ABSTRACT

Title of Thesis: Scheduling transmissions in wireless ad-hoc networks with time-varying topologies

Anna Pantelidou, Master of Science, 2004

Thesis directed by: Professor Anthony Ephremides
Department of Electrical and Computer Engineering

In wireless ad-hoc networks, signal interference and collisions from simultaneous transmissions of neighboring nodes significantly degrade throughput. Hence, it is necessary to devise scheduling policies for coordinating wireless transmissions.

In this thesis, we focus on maximum stable throughput scheduling in mobile, finite node, wireless ad-hoc networks, whose topology changes according to a stationary and ergodic process. In particular, we study the i.i.d topology case, and we extend our results to the more general case of Markov and Hidden Markov topology processes. Initially, we introduce a centralized stationary scheduling rule and then prove that it maximizes the stable throughput the network can sustain. Finally, we show through simulations that mobility of the nodes may considerably improve the network throughput and plot the corresponding results through a Monte Carlo method.

Scheduling transmissions in wireless ad-hoc networks with time-varying topologies

by

Anna Pantelidou

Thesis submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Master of Science
2004

Advisory Committee:

Professor Anthony Ephremides, Chairman/Advisor
Professor André Tits,
Professor Şennur Ulukuş

DEDICATION

To my parents, Andreas and Elma, for a lifetime of love, and continuous support in my pursuing my dreams...

ACKNOWLEDGEMENTS

First of all, I would like to express my gratitude to my advisor Anthony Ephremides for all his support and guidance throughout my studies. Working with him has been an invaluable experience, through which I acquired a lot of knowledge.

I am also thankful to Professors André Tits and Şennur Ulukuş, for agreeing to serve in on my committee and for reviewing my thesis carefully.

I am truly pleased to acknowledge my debt of gratitude to an extraordinarily large number of friends I met in College Park. I need to distinguish Vinay, for his insightful comments on technical issues, and for being a close friend in both good and bad times. I also need to thank my friend Andreas for his valuable friendship, encouragement and instruction on how to be a fighter and not give up. Moreover, I want to acknowledge Tuna, for standing by my side during my stressful days and for showing determination to efface prejudices. Although the list is huge, I would also like to mention my friends Thanos, Polyvios, Manolis, Stella, Maria T., Maria C., Azadeh, and many more.

Lastly, I should express a special thanks to my family, especially my parents, and my brothers, Manos and Andreas, for showing their love and supporting me in every way possible.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Chapter 1

Introduction

1.1 Wireless ad-hoc networks: benefits and limitations

Traditional wireless communication networks, namely cellular and satellite networks, require an infrastructure over which communication takes place. Accordingly, considerable effort and resources are required for such networks to be set up, before they can actually be used. In cases where setting up an infrastructure is a difficult or even impossible task, such as in emergency/rescue operations, military applications or disaster relief, other alternatives need to be devised.

Wireless ad-hoc networks are infrastructureless autonomous systems that have emerged to serve this need by allowing a quick network development with low equipment cost (Figure 1.1). The network is formed as soon as a collection of wireless devices, equipped with wireless communication and networking capabilities, express a wish to exchange information. They are peer-to-peer networks, i.e all the network nodes have the same capabilities and no base stations or central access points need to be involved for data exchange.

In wireless ad-hoc networks, each node is supplied with an antenna, that allows it to transmit and receive information from the other nodes. There exist more than twenty types of antennas [7]. Omni-directional antennas, also known as isotropic, have been widely used. They can radiate and

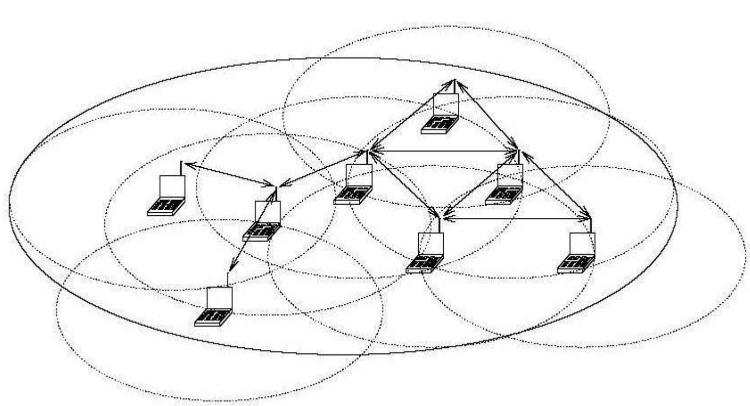

Figure 1.1: An ad-hoc wireless network

receive equally well in all directions, within a certain radius, called the range of transmission. The radius is determined by the transmission power. It should be stressed that, when a node transmits to another node, its transmission can be heard by *all* nodes that lie within transmission range. In the case of unicast, one-to-one, communication only one node will be interested in this transmission, while the rest of them will receive it as interference. The higher the transmission power, the larger the number of nodes that can be reached in a single transmission, but also the the higher the amount of interference that will be experienced by other nodes. The latter explains the need to control the transmission power and proceed in a multi-hop fashion to forward the information to its receiver. To overcome this shortcoming of isotropic antennas and tackle further challenging problems, directional and smart antennas are often used, that aim to direct transmission within a narrow cone. This technique eliminates the energy waste and the interference imposed on other nodes in the vicinity of the transmitter node ([18], [6]).

Wireless ad-hoc networks can be divided into two main categories: static and mobile. While in static wireless networks the topology is fixed and node locations cannot be updated during the course of time, in mobile networks, some or all of the nodes have movement capabilities. Mobile ad-hoc networks can be further characterized according to their mobility pattern, i.e. those where nodes move independently and those where they follow a group mobility pattern. Independent node mobility is frequently modeled by one of two widely used mobility patterns, namely the Random Way Point mobility pattern and the Random Walk mobility pattern. In this thesis, we focus on mobile ad-hoc networks, where each node is moving independently of the others, in a i.i.d or Random Walk fashion.

While ad-hoc wireless networks have the advantage of flexibility and easy deployment, their "ad-hoc" nature is the source of many limitations and challenges as well.

First of all, wireless ad-hoc networks commonly span a large geographic area. This, in conjunction with the limited transmission range through the wireless interfaces, prevents a transmitting node from reaching its intended destination in one hop. In fact, a sender and a receiver are seldom located within transmission range in this type of wireless environments. When a node chooses to transmit to an intended receiver which is not within transmission range, a multiple hop path needs to be found, so that the information gets routed through a number of intermediate "relay" nodes, until the destination is reached. Hence, routing protocols have a significant importance in wireless ad-hoc networks. However, routing decisions are hard to obtain, especially in mobile and highly dynamic networks. Node mobility might cause a path to a destination to "break", rendering obsolete the network routing information. In this case, frequent updates of the routing tables are required, which is a resource consuming procedure. In addition, unlike wireline networks, where the routing criterion is based on the shortest path, in ad-hoc networks, shortest path is not always the best selection; factors such as node mobility, power limitations of the individual nodes or interference from the transmissions of the nodes come into play and routing becomes a more challenging task.

As a second limitation, the shared wireless medium poses constraints on simultaneous transmissions. Indeed, simultaneously transmitted signals by neighboring nodes are likely to collide.

This would hinder correct reconstruction of the signals.

Thirdly, the nodes have limited battery power. Once the battery power at a node gets depleted, the node can no longer participate in either transmitting or receiving packets. As more and more nodes run out of battery, the network becomes partially or even completely disconnected, at which point communication becomes impossible.

Lastly, in dedicated link networks, the Open Systems Interconnection (OSI) [1] layer separation simplifies the implementation of different network operations by letting them occur independently of each other at different layers. For example, link scheduling can be done in the Medium Access Control (MAC) layer and is independent of finding routes, which is performed in the network layer. In wireless networks, the wireless medium is shared among users and hence some functions are better performed jointly by the different layers than independently. This has been widely studied ([5], [17]).

1.2 Link scheduling in wireless ad-hoc networks

As mentioned earlier, one of the main differences between wireless and wireline networks is the nature of the communication medium. While in wireline networks nodes are connected by dedicated channels, in wireless networks the same medium is shared by multiple users. Suppose that in a wireless network, two nodes attempt to transmit to a common network node, which is within communication range from both of them. If they transmit simultaneously, the receiver will hear the sum of the two signals and hence will not be able to correctly decode the information being transmitted. Hence, it is crucial to come up with protocols that appropriately coordinate access to

[1] A model of network architecture and a suite of protocols to implement it, developed by the International Organization for Standardization (ISO) in 1978 as a framework for international standards in heterogeneous computer network architecture.

the wireless medium to avoid collisions. One approach is to devise scheduling rules that, at every time instant, allow only a set of non-interfering nodes to transmit. These rules will attempt to optimize different performance criteria ([2], [3], [12]). Two basic approaches have been widely used: centralized and decentralized scheduling.

- **Centralized scheduling.** A centralized controller is responsible to decide which sets of network nodes are to transmit simultaneously. A simplistic scheme is to preselect a collection of sets of non-interfering links that can be activated simultaneously and then rotate among them in some predetermined fashion, i.e. Round-Robin. Clearly, one can devise a scheduler at least as good as the simplistic one, by giving it access to the network information, such as the distribution of traffic in the network and how the queue sizes at the network nodes evolve, the network topology and link availability, the remaining battery power at the nodes, etc.

- **Decentralized scheduling.** Each network node that wishes to transmit a packet is responsible for making transmission decisions that prevent the occurrence of collisions. A few examples of decentralized scheduling protocols are Carrier Sense Multiple Access (CSMA) [13], its improvement that considers collision detection (CSMA-CD) [13] and their successor, that is widely used nowadays, IEEE802.11 [4].

In this thesis, we discuss centralized scheduling in wireless ad-hoc networks with randomly varying topologies. The centralized policy that we propose is a stationary policy that purports to stabilize the network for all stabilizable arrival rates.

1.3 Previous Work

Much recent research has focused on studying the "capacity" and "throughput" of wireless ad-hoc networks. In [9] the limiting capacity of stationary wireless ad-hoc networks where the number of nodes tends to infinity, has been studied extensively. The authors first consider arbitrary networks, where node locations, destinations and traffic demands are all selected optimally, and then look at random networks, where nodes and their intended destinations are randomly chosen, i.e. independently and uniformly distributed. They consider a perfect scheduling algorithm that has complete knowledge of all nodes and traffic demands in the network, and uses this information to avoid collisions. They use such ideal scheduling scheme, where a single node attempts to communicate in a single overhearing area, to compute the optimal upper bound on the limiting capacity. In addition, they define the notion of "transport capacity" and obtain a handle on its lower bound. Their results are rather pessimistic, since they conclude that each node's throughput goes to zero as the number of nodes increases to infinity. This should be expected, since each node's throughput is limited by the amount of traffic needed to forward packets for other nodes, even in the case of optimal node placement on a disk, and optimal scheduling. What the authors of [9] also suggest is that since the network performance degrades significantly with its size, it might be beneficial to design small networks or networks where nodes may only "talk" to their neighbors. They justify the latter by showing that in a large network where nodes are restricted to communicate with their neighbors only, the data rate can be constant, independently of the network size.

The rather pessimistic result of [9] was further studied in [8], which investigates the case of a network with an infinite number of nodes moving according to random walk mobility. It is shown there that, provided the network can sustain long delays in the packet delivery to their destinations, capacity no longer converges to zero but rather to a positive constant. The authors of [8] also argue that since nodes move in a random fashion, the source or a small number of nodes through which the source relays its traffic, will eventually come close to the destination. So each packet will actually be delivered either directly or in a very small number of hops.

While in [8] and [9] the authors obtain asymptotic results on network capacity, a lot of research has been performed for networks with finite number of nodes ([10], [14], [16]). Papers [14] and

[16] focus on finite networks. In [14], the authors exhibit a centralized, optimal stable throughput scheduling policy for a finite node multi-hop constrained queuing system, with scheduling constraints, and they characterize its stability region. The constraints prevent all links from being scheduled simultaneously. In [16], the author considers a finite node network with time varying topology and no relaying and describes a centralized scheduling policy that makes use of the topology state information to stabilize all rate vectors, for which a stabilizing stationary policy exists.

Our work builds on the results of [14] and [16] to obtain a centralized maximum throughput scheduling policy under various random network configurations. We focus on stationary policies. Our work also differs significantly from [16], in that we allow relaying of the packets.

1.4 Contribution of this thesis

In this thesis, we study centralized stationary scheduling of wireless ad-hoc networks with finite number of nodes, under time-varying-topology processes. We identify an optimal, maximum throughput scheduling policy and characterize its stability region. In particular, we show that the obtained stability region is contained in a certain convex polytope while containing its interior. Finally, we verify the theoretical results by means of simulations, which we run using a C-code we developed.

The balance of this thesis is organized as follows. In Chapter 2, we introduce our model under two stationary and ergodic topology processes. Initially, we consider an i.i.d varying topology process, then a more general hidden Markov Chain. We also specify scheduling policy π_0. In Chapter 3, we show that π_0 achieves an optimal stable throughput. An introduction to the simulation tool we developed, along with our simulation results, follow in Chapter 4. The thesis concludes with three appendices on fundamentals of Markov Chains, Markov Chain Stability and basic definitions from set theory.

Chapter 2

Preliminaries

2.1 Introduction

In this chapter, we first introduce the network model to be used in our study and state the underlying assumptions that will be made throughout. Our model is inspired from that described in [14]. In particular we consider a time-varying-topology network, under two types of topology processes: an i.i.d topology process and the more general case of a stationary and ergodic Markov and Hidden Markov process. The varying topology processes will be used to capture the mobility of the nodes of the network and how this may affect the various configurations the network may evolve into. Next, we define the notion of network stability and provide intuition for how mobility may increase the maximum stable throughput a network can sustain. Finally, we introduce the stationary policy π_0, which purports to achieve a maximum throughput performance. Optimality of π_0, will be established later in Chapter 3.

2.2 Wireless network model

2.2.1 Notation and terminology

We consider a slotted time, time-varying-topology wireless ad-hoc network, operating under a TDMA based medium access scheme. The network is comprised of a finite number of N nodes, each equipped with one transceiver (transmitter/receiver pair) and an omni-directional antenna. Nodes share a common medium, hence it is essential to properly coordinate their transmissions to avoid signal interference and collisions. Towards this end, a set of **primary constraints** is imposed on the set of simultaneously transmitting nodes. These constraints dictate that:

1. A node cannot transmit and receive simultaneously.
2. At any time instant, a node cannot transmit simultaneously to multiple nodes.
3. At any time instant, a node cannot receive simultaneously from multiple nodes.

Nodes exchange datagram packets of constant packet lengths, that can be transmitted in one time slot, in a unicast fashion. We consider J distinct customer classes, each intended for a set of **exit nodes** $V_j, j = 1, \ldots, J$. The set of exit nodes is such that whenever a packet of some class reaches an exit node for this class, the packet leaves the network. The different sets V_j are allowed to overlap.

Each node is modeled as a set of J infinite buffer queues, each holding separately the packets corresponding to different customer classes. We denote by $X_{ij}(t)$ the non-negative integer queue size for class j at node i *at the end of time slot* t. In addition, $\mathbf{X}(t)$ is a queue length matrix defined as $\mathbf{X}(t) = \{X_{ij}(t), i = 1, 2, \ldots N, j = 1, 2, \ldots J\}$ and for each $j \in \{1, 2, \ldots J\}$ $\mathbf{X}^j(t)$ is a $N \times 1$ vector of all queue sizes of class j at time t, that is, $\mathbf{X}^j(t) = \{X_{ij}(t), i = 1, 2, \ldots N\}$. We denote by $\mathcal{X}$ the space of all queue size vectors.

A **link** ℓ is said to exist between nodes i_1 and i_2, if nodes i_1 and i_2 are within transmission range. We model each link as a server. In particular, a link ℓ that originates from node $s(\ell)$ and terminates at node $d(\ell)$ is a server that serves a customer from queue $s(\ell)$ and upon service completion, sends the served customer (packet) to the destination queue $d(\ell)$. All the servers are synchronized to start serving a customer at the beginning of a time slot. The primary constraints on medium access make the servers interdependent, in the sense that not all of the servers can be active at any time slot.

Node mobility causes some already established links to break, as participating nodes move far away from each other, and results in establishing new connections between nodes that move closer to each other. Hence, the mobility pattern of the nodes affects the set of possible **network topologies**, as well as their respective probability distribution. Finiteness of the number of nodes implies finiteness of the set $\mathcal{T}$ of topologies the network may evolve into, i.e. $\mathcal{T} = \{T_1, \dots, T_{N_T}\}$. We denote the topology at time slot t by $T(t) \in \mathcal{T}$. We assume that the topology process is a stationary and ergodic stochastic process and pay special attention to the i.i.d case. Each topology T_k is characterized by its set of links $L(k)$. Let L be the set of all links that appear in at least one topology, i.e. $L = \cup_{k=1}^{N_T} L(k)$. A uniform numbering of the links is used across all topologies, i.e. links are numbered from 1 to $|L|$. In other words, if link $\ell \in \{1, \dots, |L|\}$ connects nodes i_1 and i_2 under topology k, then in every topology where node i_1 is connected to i_2, the link connecting them is numbered ℓ.

Customers of each class may enter the network at any node, except for the exit nodes of the corresponding class, and at each time instant. For each customer class j, the vector of arrivals $A^j(t) = (A_{ij}(t) : i = 1, 2, \dots, N, i \notin V_j)$ is a non-negative component-wise vector. Its i^{th} element, $A_{ij}(t)$, represents the number of customers of class j arriving at queue i, during time slot t. For all queues $i : i = 1, \dots, N$ and all customer classes $j : j = 1, \dots, J$, we define $a_{ij} = E[A_{ij}(t)]$, that is assumed to be time-invariant, and we term **multi-class arrival vector** the non-negative component-wise matrix $\mathbf{a} = (a_{ij} : i = 1, \dots, N, j = 1, \dots, J)$.

To capture the dependence among link activations, due to the medium access constraints, following [14] and [16], we define a valid **activation set** for topology T_k to be a set of servers (a subset of $L(k)$), that comply with the primary constraints and are allowed to be activated simultaneously. We also define the valid **activation vector** associated to an activation set as an $|L|$ element vector with its ℓ^{th} component, $1 \leq \ell \leq |L|$, set to 1 if the ℓ^{th} server, belongs to the activation set and set to 0, otherwise. If link ℓ is not present in $T_k \in \mathcal{T}$ or medium access constraints prevent it from being activated, then the ℓ^{th} element of all the valid activation vectors for topology T_k must be set to 0. The **constraint set** $\mathcal{S}_k$ for topology $T_k \in \mathcal{T}$ is the set of all valid activation vectors associated with network topology T_k. Since the set $\mathcal{S}_k$ is determined by the primary constraints, it should be clear that for each k, if $\mathbf{c} \in \mathcal{S}_k$, all the vectors obtained by setting some of the active links of $\mathbf{c}$ to inactive, belong in $\mathcal{S}_k$ as well.

Next, we define the binary variable $E_{\ell j}(t)$, that takes values in $\{0, 1\}$ and indicates whether server ℓ is active during slot t, serving a customer of class j. The random variable $\mathbf{E}(t) = \{E_{\ell j}(t) : \ell = 1, \ldots, |L|, j = 1, \ldots, J\}$ is called a valid **multi-class activation vector** for topology $T(t)$, if the corresponding vectors $\mathbf{E}^j(t) = \{E_{\ell j}(t), \ell = 1, \ldots, |L|\}$ are such that:

1. $E_{\ell j}(t) = 0$, if $X_{s(\ell)j}(t-1) = 0$
2. $E_{\ell j}(t) = 0, \forall \ell \notin T(t)$
3. $\sum_{j=1}^{J} \mathbf{E}^j(t) \in \mathcal{S}_k,$ where k is such that $T(t) = T_k$

We denote by $\mathcal{E}$ the collection of all valid multi-class activation vectors that correspond to all possible network configurations.

Finally, we consider a centralized scheduler that at the beginning of each time slot has a complete knowledge of the current network topology and knows the queue sizes at each network node as well. At the beginning of each time slot, a scheduling decision is taken in a centralized fashion and packet transmissions are scheduled to take place. The rule which determines what this decision will be is called a **policy** . Thus, a (deterministic, stationary) scheduling policy π, is a map $\mathcal{X} \times \mathcal{T} \rightarrow \mathcal{E}$ such that, for any topology $T \in \mathcal{T}$, $\pi(\mathbf{x}, T)$ is a valid multi-class activation vector for topology T, as this vector is defined previously. The class of all stationary policies is denoted by Π.

During the course of this thesis, we mark all the quantities that depend on topology T_k by the subscript k.

2.2.2 Queue size dynamics

The queue size process, $\mathbf{X}(t)$, represents how the queues develop during the course of time for each customer class j at each network node i . Hence, the queue size at node i for class j at the end of the next time slot equals the queue size at node i for this class at the end of the current time slot, plus the external arrivals for this class at this node during the next time slot, modified by internal arrivals or departures for class j that involve node i, as decided by the scheduling rule operating on the system at the end of the current time slot. By the latter we mean customers of class j that arrive internally at node i or leave the node upon service completion, during the next time slot. We describe the queue length dynamics process for class j in vector form through the Equation 2.1:

$$\mathbf{X}^j(t+1) = \mathbf{X}^j(t) + \mathbf{R}^j\mathbf{M}(t+1)\mathbf{E}^j(t+1) + \mathbf{A}^j(t+1) \tag{2.1}$$

In Equation 2.1, $\mathbf{R}^j$ is a $N \times |L|$ matrix that we name the “combined routing matrix for class j”. It has one column for each link $\ell \in L$ and its element at the i^{th} row and ℓ^{th} column is:

$$r^j_{i\ell} = \begin{cases} 1 & \text{if } d(\ell) = i, \text{ with } i \notin V_j \\ -1 & \text{if } s(\ell) = i \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

Matrix $\mathbf{M}(t)$ is a diagonal matrix, of dimensions $|L| \times |L|$. Its ℓ^{th} diagonal element $(M(t))_\ell$ represents the binary random variable that corresponds to the successful service completion of a customer served by server ℓ during time slot t. If a customer completes service and moves from queue $s(\ell)$ to queue $d(\ell)$ (or exits the system if the node where queue $d(\ell)$ resides belongs to the class of the exit nodes for the particular customer class), then $(M(t))_\ell = 1$, otherwise $(M(t))_\ell = 0$. The latter case may occur if the ℓ^{th} link is not present in the current topology at time slot t or when although it is present, a customer did not receive full service and hence, remains in the source queue $s(\ell)$ and its service is being deferred.

When link ℓ is present in $T(t)$, the random variable $(M(t))_\ell$ may capture the link "quality". More specifically, when it takes the value 0, it may model channel errors through that link or non-interference dependent channel fading. Since we assume that elements of matrix $\mathbf{M}(t)$ are independent of each other, they cannot track interference related fading, since it would require correlations in the neighboring links quality.

In addition, as mentioned earlier, $\mathbf{E}(t)$ is a valid multi-class activation vector. The ℓ^{th} element of $\mathbf{E}^j(t)$, is never set to 1 when the queue for class j at the source node of the link is empty. This guarantees that the elements of $\mathbf{X}(t)$ are non-negative at all times.

2.2.3 Assumptions

We make a number of important assumptions about our model throughout this study.

Assumption 1 *The topology process is a stationary and ergodic process.*

In particular $p_k = P[T(t) = T_k], \forall k \in 1, \ldots, N_T$ does not depend on t. Accordingly, the distribution with which the various topologies occur may not change with time. One implication of Assumption 1 is that the the number of nodes, N is fixed, once the network is established. Therefore, new nodes cannot dynamically join the network and all network nodes have infinite battery power supply.

Assumption 2 *The union L of the links of different topologies that the network can take is such that if a customer of class j_0 reaches a queue i_0, then this customer can be forwarded to some exit node of class j_0 by passing through a sequence of n links (servers) $\{\ell_{(i)}\}_{i=1}^{n}, \subseteq L$ for some $n > 0$.*

What assumption 2 implies,is that the network is allowed to evolve into "bad"(disconnected) topologies, as long as the union of these topologies provides a way for each customer in the network to reach its destination class node.

Assumption 3 *Given the current topology $T(t)$, the binary service completion processes for different time slots t and different links $M_\ell(t), \ell = 1, \ldots, |L|$ are all independent. Furthermore, $E[\mathbf{M}(t)|T(t) = T_k]$ is independent of t, $\forall k \in \{1, \ldots, N_T\}$. We will call this expected value $\mathbf{M}_k$.*

Assumption 4 *The arrival process $A_{ij}(t)$ is independent of the current topology $T(t)$ and of the service rates of the different network links $\mathbf{M}(t)$. Also, for a given node i and a given class of customers j, the arrival process $A_{ij}(t)$ is an i.i.d process. For a given node, arrivals for different classes of customers j are independent but not necessarily identically distributed. The same holds for arrivals at different nodes for the same class of customer j. Finally, the arrival process should satisfy $E[A_{ij}^2(t)] < \infty$.*

2.3 Stability considerations

In this section, initially we give a definition of stability for irreducible and reducible Markov Chains. We then discuss whether the regions of stable arrival rates achieved by stationary policies can always be compared to determine which policy performs better in terms of maximizing the stable network throughput. In addition, we reason that we may increase the stability region of the network, by considering a time-varying-topology. The state of our system is the stochastic process $\{X(t), T(t+1)\}_{t=0}^{\infty}$, comprised by the current queue size process $\{\mathbf{X}(t)\}_{t=0}^{\infty}$ and the topology process of the next time slot $\{T(t+1)\}_{t=0}^{\infty}$. We examine two types of topology processes, namely an i.i.d process and a more general stationary and ergodic topology process. We show how we can relate the state space of our system to a Markov Chain, aiming to prove network stability.

2.3.1 Network stability and regions of stable arrival rates

Consider the system state be described by the Markov Chain $\{S(t)\}_{t=0}^{\infty}$. For the reader's ease of reference, a brief overview on Markov Chains is given in Appendix A. If this Markov chain is an irreducible Markov Chain, then stability is equivalent to ergodicity of $\{S(t)\}_{t=0}^{\infty}$ and the existence of a unique stationary distribution. Therefore, Foster's theorem, as discussed in Appendix B, can be applied to show a sufficient condition for stability. In general, however, it may not be possible to guarantee irreducibility and a more generalized definition of stability ([14], [15]) is required, as restated also in Appendix B.

Let the state space be partitioned into the classes $Y, Z_1, Z_2, Z_3, \ldots$, where $Z_i, i = 1, 2, 3, \ldots$ are sets of communicating states that are recurrent and Y is the set of all transient states. Further, let the system be in a state $S(0) = s \in Y$, at time $t = 0$. Then, the **Hitting Time** is defined as:

$$\tau_s = \begin{cases} \infty, & \text{if } S(t) \in Y, \forall t > 0 \\ min\{t > 0 : S(t) \notin Y\}, & \text{otherwise} \end{cases} \tag{2.3}$$

Definition 1 (System stability) *The system is* **stable** *if for the state process* $\{S(t)\}_{t=0}^{\infty}$ *we have:*

$$P[\tau_y < \infty] = 1, \quad \forall y \in Y \tag{2.4}$$

and all states $z \in \cup_{i=1}^{\infty} Z_i$ *are positive recurrent.*

In other words, to establish stability of the Markov Chain, one needs to show presence of two properties, namely to ensure that the Markov Chain leaves the transient states in finite time and that it enters one of the recurrent classes, each of which is positive recurrent. In particular, it follows that the system is unstable if $\cup_{i=1}^{\infty} Z_i = \emptyset$.

The *stability region* $\mathbf{C}_\pi$ of a scheduling policy π is the collection of all arrival rate vectors $\mathbf{a}$ that can be supported by π in such a way that the system is stable. The stability region of the network is the union of stability regions of all scheduling policies. In this thesis, we will focus on stationary policies only. We denote by $\mathbf{C}$ the set of arrival rate vectors that can be stabilized by some *stationary* policy. We use the notation as used in [14].

$$\mathbf{C} = \bigcup_{\pi \in \Pi} \mathbf{C}_\pi,$$

where Π is the class of all stationary policies π.

Simply stated, a scheduling policy is better than another if its stability region is larger. However, it is not always possible to compare the stability regions of two policies. Comparison is only possible if one can say that one policy dominates the other. For example, we could say that policy π_0 dominates π_1, and hence it is better, if and only if $\mathbf{C}_{\pi_1} \subset \mathbf{C}_{\pi_0}$ (Figure 2.1). If π_0 dominates π_1, the system will be stable under π_0 whenever it is stable under π_1. On the contrary, we cannot say whether π_1 or π_2 is better (Figure 2.2). Every point in the region R_1 can be achieved only by policy π_1, in R_3 only by π_2, while points in the region R_2 can be achieved by selecting either one of them.

If there is a policy which dominates all other policies, then it is an optimal policy or else a maximum throughput policy. Then, every point in the capacity region may be achieved by applying

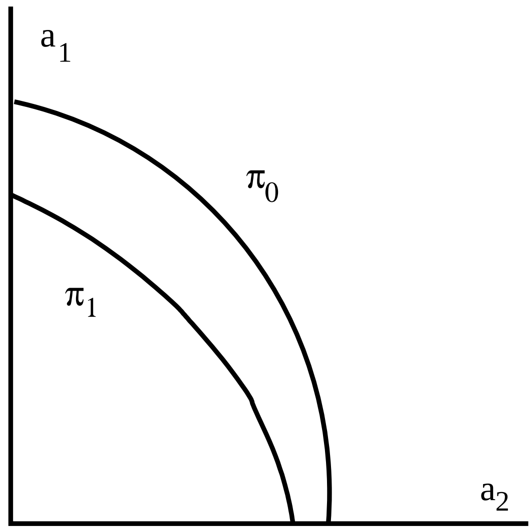

Figure 2.1: 3 Policies π_0 and π_1 can be compared. We can say that π_0 is better than π_1 since π_0 dominates π_1.

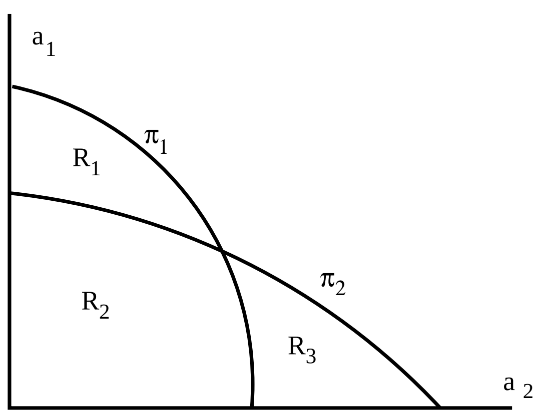

Figure 2.2: Policies π_1 and π_2 cannot be compared. There exist a set of arrival rates vectors that are stable under π_1 but not under π_2 (region R_1) and a set of arrival rate vectors that are stable under π_2 and not under π_1 (region R_3). Rates inside region R_2 are stable under both policies.

a single policy, namely the optimal one. In this thesis, we will introduce a policy of "nearly" maximum throughput. In the next chapter, it will be clearer why we use the term "nearly".

2.3.2 Motivating structure of stability regions under randomly varying topologies

For fixed multi-hop radio networks, under constraints, a scheduling policy introduced in [14] is shown to have a stability region very close to the stability region, $\mathbf{C}$, corresponding to the set of all stationary policies. Specifically, since almost all points in the stability region, except perhaps some boundary points, can be stabilized by this single scheduling policy, it is dubbed to be an optimal throughput policy.

We now consider the relationship between the stability region of a varying topology network, with the stability regions of the individual topologies. To examine this, consider the mobile network case with two topologies T_0 and T_1. There is a simple way to obtain some points *inside* the overall stability region. These points are obtained by considering the mobile network as being a time sharing of two fixed networks. Consider a scheduling policy that splits the queues at every node into two. One queue corresponds to arrivals and transmissions when the network is in state T_0 and the other corresponds to the case when the network is in state T_1. All arrivals which take place when the network is in T_0 go to its corresponding queue and the same holds for T_1. The scheduling policy when applied under T_0 schedules according to the optimal policy for T_0 applied to appropriate queues and similarly for T_1. Then clearly, the stability region achieved by this particular policy is obtained by a *fixed* convex combination of stability regions of T_0 and T_1, weighed by the stationary probabilities of the two network topologies respectively. We claim that the stability region of the mobile network can be *strictly* larger than the one obtained by the above linear combination. This claim is discussed further below.

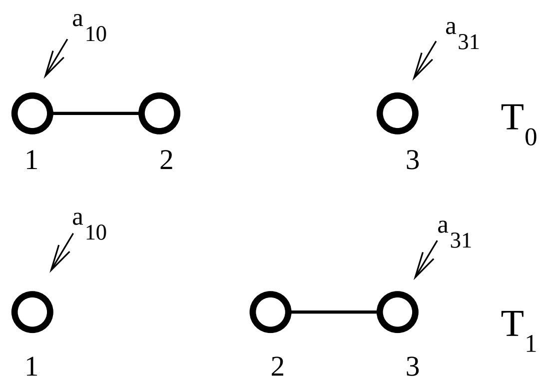

Figure 2.3: 3 node network with two network configurations. Arrivals a_{10} at node 1 are destined for node 3 and a_{31} at node 3 exit the system at node 1.

2.3.3 Mobility increases the stability region of the network

Consider the network of 3 nodes, depicted in Figure 2.3, where nodes 1 and 3 are so far apart from each other, that no matter what the location of node 2 is, node 2 can at most communicate with one of the two nodes 1 or 3. Arrivals at node 1 are to be delivered at node 3 and arrivals at node 3 exit the network at node 1. Consider two configurations of the network, (i) 1 communicates with 2 and node 3 cannot transmit or receive (ii) 2 communicates with 3 and node 1 cannot transmit or receive. Suppose that the network has a fixed topology T_0. Then since node 1 is completely disconnected from its exit node 3 no non-zero arrival rate vector can be supported. Similarly, the same holds when the network has a fixed topology T_1. Therefore, any convex combination of the two stability regions will give that the only stable arrival rate vector is the one with zero elements.

However, consider a partially connected network, that switches between the two configurations according to a stationary process. Clearly it is possible to obtain a non-zero stable arrival rate at least in one component by allowing node 2 to be used as a “relay” between nodes 1 and 3. Hence the stability region of the mobile network is strictly better than that obtained by the linear combination. We will come back to this claim in chapter 3, where the stability regions of both policies will be looked into more detail.

2.4 Topologies that vary in an i.i.d fashion

In this section, we will show that in the case of an i.i.d topology process the process $\{\mathbf{X}(t), T(t+1)\}_{t=0}^{\infty}$ that defines the state of our system is a Markov Chain. We will also elaborate why Foster's theorem cannot be applied on this Markov Chain as is and proceed to establish an extension for the present situation, that involves a Lyapunov function on $\{\mathbf{X}(t)\}_{t=0}^{\infty}$ only.

Lemma 1 *The pair $\{\mathbf{X}(t), T(t+1)\}_{t=0}^{\infty}$ with state space $(\mathcal{X} \times \mathcal{T})$, where $\{T(t)\}_{t=0}^{\infty}$ is an i.i.d process, is a Markov Chain.*

Proof: To establish that $\{\mathbf{X}(t), T(t+1)\}_{t=0}^{\infty}$ is a Markov Chain, we need to prove that the definition of Markovity holds (see Appendix A):

$$P[(\mathbf{X}(t), T(t+1))|\mathbf{X}(t-1), T(t), \mathbf{X}(t-2)T(t-1), \dots] = P[(\mathbf{X}(t), T(t+1))|(\mathbf{X}(t-1), T(t))]$$

So, we proceed as follows:

$$\begin{aligned}
& P[\mathbf{X}(t), T(t+1)|\mathbf{X}(t-1), T(t), \mathbf{X}(t-2)T(t-1), \dots] \\
= \; & P[\mathbf{X}(t)|T(t+1), \mathbf{X}(t-1), T(t), \mathbf{X}(t-2)T(t-1), \dots] \\
& P[T(t+1)|\mathbf{X}(t-1), T(t), \mathbf{X}(t-2)T(t-1), \dots] \\
= \; & P[\mathbf{X}(t)|T(t+1), \mathbf{X}(t-1), T(t)]P[T(t+1)|(\mathbf{X}(t-1), T(t))] \\
= \; & P[(\mathbf{X}(t), T(t+1))|(\mathbf{X}(t-1), T(t))] \qquad (2.5)
\end{aligned}$$

Equation 2.5 follows from the dynamics of the queue length process $\mathbf{X}(t)$ (Equation 2.1) and the i.i.d nature of the topology process. Specifically, to compute $\mathbf{X}(t)$, it suffices to know the queue lengths at time instant $t-1$, $\mathbf{X}(t-1)$ and the topology at time instant t, $T(t)$ and does not depend on other topologies and queue lengths at previous time slots. In addition, the fact that $T(t)$ is an i.i.d process allows us to write:

$$P[T(t+1)|\mathbf{X}(t-1), T(t), \mathbf{X}(t-2)T(t-1), \dots] = P[T(t+1)] = P[T(t+1)|\mathbf{X}(t-1), T(t)]$$

$\diamondsuit$

However, although $\{\mathbf{X}(t), T(t+1)\}_{t=0}^{\infty}$ is a Markov Chain, we cannot guarantee that it is irreducible. The following theorem, stated in [14], gives sufficient conditions for stability of the system according to Definition 1. It comes as a generalization of Foster's Theorem [1].

Theorem 1 *Consider a Markov Chain $\{S(t)\}$ with state space $\mathcal{S}$, a real valued, bounded from below, function $V : \mathcal{S} \to \mathbf{R}$, an $\epsilon > 0$ and a finite subset $\mathcal{S}_0$ of $\mathcal{S}$ such that:*

$$E[V(S(t+1)) - V(S(t))|S(t) = s] < -\epsilon, \quad \textit{if } s \notin \mathcal{S}_0$$

and

$$E[V(S(t+1))|S(t) = s] < \infty, \quad \forall \quad s \in \mathcal{S}_0$$

Then $\{S(t)\}_{t=0}^{\infty}$ is stable, in the sense of Definition 1.

We can also note that the above test provides only sufficient conditions. If its conditions fail to hold, we can only derive that the candidate Lyapunov function was not chosen appropriately and not that the system is unstable.

Consider the following candidate Lyapunov function $V : \mathcal{X} \times \mathcal{T} \to \mathbf{R}$, such that $V(\mathbf{x}, T) = \sum_{j=1}^{J} \sum_{i=1}^{N} \{x_{ij}\}^2$. Although it is a valid candidate, defined on the state space of our system and bounded from below by 0, we cannot establish stability by applying it on the Markov Chain $\{\mathbf{X}(t), T(t+1)\}$. This problem arises from the fact that Theorem 1 requires that we must be able to find a *finite* subset of the state space outside which the drift of the Lyapunov function is strictly negative and inside which it can be positive. Now, as some of the topologies may be completely disconnected, it appears that the candidate Lyapunov function that we have used, i.e the sum of the squares of the queue sizes, does not have a strictly negative drift for all but a finite number of states $(\mathbf{X}(t), T(t+1))$. This is so because, if T_1 is a very bad topology, for example completely disconnected, then the queue sizes under this topology can only increase (nothing gets delivered to the destination and external arrivals make the queue sizes to increase). Hence the expected value of the candidate Lyapunov function has a positive drift for the state $(\mathbf{X}(t), T_1)$ and for all values that $\mathbf{X}(t)$ can take, which are infinitely many. Hence, since the subset of states where the drift is positive is not finite, we cannot apply Theorem 1 to the Markov Chain $\{\mathbf{X}(t), T(t+1)\}$ the way it is stated above.

Hence, we shall derive a different Markov Chain, related to the state space of our system, on which Theorem 1 is applicable. Towards this end we prove Lemma 2.

Lemma 2 *Consider the Markov Chain* $\{\mathbf{X}(t), T(t+1)\}_{t=0}^{\infty}$ *with state space* $(\mathcal{X} \times \mathcal{T})$*, such that* $\{T(t)\}_{t=0}^{\infty}$ *is a finite valued i.i.d process and* $\{\mathbf{X}(t)\}_{t=0}^{\infty}$*, follows the dynamics of Equation 2.1, then the process* $\{\mathbf{X}(t)\}_{t=0}^{\infty}$ *is also a Markov Chain.*

Proof: Indeed,

$$\begin{aligned}
& P[\mathbf{X}(t+1)|\mathbf{X}(t), \mathbf{X}(t-1), \mathbf{X}(t-2), \dots, \mathbf{X}(0)] \\
= & \sum_{k=1}^{N_T} P[\mathbf{X}(t+1), T(t+1) = T_k|\mathbf{X}(t), \mathbf{X}(t-1), \mathbf{X}(t-2), \dots, \mathbf{X}(0)] \\
= & \sum_{k=1}^{N_T} P[\mathbf{X}(t+1)|T(t+1) = T_k, \mathbf{X}(t), \mathbf{X}(t-1), \dots, \mathbf{X}(0)] P[T(t+1) = T_k|\mathbf{X}(t), \dots, \mathbf{X}(0)] \\
= & \sum_{k=1}^{N_T} P[\mathbf{X}(t+1)|T(t+1) = T_k, \mathbf{X}(t)] p_k \qquad (2.6) \\
= & P[\mathbf{X}(t+1)|\mathbf{X}(t)]
\end{aligned}$$

Equation (2.6) follows from Equation 2.1 and the i.i.d nature of the topology process. Also, a useful observation is that the candidate Lyapunov function V, as defined above, does not depend on the topology process. We therefore define a different candidate Lyapunov function, $V' : \mathcal{X} \to \mathbf{R}$, such that $V'(\mathbf{x}) = \sum_{j=1}^{J} \sum_{i=1}^{N} \{x_{ij}\}^2$.

Next, we connect stability of the Markov process $\{\mathbf{X}(t)\}$ to the stability of $\{\mathbf{X}(t), T(t+1)\}$.

Theorem 2 *Consider the Markov Chain $\{\mathbf{X}(t), T(t+1)\}$ with state space $\mathcal{X} \times \mathcal{T}$, where the topology process $T(t)$ is i.i.d. Let a real valued, bounded from below, function $V' : \mathcal{X} \to \mathbf{R}$, an $\epsilon > 0$ and a finite subset $\mathcal{X}_0$ of $\mathcal{X}$ be such that:*

$$E[V'(\mathbf{X}(t+1)) - V'(\mathbf{X}(t))|\mathbf{X}(t) = \mathbf{x}] < -\epsilon, \quad \textit{if } \mathbf{x} \notin \mathcal{X}_0$$

and

$$E[V'(\mathbf{X}(t+1))|\mathbf{X}(t) = \mathbf{x}] < \infty, \quad \forall \quad \mathbf{x} \in \mathcal{X}_0$$

Then, the process $\{\mathbf{X}(t), T(t+1)\}_{t=0}^{\infty}$ is stable in the sense of Definition 1.

Proof: From Lemma 2, we know that the process $\{\mathbf{X}(t)\}_{t=0}^{\infty}$ is a Markov Chain. In view of Theorem 1 this Markov Chain is stable. In other words its transient states are exited in finite time with probability 1 and all of its recurrent classes are positive recurrent. We first show that the recurrent classes of the Markov Chain formed by the pair $(\mathbf{X}(t), T(t+1))$, are positive recurrent as follows. Since the next topology is chosen in a i.i.d fashion, $T(t+1)$ is independent of $\mathbf{X}(t)$. Also, since all the recurrent classes of $\{\mathbf{X}(t)\}_{t=0}^{\infty}$ are positive recurrent and since it exits its transient states in finite time with probability 1, the process $\{\mathbf{X}(t)\}_{t=0}^{\infty}$ must have a stationary distribution in each recurrent class. Hence, given $\mathbf{X}(t) \in Z_i, i = 1, 2, \ldots, \mathbf{X}(t)$ has a stationary distribution, say, $\mu_{Z_i}(\mathbf{x})$. Therefore $P(\mathbf{X}(t) = \mathbf{x}, T(t+1) = T_k) = \mu_{Z_i}(\mathbf{x})p_k$. Since, it has a stationary distribution the Markov Chain formed by the pair $(\mathbf{X}(t), T(t+1))$ is positive recurrent. To conclude the proof we show that the Markov Chain $\{\mathbf{X}(t), T(t+1)\}_{t=0}^{\infty}$ exits its transient states in finite time with probability 1. First, since $\{\mathbf{X}(t)\}_{t=0}^{\infty}$ exits its transient classes in finite time with probability 1, the Markov Chain formed by the pair $(\mathbf{X}(t), T(t+1))$ also leaves all the states whose first component $\mathbf{X}(t)$ belongs to the transient states of the Markov Chain $\mathbf{X}(t)$ in finite time as well. Also we have seen that all the states of $(\mathbf{X}(t), T(t+1))$, such that their first component belongs to Z_i have a stationary distribution and hence belong to positive recurrent classes. Therefore, $(\mathbf{X}(t), T(t+1))$ will also leave all its transient states in finite time with probability 1. This concludes the proof.

2.4.1 Stationary and ergodic Markov or Hidden Markov topology processes

In the previous sections, we discussed how a modified version of Foster's theorem can be applied to a network whose topology process changes in an i.i.d fashion, where the probability of topology T_k was given by p_k, $k \in \{1, \ldots, N_T\}$. However, in the general case where the topology process is a Markov or Hidden Markov stationary and ergodic process, we cannot follow the same analysis as in the i.i.d case, and apply Theorem 2, since we cannot guarantee that the process $\{\mathbf{X}(t), T(t+1)\}_{t=0}^{\infty}$ is a Markov Chain.

In this section, we involve an interleaving technique to regard a stationary and ergodic topology process as a collection of time interleaved, nearly i.i.d topology processes and show that the stability region achieved by a stationary and ergodic Markov or Hidden Markov process, is very close to the one achieved by an i.i.d topology process with the same stationary and marginal distributions. Consider a topology process that is Markov, stationary and ergodic. Let p_k denote the stationary distribution of state k for this process, i.e.

$$P[T(t) = T_k] = p_k$$

Let $\{\Theta(t)\}$ be a finite state irreducible and aperiodic Markov Chain starting in its stationary distribution $\pi(\theta)$, then for an integer t_0,

$$P(\Theta(t) = \theta, \Theta(t+t_0) = \theta') \rightarrow \pi(\theta)\pi(\theta') \text{ as } t_0 \rightarrow \infty$$

That is, as t_0 increases, the random variables, $\Theta(t)$ and $\Theta(t+t_0)$ tend to become independent.

Hence if a topology process $\{T(t)\}_{t=0}^{\infty}$ is Markov or hidden Markov derived from a finite state Markov Chain $\{\Theta(t)\}_{t=0}^{\infty}$, it will also exhibit the same behavior. That is:

$$P(T(t+t_0) = T_k, T(t) = T_{k'}) \rightarrow p_k p_{k'} \text{ as } t_0 \rightarrow \infty$$

Let the topology process $\{T(t)\}_{t=0}^{\infty}$ be Markov and $\{\mathbf{X}(t)\}_{t=0}^{\infty}$ be the queue size process. Then consider:

$$\begin{aligned} P[T(t+t_0) = T_k|\mathbf{X}(t)] &= \sum_{k'} P[T(t+t_0) = T_k, T(t+1) = T_{k'}|\mathbf{X}(t)] \\ &= \sum_{k'} P[T(t+t_0) = T_k|T(t+1) = T_{k'}, \mathbf{X}(t)]P[T(t+1) = T_{k'}|\mathbf{X}(t)] \\ &= \sum_{k'} P[T(t+t_0) = T_k|T(t+1) = T_{k'}]P[T(t+1) = T_{k'}|\mathbf{X}(t)] \end{aligned}$$

The above follows since as t_0 increases, $T(t+t_0)$ and $\mathbf{X}(t)$ become approximately independent. In addition, $P[T(t+t_0) = T_k|T(t+1) = T_{k'}]$ tends to p_k irrespective of $T_{k'}$. Hence, for large t_0, the above is approximately,

$$P[T(t+t_0) = T_k|\mathbf{X}(t)] = p_k \sum_{k'} P[T(t+1) = T_{k'}|X(t)] = p_k$$

The same equations can be written when the topology process is not Markov but hidden Markov (e.g. when each of the nodes does a random walk). If topology process $\{T(t)\}_{t=0}^{\infty}$ is derived from a Markov process $\{\Theta(t)\}_{t=0}^{\infty}$. Then, we write

$$P[T(t+t_0) = T_k|\mathbf{X}(t)] = \sum_{\theta} P[T(t+t_0) = T_k, \Theta(t+1) = \theta|\mathbf{X}(t)]$$

By carrying out the exact same analysis as above, we conclude that $P[T(t+t_0) = T_k|\mathbf{X}(t)] = p_k$ for large t_0.

Now consider the following scheduling scheme. Let time slots be partitioned into t_0 classes of the form $\Gamma_m = (m, m+t_0, m+2t_0, m+3t_0, \ldots ,)$ for $m = 0, 1, \ldots , t_0-1$. Accordingly, view the original network as a time interleaved (TDM) collection of t_0 identical networks, $\mathcal{N}_0, \ldots , \mathcal{N}_{t_0-1}$, where network $\mathcal{N}_m$ is active only for time slot $t \in \Gamma_m$. Hence, we have managed split the network into t_0 interleaved networks, all operating without interfering with each other. The topology process seen by any particular network, (of index m) is the process $\{T(m), T(m + t_0), \ldots , T(m + tt_0)\}$ which has the same marginal and stationary distribution as the original process $\{T(t)\}$, and which approximates an i.i.d process for large t_0. Consider an arrival rate $\mathbf{a}$ that is stable under an i.i.d topology process. Then, for some t_0, the individual time interleaved networks will see approximately the same topology process, and hence will be stable for this rate. Since all the t_0 networks are stable, the initial network operating under the stationary and ergodic Markov or Hidden Markov process is stable as well. Thus we proved that the stability region achieved by the i.i.d topology is a subset of the corresponding stability region when topology is Markov or Hidden Markov. In addition, we need to show that both these regions are squeezed between two sets that are essentially the same. This will follow later by Lemma 6, in chapter 3, that applies for any topology process. Therefore, for all the practical reasons, the stability region of the stationary and ergodic Markov or Hidden Markov process, and the that of the i.i.d topology with the same stationary distribution are identical.

2.5 Maximum throughput stationary scheduling policy π_0

In this thesis, we restrict our attention to a stationary policy that maximizes the stable throughput a network can support. Before formally presenting the proposed scheduling policy we define a few useful quantities. Initially, we define $(D_k(\mathbf{x}))_{\ell j}$ for each link $\ell \in L(k)$ that computes weighted differences among all queue sizes that are connected through a link(server) under topology T_k and for all customer classes:

$$(D_k(\mathbf{x}))_{\ell j} \quad = \quad \begin{cases} (x_{s(\ell)j} - x_{d(\ell)j})(m_k)_\ell, & \text{if } d(\ell) \notin V_j \\ x_{s(\ell)j}(m_k)_\ell, & \text{if } d(\ell) \in V_j, \end{cases} \tag{2.7}$$

where $(m_k)_\ell$ is the ℓ^{th} (diagonal) entry of $\mathbf{M}_k$, defined in Assumption 3.

In addition, let the weight of server ℓ, $(D_k(\mathbf{x}))_\ell$ be the maximum weighted difference in queue sizes, that is achieved for some customer class j. Hence:

$$(D_k(\mathbf{x}))_\ell = \max_{j=1,\dots,J} (D_k(\mathbf{x}))_{\ell j}. \tag{2.8}$$

We also define the weight vector of each link ℓ that is present at the current topology $T(t) = T_k$ as

$$D_k(\mathbf{x}) = \{(D_k(\mathbf{x}))_\ell : \ell = 1, \dots, |L|\}$$

Note that Equation 2.7 can be also represented in matrix form as follows:

$$(\mathbf{D}_k^j(\mathbf{x}))^T = -(\mathbf{x}^j)^T \mathbf{R}^j \mathbf{M}_k, \forall k \in \{1, \dots, N_T, \} \tag{2.9}$$

where $(\mathbf{D}_k^j(\mathbf{x}))^T$ is an $1 \times |L|$ vector.

We are now ready to define our policy π_0. We do so in the next two stages.

- **Stage** 1 Let $\hat{\mathbf{c}}_k(\mathbf{x})$ be the solution to the following maximization:

$$\hat{\mathbf{c}}_k(\mathbf{x}) = \arg\max_{\mathbf{c} \in \mathcal{S}_k} \{\mathbf{D}_k(\mathbf{x})^T \mathbf{c}\} \tag{2.10}$$

  The activation vector $\hat{\mathbf{c}}_k(\mathbf{x})$ will be the maximum weighted activation vector. Ties are resolved by selecting any one of the maximum weighted activation vectors.

- **Stage** 2 Let $(\hat{j}_k(\mathbf{x}))_\ell$ be a class for $j \in \{1, \ldots, J\}$ for which $(D_k(\mathbf{x}))_\ell = (D_k(\mathbf{x}))_{\ell j}$. Then, we define the policy π_0 as follows:

$$(\pi_0^j(\mathbf{x}, T))_\ell \;=\; \begin{cases} 1, & j = (\hat{j}_k(\mathbf{x}))_\ell,\;\; (\hat{\mathbf{c}}_k(\mathbf{x}))_\ell = 1 \text{ and } \; x_{s(\ell)j} \geq 1 \text{ for } k \\ & \text{such that } T = T_k \\ 0, & \text{else} \end{cases} \tag{2.11}$$

  Equation 2.11 is the activation rule for policy π_0. The multi-class activation vector $\mathbf{E}^j(t)$ is defined through policy π_0 as $\mathbf{E}^j(t) = \pi_0^j(\mathbf{X}(t-1), T(t))$.

The significant properties of this policy are that it is a simple, stationary policy that, as it will be shown in the next chapter, performs nearly optimally in terms of maximizing the set of stable arrival rates that the network can support. The only information that is needed in order to come to a scheduling decision is the queue sizes at each network node and the characteristics of the links available at each time slot. In addition, the scheduling is guaranteed to be collision free since it is centralized.

In the next chapter, we will characterize how the stability region of this policy looks like. We will also show that the stability regions $\mathbf{C}_{\pi_0}$ of the proposed scheduling policy π_0 and the stability region $\mathbf{C}$ of the network both lie between two convex sets, namely a convex polytope and its interior.

Chapter 3

Optimality of scheduling policy π_0

The main contribution of this Chapter is to characterize the stability region of the introduced policy and show that it is arbitrarily close to the stability region achieved by the set of all stationary policies. Initially, a few notions of flow conservation are discussed. Then, the stability region of the introduced policy is described and the policy's optimality is reasoned and proved.

3.1 Flow Conservation

In order for a network to be stable, each network node i must be able to forward all its incoming traffic in such a way that traffic reaches one of its exit nodes and therefore no traffic accumulates at i. One way to view the notion of stability is through the concept of **flow conservation**. By observing the queue dynamics equation 2.1, we see that no node except for the source and the exit node of any customer class, j, can create new packets or destroy packets. To motivate our guess for the stability region, consider a link, ℓ in the network. Let $(\mathbf{f}_k^j)_\ell$ denote the flow of class j packets that have been successfully transmitted across link $\ell \in L$. Then $\sum_{k=1}^{N_T}(\mathbf{f}_k^j)_\ell p_k$ is the average flow of customer class j through the ℓ^{th} link. As a node cannot destroy or create packets in a stable system where there is no packet accumulation, the sum of departing flows at a node for any class, must be equal to the sum of arriving flows for this class. By flow conservation, for node $i \notin V_j$,

$$\mathbf{a}_i^j = \sum_{\ell \in L: s(\ell)=i} \sum_{k=1}^{N_T} (\mathbf{f}_k^j)_\ell p_k - \sum_{\ell \in L: d(\ell)=i} \sum_{k=1}^{N_T} (\mathbf{f}_k^j)_\ell p_k = \left(- \sum_{k=1}^{N_T} \left(\mathbf{R}^j \mathbf{f}_k^j p_k \right) \right)_i .$$

Now let us consider the constraints that the flow vectors $\mathbf{f}_k^j$ should satisfy. Since $\mathbf{f}_k^j$ are flows of traffic in the network they must be positive component wise, in other words $\mathbf{f}_k^j \in \mathbf{R}_+^{|L|}, \forall j = 1, \ldots, J, \forall k = 1, \ldots, N_T$. In addition, as at each time slot the set of links activated must belong to a valid activation vector for the current topology. Therefore the vector of average transmission attempts across all links in topology k must be of the form $\sum_{\mathbf{c} \in \mathcal{S}_k} \lambda(\mathbf{c})\mathbf{c}$ where $\lambda(\mathbf{c})$ is the fraction of time slots the activation vector $\mathbf{c}$ is used while the network is in topology k and $\sum_{\mathbf{c} \in \mathcal{S}_k} \lambda(\mathbf{c}) = 1$. The flow vector $\mathbf{f}_k^j$ is the flow of successfully transmitted packets for class j. The sum $\sum_{j=1}^{J} \mathbf{f}_k^j$ is the flow vector over all links, irrespective of the customer class. This total flow vector is obtained by multiplying the vector of transmission attempts by the probability of successful transmission across each link. Hence we can argue that, $\sum_{j=1}^{J} \mathbf{f}_k^j \approx \mathbf{M}_k \sum_{\mathbf{c} \in \mathcal{S}_k} \lambda(\mathbf{c})\mathbf{c}$. With this intuitive justification, we define the following sets. We define a region of arrival rates $\mathbf{A}^\star$ as:

$$\begin{aligned} \mathbf{A}^\star \;=\; & \{\mathbf{a} = (\mathbf{a}^1, \ldots, \mathbf{a}^J) \in \mathbf{R}_+^{NJ} : \exists \mathbf{f}_k^j \in \mathbf{R}_+^{|L|}, \forall j = 1, \ldots, J, \forall k = 1, \ldots, N_T \text{ and } \delta > 1 \\ & \text{such that } \delta \sum_{j=1}^{J} \mathbf{f}_k^j \in co(\mathbf{M}_k \mathcal{S}_k) \forall k, \text{ and } \mathbf{a}^j = - \sum_{k=1}^{N_T} \mathbf{R}^j \mathbf{f}_k^j p_k, \ \forall j\} \end{aligned} \tag{3.1}$$

Let us also define another set of arrival rates, namely the region $\tilde{\mathbf{A}}$ that is:

$$\begin{aligned} \tilde{\mathbf{A}} \;=\; & \{\mathbf{a} = (\mathbf{a}^1, \ldots, \mathbf{a}^J) \in \mathbf{R}_+^{NJ} : \exists \mathbf{f}_k^j \in \mathbf{R}_+^{|L|}, \forall j = 1, \ldots, J, \forall k = 1, \ldots, N_T \\ & \text{such that } \sum_{j=1}^{J} \mathbf{f}_k^j \in co(\mathbf{M}_k \mathcal{S}_k) \forall k, \text{ and } \mathbf{a}^j = - \sum_{k=1}^{N_T} \mathbf{R}^j \mathbf{f}_k^j p_k, \ \forall j\} \end{aligned} \tag{3.2}$$

In the following section we shall argue that the sets $\mathbf{A}^\star$ and $\tilde{\mathbf{A}}$ represent the stability region of the network, under a randomly varying topology. We also define the set of arrival rates $\mathbf{A}_0$ as:

$$\begin{aligned}
&\mathbf{A}_0 = \\
&\quad \{\mathbf{a},\ \text{such that } \mathbf{a} = \sum_{k=1}^{N_T} \mathbf{a}_k p_k,\ \text{and} \\
&\quad \text{and for each } k = 1, \ldots, N_T,\ \mathbf{a}_k = (\mathbf{a}_k^1, \ldots, \mathbf{a}_k^J) \in \mathbf{R}_+^{NJ} : \exists \mathbf{f}_k^j \in \mathbf{R}_+^{|L|}, \forall j = 1, \ldots, J, \\
&\quad , \exists \delta > 1 \text{ such that } \delta \sum_{j=1}^{J} \mathbf{f}_k^j \in co(\mathbf{M}_k \mathcal{S}_k) \text{ and } \mathbf{a}_k^j = -\mathbf{R}^j \mathbf{f}_k^j,\ \forall j\}
\end{aligned}$$

The set $\mathbf{A}_0$ is obtained by a weighted average of the stability regions of individual topologies T_k. To obtain the stability region of each individual topology, the scheduling policy described in [14] is applied to each one of them, as if the network was a stationary one, taking this topology at all times.

3.2 Stating and Proving Optimality of π_0

In this section we are going exploit some set properties between the sets $\mathbf{A}^\star, \tilde{\mathbf{A}}$, the set $\mathbf{C}$ of all stable arrival rates under all stationary policies, the set $\mathbf{C}_{\pi_0}$ of stable arrival rates under the proposed policy π_0, and the set A_0 of arrival rates obtained by a convex combination of stable arrival rate regions in each network configuration.

Theorem 3 *The following properties hold:*

1. *The set $\mathbf{A}^\star$ is a convex set that is not closed.*

2. *The set of arrival rates $\tilde{\mathbf{A}}$ is a convex polytope.*

3. *The two sets $\mathbf{A}^\star$ and $\tilde{\mathbf{A}}$ are related through $\tilde{\mathbf{A}} = \overline{\mathbf{A}^\star}$.*

4. *The regions of arrival rates $\mathbf{A}^\star$, $\tilde{\mathbf{A}}$, $\mathbf{A}_0$, $\mathbf{C}$ and $\mathbf{C}_{\pi_0}$ are related through the following set inequality:*

$$\mathbf{A}_0 \overset{(1)}{\subseteq} \mathbf{A}^\star \overset{(2)}{\subseteq} \mathbf{C}_{\pi_0} \overset{(3)}{\subseteq} \mathbf{C} \overset{(4)}{\subseteq} \tilde{\mathbf{A}} \overset{(5)}{=} \overline{C_{\pi_0}} \tag{3.3}$$

Theorem 3 forms one of the core results of this thesis. It states that the capacity region achieved by any stationary policy lies between the sets $\mathbf{A}^\star$ and $\tilde{\mathbf{A}}$. The set $\tilde{\mathbf{A}}$ is argued to be a convex polytope. Hence, it will be a convex and bounded region. In addition, $\mathbf{A}^\star \subseteq \tilde{\mathbf{A}}$, where $\mathbf{A}^\star$ is a convex, but not closed, set and the two sets are related through $\tilde{\mathbf{A}} = \overline{\mathbf{A}^\star}$. Since a set and its closure don't necessarily have the same volume, one could claim that $\mathbf{A}^\star$ is much smaller from $\tilde{\mathbf{A}}$ and that we can find a policy that does at least as well as our optimal throughput scheduling policy. However, since these two sets are convex, and the set $\tilde{\mathbf{A}}$ is a convex polytope, they can differ only at the points that belong at the phases of $\tilde{\mathbf{A}}$, and hence we can conclude that they are essentially the same sets. In addition, the fact that both $\mathbf{C}$ and $\mathbf{C}_{\pi_0}$ lie between $\mathbf{A}^\star$ and $\tilde{\mathbf{A}}$, optimality of π_0 follows.

We also prove the following Lemma, that will be useful throughout our analysis.

Lemma 3 *Let the vector $\mathbf{c}$, be such that $\mathbf{c} \in co(\mathcal{S}_k)$. Then, any vector $\mathbf{a}$, that is component wise $0 \leq \mathbf{a} \leq \mathbf{c}$, must also belong to $co(\mathcal{S}_k)$.*

Proof: If the vector $\mathbf{c}$ has n non-zero elements, then $\mathbf{a}$ can have at most n non-zero elements. Without loss of generality, we sort the elements of $\mathbf{c}$ so that the first n elements are the non-zero ones. Let $\mathbf{x}^0 = \mathbf{c}$ and $\mathbf{x}^n = \mathbf{a}$, where $1 \leq n \leq |L|$.

Claim that $\mathbf{x}^k \in co(\mathcal{S}_k), \forall k = 1, \ldots, n-1$, where $\mathbf{x}^k$ is defined by:

$$\mathbf{x}^k = (a_1, a_2, \ldots, a_k, c_{k+1}, \ldots, c_{|L|}) \in co(\mathcal{S}_k)$$

.

Let us also define $\lambda_i = \frac{a_i}{c_i}$ for $i = 1, \dots, n$. Note that $\lambda_i \leq 1$ because $a_i \leq c_i, \forall i = 1, \dots, |L|$. We will show the above statement by using induction.

- (Basic Step) We know that $\mathbf{x}^0 = \mathbf{c} = (c_1, c_2, \dots, c_{|L|}) \in co(\mathcal{S}_k)$ (it is given).

- Assume that the claim holds for $k = m$ and hence $\mathbf{x}^m = (a_1, a_2, \dots, a_m, c_{m+1}, \dots, c_{|L|}) \in co(\mathcal{S}_k)$.

- We will show that the claim holds for $k = m + 1$, in other words that vector

$$\mathbf{x}^{m+1} = (a_1, a_2, \dots, a_m, a_{m+1}, c_{m+2} \dots, c_{|L|}) \in co(\mathcal{S}_k)$$

  By the induction hypothesis $\mathbf{x}^m \in co(\mathcal{S}_k)$. Therefore, it can be written as a convex combination of vectors $\mathbf{s}_i \in S_k$. By the properties of constraint set, all vectors $\mathbf{s}_i^{'}$ obtained by setting the $(m+1)^{th}$ component of $\mathbf{s}_i$ to 0, also belong in the $\mathcal{S}_k$. Therefore, it is straightforward to see that the vector $\mathbf{x}^{''}$ defined as $\mathbf{x}^{''} = (a_1, a_2, \dots, a_m, 0, c_{m+2} \dots, c_{|L|}) \in co(\mathcal{S}_k)$. Hence, $\mathbf{x}^{m+1} = (a_1, a_2, \dots, a_m, a_{m+1}, c_{m+2} \dots, c_{|L|}) = \lambda_{m+1} x^m + (1 - \lambda_{m+1}) x'' \in co(\mathcal{S}_k)$.

- When $m = n$, then $\mathbf{x}^m = \mathbf{a}$ and hence $\mathbf{a} \in co(\mathcal{S}_k)$.

This completes the proof of Lemma 3. $\diamondsuit$

3.2.1 Region $\mathbf{A}^\star$ is convex

First, we are going to show that the $\mathbf{A}^\star$ is convex and then give a counter example for not being a closed set.

We need to show that for any arrival rate vectors $\mathbf{a}_1^j$ and $\mathbf{a}_2^j$, such that $\mathbf{a}_1^j, \mathbf{a}_2^j \in \mathbf{A}^\star$, all the points that lie on the segment $\lambda \mathbf{a}_1^j + (1 - \lambda) \mathbf{a}_2^j$ belong in $\mathbf{A}^\star$ as well.

Since $\mathbf{a}_1^j, \mathbf{a}_2^j \in \mathbf{A}^\star$, we know that there exist flow vectors $\mathbf{f}1_k^j \geq 0$ and $\mathbf{f}2_k^j \geq 0$ such that for some $\delta_1, \delta_2 > 1$ they satisfy

$$\delta_1 \sum_{j=1}^{J} \mathbf{f}1_k^j \in co(\mathbf{M}_k \mathcal{S}_k)$$

and

$$\delta_2 \sum_{j=1}^{J} \mathbf{f}2_k^j \in co(\mathbf{M}_k \mathcal{S}_k)$$

Then $\mathbf{a}_1^j$ and $\mathbf{a}_2^j$ can be expressed through the equations $\mathbf{a}_1^j = \sum_{k=1}^{N_T} \mathbf{R}^j \mathbf{f}1_k^j p_k$ and $\mathbf{a}_2^j = \sum_{k=1}^{N_T} \mathbf{R}^j \mathbf{f}2_k^j p_k$.

Let us now take the convex combination of $\mathbf{a}_1^j, \mathbf{a}_2^j$ for some $\lambda, 0 \leq \lambda \leq 1$:

$$\begin{aligned}
&\mathbf{a}^j \\
&= \lambda \mathbf{a}_1^j + (1-\lambda)\mathbf{a}_2^j \\
&= -\lambda \sum_{k=1}^{N_T} \mathbf{R}^j \mathbf{f}1_k^j p_k - (1-\lambda) \sum_{k=1}^{N_T} \mathbf{R}^j \mathbf{f}2_k^j p_k \\
&= -\sum_{k=1}^{N_T} \mathbf{R}^j (\lambda \mathbf{f}1_k^j + (1-\lambda)\mathbf{f}2_k^j) p_k
\end{aligned}$$

Let $\mathbf{f}_k^j \triangleq \lambda \mathbf{f}1_k^j + (1-\lambda)\mathbf{f}2_k^j$. We must show that conditions

- $\mathbf{f}_k^j \geq 0$ and
- $\exists \delta > 1 : \delta \sum_{j=1}^{J} \mathbf{f}_k^j \in co(\mathbf{M}_k \mathcal{S}_k)$

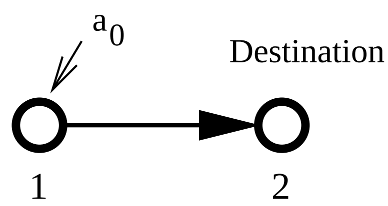

Figure 3.1: A 2 node network with a single network configuration (degenerate case of a multiple configurations network). Arrivals at node 1 need to exit the network at node 2. The set of achievable arrival rates is not a closed set.

are met. Since $\mathbf{f}1_k^j, \mathbf{f}2_k^j \geq 0$, it trivially follows that $\mathbf{f}_k^j \geq 0$ Also,

$$\begin{aligned} &\delta \sum_{j=1}^{J} \mathbf{f}_k^j \\ &= \delta \sum_{j=1}^{J} \left(\lambda \mathbf{f}1_k^j + (1-\lambda)\mathbf{f}2_k^j\right) \\ &= \lambda\delta \sum_{j=1}^{J} \mathbf{f}1_k^j + (1-\lambda)\delta \sum_{j=1}^{J} \mathbf{f}2_k^j \end{aligned}$$

Let us select $\delta = \min(\delta_1, \delta_2)$, then by Lemma 3 we can see that, $\delta \sum_{j=1}^{J} \mathbf{f}1_k^j \in co(\mathbf{M}_k \mathcal{S}_k)$, $\delta \sum_{j=1}^{J} \mathbf{f}2_k^j \in co(\mathbf{M}_k \mathcal{S}_k)$ and $\delta \sum_{j=1}^{J} \mathbf{f}_k^j \in co(\mathbf{M}_k \mathcal{S}_k)$. Therefore, $\mathbf{A}^\star$ is a convex set.

Now, we are going to present a simple example where $\mathbf{A}^\star$ is not a closed set. Consider the network depicted in Figure 3.1. It is a 2 node network with one customer class and a single network configuration. Arrivals at node 1 exit the system at exit node 2. The service rate through the single link is $\mu = 1$. The possible activation vectors are $S = \{\{0\}, \{1\}\}$ and the convex hull of S will be the closed line segment $co(S) = [0, 1]$. Let the total flow through the link be $\mathbf{f}$. Then, there should exist a $\delta > 1$ such that $\delta\mathbf{f} \in co(\mathbf{M}S)$ or $\delta\mathbf{f} \in [0, 1]$. Hence, $\mathbf{f} \in [0, 1)$, which is not a closed set. Hence the arrival rate $\mathbf{a} = R\mathbf{f}$ that is obtained through a linear mapping of $\mathbf{f}$ will not be a closed set either. This concludes the proof of Theorem 3: Part 1. $\diamondsuit$

3.2.2 Region $\tilde{\mathbf{A}}$ is a convex polytope

In this subsection, we are going to show that the set $\tilde{\mathbf{A}}$ forms a convex polytope and that it is related with $\mathbf{A}^\star$ through $\tilde{\mathbf{A}} = \overline{\mathbf{A}^\star}$.

We are now going to prove that $\tilde{\mathbf{A}}$ is a convex polytope. The idea behind this proof is to show that the corresponding f_k^j form a convex polytope and $\tilde{\mathbf{A}}$ that is related through a linear map with them is a polytope as well.

A convex polytope is a set of convex combinations of finitely many points/vectors. The constraint set $\mathcal{S}_k$ is a set of finitely many constraint vectors, $\mathbf{c}$. In addition, the set:

$$\mathbf{M}_k\mathcal{S}_k \triangleq \{\mathbf{M}_k s : s \in \mathcal{S}_k\}$$

is finite as well. Hence the set of $co(\mathbf{M}_k\mathcal{S}_k)$ is a convex polytope.

However, the set of $\mathbf{f}_k^j$ that satisfy $\sum_{j=1}^{J} \mathbf{f}_k^j \in co(\mathbf{M}_k\mathcal{S}_k)$ can be rewritten in matrix form as

$$\sum_{j=1}^{J} \mathbf{f}_k^j = \left[\begin{array}{cccc} f_k^1 & f_k^2 & \dots & f_k^J \end{array}\right] \left[\begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \end{array}\right]$$

where the vector $\left[\begin{array}{cccc} 1 & 1 & \dots & 1 \end{array}\right]$ is a linear map. Hence, by making use of Theorem 6 (See Appendix C) we obtain that the set of $\mathbf{f}_k^j, j = 1, \dots, J$ is a convex polyhedron.

To complete our proof we need to show that the set of $\mathbf{f}_k^j$ that satisfy $\sum_{j=1}^{J} \mathbf{f}_k^j \in co(\mathbf{M}_k\mathcal{S}_k)$ is a bounded set, and hence a convex polytope.

We know that $\mathbf{f}_k^j \geq 0$ and their sum is bounded, since $\sum_{j=1}^{J} \mathbf{f}_k^j \in co(\mathbf{M}_k\mathcal{S}_k)$. Hence, each one of them must be bounded as well, which proves the Claim. Moreover, since $\mathbf{R}^j, p_k$ and the sum over all k's are linear operations, we conclude that the set of $\tilde{\mathbf{A}} = \sum_{k=1}^{N_T} \mathbf{R}^j \mathbf{f}_k^j p_k$ is a convex polytope. This completes the proof of Theorem 3: Part 2. $\diamondsuit$

3.2.3 $\tilde{\mathbf{A}}$ is the closure of $\mathbf{A}^\star$

We will proceed to show that $\overline{\mathbf{A}^\star} = \tilde{\mathbf{A}}$. Towards this end, we two properties need to be proved:

1. $\tilde{\mathbf{A}} \subseteq \overline{\mathbf{A}^\star}$

2. $\overline{\mathbf{A}^\star} \subseteq \tilde{\mathbf{A}}$

1. We need to show that for every point $\mathbf{a} \in \tilde{\mathbf{A}} \Rightarrow \mathbf{a} \in \overline{\mathbf{A}^\star}$. Let $\mathbf{a} \in \tilde{\mathbf{A}}$. Consider the sequence $\mathbf{a}_n$, defined by $\mathbf{a}_n = \theta_n \mathbf{a}$, where $\theta_n \in (0,1)$ and $\lim_{n\to\infty} \theta_n = 1$. Clearly:

$$\lim_{n\to\infty} \mathbf{a}_n = \lim_{n\to\infty} \theta_n \mathbf{a} = \mathbf{a} \in \tilde{\mathbf{A}}$$

   We show that $\mathbf{a}_n \in \mathbf{A}^\star$ and hence prove the statement.

   Since $\mathbf{a} \in \tilde{\mathbf{A}}$, there must exist some flows $\mathbf{f}_k^j$ that satisfy the following:

   I $\mathbf{f}_k^j \geq 0$

   II $\sum_{j=1}^J \mathbf{f}_k^j \in co(\mathbf{M}_k \mathcal{S}_k)$

   and for each commodity j, $\mathbf{a}$ can be written as $\mathbf{a}^j = -\sum_{k=1}^{N_T} \mathbf{R}^j \mathbf{f}_k^j p_k$. Let us define the sequence $(\mathbf{f}_k^j)_n \triangleq \theta_n \mathbf{f}_k^j$. To conclude the proof we show that $(\mathbf{f}_k^j)_n$ satisfy the conditions in the definition of $\mathbf{A}^\star$ (Equation 3.1) with $\mathbf{a} = \mathbf{a}_n$. We observe that $(\mathbf{f}_k^j)_n, n = 1, 2, \ldots$ satisfy the following properties:

   I $(\mathbf{f}_k^j)_n \geq 0$

   II

$$\sum_{j=1}^J (\mathbf{f}_k^j)_n = \sum_{j=1}^J \theta_n \mathbf{f}_k^j \Rightarrow$$

$$\frac{1}{\theta_n}\sum_{j=1}^{J}(\mathbf{f}_k^j)_n = \sum_{j=1}^{J}\mathbf{f}_k^j \in co(\mathbf{M}_k\mathcal{S}_k)$$

Hence there exists a $\delta_n = \frac{1}{\theta_n} > 1$, such that:

$$\delta_n \sum_{j=1}^{J}(\mathbf{f}_k^j)_n \in co(\mathbf{M}_k\mathcal{S}_k)$$

Therefore, $\mathbf{a}_n \in \mathbf{A}^\star$ and for each customer class j can be written as:

$$\mathbf{a}_n^j = -\sum_{k=1}^{N_T}\mathbf{R}^j(\mathbf{f}_k^j)_n p_k = -\sum_{k=1}^{N_T}\mathbf{R}^j\theta_n\mathbf{f}_k^j p_k = \theta_n\mathbf{a}_n^j$$

Hence, $\mathbf{a} \in \overline{\mathbf{A}^\star}$.

2. In order to show that $\mathbf{a} \in \overline{\mathbf{A}^\star} \Rightarrow \mathbf{a} \in \tilde{\mathbf{A}}$, we will follow two steps. First we will show that that $\mathbf{A}^\star \subseteq \tilde{\mathbf{A}}$ and then that $\overline{\mathbf{A}^\star} \subseteq \tilde{\mathbf{A}}$. Let an arrival rate vector $\mathbf{a} \in \mathbf{A}^\star$. There must exist $\mathbf{f}_k^j \geq 0$ and a $\delta > 1$ such that $\delta\sum_{j=1}^{J}\mathbf{f}_k^j \in co(\mathbf{M}_k\mathcal{S}_k)$ and for each customer class j $\mathbf{a}^j = \sum_{j=1}^{J}\mathbf{R}^j\mathbf{f}_k^j p_k$. Since, $\delta\sum_{j=1}^{J}\mathbf{f}_k^j \in co(\mathbf{M}_k\mathcal{S}_k)$ and $\delta > 1$, by Lemma 3. $\sum_{j=1}^{J}\mathbf{f}_k^j \in co(\mathbf{M}_k\mathcal{S}_k)$ follows. Hence:

$$\mathbf{a} \in \mathbf{A}^\star \Rightarrow \mathbf{a} \in \tilde{\mathbf{A}}.$$

In addition, $\tilde{\mathbf{A}}$ has been proven to be a convex polytope, and therefore be a closed set. By the definition of the closure of the set, it is the smallest closed set that contains the set, which implies $\mathbf{A}^\star \subseteq \overline{\mathbf{A}^\star}$. Hence, $\overline{\mathbf{A}^\star} \subseteq \tilde{\mathbf{A}}$.

The above 2 steps completes the proof of Theorem 3: Part 3. $\diamondsuit$

We proceed to establish the set relationships among the sets $\mathbf{A}^\star$, $\tilde{\mathbf{A}}$, $\mathbf{A}_0$, $\mathbf{C}$ and $\mathbf{C}_{\pi_0}$ of Theorem 3 (Equation 3.3). Inequality (3) follows trivially, since π_0 is one policy from the set of all stationary policies $\pi \in \Pi$ and therefore its throughput $\mathbf{C}_{\pi_0}$ cannot be better than the throughput achieved by all members in Π, namely the region $\mathbf{C}$. In addition, equality (5) trivially follows once we prove that $\mathbf{A}^\star \overset{(2)}{\subseteq} \mathbf{C}_{\pi_0} \overset{(4)}{\subseteq} \tilde{\mathbf{A}}$, by taking the closure of this expression. In the sequel, the rest of the set inequalities will be shown.

3.2.4 The region of stable arrival rates under π_0 contains $\mathbf{A}^\star$

In this section, we show inequality (2) in Theorem 3 (Part 3.3). We argue that the $\mathbf{A}^\star \subseteq \mathbf{C}_{\pi_0}$ and hence, under π_0 every arrival rate vector $\mathbf{a} \in \mathbf{A}^\star$ is stable. To this respect, we are going to use a Lyapunov analysis approach. We will try to show that the conditions of Theorem 2 also hold for our model when the proposed scheduling policy is applied.

Towards thid end, we are going to use the candidate Lyapunov function V' as defined in chapter 2. In addition, we will attempt to find sufficient conditions on the arrival rates so as to ensure that the expected value of the Lyapunov function has a negative drift whenever the queue sizes are large. If we are able to find necessary and sufficient conditions, we will have characterized an optimal throughput stable scheduling policy.

We will apply this candidate Lyapunov function to the Markov Chain of the queue sizes $\{\mathbf{X}(t)\}_{t=0}^{\infty}$. In the sequence, we will show that whenever the queue sizes are large, the expected drift of the candidate Lyapunov function is negative.

Lemma 4 *For a stationary policy π, an* arbitrary *arrival rate vector $\mathbf{a}$ and a network following the queue length dynamics given by Equation 2.1, the following holds:*

$$
\begin{aligned}
& E[V'(\mathbf{X}(t+1)) - V'(\mathbf{X}(t))|\mathbf{X}(t) = \mathbf{x}] \\
& \leq \sum_{k=1}^{N_T} \left(\sum_{j=1}^{J} -2(\mathbf{D}_k^j(\mathbf{x}))^T \pi^j(\mathbf{x}, T_k) \right) p_k \\
& + \sum_{j=1}^{J} 2\mathbf{x}^{j^T}\mathbf{a}^j + \sum_{k=1}^{N_T} b_k,
\end{aligned}
\tag{3.4}
$$

where b_k are constants that do not depend on the queue sizes.

Proof:

$$E[V'(\mathbf{X}(t+1)) - V'(\mathbf{X}(t))|\mathbf{X}(t) = \mathbf{x}] \tag{3.5}$$

$$
\begin{aligned}
&= E\left[E[V'(\mathbf{X}(t+1)) - V'(\mathbf{X}(t))|\mathbf{X}(t) = \mathbf{x}, T(t+1) = T_k]|\mathbf{X}(t) = \mathbf{x}\right] \\
&= \sum_{k=1}^{N_T}\left(E[\sum_{j=1}^{J}\sum_{i=1}^{N}(X_{ij}^2(t+1) - X_{ij}^2(t))|\mathbf{X}(t) = \mathbf{x}, T(t+1) = T_k]P[T(t+1) = T_k|\mathbf{X}(t) = \mathbf{x}]\right) \\
&= \sum_{k=1}^{N_T}\left(E[\sum_{j=1}^{J}\sum_{i=1}^{N}(X_{ij}^2(t+1) - X_{ij}^2(t))|\mathbf{X}(t) = \mathbf{x}, T(t+1) = T_k]\right)p_k \\
&= \sum_{k=1}^{N_T}\left(E[\sum_{j=1}^{J}(\mathbf{X}^j(t+1)^T\mathbf{X}^j(t+1) - \mathbf{X}^j(t)^T\mathbf{X}^j(t))|\mathbf{X}(t) = \mathbf{x}, T(t+1) = T_k]\right)p_k \\
&= \sum_{k=1}^{N_T}\left(E[\sum_{j=1}^{J}(\mathbf{X}^j(t+1) - \mathbf{X}^j(t))^T(\mathbf{X}^j(t+1) + \mathbf{X}^j(t))|\mathbf{X}(t) = \mathbf{x}, T(t+1) = T_k]\right)p_k
\end{aligned} \tag{3.6}
$$

By substituting equation 2.1 into 3.6 we get:

$$
\begin{aligned}
&E[V'(\mathbf{X}(t+1)) - V'(\mathbf{X}(t))|\mathbf{X}(t) = \mathbf{x}] \\
&= \sum_{k=1}^{N_T}\left(\sum_{j=1}^{J}E\left[(\mathbf{R}^j\mathbf{M}(t+1)\mathbf{E}^j(t+1) + \mathbf{A}^j(t+1))^T\right.\right. \\
&\quad \left.\left.\left(2\mathbf{X}^j(t) + \mathbf{R}^j\mathbf{M}(t+1)\mathbf{E}^j(t+1) + \mathbf{A}^j(t+1)\right)|\mathbf{X}(t) = \mathbf{x}, T(t+1) = T_k\right]\right)p_k \\
&= \sum_{k=1}^{N_T}\left(\sum_{j=1}^{J}E[(\mathbf{R}^j\mathbf{M}(t+1)\mathbf{E}^j(t+1) + \mathbf{A}^j(t+1))^T\right. \\
&\quad \left(\mathbf{R}^j\mathbf{M}(t+1)\mathbf{E}^j(t+1) + \mathbf{A}^j(t+1)\right)|\mathbf{X}(t) = \mathbf{x}, T(t+1) = T_k] \\
&+ \left.\sum_{j=1}^{J}E[2(\mathbf{R}^j\mathbf{M}(t+1)\mathbf{E}^j(t+1) + \mathbf{A}^j(t+1))^T\mathbf{X}^j(t)|\mathbf{X}(t) = \mathbf{x}, T(t+1) = T_k]\right)p_k
\end{aligned} \tag{3.7}
$$

For each of the N_T terms of Equation 3.7 correspond two inner summations. The first inner summation of 3.7 can be bounded by a constant, say b_k since it only contains entries from $\mathbf{R}^j$ and $\mathbf{A}(\cdot)$, which are fixed given the topology and don't vary with the queue sizes, and $\mathbf{E}(\cdot)$, which depends on the queue sizes but is bounded for all topologies by 1 component wise.

Now let us look at the second inner summation of equation 3.7, which corresponds to topology,

$T_k \in \mathcal{T}$:

$$\begin{aligned}
&\left(\sum_{j=1}^{J} E\left[2(\mathbf{R}^j\mathbf{M}(t+1)\mathbf{E}^j(t+1) + \mathbf{A}^j(t+1))^T\mathbf{X}^j(t)|\mathbf{X}(t) = \mathbf{x}, T(t+1) = T_k\right]\right) p_k \\
&= \left(\sum_{j=1}^{J} E\left[2(\mathbf{R}^j\mathbf{M}(t+1)\mathbf{E}^j(t+1))^T\mathbf{X}^j(t)|\mathbf{X}(t) = \mathbf{x}, T(t+1) = T_k\right]\right) p_k \\
&+ \left(\sum_{j=1}^{J} E\left[2(\mathbf{A}^j(t+1))^T\mathbf{X}^j(t)|\mathbf{X}(t), T(t+1) = T_k\right]\right) p_k \\
&= \left(\sum_{j=1}^{J} 2(\mathbf{x}^j)^T\mathbf{R}^j\mathbf{M}_k\pi^j(\mathbf{x}, T_k)\right) p_k \\
&+ \left(\sum_{j=1}^{J} 2(\mathbf{x}^j)^T\mathbf{a}^j\right) p_k,
\end{aligned}$$

where we have used that $\pi^j(\mathbf{x}, T_k) = E[\mathbf{E}^j(t+1)|\mathbf{X}(t) = \mathbf{x}, T(t+1) = T_k]$ and in view of Assumption 3 and the fact that $\mathbf{M}(t)$ is independent of the queue sizes, $\mathbf{M}_k = E[\mathbf{M}(t+1)|\mathbf{X}(t) = \mathbf{x}, T(t+1) = T_k]$. The elements of matrix $\mathbf{M}_k$, are the probabilities $(m_k)_\ell$ that represent the probability that a transmission goes through link ℓ. So, equation 3.7 becomes:

$$\begin{aligned}
&E[V'(\mathbf{X}(t+1)) - V'(\mathbf{X}(t))|\mathbf{X}(t) = \mathbf{x}] \\
&\leq \sum_{k=1}^{N_T} \left(\sum_{j=1}^{J} 2(\mathbf{x}^j)^T\mathbf{R}^j\mathbf{M}_k\pi^j(\mathbf{x}, T_k)\right) p_k \\
&+ \sum_{j=1}^{J} 2(\mathbf{x}^j)^T\mathbf{a}^j \left(\sum_{k=1}^{N_T} p_k\right) \\
&+ \sum_{k=1}^{N_T} b_k
\end{aligned} \tag{3.8}$$

Equation 3.8 can be rewritten by substituting Equation 2.9 into the equivalent entries for $(\mathbf{x}^j)^T\mathbf{R}^j\mathbf{M}_k$, as:

$$
\begin{aligned}
&E[V'(\mathbf{X}(t+1)) - V'(\mathbf{X}(t))|\mathbf{X}(t) = \mathbf{x}] \\
&= \sum_{k=1}^{N_T} \left(\sum_{j=1}^{J} -2(\mathbf{D}_k^j(\mathbf{x}))^T \pi^j(\mathbf{x}, T_k) \right) p_k \\
&+ \sum_{j=1}^{J} 2\mathbf{x}^{jT} \mathbf{a}^j + \sum_{k=1}^{N_T} b_k
\end{aligned}
\tag{3.9}
$$

$\diamondsuit$

Now we impose the condition that the arrival rates belong to $\mathbf{A}^\star$ and examine the system under the policy π_0. Then we can derive the following result.

Lemma 5 *Consider some stationary policy π, then for all arrival rate vectors $\mathbf{a} \in \mathbf{A}^\star$ and a network following the queue length dynamics given by Equation 2.1, the following holds:*

$$
\begin{aligned}
&E[V'(\mathbf{X}(t+1)) - V'(\mathbf{X}(t))|\mathbf{X}(t) = \mathbf{x}] \\
&\leq 2\sum_{k=1}^{N_T} \left(\sum_{j=1}^{J} \left(-(\mathbf{D}_k{}^j(\mathbf{x}))^T \pi^j(\mathbf{x}, T_k)\right) + (\mathbf{D}_k(\mathbf{x}))^T \sum_{i=1}^{|\mathcal{S}_k|} (\lambda_k)_i (\mathbf{c}_k)_i \right) p_k \\
&+ \sum_{k=1}^{N_T} b_k
\end{aligned}
\tag{3.10}
$$

Proof: By the flow conservation constraints described earlier and since $\mathbf{a} \in \mathbf{A}^\star$, we have that

$$
\exists \delta > 1 : \delta \sum_{j=1}^{J} \mathbf{f}_k^j \in co(\mathbf{M}_k \mathcal{S}_k)
$$

and by Equation 3.1

$$
\mathbf{a}^j = -\sum_{k=1}^{N_T} \left(\mathbf{R}^j \mathbf{f}_k^j p_k\right), \forall j = 1, \dots, J
$$

In the sequel, we prove that this arrival rate vector $\mathbf{a}$ is stable.

From Equation 3.1, we know that we may find $(\gamma_k)_i \geq 0, \forall i = 1, \dots, |\mathcal{S}_k|$, that satisfy $\sum_{i=1}^{|\mathcal{S}_k|} (\gamma_k)_i = 1$ and are such that

$$
\delta \sum_{j=1}^{J} \mathbf{f}_k^j = \mathbf{M}_k \sum_{i=1}^{|\mathcal{S}_k|} (\gamma_k)_i (\mathbf{c}_k)_i \text{ ,where } (c_k)_i \in \mathcal{S}_k,
$$

In the Equation above, $(\mathbf{c}_k)_i \in \mathcal{S}_k$ are valid activation vectors under topology T_k.

Since $\delta \sum_{j=1}^{J} \mathbf{f}_k^j \in co(M_k \mathcal{S}_k)$, it follows from Lemma 3 that any vector smaller than that component wise will also be in this Convex Hull. Hence, so will $\sum_{j=1}^{J} \mathbf{f}_k^j$.

As $\delta > 1$, we can obtain coefficients $(\lambda_k)_i = \frac{(\gamma_k)_i}{\delta} \geq 0, \quad \forall i \in \{1, \ldots, |\mathcal{S}_k|\}$ such that:

$$\sum_{j=1}^{J} \mathbf{f}_k^j = \mathbf{M}_k \sum_{i=1}^{|\mathcal{S}_k|} (\lambda_k)_i (\mathbf{c}_k)_i \text{ where } (\mathbf{c}_k)_i \in \mathcal{S}_k \text{ and } \sum_{i=1}^{|\mathcal{S}_k|} (\lambda_k)_i < 1,$$

Moreover, from Equation 3.1, we can write

$$\sum_{j=1}^{J} 2(\mathbf{x}^j)^T \mathbf{a}^j = -\sum_{k=1}^{N_T} \left(\sum_{j=1}^{J} 2(\mathbf{x}^j)^T \mathbf{R}^j \mathbf{f}_k^j \right) p_k \tag{3.11}$$

and by substituting Equation 2.9 into 3.11 we get that:

$$\sum_{j=1}^{J} 2(\mathbf{x}^j)^T \mathbf{a}^j = 2 \sum_{k=1}^{N_T} \left(\sum_{j=1}^{J} \mathbf{D}_k{}^j (\mathbf{x})^T \mathbf{M}_k^{\dagger} \mathbf{f}_k^j \right) p_k \tag{3.12}$$

In the equation above, $\mathbf{M}_k^{\dagger}$ is the *pseudoinverse* of the diagonal matrix $\mathbf{M}_k$. Since $\mathbf{M}_k$ is a diagonal matrix, its pseudo-inverse is a diagonal matrix as well, with diagonal entries:

$$(m_k^{\dagger})_\ell = \begin{cases} (m_k^{-1})_\ell & \text{if } (m_k)_\ell \neq 0 \\ 0 & \text{if } (m_k)_\ell = 0 \end{cases}$$

Therefore we can argue as follows:

$$\begin{aligned}
&\sum_{j=1}^{J} 2(\mathbf{x}^j)^T \mathbf{a}^j \\
&\quad \leq \; 2 \sum_{k=1}^{N_T} \left(\max_{j=1,\ldots,J} \left(\mathbf{D}_k{}^j (\mathbf{x})^T \right) \mathbf{M}_k^{\dagger} \sum_{j=1}^{J} \mathbf{f}_k^j \right) p_k \\
&\quad < \; 2 \sum_{k=1}^{N_T} \left((\mathbf{D}_k(\mathbf{x}))^T \mathbf{M}_k^{\dagger} \mathbf{M}_k \sum_{i=1}^{|\mathcal{S}_k|} (\lambda_k)_i (\mathbf{c}_k)_i \right) p_k \\
&\quad = \; 2 \sum_{k=1}^{N_T} \left((\mathbf{D}_k(\mathbf{x}))^T \sum_{i=1}^{|\mathcal{S}_k|} (\lambda_k)_i (\mathbf{c}_k)_i \right) p_k
\end{aligned} \tag{3.13}$$

The equality in Equation 3.13 is justified as follows. The entry $\sum_{i=1}^{|\mathcal{S}_k|}(\lambda_k)_i(\mathbf{c}_k)_i$ is the sum over all possible valid activation vectors. They must have 0s, for each link that is not available under topology T_k. In addition, $\mathbf{M}_k^{\dagger}\mathbf{M}_k$ is a diagonal matrix. Its diagonal elements are 1 if $(m_k)_\ell \neq 0$ and 0 if $(m_k)_\ell = 0$, in which case the ℓ^{th} entry of c will be 0 as well. Hence:

$$\mathbf{M}_k^{\dagger}\mathbf{M}_k \sum_{i=1}^{|\mathcal{S}_k|}(\lambda_k)_i(\mathbf{c}_k)_i = \sum_{i=1}^{|\mathcal{S}_k|}(\lambda_k)_i(\mathbf{c}_k)_i$$

and Equation 3.13 follows. From equations 3.13 and 3.4, we can write:

$$\begin{aligned} & E[V'(\mathbf{X}(t+1)) - V'(\mathbf{X}(t))|\mathbf{X}(t) = \mathbf{x}] \\ \leq \;& 2\sum_{k=1}^{N_T}\left(\sum_{j=1}^{J}\left(-(\mathbf{D}_k{}^j(\mathbf{x}))^T\pi_0^j(\mathbf{x},T_k)\right) + (\mathbf{D}_k(\mathbf{x}))^T\sum_{i=1}^{|\mathcal{S}_k|}(\lambda_k)_i(\mathbf{c}_k)_i\right)p_k \\ + \;& \sum_{k=1}^{N_T} b_k \end{aligned} \tag{3.14}$$

$\diamondsuit$

Next we proceed to show the remaining part of inequality (2) in Theorem 3. Examining the definition of π_0 in Equation 2.7, we can see that, for all $j = 1, 2, \ldots, J$,

$$\mathbf{D}_k{}^j(\mathbf{x})^T\pi_0^j(\mathbf{x},T_k) = \mathbf{D}_k(\mathbf{x})^T\pi_0^j(\mathbf{x},T_k).$$

This equivalence results from the fact that any element of $\pi_0^j(\mathbf{x},T_k)$ is non-zero only when the corresponding element of $\mathbf{D}_k{}^j(\mathbf{x})$ equals $\mathbf{D}_k(\mathbf{x})$, in other words achieves the maximum of Equation 2.8). Hence we have,

$$\sum_{j=1}^{J}\mathbf{D}_k(\mathbf{x})^T\pi_0^j(\mathbf{x},T_k) = \sum_{j=1}^{J}\mathbf{D}_k^j(\mathbf{x})^T\pi_0^j(\mathbf{x},T_k) = \max_{\mathbf{c}\in\mathcal{S}_k}\{\mathbf{D}_k(\mathbf{x})^T\mathbf{c}\}. \tag{3.15}$$

and Equation 3.10 becomes:

$$
\begin{aligned}
& E[V'(\mathbf{X}(t+1)) - V'(\mathbf{X}(t))|\mathbf{X}(t) = \mathbf{x}] \\
\leq & \sum_{k=1}^{N_T} \left[\left(-2 \max_{\mathbf{c} \in \mathcal{S}_k} \left(\mathbf{D}_k(\mathbf{x})^T \mathbf{c} \right) + 2(\mathbf{D}_k(\mathbf{x}))^T \sum_{i=1}^{|\mathcal{S}_k|} (\lambda_k)_i (\mathbf{c}_k)_i \right) p_k + b_k \right] \\
\leq & \sum_{k=1}^{N_T} \left[\left(-2 \max_{\mathbf{c} \in \mathcal{S}_k} \left(\mathbf{D}_k(\mathbf{x})^T \mathbf{c} \right) + 2 \max_{\mathbf{c} \in \mathcal{S}_k} \left((\mathbf{D}_k(\mathbf{x}))^T \mathbf{c} \right) \sum_{i=1}^{|\mathcal{S}_k|} (\lambda_k)_i \right) p_k \right. \\
+ & \left. b_k \right] \\
= & \sum_{k=1}^{N_T} \left[\left(-2 \left(1 - \sum_{i=1}^{|\mathcal{S}_k|} (\lambda_k)_i \right) \max_{\mathbf{c} \in \mathcal{S}_k} \left((\mathbf{D}_k(\mathbf{x}))^T \mathbf{c} \right) \right) p_k + b_k \right]
\end{aligned}
\tag{3.16}
$$

We need to show that whenever the queue sizes increase above some threshold, the right hand side of Equation 3.16 becomes strictly negative. This would guarantee that the drift of the expected value of the Lyapunov function becomes strictly negative and bounded away from 0 for large queue sizes. Hence, we proceed to prove that whenever $V'(\mathbf{x}) \geq v$, then it must hold that for at least one topology k the

$$
\max_{\mathbf{c} \in \mathcal{S}_k} \left(\mathbf{D}'_k(\mathbf{x})^T \mathbf{c} \right) > h(v),
$$

where $h(v)$ is a positive, increasing and unbounded function of v.

Let the queue sizes be large enough so that $V'(\mathbf{x}) \geq v$. Then,

$$
\begin{aligned}
v \leq & \; V'(\mathbf{x}) \\
= & \sum_{j=1}^{J} \sum_{i=1}^{N} \{x_{ij}\}^2 \\
\leq & \sum_{j=1}^{J} N \max_{i=1,\dots,N} \{x_{ij}\}^2 \\
\leq & \; NJ \max_{i=1,\dots,N} \max_{j=1,\dots,J} \{x_{ij}\}^2
\end{aligned}
\tag{3.17}
$$

Therefore,

$$\max_{i=1,\dots,N} \max_{j=1,\dots,J} \{x_{ij}\} \geq \sqrt{\frac{v}{JN}}. \tag{3.18}$$

Let us suppose now that the queue that achieves this maximum value is the $i^\star$ and this happens for customers of class $j^\star$. In other words:

$$(i^\star, j^\star) = arg \max_{i=1,\dots,N, j=1,\dots,J} \{x_{ij}\}$$

The queue size that corresponds to node $i^\star$ and class $j^\star$ is $\mathbf{x}_{i^\star,j^\star}$. By making use of the assumption on the topology process (Assumption 2), we know that if we take the union of all possible topologies there will exist n links $\ell_i \in \bigcup_{k=1}^{N_T} L(k)$, for some $n > 0$, such that for some nodes $i^\star, i_1, \dots, i_n$ $s(\ell_1) = i^\star, s(\ell_{m+1}) = i_m$ and $d(\ell_{m+1}) = i_{m+1}, m, \in \{1, \dots, n-1\}$ and $i_n \in V_{j^\star}$, where $n \leq N$. Hence, $x_{i^\star,j^\star}$ can be rewritten to the equivalent:

$$x_{i^\star j^\star} = (x_{i^\star j^\star} - x_{i_1 j^\star}) + (x_{i_1 j^\star} - x_{i_2 j^\star}) + \dots + (x_{i_{n-1} j^\star} - x_{i_n j^\star}) + x_{i_n j^\star}$$

Therefore we have,

$$\begin{aligned} \Delta = & \max_{m=1,\dots,n-1} \{x_{i^\star j^\star} - x_{i_1 j^\star}, x_{i_m j^\star} - x_{i_{m+1} j^\star}, x_{i_n j^\star}\} \\ \geq & \frac{x_{i^\star j^\star}}{n} \\ \geq & \frac{x_{i^\star j^\star}}{N} \\ \geq & \frac{1}{N}\sqrt{\frac{v}{JN}} \end{aligned} \tag{3.19}$$

Let the above maximum be achieved for some link $\ell_{(k')} \in L(k')$ for topology $T_{k'}$. Now let us look at topology $T_{k'}$ and look at link $\ell_{(k')}$. Let $i_1 = s(\ell_{(k')})$ and $i_2 = d(\ell_{(k')})$. Let also $\Delta = x_{i_1 j^\star} - x_{i_2 j^\star}$, for the customer class $j^\star$ described above. Then we get:

$$\begin{aligned} \Delta = & \; x_{i_1 j^\star} - x_{i_2 j^\star} \\ \leq & \max_{j=1,\dots,J} \{x_{i_1 j} - x_{i_2 j}\} \end{aligned}$$

Therefore for topology $T_{k'}$, we must have:

$$\begin{aligned}
&\max_{\mathbf{c}\in\mathcal{S}_{k'}}\left(\mathbf{D}_{k'}(\mathbf{x})^T\mathbf{c}\right) \\
&\qquad \geq \max_{\mathbf{c}\in\mathcal{S}_{k'}}\left((\mathbf{D}_{k'}^{j^\star}(\mathbf{x}))^T\mathbf{c}\right) && (3.20)\\
&\qquad \geq \Delta m_{k'\ell_{k'}} && (3.21)\\
&\qquad \geq \frac{m_{k'\ell_{k'}}}{N}\sqrt{\frac{v}{JN}} && (3.22)
\end{aligned}$$

Inequality 3.20 follows from the fact that $D_{k'\ell}(\mathbf{x}) \geq D_{k'\ell j^\star}(\mathbf{x})$. Inequality 3.21 follows from the fact that any active single link (in this case $\ell_{k'}$,) is a valid activation set in itself. The inequality 3.22 come from equation 3.19.

Hence we obtain that:

$$V'(\mathbf{x}) \geq v \Rightarrow -\max_{\mathbf{c}\in\mathcal{S}_{k'}}\left(\mathbf{D}_{k'}(\mathbf{x})^T\mathbf{c}\right) \leq -\frac{m_{k'\ell_{k'}}}{N}\sqrt{\frac{v}{JN}}$$

And therefore we have:

$$\begin{aligned}
&E[V'(\mathbf{X}(t+1)) - V'(\mathbf{X}(t))|\mathbf{X}(t)=\mathbf{x}] \\
&\leq \sum_{k=1}^{N_T}\left(-2\left(1-\sum_{i=1}^{|\mathcal{S}_k|}(\lambda_k)_i\right)\max_{\mathbf{c}\in\mathcal{S}_k}\left((\mathbf{D}_k(\mathbf{x}))^T\mathbf{c}\right)\right)p_k + \sum_{k=1}^{N_T} b_k \\
&\leq -2\left(1-\sum_{i=1}^{|\mathcal{S}_{k'}|}(\lambda_{k'})_i\right)\max_{\mathbf{c}\in\mathcal{S}_{k'}}\left((\mathbf{D}_{k'}(\mathbf{x}))^T\mathbf{c}\right)p_{k'} + \sum_{k=1}^{N_T} b_k \\
&\leq -2\left(1-\sum_{i=1}^{|\mathcal{S}_{k'}|}(\lambda_{k'})_i\right)\frac{m_{k'\ell_{k'}}}{N}\sqrt{\frac{v}{JN}}p_{k'} + \sum_{k=1}^{N_T} b_k \\
&\leq -2\min_{k=1,\dots,N_T}\left\{1-\sum_{i=1}^{|\mathcal{S}_k|}(\lambda_k)_i\right\}\frac{m_{(k')\ell_{k'}}}{N}\sqrt{\frac{v}{JN}}p_{k'} + \sum_{k=1}^{N_T} b_k \\
&\leq -2\min_{k=1,\dots,N_T}\left\{1-\sum_{i=1}^{|\mathcal{S}_k|}(\lambda_k)_i\right\}\frac{1}{N}\sqrt{\frac{v}{JN}}(\min_{k=1,\dots,N_T} p_k)(\min_{k=1,\dots,N_T,\ell_k\in L(k)}\{m_{k\ell_k}\}) \\
&+ \sum_{k=1}^{N_T} b_k \qquad (3.23)
\end{aligned}$$

Each of the above minimizations are achieved at strictly positive values. Hence, we observe that if we select v sufficiently large, we can make the right hand side of Equation 3.23 strictly negative and bounded away from zero. The variable v must be selected in such a way so that the first term (negative) must overcome the summation of the 2 positive terms. Then, we have proved that

For the mobile network with i.i.d topology processes, scheduled by policy π_0, and for arrival rates satisfying Equation 3.1, there exist a $v > 0$ and $\epsilon > 0$ such that, if $V'(\mathbf{x}) \geq v$ then

$$E[V'(\mathbf{X}(t+1)) - V'(\mathbf{X}(t))|\mathbf{X}(t) = \mathbf{x}] \leq -\epsilon \tag{3.24}$$

Also, for any finite arrival rate vector $\mathbf{a}$, it is straightforward to see that,

$$E[V'(\mathbf{X}(t+1))|\mathbf{X}(t) = \mathbf{x}] < \infty, \forall \mathbf{x} \in \mathcal{X}.$$

Hence the time varying network, in which topologies change in a i.i.d fashion is stable for these arrival rates. $\diamondsuit$

An interesting fact that emerges from the above analysis is that the above holds under relaxed conditions on the network connectivity. It is not necessary that all the individual topologies have to be connected for stability. Assumption 2 concerning connectivity of the union of the topologies is sufficient. With a non-zero probability the network can become completely disconnected and remain stable, provided the arrival rate vectors satisfy the constraints of equation 3.1.

3.2.5 The region of stable arrival rates under any policy is contained in $\tilde{\mathbf{A}}$

In this section we are going to prove that inequality (4) holds, i.e that any arrival rate that is stable under some stationary policy must belong in the set $\tilde{\mathbf{A}}$. Towards this end we state and prove the following lemma.

Lemma 6 *If the system is stable under some stationary policy π, i.e. $\mathbf{a} \in \mathbf{C}$, then $\mathbf{a} \in \tilde{\mathbf{A}}$.*

Proof: Consider an arrival rate vector $\mathbf{a} \in \mathbf{C}$, *i.e* the system be stable under some stationary policy $\tilde{\pi}$. If the Markov Chain $\{\mathbf{X}(t), T(t+1)\}_{t=0}^{\infty}$ that represents the state of our system, can either be in a positive recurrent state of some recurrent class or in a transient state. Suppose that the system starts in some positive recurrent class, then it will remain there forever. Suppose, on the other hand, that the system starts in one of the transient states. Then, by Definition 1, the system will have to leave the set of transient states in finite time with probability 1 and land itself into one of the positive recurrent classes from $Z = \cup_{i=1}^{\infty} Z_i$. Hence, without loss of generality we may assume that the Markov Chain $\{\mathbf{X}(t), T(t+1)\}_{t=0}^{\infty}$ will be restricted to one of the recurrent classes $Z_i, i = 1, 2, \ldots$, that is positive recurrent and hence ergodic.

By Assumption 3, the service completion matrix of the servers, $\mathbf{M}(t)$, given the current topology $T(t)$ is independent of $\mathbf{X}(0), \ldots, \mathbf{X}(t-1)$ because they only depend on the current topology. Hence we can show the following simple claim.

Claim 1 *The triplet* $\{\mathbf{X}(t), T(t+1), \mathbf{M}(t+1)\}_{t=0}^{\infty}$ *is a Markov Chain.*

Proof:

$$
\begin{aligned}
&P[\mathbf{X}(t), \mathbf{M}(t+1), T(t+1) | \mathbf{X}(t-1), \mathbf{M}(t), T(t), \mathbf{X}(t-2), \mathbf{M}(t-1), T(t-1) \ldots] = \\
&= P[\mathbf{M}(t+1) | \mathbf{X}(t), T(t+1), \mathbf{X}(t-1), \mathbf{M}(t), T(t), \mathbf{X}(t-2), \mathbf{M}(t-1), T(t-1) \ldots] \\
&= P[\mathbf{M}(t+1) | T(t+1)] P[\mathbf{X}(t), T(t+1) | \mathbf{X}(t-1), \mathbf{M}(t), T(t)] \\
&= P[\mathbf{M}(t+1) | \mathbf{X}(t), T(t+1), \mathbf{X}(t-1), \mathbf{M}(t), T(t)] P[\mathbf{X}(t), T(t+1) | \mathbf{X}(t-1), \mathbf{M}(t), T(t)] \\
&= P[\mathbf{X}(t), \mathbf{M}(t+1), T(t+1) | \mathbf{X}(t-1), \mathbf{M}(t), T(t)]
\end{aligned}
$$

Therefore, $\{\mathbf{X}(t), T(t+1), \mathbf{M}(t+1)\}_{t=0}^{\infty}$ is a Markov Chain, which is ergodic if $\{\mathbf{X}(t), T(t+1)\}_{t=0}^{\infty}$ is ergodic. Now consider the quantity $\mathbf{F}^j(t) = \mathbf{M}(t)\tilde{\pi}^j(\mathbf{X}(t-1), T(t))$. Let us also define

$$\mathbf{f}_k^j \triangleq E[\mathbf{F}^j(t) | T(t) = T_k]$$

We are going to show that $\mathbf{f}_k^j$ satisfy the conditions in the definition of $\tilde{\mathbf{A}}$ and thus prove that $\mathbf{a} \in \tilde{\mathbf{A}}$. Consider some node i and some customer class j. Since the system is stable, packets do not accumulate at any node. Therefore, the limiting average number of external arrivals at node i of class j, in addition with the internal arrivals of class j due to scheduling, are equal to the average number of internal departures of customers of class j. Let i be such that $i \notin V_j$. Then:

$$
\begin{aligned}
a_i^j = & \\
\stackrel{a.s}{=} & \lim_{\tau\to\infty} \{\frac{1}{\tau}\sum_{t=1}^{\tau} \mathbf{A}_i^j(t)\} \text{By stationarity of arrival process} \\
= & \lim_{\tau\to\infty} \frac{1}{\tau}\sum_{t=1}^{\tau} \left[\sum_{\ell:s(\ell)=i} F_\ell^j(t) - \sum_{\ell:d(\ell)=i} F_\ell^j(t) \right] \\
= & \lim_{\tau\to\infty} \frac{1}{\tau}\sum_{t=1}^{\tau} \left[-\sum_{\ell:s(\ell)=i} R_{i\ell}^j F_\ell^j(t) - \sum_{\ell:d(\ell)=i} R_{i\ell}^j F_\ell^j(t) \right] \\
= & \lim_{\tau\to\infty} -\frac{1}{\tau}\sum_{t=1}^{\tau} \left[\sum_{\ell\in L} R_{i\ell}^j F_\ell^j(t) \right]
\end{aligned}
$$

In vector notation we will have,

$$
\begin{aligned}
\mathbf{a}^j \stackrel{a.s.}{=} & \lim_{\tau\to\infty} -\frac{1}{\tau}\sum_{t=1}^{\tau} \left[\mathbf{R}^j\mathbf{F}^j(t)\right] \\
= & -\mathbf{R}^j \lim_{\tau\to\infty} \frac{1}{\tau}\sum_{t=1}^{\tau} \left[\mathbf{F}^j(t)\right]
\end{aligned}
\tag{3.25}
$$

The quantity $\sum_{t=1}^{\tau} \mathbf{F}^j(t)$ indicates how many customers of class j have crossed each server during time $[1,t]$. Since $\{\mathbf{X}(t-1), \mathbf{M}(t), T(t)\}_{t=0}^{\infty}$ is an ergodic Markov Chain, any function of an ergodic Markov Chain is ergodic well, we know that $\mathbf{F}^j(t)$ will be ergodic. Hence:

$$
\frac{1}{\tau}\sum_{t=1}^{\tau} \mathbf{F}^j(t) \stackrel{a.s}{\to} E[\mathbf{F}^j(t)]
$$

We have that:

$$\begin{aligned} E[\mathbf{F}^j(t)] &= E[E[\mathbf{F}^j(t)|T(t)]] \\ &= \sum_{k=1}^{N_T} E[\mathbf{F}^j(t)|T(t)=T_k]p_k \\ &= \sum_{k=1}^{N_T} \mathbf{f}_k^j p_k \end{aligned}$$

and Equation 3.25 becomes:

$$\begin{aligned} \mathbf{a}_i^j \overset{a.s}{\longrightarrow} & -\mathbf{R}^j E[\mathbf{F}^j(t)] \text{ by ergodicity} \\ = & -\sum_{k=1}^{N_T} \mathbf{R}^j \mathbf{f}_k^j p_k \end{aligned}$$

Now we need to check whether all the conditions on $\mathbf{f}_k^j$ in the definition of $\tilde{\mathbf{A}}$ are satisfied.

- $\mathbf{f}_k^j = E[\mathbf{M}(t)\tilde{\pi}^j(\mathbf{X}(t-1), T(t))|T(t) = T_k] \geq 0$, as an expectation of a non-negative quantity.

- Consider

$$\begin{aligned} \sum_{j=1}^{J} \mathbf{f}_k^j &= \sum_{j=1}^{J} E[\mathbf{M}(t)\tilde{\pi}^j(\mathbf{X}(t-1), T(t))|T(t) = T_k] \\ &= \sum_{j=1}^{J} \mathbf{M}_k E[\tilde{\pi}^j(\mathbf{X}(t-1), T_k)|T(t) = T_k] \qquad (3.26) \\ &= \mathbf{M}_k E\left[\sum_{j=1}^{J} \tilde{\pi}^j(\mathbf{X}(t-1), T_k)|T(t) = T_k\right] \end{aligned}$$

  Equation 3.26 follows from Assumption 3. In addition, since $\tilde{\pi}$ is a valid scheduling policy,

$$\sum_{j=1}^{J} \tilde{\pi}^j(\mathbf{X}(t-1), T_k) \in \mathcal{S}_k$$

  and therefore

$$\mathbf{M}_k E\left[\sum_{j=1}^{J} \tilde{\pi}^j(\mathbf{X}(t-1), T_k)|T(t) = T_k\right] \in co(\mathbf{M}_k \mathcal{S}_k).$$

  Hence: $\sum_{j=1}^{J} \mathbf{f}_k^j \in co(\mathbf{M}_k \mathcal{S}_k)$.

Therefore, we showed that there exist flows $\mathbf{f}_k^j \geq 0$, that satisfy $\sum_{j=1}^{J} \mathbf{f}_k^j \in co(\mathbf{M}_k \mathcal{S}_k)$, and such that $\mathbf{a}^j = -\sum_{k=1}^{N_T} \mathbf{R}^j \mathbf{f}_k^j p_k, \forall j$. This implies that $\mathbf{a} \in \tilde{\mathbf{A}}$, which completes the proof. $\diamondsuit$

So, we proved that the arrival rate vector $\mathbf{a}$ must belong in $\mathbf{a} \in \tilde{\mathbf{A}}$. We can conclude therefore, that the region $\tilde{\mathbf{A}}$ contains indeed the network stability region obtained by the set of all stationary scheduling policies.

3.2.6 Time sharing based policy

In this section we are going to show that a policy that performs time sharing between network configurations is suboptimal with respect to maximizing the network's stable throughput and that the proposed scheduling policy performs at least as well. In other words, we will show that equality (1) of Equation 3.3 holds.

Clearly, $\mathbf{A}_0$ are the arrival rates obtained by time sharing of individual networks. Now we proceed to show that the capacity region is larger than the linear combination of stability regions of individual topologies.

Claim 2 *Claim* $\mathbf{A}_0 \subseteq \mathbf{A}^\star$

It is easy to see that if $\mathbf{a} \in \mathbf{A}_0$, then $\mathbf{a} = \sum_{k=1}^{N_T} \mathbf{a}_k p_k$, $\mathbf{a}_k \in CapacityRegion(T_k)$. Therefore

$$\mathbf{a}_k^j = -\mathbf{R}^j \mathbf{f}_k^j, \tag{3.27}$$

which follows from [14], and hence:

$$\mathbf{a}^j = -\sum_{k=1}^{N_T} \mathbf{R}^j \mathbf{f}_k^j p_k \tag{3.28}$$

and conditions of belonging to $\mathbf{A}^\star$ are satisfied. Therefore:

$$\mathbf{a} \in \mathbf{A}_0 \Rightarrow \mathbf{a} \in \mathbf{A}^\star$$

and hence:

$$\mathbf{A}_0 \subseteq \mathbf{A}^\star$$

We will show later, through the example of section 3.3, that the region of arrival rates $\mathbf{A}_0$ can be a strict subset of the region $\mathbf{A}^\star$

3.3 Example of Mobile network with two configurations

Consider a 3 node network with two configurations T_0 and T_1 as shown in Figure 3.2. The network exists in one of the two configurations at any time and switches between the two according to a two state homogeneous Markov Chain with equiprobable stationary distribution ($p_0 = p_1 = 0.5$). Let us consider perfect links that allow perfect communication, in other words links for which the service rate of the corresponding server is 1. In network configuration T_0 the network is such that a perfect communication is possible only from nodes 1 to 2 and no communication is possible between nodes 2 and 3. In network configuration T_1, perfect communication is only possible from node 2 to 3. Assume that there is a single class of customers that will exit the system as soon as they reach node 3. Let a_1, a_2 denote the arrival rates at nodes 1 and 2 for node 3 . As there is only one destination class, we shall drop the superscript j indicating the destination class from the discussion in this section. As there is only one possible active link in both configurations, the activations sets $\mathcal{S}_0$ and $\mathcal{S}_1$ contain singleton elements, that is, $\mathcal{S}_0 = \{[1,0]\}$ and $\mathcal{S}_1 = \{[0,1]\}$

Then the matrix $\mathbf{R}$ is a 3×2 matrix indicating the routing matrix under the union of topologies. Each row of the matrix corresponds to a node. Each column corresponds to a link, $\ell \in L$. Then we have $\mathbf{R} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}$. Consider the stability region of configuration T_0. Clearly the only flow vector $\mathbf{f}_0$ such that $-\mathbf{R}\mathbf{f}_0 \geq 0$ and $(\mathbf{f}_0)_2 = 0$ is the vector $[0,0]$. Hence the stability region for

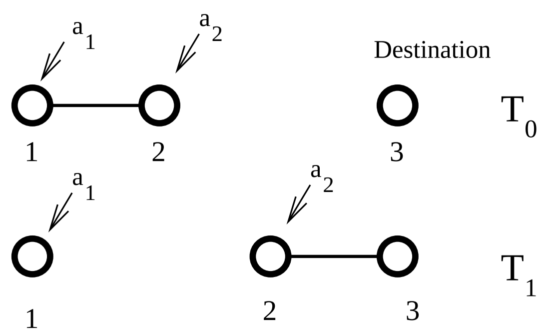

Figure 3.2: A 3 node network with two network configurations, T_0 and T_1. Arrivals occur at nodes 1 and 2 and will exit the network at node 3.

configuration T_0 contains only the point $[a_1, a_2]^T = [0, 0]^T$. This is also obvious by looking at the network configuration, as no positive arrival rates to destination 3 are stable.

Similarly, consider the stability region of configuration T_1. Any valid flow vector for configuration T_1, $\mathbf{f}_1$ which gives $-\mathbf{R}\mathbf{f}_1 \geq 0$ must be of the form $[0, \alpha]^T$ where $0 \leq \alpha < 1$. Hence the capacity region for network T_1 contains points of the form $-\mathbf{R}[0, \alpha]^T = [0, \alpha]^T$. Therefore, the stability region or configuration T_1 contains only a line $[a_1, a_2]^T \in \{[0, \alpha]^T : 0 \leq \alpha < 1\}$.

So, we deduce that the region $\mathbf{A}_0$ obtained by time sharing of the two networks is given by $[a_1, a_2]^T \in \{0.5[0, 0]^T + 0.5[0, \alpha]^T : 0 \leq \alpha < 1\}$, which is a vertical line of length 0.5 through the origin.

Now consider the achievable region $\mathbf{A}^\star$ for this network.

It is given as:

$$\mathbf{A}^\star = \{[a_1, a_2]^T : [a_1, a_2]^T = -\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} (0.5\mathbf{f}_0 + 0.5\mathbf{f}_1) \geq 0\} \tag{3.29}$$

where each element of $\mathbf{f}_{(i)}, i = 0, 1$ lies in $[0, 1)$. Also, by constraints on the activation set, $\mathbf{f}_0$ must be of the form $[\alpha, 0]^T$ and $\mathbf{f}_1$ must be of the form $[0, \beta]^T$. It is straightforward to see that the

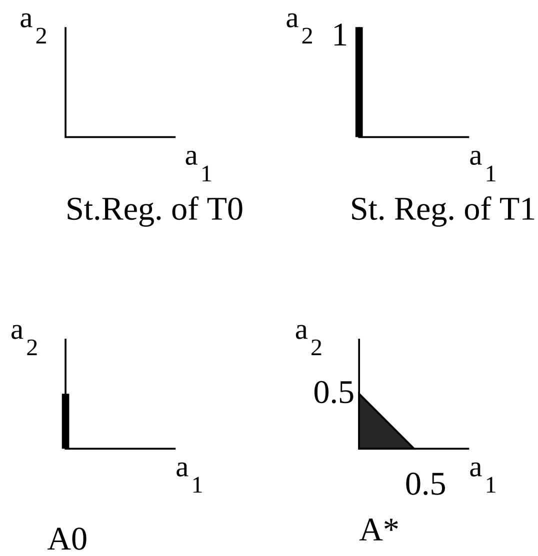

Figure 3.3: Stability regions of each configuration, sets $\mathbf{A}_0$ and $\mathbf{A}^\star$ for the 3 node network.

elements of $\mathbf{A}^\star$ are of the form $[0.5\alpha, -0.5\alpha + 0.5\beta, 0]$ where $0 \leq \alpha, \beta < 1$. The necessary and sufficient additional condition for these entries to be nonnegative is $\beta \geq \alpha$, we can ensure that each of the elements is nonnegative. Figure 3.3 shows $\mathbf{A}^\star$. Clearly this region is a strict superset of the region $\mathbf{A}_0$ and positive arrival rates on node 1 can be delivered to node 3 by using node 2 as a relay.

As an evidence of what we argued earlier, note that, for any $\alpha > 0$, the term $-\mathbf{R}\mathbf{f}_0$ has a negative element, which indicates a nonzero "departure rate" at node 2, for destination node, in configuration T_0.

3.4 Conclusions

In this chapter we found an optimal stable scheduling policy for a network which changes its topology according to an i.i.d topology process and obtained the achievable rate region. By the discussion of stationary and ergordic topology processes, discussed in chapter 2, if follows that this region is achievable in this more general case as well. Then we showed that any rate outside the specified region is unstable, which implies that indeed this is the stability region of the network. Moreover, we elaborated that the capacity region of the mobile network is strictly better than the one resulting from a linear combination of stationary networks.

Chapter 4

Simulations Results

In this chapter, we introduce our simulation tool and present simulation results for networks whose topology varies in a random fashion. Our ultimate aim is to verify that mobility increases the maximum stable throughput that the network can sustain. In other words, even when nodes move and the network evolves exclusively within disconnected topologies, we exhibit that the proposed scheduling policy achieves a throughput that is strictly positive, provided that Assumption 2 is satisfied.

4.1 Simulation Tool

The simulating tool has been developed in C language. It simulates the scheduling policy of section 2.5, operating on a mobile environment. It is able to model mobility that results in 2 types of topology processes: Topologies that change in a i.i.d manner and a hidden Markov stationary and ergodic topology process, i.e. when nodes move based on a random walk mobility pattern. We focus our attention in i.i.d varying topologies. In addition, although our tool can model servers (links) that fail to serve the customer under service, we restrict our attention to perfect servers. This assumption makes it more intuitive to understand and explain the simulation results. Finally, the arrival process is Bernoulli.

4.2 Simulation Algorithm

We are using a "Monte Carlo" [1] based simulation procedure in order to characterize an arrival rate vector as stable or unstable. The basic idea of our algorithm is to start from large valued arrival rate vectors, that we expect to be unstable, and decrease one coordinate by a step, at each simulation run. So, as long as the arrival rate vector is unstable, one of its coordinates is decreased again, and the procedure is repeated until a stable arrival rate vector is found. Each simulation run is bounded by a maximum time slot, at which point the corresponding arrival rate vector is accounted for stability. In our simulations, stability is estimated through a threshold criterion. In other words, if one or more of the network queues exceeds a threshold value, then the corresponding arrival rate vector is characterized as "unstable".

We ran our simulations, for a simulation run of size 3000 time slots, a step of value 0.05 and a threshold for stability check set to 50 packets/queue. Moreover, in the figures that follow, the obtained arrival rates are characterized and separated by plotting the stable arrival rates in red "$*$" and the unstable ones, by blue dots.

4.3 Network Scenarios

In the sequel, we will show through a set of examples what is the maximum stable throughput a network can sustain, when the scheduling is performed according to the proposed scheduling policy. We are using small networks that will help us explain and understand the obtained results

[1] Monte Carlo is a stochastic technique, which solves a mathematical problem, usually too complicated to be solved analytically. It is called stochastic since it generates suitable random numbers and uses probability statistics to come up with an answer. The random selection process is repeated many times and each time a new scenario and a solution to the problem are created. The collection of all scenarios, give a range of possible solutions that can be characterized by the properties they satisfy and that have different probabilities of occurrence.

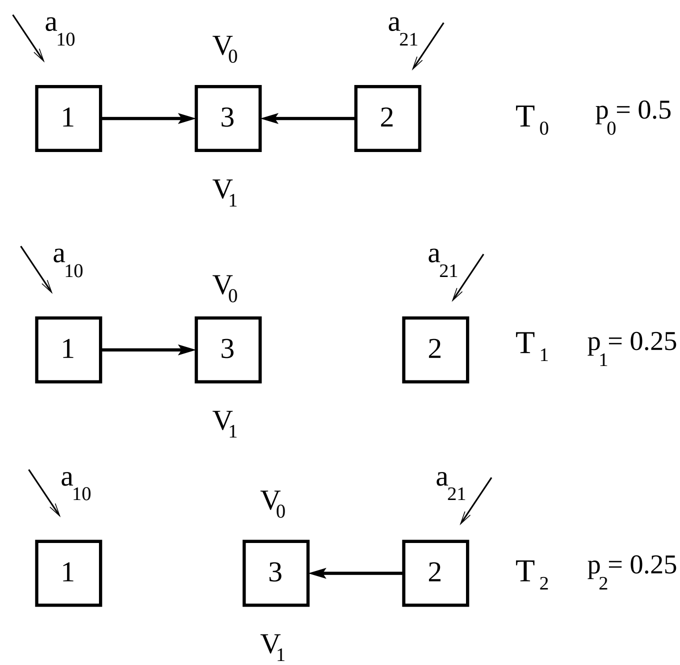

Figure 4.1: A 3 node network that switches among 3 topologies. Traffic of rate a_{10} at node 1 and of rate a_{21} at node 2 are both to be delivered to exit node 3.

by using intuitive arguments.

4.3.1 A 3 node network that switches among 3 topologies

In this section, we examine what the stability region of the network depicted in Figure 4.1 looks like. In this network, we consider two customer classes. Customer class 0, of rate a_{10}, enters the network at node 1 and exits the network at node 3. Similarly, customer class 1, of rate a_{21}, enters the network at node 2 and exits at node 3. Both nodes 1 and 2 compete to send their traffic to node 3, but due to the medium access primary constraints, not both of them can transmit simultaneously to their intended exit nodes. The network switches among 3 topologies, T_0, T_1 and T_2, with stationary probabilities $p_0 = 0.5$, $p_1 = 0.25$ and $p_2 = 0.25$.

Let us first examine the network throughput of a fixed network taking one of topologies $T_i, i =$

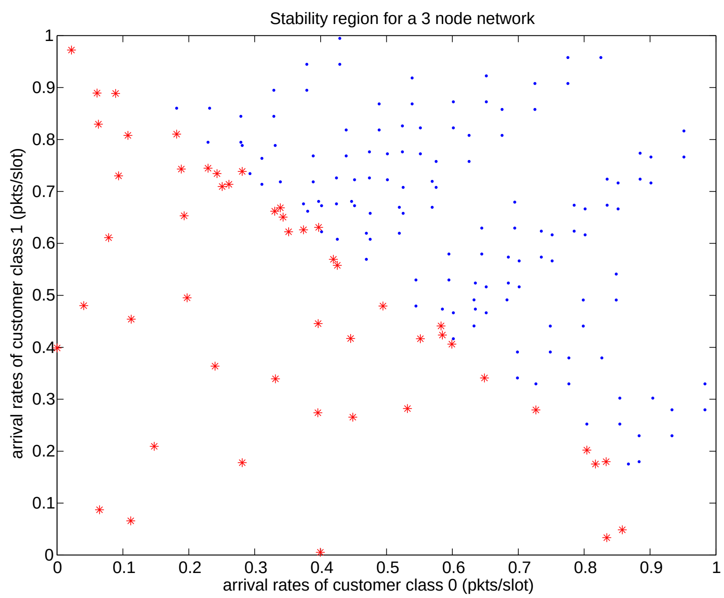

Figure 4.2: Maximum stable throughput of a stationary network, that has topology T_0. Arrivals of rates a_{10} and a_{21} occur at nodes 1 and 2 respectively and are intended for exit node 3.

$0, 1, 2$. The maximum stable throughput, obtained by our proposed scheduling policy, for a stationary network of topology T_0, T_1 and T_2 is depicted in Figures 4.2, 4.3 and 4.4 respectively. In Figure 4.2 the set of stable arrival rates are the ones that lie in the positive quadrant and are upper bounded by the line $a_{10} + a_{21} = 1$. Else, the arrival rate region is obtained by the intersection of half spaces $a_{10} \geq 0$, $a_{21} \geq 0$, $a_{10} \leq 1$, $a_{21} \leq 1$ and $a_{10} + a_{21} = 1$. The boundary point $(1, 0)$ is obtained by setting the rate $a_{21} = 0$ at node 2, which would allow node 1 to send packets at node 3 with a rate of 1. Similarly, the point $(0, 1)$ is obtained by setting $a_{10} = 0$ and allowing node 2 to transmit at a maximum rate of 1. When the network topology is T_1, node 2 is not connected to its exit node at any time instant, hence the maximum stable arrival rate that it can deliver is 0 (Figure 4.3). The same observation holds for node 1 under topology T_2 (Figure 4.4). When the network switches among topologies, the corresponding maximum stable throughput region is depicted in Figure 4.5. Now, both nodes 1 and 2 can reach their exit node for a maximum of 0.75 fraction of

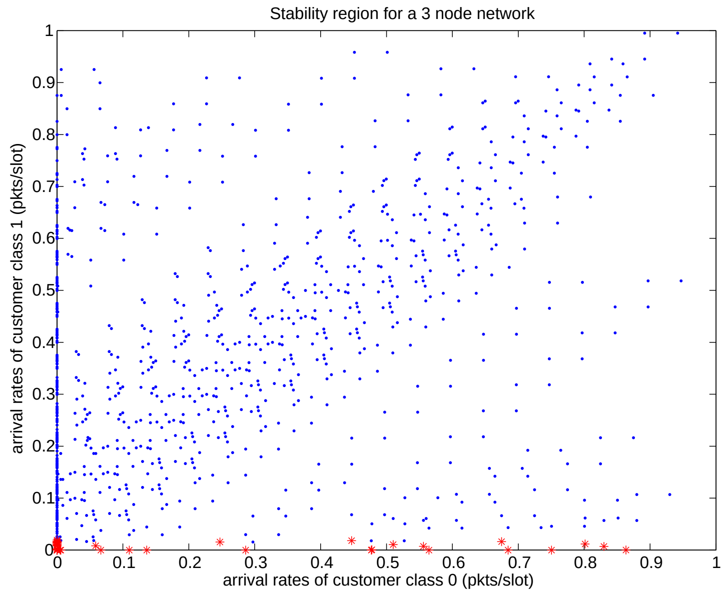

Figure 4.3: Maximum stable throughput of a stationary network, that has topology T_1. Arrivals of rates a_{10} and a_{21} occur at nodes 1 and 2 respectively and are intended for exit node 3.

time. Therefore, the stability region is obtained as an intersection of the half spaces $a_{10} \leq 0.75$, $a_{21} \leq 0.75$, $a_{10} + a_{21} \leq 1$, $a_{10} \geq 0$ and $a_{21} \geq 0$.

4.3.2 3 nodes in tandem

In this network scenario, we revisit the example discussed in section 3.3 (Figure 4.6). Traffic for customer class 0, of rate a_{10}, enters the network at node 1 and traffic for customer class 1, of rate a_{21} enters the network at node 2, both having 3 as their exit node. The network switches between topology T_0 and T_1, that have stationary probabilities p and $1-p$ respectively. We look at the throughput region of this network, for different values of the stationary distribution p, namely $p = 0.5$, $p > 0.5$ and $p < 0.5$. We observe that when the network takes topology T_0 the maximum stable throughput it can sustain is 0, since the exit node can never be reached (Figure 4.7). When the network operates under topology T_1, traffic that enters the network at node 1 can never reach

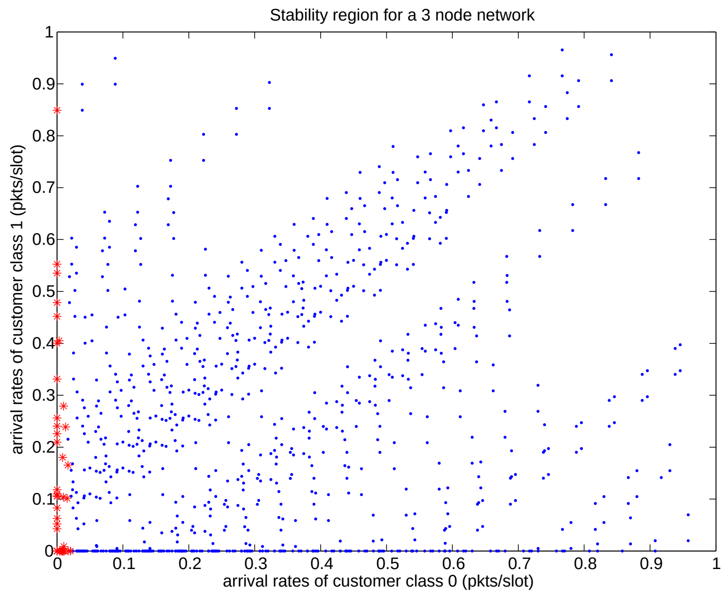

Figure 4.4: Maximum stable throughput of a stationary network, that remains in topology T_2 at all times. Arrivals of rates a_{10} and a_{21} occur at nodes 1 and 2 respectively and are intended for exit node 3.

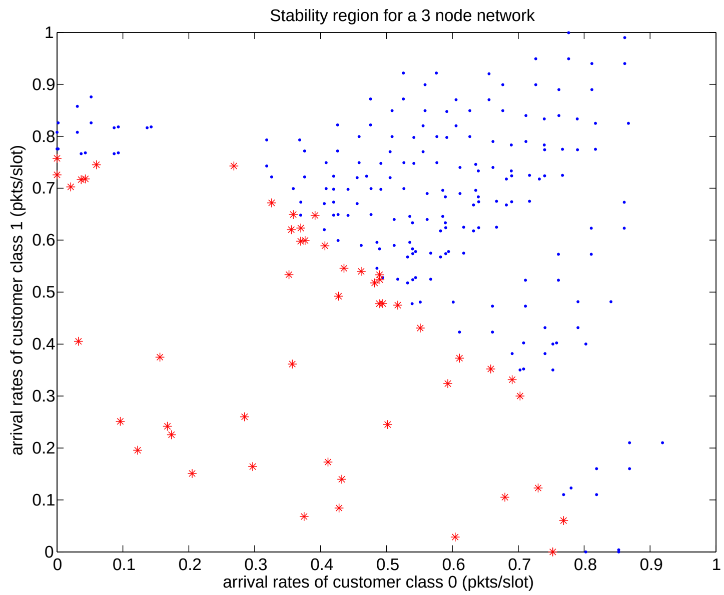

Figure 4.5: Maximum stable throughput region for the i.i.d topology varying network of Figure 4.1.

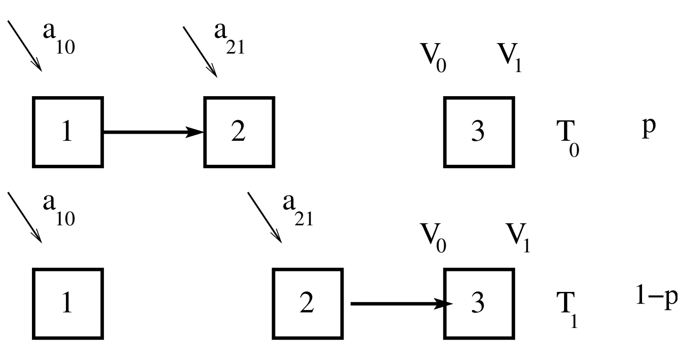

Figure 4.6: A 3 node network in tandem. Traffic is at both nodes 1 and 2 is intended for exit node 3.

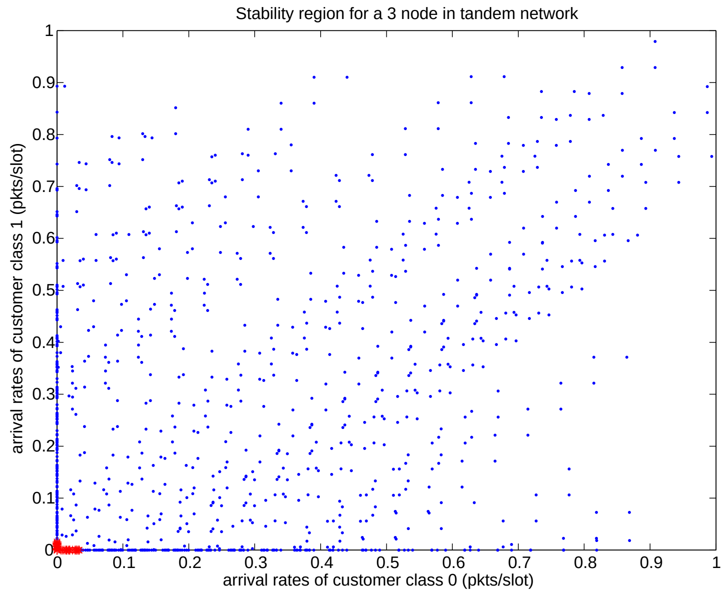

Figure 4.7: Network operates in topology T_0. The exit node 3 is isolated at all times and hence the maximum stable throughput is 0 for both types of traffic.

exit node 3, hence the maximum stable throughput for this rate is $a_{10} = 0$. However, this allows node 2 to transmit at the maximum rate of 1. (Figure 4.8)

- $p = 0.5$

  The maximum throughput region when the network alternates topologies, staying at each one of them equal amount of time is depicted in Figure 4.9. The stability region will be bounded by the intersection of half spaces $a_{10} \geq 0$, $a_{21} \geq 0$, $a_{10} \leq 1$, $a_{21} \leq 1$ and $a_{10} + a_{21} \leq 0.5$. The plot of Figure 4.9 is fairly intuitive. Since the link to exit node 3 is available only 0.5 fraction of time, 0.5 will be the upper bound on stable arrival rates for each type of traffic. A rate of $a_{21} = 0.5$ can be achieved when $a_{10} = 0$. In addition, since node 1 is 2 hops away from the exit node of its traffic, the maximum throughput it can deliver can be 0.5, which is achievable when there are no arrivals at node 2.

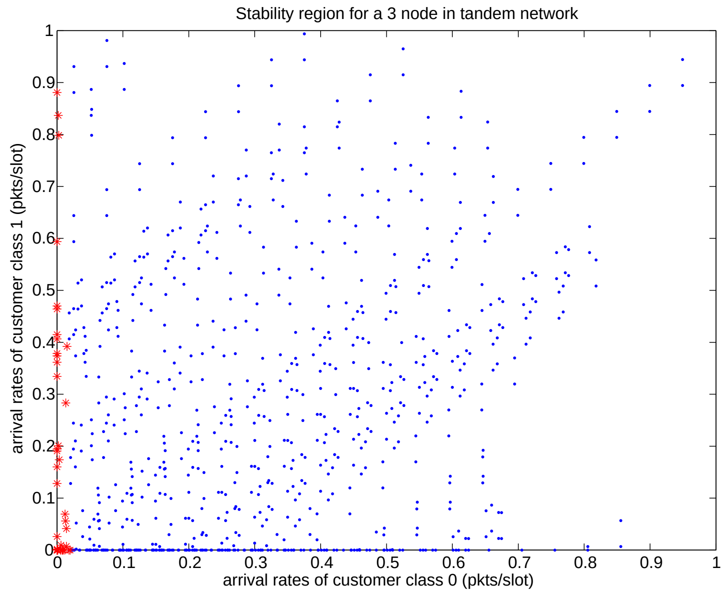

Figure 4.8: Network is in topology T_1. Only packets of of traffic that enters the network at node 1 can reach exit node 3.

- $p > 0.5$

  When topology T_0 occurs more often than topology T_1, e.g. when $p = 0.75$, the stability region of the network is as pictured in Figure 4.10. Since the link to the exit node 3 is only available for 0.25 fraction of time, the maximum stable arrival rate for either traffic type will be upper bounded by 0.25.

- $p < 0.5$

  Topology T_0 occurs less often than topology T_1, e.g. with stationary probability 0.25. The stability region for this case is depicted in Figure 4.11. Since topology T_1 occurs with probability 0.75, the link to exit node 3 is available for 0.75 fraction of time. Hence, node 2 can deliver a maximum stable rate of $a_{21} = 0.75$, when node 1 is silent. On the contrary, a stable rate a_{10} cannot increase more than 0.25, since only for this fraction of time node 1 is allowed

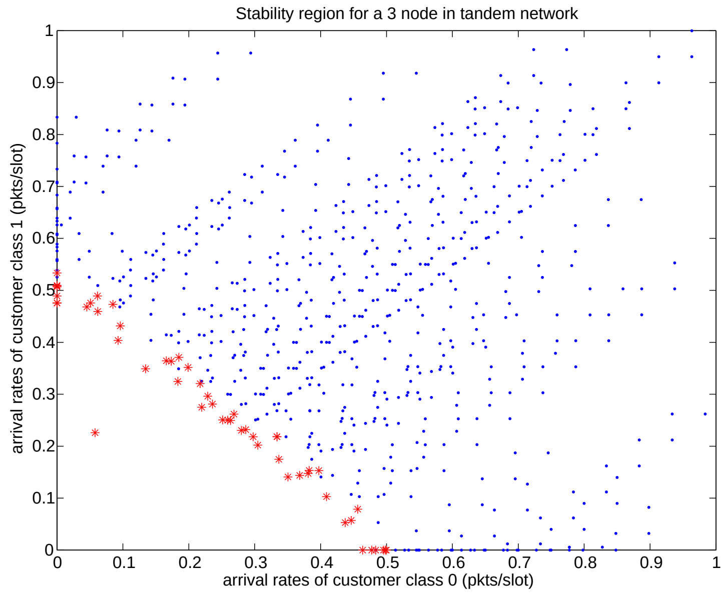

Figure 4.9: Stability region of the 3 nodes in tandem network, when topologies T_0 and T_1 occur equally likely.

to relay its traffic and hence no more than 25% of packets can reach node 3 from node 1.

A similar topology of 3 nodes in tandem, with different traffic characteristics is studied next (Figure 4.12). It is easy to see, that if the network takes exclusively either topology T_0 or T_1, the network throughput would be 0 (Figures 4.13 and 4.14). On the other hand, by allowing the network to switch between the 2 topologies a throughput strictly greater than 0 is achieved. We will again examine the throughput region of this network for different values that the stationary distribution the topology process can take, namely $p = 0.5$, $p > 0.5$ and $p < 0.5$.

- $p = 0.5$

  The maximum stable throughput region of a network that switches between topologies T_0 and T_1, while staying at each one of them equal amount of time, is depicted in Figure 4.15. The stability region of Figure 4.15 is bounded by the intersection of half spaces $a_{10} \geq 0$,

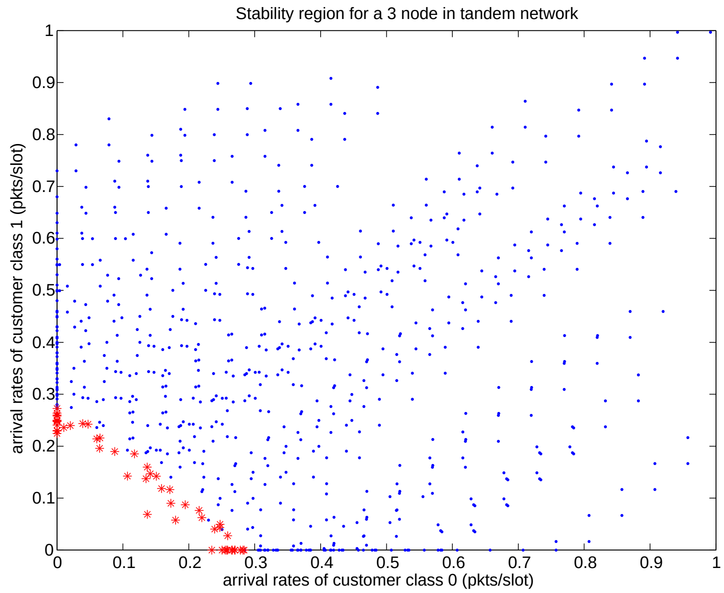

Figure 4.10: Stability region of the 3 nodes in tandem network, when topology T_0 occurs with probability 0.75 and T_1 with probability 0.25.

$a_{31} \geq 0$, $a_{10} \leq 1$, $a_{31} \leq 1$ and $a_{10} + a_{31} \leq 0.5$. Although, the mobile network has the same stability region as the one of Figure 4.9, when topologies T_0 and T_1 occur equally likely, we observe that mobility has benefited the stability of this network even more, since the individual topologies T_0 and T_1 have throughput strictly zero.

- $p > 0.5$ Now, we look at the case where topology T_0 occurs more often than topology T_1, e.g. with probability 0.75. The stability region of this network is presented in Figure 4.16. For this mobile network, the maximum stable arrival rate a_{10} is 0.25, since although node 1 may relay its packets to node 2 for 0.75 fraction of time, the link to the exit node is only available for 0.25 fraction of time. Furthermore, the maximum arrival rate a_{31} is also 0.25 since node 3 can only relay its traffic to node 2 for 0.25 fraction of time. From the above, we obtain that the stability region must be obtained by the intersection of the half spaces $a_{10} \geq 0$, $a_{31} \geq 0$,

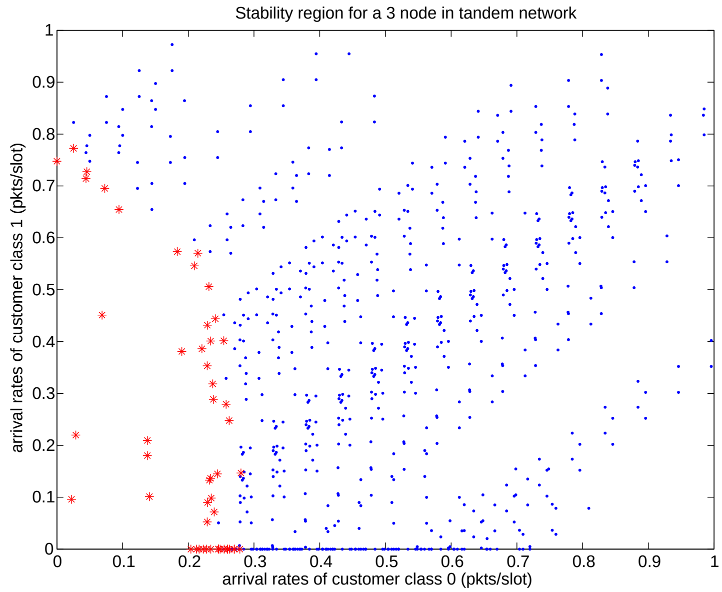

Figure 4.11: Stability region of the 3 nodes in tandem network, when topology T_0 occurs with probability 0.25 and T_1 with probability 0.75.

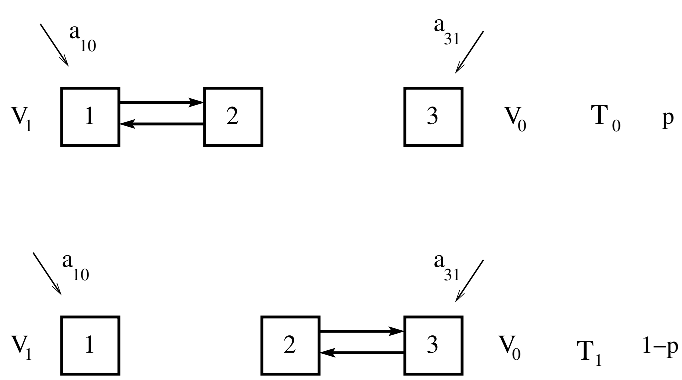

Figure 4.12: A 3 node network in tandem. Traffic, of customer class 0 at node 1 is intended for node 3 and traffic, of customer class 1 at node 3 is to be delivered at node 1.

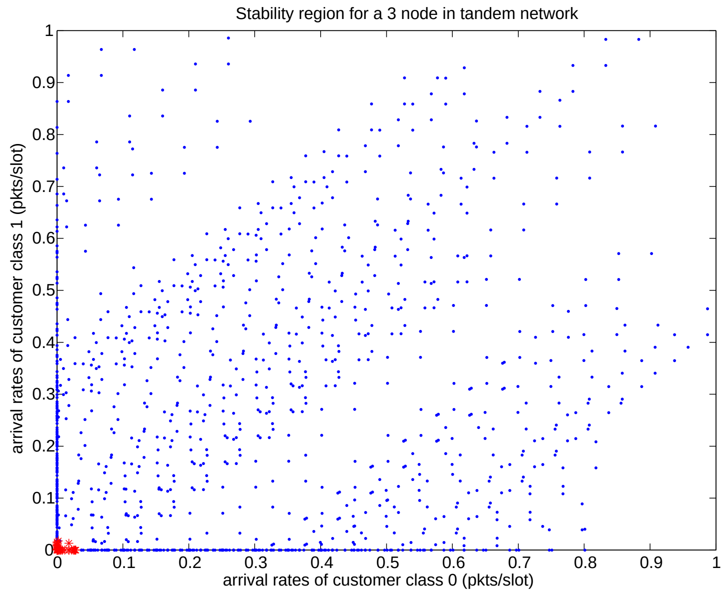

Figure 4.13: The maximum stable throughput of a stationary network that operates under topology T_0 at all time slots is zero. This is because the exit nodes are isolated from all the traffic sources.

$a_{10} \leq 0.25$, $a_{31} \leq 0.25$ and $a_{10} + a_{31} \leq 0.25$, as is also verified by Figure 4.16.

- $p < 0.5$ In case that topology T_0 occurs with less frequently than topology T_1, the stability region will be exactly the same as the previous case of Figure 4.16. This is because of symmetry and is shown in Figure 4.17.

4.3.3 4 node network

In this section we will look at a network of 4 nodes that switches between 2 topologies, as can be seen in Figure 4.18. There exist 2 customer classes, one of rate a_{10} arriving at node 1 and exit node 3 and one at node 4, with rate a_{41} and exit node 2. We observe that in both topologies, the nodes where traffic enters the network are disconnected from the exit nodes. Therefore, the total network throughput, as depicted in Figures 4.19 and 4.20 is 0.

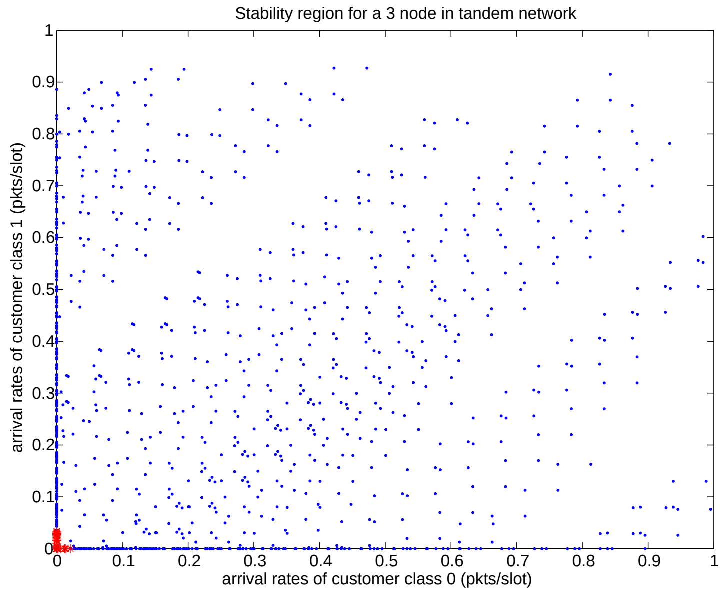

Figure 4.14: The maximum stable throughput of a stationary network that operates under topology T_1 at all time slots is zero. This is because the exit nodes are isolated from all the traffic sources.

We will show that although the individual networks are "bad" disconnected networks, the switched network, that alternates between topologies T_0 and T_1, achieves a throughput region that has positive area. In the sequel, we obtain the throughput regions of the switched network, that alternates between topologies T_0 and T_1, for different values of the stationary probability distribution under which these topologies occur. Specifically, we will look into three cases, namely when $p = 0.5$. $p > 0.5$ and $p < 0.5$

- $p = 0.5$

  The throughput region of the network of Figure 4.18, when each topology T_0 and T_1 occur equally probably, is depicted in Figure 4.21.

- $p > 0.5$

  When topology T_0 occurs more often than T_1, then the stability region of the network, as

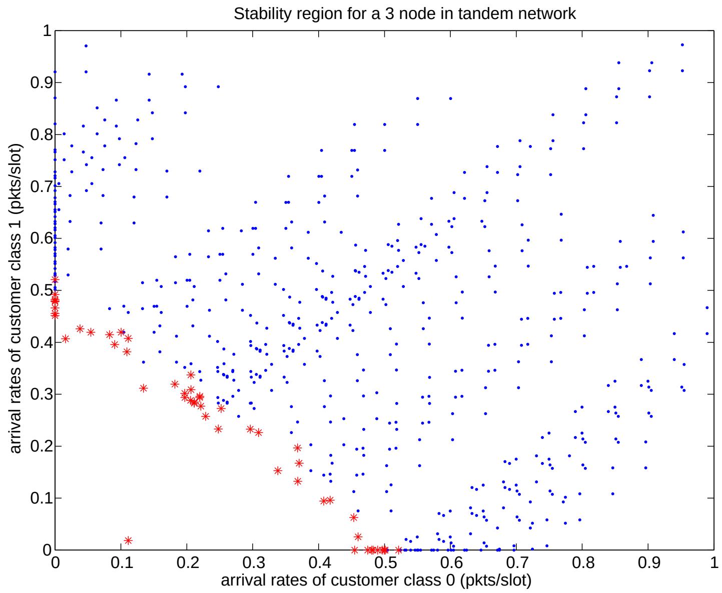

Figure 4.15: Stability region of the 3 nodes in tandem network, when topologies T_0 and T_1 occur equally likely.

obtained by our simulations, is depicted in Figure 4.22 (which corresponds to $p = 0.75$).

- $p < 0.5$ When topology T_0 occurs less often than T_1, then the stability region of the network is shown in Figure 4.23(which corresponds to $p = 0.25$).

4.3.4 4 nodes on a ring

In this section, we are going to analyze the throughput region of a network comprised, by 4 nodes that reside on a ring of radius r. There exist 2 customer classes. Customer class 0, of rate a_{10}, arriving at node 1, with exit node 3 and customer class 1, of rate a_{41}, arriving at node 4, with exit node 2. The positions of all the nodes, at each time instant, can be determined as long as one node's position is completely known. To illustrate this, let a node be located at angle $\theta(t) \in [0, \pi/2], \forall t \geq 0$. The rest of the nodes will be placed at positions $\pi - \theta(t)$, $\pi + \theta(t)$ and $2\pi - \theta(t)$. Hence, each

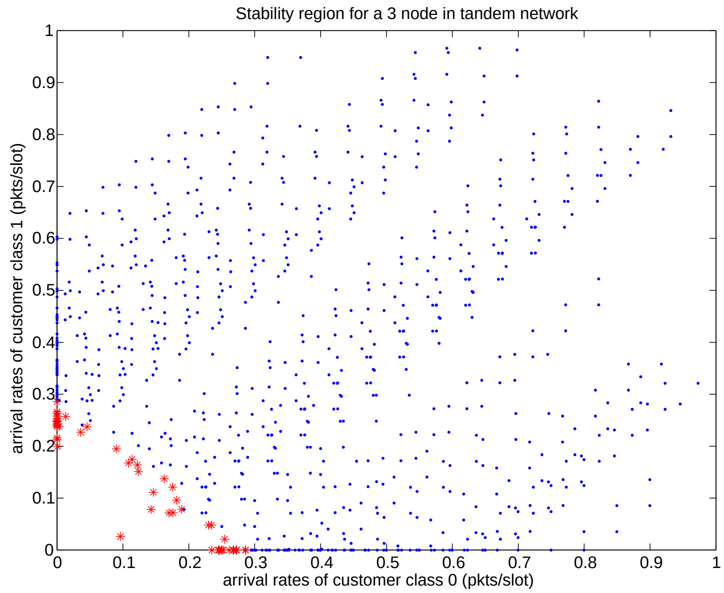

Figure 4.16: Stability region of the 3 nodes in tandem network, when topology T_0 occurs more frequently than T_1 (with stationary probability 0.75).

quadrant is allocated to exactly one node. Let these nodes be numbered from 1 to 4, as shown in Figure 4.24. Consider also that the power at the nodes is such, that a pair of nodes communicates if their distance is at most R. We assume that the radius of the ring is large enough to prevent a node from communicating with all others at all time slots. As the value of $\theta(t)$ varies, the different topologies that the network evolves into, vary as well.

We are going to look at the case where $r > \frac{R}{\sqrt{2}}$. If $\theta(t)$ is small enough, then the network nodes that may communicate with each other will be the pair of nodes 1 and 2, as well as nodes 3 and 4. This would put a constraint on the distance between these nodes to be less than the range of communication, namely $2rsin(\theta(t)) \leq R$ or else $\theta(t) \leq sin^{-1}(\frac{R}{2r})$. To this respect, we define the quantity $\theta_0 = sin^{-1}(\frac{R}{2r})$. Hence, as soon as $\theta(t)$ increases and becomes larger θ_0, the distance between communicating nodes 1 and 2 and 3 and 4 becomes larger than R, and hence they may not communicate any more. More specifically, the network will be completely disconnected for

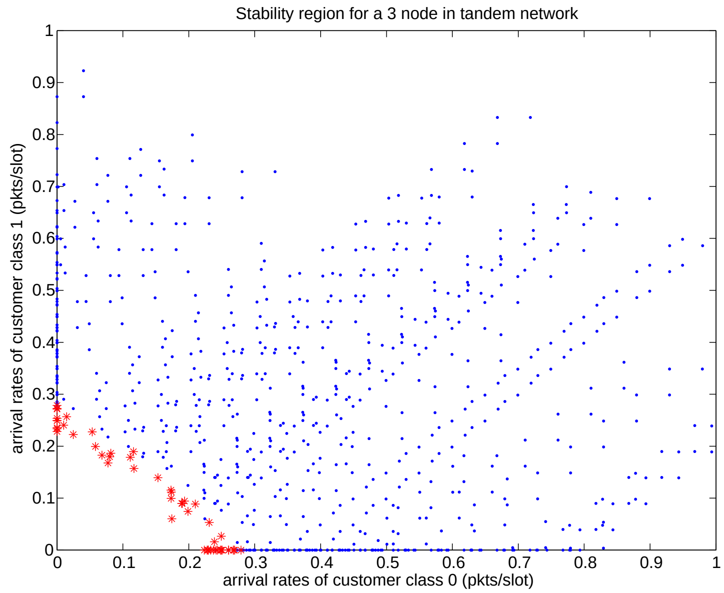

Figure 4.17: Stability region of the 3 nodes in tandem network, when topology T_0 occurs less frequently than T_1 (with stationary probability 0.25).

values of $\theta(t)$, such that $\theta_0 < \theta(t) < \pi/2 - \theta_0$. Further increase in $\theta(t)$, i.e. $\pi/2 - \theta_0 < \theta < \pi/2$ results in a different set of communicating nodes, namely 1 communicates with 4 and 2 with 3. By considering a uniform distribution on $\theta(t)$, we observe that topologies T_0 and T_1 occur each with equal probability $p = \frac{4\theta_0}{2\pi}$ and T_2 occurs with probability $1 - 2p$. So we have the following set of possible network topologies:

- $0 \leq \theta(t) \leq \theta_0$: Topology T_0 is present (Figure 4.25).
- $\theta_0 < \theta(t) < \pi_2 - \theta_0$: Topology T_2 is present (Figure 4.26).
- $\theta_0 \leq \theta(t) \leq \pi/2$: Topology T_1 is present.(Figure 4.27).

Let us now consider that $p = 1/4$. Then, the maximum throughput scheduling policy for this network achieves the stability region of Figure 4.28.

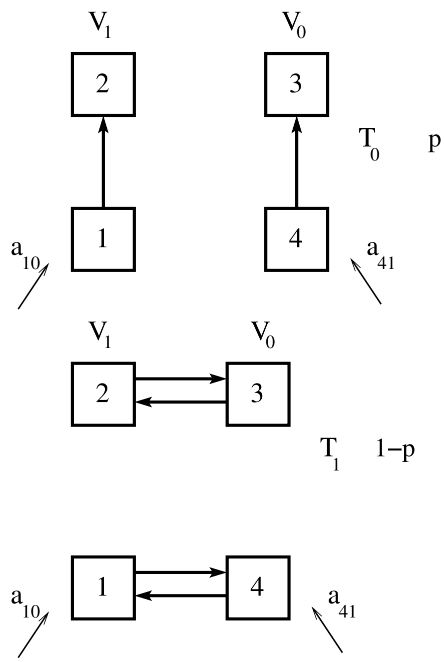

Figure 4.18: A 4 node network that takes two topologies, T_0 and T_1 with probabilities p and $1-p$ respectively. Traffic of rate a_{10} arrives at node 1 with exit node 3 and traffic of with rate a_{41} arrives at node 4 with exit node 2.

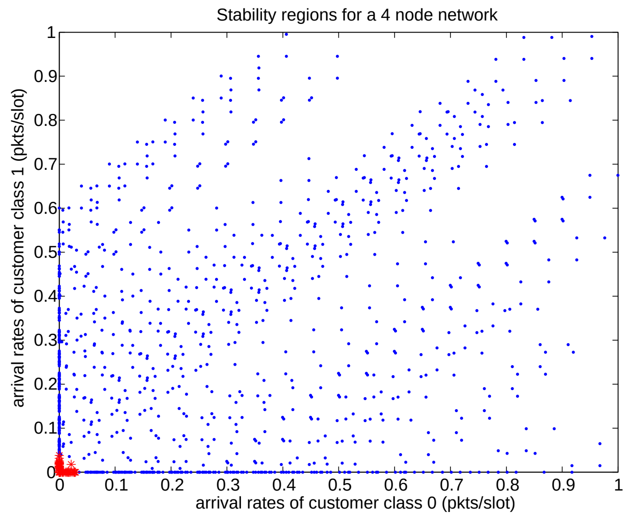

Figure 4.19: Under topology T_0, the nodes at which arrivals occur are disconnected from the exit nodes at all times. Hence, the maximum throughput that can be delivered is 0.

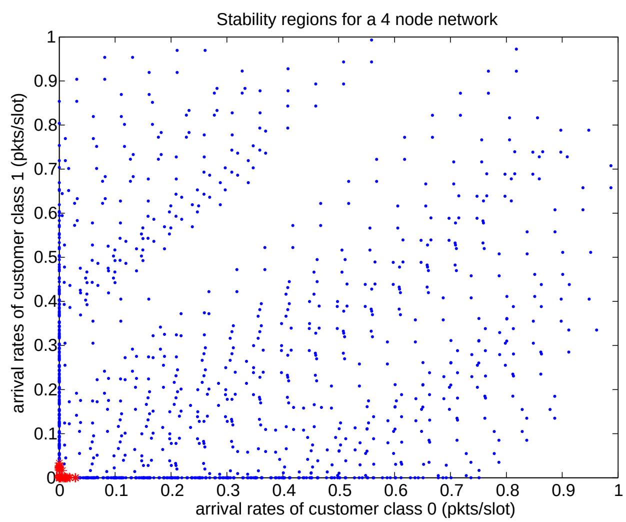

Figure 4.20: Under topology T_1, the nodes at which arrivals occur are disconnected from the exit nodes at all times. Hence, the maximum throughput that can be delivered is 0.

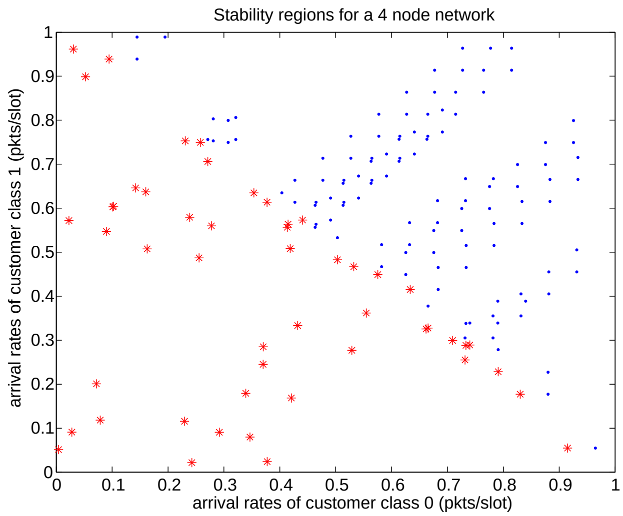

Figure 4.21: Stability region of a 4 node network, when the two topologies, T_0 and T_1 occur equally likely ($p = 0.5$).

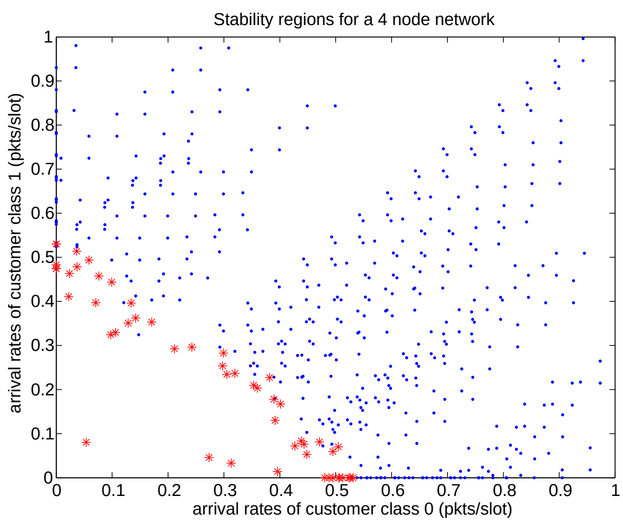

Figure 4.22: Stability region of a 4 node network, when topology T_0 occurs more often, namely with probability 0.75, and T_1 with probability 0.25.

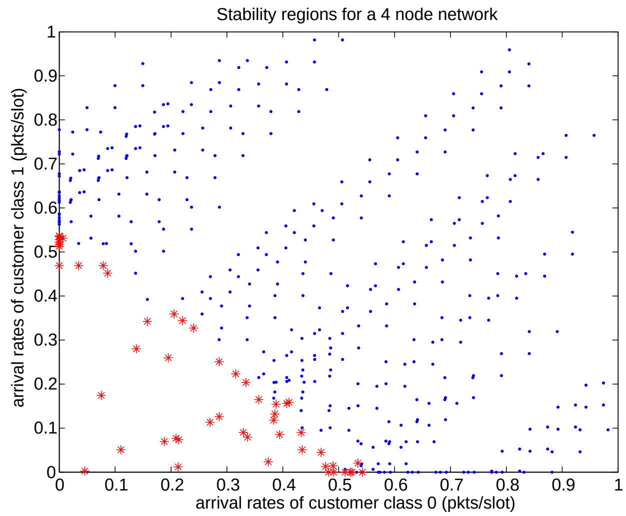

Figure 4.23: Stability region of a 4 node network, when topology T_0 occurs less often, namely with probability 0.25, and T_1 with probability 0.75.

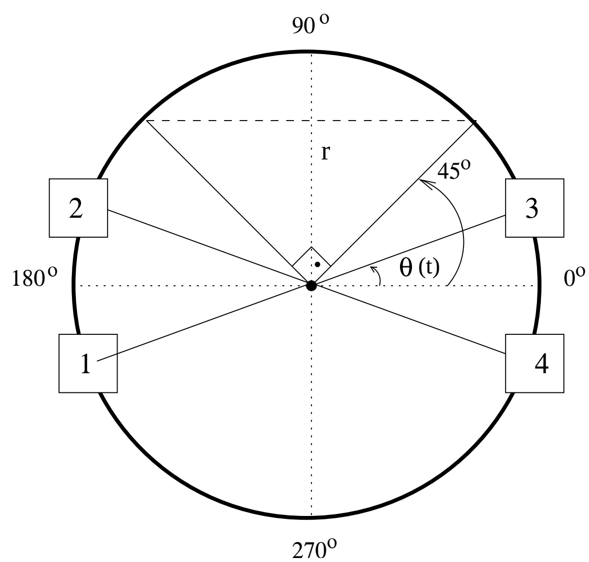

Figure 4.24: A network of 4 nodes, residing on a ring of radius r.

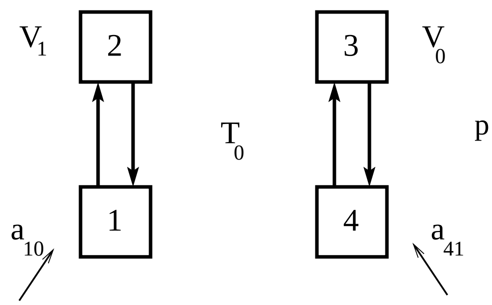

Figure 4.25: Topology T_0 is present. The nodes that are able to communicate with each other are nodes 1 and 2 and nodes 3 and 4.

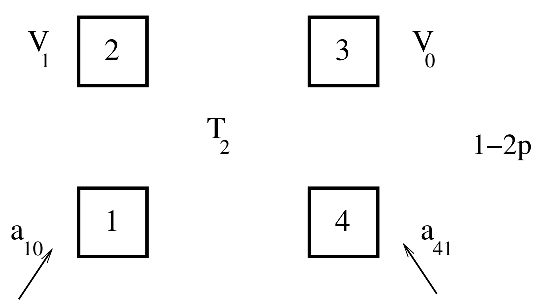

Figure 4.26: Topology T_2 is present. All network nodes are disconnected from each other.

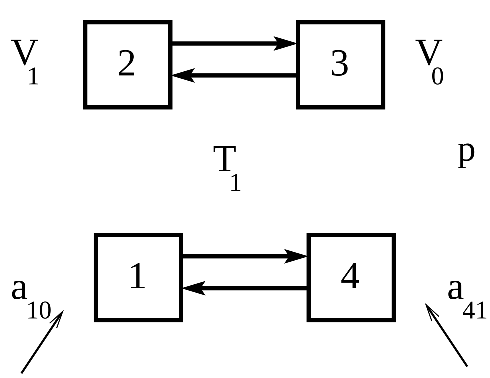

Figure 4.27: Topology T_1 is present. Nodes 2 and 3 communicate and nodes 1 and 4 communicate.

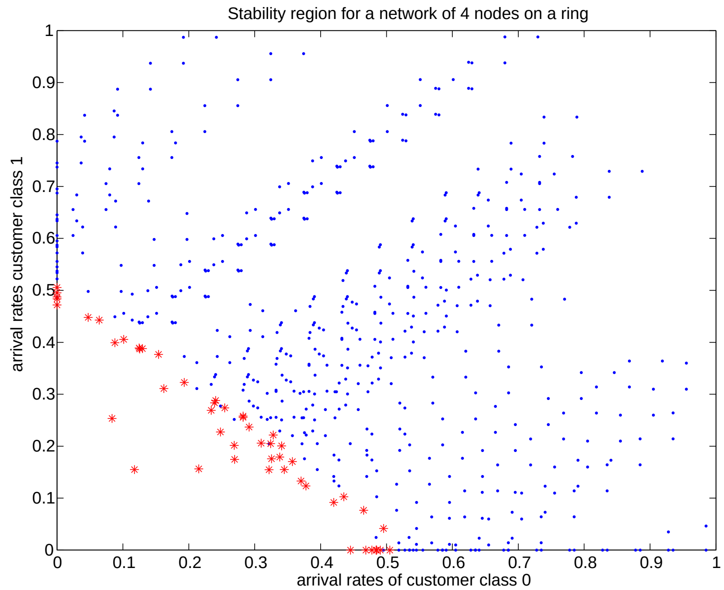

Figure 4.28: Stability region of the network consisting of 4 nodes on a ring, when the occurrence probability of topologies T_0 and T_1 is $p = 1/4$.

4.4 Conclusions

In this chapter we introduced our simulation tool and presented simulation results on what is the maximum stable throughput achieved by the scheduling policy we introduced, when operating on a set of networks. The networks we simulated are not complicated. However, they are selected carefully to depict in a clear way, how mobility may increase the stable traffic a network can handle.

Chapter 5

Appendix A: Markov Chains

In this Chapter, we are going to give a brief overview on Markov Chains ([11], [1]). First of all, we need to introduce the notion of a stochastic process. A **stochastic process** $\{X(t), t \in \mathcal{T}\}$ is a family of random variables, indexed by a variable $t \in \mathcal{T}$, i.e $X(t)$ is a random variable. When $\mathcal{T}$ is a countable set, then the stochastic process is said to be **discrete-time process** and when $\mathcal{T}$ is an interval of the real line, the stochastic process is said to be **continuous-time process**.

In this Thesis, we will restrict ourselves in discrete-time discrete state space stochastic processes $\{X_n, n = 0, 1, \dots\}$. Without loss of generality the state space takes values in the non-negative integers $\{0, 1, \dots\}$. For example, $X_n = y$, means that the stochastic process is in state y at instant n.

5.1 Markov Chains

A **Markov Chain** is a stochastic process $\{X_n, n = 0, 1, \dots\}$ for which given the present, the evolution of the process becomes conditionally independent of the past. In other words:

$$P[X_{n+1} = z | X_n = y, X_{n-1} = x_{n-1}, \dots, X_0 = x_0] = P[X_{n+1} = z | X_n = y] = P_{yz}(n) \qquad (5.1)$$

for all $x_0, x_1, \dots, x_{n-1}, y, z$ and for all $n \geq 0$, where

$P_{yz}(n) \geq 0, \forall y, z \in \mathcal{X}, \forall n \geq 0$ and $\sum_{z=1}^{\infty} P_{yz}(n) = 1,$ for $y = 0, 1, \ldots, \forall n \geq 0.$

It is due to these properties that X_n is called the state of the process at time n. The set of all possible values for X_n is the **state space**, $\mathcal{X}$, of the Markov Chain.

If, furthermore, the Markov Chain is time homogeneous, in other words the transition probability does not vary with the time index, we get:

$$P[X_{n+1} = z | X_n = y] = P[X_1 = z | X_0 = y] = P_{yz} \tag{5.2}$$

Then:

$$P[X_{n+1} = z | X_n = y, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0] = P_{yz} \tag{5.3}$$

The **transition probability matrix**,$\mathbf{P}$, is a matrix with elements the transition probabilities P_{yz}. Hence in the y^{th} row and z^{th} column, the element will represent the probability that the next state will be z when the current state is y.

We will focus our attention on time homogeneous stochastic processes.

5.2 Classification of States

Let $y, z \in \mathcal{X}$ be two states of the Markov Chain $\{X_n, n = 0, 1, \ldots\}$ with transition probability matrix $\mathbf{P}$. State z is said to be **accessible** from state y if for some $n \geq 0, (\mathbf{P}^n)_{yz} > 0$. We say that states y, z **communicate** if they are accessible to each other, in other words there exists a positive probability to go from state y to state z and from z to y. Two communicating states are said to belong to the same **class**. A Markov Chain is said to be **irreducible** if it has a single class of communicating states. Any state of a Markov Chain can be further classified into being transient or recurrent. Let P_y denote the probability that starting at state y the process will ever re-enter y. State y is said to be **recurrent** if $P_y = 1$ and **transient** if $P_y < 1$. Recurrency of state y implies that it will be visited infinitely often. This results from the fact that starting from y, y will be visited again with probability 1 and by the definition of the Markov Chain the process will be restarting each time state y is visited, meaning that y will be revisited in the future as well. On the other hand, if a state is transient, there is a positive probability that it will never be visited in the future. Let $1-P_y$ be this probability. Then the probability that starting at state y, the Markov Chain will spend at y exactly $n \geq 1$ time periods (or revisit y, exactly $n-1$ time periods) will be geometrically distributed according to $P_y^{n-1}(1-P_y)$, with mean $1/(1-P_y)$ (finite). Hence, we can also say that state y is recurrent if and only if starting at state y the expected number of time periods that the Markov Chain will spend at state y will be infinite. This, implies ([11]) that state y is **recurrent** if $\sum_{n=1}^{\infty} P_{yy}^n = \infty$ and **transient** if $\sum_{n=1}^{\infty} P_{yy}^n < \infty$. A recurrent state is **positive recurrent** if given that the process starts at state y, the expected time to return to y is finite. If a recurrent state is not positive recurrent, it is **null recurrent**.

5.3 Stopping Time

Let the stochastic process $\{X_n, n = 0, 1, \ldots\}$, we will say that the random variable τ_s will be a **stopping time** with respect to the sequence $\{X_n, n = 0, 1, \ldots\}$ if the occurrence of the event $\{\tau_s = n\}$ can be determined completely by looking only at the realization of the process up to time n, i.e. $\{\tau_s = n\}$ is a function of $\{X_n, n = 0, 1, \ldots\}$.

5.3.1 Hitting Time

Let us divide the state space of our Markov Process in a partition of sets, $\mathcal{X}_i$ such that $\cup_{i=1}^{\infty}\mathcal{X}_i = \mathcal{X}$. Consider also, without loss of generality, that at time $n = 0$, the system is in state $X \in \mathcal{X}_1$, then the **Hitting Time** is given as a function of the stopping time defined below:

$$\tau_x = \begin{cases} \infty, & \text{if } X_n \in \mathcal{X}_1, \forall n > 0 \\ min\{n > 0 : X_n \notin \mathcal{X}_1\}, & \text{otherwise} \end{cases} \tag{5.4}$$

Chapter 6

Appendix B: Markov Chain Stability

In this Chapter, we will discuss stability issues in (time homogeneous) Markov Chains as presented in [1].

6.1 Stability of irreducible Markov Chains

In the case of an aperiodic, irreducible Markov Chain if all states are positive recurrent then the Markov Chain is ergodic. Also, consider a Markov Chain that is irreducible and aperiodic. If this Markov Chain is ergodic as well, this implies positive recurrence. Hence, for the irreducible Markov Chain case, stability is equivalent to ergodicity. Theorem 4 gives sufficient conditions for positive recurrence, and hence stability, for irreducible Markov Chains.

Theorem 4 (Foster's Theorem) *Consider an irreducible Markov Chain $X_n, n = 0, 1, \ldots$ with state space $\mathcal{X}$, a real valued, bounded from below, function $V : \mathcal{X} \to \mathbf{R}$, an $\epsilon > 0$ and a finite subset $\mathcal{X}_0$ of $\mathcal{X}$ such that:*

$$E[V(X_{n+1}) - V(X_n)|X_n = x] < -\epsilon, \quad \textit{if } x \notin \mathcal{X}_0$$

and

$$E[V(X_n)|X_n = x] < \infty, \quad \forall \quad x \in \mathcal{X}_0$$

Then, the corresponding time-homogeneous Markov Chain is positive recurrent.

Note that if the Markov Chain is finite and irreducible, then there will exist a single class of states, namely all states will be positive recurrent. In this case, the Markov Chain will be stable.

6.2 Stability of reducible Markov Chains

Consider the reducible Markov Chain $\{X_n, n = 0, 1, \ldots\}$ and partition its state space into the classes $Y, Z_1, Z_2, Z_3, \ldots$, where $Z_i, i = 1, 2, 3, \ldots$ are sets of communicating states that are recurrent and Y is the set of all transient states. While in the case of irreducible Markov Chains stability is equivalent to positive recurrence, in the case of reducible Markov Chains we can define a system stability as follows([14]).

Let the Markov Chain at time $n = 0$ be in a state $\mathbf{X}(0) = \mathbf{x} \in Y$, where Y is the set of transient states. Then the system will be stable if Definition 2 ([14]) holds:

Definition 2 (System stability) *The system is* **stable** *if for the state process X_n we have:*

$$P[\tau_y < \infty] = 1, \quad \forall y \in Y \tag{6.1}$$

and all states $z \in \cup_{i=1}^{\infty} Z_i$ are positive recurrent,

where τ_y is a Hitting Time as presented in Appendix A(Equation 5.4). An extension to Theorem 4, that gives sufficient conditions, about stability of a reducible Markov Chain is defined also in [14] and [15]:

Theorem 5 *Consider a Markov Chain $\{X_n, n = 0, 1, \dots\}$ with state space $\mathcal{X}$, a real valued, bounded from below, function $V : \mathcal{X} \to \mathbf{R}$, an $\epsilon > 0$ and a finite subset $\mathcal{X}_0$ of $\mathcal{X}$ such that:*

$$E[V(X_{n+1}) - V(X_n)|X_n = x] < -\epsilon, \quad \textit{if } x \notin \mathcal{X}_0$$

and

$$E[V(X_{n+1})|X_n = x] < \infty, \quad \forall \quad x \in \mathcal{X}_0$$

Then, the Hitting Time τ_x, defined in Appendix A satisfies:

$$P[\tau_x < \infty] = 1, \quad \forall x \in Y, \tag{6.2}$$

and all the recurrent classes of this Markov Chain are positive recurrent, i.e. $\{X_n\}$ is stable.

Chapter 7

Appendix C: Definitions on sets

A set, S, is said to be **convex** if $\lambda s_1 + (1-\lambda)s_2 \in S$, for all $s_1, s_2 \in S, 0 \leq \lambda \leq 1$. A set S is said to be **open**, if for any given point $s \in S$ and for some $\epsilon > 0$ arbitrarily small, the ball, centered at s, with radius ϵ, is contained in S. The set is said to be **closed** if for any converging sequence defined in S, the limit of the sequence will be in S as well. Equivalently, a set is closed if and only if it's complement is an open set. A point s belongs to the **closure** of a set $\overline{S}$ if for some converging sequence s_n in S the limit is $\lim_{n\to\infty} s_n = s$. Equivalently, the closure of a set S is defined as the intersection of all closed sets containing S and is necessarily a closed set. Let the set $S \subseteq \mathbf{R}^n$. The convex combination of elements $s_i \in S$ is the element $s = \sum_i \lambda_i s_i$, where $\lambda_i \geq 0, \forall i$ and $\sum_i \lambda_i = 1$. The convex hull of S, denoted as $co(S)$, is the set of all points that can be expressed as a convex combination of elements in S. Note that the convex hull of a set S is convex. The convex hull of S is the smallest convex set that contains S. A set $S \subseteq \mathbf{R}^n$ is called a **convex polyhedron** if there exists an $m \times n$ matrix $\mathbf{T}$ and a vector $\mathbf{r} \in \mathbf{R}^m$, such that $S = \{x : \mathbf{T}x \leq \mathbf{r}\}$. In other words, a convex polyhedron is the intersection of finitely many **half spaces**. A set $P \subseteq \mathbf{R}^n$ is called a **convex polytope** if $\exists S \subseteq \mathbf{R}^n$, where S is finite, such that $P = co(S)$. A convex polyhedron that is also bounded, is a convex polytope.

Theorem 6 *Let P be a convex polytope and L be a linear map. Then the* **linear pre-image** *of P,* *where $LP(P)$ is defined as:*

$$LP(P) = \{x : Lx \in P\},$$

is a convex polyhedron.

Theorem 7 *Let a convex polytope P and L be a linear map. Then the* **linear image** *of P*

$$\{Lx : x \in P\}$$

is a convex polytope.

BIBLIOGRAPHY

[1] Pierre Brémaud. *Markov Chains: Gibbs Fields, Monte Carlo Simulation and Queues*. Springer, 1999.

[2] I. Chlamtac and A. Lerner. Fair algorithm for maximal link activation in multihop packet radio networks. *IEEE Transactions on Communications*, 35(7):739–746, July 1987.

[3] I. Cidon and M. Sidi. Distributed assigned algorithms for multihop packet-radio networks. 38(10), October 1989.

[4] Wim Diepstraten and Phil Belanger. 802.11 tutorial: Mac entity. Lucent Technologies, Aironet Wireless Communications.

[5] W. Yuen et al. A simple but effective cross-layer networking system for mobile ad hoc networks. In *Proc. IEEE PIMRC*, 2002.

[6] Z. Huang, C. Shen, C. Srisathapornphat, and C. Jaikaeo. Topology control for ad hoc networks with directional antennas. In *Proceedings IEEE Int. Conference on Computer Communications and Networks*, 2002.

[7] J.D. Kraus and R.J. Marhefka. *Antennas: for All Applications*. McGraw-Hill, 3rd edition, New York, 2002.

[8] Grossglauser M. and Tse D. Mobility increases the capacity of ad-hoc wireless networks. In *Proceedings IEEE, Infocom*, April 2001.

[9] Gupta P. and Kumar P.R. The capacity of wireless networks. *IEEE Transactions on Information Theory*, 46(2), March 2000.

[10] A. Puri and P. Varaiya. Routing and scheduling in multi-hop wireless networks. In *Vehicular Technology Conference (VTC)*, 2003.

[11] Sheldon M. Ross. *Introduction to Probability Models (eighth edition)*. Academic Press, 2003.

[12] J. A. Silvester. Perfect scheduling in multihop broadcast networks. In *Proceedings ICCC, London, England*, September 1992.

[13] Sami Tabbane. *Handbook of Mobile Radio Networks*. Artech House Publisher, 2000.

[14] L. Tassiulas and A. Ephremides. Stability properties of constrained queueing systems and scheduling policies for maximum throughput in multihop radio networks. *IEEE Transactions on Automatic Control*, 37(12), December 1992.

[15] Leandros Tassiulas. *Dynamic link Activation Scheduling in Multihop Radio Networks with Fixed or Changing Connectivity*. PhD thesis, University of Maryland-College Park, 1991.

[16] Leandros Tassiulas. Scheduling and performance limits of networks with constantly changing topology. *IEEE Transactions on Information Theory*, 43(3), May 1997.

[17] S. Toumpis and A. J. Goldsmith. Performance, optimization, and cross-layer design of media access protocols for wireless ad hoc networks. In *Proc. IEEE ICC*, 2003.

[18] Su Yi, Yong Pei, and Shivkumar Kalyanaraman. On the capacity improvement of ad hoc wireless networks using directional antennas. In *Proceedings International Symposium on Mobile Ad Hoc Networking and Computing*, 2003.