré group[30] with mass m_α and spin $L(\alpha)$ (the energy factor in Eq. (4.19) is needed to make $U(\Lambda)$ unitary consistent with (4.8)). Since $N(\alpha)$ is linearly dependent on $L(\alpha)$ [29] the mass-squared spectrum is linearly dependent on the spin. $\Phi(X,\xi)$ is the free field for the infinity of mesons with all possible quantum numbers α. By combining Eqs. (2.26), (4.15) and (2.19) we can easily show that

$$U(\Lambda)\quad \Phi(X,\xi)\; U^\dagger(\Lambda) = \Phi(\Lambda X,\Lambda\xi) \quad (4.20)$$

for any proper Lorentz transformation Λ.

Eqs. (4.12) and (4.20) define the action of continuous Poincaré transformations on $\Phi(x,\xi)$. For completeness we now discuss the discrete transformations parity and time reversal. For a free field the discrete transformations are of purely mathematical interest, but the formalism will have physical consequences when meson interactions are introduced.

Note that the field $\Phi_\alpha(X,\xi)$ is not coupled to any other field $\Phi_\beta(Y,\eta)$ $(\alpha \neq \beta)$ by the Euler-Lagrange equation or the Heisenberg equation. $\Phi_\alpha(X,\xi)$ is only coupled to $\Phi_\beta(Y,\eta)$ for $E(\alpha) = E(\beta)$, $L(\alpha) = L(\beta)$, by a proper Lorentz transformation. Then the parity and time reversal transformations may be defined differently for fields with different $E(\alpha)$ or $L(\alpha)$ and still give a parity and time reversal invariant theory. In the lagrangian formulation this may require replacing $\mathcal{L}_0(X,\xi)$ by a separate free lagrangian (with the same form as $\mathcal{L}_0$) for each $E(\alpha)$ and $L(\alpha)$, but that does not change the free field theory of $\Phi(X,\xi)$.

The parity operator is denoted by $\mathcal{P}$, the time reversal operator by $\mathcal{T} = \mathcal{U}K$ where $\mathcal{U}$ is unitary and K complex conjugates all c-numbers. With the proper choice of phase for $\mathcal{P}$,

$$\mathcal{P}|\vec{P}, \alpha\rangle = \pm |-\vec{P}, \alpha\rangle \qquad (4.21)$$

and therefore $\mathcal{P}\, a_\alpha(\vec{P})\, \mathcal{P}^{-1} = \pm a_\alpha(-\vec{P})$,

where the $(\pm)$ sign is the intrinsic parity of the meson with quantum numbers α. Since $\Phi(X,\xi)$ is hermitian, the phase of $\mathcal{U}$ can be adjusted

$$\mathcal{U} a_\alpha(\vec{P})\, \mathcal{U}^{-1} = \pm a_\alpha(-\vec{P}) \ . \qquad (4.22)$$

Let $\mathcal{P}$, $\mathcal{T}$ operate on $\Phi_\alpha(X,\xi)$. The wavefunctions can be transformed by Eq. (2.32) and (2.33) with the result that

$$\mathcal{P}\Phi_\alpha(X,\xi)\mathcal{P}^{-1} = \pm (-1)^{N(\alpha)}\, \Phi_\alpha(\mathcal{P}X, \mathcal{P}\xi) \qquad (4.23)$$

and

$$\mathcal{T}\Phi_\alpha(X,\xi)\mathcal{T}^{-1} = \pm \Phi_\alpha(-\mathcal{P}X, -\mathcal{P}\xi) \ . \qquad (4.24)$$

For example, we can give all the mesons positive intrinsic parity,

$$\mathcal{P}\, a_\alpha(\vec{P})\, \mathcal{P}^{-1} = a_\alpha(-\vec{P})$$

i.e.

$$\mathcal{P}\Phi_\alpha(X,\xi)\mathcal{P}^{-1} = (-1)^{N(\alpha)}\Phi_\alpha(\mathcal{P}X, \mathcal{P}\xi) \ . \qquad (4.25)$$

With this assignment of parities, $\Phi(X,\xi)$ is neither even or odd under a transformation by P. Or we can make the intrinsic parity equal to the parity of the wavefunction $\phi_\alpha(p,\xi)$, viz $(-1)^{N(\alpha)}$ so that

$$P\, a_\alpha(\vec{P})\, P^{-1} = (-1)^{N(\alpha)}\, a_\alpha\, (-\vec{P})$$

and

$$P\Phi_\alpha(X,\xi)P^{-1} = \Phi_\alpha(PX,\ P\xi). \qquad (4.26)$$

This is especially attractive mathematically, because Eq. (4 .26) gives $\Phi(X,\xi)$ a definite parity:

$$P\Phi(X,\xi)P^{-1} = \Phi(PX,\ P\xi)\ . \qquad (4.27)$$

Of course, other parity assignments besides Eq.(4 .25) or Eq.(4 .26) are possible. We may define U so that $\Phi(X,\xi)$ is even under transformation by PT.

The fundamental relations of a quantum theory are the commutation relations. For a one-coordinate local free scalar field $\Phi(x)$, the basic commutator is

$$[\Phi(x),\ \Phi(y)] = i\Delta(x-y;\ m) \quad , \qquad (4.28)$$

where $\Delta(x-y\ ;m) = i\int \frac{dp}{(2\pi)^3}\,\delta(p^2-m^2)\,\varepsilon(p^0)e^{ip\cdot(x-y)}$.

For the two-coordinate free field $\phi_k(X,\xi)$, the corresponding commutator is

$$[\Phi_k(X,\xi),\ \Phi_\ell(Y,\eta)] = \delta_{k\ell}\int \frac{d\vec{P}\ d\vec{Q}}{(2\pi)^3\,(4P^0_{(k)}Q^0_{(\ell)})^{1/2}}\,\phi_k(P,\xi)\phi_\ell(Q,\eta)$$

$$\cdot\ [\,e^{i(Q\cdot Y-P\cdot X)}\,\delta(\vec{P}-\vec{Q}) - e^{i(P\cdot X-Q\cdot Y)}\,\delta(\vec{P}-\vec{Q})\,]$$

$$= i\delta_{k\ell}D(X-Y,\ \xi,\eta;\ m_k)\ , \qquad (4.29)$$

where $D(X-Y,\ \xi,\eta;\ m_k) = i\int \frac{dP}{(2\pi)^3}\ \delta(P^2-m_k^2)\varepsilon(P^0)e^{iP\cdot(X-Y)}\ \phi_k(P,\xi)\phi_k(P,\eta)$.

(no sum over k) (4 .30)

The local field commutator has the property of microcausality, that is it vanishes for $x \sim y$. The two-coordinate field commutator does not share this property;

$$D(X-Y, \xi,\eta; m_k) \neq 0 \text{ for arbitrary } \xi,\eta \text{ with } X \sim Y. \tag{4.31}$$

Only special values of ξ,η make D vanish with $X \sim Y$; for example $\xi^0=\eta^0 = 0$ and $X^0 = Y^0$ makes D vanish.

The mathematical reason for this "acausal" behavior of D is that D contains the oscillator wavefunctions, making the integrand of Eq. (4.30) contain an even function of P^0 (for $X^0=Y^0$); compare this with the Δ function, with an integrand odd in P^0 ($X^0=Y^0$). The physical reason for the commutator in Eq. (4.29) to be "acausal" (in the external coordinate) is that the two quarks inside the meson are in relative but bound state motion, i.e. the wavefunction $\phi_k(P,\xi)$ is a standing wave in ξ-space. The amplitudes of the standing wave for different quark separations ξ are correlated through the relative equation of motion, Eq. (2.4). If mesons really have a stable space-time extended structure, then it is plausible that whatever mathematical object represents the extended particle (in this theory it is the wavefunction) will show some "acausal" correlations between space-like separated points. In this regard, notice that if the commutator in Eq. (4.29) is integrated over all values of (ξ,η), i.e. if it is averaged over all the relative quark motion, then it vanishes for $X \sim Y$.

Microcausality plays an important role in the theory of interacting one-coordinate local fields. Microcausality makes dynamical quantities built from the fields, for example the hamiltonian density, commute between space-like separated points. An application of this is the proof that the Tomonaga-Schwinger equation is integrable[37]. Microcausality is equivalent to the covariance of the T-product,

$$T[\Phi(X)\ \Phi(Y)] \equiv \theta(X^0-Y^0)\ \Phi(X)\ \Phi(Y) + \theta(Y^0-X^0)\Phi(Y)\Phi(X)\ . \tag{4.32}$$

Microcausality in this form is crucial to the construction of the S-matrix by a covariant perturbation series. The "acausality" (in the external coordinate) of the $\Phi(X,\xi)$ commutator, Eq. (4 .29), has the serious consequence that the corresponding T-product

$$T[(\Phi_k(X,\xi)\ \Phi_\ell(Y,\eta)] \equiv \theta(X^0-Y^0)\Phi_k(X,\xi)\Phi_\ell(Y,\eta) + \theta(Y^0-X^0)\Phi_\ell(Y,\eta)\Phi_k(X,\xi) \qquad (4\ .33)$$

is not covariant (however, the T-product integrated over all values of (ξ,η) is indeed covariant).

The field $\Phi(X,\xi)$ describes a relativistic two-quark system. To understand the physics of the system we have separated the free-particle motion of the system as a whole, with respect to some external coordinate frame, from the internal structure of the system, which is the dynamical motion of one quark with respect to the other described by the wavefunction $\phi_k(P,\xi)$. This separation has been carried out mathematically by introducing external and internal variables. X and its Fourier transform P describe free particles (really the two-quark systems), and (X,P) are sufficiently separated from ξ for us to construct a second quantized multiparticle formalism in the (X,P) variables. The lack of microcausality in Eq. (4 .29) means that although (X,P) have free particle kinematics they are not really the variables of a free quantum field. This is just a mathematical problem -- (X,P) are not separated from ξ enough to be suitable variables of a free quantum field, so we need a new decomposition of Φ with better separation of external and internal degrees of freedom.

Suppose the definition of the wavefunction $\phi_k(P)$ is extended to arbitrary timelike momenta P, i.e. to momenta off the mass shell, $P^2 \neq m_k^2$, as well as on the mass shell. For now, make Eq. (2 .18) the only condition on $\phi_k(P)$ with P arbitrary (but time-like). Then we can write

$$\Phi_k(X,\xi) = \frac{1}{(2\pi)^3} \int \frac{d\vec{P}}{(2P_0)^{\frac{1}{2}}} \, dQ \, [e^{iP\cdot X} \, a_k^{\dagger}(\vec{P}) + e^{-iP\cdot X} \, a_k(\vec{P})]$$

$$\cdot \phi_k(Q,\xi) \;\; \delta(P-Q)\theta(Q^2) \;\; . \qquad (4.34)$$

Writing the delta function as

$$\delta(P-Q) = \frac{1}{(2\pi)^4} \int d\tilde{X} \; e^{i\tilde{X}.(P-Q)}$$

and applying Eq. (2.18) to (4.34) puts $\Phi_k(X,\xi)$ into the form

$$\Phi_k(X,\xi) = \frac{1}{(2\pi)^{3/2}} \int d\tilde{X} \; \frac{d\vec{P}}{(2P_0)^{\frac{1}{2}}} \left[e^{iP\cdot(X+\tilde{X})} \; a_k^{\dagger}(\vec{P}) + e^{-iP\cdot(X+\tilde{X})} a_k(\vec{P}) \right]$$

$$\cdot \int \frac{dQ}{(2\pi)^4} \; e^{-i\tilde{X}.Q} \;\; \phi_k(Q,\xi) \;\; \theta(Q_2) \;\; . \qquad (4.35)$$

Define

$$f_k(X,\xi) \;\; \equiv \;\; \frac{1}{(2\pi)^4} \int dQ e^{-iX\cdot Q} \;\; \phi_k(Q,\xi)\theta(Q^2) \qquad (4.36)$$

and

$$\Phi_k(X) \;\equiv\; \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{P}}{(2P_0)^{\frac{1}{2}}} \left[e^{iP\cdot X} \; a_k^{\dagger}(\vec{P}) + e^{-iP.X} \; a_k(\vec{P}) \right] . \qquad (4.37)$$

Then

$$\Phi_k(X,\xi) = \int d\tilde{X} \;\; \Phi_k(X+\tilde{X}) f_k(\tilde{X},\xi)$$

$$= \int d\tilde{X} \; \Phi_k(\tilde{X}) f_k(\tilde{X}-X,\xi) \;\; . \qquad (4.38)$$

$\Phi_k(X)$ is on ordinary local boson quantum field with mass m_k and quantum numbers k; it satisfies Eqs. (4.5) and (4.28). The whole second quantized formalism for the free mesons (two-quark systems) is contained in $\Phi_k(X)$. $f_k(X,\xi)$ is a real function (possibly a generalized function). All the information about the internal structure of the meson is contained in f_k. $\Phi_k(X,\xi)$ is the convolution, in the external variable, of $\Phi_k(X)$ with $f_k(X,\xi)$, i.e. it is a superposition of $\Phi_k(X)$ onto itself or a smearing of $\Phi_k(X)$ by $f_k(X,\xi)$. With Eq. (4.38) the commutator of Eq. (.29) becomes

$$[\Phi_k(X,\xi)\ \Phi_\ell(Y,\eta)] = \int d\tilde{X}d\tilde{Y}\ [\Phi_k(\tilde{X}),\ \Phi_\ell(\tilde{Y})] \cdot f_k(\tilde{X}-X,\ \xi)\ f_\ell(\tilde{Y}-Y,\eta)$$

$$= i\delta_{k\ell} \int d\tilde{X}d\tilde{Y}\ \Delta(\tilde{X}-\tilde{Y};\ m_k) f_k(\tilde{X}-X,\ \xi)\ f_\ell(\tilde{Y}-Y,\ \eta) \quad . \qquad (4\ .39)$$

Eqs. (4 .38) and (4 .39) show that the free meson field is better separated from the internal wavefunction by $\tilde{X}$ than by X. The field commutator is microcausal in $(\tilde{X},\tilde{Y})$, because $(\tilde{X},\tilde{Y})$ are the arguments of ordinary free local fields. Therefore, time ordering applied to $(\tilde{X},\tilde{Y})$ is covariant. This is the minimum separation we need to build the S-matrix. Eq. (4.38) is the free meson field representation we will use to construct the quantum field interaction of mesons.

To make $f_k(X,\xi)$ well-defined a specific form for $\phi_k(P)$ off mass shell must be put into Eq. (4 .36). $\phi_k(P)$ does not have a unique off mass shell extension. But there is a unique extension that satisfies Eqs. (2.4), (2.18), (2.19) and (2.22) (which are the defining equations for $\phi_k(P)$ on the mass shell) for arbitrary time-like P: $\phi_k(P)$ off mass shell has the same form as on mass shell, but $m_k{}^2$ is replaced by P^2. With this extension of $\phi_k(P)$ off mass shell, $\Phi_k(X,\xi)$ in the form of Eq. (4.38) manifestly satisfies the field equations, Eqs. (2.2) and (2.23), as well as the Heisenberg equation of motion, Eq. (4.5). Physically this extension means the off mass shell meson has the same internal structure as a real (on mass shell) meson. For the remainder of the paper $\phi_k(P)$ means this extension of the on mass shell wavefunction. Some mathematical properties of $\phi_k(P)$ and f_k are discussed in Appendix B.

Of course the discussion above can also be done with the orbital basis wavefunctions. $\phi_\alpha(P)$ off mass shell is defined by Eq. (2 .25), so that

$$f_\alpha(X,\xi) = \frac{1}{(2\pi)^4} \int dQ e^{-iX\cdot Q} \phi_\alpha(Q,\xi)\theta(Q^2) \quad , \tag{4.40}$$

$$\Phi_\alpha^{(-)}(X) \equiv \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{P}}{(2P_0)^{\frac{1}{2}}} e^{+iP\cdot X} a_\alpha(\vec{P})^\dagger = \left[\Phi_\alpha^{(+)}\right]^\dagger , \tag{4.41}$$

and

$$\Phi_\alpha(X,\xi) = \int d\tilde{X}[\Phi_\alpha^{(+)}(\tilde{X}) f_\alpha(\tilde{X}-X, \xi) + \Phi_\alpha^{(-)}(\tilde{X}) f_\alpha^*(\tilde{X}-X,\xi)] \quad . \tag{4.42}$$

V. MESON INTERACTIONS

The two-coordinate field $\Phi(X,\xi)$ describes freely moving mesons. The free field theory explains the static properties of mesons, through the internal quark motion wavefunction(or, in a more realistic version, the internal motion-spin-SU(3) wavefunction). Now that we have a consistent free field theory, we can move on towards our real goal--a physically consistent theory of interacting mesons. In order to use the insights and techniques of ordinary QFT we will produce interactions by the standard method of modifying the lagrangian with terms of third order or higher in the field operators. We want the resulting lagrangian to be physically reasonable (consistent with the usual conservation laws, etc.) and to generate the interaction of mesons, i.e. we want the interacting field to have the same particle interpretation as the free field. For example, single meson states created by the interacting field should be equivalent to single meson states of the free field. The interaction will perturb (perhaps very strongly) the free field, but the interacting field should not be a radically different mathematical object from the free field. These requirements will be satisfied if we maintain in the interacting theory the following principles, which were essential to the success of the free theory:

1) separation of internal and external parts of the field, including separation of the dynamics,

2) use of the covariant harmonic oscillator wavefunctions to describe internal quark motion probabilistically, and

3) second quantization of the external field part.

Given these principles, the fundamental question is: what is the role of the internal versus external variable in the interaction?

Different interaction terms, representing different answers to this question, may generate physically very different interactions and may even require different methods of quantization, commutation relations, etc. We will not seek a definitive answer to this broad question. Our goal is more modest. We will work out one concrete example, to show that an interacting two-coordinate field theory is possible and physically interesting. We have the successful meson decay rate calculations of Kim and Noz[22] to guide us. Because their interaction model is simple and gives physically interesting results, it is the starting point for our interacting field theory.

In this Section we review the formalism and interpretation of the Kim-Noz interaction. Then we generalize it and derive the equation of motion for the second quantized interacting meson field. We conclude this section with a discussion of the interaction picture for this theory.

In the Kim-Noz model the amplitude for decay of a meson with internal state k into mesons with internal states ℓ,m is

$$M_{k\ell m} = \frac{g}{6} \int dx_1\, dx_2\, dx_3\, \Phi_k^c(x_1,x_2)\, \Phi_\ell^c(x_2, x_3)\Phi_m^c(x_3, x_1)$$

$$= \frac{g}{6} \int dXdYdZ\, \Phi_k^c(X,Z-Y)\Phi_\ell^c(Y, X-Z)\, \Phi_m^c(Z, Y-X) \tag{5.1}$$

where Φ_k^c is the classical solution to Eq. (2 .2) and g/6 is the coupling constant. Figure 2 represents M as a three quark vertex; each meson has two quark lines. Kim and Noz[22] calculated M for

$$\Phi_k^c(X,\xi) = e^{-iP\cdot X}\, \phi_k(P,\xi) \tag{5.2}$$

and all meson momenta going into the vertex. If $M_{k\ell m}$ is summed over all permutations of k,ℓ,m then the symmetric amplitude is (up to trivial factors)

$$M_{(klm)} = \frac{g}{6} \langle 0| : \int dXdYdZ\Phi(X,Z-Y)\,\Phi(Y,X-Z)\,\Phi(Z,Y-X) : |P_{(k)},k;\, P_{(\ell)},\ell;\, P_{(m)},m\rangle \tag{5.3}$$

where Φ is the second quantized free meson field. In this form the Kim-Noz model is a generalization of $:\phi^3:$ field theory.

The new feature of our field theory is the internal coordinate ξ, so we should illuminate the role of internal motion in the interaction. Consider Eq. (5.1). The classical field $\phi_k^c(x,\xi)$ is the probability amplitude that two quarks are in relative oscillator state k, in space-time position $x_1 = X + \xi$, $x_2 = X - \xi$. Then the integral in M may be interpreted as the *relativistic probability amplitude* that six quarks of mesons k,ℓ,m overlap in spacetime position as in Figure 3, or the probability amplitude for oscillator k to fission into oscillators ℓ,m. The three meson vertex is also a three quark vertex, Figure 2, so the integral may also be interpreted as the probability amplitude for three quarks to change spacetime direction and still move with pairwise harmonic motion. If we transform the wavefunctions to momentum space M is rewritten as

$$M_{(k\ell m)}(P_1,P_2,P_3) = \frac{g\,\delta(P_1+P_2+P_3)}{16(2\pi)^{9/2}\sqrt{8P_1^\circ P_2^\circ P_3^\circ}} \int dq\, \tilde{\phi}_k(P_1,q)\,\tilde{\phi}_\ell(P_2,q+P_3)\,\tilde{\phi}_m(P_3,q-P_2) \tag{5.4}$$

where

$$\tilde{\phi}_k(P,\xi) \equiv \int d\xi\, e^{iq\cdot\xi}\, \phi_k(P,\xi). \tag{5.5}$$

$\tilde{\phi}_k(P,q)$ is the probability amplitude that two quarks in relative oscillator state k have momenta $\frac{1}{2}(P \pm q)$. Then the momentum overlap integral in Eq. (5.4) may be interpreted as the probability amplitude for three quarks to move freely in and out of relative oscillator states, Figure 4. Of course, these interpretations rely on the fact that the harmonic oscillator wavefunctions we use carry a covariant probability interpretation.

We can emphasize the similarities and differences of this interaction with the usual trilinear local QFT if we rewrite M once more:

$$M_{(k\ell m)}\ (P_1,P_2,P_3) = \frac{g(2\pi)^4\delta(P_1+P_2+P_3)}{(2\pi)^{9/2}\sqrt{8P_1^\circ P_2^\circ P_3^\circ}}\ V_{k\ell m}\ (P_1,\ P_2,\ P_3) \qquad (5.6)$$

where $V_{k\ell m} \equiv \frac{1}{16}\int d\xi d\eta \phi_k(P_1,\xi)\phi_\ell(P_2,\eta)\ \phi_m(P_3,-\xi-\eta)e^{i(P_2\cdot\xi-P_1\cdot\eta)}$

$$= \frac{1}{16(2\pi)^4}\int dq\ \tilde{\phi}_k(P_1,q)\ \tilde{\phi}_\ell(P_2,q+P_3)\tilde{\phi}_m(P_3,q-P_2)\quad . \qquad (5.7)$$

V is the *bare vertex function*; Eq. (5.6) expresses M as V multiplied by the kinematic factors appropriate to a diagram with three external lines. The transformation rule for wavefunctions, Eq. (2.26), gives V the correct transformation rule for a three particle vertex function:

$$V_{\alpha\beta\gamma}(\Lambda P_1,\Lambda P_2,\Lambda P_3) = Q_{\alpha\sigma}\ [M(\Lambda,P_1)]Q_{\beta\mu}[M(\Lambda,P_2)]Q_{\gamma\nu}[M(\Lambda,P_3)]$$
$$\cdot V_{\sigma\mu\nu}(P_1,\ P_2,\ P_3). \qquad (5.8)$$

This is what we expect from an ordinary theory with trilinear coupling between the fields of mesons k,ℓ,m. But since V is a probability overlap integral, the strength of the vertex depends on the space-time extended structure and internal quark motion of mesons k,ℓ,m. If the meson momenta are put off the mass shell the vertex strength is momentum dependent as well. So V is similar in its function but radically different in its structure from the bare vertex of ordinary field theories.

Our goal is an interacting field theory whose lowest order amplitudes are similar in structure to the Kim-Noz amplitudes $M_{(k\ell m)}$. But before we can put the Kim-Noz interaction into our theory we need a formalism more general than free field theory. We will use the simplest generalization--we replace the free local field $\Phi_k(X)$ by an interacting field $\Phi'_k(X)$. We take the interacting two-coordinate meson field to be

$$\Phi'_k(X,\xi) \equiv \int d\tilde{X}\ \Phi'_k(\tilde{X})\ f_k(\tilde{X}-X,\ \xi), \qquad (5.9)$$

$$\Phi'(X,\xi) = \sum_k \Phi'_k(X,\xi).$$

We are familiar with the functions f_k from Sec. 4. In order to give a probability interpretation to the CHO wavefunction in f_k, we must exclude time coordinate excitations, so we impose on Φ' the subsidiary condition

$$\partial^\mu_X(\partial_{\xi\mu} + \frac{\omega}{2}\xi_\mu)\ \Phi'(X,\xi) = 0. \qquad (5.10)$$

Compare this with Eq. (2.23).

We have given the interacting meson the same internal structure as the free meson, since both are described by the same wavefunction $\phi_k(P,\xi)$ which satisfies Eqs.(2.4) and (2.22). The physical significance of this is that excitations of the internal time coordinate do not enter the wavefunction during the interaction. Previous investigations have shown ϕ_k to give a good explanation of hadron spectroscopy[8,21,38]. The physics of ϕ_k is well understood, but we do not know what physics, if any, the time excitations correspond to. So we have eliminated them from the interaction by imposing the subsidiary condition Eq.(5.10).

Equation (5.9) is the most general representation of the interacting field consistent with the three quantization principles discussed above, because the functions $e^{iP.X}\ \phi_k(P,\xi)$ for arbitrary P (including P off the free meson mass shell) and k form a complete set. The only restriction our quantization principles put on Φ' is the subsidiary condition. $\Phi_k'(X,\xi)$ can be expanded in the hermite basis as

$$\Phi_k'(X,\xi) = \int dP \left[e^{-iP.X}\ a_k'(P) + e^{iP.X}\ a_k'^{\dagger}(P) \right] \phi_k(P,\xi)\theta(P^2). \quad (5.11)$$

With

$$\Phi_k'(X) = \int dP \left[e^{-iP.X}\ a_k'(P) + e^{iP.X}\ a_k'^{\dagger}(P) \right] \quad (5.12)$$

we always can represent $\Phi_k'(X,\xi)$ according to Eq. (5.9). Of course, we can repeat this analysis with the Hermite basis function ϕ_k replaced by the orbital basis functions ϕ_α and the fields Φ_k' replaced by Φ_α'. We assume that under a change of coordinate frame $\Phi_k'(X)$ transforms such that

$$\sum_{L(\alpha)\text{fixed}} \Phi_\alpha'(X,\xi)$$

is a Lorentz scalar.

We are still free to specify the dynamics of $\Phi_k'(X)$. Motivated by the model calculation of Kim and Noz[22] we want the second quantized fields to interact with trilinear couplings while the interaction vertex is determined by the overlap of wavefunctions. Specifically, we choose the interaction lagrangian density as

$$L_1(X,Y,Z) = \frac{1}{2}[\Phi_k'(X)\Phi_\ell'(Y)\Phi_m'(Z) + \Phi_m'(Z)\Phi_\ell'(Y)\Phi_k'(X)]F_{k\ell m}(X,Y,Z) \quad (5.13)$$

(summation convention assumed from now on), where the function

$$F_{k\ell m}(X,Y,Z) \equiv \int d\tilde{X}d\tilde{Y}d\tilde{Z} f_k(X-\tilde{X},\tilde{Z}-\tilde{Y})\; f_\ell(Y-\tilde{Y},\tilde{X}-\tilde{Z})\; f_m(Z-\tilde{Z},\tilde{Y}-\tilde{X}) \;. \qquad (5.14)$$

$F_{k\ell m}$ is real and has the following symmetry properties:

$$F_{k\ell m}(X + A,\; Y + A,\; Z + A) = F_{k\ell m}\,(X,Y,Z) \qquad (5.15)$$

and

$$F_{k\ell m}\,(X,Y,Z) = (-1)^{N(k)+N(\ell)+N(m)}\; F_{\ell k m}\,(Y,\; X,\; Z). \qquad (5.16)$$

The total action J is

$$J = \int dXd\xi\; L_0(\Phi',\partial^\mu\Phi') + \frac{g}{6}\int dXdYdZ\; L_1(\Phi') \;.$$

Variation of J gives the Euler-Lagrange equation of motion:

$$(\Box_X + \Box_\xi - \frac{\omega^2}{4}\,\xi^2 + m_0^{\,2})\Phi'(X,\xi) = \frac{g}{2}\int dY\Phi'(Y,X-Y-\xi)\Phi'(Y+\xi,Y-X)$$
$$+ \text{ hermitean conjugate.} \qquad (5.17)$$

We define the full hamiltonian H as

$$H \equiv \int dXd\xi\; \frac{\partial L}{\partial(\partial_{X_0}\Phi')}\;\partial_{X_0}(X,\xi)$$
$$- \frac{g}{18}\int dXdYdZ\; [L_1(X,Y,Z) + L_1(Y,X,Z) + L_1(Y,Z,X)] \qquad (5.18)$$
$$= H_0(\Phi') + H_1(\Phi') \;.$$

Then the time development of $\Phi'_k(X)$ is given by the Heisenberg equation of motion,

$$i\,\frac{\partial\Phi'_k(X)}{\partial X_0} = [\Phi'_k(X),\, H] \;. \qquad (5.19)$$

This defines our theory. H_1 is hermitean and trilinear in the fields, but due to the symmetry of $F_{k\ell m}$ there is no coupling between fields Φ'_k, Φ'_ℓ, Φ'_m when $N(k) + N(\ell) + N(m)$ = odd integer. The theory is clearly invariant under translations and proper Lorentz transformations. It is also space- and time-reversal invariant for a wide variety of choices for the intrinsic parity of each Φ'_k; for example, Eqs. (4.25) and (4.26) are both acceptable.

Before going further we must point out some problems with the interacting field formalism we have outlined above. In an ordinary local field theory the canonical commutation relations (CCR's) imply the equivalence of the Heisenberg and Euler-Lagrange equations of motion[35]. But there are no CCR's for our field theory, and even if there were we could not prove the equivalence of Eqs. (5.17) and (5.19). To prove these equations are compatible we need a complete solution to the dynamics, because H_1 contains fields at different times. Also, we cannot prove that the subsidiary condition is consistent with either equation of motion. But similar deep questions about local QFT are also unanswered. We assume that our method of quantizing an interacting theory, which is physically reasonable, is also mathematically consistent. In any case, we do not use the Euler-Lagrange equations any further.

The usual assumptions of QFT can be generalized to the infinite collection of fields $\Phi_k'(X)$. We assume there is a single unique vacuum state, good for all the fields. We assume every Φ_k' has a single meson state with mass m_k. We assume the usual asymptotic condition[34] holds for every $\Phi_k'(X)$. Then free mesons are asymptotic "in" and "out" states of $\Phi_k'(X)$. We assume the "in" and "out" states of all k are complete sets. With the reduction formalism any S-matrix element is related to a vacuum expectation value of a product of Heisenberg fields.

To develop a perturbation series for the S matrix it is customary to transform the fields to the interaction picture[34], which we proceed to do. We define the U matrix as

$$U(t,0) \equiv e^{iH_0 t} \; e^{-iHt} \; , \quad U(t_1,t_2) = U(t_1,0)U^{\dagger}(t_2,0). \tag{5.20}$$

We assume that any Heisenberg operator $A'(t,\vec{x})$ built from the fields can be related to the corresponding free operator $A(t,\vec{x})$ by

$$A'(t,\vec{x}) = U^{\dagger}(t,0)\; A(t,\vec{x})\; U(t,0) \; , \tag{5.21}$$

where $$A(t,\vec{x}) = e^{iH_0 t} \quad A'(0,\vec{x})e^{-iH_0 t} \quad . \tag{5.22}$$

The U matrix solves the equation

$$i\,\frac{dU(t,0)}{dt} = \left[e^{iH_0 t}\; H_1 \quad e^{-iH_0 t}\right] U(t,0) \quad , \tag{5.23}$$

subject to the initial condition $U(0,0) = 1$. The S matrix is

$$S = \frac{U(\infty,-\infty)}{\langle 0|U(\infty,-\infty)|0\rangle} \quad . \tag{5.24}$$

Clearly it is unitary because H_0, H_1 are hermitean.

The interaction picture simplifies ordinary QFT because $e^{iH_0 t}\; H_1 \; e^{-iH_0 t}$ becomes a functional of free fields. The situation is more complicated in our theory. If we use Eq. (5.18) to evaluate H_1 at $X^0 = 0$, then

$$e^{iH_0 t}\; H_1 e^{-iH_0 t} = \frac{-g}{36} \int d\vec{X}\; \Phi_k(t,\vec{X})U(t,Y^0)\Phi_\ell(Y)\; U(Y^0,Z^0)\Phi_m(Z)U(Z^0,t)$$
$$\cdot F_{k\ell m}((t,\vec{X}),\; Y,\; Z) + 5 \text{ similar terms.} \tag{5.25}$$

The fields in the integral are free, but the unknown operator U is also present. Eq. (5.23) is really a nonlinear integrodifferential equation for U.

6. S MATRIX IN PERTURBATION THEORY

Our goal is to approximate S matrix elements by a power series in the coupling constant g. In ordinary QFT we do this by iteratively solving Eq. (5.23), like a Volterra integral equation. Eq. (5.25) seems to imply that the $\mathcal{O}(g)$ iteration of our theory requires U(t,0) for all t. If this really was the case then a perturbation theory based on the interaction picture would not be practical. But there are two features of the theory that alter this conclusion.

First, most of the contribution to the integral of Eq. (5.25) comes from the three fields evaluated at almost the same time. To see why, notice that (up to trivial factors)

$$F_{k\ell m}(X,Y,Z) \sim \int dPdQdR\, e^{i(X\cdot P+Y\cdot Q+Z\cdot R)}\, \delta(P+Q+R) V_{k\ell m}(P,Q,R) \qquad (6.1)$$

From the discussion in Appendix B, $V_{k\ell m}$ is nonzero only for $|P-Q| \lesssim \omega$ and $|P-R| \lesssim \omega$, which implies that $F_{k\ell m}$ is nonzero only for $|X-Y| \lesssim \omega^{-1}$, $|X-Z| \lesssim \omega^{-1}$. Thus, the interaction is significant only when mesons k,ℓ,m overlap in spacetime (ω^{-1} ~ size of meson), so the iteration of Eq. (5.23) only requires U(t,0) for small t, i.e. for U "close" to 1.

Second, we have not yet normal ordered H_1. Such normal ordering is trivial in ordinary QFT; it eliminates emabrrassing infinities in vacuum expectation values that should be classically interpretable, and it eliminates the divergent tadpole diagrams from perturbation theory[34]. Since our H_1 contains interacting fields at different times we do not know what prescription to use for normal ordering. But at the level of Eq. (5.25) we should use some prescription; if we do not, then a perturbation series will include tadpoles (probably divergent). Any normal ordering will be symmetric (thus obviating the need to explicitly symmetrize in Eqs. (5.13) and (5.18)) and therefore the ordering of Φ_k, Φ_ℓ, Φ_m in Eq. (5.25) should not be very important.

These features of the theory suggest that we can approximate $e^{iH_0t} H_1 e^{-iH_0t}$ by eliminating the U matrix in Eq. (5.25) and then normal ordering the free fields, i.e. setting

$$U(t,Y^0) \approx U(Y^0,Z^0) \approx U(Z^0,t) \approx 1 \qquad (6.2)$$

and

$$e^{iH_0t} H_1 e^{-iH_0t} \approx \frac{-g}{6} \int d\vec{X}dYdZ{:}\phi_k(t,\vec{X})\Phi_\ell(Y)\Phi_m(Z){:}F_{k\ell m}(t\vec{X},Y,Z). \qquad (6.3)$$

We can solve the U matrix equation iteratively in this approximation by regarding $:\Phi_k\Phi_\ell\Phi_m:F_{k\ell m}$ as a hamiltonian density depending on one time variable (ignoring time differences among the fields). The resulting U matrix is the sum of time ordered products of such densities integrated over all variables. Because we have normal ordered the fields in the density, we tentatively extend the time ordering of n densities to the time ordering of 3n fields (because previous normal ordering removes all significance from time ordering among fields in the same density). Then we (tentatively) calculate the U matrix as

$$U(\infty,-\infty) = 1 + \sum_{n=1}^{\infty} \left(\frac{-ig}{6}\right)^n \frac{1}{n!} \int dX_1 \ldots dZ_n \quad T[{:}\Phi_a(X_1)\Phi_b(Y_1)\Phi_c(Z_1){:}\ldots$$

$$\ldots{:}\Phi_k(X_n)\Phi_\ell(Y_n)\Phi_m(Z_n){:}]F_{abc}(X_1, Y_1, Z_1)\ldots F_{k\ell m}(X_n,Y_n,Z_n) \quad . \qquad (6.4)$$

T denotes time ordering of the free fields after normal ordering, i.e. the fields in [ ] above are split into positive and negative frequency parts, normal ordered in each triplet of fields, and then reordered so that every operator stands to the left of all operators in other triplets that have an earlier time variable.

Our fundamental approximation is to neglect time differences between fields in H_1 for the purpose of iterating the U matrix equation. Our motivation is physical, and with this approximation we have made a great simplification in the U matrix without much attention to mathematical rigor. But our method is not as drastic as it may seem. Eq.(6 .4) preserves the Lorentz covariance of the theory, because the time ordering of free local fields and the overlap functions $F_{k\ell m}$ are covariant. We will verify this in the S matrix elements below. In Section 7 we will establish the unitarity of our approximation, provided that we consistently apply this approximation in the time ordering prescription of Eq(6 .4).

Our immediate task is to reduce Eq. (6 .4) to diagramatic rules of calculation. We denote the time ordered contraction of free local fields by

$$\overbrace{\Phi_k(X)\Phi_\ell}(Y) \equiv \langle 0|T[\Phi_k(X)\Phi_\ell(Y)]|0\rangle = i\delta_{k\ell}\,\Delta_F(X-Y;m_k) = \frac{i\delta_{k\ell}}{(2\pi)^4}\int dP\,\frac{e^{iP.(X-Y)}}{P^2-m_k+i\varepsilon}\ . \qquad (6\ .5)$$

It follows from Wick's theorem that

$$U(\infty,-\infty) = 1+\sum_{n=1}^{\infty}\left(\frac{-ig}{6}\right)^n \frac{1}{n!}\int dX_1\ldots dZ_n\Big\{:\Phi_a(X_1)\ldots\Phi_m(Z_n): + :\Phi_a(X_1)\Phi_b(Y_1)\overbrace{\Phi_c(Z_1)\Phi_d}(X_2)\ldots\Phi_m(Z_n): + \ldots \text{ all allowed contractions}\Big\}$$

$$\cdot F_{abd}(X_1,Y_1,Z_1)\cdots F_{k\ell m}(X_n,Y_n,Z_n)\ . \qquad (6\ .6)$$

The bracket { } above does not contain contractions between fields of the same triplet $:\Phi_k\Phi_\ell\Phi_m:$.

Let us consider a single contraction summed over all quantum numbers n on the contracted field $\Phi_n(Z)$ and integrated over the contracted variables:

$$\int dZ\, dA\, F_{k\ell n}\ (X,Y,Z)\ \overbrace{\Phi_n(Z)\, \Phi_n}(A)\ F_{nrs}\ (A,B,C)$$

$$= i\int dZdAd\tilde{X}d\tilde{Y}d\tilde{Z}d\tilde{A}d\tilde{B}d\tilde{C}\ f_k(X-\tilde{X},\ \tilde{Z}-\tilde{Y})\ f_\ell(Y-\tilde{Y},\ \tilde{X}-\tilde{Z})\ f_n(Z-\tilde{Z}\ \tilde{Y}-\tilde{X})$$

$$\cdot\Delta_F\ (Z-A;m_n)f_n(A-\tilde{A},\ \tilde{C}-\tilde{B})\ f_r\ (B-\tilde{B},\ \tilde{A}-\tilde{C})\ f_s\ (C-\tilde{C},\ \tilde{B}-\tilde{A})$$

$$= \frac{i}{(2\pi)^{12}}\int d\tilde{X}d\tilde{Y}d\tilde{Z}d\tilde{A}d\tilde{B}d\tilde{C}dZdAdPdQdQ'\ [f_kf_\ell f_rf_s\ \frac{e^{iP\cdot(Z-A)}}{P^2-m_n^2+i\varepsilon}$$

$$\cdot e^{-iQ(Z-\tilde{Z})}\ \theta(Q^2)\phi_n(Q,\ \tilde{Y}-\tilde{X})e^{-iQ'(A-\tilde{A})}\ \theta(Q'^2)\phi_n(Q',\ \tilde{C}-\tilde{B})] \qquad (6.7)$$

(after using Eq. (4 .36)).

When we carry out the Z, A integrations we get momentum delta functions, which allows us to carry out the P, Q' integrations; Eq. (6.7) becomes

$$\frac{i}{(2\pi)^4}\int d\tilde{X}d\tilde{Y}d\tilde{Z}d\tilde{A}d\tilde{B}d\tilde{C}\ f_kf_\ell f_rf_s\left[\int dQ\ \frac{\phi_n(Q,\tilde{Y}-\tilde{X})\phi_n(Q,\tilde{C}-\tilde{B})e^{iQ(\tilde{Z}-\tilde{A})}}{Q^2-m_n^2+i\varepsilon}\ \theta(Q^2)\right] \qquad (6.8)$$

The contraction of any two fields in Eq. (6 .4) produces the factor in brackets [ ] above. Some properties of this factor are discussed in Appendix B.

In Eq. (6.7) we only integrate over the contracted variables Z,A. Variables X,Y,B,C are nicely separated, each in its own f_k function in Eq. (6.8). If the fields Φ_k, Φ_ℓ, Φ_r, Φ_s are also contracted away (possibly with some other fields not present in

Eq. (6.7)) then their contractions (after integration) will give another factor like [ ] in Eq. (6.8), with the appropriate variables in the oscillator wavefunctions.

When we calculate a matrix element of $U(\infty,-\infty)$ between free meson states, the uncontracted fields annihilate the mesons. By writing the F_{abc} functions as the overlap integral of three f functions and integrating over the dummy variables of the normal ordered (uncontracted) fields it follows that every state $|P,a\rangle$ is annihilated by a $\Phi_a(X,\xi)$ field, leaving a factor of $\frac{1}{(2\pi)^{3/2}(2P_0)^{\frac{1}{2}}}\phi_a(P,\xi)e^{-iP\cdot X}$.

Now we see the structure of the matrix elements of any $O(g^n)$ term in the Wick expansion of $U(\infty,-\infty)$, Eq. (6.6). After contracting fields and annihilating all initial and final particles we are left with 3n oscillator wavefunctions. Each wavefunction carries a momentum variable (possibly a dummy momentum to be integrated over, as in Eq. (6.8)) and spacetime variables. The spacetime variables are the tilded variables of Eq. (5.14); they are grouped into triplets by the $F_{k\ell m}$ functions. All these variables are integrated over, but the integration for each triplet may be done separately. The result of one integration is

$$\int dXdYdZ\ e^{-i(P\cdot X+Q\cdot Y+R\cdot Z)}\ \phi_a(P,Z-Y)\phi_b(Q,X-Z)\phi_c(R,Y-Z)$$

$$= (2\pi)^4\ \delta(P+Q+R)\ V_{abc}\ (P,Q,R), \qquad (6.9)$$

the bare vertex function. Some or all the momenta in V_{abc} may be virtual momenta that come from contractions, as in Eq. (6.8); the virtual momenta and their Feynman denominators must be integrated over.

If we use the orbital basis of wavefunctions instead of the Hermite basis, we must use a different vertex function because the wavefunction associated with the positive frequency part of the field is $\phi_\alpha(P)$, while the negative frequency part of the field has $\phi^*_\alpha(P)$. In the orbital basis we denote momenta leaving a vertex by a superscript, momenta entering a vertex by a subscript. For example, $V^\alpha{}_{\beta\gamma}(-P,Q,R)$ means the overlap integral of states (Q,β) and (R,γ) entering and (P,α) leaving the vertex, Figure 5:

$$V^\alpha{}_{\beta\gamma}(-P,Q,R) = \frac{1}{16}\int d\xi d\eta \; e^{i(Q\cdot\xi+P\cdot\eta)} \; \phi^*_\alpha(P,\xi)\phi_\beta(Q,\eta)\phi_\gamma(R,-\xi-\eta) \quad (6.10)$$

Now we can write graphical rules for the nth order S matrix element in our approximation, Eq. (6.4). Draw all connected graphs with n vertices and the appropriate number of external lines. With each line and vertex is associated a factor in momentum space, Figure 6.

We integrate $\int \frac{dQ}{(2\pi)^4}$ over all virtual momenta, and sum over all possible quantum numbers carried by internal lines. Note that the factor $\frac{1}{6^n\, n!}$ in Eq. (6 .4) drops out because each graph corresponds to 6^n n! terms in Eq. (6 .6) that differ only by a reordering of the triplets $(X_1,Y_1,Z_1)\ldots(X_n,Y_n,Z_n)$ or a reordering among variables inside each triplet.

With the normalization adopted in Sec. 4 , the S matrix element <S> is related to the differential scattering cross section $d\sigma$ for two initial mesons as follows:

$$\langle S\rangle = 1 - i(2\pi)^4\ \delta(\text{total momentum})\ \langle T\rangle$$

$$d\sigma = \frac{(2\pi)^{10}}{|\vec{V}_1-\vec{V}_2|}\, s\delta(\text{total momentum})\,|\langle T\rangle|^2\ d(\text{final momenta}), \qquad (6\ .11)$$

where $|\vec{V}_1-\vec{V}_2|$ is the relative speed of the initial particles and $s = \prod_j \frac{1}{\nu_j!}$ is the statistical factor for an initial or final state with ν_j particles of type j. The rate $d\Gamma$ for a single meson to decay is

$$d\Gamma = (2\pi)^7 s\ \delta(\text{total momentum})\ |\langle T\rangle|^2\ d(\text{final momenta})\ . \qquad (6\ .12)$$

The Lorentz covariance of the S matrix can be verified explicitly. The graphical rules give us $-i(2\pi)^4\delta(\text{total momentum})\ \langle T\rangle$. If we Lorentz transform the initial and final mesons and then calculate <T> we must put in kinematic factors and a rotation matrix $Q_{\alpha\beta}[M]$ for each external meson (see Eq. (4 .19)). All other factors in momentum space are Lorentz scalars, because for any Lorentz transormation Λ,

$$\sum_{\alpha} \phi^*_\alpha(P,\xi)\ \phi_\alpha(P,\eta) = \sum_{\alpha} \phi^*_\alpha(\Lambda P,\ \Lambda\xi)\phi_\alpha(\Lambda P,\ \Lambda\eta), \tag{6.13}$$

where the sum runs over α with N(α), L(α) fixed. It follows that

$$\langle \Lambda P_\alpha,\alpha;\ldots\Lambda P_\beta,\beta|S|\Lambda P_\mu,\mu;\ldots\Lambda P_\nu,\nu\rangle$$

$$= \frac{P^\circ_\alpha\ldots P^\circ_\beta P^\circ_\mu\ldots P^\circ_\nu}{\Lambda P^\circ_\alpha\ldots\Lambda P^\circ_\beta\Lambda P^\circ_\mu\ldots\Lambda P^\circ_\nu}\ Q^*_{\alpha\alpha'}[M(\Lambda,P_\alpha)]\cdots Q^*_{\beta\beta'}[M(\Lambda,P_\beta)]Q_{\mu\mu'}[M(\Lambda,P_\mu)]$$

$$\cdots Q_{\nu\nu'}[M(\Lambda,P_\nu)]\langle P_{\alpha}',\alpha';\ldots P_{\beta}',\beta'|S|P_{\mu}',\mu';\ldots P_{\nu}',\nu'\rangle \tag{6.14}$$

and therefore

$$U(\Lambda)\ SU^{-1}(\Lambda) = S\ . \tag{6.15}$$

Note that if $|\langle T\rangle|^2$ is summed over initial and final spins, we get a Lorentz scalar function of external momenta multipled by the phase space factors $\frac{1}{2P_0}$. Of course, this Lorentz covariance is achieved only because we use Lorentz covariant wavefunctions.

To gain further insight into our theory, let us consider the second order graphs for two initial and final mesons. These are the annihilation graph, Figure 7, and the exchange graphs, Figure 8.

$$\langle P_{(\mu)},\mu;\ P_{(\nu)},\nu|S|P_{(\lambda)},\lambda;P_{(\sigma)},\sigma\rangle = \frac{-ig^2\delta(P_{(\mu)}+P_{(\nu)}-P_{(\lambda)}-P_{(\sigma)})}{(2\pi)^2\ 4\sqrt{P^\circ_{(\mu)}P^\circ_{(\nu)}P^\circ_{(\lambda)}P^\circ_{(\sigma)}}} \tag{6.16}$$

$$\cdot\Bigg[\ \frac{V_{\lambda\sigma}{}^{\alpha}(P_{(\lambda)},P_{(\sigma)},-P_{(\mu)}-P_{(\nu)})V_{\alpha}{}^{\mu\nu}(P_{(\mu)}+P_{(\nu)},-P_{(\mu)},-P_{(\nu)})}{(P_{(\mu)}+P_{(\nu)})^2-m_\alpha^2}$$

$$+\ \Theta[(P_{(\lambda)}-P_{(\mu)})^2]\frac{V^{\mu}{}_{\lambda}{}^{\alpha}(-P_{(\mu)},P_{(\lambda)},P_{(\mu)}-P_{(\lambda)})V_{\alpha}{}^{\nu}{}_{\sigma}(P_{(\lambda)}-P_{(\mu)},-P_{(\nu)},P_{(\sigma)})}{(P_{(\mu)}-P_{(\lambda)})^2-m_\alpha^2}$$

$$+\ \Theta[(P_{(\lambda)}-P_{(\nu)})^2]\frac{V^{\nu}{}_{\lambda}{}^{\alpha}(-P_{(\nu)},P_{(\lambda)},P_{(\nu)}-P_{(\lambda)})V_{\alpha}{}^{\mu}{}_{\sigma}(P_{(\lambda)}-P_{(\nu)},-P_{(\mu)},P_{(\sigma)})}{(P_{(\lambda)}-P_{(\nu)})^2-m_\alpha^2}\ \Bigg]$$

$$+\mathcal{O}(g^3)\ .$$

The first term in the bracket { } is the annihilation contribution; the second and third terms are exchange contributions.

Every expression in Eq. (6.16) has a simple physical interpretation. For example, in the annihilation term $V_{\lambda\sigma}{}^{\alpha}(P_{(\lambda)},P_{(\sigma)},-P_{(\mu)}-P_{(\nu)})$ is the probability amplitude that the incoming mesons $|P_{(\lambda)},\lambda>$ and $|P_{(\sigma)},\sigma>$ form the virtual meson with momentum $P_{(\mu)}+P_{(\nu)}$ and quantum numbers α. This virtual meson is a virtual bound state of two quarks described by the wavefunction $\phi_{\alpha}(P_{(\mu)}+P_{(\nu)})$. Although this virtual meson is off the mass shell, in its rest frame it has the same internal structure (wavefunction) as a physical meson with the same quantum numbers. The Feynman denominator propagates the virtual meson to the next vertex. $V_{\alpha}{}^{\mu\nu}$ is the probability amplitude that the virtual state becomes the outgoing mesons $|P_{(\mu)},\mu>$ and $|P_{(\nu)},\nu>$.

We interpret the second and third terms in { } as the exchange of a virtual meson between the vertices. The theta functions on the exchange terms arise because virtual bound states with spacelike momentum do not exist in our theory. We excluded such states because if we extend the internal wavefunction $\phi_{\alpha}(Q)$ off mass shell to spacelike Q we do not know how to give this wavefunction a covariant probability interpretation (or, in fact, any reasonable physical interpretation). Since the wavefunctions and vertex functions are only defined for timelike momenta, we could drop the theta functions from our formulae; we keep them only to remind us of this new feature in our theory. Since our graphs have no spacelike lines, for elastic scattering with $\lambda=\mu$ and $\sigma=\nu$ the graph in Fig. 8a does not contribute, and the graph in Fig. 8b contributes only when $\lambda \neq \sigma$ and only in a limited phase space.

We close this Section with two remarks about the relation between our theory of extended mesons and ordinary local QFT. First, at the level of practical S matrix calculation our field theory has many formal similarities with the standard field theory. In fact, if we replace the internal wavefunction $\phi_k(P,\xi)$ by $\delta(\xi)$ then we recover ordinary $:\phi^3:$ (Hurst-Thirring) QFT in Eq. (5.18) or Eq. (6.4) (after we replace the triple integral for $F_{k\ell m}$ by a double integral, so as not to integrate $\Phi(x)$ over all x twice). So our theory is more general than the regular QFT since in this sense it includes the usual theory as a special case. Second, if we reduce the size of the bound state we do not approach ordinary local QFT in the limit that meson size $\rightarrow$ zero; our field theory always exhibits some aspect of the composite nature of the meson. For example, if $\omega \rightarrow \infty$ the bound state shrinks to a point, but the bare vertex function $V_{k\ell m}$ does not approach a constant because the bound state wavefunction is defined for timelike momentum only and also the Lorentz contraction effect between wavefunctions (Fig. 1) is ω independent[39]. The ordinary field theory is a special case but not a limiting case of our theory. Our field theory shows a fundamental difference between the dynamics of bound states (regardless of size) and of elementary point particles.

7. UNITARITY

The exact S matrix, Eq.(5.24), is unitary because H_0 and H_1 are hermitean hamiltonians. But we do not know how to solve the exact theory, even perturbatively, because(unlike ordinary QFT)we have a nonlinear intergrodifferential equation for the U matrix, Eqs.(5.23) and (5.25). Using our physical insight we have replaced the exact U matrix equation with the approximate Eq.(6.3) and we have calculated U approximately by Eq.(6.4). We will soon see that naive application of Eq.(6.4) seems to violate unitarity. This apparent nonunitarity arises because Eq.(6.4) is not completely consistent with the physical principle of our approximation. When we fully implement the physical approximation we will find the S matrix is indeed unitary.

We begin the analysis of unitarity with some general formalism. For any theory we assume S can be expanded in powers of the coupling constant g:

$$S = \sum_{n=0}^{\infty} \frac{g^n}{n!} \int dx_1 \ldots dx_n \; S(x_1, \ldots, x_n) \quad , \tag{7.1}$$

where $S_0 = 1$ and S_n is a symmetric operator function of its arguments. In the Hurst-Thirring $:\phi^3:$ local QFT

$$S_n(x_1, \ldots, x_n) = i^n \; T(:\phi^3(x_1): \ldots :\phi^3(x_n):) \; . \tag{7.2}$$

In our extended meson theory, x_i represents the triplet

(X_i, Y_i, Z_i) and

$$S_n(x_i, \ldots, x_n) = (\tfrac{i}{6})^n \, T\,(:\Phi^3(x_1): \ldots, :\Phi^3(x_n):)$$
$$F(x_1) \ldots F(x_n), \qquad (7.3)$$

where $:\Phi^3(x_1):$ means $:\Phi_a(X_i)\Phi_b(Y_i)\Phi_c(Z_i):$, $F(x_i)$ means $F_{abc}(X_i, Y_i, Z_i)$ and Eq. (7.3) includes implicitly a sum over the subscripts a,b,c, etc. In this notation the unitarity of S is expressed as

$$SS^{\dagger} = 1 = \sum_{m,n=0}^{\infty} \frac{g^n}{m!(n-m)!} \int dx_1 \ldots dx_n \, S_m(x_1, \ldots, x_m)$$
$$\cdot \, S^{\dagger}_{n-m}(x_{m+1}, \ldots, X_n). \qquad (7.4)$$

The $O(g^n)$ term for $n > 0$ must vanish[33]:

$$\sum_{m,n=0}^{\infty} \frac{1}{m!(n-m)!} \int dx_1 \ldots dx_n \, S_m(x_1, \ldots, x_m)$$
$$S^{\dagger}_{n_m}(x_{m+1}, \ldots, x_n) = 0. \qquad (7.5)$$

To symmetrize the integrand, we introduce the symbol

$$P(\tfrac{m}{n})$$

which represents the sum over the $\frac{n!}{m!(n-m)!}$ ways of dividing the set of points $\{x_1, \ldots, x_n\}$ into two subsets of m and n-m points. Then we we can rewrite Eq. (7.5) in the symmetric form

$$\int \sum_{m=}^{n} P(\tfrac{m}{n}) \, S_m(x_1, \ldots, x_m) \, S^{\dagger}_{n-m}(x_{m+1}, \ldots, x_n) = 0. \qquad (7.6)$$

In ordinary QFT we can set the symmetric integrand equal to zero

$$S_n(x_1, \ldots, x_n) + S_n^\dagger(x_1, \ldots, x_n) + \sum_{m=1}^{n-1} P\left(\tfrac{m}{n}\right) S_m(x_1, \ldots, x_m)\, S_{n-m}^\dagger(x_{m+1}; \ldots, x_n) = 0. \qquad (7\ .7)$$

At the nth order of perturbation theory the unitarity condition is that Eqs. (7 .7) or (7 .6) are satisfied.

It is easy to relate Eq. (7 .7) to the standard analysis of unitarity which relies on physical intermediate states. When we apply Wick's theorem to the fields in $S_m S_{n-m}^\dagger$ we pick up some normal (not time ordered) contractions. For example, the normal contraction of a hermitean scalar field $\phi(x)$ is

$$\underbrace{\phi(x)\phi(y)} \equiv \langle 0|\phi(x)\phi(y)|0\rangle = \Delta_+(x-y;m), \qquad (7\ .8)$$

where $$\Delta_+(x-y;m) = \int \frac{dp}{(2\pi)^3}\, \delta(p^2-m^2)\theta(p^0)e^{-ip\cdot(x-y)}\ . \qquad (7\ .9)$$

The structure of Δ_+ makes it clear why these contractions correspond to converting some internal lines in a Feynman diagram into on mass shell, positive energy (i.e. physical) intermediate states.

Next, we compare the unitarity properties of Hurst-Thirring QFT with our field theory by applying Eq. (7 .7) to Eqs. (7 .2), (7 .3). We begin with the simplest case, the second order tree diagram of Figure 9. For $:\phi^3:$ theory the unitarity condition is

$$(i)^2 :\phi(x_1)\phi(x_1)\phi(x_2)\phi(x_2): \left[\overbrace{\phi(x_1)\phi(x_2)} - \underbrace{\phi(x_1)\phi(x_2)}\right] \qquad (7\ .10)$$

$$+ \text{ hermitean conjugate} = 0$$

Since $$\overbrace{\phi(x_1)\phi(x_2)} = \theta(x_1^0-x_2^0)\underbrace{\phi(x_1)\phi(x_2)} + \theta(x_2^0 - x_1^0)\underbrace{\phi(x^2)\phi(x_1)} \qquad (7\ .11)$$

and $$i\Delta(x-y;m) = \underbrace{\phi(x)\phi(y)} - \underbrace{\phi(y)\phi(x)}\ , \qquad (7\ .12)$$

the unitarity condition for the second order tree is

$$(i)^2 :\phi\phi\phi\phi: \left[\theta(x_2^0 - x_1^0)\ i\Delta(x_2-x_1;m) \right] + \text{hermitean conjugate} = 0. \qquad (7.12)$$

Eq. (7.12) is satisfied trivially because the Δ function is pure real. For our extended meson theory the unitarity condition is formally identical,

$$(i)^2 \text{"}:\Phi(x_1)\Phi(x_1)\Phi(x_2)\Phi(x_2): \left[\overbrace{\Phi(x_1)\Phi(x_2)} - \underbrace{\Phi(x_1)}\Phi(x_2) \right]\text{"} F(x_1)F(x_2) \qquad (7.13)$$
$$+ \text{ hermitean conjugate} = 0$$

where we are using an abbreviated notation (indicated by " ") in which $\Phi(x_i)$ means $\Phi_a(X_i \text{ or } Y_i \text{ or } Z_i)$, $\overbrace{\Phi(x_i)\Phi(x_j)}$ means $\overbrace{\Phi_a(X_i \text{ or } Y_i \text{ or } Z_i)\Phi_a(X_j}$ or Y_j or $Z_j)$, a sum over permutations of the variables is implied, and each variable occurs once and only once in each operator (e.g. $:\Phi_a(X_1)\Phi_b(Y_1)\Phi_k(X_2)\Phi_\ell(Y_2):\overbrace{\Phi_c(X_1)\Phi_c}(Z_2)$ is not allowed). Eq. (7.13) is satisfied because F_{abc} and Δ are pure real functions. Therefore, the second order tree diagram of our theory, Figures 7 and 8, is unitary.

This reasoning may be extended to the nth order tree diagram. It is not hard to show that in our theory a tree diagram of any order satisfies the corresponding unitarity condition.

Problems with unitarity occur in loop diagrams. To illustrate the problem, we consider the second order loop Figure 10. In Hurst-Thirring QFT this diagram is logarithmically divergent, but this will not affect our result, while using a higher order (convergent) loop would needlessly complicate our discussion. The unitarity condition is

$$(i)^2:\phi(x_1)\phi(x_2): \left[\overbrace{\phi(x_1)\phi}(x_2)\overbrace{\phi(x_1)\phi}(x_2) - \phi\underbrace{(x_1)}\phi(x_2)\phi\underbrace{(x_1)}\phi(x_2) \right]$$
$$+ \text{ hermitean conjugate} = 0. \qquad (7.14)$$

The bracket [ ] above is equal to

$$\theta(x_2^0-x_1^0)\{\underbrace{\phi(x_2)\phi(x_1)}\underbrace{\phi(x_2)\phi(x_1)}-\underbrace{\phi(x_1)\phi(x_2)}\underbrace{\phi(x_1)\phi(x_2)}\}. \qquad (7.15)$$

Since $\underbrace{\phi(x)\phi(y)}^{*} = \underbrace{\phi(y)\phi(x)}$, the bracket { } above is pure imaginary, and therefore Eq. (7 .14) is satisfied.

In our theory, the unitarity condition for the second order loop is

$$(i)^2\ {}'':\Phi(x_1)\Phi(x_2):\left[\overbrace{\Phi(x_1)\Phi(x_2)}\overbrace{\Phi(x_1)\Phi(x_2)}-\underbrace{\Phi(x_1)\Phi(x_2)}\underbrace{\Phi(x_1)\Phi(x_2)}\right]''F(x_1)F(x_2)$$

$$+ \text{ hermitean conjugate } = 0. \qquad (7.16)$$

The abbreviated notation here stands for a sum over many terms with similar structure. To prevent any confusion at this stage, we pick out one term of this sum, viz.

$$(i)^2:\Phi_a(X_1)\Phi_k(X_2):\left[\overbrace{\Phi_b(Y_1)\Phi_b(Y_2)}\overbrace{\Phi_c(Z_1)\Phi_c(Z_2)}-\underbrace{\Phi_b(Y_1)\Phi_b(Y_2)}\underbrace{\Phi_c(Z_1)\Phi_c(Z_2)}\right]$$

$$\cdot F_{abc}(X_1,Y_1,Z_1)F_{kbc}(X_2,Y_2,Z_2) + \text{ hermitean conjugate.} \qquad (7.17)$$

There are 35 other terms that differ from (7 .17) by permutation inside the triplets (X_1,Y_1,Z_1) and (X_2,Y_2,Z_2).

The bracket [ ] above can be written as

$$\theta(Y_1^0-Y_2^0)\theta(Z_2^0-Z_1^0)\underbrace{\Phi_b(Y_1)\Phi_b(Y_2)}\{\underbrace{\Phi_c(Z_2)\Phi_c(Z_1)}-\underbrace{\Phi_c(Z_1)\Phi_c(Z_2)}\}$$

$$+\ \theta(Y_2^0-Y_1^0)\theta(Z_1^0-Z_2^0)\underbrace{\Phi_c(Z_1)\Phi_c(Z_2)}\{\underbrace{\Phi_b(Y_2)\Phi_b(Y_1)}-\underbrace{\Phi_b(Y_1)\Phi_b(Y_2)}\}$$

$$+\ \theta(Y_2^0-Y_1^0)\theta(Z_2^0-Z_1^0)\{\underbrace{\Phi_b(Y_2)\Phi_b(Y_1)}\underbrace{\Phi_c(Z_2)\Phi_c(Z_1)}-\underbrace{\Phi_b(Y_1)\Phi_b(Y_2)}\underbrace{\Phi_c(Z_1)\Phi_c(Z_2)}\}. \qquad (7.18)$$

The third term in (7 .18) is pure imaginary and gives no contribution to (7 .17) when we take its hermitean part. The first and second terms do give a contribution, viz.

$$:\Phi_a(X_1)\Phi_k(X_2):F_{abc}(X_1,Y_1,Z_1)\ F_{kbc}(X_2,Y_2,Z_2)$$

$$\cdot[\Delta_{ret}(Y_1-Y_2;m_b)\Delta_{ret}(Z_2-Z_1;m_c) + \Delta_{adv}(Y_1-Y_2;m_b)\Delta_{adv}(Z_2-Z_1;m_c)]\ , \qquad (7.19)$$

where

$$\Delta_{ret}(x) = -\Theta(x^0)\ \Delta(x),\ \Delta_{adv}(x) = \Theta(-x^0)\Delta(x). \qquad (7.20)$$

Expression (7 .19) does not satisfy Eq. (7.16), so the second order loop in our theory naively violates unitarity. In the abbreviated notation the sum of all 36 terms on the left side of (7 .16) is

$$":\Phi(x_1)\Phi(x_2):F(x_1)F(x_2)[\Delta_{ret}(x_1-x_2)\Delta_{ret}(x_2-x_1)$$

$$+ \Delta_{adv}(x_1-x_2)\Delta_{adv}(x_2-x_1)]". \qquad (7.21)$$

The hermitean terms in expression (7 .18) do not occur in $:\phi^3:$ theory because the point interaction of $:\phi^3:$ theory sets $X_i = Y_i = Z_i$ in (7 .18), which makes those terms vanish. But our theory has a non-pointlike interaction in which the real and/or virtual states reach the vertex at three different times. For a tree diagram the naive application of the time ordering prescription in Eq. (6 .4) gives only one time relation between any two vertices, so the diagram is formally identical to the $:\phi^3:$ case. The existence of the internal time coordinate in the meson, which makes the vertex spread in time, has no effect in the tree diagram, so the trees are unitary. But for a loop diagram, naively time ordering the fields in Eq. (6 .4) gives several time relations between some vertices. If the vertices overlap in time, then time ordering between the vertices is ambiguous -- one vertex may be both earlier and later than another. In this condition a "loop" does not appear to act physically like a loop, because the two lines of the "loop" may propagate both virtual states forward in time (or both backward in time); see Figure 11. This is the extra contribution to loop diagrams, not present in theories like $:\phi^3:$ with point vertices. The inconsistency of expressions (7.16) and (7.21) is the result of our naive approach to the time ordering in Eq. (6.4).

Let us recall our approximation scheme for the U matrix. To make Eq. (5.23) the basis of a perturbation theory we had to linearize it by removing U from the integrand in Eq. (5.25). Then we iterated Eq. (5.23), regarding "$\Phi\Phi\Phi F$" as a point interaction. *Therefore, in our approximation propagation between vertices ignores the time spread of the vertices.* So to be physically consistent the time ordering prescription of Eq. (6.4) should implicitly neglect the extra contributions like expression (7.21) which come from ambiguous time orderings as in Figure 11. Then our approximation, carefully executed, preserves the unitarity of S.

In performing calculations of S in this approximation, we can carry out the naive time ordering of fields in Eq.(6.4) as an intermediate step by applying the diagramatic rules of Sec. 6, and then subtract out the contribution arising from the overlaping of vertices in time. In most cases we can write down this subtraction simply by inspection of the possible time ordering ambiguities in the loop diagrams. For example, in the third order loop diagrams we must subtract from the S matrix

$$(i)^3 \text{"}:\Phi(x_1)\Phi(x_2)\Phi(x_3):F(x_1)F(x_2)F(x_3)[\theta(x_1-x_2)\theta(x_2-x_3)\theta(x_3-x_1)$$

$$\Delta_+(x_1-x_2)\Delta_+(x_2-x_3)\Delta_+(x_3-x_1) + \theta(x_2-x_1)\theta(x_3-x_2)\theta(x_1-x_3)$$

$$\Delta_-(x_1-x_2)\Delta_-(x_2-x_3)\Delta_-(x_3-x_1)]\text{"}$$

for Figure 12a and

$$(i)^3 \text{"}:\Phi(x_1)\Phi(x_3)\Phi(x_3):[\theta(x_1-x_2)\theta(x_2-x_1)\theta(x_2-x_3)\Delta_+(x_1-x_2)\Delta_+(x_2-x_1)$$

$$\Delta_+(x_2-x_3) + \theta(x_2-x_1)\theta(x_1-x_2)\theta(x_3-x_2)\Delta_-(x_1-x_2)\Delta_-(x_2-x_1)$$

$$\Delta_-(x_2-x_3)]\text{"}$$

for Figure 12b. For the box diagram in two meson scattering the subtraction is somewhat more complicated to write but the principle is the same as in the simpler second and third order loops.[1] Physically this approximation is good when the main contribution to the diagram comes from vertices well separated in time, which is when the virtual states in the diagram are mainly long-lived (compared to the mesonic time ω^{-1}), that is when they are mainly near the mass shell (compared to the spacing ω between mesonic levels).

We have developed an approximation scheme to calculate the S matrix which is consistent with Lorentz covariance and unitarity. In this scheme the extended structure of the mesons manifests itself in the vertex function, but the vertices act like points when they connect to propagators. To make further progress we must incorporate the effects of extended meson size on propagations. This requires that we deal with the nonlinear effects of propagation over internal time intervals inside the vertex, Eq. (5.25). These effects become important when the spacetime separation between vertices is of the same order as the intrinsic size of the vertex ω^{-1}, i.e. when the vertices overlap. Notice that neglecting these nonlinearities has given us exact Feynman propagation between vertices (cf. the graphical rules of Sec. 6). We conjecture that the exact nonlinear theory of S, based on Eq. (5.25), will modify the propagator at short distances.

8. SUMMARY AND OUTLOOK

Our research grew out of attempts to relate several outstanding problems of quantum theory to some more recent discoveries in the subnuclear world, in the hope of making progress on the theoretical and phenomenological fronts simultaneously. The theoretical questions we are refering to are the problem of a relativistic interpretation of probability and the problem of the meaning of extended size and localization in relativistic quantum theory. Of course, these problems were recognized at the birth of quantum mechanics; their relevance has not diminished with age [24]. In particle physics we are very interested in understanding the enormous success of SU(3) and other symmetry schemes based on the quark picture(such as SU(6)) for hadrons as well as the growing body of evidence for the non-point nature of hadrons(for example, the evidence from inclusive lepton reactions for pointlike constituents inside the nucleon).

The mesonic field theory we have developed is a model applicable to all these problems and more. The field has a new dynamical degree of freedom ξ, the spacetime separation of the two constituents of the meson. The harmonic oscillator wavefunction is a relativistic probability amplitude and a description of a non-point meson localized in spacetime; it also gives a good description of mesonic spectroscopy and static properties. The theory separates external from internal variables, making most of the usual formalism of QFT applicable to the external field. The theory is in lagrangian form, with the usual connection between symmetries of L and conservation laws. The self interaction of the field describes meson scattering in two senses: (1) the external degrees of freedom(associated with the meson as a whole) interact, and (2) the S matrix is calculated as a perturbation around states that are already mesons(bound systems of two quarks). This seems superior to theories which perturb around free quarks and model the self interaction of quarks(especially since there is no evidence that quarks can be free). The internal and external interactions are related by the bare vertex function $V_{k\ell m}$ or its fourier transform $F_{k\ell m}$. The theory is consistent with the accepted principles of covariance, causality and unitarity in QFT.

The most successful quantum field theory is Quantum Electrodynamics for leptons. Its success is famous and unrivaled by anything else in physics. QED is a field theory of point particles, and there is no firm evidence to shake our belief that lepton and photon are pure point particles. QFT has been much less successful in explaining the dynamics of hadrons, while there is ample evidence for their non-point character. Our mesonic field theory is interesting not because of its possible phenomenological accuracy(although we seek such accuracy), but rather because it directly incorporates our fundamental physical ideas about internal hadron structure into the convenient calculational formalism of QFT. We feel this is a significant advance for QFT.

Our field theory suggests several new lines of investigation. The perturbation theory is a systematic extension of standard quark model calculations(O(g) of our theory), so we could make the theory more realistic(by incorporating quark spin and SU(3) wavefunctions) and explore the accuracy of the theory in higher orders of g. Since the vertices are complicated functions of momentum and we cannot use residue calculus to evaluate the integrals, detailed calculation of high order S matrix elements seems very formidable. But the form of the vertex function gives a natural cutoff at high momentum(see Appendix B), so the convergence properties of the perturbation series(if we could calculate it)might be much better than in ordinary QFT. This possibility alone justifies further examination of our theory, independently of predictive accuracy. Since the experimental data on meson-meson scattering is indirect and not very certain, it is of interest to extend our theory to include baryons with meson-baryon interactions in order to compare the theory with data on baryon-baryon and baryon-meson scattering and baryon decay. A preliminary calculation that may be the lowest order in such a baryonic field theory has already been done[41].

Our work can also be extended by a better analysis and approximation of the nonlinear integrodifferential equation for the S matrix, Eqs.(5.23) and (5.25). This equation can be a testing ground for new ideas on the relation between ordinary QFT and more general field theories It has already shown itself to be a good laboratory to explore the difference between point and extended particles.

APPENDIX A: HELICITY BASIS

For some applications the orbital basis of wavefunctions is not the best, but we can easily transform to other bases. Here we explicitly construct the helicity basis.

We define a new complementary set $\{C'(P)\}$ where $C'(P)$ is a Lorentz transformation along the 3-axis followed by a rotation around an axis in the 1-2 plane.

$$C'(P) = \begin{bmatrix} (1-\frac{P^3}{|\vec{P}|})\frac{P_2^2}{P_1^2+P_2^3} + \frac{P^3}{|\vec{P}|} & -(1-\frac{P^3}{|\vec{P}|})\frac{P_1P_2}{P_1^2+P_2^2} & \frac{P^0P^1}{m|\vec{P}|} & \frac{P^1}{m} \\ -(1-\frac{P^3}{|\vec{P}|})\frac{P_1P_2}{P_1^2+P_2^2} & (1-\frac{P^3}{|\vec{P}|})\frac{P_1^2}{P_1^2+P_2^2} + \frac{P^3}{|\vec{P}|} & \frac{P^0P^2}{m|\vec{P}|} & \frac{P^2}{m} \\ -\frac{P^1}{|\vec{P}|} & \frac{-P^2}{|\vec{P}|} & \frac{P^0P^3}{m|\vec{P}|} & \frac{P^3}{m} \\ 0 & 0 & \frac{|\vec{P}|}{m} & \frac{P^0}{m} \end{bmatrix} \tag{A.1}$$

In the rest frame the helicity functions $\phi'_\alpha(\underset{o}{P})$ and the orbital functions $\phi_\alpha(\underset{o}{P})$ are the same (see Eq. (2 .25)),

$$\phi'_\alpha(\underset{o}{P}) \equiv U_{\alpha k}\, \phi_k(\underset{o}{P}) \; . \tag{A.2}$$

In an arbitrary frame, for $P = C'(P)\underset{o}{P}$,

$$\phi'_\alpha(P,\ \xi) = \phi'_\alpha(\underset{o}{P},\ C'(P)^{-1}\xi). \tag{A.3}$$

Under a Lorentz transformation Λ,

$$\phi'_\alpha(\Lambda P,\ \Lambda\xi) = Q_{\alpha\beta}\,[M'(\Lambda,P)]\ \phi'_\beta\ (P,\ \xi) \tag{A.4}$$

where $\quad M'(\Lambda,P) \equiv C'(\Lambda P)^{-1}\,\Lambda C'(P).$ $\qquad$ (A.5)

When Λ is a rotation R the Wigner rotation $M'(R,P)$ is a rotation around the 3-axis[30] by angle $\beta(R,P)$.

Then

$$\phi'_\alpha(RP,\ R\xi) = e^{iM(\alpha)\beta(R,P)}\ \phi'_\alpha(P,\xi) \tag{A.6}$$

where $M(\alpha)$ is the 3-component of angular momentum of $\phi_\alpha(\underset{o}{P})$. If R is a rotation around $\vec{P}$ by angle θ, then $\beta(R,P) = \theta$. The bases ϕ_α and ϕ'_α are related by

$$\phi'_\alpha(P) = Q_{\alpha\beta}[R(P)^{-1}]\phi_\beta(P)\ , \tag{A.7}$$

where $R(P)$ is the rotation that transforms the 3-direction into the $\vec{P}$ direction. The Wigner rotations M and M' are related by

$$M'(\Lambda,P) = R^{-1}(\Lambda P)M(\Lambda,P)R(P). \tag{A.8}$$

Naturally, all these formulae are equally valid for on- and off-mass shell wavefunctions.

We may define helicity creation and annihilation operators as

$$a'_\alpha(P) \equiv Q_{\beta\alpha}[R(P)]\ a_\beta(P)\ . \tag{A.9}$$

Then

$$a_\alpha(P)\phi_\alpha(P) = a'_\alpha(P)\phi'_\alpha(P). \tag{A.10}$$

A single meson state in the helicity basis is

$$|P,\alpha>' \equiv a'^{\dagger}_\alpha(P)|0> = Q^*_{\beta\alpha}[R(P)]|P,\beta>. \tag{A.11}$$

For an arbitrary Lorentz transformation Λ the unitary operator $U(\Lambda)$ defined by Eq. (4.19) transforms the helicity creation operators as follows:

$$U(\Lambda)a'^{\dagger}_\alpha(P)\ U^\dagger(\Lambda) \equiv \left[\frac{(\Lambda P_{(\alpha)})^\circ}{(P_{(\alpha)})^\circ}\right]^{1/2} a'^{\dagger}_\beta(\Lambda P)\ Q^*_{\beta\alpha}\ [M'(\Lambda,P)]. \tag{A.12}$$

For a rotation R, Eq. (A.12) simplifies to

$$U(R)a'^{\dagger}_\alpha(P)U^\dagger(R) = e^{-iM(\alpha)\beta(R,P)}a'^{\dagger}_\alpha(RP)\ . \tag{A.13}$$

APPENDIX B: OFFSHELL CHO AND VERTEX FUNCTIONS

The oscillator wavefunction $\phi_k(Q)$ for arbitrary timelike Q is defined by Eqs. (2.15), (2.17) and (2.18) with m_k replaced by Q^2. The subscript k denotes all quantum numbers needed to identify the state, viz. (n_1, n_2, n_3) in Eq. (2.15). $\phi_k(Q)$ satisfies Eq. (2.22). $\phi_\alpha(Q)$ is defined by Eq. (2.25). The subscript α denotes (E, L, M) where E is defined by Eq. (2.12) and where L is the total and M is the 3-component of angular momentum of $\phi_\alpha(Q)$ in the frame where $\vec{Q} = 0$. It follows immediately from the definition that $|\phi_k(Q,\xi)|$ for $Q^2>0$ is bounded as a function of Q and ξ, with the bound depending on k. We define $\phi_k(Q) = 0$ for $Q^2<0$.

In its rest frame $(\vec{Q} = 0)$ $\phi_k(Q)$ contains the gaussian factor

$$\exp\{-\frac{\omega}{4}[\xi_o^2 + \xi_1^2 + \xi_2^2 + \xi_3^2]\} \ . \tag{B.1}$$

In an arbitrary Lorentz frame the gaussian becomes

$$\exp\{-\frac{\omega}{4}[-\xi^2 + \frac{2(Q\cdot\xi)^2}{Q^2}]\} \ . \tag{B.2}$$

Compare this with the gaussian factor on the mass shell,

$$\exp\{-\frac{\omega}{4}[-\xi^2 + \frac{2(Q\cdot\xi)^2}{m_k^2}]\}. \tag{B.3}$$

Expression (B.2) is an analytic function of Q everywhere except at $Q^2 = 0$, where (B.2) has an essential singularity.

We have not defined $\phi_k(Q)$ for $Q^2 = 0$. However, we note that, for $\xi\cdot Q \neq 0$ and $\vec{Q} \neq 0$, as $Q^o \to |\vec{Q}|$ along the real line

$$\lim_{Q^o\to Q^2} \left(\frac{1}{Q^2}\right)^n \exp\{-\frac{\omega}{4}[-\xi^2 + \frac{2(Q\cdot\xi)^2}{Q^2}]\} = 0 \tag{B.4}$$

for all n. The momentum space wavefunction $\tilde{\phi}_k(Q,q)$ has similar behavior as $Q^o \to |\vec{Q}|$, because the fourier transform of a harmonic oscillator wavefunction is (up to a scale transformation) the same wavefunction.

The off mass shell wavefunction is used only to calculate the off mass shell vertex function

$$V_{k\ell m}(P,Q,R) = \frac{1}{16(2\pi)^4}\int dq\,\tilde{\phi}_k(P,q)\tilde{\phi}_\ell(Q,q+R)\tilde{\phi}_m(R,q-Q) \tag{B.5}$$

with $P+Q+R = 0$. For timelike momenta we can calculate $V_{k\ell m}$ in a frame where $\vec{R} = 0$. Equation (B.4) implies that as P^2 or $Q^2 \to 0$ ($\vec{P} \neq 0$, $\vec{Q} \neq 0$) the wavefunctions $\tilde{\phi}_k$, $\tilde{\phi}_\ell$ are large only near the null plane while by (B.1) the gaussian in ϕ_m is symmetric around 0; see Fig. 13. Then as $P^0 \to |\vec{P}|$ or $Q^0 \to |\vec{Q}|$ ($\vec{P} \neq 0$, $\vec{Q} \neq 0$) along the real line

$$V_{k\ell m}(P,Q,R) \to 0 . \tag{B.6}$$

Therefore we define, for $P^2 = 0$ ($\vec{P} \neq 0$) and Q^2 or $R^2 > 0$

$$V_{k\ell m}(P,Q,R) \equiv 0. \tag{B.7}$$

In the frame where $\vec{R} = 0$, $\phi_m(R,\xi)$ is really independent of R, and therefore we define for $P^2 \geq 0$ and $P \neq 0$

$$V_{k\ell m}(P,-P,0) \equiv V_{k\ell m}(P,-P,R_o) \tag{B.8}$$

where R_o is any timelike momentum with $\vec{R}_o = 0$. The limiting value of $V_{k\ell m}(P,Q,R)$ as $P,Q,R \to 0$ depends on the way the momenta approach zero. However, $V_{k\ell m}(P,Q,R)$ is bounded as the momenta approach zero, so we may arbitrarily set

$$V_{k\ell m}(0,0,0) \equiv 0 \tag{B.9}$$

without affecting the result of any of our calculations.

We adopt the definition, consistent with Eq.(B.7), that $\phi_k(Q) = 0$ for $Q^2 = 0$, $Q \neq 0$. There still is an ambiguity in $\phi_k(0)$, but this does not affect any of our calculations. Now $|\phi_k(Q,\xi)|$ is well defined and bounded for all ξ and all $Q \neq 0$, so f_k defined by Eq.(4.36) is well defined as a tempered distribution [42].

Whenever we contract two local fields in the S matrix we encounter the integral

$$\frac{1}{(2\pi)^4} \int dQ \, \frac{e^{iQ\cdot X} \, \theta(Q^2) \, \phi_k(Q,\xi)\phi_k(Q,\eta)}{Q^2 - m_k^2 + i\varepsilon} \equiv D_F(X,\xi,\eta;\, m_k) \, . \tag{B.10}$$

In ordinary field theory the corresponding integral is

$$\frac{1}{(2\pi)^4} \int dq \, \frac{e^{iq\cdot x}}{q^2 - m^2 + i\varepsilon} = \Delta_F(x). \tag{B.11}$$

In momentum space Δ_F and the delta function are well-defined as tempered distributions. Since $\phi_k(Q)$ is bounded, D_F is well defined as a tempered distribution[42].Formulae involving Δ_F may be evaluated with the residue calculus, but formulae with D_F cannot be evaluated the same way, because in momentum space $D_F(Q,\xi,\eta;\, m_k)$ is not an analytic function of Q. This is because of the $\theta(Q^2)$ and also the essential singularity of $\phi_k(Q)$ at $Q^2 = 0$. The large Q^2 effects of D_F cannot be more singular than the large Q^2 effects of Δ_F, because ϕ_k is bounded.

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arguments above) that a consistent field theory might be possible by second quantizing only the external-variable part of $\Phi(X,\xi)$ was suggested to the author by Y. S. Kim in a private communication. Kim's conjecture was based on his observation that second quantization (at least in spinor field theory) is needed to eliminate the paradoxical negative energy states, but the analogous negative "energies" E(k) for the wavefunction are eliminated by the subsidiary condition, Eq. (2.22), without second quantization.

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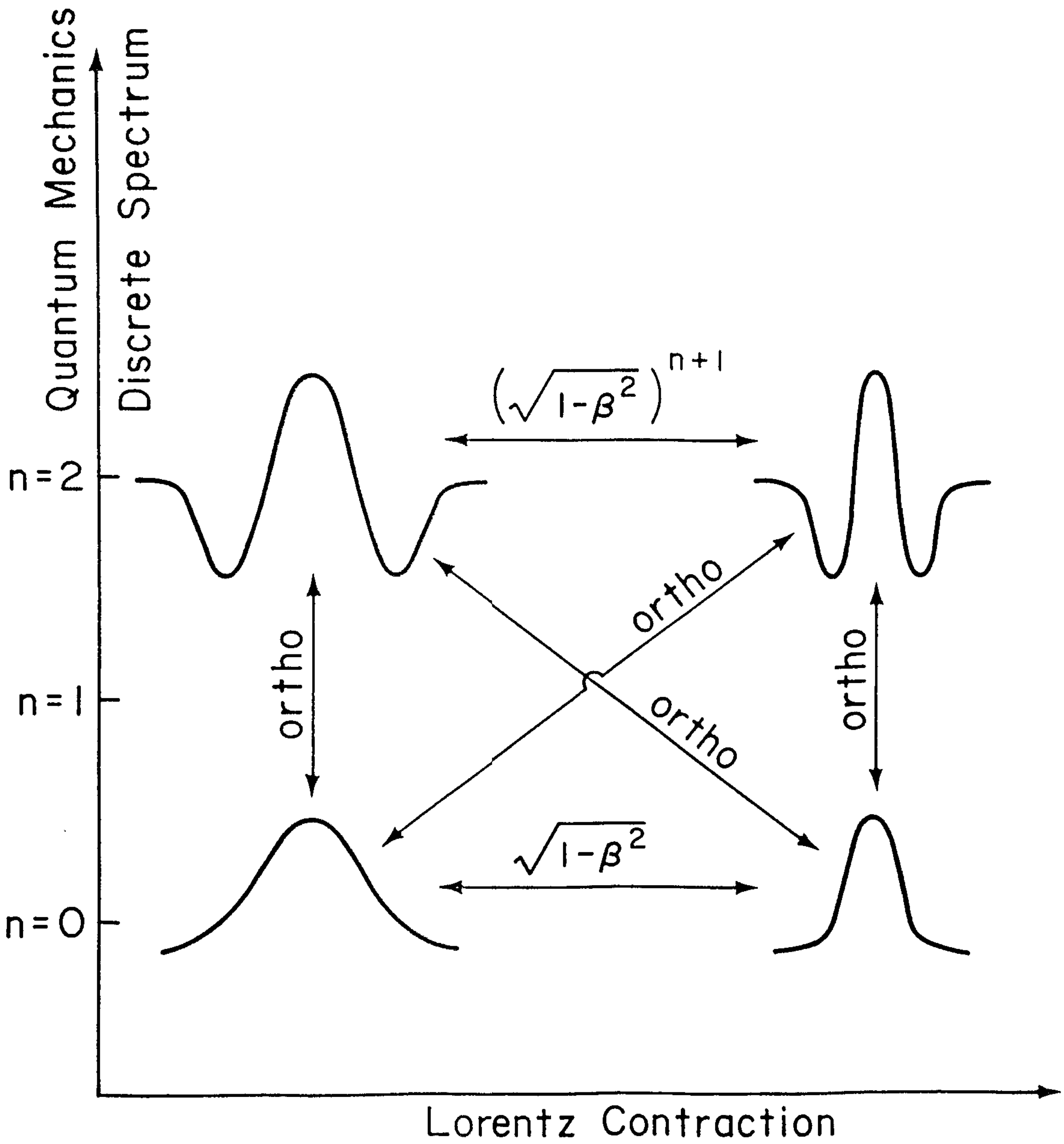
FOOTNOTES

1. The author acknowledges R. Brandt for pointing out to him the importance of the box diagram for unitarity.

FIGURE 1


Quantum Mechanics
Discrete Spectrum
n=2
n=1
n=0
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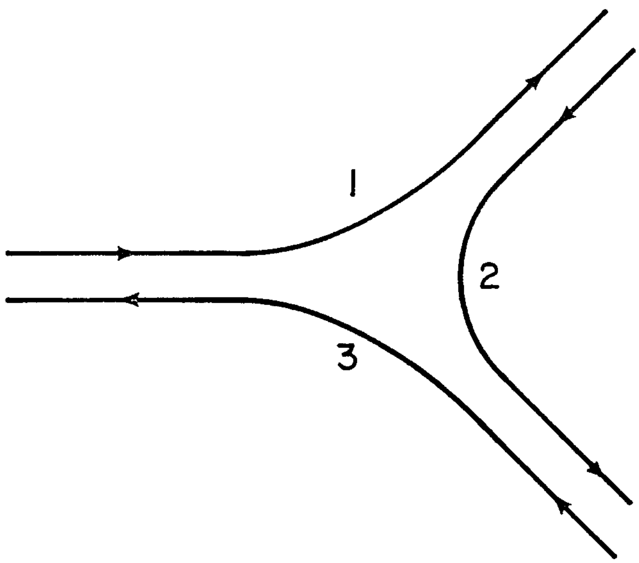
ortho
ortho
ortho
ortho
$\sqrt{1-\beta^2}$
Lorentz Contraction

FIGURE 2



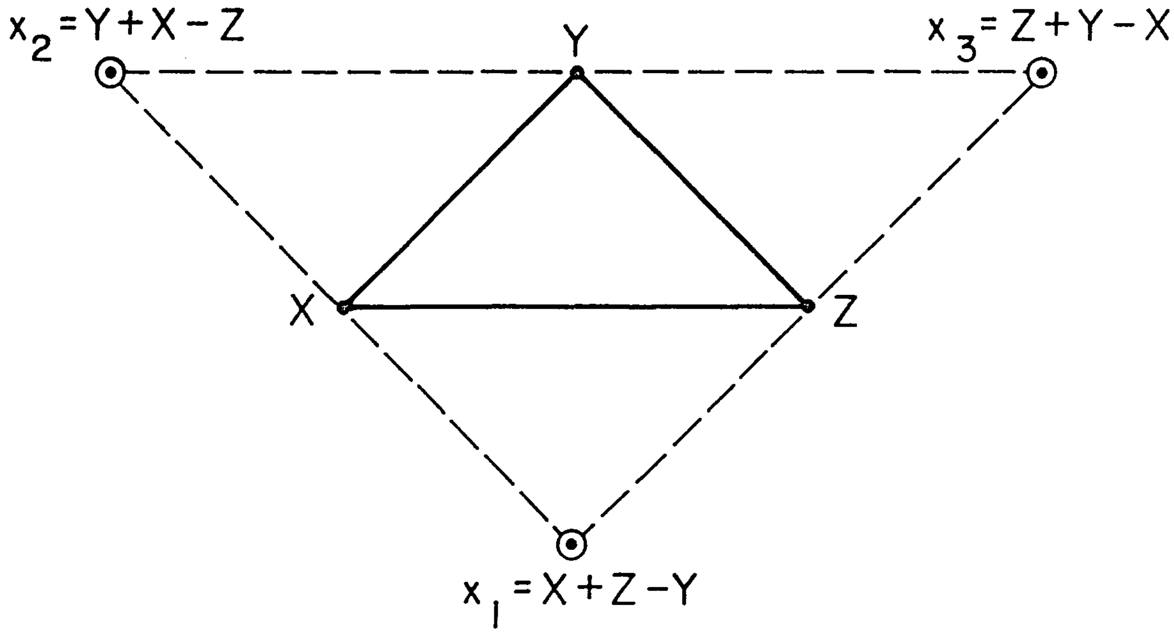


FIGURE 3

FIGURE 4

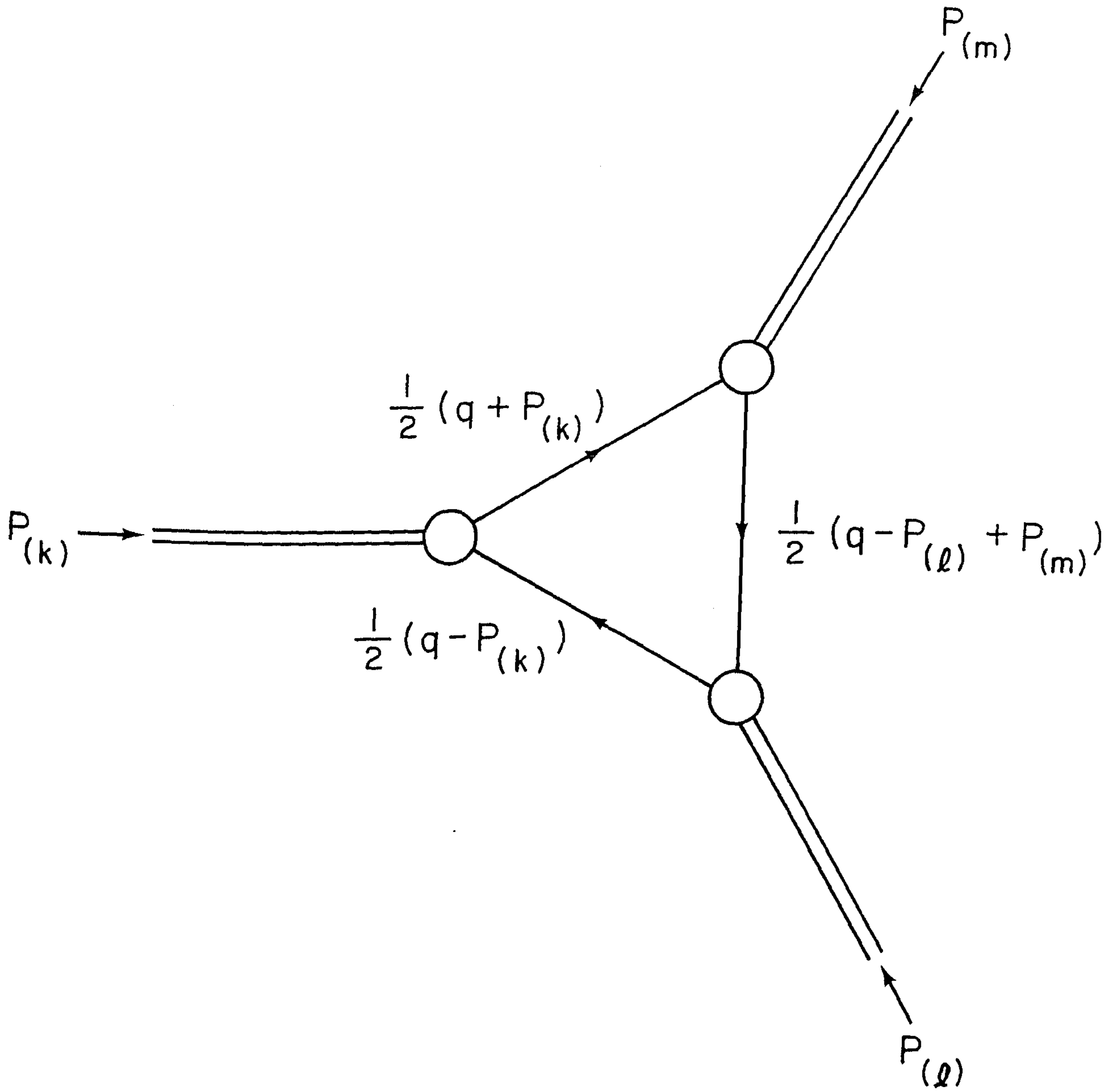
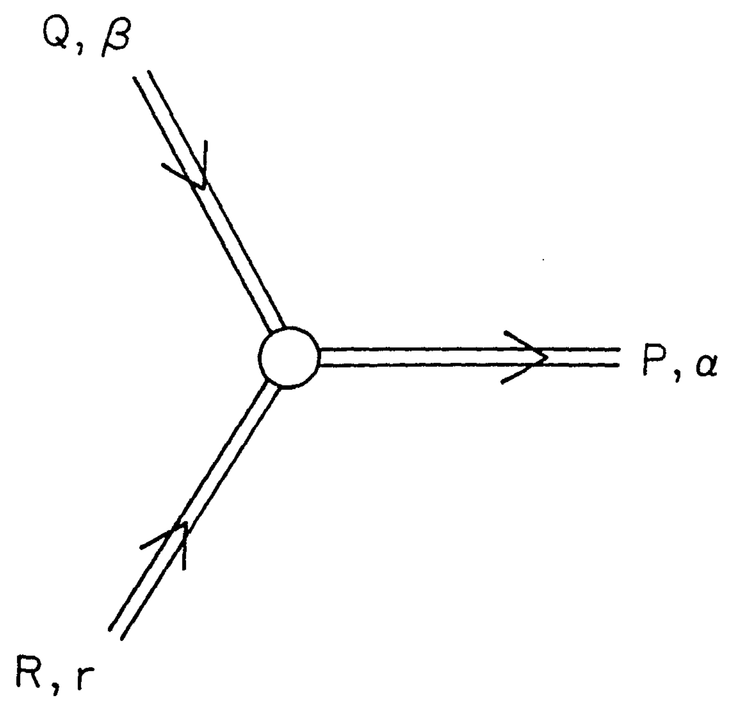
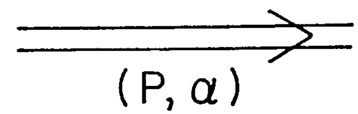


FIGURE 5

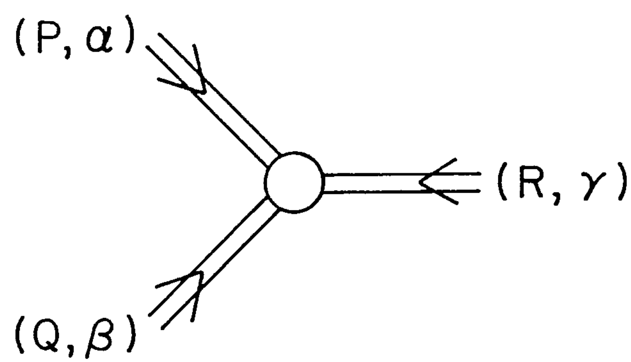


External line
(momentum P,
quantum numbers α)



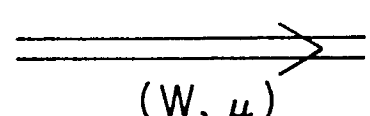
$$\frac{1}{(2\pi)^{3/2}\sqrt{2P^{o}}}$$

Vertex



$$igV_{\alpha\beta\gamma}(P,Q,R)(2\pi)^{4}\delta(P$$

Internal line
(virtual momentum W,
quantum numbers μ)



$$\frac{i\,\theta\,(W^{2})}{W^{2}-m_{\mu}^{2}+i\epsilon}$$

FIGURE 6

FIGURE 7

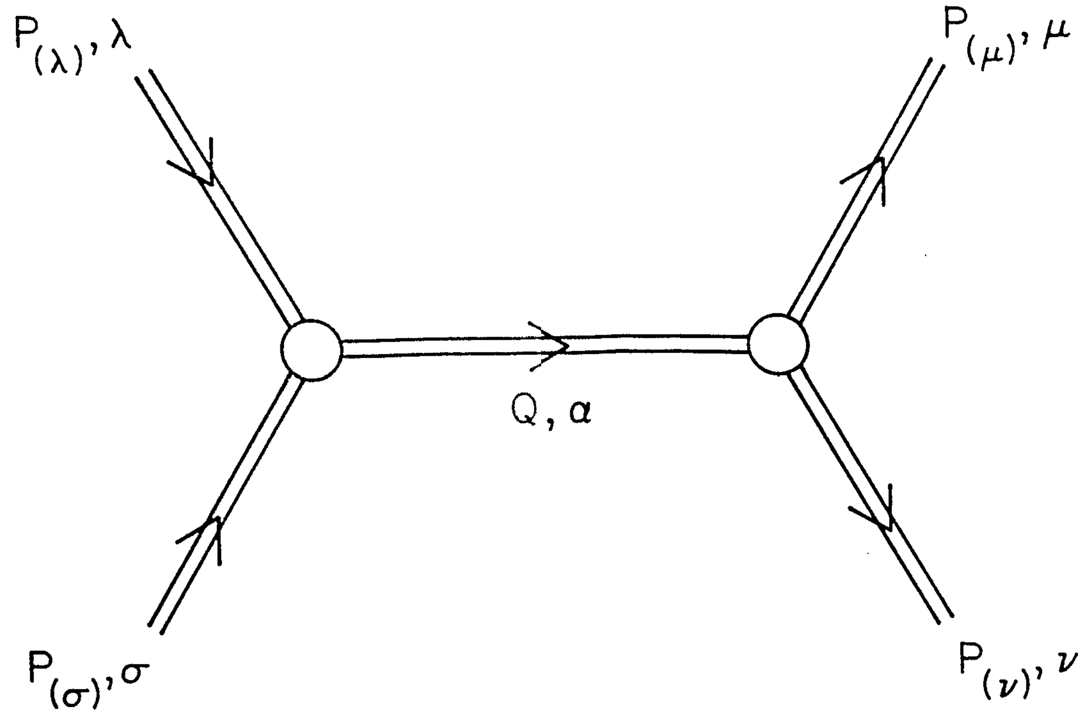


FIGURE 8

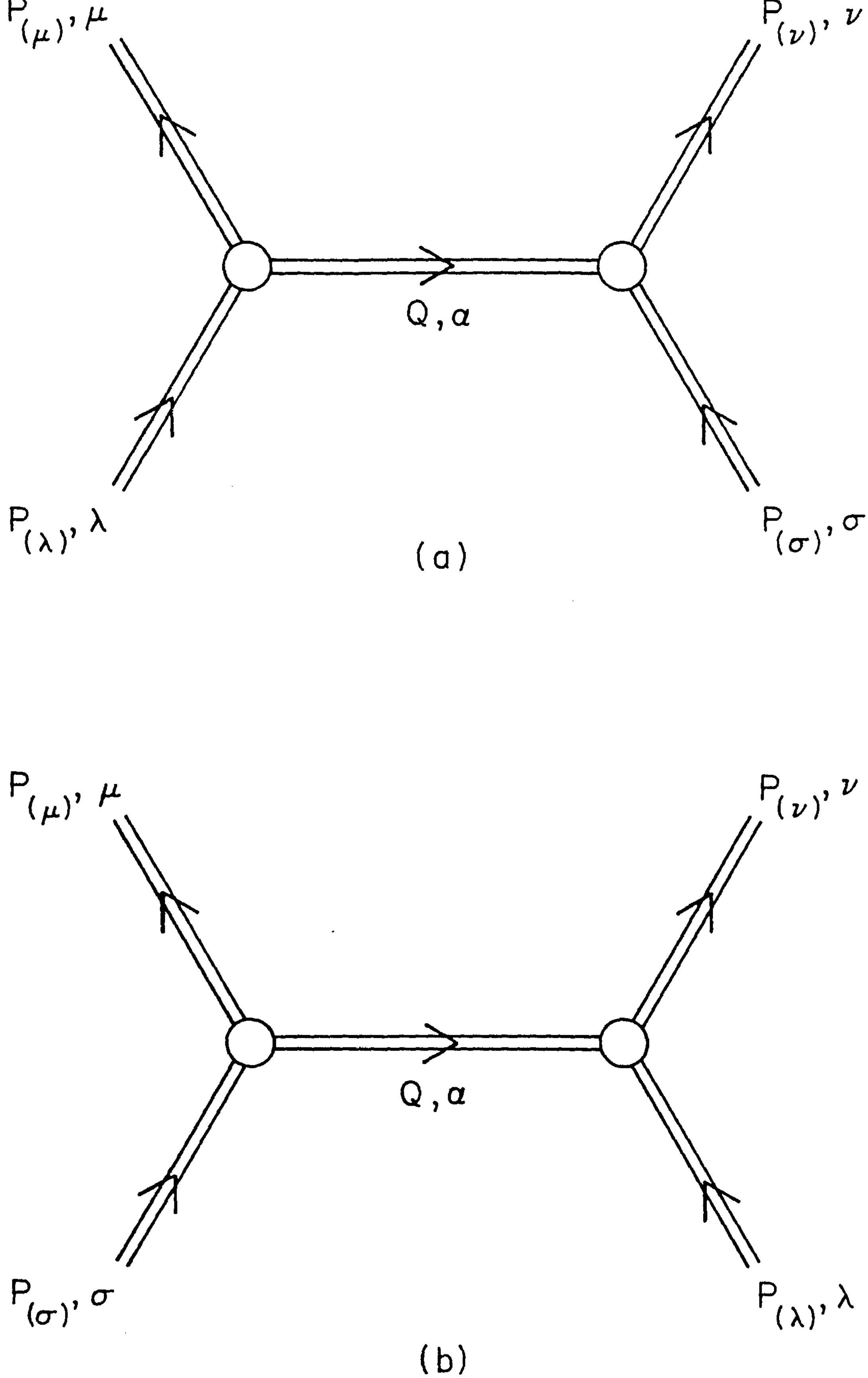

$P_{(\mu)}, \mu$
$P_{(\nu)}, \nu$
Q, α
$P_{(\lambda)}, \lambda$
$P_{(\sigma)}, \sigma$
(a)
$P_{(\mu)}, \mu$
$P_{(\nu)}, \nu$
Q, α
$P_{(\sigma)}, \sigma$
$P_{(\lambda)}, \lambda$
(b)

FIGURE 9

FIGURE 10

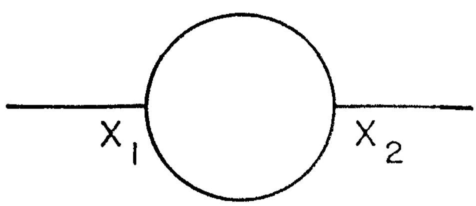


FIGURE 11

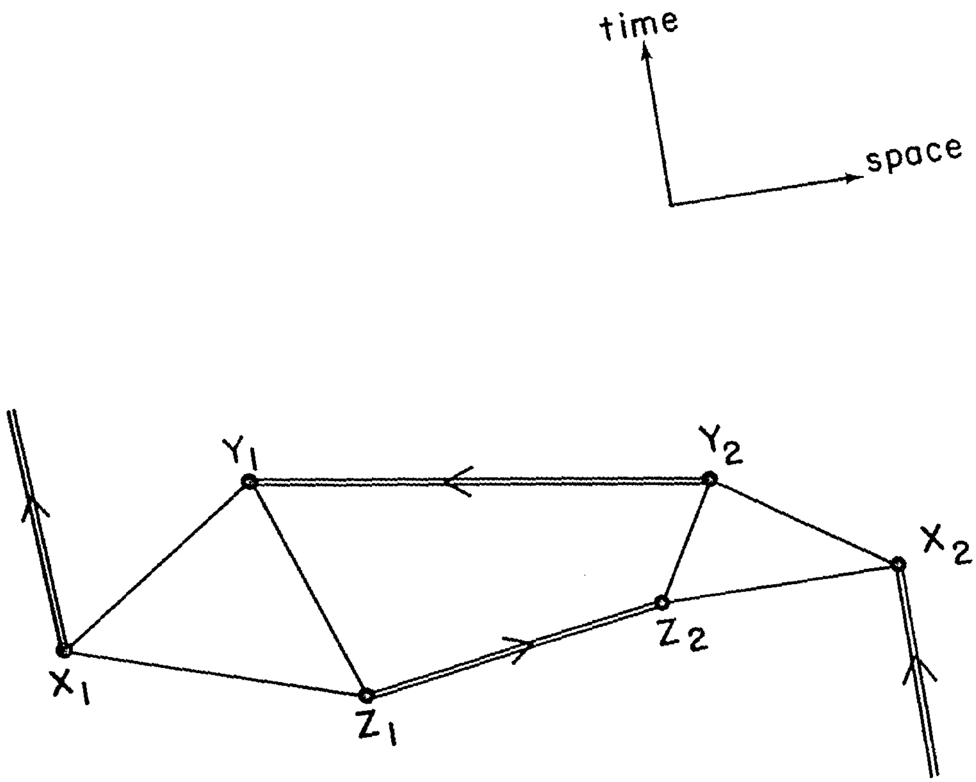


FIGURE 12

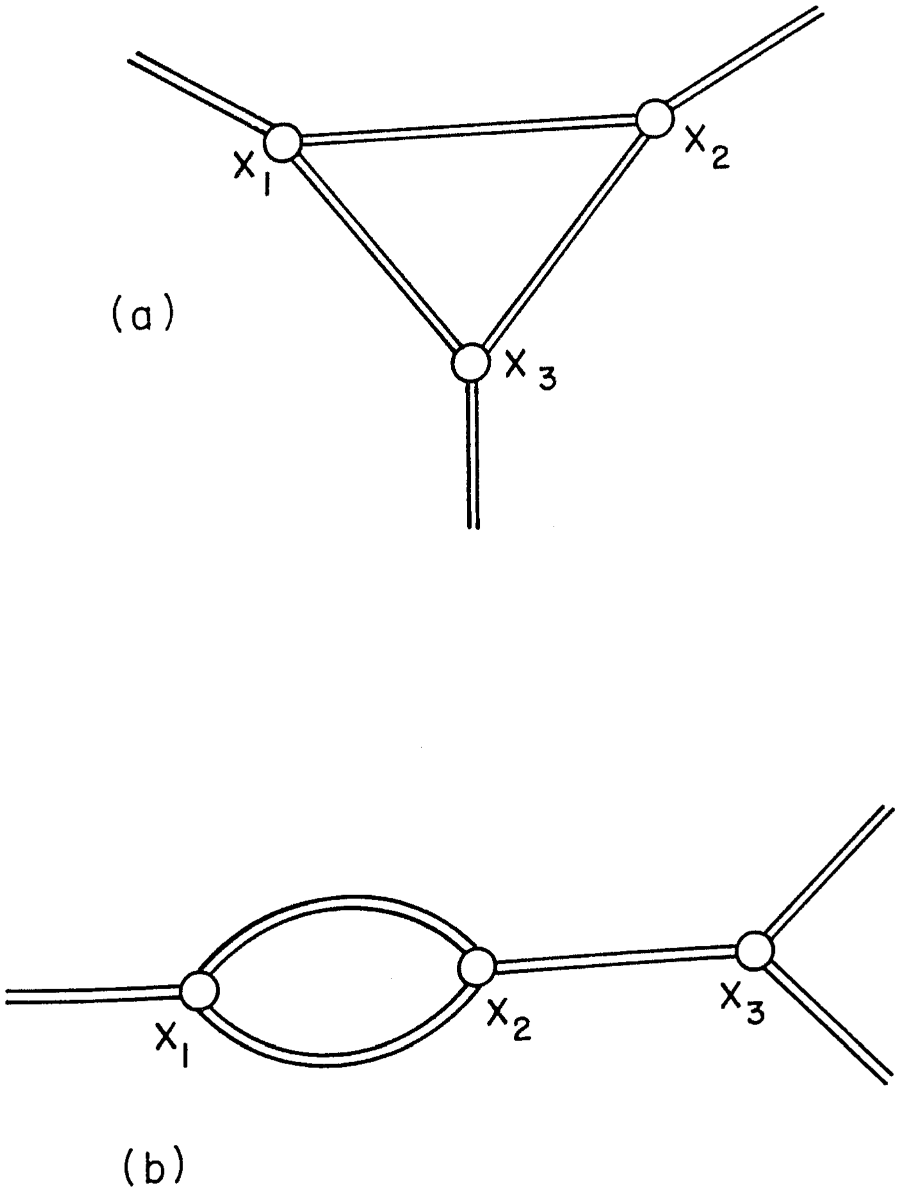


FIGURE 13

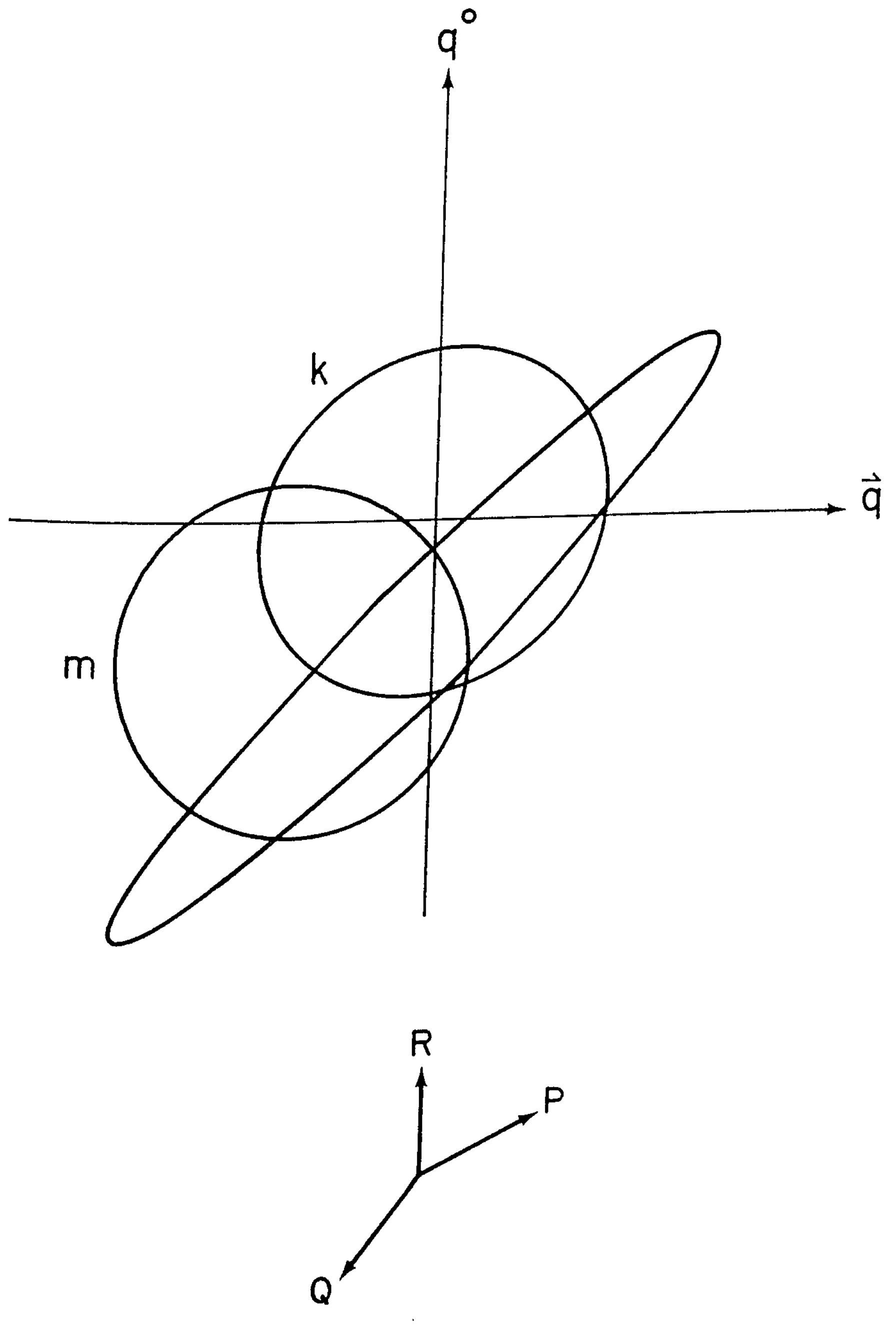


FIGURE CAPTIONS

1. Orthogonality relations and Lorentz contraction properties of covariant harmonic oscillators. "Ortho" means orthogonal. A similar figure has been published in Phys: Rev. D12,129(1975).

2. Three meson vertex of Kim and Noz[24] as a three quark vertex.

3. Spacetime overlap of six quarks in three mesons. Black dots locate the meson centers, circles locate the quarks.

4. Three meson vertex in momentum space. This figure shows the momenta of the participating quarks.

5. Three meson vertex diagram representing the bare vertex function $V^{\alpha}{}_{\beta\gamma}(-P,Q,R)$ for $P = Q + R$.

6. S matrix diagrams and the corresponding momentum space factors.

7. Second order annihilation graph for two meson scattering.

8. Second order exchange graphs for two meson scattering.

9. Second order tree diagram in $:\phi^3:$ theory.

10. Second order loop diagram in $:\phi^3:$ theory.

11. Second order loop diagram of extended vertices. When the vertices overlap, time ordering can become ambiguous and a virtual state can travel forward (or backward) in time around the whole loop.

12. Third order loop diagrams in extended meson theory.

13. Overlap of momentum space wavefunctions for $V_{k\ell m}(P,Q,R=-P-Q)$ in the frame where $\vec{R} = 0$. As mesons k,ℓ move faster they are Lorentz contracted toward the null plane. $V_{k\ell m}$ receives contributions mainly from the small region where all mesons overlap.