# ABSTRACT

Title of thesis: TWO APPLICATIONS INVOLVING THE ANALYTIC HIERARCHY PROCESS

Brian David Alford, Master of Science, 2004

Thesis directed by: Professor Bruce Golden
Department of Decision and Information Technologies

The analytic hierarchy process (AHP) is a popular tool used in decision making for ranking alternatives based on quantitative and qualitative criteria. In this thesis, we investigate two applications involving the AHP: determining the greatest sports records and generating priority vectors for inconsistent interval judgments. We determine rankings of the greatest active single-season, career, and single-event sports records by applying the ratings mode of the AHP. In addition, we present an extension to a linear programming method used for generating priority vectors for interval pairwise comparison matrices. By introducing multiplicative stretch factors for each interval comparison, our linear programming method with stretching can be used to solve problems when inconsistent interval judgments are present. We describe the linear programming method, apply it to three problems, and compare its performance to other methods for solving inconsistent interval AHP problems.

# TWO APPLICATIONS INVOLVING THE ANALYTIC HIERARCHY PROCESS

by

Brian David Alford

Thesis submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Master of Science
2004

Advisory Committee:

Professor Bruce Golden, Chair/Advisor
Professor Edward Wasil
Professor Peter Wolfe

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# Chapter 1

# Introduction

Decision making is a part of everyone's daily life. We are always making conscious and subconscious decisions that have many different long- and short-term consequences. For complex decisions, we often consult many sources, gather information, and make observations to aid our selection. One tool that can assist with the decision-making process is the analytic hierarchy process (AHP). This thesis studies two problems involving the AHP. First, we examine well-known great sports records and use the AHP to determine the greatest single-season, career, and single-event sports records. Second, we investigate a new method for evaluating pairwise comparison matrices in the AHP that contain interval judgments.

Several questions come to mind when we consider ranking the greatest sports records of all time. How can we compare different types of performances from a single sport? Can we determine whether Nolan Ryan's single-season strikeout record is better than Ty Cobb's career batting average? How can records from different sports be compared when the environment among sports is very different? Can we compare records such as John Stockton's career assist record to Pete Sampras's 14 grand slam tennis titles? Questions such as these abound in living rooms, at bars and pubs, on Internet message boards, and on sports media shows across the country. Why not make a formal, quantitative attempt to answer these questions?

The world of sports is a good candidate for a study in decision making with the AHP for several reasons. First, modern sports have been intertwined with numbers and statistical measurements since the early twentieth century. In almost every sport played today, winning involves maximizing or minimizing some measurable quantity, for example, running the fastest time, scoring the most points or goals, or shooting the lowest score. It is a world in which performances can be measured, recorded, and compared against others objectively. It is through numbers that we remember some of the greatest athletes in sports history: Joe DiMaggio (56), Wilt Chamberlain (100), and Wayne Gretzky (99), to name three [5].

Second, the sports environment lends itself to being an area that promotes many different opinions. People are often naturally drawn to favor one sport over another. Fans of one team are diametrically opposed to followers of their team's rival. Sports evoke some of the deepest emotions within fans and hence, a lot of subjectivity regarding sports exists today (for example, see any Internet message board such as CBS.Sportsline.com [6]).

Third, many individuals have an interest in a sport of some kind, be it as a participant or fan. The act of playing, watching, or reading about sports can occupy a considerable amount of leisure time for the average person. The discussion of the greatest athletes and athletic feats permeates all walks of life: rich and poor, young and old. For example, a recent article in the KidsPost section of the *Washington Post* [4] highlighted some of the athletes associated with a variety of sports records. It discussed athletes such as Paul Hornung and his single-season points record in football to Wilt "the Stilt" Chamberlain and his single-season scoring and rebounding records in basketball. A study of the greatest sports records would be of general interest to a large number of people from all age groups.

Taken together, these three reasons (objective data, subjective data, and a large following) indicate that the world of sports would be receptive to a study that determines the greatest sports record.

Ambiguity and uncertainty play an influential role in decision making. Often, we are faced with situations where we must weigh many different feelings and opinions before we can make a decision. In some instances, we encounter a decision but are hampered by indecisiveness. We attempt to evaluate the factors involved in the selection but remain uncertain about the best choice to make. Other times, we become involved in making a decision with a heterogeneous group of people. Each person in the group has different preferences that need to be considered before the decision can be made. In such cases, we would like to account for the ambiguity and uncertainty in our decision-making method.

Comparing different solutions to a problem two-at-a-time is one way to break down a complex decision into several smaller, more manageable decision problems. Performing pairwise comparisons, or measurements of the relative importance of one alternative to another, is a key component of the analytic hierarchy process. The AHP requires the decision maker (DM) to assign a numerical value to a comparison of two items to capture his or her preferences between each pair of alternatives. Sometimes, when faced with a large amount of ambiguity, the DM might not be able to specify an exact value for these comparisons. Instead of single numbers, interval pairwise comparisons can be used by a DM to express this uncertainty with his or her actual judgments.

Using the AHP to make decisions whenever interval pairwise comparisons have been made is an active area of research in management science. Techniques based on linear programming [3, 19, 23, 26] and nonlinear optimization [26, 34] to solve interval AHP problems have been proposed in the literature. Each of these techniques has its own limitation when interval pairwise comparisons are used (more about this in Chapter 4). In this thesis, we introduce a linear programming model with stretching (LP-S) that generates solutions for all types of AHP problems involving interval pairwise comparison. We demonstrate that the LP-S does not have the same limitations as other interval techniques. The LP-S method provides an efficient, practical way for generating priority vectors.

The rest of this thesis is divided into four chapters. Chapter 2 introduces the analytic hierarchy process and describes its application to decision-making problems. Chapter 3 discusses single-season, career, and single-event sports records and applies the AHP to determine the greatest sports records in each category. In Chapter 4, we describe a new technique for generating weights for interval pairwise comparison matrices and focus on solving AHP problems with inconsistent interval judgments. Chapter 5 concludes by summarizing the thesis and indicating areas for future work.

# Chapter 2

# Introduction to the Analytic Hierarchy Process

Making complex decisions whenever both quantitative and qualitative information are present can be a difficult task. How can we accurately choose between a set of items when considering both objective and subjective factors that influence the overall decision? In this chapter, we provide a brief introduction to the analytic hierarchy process, a well-known method for solving decision-making problems. We give a short description of the AHP and describe how it can be used for making decisions. In addition, we provide the mathematical framework underlying the eigenvector method (EM), which is the most widely used method for finding a priority vector in the AHP.

The last section of this chapter introduces two modifications to traditional AHP problems: the ratings mode and interval pairwise comparisons. In studies when the number of alternatives (items that the decision maker is choosing among) is large, it is often convenient to break the problem down into smaller pieces. The ratings mode provides a simple approach to solving problems with many alternatives and decision factors. Interval pairwise comparisons are used when a decision maker wishes to specify an interval of values for each comparison rather than assign a single value for a comparison.

## 2.1 The Analytic Hierarchy Process

The analytic hierarchy process (AHP) is a widely-used technique for comparing a set of alternatives with respect to an overall goal [31]. It is a popular tool used by decision makers when the choice of alternatives is influenced by both quantitative and qualitative data. The AHP relies on the ability of the decision maker to decompose the main problem into a hierarchy of smaller decision problems that consist of different objective and subjective factors that work together to influence the overall goal. The overall result of using the AHP is a priority vector that provides a ranking of the different alternatives under consideration.

There are four main steps to the analytic hierarchy process: building the hierarchy, making pairwise comparisons, generating priority vectors, and synthesizing with respect to the overall goal. The following subsections introduce and illustrate these steps.

### 2.1.1 Hierarchies

Building the hierarchy is often the most challenging of the four main steps in the AHP. Creating the hierarchy requires an intuitive feel for the various factors and subfactors that directly influence the overall goal as well as an ability to identify alternatives suitable for accomplishing the goal. The hierarchy must be designed so that these alternatives are accurately evaluated on their ability to satisfy the overall goal. Both of these tasks require the DM to be extremely knowledgeable and familiar with all facets of the problem.

The hierarchy starts at the top by clearly stating the goal of the problem. Directly beneath this goal are the primary criteria to be considered when making the decision. In Figure 2.1, we see that the overall goal is listed at the top of the hierarchy and is broken down into three key criteria that directly influence the goal above them. These criteria can also be called factors, and in this thesis, we use the two terms interchangeably.

These criteria can be further broken down into subcriteria. For example, in Figure 2.1, Criterion 1 is broken down into two subcriteria, while Criterion 3 is broken down into three subcriteria. In general, there is no limit to the size and number of levels within the hierarchy, although, as a practical matter, there are usually only two or three levels of criteria and subcriteria beneath the overall goal. Ideally, the hierarchy should be large enough to capture all important criteria involved in the decision-making process but small enough for the problem to remain manageable and meaningful.

At the bottom level of the hierarchy, the alternatives are listed beneath the subcriteria and are connected to each one. We see the lines extending from Subcriterion 1 down to all five alternatives. This indicates that the DM compares all five alternatives with respect to Subcriterion

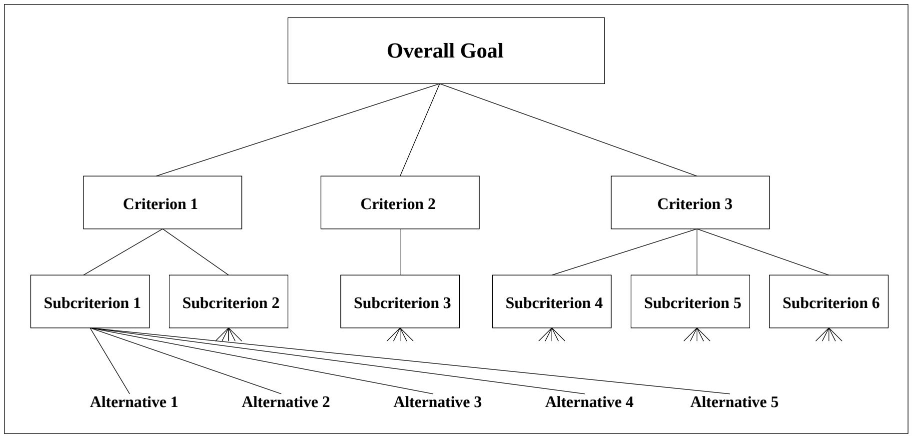


Figure 2.1: General hierarchy structure

1. The lines extending from the other four subcriteria indicate the comparisons also made between the five alternatives beneath each subcriterion.

### 2.1.2 Pairwise Comparisons

The analytic hierarchy process relies on pairwise comparisons to evaluate the importance of the criteria, subcriteria, and alternatives. With pairwise comparisons, there is no need for the DM to choose an ordering for all the criteria or alternatives at once; he or she only needs to compare the relative importance of one criterion or alternative to another. Saaty [31] has argued that making judgments in a pairwise fashion puts the problem into a comfortable form that can be handled easily. Comparing alternatives in a pairwise fashion allows us to reduce the problem from evaluating many choices to evaluating only two at a time.

The comparison process moves from the top of the hierarchy down. The criteria beneath the goal are pairwise compared, followed by the subcriteria beneath each criterion. At the bottom of the hierarchy, the alternatives are then compared relative to the subcriteria. For example, in Figure 2.1, Criterion 1 through Criterion 3 are first compared with respect to the goal. Subcriterion 1 and Subcriterion 2 are then compared relative to Criterion 1. Finally, Alternative 1 through Alternative 5 are compared with respect to Subcriterion 1. This is completed for the entire hierarchy.

The pairwise comparisons made by the DM are assigned numerical values based on the 1 to 9 scale recommended by Saaty [31]. These comparisons represent a ratio of the weight assigned to one alternative versus the weight assigned to another. For example, if alternative one is three times as important as alternative three with respect to the current subcriterion, then the DM would assign the value of three to this ratio. Comparing all alternatives allows the decision maker to construct a pairwise comparison matrix to store these judgments for each subfactor. In a pairwise comparison matrix, the $(i, j)$ entry represents the DM's opinion on the relative strength of alternative $i$ to alternative $j$ with respect to that subcriterion.

Pairwise comparison matrices are positive reciprocal matrices. The weights of the alternatives are all positive, and the comparisons of one alternative to another satisfies $\frac{1}{9} \leq a_{ij} \leq 9,\ a_{ij} \neq 0 \quad \forall i, j$. The comparison of alternative $j$ to alternative $i$ will always be the reciprocal of the comparison of alternative $i$ to alternative $j$. This reduces the work of the decision maker; only the upper triangular portion of a pairwise comparison matrix needs to be filled in. A typical pairwise comparison matrix has the following form:

$$A = \begin{bmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ \frac{1}{a_{12}} & 1 & a_{23} & \dots & a_{2n} \\ \frac{1}{a_{13}} & \frac{1}{a_{23}} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{1}{a_{1n}} & \frac{1}{a_{2n}} & \dots & \dots & 1 \end{bmatrix}. \tag{2.1}$$

### 2.1.3 Determining the Priority Vector

The goal of the AHP is to use the pairwise comparison matrices to determine the values for the weights of the criteria and alternatives. The positive reciprocal nature of a pairwise comparison matrix can be used to determine a priority vector that contains these weights.

An important property we would like each pairwise comparison matrix to possess is consistency. The pairwise comparison matrix $A$ is called consistent if $a_{ij} \cdot a_{jk} = a_{ik}$ for all $i, j, k = 1, 2, \dots, n$. For consistent matrices, each matrix element in $A$ then represents an exact ratio of the weights assigned to each alternative. If we define $w_i$ to be the relative weight of criterion or alternative $i$, a consistent pairwise comparison matrix has the property that $a_{ij} = \frac{w_i}{w_j} \quad \forall i, j$. A consistent pairwise comparison matrix $A$ with these entries has the form of (2.2).

$$A = \begin{bmatrix} \frac{w_1}{w_1} & \frac{w_1}{w_2} & \frac{w_1}{w_3} & \cdots & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & \frac{w_2}{w_2} & \frac{w_2}{w_3} & \cdots & \frac{w_2}{w_n} \\ \vdots & \vdots & \ddots & & \vdots \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \frac{w_n}{w_3} & \cdots & \frac{w_n}{w_n} \end{bmatrix}. \tag{2.2}$$

Perfectly consistent pairwise comparison matrices have rank one, and this allows us to extract a priority vector. Using the properties possessed by a consistent pairwise comparison matrix, we can find a priority vector using simple algebraic manipulation.

$$\begin{aligned} a_{ij}\frac{w_j}{w_i} &= 1 & i, j &= 1, 2, \ldots, n \\ \sum_{j=1}^{n} a_{ij}\frac{w_j}{w_i} &= n & i &= 1, 2, \ldots, n \\ \sum_{j=1}^{n} a_{ij}w_j &= nw_i & i &= 1, 2, \ldots, n\ . \end{aligned}$$

If we denote the priority vector by $\mathbf{w} = [w_1 \quad w_2 \quad \ldots \quad w_n]^T$, then we can rewrite this system of equations in matrix-vector form. This gives us the familiar eigenvalue equation

$$A\mathbf{w} = n\mathbf{w}. \tag{2.3}$$

Given a perfectly consistent pairwise comparison matrix $A$, the right eigenvector of $A$ is composed of a set of weights that are derived directly from the comparison ratios. Normalizing this eigenvector so that its elements sum to one gives a unique set of weights for the alternatives.

In practice, the decision maker is typically not perfectly consistent in making pairwise comparisons. Inconsistencies occur whenever $a_{ij} \cdot a_{jk} \neq a_{ik}$, and we usually allow a small amount of inconsistency in making comparisons.

The presence of inconsistencies implies that each $(i, j)$ entry of $A$ is actually an approximation to the ratio of the weight of alternative $i$ to the weight of alternative $j$. Thus, $A$ is no longer of rank one, and more than one nonzero eigenvalue might be present. When inconsistencies exist, Saaty [31] has shown that determining the priority vector for any pairwise comparison matrix involves solving the altered eigenvalue problem

$$A\hat{\mathbf{w}} = \lambda_{max}\hat{\mathbf{w}}, \tag{2.4}$$

where $\hat{\mathbf{w}}$ is an approximation to the underlying true priority vector, and $\lambda_{max}$ is the maximum eigenvalue of $A$. This method is called the eigenvector method (EM). Saaty has shown that $\lambda_{max} \geq n$, with equality holding in the purely consistent case. Whenever $\lambda_{max}$ is close to $n$ we typically find $\hat{\mathbf{w}}$ to be a relatively good approximation to $\mathbf{w}$. It is this vector $\hat{\mathbf{w}}$ that gives us the priority vector for a pairwise comparison matrix.

A measure of how "close" $\lambda_{max}$ is to $n$ is the consistency index (CI), given by:

$$CI = \frac{\lambda_{max} - n}{n - 1}. \tag{2.5}$$

A common practice is that as long as the CI is less than 0.1, we can accept the priority vector as a good approximation. Whenever the CI is greater than 0.1, the pairwise comparison matrix might contain some contradictory information. A few individual pairwise comparisons might need to be adjusted by the decision maker to make the judgments more consistent.

### 2.1.4 Hierarchical Composition

Employing the eigenvector method on a pairwise comparison matrix returns the weights needed to determine the final alternative rankings. The principle of hierarchical composition is used to find the overall priority vector. The total weight assigned to an alternative is found by tracing the paths that lead from the goal down to the alternative, multiplying the weights of the branches in the path to determine the weight of the path, and adding these path weights together. We illustrate this concept in Figure 2.2.

## 2.2 Variations of the Standard AHP

The previous section presents the standard version of the analytic hierarchy process. Some situations require a variant of the AHP to determine the assignment of weights to each alternative. The following two subsections discuss two possible variants.

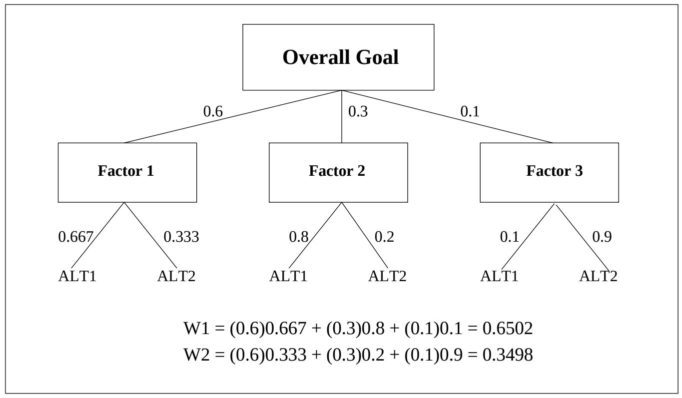


Figure 2.2: Hierarchical composition for the standard AHP

### 2.2.1 AHP with Ratings Mode

The standard AHP relies on individual pairwise comparisons in order to generate priority weights for the alternatives. When the number of alternatives is large, the task of performing individual pairwise comparisons can be time consuming. In order to simplify the process, each subfactor is further divided several different categories or ratings.

Beneath each subfactor, we first list several different ratings. These ratings are pairwise compared to one another, and the eigenvector of weights for the ratings is determined. In contrast to the normalization used by the normal EM described in the previous section, the eigenvector is normalized so that its largest weight is equal to one. The alternatives are not pairwise compared. Instead, an alternative is assigned a rating and receives the weight of that rating. Hierarchical composition is then used to determine the overall weights. In contrast to the standard AHP, the overall alternative weights do not sum to one when the ratings mode is used.

To illustrate how an overall weight is generated for an alternative when ratings are used, we provide an example. Consider Alternative 1 for the illustration given in Figure 2.3. Suppose that we rate Alternative 1 to be “High” for Subfactor A, “Good” with respect to Subfactor B, “Above Average” with respect to Subfactor C, and “Low” for Subfactor D. The weight assigned to Alternative 1 is then calculated in the following manner: $w_1 = 1(0.75)(0.6) + 0.3(0.25)(0.6) + 1(0.8)(0.4) + 0.154(0.2)(0.4) = 0.827$. The weights for other alternatives are found in a similar

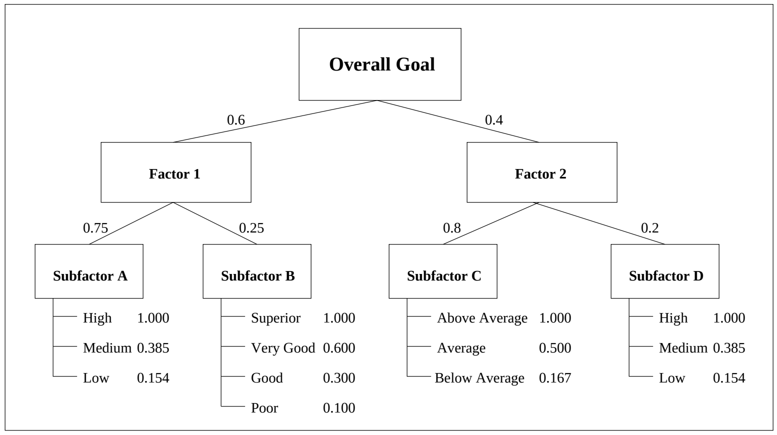


Figure 2.3: Hierarchical composition for the AHP with ratings mode

fashion.

### 2.2.2 Interval Pairwise Comparison Matrices

Sometimes, it is difficult for a decision maker to identify a single pairwise comparison value between alternatives. An interval of comparison values might then be specified by the DM instead of a single number. Interval judgments are most commonly employed in the AHP to express the uncertainty of the DM concerning a judgment.

For interval judgments, the comparison of alternative $i$ to alternative $j$ requires that the decision maker create a nonnegative lower bound ($l_{ij}$) and an upper bound ($u_{ij}$) to be placed on the relative strength of alternative $i$ to alternative $j$, yielding $l_{ij} \leq \frac{w_i}{w_j} \leq u_{ij}$. This process is continued until all judgments are assigned interval bounds. Interval pairwise comparison matrices are represented in the following form:

$$B = \begin{bmatrix} 1 & [l_{12}, u_{12}] & \dots & [l_{1n}, u_{1n}] \\ [l_{21}, u_{21}] & 1 & \dots & [l_{2n}, u_{2n}] \\ \vdots & \vdots & \ddots & \\ [l_{n1}, u_{n1}] & [l_{n2}, u_{n2}] & & 1 \end{bmatrix}. \tag{2.6}$$

As with non-interval matrices, the interval matrices must preserve the reciprocal nature of the comparisons. For an interval pairwise comparison matrix, this implies that $u_{ji} = \frac{1}{l_{ij}}$ and $l_{ji} = \frac{1}{u_{ij}}$ for all $i, j$ with $i \neq j$. Similar to the traditional AHP, the upper (or lower) triangular portion of the pairwise comparison matrix contains all necessary information needed to determine a priority vector.

At each level of the interval AHP, we generate a suitable ranking of the alternatives that reflects the preferences of the DM at that level. The process of determining a priority vector is different from the standard AHP. The eigenvector method cannot be applied directly to an interval pairwise comparison matrix for obvious reasons. For interval AHP problems, we attempt to find a priority vector **w** such that all ratios of the elements in **w** fall within the interval bounds given by the pairwise comparisons. In general, there may be an infinite number of sets of weights satisfying these interval bounds. The intent of interval AHP methods is to determine a way of finding a priority vector that "best" represents the opinions of the DM.

Currently, there are several techniques for generating a set of weights for an interval pairwise comparison matrix. One promising technique that is used for solving a variety of AHP problems (including ones with interval judgments) is a linear programming (LP) approach recently introduced by Chandran et al. [7]. In Chapter 4, this approach is presented, discussed, and expanded so that it can handle all types of interval pairwise comparisons.

# Chapter 3

# Application of the AHP - Greatest Sports Records

## 3.1 Sports Records Introduction

The questions posed in the Introduction were the same questions that Golden and Wasil [16] asked themselves for many years. The friendly discussions and debates they had between one another led them to produce their first study on the subject of determining the greatest sports records in 1987. At that time, the application of a decision-making technique to an area as debated as sports records was appealing to many within the general public. It was very well-received by academics and sports followers alike, generating a fair amount of attention in a variety of national media. For example, the study was the subject of Frank Deford's commentary on NPR and Dave Krieger's article in the *Chicago Tribune* [22], and it served as a reference for Jack McCallum's exposition on sports records in *Sports Illustrated* [25]. Many people, including the authors, enjoyed studying and analyzing a subject that has been a part of most of their lives since childhood. Why, then, do we now produce another study of the greatest sports records?

The 1987 study [16] was limited with respect to the number of records and sports considered. In total, 23 records from five different sports were examined in the entire study. Since the 1987 work, we have identified other records that we feel deserve to be measured against those 23. In his 2000 thesis [29], Richardson expanded the 1987 study by considering 42 records. In this study, we attempt to rank more records by using a slightly different technique than those employed in the studies of Golden and Wasil [16] and Richardson [29].

Our desire to expand the sports record study is based on one main reason–the environment surrounding sports is very dynamic. The sports climate is constantly evolving with rule changes, the advent of new technology, player trades between teams, and so forth. The well-known, sometimes overused saying that "records are made to be broken" embodies the dynamic of sports records succinctly and accurately. While sports records may not change from one day to the next, their values and holders are not carved into stone. New athletes emerge and challenge the records; some fall short of breaking a record, while others succeed in establishing new marks of their own. Given some of the amazing sports occurrences of the past 17 years, we felt there was a need to update the study to consider the new records that have been established since the release of the 1987 and 2000 studies. This need, combined with our own interests in comparing these new marks against those set decades ago, inspired us to complete a new study.

### 3.1.1 Selection of Sports Records

Deciding which records to consider in this study was itself a difficult process. We have selected a number of records from a variety of sports that we feel stand out above other records– they are "great" rather than just "good" records. However, the nature of this study requires some restrictions on these selections. The records we study are from professional sports played in the United States that have a considerable public following and have quantifiable statistics. We examine sports records from seven sports: baseball, football, basketball, golf, hockey, tennis, and track and field.

Most of the major statistics in these sports are tied to some quantity that can compared to other performances. For this reason alone, we are unable to include some of the best sporting moments in history. One might argue that Don Larsen's perfect game in the 1956 World Series, the third great Muhammad Ali - Joe Frazier fight (the "Thrilla in Manilla"), and Joe Montana's pass to Dwight Clark's in the back of the end zone to win the NFC Championship game in 1982 represent three examples of the most memorable moments in modern sports history. However, all are phenomenal sports *moments*, not records. It is difficult to objectively compare these performances to other similar pitching outings, boxing matches, or touchdown catches. Thus, we do not consider performances such as these in our study.

In addition, we are restricted to select records where data are accurately recorded and can be obtained easily through reliable sources. Baseball records have been kept officially since the early twentieth century and can be found for just about any player in virtually any category. One could look up the batting average of an "Average Joe" from the 1920-something season simply by visiting Major League Baseball's web site [27]. In contrast, there are hundreds of professional soccer leagues worldwide, making data gathering for soccer a cumbersome task. Although FIFA [15] acts as the general ruling body for the sport, no one organization keeps track of all statistical information regarding goals, assists, and saves that are needed for objective comparisons of soccer records. There have been some amazing soccer performances, but there is no easy way for us to obtain meaningful soccer data. Thus, the difficulty required in obtaining accurate, quantitative information has guided us in selecting records for this study.

We only consider active sports records for our comparisons; no previously-held record, no matter how significant or impressive, is studied here. When Bob Beamon long jumped over 29 feet in 1968, his jump was viewed as super-human because the world long jump record at the time was just under 27 feet, 5 inches. In 1991, though, Beamon's record fell to Mike Powell by .05 meters. Due to the magnitude of Beamon's improvement over the previous mark (he jumped past the previous record by nearly two feet) and the circumstances surrounding the jump (he experienced a minor seizure after his jump), his performance will no doubt be more remembered over time than that of Powell's, even though Powell's record is clearly a better jump. Another impressive record was Babe Ruth's single-season slugging average (0.847) set in 1920; this record was selected as the greatest single-season record in the 1987 study. In 2001, Barry Bonds had a slugging average of 0.863 and bettered Ruth's mark. Beamon and Ruth, however, are no longer are the holders of their respective records. No matter how impressive their records may have been, they are not active records for consideration in this study.

To achieve meaningful comparisons, we did not compare all of the records with a single analysis. Rather, we continued in the same spirit as the previous studies and broke the overall goal of choosing the greatest sports record down into three manageable problem categories: single-season, career, and single-event records. We selected 19 single-season records, 20 career records, and six single-event records for the study. These records are given in Table 3.1 (single-season

| **Record** | **Holder (2004)** | **Value** | **Year** | **Identifier** |
|---|---|---|---|---|
| | | **Baseball** | | |
| Consecutive-game hitting streak | Joe DiMaggio | 56 | 1941 | DIMAGGIO |
| Lowest earned run average | Bob Gibson | 1.12 | 1968 | GIBSON |
| Stolen bases | Rickey Henderson | 130 | 1982 | HENDRSON |
| Batting average | Rogers Hornsby | 0.424 | 1924 | HORNSBY |
| Home runs | Barry Bonds | 73 | 2001 | BONDSHR |
| Slugging average | Barry Bonds | 0.863 | 2001 | BONDSSA |
| Strikeouts | Nolan Ryan | 383 | 1973 | RYAN |
| Runs batted in | Hack Wilson | 191 | 1930 | WILSON |
| | | **Basketball** | | |
| Average assists per game | John Stockton | 14.5 | 1989-90 | STOCKTON |
| Average rebounds per game | Wilt Chamberlain | 27.2 | 1960-61 | WILTREB |
| Average points per game | Wilt Chamberlain | 50.4 | 1961-62 | WILTSAVG |
| | | **Football** | | |
| Rushing yards | Eric Dickerson | 2105 | 1984 | DICKRSON |
| Points | Paul Hornung | 176 | 1960 | HORNUNG |
| Touchdown passes | Dan Marino | 48 | 1984 | MARINO |
| Touchdowns | Priest Holmes | 27 | 2003 | HOLMES |
| | | **Golf** | | |
| Consecutive tournament victories | Byron Nelson | 11 | 1945 | NELSON |
| | | **Hockey** | | |
| Assists | Wayne Gretzky | 163 | 1985-86 | GRETZKYA |
| Goals | Wayne Gretzky | 92 | 1981-82 | GRETZKYG |
| Points | Wayne Gretzky | 215 | 1985-86 | GRETZKYP |

Table 3.1: Active single-season sports records as of the end of the 2003-2004 season

records), Table 3.2 (career records), and Table 3.3 (single-event records).

### 3.1.2 Data Collection and Comments

We gathered data for this study from many reputable sources. Since most ruling bodies maintain an accurate collection of current historical and statistical information on the Internet, we obtained most data from each individual sport's official web site. In some instances, we consulted other sports web sites and a small collection of sports almanacs for verification purposes, especially when the official site did not contain all information needed.

We note that the timing of the study dictated exactly what data were obtained. All data are current through the end of the most recently completed season as of August 2004. Golf, baseball, football, and tennis records and information are accurate through the end of 2003, while hockey and basketball records and data are accurate through the 2003-04 regular season. Every record

| Record | Holder (2004) | Value | Years | Identifier |
|---|---|---|---|---|
| | **Baseball** | | | |
| Home runs | Hank Aaron | 755 | 1954 - 1976 | AARON |
| Batting average | Ty Cobb | 0.367 | 1905 - 1928 | COBB |
| Stolen bases | Rickey Henderson | 1406 | 1980 - 2003 | HENDRSON |
| Consecutive games | Cal Ripken Jr. | 2632 | 1982 - 1998 | RIPKEN |
| Hits | Pete Rose | 4256 | 1963 - 1986 | ROSE |
| Slugging average | Babe Ruth | 0.690 | 1914 - 1935 | RUTH |
| Strike outs | Nolan Ryan | 5714 | 1966 - 1993 | RYAN |
| | **Basketball** | | | |
| Points | Kareem Abdul-Jabbar | 38387 | 1968 - 1989 | JABBAR |
| Points per game | Michael Jordan | 30.1 | 1984 - 2003 | JORDWILT |
| | Wilt Chamberlain | 30.1 | 1959 - 1973 | |
| Rebounds | Wilt Chamberlain | 23924 | 1959 - 1973 | WILTREB |
| Assists | John Stockton | 15806 | 1984 - 2003 | STOCKTON |
| | **Football** | | | |
| Points | Gary Anderson | 2346 | 1982 - Present | ANDERSON |
| Passing yards | Dan Marino | 61243 | 1984 - 2000 | MARINO |
| Rushing yards | Emmitt Smith | 17418 | 1990 - Present | SMITH |
| Consecutive games with a touchdown pass | Johnny Unitas | 47 | 1956 - 1960 | UNITAS |
| | **Golf** | | | |
| Major professional victories | Jack Nicklaus | 18 | 1959 - 1986 | NICKLAUS |
| Major professional victories | Patty Berg | 15 | 1940 - 1962 | BERG |
| | **Hockey** | | | |
| Points | Wayne Gretzky | 2857 | 1979 - 1999 | GRETZKY |
| | **Tennis** | | | |
| Grand slam singles titles | Pete Sampras | 14 | 1988 - 2002 | SAMPRAS |
| Grand slam singles titles | Margaret Smith Court | 24 | 1962 - 1977 | COURT |

Table 3.2: Active career sports records through the end of the 2003-2004 season

in each of the three categories is based on the standard regular season played by all participants in the sport. As is the standard practice in determining records, post-season performances were excluded from consideration.

One key point must be mentioned regarding our selection of baseball records and all baseball data. We considered only Major League performances from the 1920 season onward when determining baseball records and gathering statistical data. Following the 1919 season, several fundamental rule changes drastically affected the game and therefore had an enormous impact on players' statistics. With the outlaw of "trick pitches" and the decision to use a larger number

| Record | Holder (2004) | Value | Day | Identifier |
|---|---|---|---|---|
| | | **Basketball** | | |
| Points | Wilt Chamberlain | 100 | 03/02/62 | WILTPTS |
| Rebounds | Wilt Chamberlain | 55 | 11/24/60 | WILTREB |
| | | **Football** | | |
| Points | Ernie Nevers | 40 | 11/28/29 | NEVERS |
| Yards rushing | Jamal Lewis | 295 | 09/14/03 | LEWIS |
| Yards passing | Norm Van Brocklin | 554 | 09/28/51 | VBRKLN |
| | | **Track and Field** | | |
| Long jump (m) | Mike Powell | 8.95 | 08/30/91 | POWELL |

Table 3.3: Active single-event records through the end of the 2003-2004 season

of new "lively baseballs" in each game, the sport from 1919 on changed from favoring pitchers to favoring hitters. In his recent book [35], Russell Wright discussed the sharp discontinuity in overall earned run average (ERA), batting average, and stolen bases that occurred between the 1919 and 1920 baseball seasons brought about by these changes. A simple illustration of this fact can be seen by examining the ERA of the Hall of Famer Walter Johnson, whose career spanned this time. From 1907 to 1919, Johnson had only two seasons with an ERA greater than two (the maximum of which was 2.22). However, after the rule changes, from 1920 until his retirement in 1927, he had only two seasons with an ERA of under three (the minimum of which was 2.72). Baseball was a different game after 1919. Therefore, we do not consider records or performances that occurred before this year.

Because of this fact, sports aficionados will notice the omission of one impressive career record: Cy Young's 511 career wins. This record has survived for close to a century and will most likely stand forever. Cy Young pitched from 1890 to 1911 when the game of baseball was dramatically different. Today, only four active pitchers have close to *half* as many wins as Cy Young (Roger Clemens, Greg Maddux, Tom Glavine, and Randy Johnson), and each of these men has been pitching in the Major Leagues for almost 20 years. However, pitching in the twenty-first century is significantly different than pitching in the late nineteenth and early twentieth centuries. Though it may be one of the longest lasting records, we are simply unable to consider Cy Young's career win record as a part of this study.

The golf and tennis records were based on the four recognized major and grand slam tournaments in these two sports, respectively. In men's golf, there are four tournaments–the Masters, the U.S. Open, the British Open, and the PGA Championship–commonly referred to as the majors. The four grand slam tennis events are the same for both men and women: the Australian Open, the French Open, Wimbledon, and the U.S. Open. We used these groups of tournaments to determine the victory totals for the four golf and tennis records.

### 3.1.3 Sports Methodology

The subject of great sports records and sports performances has inspired books [11, 30], columns in magazines [25], newspaper articles [4, 9, 10, 22, 28], and postings on Internet web sites and discussion boards [12, 13, 14, 18]. Most of the works and surveys that make an attempt to rank sports records rarely use any kind of formal methodology. They often are solely built around the opinions of the writer(s) and thus tend to become rather subjective. Since the sports world is built on quantifiable data, a study ranking great records of different sports needs to use some method that accounts for objective data. In this study, we use the analytic hierarchy process (AHP) to capture both objective and subjective qualities of the records.

Following the methods described in Chapter 2, there are four main steps in applying the analytic hierarchy process to sports records: 1) building a hierarchy for each of the three record categories, 2) making the appropriate pairwise comparisons, 3) generating priority vectors, and 4) synthesizing the local weights with respect to the overall goal.

Creating the hierarchy is often the most difficult step in the process. Building the hierarchy for each record category required us to pinpoint different factors and subfactors that are good measures of the "greatness" of sports records. When the first study was carried out in the mid 1980s, the decomposition of the problem into these factors and subfactors was carefully thought through. Since our feelings towards the qualities possessed by great records has not changed over the past two decades, we chose to use the same hierarchies created in the 1987 study for this analysis. Sports have changed over time, but our "yard stick" for greatness remains unchanged. Each hierarchy and the subfactors that influence the goal are described in detail in the sections that follow.

At this point in the standard AHP for our sports problem, every record would be pairwise compared to the others at the subfactor level. Since the sample of single-event records we selected was small in number (six), we were able to use this technique for the analysis of single-event records. For single-season and career records, however, the number of alternatives we considered was rather large, making the task of comparing alternatives time consuming. In this case, the comparisons can be simplified by further dividing each subcriteria into a number of different categories or ratings. Each record is placed in a category with other records that have similar qualities (for subjective subfactors) or possess a similar value (for objective subfactors) with respect to the subcriterion in question.

For the assignment of local priority weights within the single-season and career categories, two techniques can be used: link elements or ratings mode. The link elements method requires the DM to first pairwise compare the alternatives in each category beneath each subcriterion. The categories are then "linked" together using one alternative in each category. Hierarchical synthesis is then performed as for the standard AHP. Though the link elements feature uses individual pairwise comparisons for the assignment of a priority vector, it requires far fewer comparisons than the standard AHP. Richardson's thesis [29] employed this technique for comparing the different sports records.

The alternative to the link elements feature is the ratings mode introduced in Chapter 2. In this study, we chose to use the ratings mode, since the assignment of weights to each alternative then is much easier than for the link elements method. Instead of pairwise comparing the alternatives directly, each alternative is assigned a rating beneath each subcriterion. The ratings beneath each subfactor are pairwise compared to one another and weights are generated just as for the rest of the hierarchy. For the alternative weights, each alternative receives the weight assigned to the rating.

Our use of the ratings mode has its advantages and disadvantages. Since each record is not pairwise compared with the other records, using ratings sacrifices some of the level of detail present in each subcriteria. For each rating, we make no distinction between the records within that rating, even when it is sometimes possible. However, ratings allow us to not be overly specific for the subjective subfactors. We are not required to determine an exact way to measure aspects of records that are not well-quantified in nature (e.g., the degree of "glamorousness"). Ratings also allow us (or a reader) to easily compute how records not included in the study might stand up against the others. Using weights for the hierarchy and ratings, one simply needs to determine which ratings would be assigned to the new record for each subfactor and synthesize the contributions from each of these ratings to find the new record's ranking.

The final two steps in applying the AHP to the sports record problem were carried out using Expert Choice. Expert Choice is a graphical software package used for solving problems in the AHP, providing tools for hierarchy creation, priority vector generation, and overall synthesis. After all the comparisons are entered into the hierarchy, Expert Choice employs Saaty's eigenvector method (EM) [31] to determine the local weights. A multiplicative synthesis, much like finding probabilities using a tree diagram, is carried out to determine and display the overall rankings of the records.

With the framework for the problem in place, we can begin our critique of the great records listed in the previous tables. In the sections that follow, we apply the analytic hierarchy process to the sports record problem. In Section 3.2, we focus on single-season records. In Section 3.3, we cover career records, and in Section 3.4, we analyze the single-event records.

## 3.2 Single-Season Sports Records

Our first study analyzes records that were established in a single season. Season records represent outstanding performances occurring game after game throughout a single year. They indicate a level of season-long achievement and consistency never matched by any athlete in his or her respective sport.

In our attempt to analyze as many records in as many different sports as possible, we have carefully chosen marks that are popular and represent truly great achievements, regardless of the sport. These 19 records are listed in Table 3.1. Nearly every record we considered was based on an entire season's worth of competition, be it a tally of the total number of a certain statistic (e.g., hockey goals) or a per-game average (e.g., average points per game in basketball). We note that two records are exceptions: Joe DiMaggio's 56-game hitting streak and Byron Nelson's streak of 11 consecutive professional victories. These two records are season marks due to their length; both streaks were set during the course of one season's play.

### 3.2.1 Season Hierarchy

Following the four AHP steps described earlier, determining the best single-season sports record began with constructing a hierarchy that takes into account the various factors that influence a record's greatness. In determining the criteria for the hierarchy, it was important that the metrics accurately reflect the significance of each single-season record. The single-season hierarchy created for the 1987 study served this purpose well, and we chose to use the same one for this study. Our single-season hierarchy contains three branches directly beneath the main goal: Duration of Record, Incremental Improvement, and Other Record Characteristics.

Duration of Record captures a time element possessed by each record through the subcriteria Years Record has Stood (YRS) and Years Record is Expected to Stand (YRES). Incremental Improvement allows quantitative comparisons between the athlete's performance and those of other athletes through the subcriteria Percent over Previous Record (PPR) and Percent over Contemporary Mark (POC). The Other Record Characteristics factor captures the subjective qualities of

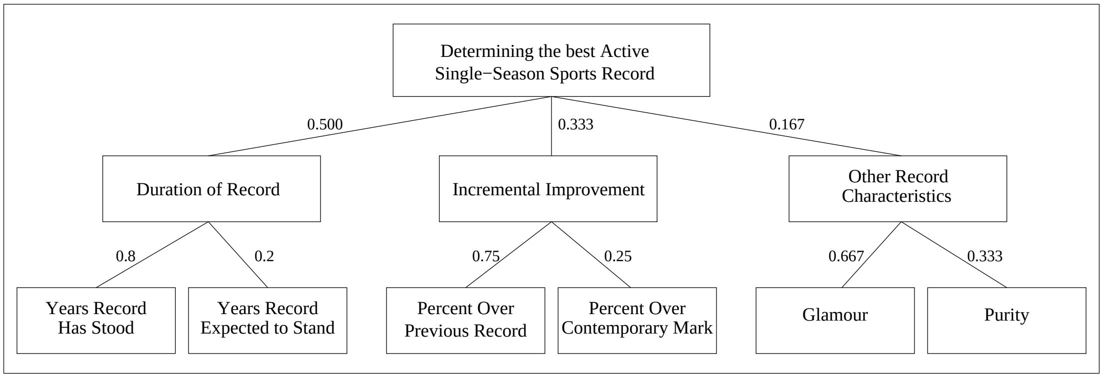


Figure 3.1: Single-season hierarchy with weights

| | Duration | Improvement | Other | **Priority** |
|---|---|---|---|---|
| Duration | 1 | $\frac{3}{2}$ | 3 | 0.500 |
| Improvement | | 1 | 2 | 0.333 |
| Other | | | 1 | 0.167 |

Table 3.4: Pairwise comparisons and weights for the first level criteria of the single-season hierarchy

each record through the subcriteria Glamour and Purity. This is illustrated in Figure 3.1.

After the hierarchy was constructed, we made pairwise comparisons between the factors at each level of the hierarchy. These comparisons were carefully considered, as they are very influential in generating overall weights. The hierarchy comparisons express our feelings towards the relative importance each main factor plays in determining the greatest single-season records. These judgments have not changed in the past 17 years. Therefore, we used exactly the same comparison matrices for the factor and subfactor levels. In Tables 3.4 and 3.5, we show the pairwise comparison matrices associated with the factor and subfactor comparisons, along with the priority vectors generated from these matrices. We used Expert Choice to generate weights for each matrix at each level in the hierarchy. These weights are also displayed on the hierarchy in Figure 3.1.

We now explore each of these subfactors with a little more detail. We provide descriptions of each subfactor and describe how each one impacts the greatness of each single-season record.

| | | | **Priority** | | | | **Priority** | | | | **Priority** |
|---|---|---|---|---|---|---|---|---|---|---|---|
| YRS | 1 | 4 | 0.800 | PPR | 1 | 3 | 0.750 | Glamour | 1 | 2 | 0.667 |
| YRES | | 1 | 0.200 | POC | | 1 | 0.250 | Purity | | 1 | 0.333 |

Table 3.5: Pairwise comparisons and weights for the second level criteria of the single-season hierarchy

### 3.2.2 Duration

- Years Record has Stood

The most important characteristic that distinguishes between good and great records is durability, or, how long a record has been able to stand the test of time. A record that has stood for several decades deserves serious consideration as a great record–it has survived many challenges over the years. A new record that has been established only recently has yet to prove its long-term value. Hack Wilson's runs batted in (RBI) record is a good example of being a durable record. This record has lasted since the depression era (1930) despite recent challenges by Manny Ramirez (165 in 1999), Sammy Sosa (160 in 2001), and Juan Gonzalez (157 in 1999), who had a good chance to break Wilson's record when he had 101 RBIs at the mid-season break. The game of baseball now has nearly double the number of teams and players than in the 1930s, yet Wilson's record has not been broken. On the other hand, the single-season home run and slugging average records set by Barry Bonds in 2001 have only lasted for two full seasons. Bonds's records have not yet demonstrated that they are durable records.

The Years Record has Stood subcriterion incorporates the objective quality of durability into our study. We calculate the years each record has stood by subtracting the year in which the record was set from the year of the most recently completed season (2003 for football, golf, and baseball; 2004 for basketball and hockey). These data, sorted in descending order by years stood, are contained in Table 3.6.

We see that the the records fall into four natural ratings, based on whether the record has lasted (a) greater than 50 years, (b) greater than 25 but less than 50 years, (c) greater

| Identifier | Years Stood |
|---|---|
| Over 50 years | |
| HORNSBY | 79 |
| WILSON | 73 |
| DIMAGGIO | 62 |
| NELSON | 58 |
| 25 years to less than 50 years | |
| HORNUNG | 43 |
| WILTREB | 43 |
| WILTSAVG | 42 |
| GIBSON | 35 |
| RYAN | 30 |
| 10 years to less than 25 years | |
| GRETZKYG | 22 |
| HENDRSON | 21 |
| MARINO | 19 |
| DICKRSON | 19 |
| GRETZKYA | 18 |
| GRETZKYP | 18 |
| STOCKTON | 14 |
| Less than 10 years | |
| BONDSHR | 2 |
| BONDSSA | 2 |
| HOLMES | 0 |

Table 3.6: Number of years elapsed since each single-season record was established

than 10 but less than 25 years, or (d) less than 10 years.

- Years Record is Expected to Stand

A second influential factor affecting the greatness of a record is how long we perceive that it could last. Expected longevity is an important quality of a great record as it measures a record's potential for withstanding future challenges. Recent impressive performances or the emergence of a promising athlete provide an insight to the potential susceptibility of a record. For example, recent seasons of Randy Johnson suggest that Nolan Ryan's single-season strikeout record could possibly be surpassed in the near future. The fact that no one in recent years has stolen half as many bases in a season as that of Rickey Henderson's 138 stolen bases in 1982 leads us to believe that his record will last for many years.

The Years Record is Expected to Stand subfactor was split into three ratings: records expected to last (a) more than 20 years, (b) at least 10 but less than 20 years, or (c) less

| More than 20 years | 10 to 20 years | Less than 10 years |
|---|---|---|
| DIMAGGIO | HENDRSON | BONDSHR |
| GIBSON | STOCKTON | DICKRSON |
| HORNSBY | WILSON | HOLMES |
| WILTREB | BONDSSA | MARINO |
| WILTSAVG | GRETZKYP | RYAN |
| NELSON | GRETZKYG | HORNUNG |
| GRETZKYA | | |

Table 3.7: Three ratings groups for years each single-season record is expected to stand

than 10 years. To assign each record to a ratings group, we examined the best athletic performances in each category from 1994 to 2004. Data regarding these challenges are contained in Table A.1a through Table A.1e in Appendix A. These data provide a way for us to subjectively evaluate how long we expect each record to remain active. Table 3.7 displays our judgments.

### 3.2.3 Incremental Improvement

- Percent over Previous Record

The percentage by which a record broke the previous record is another important measure of the greatness of a record. This subcriterion distinguishes those records that on a percentage basis greatly improve on a previous record from those that barely beat the existing record. During the 1985-86 hockey season, Wayne Gretzky had enough assists alone to break George Esposito's record of 152 points set during the 1970-71 season. Adding the points Gretzky received from goals in 1985-86 gave him a total of 215, almost a 42% improvement over Esposito's record. In contrast, the 27 touchdowns Priest Holmes had during the 2003-04 football season added only one to Marshall Faulk's 26 touchdowns in the 2000-01 season, only about a 4% improvement.

The percentages for this subfactor are calculated by subtracting the previous record's value from the current record's value, dividing the result by the previous record's value, and multiplying this quantity by 100. In Table 3.8, we display the records and their percentages over the previous record, sorted in descending order. Again, following natural breaks in the

| Identifier | Value | Year | Previous Holder | Previous Value | Percent over Previous Value |
|---|---|---|---|---|---|
| 50% or greater | | | | | |
| NELSON | 11 | 1923 | Walter Hagen | 4 | 175.00 |
| WILTSAVG | 50.4 | 1958-59 | Bob Pettit | 29.2 | 72.60 |
| GRETZKYA | 163 | 1970-71 | Bobby Orr | 102 | 59.80 |
| 30% to less than 50% | | | | | |
| GRETZKYP | 215 | 1970-71 | Phil Esposito | 152 | 41.45 |
| DIMAGGIO | 56 | 1922 | George Sisler | 41 | 36.59 |
| MARINO | 48 | 1961 | George Blanda | 36 | 33.33 |
| GIBSON | 1.12 | 1943 | Spud Chandler | 1.64 | 31.71 |
| 15% to less than 30% | | | | | |
| HORNUNG | 176 | 1942 | Don Hutson | 138 | 27.54 |
| GRETZKYG | 92 | 1970-71 | Phil Esposito | 76 | 21.05 |
| WILTREB | 27.2 | 1958-59 | Bill Russell | 23 | 18.26 |
| 5% to less than 15% | | | | | |
| HENDRSON | 130 | 1974 | Lou Brock | 118 | 10.17 |
| WILSON | 191 | 1927 | Lou Gehrig | 175 | 9.14 |
| DICKRSON | 2105 | 1973 | O.J. Simpson | 2003 | 5.09 |
| Less than 5% | | | | | |
| STOCKTON | 14.5 | 1984-85 | Isiah Thomas | 13.9 | 4.32 |
| BONDSHR | 73 | 1998 | Mark McGwire | 70 | 4.29 |
| HOLMES | 27 | 2000 | Marshall Faulk | 26 | 3.85 |
| BONDSSA | 0.863 | 1920 | Babe Ruth | 0.847 | 1.89 |
| HORNSBY | 0.424 | 1922 | George Sisler | 0.420 | 0.95 |
| RYAN | 383 | 1965 | Sandy Koufax | 382 | 0.26 |

Table 3.8: Percent over previous single-season record

data, we divided this subfactor into five ratings: (a) 50% or better, (b) at least 30% but less than 50%, (c) greater than 15% but less than 30%, (d) greater than 5% but less than 15%, or (e) less than 5% over the previous mark. These groups are shown in Table 3.8.

- Percent over Contemporary Mark

  Some records reflect a growing trend in the sport rather than a single great performance (e.g., an emphasis on hitting more home runs). The Percent over Contemporary Mark subfactor takes into account how a record compares to other marks from the year in which the record was set. If a record is broken in a year when multiple athletes have outstanding performances and possibly multiple people break the previous record, it might indicate that the record is not that great. For example, in 1998, both Mark McGwire and Sammy Sosa surpassed Roger Maris's single-season home run record. Although breaking Maris's record

| Identifier | Value | Year | Contemporary | Contemporary Mark | Percent over Contemporary |
|---|---|---|---|---|---|
| 55% or greater | | | | | |
| NELSON | 11 | 1945 | Sam Snead | 3 | 266.67 |
| DIMAGGIO | 56 | 1941 | Bruce Campbell | 27 | 107.41 |
| GRETZKYA | 163 | 1985-86 | Mario Lemieux | 93 | 75.27 |
| HENDRSON | 130 | 1982 | Tim Raines | 78 | 66.67 |
| WILTSAVG | 50.4 | 1961-62 | Walt Bellamy | 31.6 | 59.49 |
| 40% to less than 55% | | | | | |
| GRETZKYP | 215 | 1985-86 | Mario Lemieux | 141 | 52.48 |
| MARINO | 48 | 1984 | Dave Krieg | 32 | 50.00 |
| RYAN | 383 | 1973 | Bert Blyleven | 258 | 48.45 |
| GRETZKYG | 92 | 1981-82 | Mike Bossy | 64 | 43.75 |
| HORNUNG | 176 | 1960 | Gene Mingo | 123 | 43.09 |
| 20% to less than 40% | | | | | |
| HOLMES | 27 | 2003 | Ahman Green | 20 | 35.00 |
| GIBSON | 1.12 | 1968 | Luis Tiant | 1.61 | 30.43 |
| STOCKTON | 14.5 | 1989-90 | Magic Johnson | 11.5 | 26.09 |
| DICKRSON | 2105 | 1984 | Walter Payton | 1684 | 25.00 |
| Under 20% | | | | | |
| BONDSSA | 0.863 | 2001 | Sammy Sosa | 0.737 | 17.10 |
| BONDSHR | 73 | 2001 | Sammy Sosa | 64 | 14.06 |
| WILTREB | 27.2 | 1960-61 | Bill Russell | 23.9 | 13.81 |
| HORNSBY | 0.424 | 1924 | Babe Ruth | 0.378 | 12.17 |
| WILSON | 191 | 1930 | Lou Gehrig | 174 | 9.77 |

Table 3.9: Percent over contemporary mark for single-season records

was an amazing feat for both McGwire and Sosa, the fact that more than one person hit more than 61 homers in 1998 when no one person had in the previous 37 years suggests maybe something was special about that year. It detracted somewhat from McGwire's record's greatness.

The data for this subcriterion are contained in Table 3.9. For each record, we give the second best performance in the year that the record was established. These records are broken down into four ratings groups based on the percentage that the record improved a contemporary's mark: (a) greater than 55%, (b) at least 40% but less than 55%, (c) at least 20% but less than 40%, or (d) less than 20% over the contemporary mark. These groups are indicated in Table 3.9.

| Well Known | Known | Not Always Known | Not Known |
|---|---|---|---|
| DIMAGGIO | DICKRSON | HORNSBY | HORNUNG |
| BONDSHR | GRETZKYP | HENDRSON | WILTREB |
| | WILTSAVG | WILSON | GIBSON |
| | GRETZKYG | MARINO | STOCKTON |
| | | GRETZKYA | BONDSSA |
| | | HOLMES | NELSON |
| | | RYAN | |

Table 3.10: Four ratings groups for glamour - single-season records

### 3.2.4 Other Record Characteristics

- Glamour

Public perception is another important factor that we need to take into account. Records that are easily recognized by an average member of the sports community are likely to be seen as superior to records that lie in obscurity. For example, most people who have seen a baseball game could name the current single-season home run record holder, Barry Bonds. Very few of them could also name the single-season slugging average record holder (it's also Barry Bonds), much less provide a definition for slugging average. Both records are exceptional, but Barry Bonds will be remembered for hitting baseballs out of SBC Park into McCovey Cove, not averaging 0.863 bases per at bat in 2001.

The Glamour subcriterion provides a subjective measure of how well-known a record is. Four ratings are used to describe the level of glamour associated with a record: (a) records that are well known by most people, (b) records that are known by avid fans but only might be identified by an above-average person, (c) records that are not always known by avid fans and not known by the average person, and (d) records that are not known or easily identified by most people. In Table 3.10 we show the four groups.

- Purity

Some records are set entirely by the play of one individual, while other records depend on assistance from teammates and, sometimes, opposing players. It was solely the responsibility of Rogers Hornsby to be an efficient hitter each at-bat. He might have bene-

| Not Aided | Slightly Aided | Greatly Aided |
|---|---|---|
| DIMAGGIO | NELSON | HOLMES |
| GIBSON | WILTREB | DICKRSON |
| RYAN | STOCKTON | MARINO |
| HENDRSON | GRETZKYA | HORNUNG |
| HORNSBY | GRETZKYP | WILSON |
| BONDSHR | GRETZKYG | WILTSAVG |
| BONDSSA | | |

Table 3.11: Three ratings groups for purity - single-season records

fited at times by facing weak pitchers, but it was still up to him to get hits. Dan Marino's single-season touchdown passes record required not only great execution on his part, but also excellent pass protection, skilled receivers, and (perhaps) poor pass defenses.

The Purity subcriterion reflects whether or not an individual was aided in setting a record. We divided this subcriterion into three ratings: (a) records not aided by others, (b) records slightly aided by others, and (c) those greatly aided by others. The rating assigned to each record is given in Table 3.11.

### 3.2.5 Single-Season Results

Once we had assigned all records to an appropriate rating for the six subfactors, we pairwise compared the ratings to generate the local priority weights. The pairwise comparison matrices using these judgments along with the priority vector generated for each one are contained in Section A.1.2 of Appendix A. Using hierarchical synthesis, we obtained the weights for the 19 single-season records. The overall rankings for single-season records are given in Figure 3.2.

Three records stand out in Figure 3.2: Byron Nelson's 11 consecutive professional victories, Joe DiMaggio's 56-game hitting streak, and Wilt Chamberlain's 50.4 points per game average. There is a "dead heat" between Nelson and DiMaggio's streaks for the greatest single-season sports record with only a few thousandths of a point separating the two records. Wilt Chamberlain's 50.4 points per game average in the 1961-62 season was a close third. The weights for the remaining 16 records decrease somewhat continuously towards zero.

Although Nelson's record is less known than DiMaggio's record, most sports aficionados

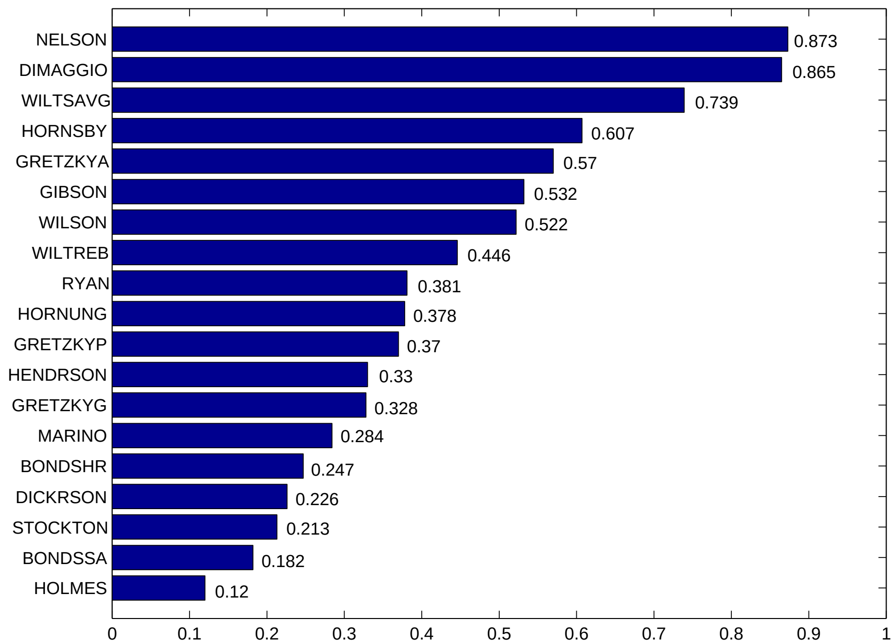


Figure 3.2: Overall rankings of the 19 single-season records

will not dispute the greatness of these two records. In comparison to the other season marks, these two records required a greater amount of consistency from these two athletes over a season to set the record. For the other single-season marks, a poor performance one night could be offset by a great performance sometime later in the season. Since Nelson and DiMaggio's records were both streaks, an off night or an off weekend simply did not take place.

The records of Nelson and DiMaggio will likely last for a long time for several reasons. When both marks were being established in the 1940s, they were generally well-followed by fans, and their performances were popularized by the media. No doubt, the pressures that surrounded these two athletes were significant and affected their performances to some degree. The short essays in *Sacred Records* [11] give a few accounts of some the extreme pressures placed on these two men by the general public. DiMaggio's friend Lefty Gomez would often help him elude the public to try to avoid the extra attention his streak was attracting. The amount of media attention these two records received in the 1940's would likely be dwarfed by the present-day media flurry that would occur should either streak be challenged today. For example, when a baseball player approaches hitting in 30 consecutive games, his progress is already closely monitored by national media (e.g., Luis Castillo's 35-game hit streak in 2002 or Albert Pujols' 30-game streak in 2003). One can only guess what might occur with the media should someone approach the 40- to 50-game marks. Most likely, sports networks such as ESPN and Fox would have hourly updates of the player's at-bats, akin to the McGwire-Sosa-Bonds home run craze of the past decade or Ichiro Suzuki's pursuit of George Sisler's single-season hits record during the late 2004 season. Public pressures would likely be more burdensome on the athlete than they were 60 years ago.

One reason that Nelson's record is practically untouchable is that winning 11 tournaments in one year itself is considered highly improbable for a professional golfer. When Tiger Woods had nine wins in 2000, it was the most wins by a golfer in a season in half a century–Sam Snead had 11 wins in 1950. As unlikely is winning 11 events, having those 11 wins occur in 11 straight tournaments is nearly impossible. Nelson himself recently noted the difficulties facing potential challengers to his record today [24]. During the 1945 season, Nelson started his streak by winning the Miami International Four Ball in March. He won the next 10 events in which he competed until his streak was broken in the middle of August. His streak was aided by a few extended breaks in the tournament schedule, one as long as two months (from early April to early June) [8]. In addition, he skipped the St. Paul Open in late July. Tournaments on the PGA Tour today are played almost every week from mid-January through November, and most top golfers today only win two or three times a year. (Recent exceptions are Vijay Singh's nine wins this year and Tiger Woods's nine wins in 2000 and eight wins in 1999.) Winning 11 times in a row in a single season is out of reach for today's top professional golfers. Thus, Nelson's mark will likely never be equaled again.

Among the remaining records, there are no big surprises. Wilt Chamberlain's records for the average number of points per game and the average number of rebounds per game were both ranked high in the study. These two records fare well because they improve upon the previous record and contemporary marks by a considerable amount. Wayne Gretzky's single-season assist record fared the highest of his three records because of the same two subcriteria. The baseball records of Hornsby, Gibson, and Wilson are rated high because they have lasted for many years and should last several years into the future.

It might seem surprising that Barry Bonds's home run record ranks low on our list. This record has already been mentioned as one of the greatest records of all-time [10, 12, 20]. With our criteria, it does not receive a large weight since it was only set three years ago. This record has yet to be given time for challengers to show just how great a feat hitting 73 home runs may very well be. When McGwire hit 70 homers in 1998, Barry Bonds was hitting around 40 home runs in a season. Hardly anyone thought that Bonds would be the one breaking McGwire's record, even if they thought that McGwire's mark could be broken. In addition, Bonds's home run record does not fare particularly well in other categories. For example, on a percentage basis, he only slightly improved McGwire's 70 home runs. Sammy Sosa's 64 home runs in 2001 were close to Bonds's 73, lowering Bonds's mark with respect to contemporary performances. Thus, given our criteria, Bonds' record does not rank as one of the top records.

As of November 2004, only one record appears in jeopardy of being broken by a performance in the 2004-05 season: Dan Marino's 48 touchdown passes in a single-season. Through only eleven games, Peyton Manning nine more touchdown passes (41) than the league leader did in the 2003-04 season (Brett Favre, 32). During his last two games in November of 2004, Manning threw a very impressive 10 touchdown passes. At his current pace, he could reach as many as 60 touchdown passes and easily surpass Marino's 20-year-old record.

### 3.2.6 Comparison of Rankings to the Results of Previous Studies

In comparing our single-season record rankings to the results of Golden and Wasil [16] and the results of Richardson's thesis [29], we notice several similarities. While we cannot compare the exact weights associated with each record in those studies to the weights here, we can still observe the ordering of the records in those studies. In general, the rankings among all three studies are consistent with one another. Very little shuffling occurs among the records that the studies have in common. Only newly-set records make substantial moves in the rankings.

In Richardson's thesis, the single-season records of Nelson and DiMaggio were also at the top of the single-season rankings. However, the gap between Nelson and DiMaggio given in our study narrows. This is attributable mostly to the use of ratings for the Percent over Contemporary Mark. Nelson's 267% improvement over Sam Snead's three consecutive wins in 1945 is more than double DiMaggio's 107% improvement over Bruce Campbell's 27-game hitting streak in 1941. For direct comparisons, Nelson's record should receive more weight than DiMaggio's. With the use of ratings, Nelson and DiMaggio receive the same weight with respect to this subfactor because they both greatly improved their contemporaries' marks. The 1987 study did not include Nelson's record.

Babe Ruth's single-season slugging average record ranked third in Richardson's list of single-season records and finished first in Golden and Wasil's study. As mentioned earlier, Barry Bonds surpassed Ruth's slugging average in 2001. Here, Bonds's slugging average record finishes next-to-last in the rankings. Due to its short duration, its small relative improvement over Ruth's record, and Sammy Sosa's good slugging average in 2001, Bonds's slugging average record does not fare as well as did Babe Ruth's slugging average record in the two previous studies.

In our study, one of the biggest movers up the single-season rankings compared to Richardson's thesis is Wayne Gretzky's single-season assists record. Since only one player in the past five years has had even half of the number of assists that Gretzky had during the 1985-86 season, we now believe Gretzky's record will remain unchallenged for the next 20 or so years. Gretzky's assist record has moved above Hack Wilson's RBI record, Bob Gibson's ERA record, and Wilt Chamberlain's average rebounds per game record.

## 3.3 Career Sports Records

In our second study, we examined the ultimate measure of a player's consistency, career records. Often, career records are a better overall indicator of an athlete's true talent than single-season records because they require the athlete to not only produce outstanding performances during one season, but to also repeat this performance year after year. Most career records we examined consist of statistical measurements that span an athlete's entire career, from rookie season to retirement (e.g., scoring average). We also considered some career records that are based on only a few years of one's career (e.g., consecutive games played). The 20 career records that we studied are given in Table 3.2.

### 3.3.1 Career Hierarchy

As with the season records, building the career hierarchy was the first task. We were satisfied with the career hierarchy from the previous studies and therefore used the same hierarchy found in Richardson's thesis [29] and Golden and Wasil's paper [16]. We note that the career hierarchy is almost identical to the single-season hierarchy, with the one exception being the Incremental Improvement factor. Since career records are set over long durations, anomalies that might occur during a single season tend to average out over time. The need to compare career statistics relative to an athlete's contemporaries (which is not practical with regards to careers) is not accounted for in the career hierarchy. Thus, we have only one subfactor beneath Incremental Improvement, that is, Percent over Second Best. The Duration and Other Record Characteristics branches are exactly as described in the single-season section.

The pairwise comparison matrices for the first and second levels of the hierarchy are the same for career and single-season records. These matrices, along with the hierarchy weights generated from them, are contained in Table 3.4 and Table 3.5. The hierarchy with these weights is illustrated in Figure 3.3.

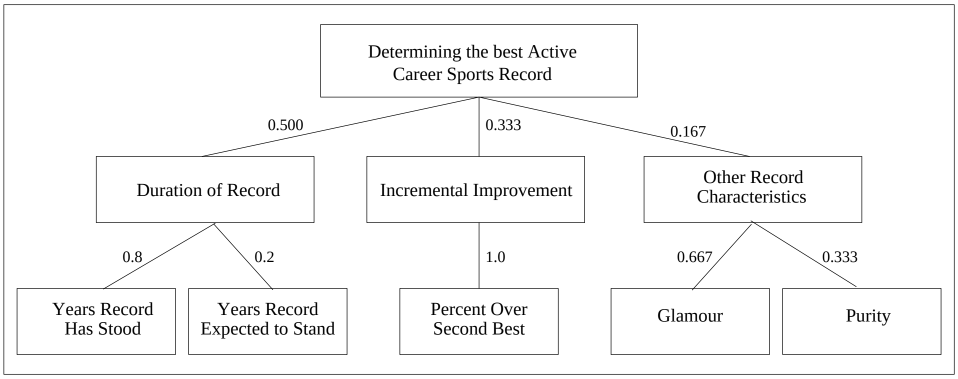


Figure 3.3: Career hierarchy with weights

### 3.3.2 Duration

- Years Record has Stood

Great career records stand the test of time. Career records established decades ago have been able to demonstrate that their marks have the ability to last over time, while newly-established records have yet to show that they are not simply the result of a growing trend. As with season records, this subfactor takes into account how long each career mark has lasted in its sport.

The calculation of how long each record has stood was tricky for career records. For many of the records, the new record-setter passed the previous record somewhat late in his or her career and continued to add to his or her record for a few more seasons until retirement. However, instead of using the exact year in which the new record was established, we used the athlete's retirement year as the starting point for the measurement of how long the record has stood. This choice was made for two reasons. First, some records are based on the average over an entire career. Simply exceeding the previous average does not guarantee that the athlete will hold the record when he retires. Second, determining exactly when the record-breaking activity occurred might not be possible or very easily identified for each record. To maintain consistency, we used the retirement year for the calculation of this subfactor.

There were three exceptions. Over the past two decades, Jack Nicklaus continued to

| Identifier | Years Stood |
|---|---|
| Over 50 years | |
| COBB | 75 |
| RUTH | 68 |
| 25 years to less than 50 years | |
| UNITAS | 43 |
| BERG | 41 |
| WILTREB | 31 |
| AARON | 27 |
| COURT | 26 |
| 10 years to less than 25 years | |
| NICKLAUS | 17 |
| ROSE | 17 |
| JABBAR | 15 |
| RYAN | 10 |
| Under 10 years | |
| RIPKEN | 5 |
| GRETZKY | 5 |
| MARINO | 3 |
| SAMPRAS | 1 |
| JORDWILT | 1 |
| STOCKTON | 1 |
| ANDERSON | 0 |
| HENDRSON | 0 |
| SMITH | 0 |

Table 3.12: Years each career record has stood

play in some of the majors each year, but the last time he was a serious contender in a championship was when he won the Masters in 1986. Thus, we used 1986 as his "retirement" year. The other two records were unique because they were streaks: Johnny Unitas' touchdown passing record and Cal Ripken's consecutive game streak. They both continued to play their sport for several more years, yet they never added to their record after once their streak was over. Thus, we used the year in which each streak ended when computing the years stood for these two records.

The data show that the records fall into several natural groups, and at these breaks we grouped the records into four ratings based on the length of time elapsed since the record was set: (a) greater than 50 years, (b) greater than 25 but less than 50 years, (c) greater than 10 but less than 25 years, and (d) less than 10 years. These groupings, along with the data for this subfactor, are contained in Table 3.12.

| Greater than 20 years | 10 to 20 years | Less than 10 years |
|---|---|---|
| COBB | JORDWILT | AARON |
| RUTH | STOCKTON | JABBAR |
| RYAN | UNITAS | ANDERSON |
| HENDRSON | NICKLAUS | SMITH |
| ROSE | | MARINO |
| WILTREB | | SAMPRAS |
| GRETZKY | | BERG |
| RIPKEN | | |
| COURT | | |

Table 3.13: Three ratings groups for years each career record is expected to stand

- Years Record is Expected to Stand

  The Years Record is Expected to Stand subfactor represents the same measure for career records as that used for season records. It is a subjective evaluation of how long we feel each record will last. To make these decisions, we gathered data on several potential athletes that could become record setters in the future. These data are given in Table A.2a, Table A.2b, and Table A.2c in Appendix A.

  Many of the career records are expected to last a very long time due to either the lack of any realistic present-day challengers or the fact that some records simply require many years of excellent play by an athlete before the athlete is able to amass enough statistics to break the current record. As with the single-season records, we have grouped the 20 career records into three ratings based on whether the record is expected to last (a) at least 20 years, (b) more than 10 years but less than 20 years, and (c) less than 10 years. These groupings are given in Table 3.13.

### 3.3.3 Incremental Improvement

- Percent Over Second Best

  We also measure the amount by which a career record is better than its closest competitor. In the Percent over Second Best subfactor, we compare the value associated with each record to the value of the athlete with a total that is closest to the record's value. Unlike season records, it is important that we measure career records against all challenges to the

| Identifier | Value | Second Best | Value | Percent over second best |
|---|---|---|---|---|
| 40% or greater | | | | |
| NICKLAUS | 18 | Walter Hagen | 11 | 63.64 |
| UNITAS | 47 | Dan Marino | 30 | 56.67 |
| STOCKTON | 15806 | Mark Jackson | 10334 | 52.95 |
| GRETZKY | 2857 | Mark Messier | 1887 | 51.40 |
| HENDRSON | 1406 | Lou Brock | 938 | 49.89 |
| 15% to 40% | | | | |
| RYAN | 5714 | Steve Carlton | 4136 | 38.15 |
| RIPKEN | 2632 | Lou Gehrig | 2130 | 23.57 |
| MARINO | 61243 | John Elway | 51475 | 18.98 |
| SAMPRAS | 14 | Roy Emerson | 12 | 16.67 |
| BERG | 15 | Mickey Wright | 13 | 15.38 |
| 5% to 15% | | | | |
| WILTREB | 23924 | Bill Russell | 21620 | 10.66 |
| COURT | 24 | Steffi Graf | 22 | 9.09 |
| RUTH | 0.69 | Ted Williams | 0.634 | 8.83 |
| AARON | 755 | Babe Ruth | 714 | 5.74 |
| Less than 5% | | | | |
| SMITH | 17418 | Walter Payton | 16726 | 4.14 |
| JABBAR | 38387 | Karl Malone | 36928 | 3.95 |
| ANDERSON | 2346 | Morton Andersen | 2259 | 3.85 |
| COBB | 0.367 | Rogers Hornsby | 0.358 | 2.51 |
| ROSE | 4256 | Ty Cobb | 4191 | 1.55 |
| JORDWILT | 30.1 | | | 0 |

Table 3.14: Second-best performance to each of the career records

record, both before and after the record has been established. Consider Kareem Abdul Jabbar's career points record. Though Jabbar's total number of points was almost 7000 points better than any player's point total before him (a 22% improvement over Wilt Chamberlain), he is now only slightly ahead of Karl Malone. The proximity to Malone's point total reduces the greatness of his record.

We searched the Internet and through sports almanacs to find the second-best career performances for each career record. We show this data in Table 3.14. As seen in the table, we used four ratings based on the percentage over the second best mark: (a) more than 40%, (b) greater than 15% but less than 40%, (c) between 5% and 15%, or (d) less than 5%.

We gave Michael Jordan's 30.1 career scoring average record a value of zero for the Percent over Second Best subfactor. We artificially set this percentage to zero because two different people hold the average points per game record. Records that stand alone, far ahead

| Well Known | Known | Not Always Known | Not Known |
|---|---|---|---|
| AARON | RYAN | MARINO | RUTH |
| ROSE | JABBAR | JORDWILT | COBB |
| NICKLAUS | HENDRSON | STOCKTON | WILTREB |
| GRETZKY | SMITH | UNITAS | ANDERSON |
| | SAMPRAS | | BERG |
| | RIPKEN | | COURT |

Table 3.15: Four ratings groups for glamour - career records

of any other performance, are much more indicative of a great achievement than those records that are held by multiple players. The fact that Michael Jordan and Wilt Chamberlain both hold a 30.1 per game scoring average makes it less impressive than the other 19 career records. One interesting bit of information is that Michael Jordan's second return from retirement hurt his points per game average somewhat. At the time of Richardson's thesis, Jordan held a 31.5 point per game scoring average. When he returned to play with the Washington Wizards for two seasons, his career average decreased to 30.1 points per game, bringing his average back into a tie with Wilt Chamberlain.

### 3.3.4 Other Record Characteristics

- Glamour and Purity

  The subfactors Glamour and Purity capture the same quality of greatness of career records as they do for season records. They measure the degree to which each record is known (Glamour) and the extent to which each record is a reflection of the athlete's individual performance (Purity). The four ratings used for Glamour are given in Table 3.15, while the three ratings for Purity are given in Table 3.16.

### 3.3.5 Career Results

In order to determine the local weights beneath each subfactor, we performed the comparisons between each pair of ratings as described in the methodology section. The matrices containing these comparisons are located in Appendix A (Section A.2.2). After making these comparisons, we used Expert Choice to generate the priority vector for each subfactor and to synthesize the overall

| Not Aided | Slightly Aided | Greatly Aided |
|---|---|---|
| AARON | GRETZKY | ANDERSON |
| COBB | JABBAR | WILTREB |
| HENDRSON | JORDWILT | MARINO |
| RIPKEN | SMITH | UNITAS |
| ROSE | STOCKTON | |
| RUTH | SAMPRAS | |
| RYAN | COURT | |
| NICKLAUS | BERG | |

Table 3.16: Three ratings groups for purity - career records

weights. In Figure 3.4, we display the rankings of the career records.

There were four records with a score greater than 0.600 at the top of the list of great career records. Babe Ruth's 0.690 career slugging average had the highest score, followed by Wayne Gretzky's 2857 hockey points, Jack Nicklaus's 18 major tournament victories, and Ty Cobb's 0.367 career batting average. Rickey Henderson's 1406 stolen bases and Johnny Unitas' 47 consecutive games with a touchdown pass were near the top with scores just under 0.600.

In examining the overall results, there appears to be three distinct levels within the career records: an upper level (records RUTH through UNITAS), a middle level (from STOCKTON through WILTREB), and a lower level (from SAMPRAS through ANDERSON). The records in the upper level soundly beat the closest marks. Four of the six records in the upper level were better than the second place mark by nearly 50 percent or more. The two records that improved the second place mark by only a small amount, RUTH and COBB, were given the biggest boost by the duration that each record has lasted. RUTH and COBB have both lasted more than 25 years longer than the rest of the career records; they both have been around for nearly three-quarters of a century. The GRETZKY, HENDRSON, and NICKLAUS records were assigned the lowest two ratings for the Years Record has Stood, getting very little contribution from this subfactor.

Babe Ruth's career slugging average is a great record. A very small number of players each baseball season have slugging percentage greater than 0.690, and maintaining that high of an average for an entire career can be accomplished by only the best sluggers in the history of the game. Barry Bonds now holds the record for the best single-season slugging average, and over the last few seasons, he has consistently had a slugging average between 0.700 and 0.900. However,

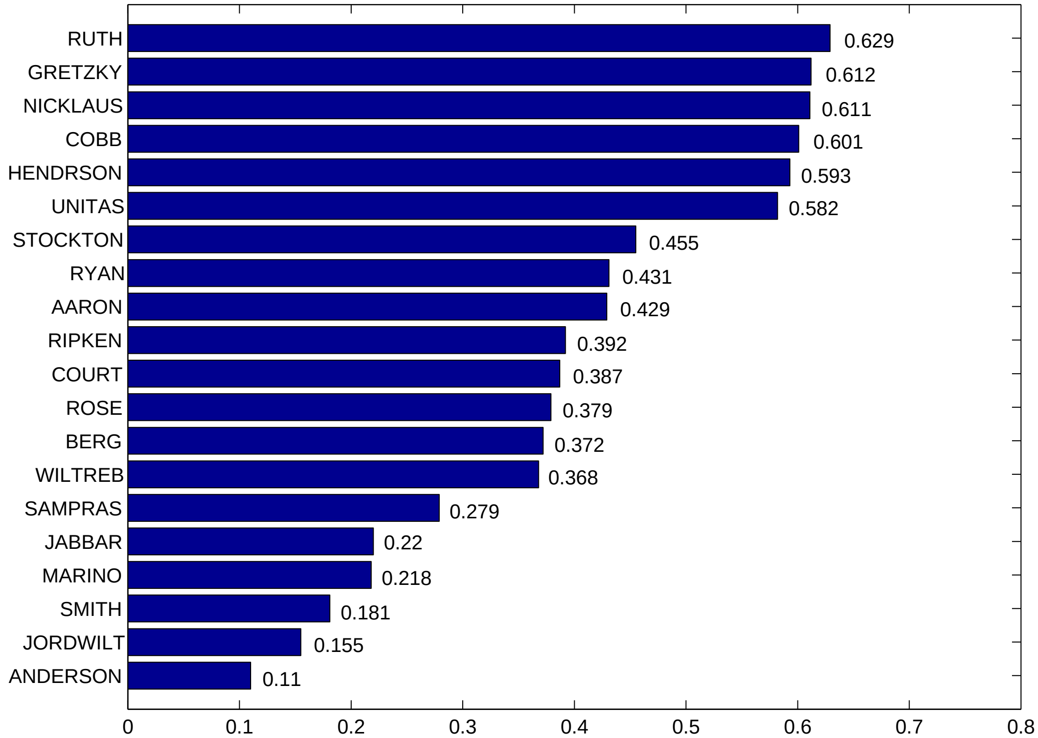


Figure 3.4: Overall rankings of the 20 career records

early in his career, Bonds did not have a slugging percentage close to what he had over the past several seasons. This will hurt his chances of surpassing Ruth's career record.

Wayne Gretzky's points total record is a very recent record that fared well in the study. Gretzky retired in 1999, yet, his record still finishes in second place. Gretzky's record is among the best records with respect to several subfactors including Percent over Second and Years Record is Expected to Stand. His point total is so far ahead of the second place total set by Mark Messier that the points associated with his total number of assists alone (1963) is greater than Messier's entire point total. In addition, the highest single-season point total scored by anyone in the NHL has barely exceeded 100 points over the past several years. Given the length of time required to amass 2,800 points, it seems that Gretzky's record is likely to last more than 20 years into the future. Though Gretzky's record does not receive much weight with respect to duration, it benefits greatly from several other subfactors and is worthy of its position near the top of the rankings.

Several of the popular career records could have new holders in the near future. Many fans expect Barry Bonds to continue playing and surpass Hank Aaron's home run record within the next two seasons. With eight victories in the majors before the age of 30, Tiger Woods could challenge Jack Nicklaus's hold on the record for the most men's majors. Despite his current drought, should Tiger have another run like his 2000 through 2002 seasons in which he won half of the majors in those years (six), he could end up surpassing Nicklaus in the next decade. Kareem Abdul Jabbar's basketball points record is also within reach, as it could be passed by Karl Malone if continues to play and remain healthy. As of the 2004-05 football season, Gary Anderson is still kicking for the Tennessee Titans. Should he retire at the end of this season, his career football points record could be broken next year by Morten Andersen, who is less than 100 points behind Anderson.

Some records could move slightly higher or lower in the rankings because of athletic performances occurring during the 2004 season that might change the rating assigned to each record. Emmitt Smith continues to add to his career rushing total, and, during the 2004-05 season, he already has enough yards to make his total over five percent greater than Walter Payton's second best total of 16,726 yards. Brett Favre now holds the second-best mark for consecutive games with a touchdown pass after he threw for a touchdown in his first eleven games of the 2004-05 season, giving him 36 through the end of November, 2004. When the study is updated to include the present season, Johnny Unitas' record will fall with respect to the Percent over Second Best subfactor and therefore also fall in the overall rankings. The second best hockey point total is now held by an active player (Mark Messier). However, in 2004, the players were locked out by management, and the 2004-05 NHL season seemed unlikely as of late November, 2004. Gretzky's record is likely to retain the same rating for Percent over Second Best.

The two records set by women fared rather well in our overall rankings. Margaret Smith Court's 24 tennis grand slam singles titles and Patty Berg's 15 major professional victories both finished in the middle tier of records. Both records finished high with respect to duration; they have been around for over a quarter of a century. They face different challenges in the future, however. With Steffi Graf's retirement, Court's record should last for many years to come. Though several promising young women in tennis today have the ability to win multiple grand slam tournaments, the off-court distractions of becoming a sports icon will likely prevent any current player from winning more than 24 grand slam titles. On the other hand, Berg's record seems within reach of Annika Sorenstam. Currently, Sorenstam has half as many major victories as did Berg. If she continues to lead the LPGA tour as she has in the past several years, with another 10 years or so of competitive golf left in her career, she could challenge Berg's record.

### 3.3.6 Comparison of Rankings to the Results of Previous Studies

In general, the records near the top of Richardson's work and the 1987 study are the same as those presented here. However, there are a few differences in the order at the top of the rankings in this study, Richardson's thesis [29], and the study of Golden and Wasil [16]. Our study ranks Babe Ruth's career slugging average as the greatest career record, while both Richardson and Golden and Wasil have Ruth's record as a close second. Ty Cobb's career batting average tops Richardson's career list, while Johnny Unitas' record for consecutive games with a touchdown pass was deemed the greatest by Golden and Wasil. In our rankings, Cobb's record finished fourth, while Unitas' record finished sixth.

There are very few differences between Cobb's record and Ruth's record. They both have lasted for many decades, are expected to last for several decades to come, and are representative of individual, rather than team-aided, performances. The only subfactor where Cobb's record prevails over Ruth's is duration; Cobb's career ended seven years earlier than Ruth's career. However, these records were both assigned the same rating because they have lasted significantly longer than the other career records. Cobb's record finishes lower than Ruth's because it is only 2.5% over Rogers Hornsby's 0.358 career batting average, while Ruth's record is almost 9% greater than Ted Williams's 0.634 career slugging average. For this subfactor, COBB and RUTH were assigned different ratings. Therefore, Ruth's record beats Cobb's record in the final analysis.

Compared to Richardson's results, Wayne Gretzky's point record and Rickey Henderson's career stolen base record each moved up several places in our rankings. Gretzky's record placed second, only slightly ahead of Nicklaus's majors, while Henderson's record moved ahead of Unitas' record, Nolan Ryan's career strikeout record, and Hank Aaron's career home run record. Henderson's record improved primarily because he continued to add to his career total until the end of the 2003 season, and his record has become as well known as some of the other notable baseball records. Gretzky did not hold the hockey career points record in 1987. At that time, Gordie Howe's hockey points record was good enough to rank fourth out of ten career records in Golden and Wasil's study. Gretzky's career performance surpassed Howe by over 1000 points, more than a 50% increase, and that has placed this record high in the overall rankings.

Most of the other records fared about the same in these new rankings as they did in the previous two studies. Many records stayed in the same position relative to other career records, while the few records that have been broken over the last several years have fallen near the bottom. Overall, the studies are consistent in their rankings of the greatest career records.

## 3.4 Single-Event Sports Records

The final category of sports records that we considered were single-event records, or records that arise from a single unit of competition. A single unit of competition depends on the sport; it could be a single game, a single sprint, a single jump, and so forth. Notably, all single-event records arise from activity occurring on a single day.

Selecting single-event records was more difficult than for season and career records. Many of the single-event records are obscure and unknown, remaining unrecognized by the general public until an athlete ties or breaks the record. For example, while the single-game record for yards gained rushing in a football game is well known, the record for the number of rushing attempts in a game by a single running back will likely only become known if it is broken by an athlete today.

Some records that could potentially be included are broken on a somewhat regular basis. A record that is broken often is not that great of a record because it would not be expected to last for a long period of time. It seems that at least a few world records in swimming are set at each major worldwide competition. For example, at the 2004 Summer Olympics, Michael Phelps set the world record in the 400-meter individual medley for the fifth time in two years [1]. Including records such as Phelps's in this study, much less keeping track of the active holder and time of other records that change often, is simply not always feasible.

Other popular records that we might consider are held by numerous individuals. Examples of these include the number of home runs in a baseball game, the number of touchdown passes in a football game, and most free throws made in one quarter of an NBA basketball game. Twelve baseball players have hit four home runs in a single nine-inning baseball game. Since so many players share the record, hitting four home runs in a game does not distinguish itself as a great record. Such records do not fit well into our study.

We used many of the same single-event records as Richardson did in his thesis [29]. These six records are listed in Table 3.3.

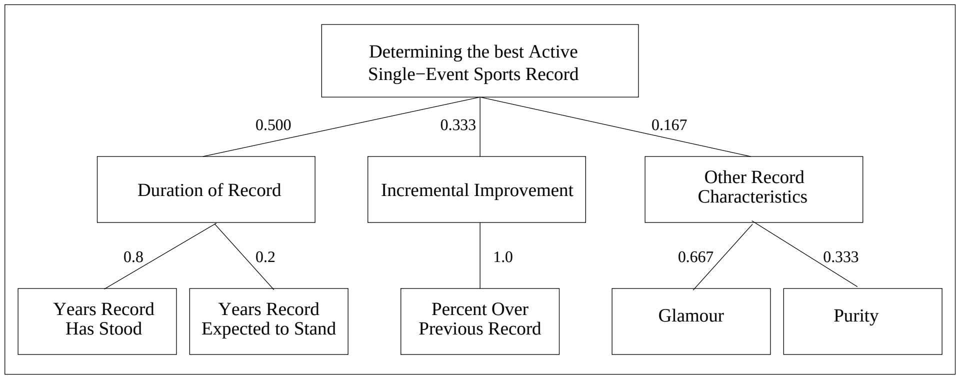


Figure 3.5: Single-event hierarchy with weights

### 3.4.1 Single-Event Hierarchy

The single-event hierarchy also has the same structure as the single-season hierarchy, with the only exception being the exclusion of the Percent over Contemporary Mark subfactor beneath Incremental Improvement. The pairwise comparison matrices used to generate the hierarchy weights weights follow directly from the single-season and career record categories. The single-event hierarchy with weights is shown in Figure 3.5.

For the small number of single-event records, we were able to directly pairwise compare the records with respect to each subfactor, as described in Subsection 3.1.3. Thus, we employed the traditional AHP to generate weights for each subfactor.

### 3.4.2 Duration

- Years Record has Stood

  The Years a Record has Stood subfactor measures the length of time elapsed since each single-event record was set. A record such as Ernie Nevers' single-game points record for football has lasted for three quarters of a century. For nearly 75 years, his record has demonstrated that scoring 40 points in a football game is a great accomplishment. In Table 3.17, we give the calculation for the number of years each record has lasted.

  The pairwise comparisons for this subfactor were determined by creating a ratio for each pair of alternatives based on the years the record has lasted. Since Jamal Lewis's record

| Identifier | Years Stood |
|---|---|
| NEVERS | 74 |
| VBRKLN | 52 |
| WILTREB | 43 |
| WILTPTS | 41 |
| POWELL | 12 |
| LEWIS | 0 |

Table 3.17: Number of years each single-event record has lasted

was just set last year (2003), we assigned a value of nine to the pairwise comparisons with this record. The pairwise comparison matrix and its priority vector can be found in Appendix A in Table A.4.

- Years Record Expected to Stand

This subfactor captures our subjective feelings as to how long we believe each record will last. We collected data for each record regarding top performances over the past 10 years. In Table A.3 in Appendix A, we document several recent single-event performances from 1994-2004 that have challenged each current single-event record.

By examining these values, we judged how long we though each record would last and compared the records. The pairwise comparison matrix that was created from these comparisons is given in Table A.5.

### 3.4.3 Incremental Improvement

- Percent over Previous Record

The amount by which the current record surpasses the previous mark plays an important role in evaluating a record's greatness. Simply scoring one more point or running a tiny fraction of a second faster than the previous record's value is not as impressive as performances that greatly surpass the previous record on a percentage basis.

In Table 3.18, we give the value of each previous record holder, along with the percentage improvement of the current record's value over the previous value. After computing the percentages for this subfactor, we constructed the pairwise comparison matrix. This matrix

| Identifier | Value | Year | Previous Holder | Previous Record | Percent Over Previous Record |
|---|---|---|---|---|---|
| WILTPTS | 100 | 1960 | Elgin Baylor | 71 | 40.85 |
| NEVERS | 40 | 1922 | James Conzelman | 30 | 33.33 |
| VBRKLN | 554 | 1949 | Johnny Lujack | 468 | 18.38 |
| WILTREB | 55 | 1960 | Bill Russell | 51 | 7.84 |
| LEWIS | 295 | 2000 | Corey Dillon | 278 | 6.12 |
| POWELL | 8.95 | 1968 | Bob Beamon | 8.9 | 0.56 |

Table 3.18: Single-event previous record holders

is given in Table A.6.

### 3.4.4 Other Record Characteristics

- Glamour and Purity

  The Glamour and Purity subcriteria both follow from the two previous record categories. Some records are more recognizable than others by the average sports fan and are more well known. In addition, each record provides a different indicator of an individual's talent than other records. These two subcriteria attempt to capture these qualities.

  In the pairwise comparisons of the different records, we carefully evaluated the degree to which each record is "glamorous" and "pure." For these subcriteria, we assigned numerical values for the comparison of the records with respect to these qualities. The pairwise comparison matrices we generated, along with the weights derived from each matrix, are given in Appendix A in Tables A.7 and A.8.

### 3.4.5 Single-Event Results

After the pairwise comparison matrices for each subfactor were generated, the local weights were computed and synthesized using Expert Choice. This gave us the overall rankings for the single-event records. Unlike the single-season and career records, these weights were normalized so that they sum to one. In Figure 3.6, we display the overall rankings for the single-event records.

Similar to the single-season category, two records finished very close to each other, and far ahead of the other records. Wilt Chamberlain's 100 point performance in 1960 finished slightly

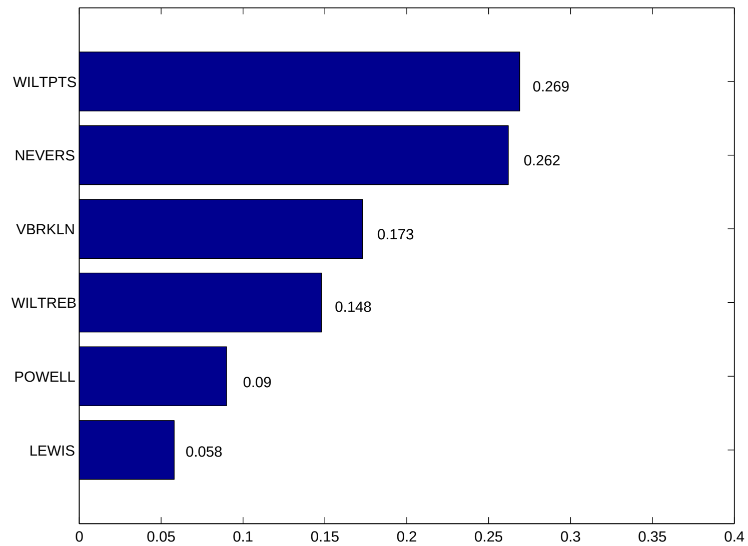


Figure 3.6: Overall rankings of single-event records

ahead of Ernie Nevers' 40 point football game in 1929 to be the greatest single-event sports record. The remaining four records finished considerably further behind these two.

Both Chamberlain's record and Nevers' record look fairly safe given the present state of each athlete's respective sport. No one in the past decade has put up a close challenge to either record. Even with the introduction of the 3-point line in the NBA in 1979 and the emergence of some the most outstanding individual basketball players to ever play the game, Chamberlain's record has breezed through the past several decades virtually unchallenged. Nevers' record is in a similar position. When Nevers set his record, he had the opportunity to contribute points as both a running back and as a place kicker; scoring his 40 points came about through six touchdowns (TDs) and four point-after touchdown attempts. Now, obtaining five TDs in a game is something that happens on average only once every couple of seasons. Acquiring the seven TDs or six TDs with three two-point conversions in a game to surpass Nevers would be a very rare feat.

Mike Powell's record received a lower weight in our ranking than it likely deserves. All other records we consider in the study depend on an athlete's performance on a team or against other competitors. Powell's long jump record is the only record that is limited by natural human ability alone. In 1991, Powell jumped a mere 2 inches further than Bob Beamon's 29 foot, 2.5

inch jump. Because of this small improvement, his record suffers significantly in the Percent over Second Best subcriterion. His record is very impressive nonetheless, as only Powell and Beamon have ever long jumped over 29 feet. In addition, Powell's record looks to be safe in the near future. In the 2004 Summer Olympic games, the gold medal winning jump was over a foot behind Powell's leap. Powell's jump speaks much of his inherent talent, and his record likely deserves to be ranked higher.

#### 3.4.6 Comparison to Rankings from Previous Studies

The single-event study is almost identical to the single-event records study from Richardson's thesis. Only the single-game football rushing record has been broken in the past five years, but it was broken two times: first by Corey Dillon in 2001 and then by Jamal Lewis in 2003. Several of the entries for the pairwise comparison matrices in this thesis were used directly from Richardson's study; in some categories, nothing had changed since his study was performed.

This study omitted one record that was considered by Richardson's thesis, Susie O'Neill's world record in the 200m butterfly. At the time of the Richardson's study, Susie O'Neill had just broken Mary Meagher's 18 year old 200m butterfly record, and her performance was a very memorable moment in the world of women's swimming. O'Neill's record was broken by three hundredths of a second in 2002 by Otylia Jedrzejczak. This new performance was not as significant as was O'Neill's, and thus we decided to not include it in this study.

With this omission and Jamal Lewis's new record, the overall weights assigned to the records changed slightly. The overall order remained the same, though NEVERS now has gained ground and has a closer overall weight to WILTPTS. The remaining records have weights similar to those given by Richardson's thesis, with Lewis's record receiving just over half of the weight Walter Payton's record had in 2000.

# Chapter 4

## Interval AHP Introduction

The use of interval judgments in the analytical hierarchy process (AHP) has begun to grow in many decision-making environments, providing an effective means for modeling the presence of uncertainty in a decision maker's (DM) judgments or representing comparisons in group decision making. The most effective method for generating weights when interval comparisons are used, however, is still under debate. The interval AHP problem is relatively new–only in the past decade has it received much attention in the literature. Arbel and Vargas [3] produced one of the first methods to solve interval AHP problems when they introduced Preference Programming and Preference Simulation just over a decade ago. Since then, several attempts to either adapt traditional AHP methods or invent new mathematical approaches have been proposed to address the interval problem. Typical techniques usually employ a variant of the eigenvector method, a statistical/simulation based approach, fuzzy set logic, or mathematical programming to generate a priority vector from these judgments.

One method, a linear programming (LP) method recently introduced by Chandran et al. [7], provides a promising approach for solving a variety of AHP problems. As demonstrated in that paper, this LP model is very robust; it can be utilized for traditional, interval, and mixed (a combination of traditional and interval) pairwise comparison matrices. In this thesis, however, our focus is to further investigate only one of these problems: solving the AHP with interval judgments. In particular, we concentrate on problems with inconsistent interval pairwise comparison matrices, or problem instances where there is no set of weights that satisfy all interval judgments. This is an extension of the method in Chandran et al. [7].

Inconsistencies in the AHP are not uncommon, nor are they necessarily seen as poor judgments by the decision maker. Rather, they reflect the reality that uncertainty is widely prevalent in decision making. Often, a DM is bombarded by vast amounts of information at once, making the process of determining the relative strength of one alternative over another difficult. This confusion can obscure the comparison process, potentially giving rise to small contradictions in the pairwise comparisons. Inconsistencies in the comparisons can occur, and they are not always easily detected. Thus, any successful interval method for finding priority vectors should be able to address their presence.

In its current form, the LP model [7] is unable to properly cope with inconsistencies. Whenever an inconsistent interval pairwise comparison matrix is used with the LP method, the linear program becomes infeasible. To address this shortcoming, we present a small revision to the model. By introducing stretch factors for each interval and by adding an introductory "expansion" stage, we extend the LP method so that it first finds the minimal stretching required to reach feasibility. This expanded solution space (with some intervals widened) becomes nonempty. Having feasibility ensured for these stretched intervals, we then proceed with the two-stage LP model.

The sections that follow introduce this new technique, demonstrate it on several sample (inconsistent) interval pairwise comparison matrices, and compare it to other interval methods.

## 4.1 A Linear Programming Technique for Interval AHP Problems

Interval comparisons have begun to become more widely used in the analytic hierarchy process. As described in Subsection 2.2.2, interval judgments are captured in a pairwise comparison matrix having the form:

$$B = \begin{bmatrix} 1 & [l_{12}, u_{12}] & \dots & [l_{1n}, u_{1n}] \\ [l_{21}, u_{21}] & 1 & \dots & [l_{2n}, u_{2n}] \\ \vdots & \vdots & \ddots & \\ [l_{n1}, u_{n1}] & [l_{n2}, u_{n2}] & & 1 \end{bmatrix}. \tag{4.1}$$

The interval limits in the lower triangular portion of (4.1) come from the reciprocals of the interval bounds in the upper triangular entries. Priority vectors for interval matrices can be determined in a number of ways. The following two subsections present one of these methods.

### 4.1.1 LP Method for Intervals

One technique used to find a priority vector for AHP problems is based on the LP formulation presented by Chandran et al. [7]. The LP method is centered around the idea that the best priority vector will minimize the error $\epsilon_{ij}$ between the actual weight ratios and the decision maker's judgments $a_{ij}$. The errors are related to each entry of the pairwise comparison matrix through the multiplicative relationship $\frac{w_i}{w_j} = a_{ij}\epsilon_{ij}$ for every $i, j$ with $i \neq j$. The errors assume positive values relatively close to one, with values greater than (less than) one indicating underestimated (overestimated) measurements. A logarithmic transformation of this error relationship and a change of variables yields a set of simple, straightforward linear equalities and inequalities. These equations become constraints in a two-stage LP that is used to determine an optimal priority vector.

When considering the presence of interval pairwise comparisons, measuring the error in each judgment ratio $\frac{w_i}{w_j}$ is not explicit since the comparisons are intervals of the form $[l_{ij}, u_{ij}]$ rather than single numbers. Therefore, in order to compute the error associated with each judgment, the weight ratios are examined relative to the geometric mean of the interval endpoints through the equation

$$\frac{w_i}{w_j} = \epsilon_{ij}\sqrt{l_{ij}u_{ij}} \quad . \tag{4.2}$$

Following the non-interval case, error values greater than one are indicative of potentially underestimated interval bounds, while error values less than one indicate overestimated interval bounds.

The goal of the LP method is to minimize the total error in the pairwise comparisons. Comparing these errors relative to the geometric mean of the interval bounds, rather than arithmetic mean, is important. The geometric mean is desired whenever interval judgments are used because it preserves the reciprocal nature of the pairwise comparison matrix. Since $u_{ji} = \frac{1}{l_{ij}}$ and $l_{ji} = \frac{1}{u_{ij}}$,

$$a_{ij} = \sqrt{l_{ij}u_{ij}} = \left(\sqrt{\frac{1}{u_{ij}} \cdot \frac{1}{l_{ij}}}\right)^{-1} = \frac{1}{\sqrt{l_{ji}u_{ji}}} = \frac{1}{a_{ji}}.$$

Because of this relationship, the errors are also reciprocal in nature, that is, $\epsilon_{ji} = \frac{1}{\epsilon_{ij}}$. If we were to compute the errors relative to the midpoint of each interval, the reciprocal nature of the error would not be present. Since the goal of our method is to minimize a measure of total error, the weights found when using the arithmetic mean would in turn be dependent on the order in which the alternatives were assigned and compared. Using information from two different pairwise comparison matrices representing the same comparison information might then result in two different priority vectors.

To demonstrate this concept on a small scale, consider the following problem in the AHP. Suppose that we have two alternatives, 1 and 2, and we believe that the relative importance of alternative 1 to alternative 2 lies in the interval $[2, 4]$. To find a solution close to the arithmetic mean of this interval, we derive the (normalized) weights $w_1 = 0.750$ and $w_2 = 0.250$. The error value in this ratio is equal to one since $\frac{w_1}{w_2} = \frac{0.750}{0.250} = \frac{2+4}{2}$, indicating $\epsilon_{ij} = 1$. Using the same comparison information, we also know that the relative importance of alternative 2 to alternative 1 lies in the interval $[\frac{1}{4}, \frac{1}{2}]$. If we attempt to find a solution close to the arithmetic mean of this interval, we derive the (normalized) weights $w_1 = 0.7273$ and $w_2 = 0.2727$. The error value associated with this ratio is also equal to one. This presents us with two different solutions for our comparisons that possess the same total error. Which set of weights do we choose?

Rather than use the error relationship in Equation (4.2) for appropriate values of $i$ and $j$, which is a nonlinear equation in the three unknowns $w_i, w_j$, and $\epsilon_{ij}$, we take the natural logarithm of both sides of the equation. In Equation (4.3), we give the transformed version.

$$\ln w_i - \ln w_j = \ln \epsilon_{ij} + \ln \sqrt{l_{ij} u_{ij}} \quad . \tag{4.3}$$

By introducing new variables in this transformed space, i.e., $x_i = \ln w_i$ and $y_{ij} = \ln \epsilon_{ij}$, the error relationship is linear.

In addition to these error equalities, we develop other constraints by utilizing the interval judgments directly. Splitting each interval inequality of the form $l_{ij} \leq \frac{w_i}{w_j} \leq u_{ij}$ into a pair of inequalities yields two constraints for each $i, j$:

$$l_{ij} \leq \frac{w_i}{w_j} \qquad \text{and} \qquad u_{ij} \geq \frac{w_i}{w_j}. \tag{4.4}$$

Other methods [19, 26] choose to linearize these constraints by multiplying both sides of the inequalities by the weight $w_j$. We, however, transform both inequalities to work in the natural logarithm space. This gives the following additive constraints for any $i, j$ with $i \neq j$:

$$\ln l_{ij} \leq \ln w_i - \ln w_j \qquad \text{and} \qquad \ln u_{ij} \geq \ln w_i - \ln w_j.$$

The LP method has two separate stages. The objective functions for both stages examine the error between the weight ratios and the geometric means of the intervals. Since each error $\epsilon_{ij}$ assumes values in an interval around one, the corresponding $y_{ij} = \ln \epsilon_{ij}$ can assume either positive or negative values. However, since the errors are reciprocal, $y_{ij} = -y_{ji}$. By defining $z_{ij} = |y_{ij}|$, we can capture the total error in a priority vector by summing up the $z_{ij}$ variables.

The objective of the LP method's first stage is to minimize the sum of the positive errors in the priority vector. That is, it aims to minimize the sum of the $z_{ij}$'s. This objective can be thought of as a minimization of the sum of the overestimated comparisons. In the original non-transformed space, it corresponds to minimizing the product of the errors greater than or equal to one.

We combine the constraints that are outlined previously with nonnegativity restrictions on each $z_{ij}$. Since half of a pairwise comparison matrix contains all the comparison information needed to create a priority vector, the values of $i$ and $j$ range over the upper triangular entries in the pairwise comparison matrix. This gives us a linear program to solve for the priorities.

- **Stage 1:** Minimize the total sum of the error in the priority vector.

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n-1} \sum_{j>i} z_{ij} & & \\
\text{s.t.} \quad & x_i - x_j - y_{ij} = \ln \sqrt{l_{ij} u_{ij}} & \forall\, i, j \quad \text{with} \quad i \neq j & \qquad (4.5)\\
& z_{ij} - y_{ij} \geq 0 & \forall\, i < j & \qquad (4.6)\\
& z_{ij} - y_{ji} \geq 0 & \forall\, i < j & \qquad (4.7)\\
& x_i - x_j \geq \ln l_{ij} & \forall\, i < j & \qquad (4.8)\\
& x_i - x_j \leq \ln u_{ij} & \forall\, i < j & \qquad (4.9)\\
& x_1 = 0 & & \qquad (4.10)\\
& z_{ij} \geq 0 & \forall\, i < j & \qquad (4.11)\\
& x_i, y_{ij} \text{ unrestricted} & \forall\, i < j & \qquad (4.12)
\end{aligned}$$

The constraints in Equations (4.5), (4.8), and (4.9) come directly from the previous discussion. Equations (4.6) and (4.7) represent $z_{ij} = |y_{ij}|$ since one member of each $(y_{ij}, y_{ji})$ pair is positive, and the sum of the $z_{ij}$ are minimized. The constraint requiring $x_1 = 0$ is arbitrary. Since an infinite number of solutions for this LP exist (the weights are not unique, only their ratios are), we choose the solution where $w_1 = 1$, without loss of generality.

It is possible for Stage 1 to have multiple optimal solutions, representing different sets of weight ratios that all have the same total error. However, multiple sets of weights are not desired; we prefer to have a single set of weights for a pairwise comparison matrix. To determine which solution from this set is the best, we need to do more work. We create a stage to follow Stage 1 to select a solution from the Stage 1 optimal solutions whose single maximum error $z_{\max}$ is minimal.

The second stage constraints follow directly from the first stage with the added requirement that only Stage 1 optimal solutions are feasible in Stage 2. Letting $z^*_{\text{sum}}$ denote the Stage 1 optimal objective value, we present the second stage.

- **Stage 2:** From the set of solutions that minimize the sum of the total error, find the solution that has the minimum individual error maximum.

$$\begin{aligned} \min \quad & z_{\max} \\ \text{s.t.} \quad & \sum_{i=1}^{n-1} \sum_{j>i} z_{ij} = z^*_{\text{sum}} && (4.13) \\ & z_{\max} \geq z_{ij} \qquad \forall\, i < j && (4.14) \\ & \text{constraints (4.5)-(4.12).} \end{aligned}$$

The two-stage method is discussed in further detail and illustrated on several interval and mixed pairwise comparison matrices in [7].

### 4.1.2 Extending the LP Method for Inconsistent Intervals

In its current form, the two-stage method has one big limitation. The presence of tight intervals and inconsistencies in the interval judgments could pose problems for the linear program. In these two cases, the convex set bounded by the interval constraints in Equations (4.8) and (4.9) might cause the solution space to be empty, yielding no feasible priority vectors for the linear program. To an LP solver, this infeasibility is the "solution" to the first stage. In reality, though, the infeasibility of the linear program only indicates that there is no priority vector that can completely satisfy all of the interval judgments.

Inconsistent intervals should not prevent us from finding at least a candidate priority vector. We would naturally prefer *any* solution (albeit a somewhat inconsistent one) to a warning about an infeasible solution. Rather than receive such a message, we wish to have our method find solutions that are closest to satisfying the interval bounds. From this set of close solutions, we wish to select the solution that also minimizes the same total error and $z_{\max}$ as in the stages for the normal interval LP method.

To allow solutions that do not satisfy the intervals, we must soften the constraints that are too restrictive. The intervals need to have the ability to expand so that weight ratios can violate one (or more) of the interval bounds as needed. To facilitate this expansion, we introduce multiplicative "stretch factors" to the upper and lower interval bound inequalities. By multiplying each lower and upper bound by designated stretch factors $\lambda_{ij}$ and $\gamma_{ij}$ such that $\lambda_{ij} \leq 1$ and $\gamma_{ij} \geq 1$, respectively, the upper (lower) bound is given the ability to increase (decrease) and incorporate solutions with weight ratios outside the original intervals. These stretch factors alter the inequalities given in Equation (4.4).

$$\frac{w_i}{w_j} \geq l_{ij}\lambda_{ij} \qquad \text{and} \qquad \frac{w_i}{w_j} \leq u_{ij}\gamma_{ij}. \tag{4.15}$$

Each set of weights now has to satisfy these softer intervals, not the hard constraints. To be consistent with our error measurements, we would like the stretch factors to also satisfy a reciprocal relationship. When the stretch factors satisfy the relationship $\lambda_{ij} = \frac{1}{\gamma_{ij}}$, the geometric mean of each interval is preserved. Keeping the geometric mean unchanged preserves the DM's preference structure and also allows the error in the soft intervals to be measured in the same manner as in the two-stage interval LP method.

Interval stretching whenever inconsistent interval judgments are present is not a new concept. Leung and Cao [23] proposed using multiplicative factors $(1 + \delta)$ and $(1 - \delta)$ for the upper and lower interval bounds for a given tolerance parameter $\delta$. Islam et al., [19] and Mikhailov [26] both proposed additive modifications to the interval bounds. None of these methods impose any extra restrictions on how these intervals are expanded.

In the transformed space, the reciprocal nature of the stretch factors is advantageous. Since we have $\ln \lambda_{ij} = \ln \frac{1}{\gamma_{ij}} = -\ln \gamma_{ij}$, we need to introduce only one stretch factor for each pair of interval constraints for the formulation. Letting $g_{ij} = \ln \gamma_{ij}$ and using $x_i = \ln w_i$, the constraints from (4.15) become additive in the transformed space, yielding

$$x_i - x_j \geq \ln l_{ij} - g_{ij} \qquad \text{and} \qquad x_i - x_j \leq \ln u_{ij} + g_{ij}. \tag{4.16}$$

These equations can be directly compared to the constraints (4.8) and (4.9) in the original formulation. Since $g_{ij} \geq 0 \quad \forall i, j$, the new solution space is nonempty for any set of interval bounds, consistent or inconsistent. There will always exist $g_{ij}$'s large enough to accommodate any inconsistencies in the judgments.

While stretching allows us to obtain a set of weights for any set of intervals, the added flexibility in the intervals should not be used arbitrarily. Allowing the intervals to stretch too much might seriously disregard the original judgments of the decision maker. Thus, solutions closer to satisfying the original intervals are preferred to those further away. In order to find these solutions, we introduce a preliminary stage (with a function similar to that of the Phase 0 for the simplex method) that we call Stage 0. This stage seeks the total minimum interval stretching needed to obtain a feasible set of weights. This is accomplished by using the new interval constraints (4.16) and an objective function that minimizes the sum of the transformed stretch factors $g_{ij}$ (the product of the $\gamma_{ij}$'s in the non-transformed space). The output from this stage is a set of solutions that become feasible when the interval limits are changed by this minimum amount. From this set of solutions, we proceed as in Stage 1 and Stage 2 to minimize the total error in the priority vector. The entire three-stage formulation follows:

- **Stage 0:** Find the closest feasible solution(s) to the original simplex. Call the optimal value of this stage $g^*_{\text{sum}}$

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n-1}\sum_{j>i} g_{ij} \\
\text{s.t.} \quad x_i - x_j - y_{ij} &= \ln\sqrt{l_{ij}u_{ij}} && \forall i,j \quad \text{with} \quad i \neq j && (4.17)\\
z_{ij} - y_{ij} &\geq 0 && \forall\, i<j && (4.18)\\
z_{ij} - y_{ji} &\geq 0 && \forall\, i<j && (4.19)\\
x_i - x_j + g_{ij} &\geq \ln l_{ij} && \forall\, i<j && (4.20)\\
x_i - x_j - g_{ij} &\leq \ln u_{ij} && \forall\, i<j && (4.21)\\
x_1 &= 0 && && (4.22)\\
z_{ij}, g_{ij} &\geq 0 && \forall\, i<j && (4.23)\\
x_i, y_{ij} & \text{ unrestricted} && \forall\, i<j && (4.24)
\end{aligned}
$$

- **Stage 1:** Proceed as before (see the earlier Stage 1) using the total minimal inflation value ($g^*_{\text{sum}}$) needed to obtain feasibility (added as constraint (4.25)). Denote the optimal solution to Stage 1 as $z^*_{\text{sum}}$

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n-1}\sum_{j>i} z_{ij} \\
\text{s.t.} \quad & \sum_{i=1}^{n-1}\sum_{j>i} g_{ij} = g^*_{\text{sum}} && (4.25)\\
& \text{constraints (4.17)-(4.24).}
\end{aligned}
$$

- **Stage 2:** Continue as in the earlier Stage 2 by minimizing $z_{\max}$, again accounting for the minimal stretching required.

$$
\begin{aligned}
\min \quad & z_{\max} \\
\text{s.t.} \quad & \sum_{i=1}^{n-1}\sum_{j>i} g_{ij} = g^*_{\text{sum}} \\
& \sum_{i=1}^{n-1}\sum_{j>i} z_{ij} = z^*_{\text{sum}} && (4.26)\\
& z_{\max} \geq z_{ij} \qquad \forall\, i<j && (4.27)\\
& \text{constraints (4.17)-(4.24).}
\end{aligned}
$$

With the completion of the three-stage process, a priority vector is given that contains the ranking of the alternatives derived from the expanded interval comparisons. We refer to the three-stage method as the LP with stretching, or LP-S.

### 4.1.3 Advantages of the LP-S

The original paper on the LP method contains an in-depth illustration of the benefits of the two-stage method described in Subsection 4.1.1. Fortunately, the three-stage approach follows as a natural extension of the two-stage method. After executing Stage 0, formulations for the two remaining stages are almost identical to the original version. Allowing interval inflation adds to the LP method's robustness; the method can be applied to interval pairwise comparison matrices regardless of whether the comparisons are consistent or inconsistent. If the optimal objective function value of Stage 0 is zero, the remaining two stages reduce to the original LP method. Consequently, Stage 0 serves as a sort of consistency test–a preliminary way for determining whether or not interval stretching is necessary.

The LP-S method is also practical in nature. The idea behind measuring a priority vector's error relative to the comparisons is novel and straightforward. The interval stretching we present is simple, but it performs well for finding solutions closest to the region defined by the original intervals. The LP-S method can be easily implemented, and problems can be solved using freely available optimization software. In addition, using linear programming allows the model to be easily customized to incorporate specific preferences of the decision maker.

For example, if the DM decides that some intervals are more important than others, limitations on the amount of stretching $g_{ij}$'s can be added directly as constraints to force particular interval comparisons to be satisfied. If the decision maker determines that he or she wants the $\frac{w_2}{w_3}$ ratio to lie inside $[l_{23}, u_{23}]$, then he or she could add a constraint to make the stretch factor associated with this interval equal to zero ($g_{23} = 0$). Also, "stretching costs" can be incorporated into the Stage 0 objective function to penalize the objective when particular intervals are stretched. Instead of forcing the (2,3) interval to be satisfied through a constraint, the coefficient on $g_{23}$ in the objective function can be increased so that the objective value will be greater when the (2,3) interval is stretched instead of the other intervals. Either of these two techniques can easily be used to include the preferences of the decision maker in ways other than the pairwise comparisons.

We realize that our formulation contains a rather large number of variables relative to some of the the other interval methods. Some users might argue that the requirement of running three separate stages can become cumbersome and complicated for generating solutions. However, once the formulation is implemented one time for computational purposes, changing from one problem to the next demands little to no more work than any of the other methods. The three stages are nearly identical, so all the constraints can be implemented in a single text file with the three objective functions listed at the top of the file. Changing from one stage to the next only requires the appropriate objective function for the stage in question to be uncommented, the other two objective functions to be commented so that they are not in use, and the previous stage's optimal objective value to be input as a constraint. The use of multiple objectives allows us to be more critical of solutions than a single-stage LP. Furthermore, the use of three stages helps to reduce the possibility of multiple optimal solutions. It simply gives better solutions than single-stage methods.

The sections that follow present several interval pairwise comparison matrices and give illustrations of our method in order to provide some justification for the above claims.

## 4.2 Simple Computational Example of LP-S Method

We first wish to present an example to show the entire LP-S formulation for a sample interval pairwise comparison matrix. We present an infeasible interval pairwise comparison matrix based on the interval pairwise comparison matrix first introduced by Arbel and Vargas [3]. By arbitrarily altering the $a_{14}$, $a_{23}$, and $a_{24}$ entries in that matrix, the LP model (without stretching) has no feasible solutions, indicating the presence of an inconsistent set of intervals. This pairwise matrix is given in (4.28).

$$A = \begin{bmatrix} 1 & [2,5] & [2,4] & [1,2] \\ & 1 & [2.5,3] & [1,1.5] \\ & & 1 & [\frac{1}{2},1] \\ & & & 1 \end{bmatrix}. \tag{4.28}$$

The matrix $A$ does not appear to have any obvious inconsistencies. No entry stands out as being incompatible with the other intervals. However, the presence of several tight intervals confines the weight ratios to a small range of values that do not all agree with one another.

The formulation for the LP-S method takes the entries in the upper triangular portion of $A$ and creates the three linear programs as described in Section 4.1.2. We entered the formulation into a text file for implementation in a linear optimization software package. The number of variables and constraints is reasonably small for any pairwise comparison matrix acceptable for the AHP. Thus, we were able to use the free optimization software program LINDO for generating the weights for our method.

We have presented the formulation for this problem in its entirety with all constraints explicitly detailed in Section B.1 in Appendix B. Upon running Stage 0, we found an optimal objective function value of $\sum\sum g_{ij} = 0.2232$, indicating that some stretching is needed to obtain a feasible set of weights. Using this value as a constraint in Stage 1, the optimal total error was $\sum\sum z_{ij} = 1.811$. Finally, using both of these objective values as constraints in Stage 2, we obtained the optimal $z_{\max}$ for Stage 2 of 0.661 and the optimal priority vector for this pairwise comparison matrix: $\mathbf{w} = [0.423 \quad 0.259 \quad 0.106 \quad 0.212]^T$.

Along with the formulation in Appendix B, we have included the output found by executing LINDO for the three stages. In comparing the output from Stage 0 with Stage 1, we notice the impact of having multiple stages. Stage 0 terminates with an optimal value of 0.2232, and the solution given by Stage 0 has only one nonzero stretch factor ($g_{23} = 0.2232$). Once Stage 1 is completed, however, the optimal solution requires two intervals to be stretched ($g_{12} = 0.2028$ and $g_{23} = 0.0204$). Stage 0 has multiple optimal solutions, but Stage 1 reduces the number of optimal solutions. Stage 2 further ensures that the solution found from Stage 1 minimizes the maximum single error $z_{\max}$.

## 4.3 Computational Comparisons

At the present time, there are only a few methods that can use an interval pairwise comparison matrix as its input and find a priority vector from the intervals. In addition to the LP-S method, two methods appear to be successful at handling several varieties of pairwise comparison matrices: Lexicographic Goal Programming (LGP) introduced by Islam et al. [19], and Fuzzy Preference Programming (FPP) introduced by Mikhailov [26]. The two subsections that follow present a pair of computational experiments in which these three methods are compared with respect to inconsistent interval AHP.

### 4.3.1 Computational Comparison I - Interval Stretching

In this subsection, we use the LP-S and four other methods to generate priority vectors from a sample pairwise comparison matrix and compare these solutions with one another. For this comparison, we use the same $4 \times 4$ inconsistent interval pairwise comparison matrix (4.29) introduced by Kress [21] and studied further by Islam et al. [19]. We denote this by $B$.

$$B = \begin{bmatrix} 1 & [1,2] & [1,2] & [2,3] \\ & 1 & [3,5] & [4,5] \\ & & 1 & [6,8] \\ & & & 1 \end{bmatrix} \tag{4.29}$$

In addition to the LP-S, LGP, and FPP methods, we used two traditional non-interval methods for the computation of priority vectors: the eigenvector method (EM), and the logarithmic least-squares (LLS) method. The weights for LP-S were found by optimizing the three stages in our formulation, while the weights generated by the the LGP method were taken directly from [19]. For the FPP method, linear membership functions were used to create the formulation (which is a linear program), and LINDO was used to perform the optimization. In order to employ the

| **Method** | $w_1$ | $w_2$ | $w_3$ | $w_4$ | Objective 0 | Objective 1 |
|---|---|---|---|---|---|---|
| LP-S | 0.3252 | 0.3636 | 0.2299 | 0.0813 | 1.792 | 2.740 |
| LGP | 0.3030 | 0.4545 | 0.1515 | 0.0910 | 1.792 | 3.197 |
| FPP | 0.318 | 0.4318 | 0.1932 | 0.0568 | 2.211 | 3.4225 |
| EM | 0.3016 | 0.4006 | 0.2279 | 0.0699 | 1.928 | 3.055 |
| LLS | 0.3127 | 0.3932 | 0.2229 | 0.0712 | 1.891 | 2.959 |

Table 4.1: Priority vectors generated from $B$ and $\tilde{B}$ for the five methods

EM and LLS method, we created a non-interval matrix $\tilde{B}$ where each $(i, j)$ entry of $\tilde{B}$ was the geometric mean of the $(i, j)$ interval bounds in $B$. These two techniques were used primarily for general comparison purposes. For the three linear programming methods (LP-S, LGP, and FPP), we have included the formulations created for this pairwise comparison matrix in Appendix B (Section B.2).

In Table 4.1, we provide the priority vectors generated by the five methods. Columns two through five contain the weights of the four alternatives. The Objective 0 column (representing $\sum\sum g_{ij}$) gives the Stage 0 objective function value for each solution, the implied sum of the stretch factors associated with each solution. The Objective 1 column (representing $\sum\sum z_{ij}$) in the table gives the value of the sum of the Stage 1 errors for each priority vector. For the LP-S method, the values used to populate the table came directly from running the three stages in LINDO and observing the optimal objective function values. In order to find the Stage 0 and Stage 1 values for the other four methods, we inserted each weight into the LP-S formulation as a constraint and observed the objective function values from Stages 0 and 1 induced by these solutions.

In comparing the solutions obtained by these methods, we see that the values assigned to weights $w_1$ and $w_4$, in general, differ by only two to four hundredths of a point, while weights $w_2$ and $w_3$ vary quite substantially among the methods. Though the solutions differ, all agree on the rank ordering: $w_2 > w_1 > w_3 > w_4$. In addition, the total error ($\sum\sum z_{ij}$) for all other solutions is greater than the error for the LP-S solution. The weights associated with LP-S have a Stage 0 objective function value less than or equal to the sum implied by the weights of the other four methods, indicating a smaller amount of stretching. Indeed, our solution does satisfy the two objectives sought by the three-stage approach: minimal error with minimal stretching.

| **Method** | $\gamma_{12}$ | $\gamma_{13}$ | $\gamma_{14}$ | $\gamma_{23}$ | $\gamma_{24}$ | $\gamma_{34}$ |
|---|---|---|---|---|---|---|
| LP-S | 1.118 | 1 | 1.333 | 1 | 1.897 | 2.121 |
| LGP | 1.5 | 1 | 1.1099 | 1 | 1 | 3.604 |

Table 4.2: Numerical values of the six stretch factors for the LP-S and LGP solutions

One observation of particular interest is that the LP-S and LGP solutions have the same Stage 0 optimal value. The LGP method also attempts to stretch the interval bounds to obtain feasible solutions by employing the principles of goal programming to generate weights [19]. The LGP begins with the inequalities from Equation (4.4) and adds a pair of nonnegative deviation variables to each inequality. With these new variables, the LGP can find solutions that violate the interval comparisons. The objective function of the LGP minimizes the sum of these deviation variables in an attempt to find the solution closest to satisfying the interval comparisons. This formulation can be seen in detail in Subsection B.2.2 of Appendix B.

The goal of the LGP is rather similar to the objective of our Stage 0. The LGP finds a solution that satisfies the expanded intervals but goes no further. No attempt is made to measure the "quality" of this priority vector. Therefore, the optimal amount of stretching that is returned by LGP is not guaranteed to give a unique set of deviations nor a unique priority vector. Sometimes, different combinations of intervals can be expanded so that they yield different feasible sets of weights satisfying the same total optimal stretching.

The matrix $B$ in (4.29) is one example where this occurs. In Table 4.1, we see that both the LP-S and LGP methods return the same value for the minimum total stretching, but the two priority vectors are quite different, especially for $w_2$. This illustrates that there are at least two (and likely more) ways in which the interval deviations can be distributed, each yielding a different set of expanded intervals and a different priority vector. We observe these differences by examining the values of the stretch factors associated with both solutions, given in Table 4.2.

The different values for the individual $\gamma_{ij}$'s associated with the LP-S and LGP solutions were found from our optimization. The LP-S values came directly from the output of the Stage 2 values. Since the LGP does not use multiplicative stretch factors, we inserted the optimal LGP solution into the LP-S formulation and observed the implied value for each stretch factor. These values were then verified through a comparison with the optimal solution given in Islam et al. [19]. The LP-S solution has a lower total error relative to the geometric mean, and it achieves this through the alteration of four different intervals. The LGP solution requires a fewer number of intervals to be inflated, but it possesses a greater total error.

The reason for the differences in the two solutions becomes apparent upon examining the LGP formulation. The LGP method does not account for the possibility of multiple optimal interval expansions. It finds the minimum total amount of alterations required for feasibility, but having obtained its solution, it does not go further to measure how "good" the solution is. Therefore, the LGP method, though it provides a solution that is as close to satisfying the original interval bounds as the solution given by the LP-S method, does not detect multiple optimal solutions nor discriminate amongst them. LP-S, through Stage 1, addresses this situation by providing a way for the best solution to be found, given these minimal changes. The LP-S method not only finds a close solution, but it also locates the closest solution with minimal error. This sample problem demonstrates our model's effectiveness and illustrates the benefits of having multiple stages for the generation of a priority vector.

#### 4.3.2 Computational Comparison II

For the second comparison, we used the relatively new Fuzzy Preference Programming (FPP) method [26]. In recent years, fuzzy set logic has begun to receive a fair amount of attention in the decision-making world. Fuzzy Preference Programming is based on using this logic to treat each interval as a separate fuzzy set. Fuzzy sets are quite different from classical sets. Elements can be partially in a set, and the degree to which they are included in a set is determined by a membership function. This membership function assigns membership values to its elements based on how "close" an element is to being in the classical set.

For the FPP, the membership value expresses the degree to which a ratio of weights $\frac{w_i}{w_j}$ lies inside the $i, j$ interval. The membership value, denoted by $\lambda$, is determined by evaluating the membership function at the value of the ratio. In the FPP, the membership functions are linear or nonlinear piecewise monotonic functions that are maximized at the center of each interval. With additive deviation parameters appended to the upper and lower bounds of each interval, each membership function can allow solutions that are outside the interval bounds.

Using a membership function for each interval in the upper triangular portion of a pairwise comparison matrix, the FPP method results in a mathematical program to find a priority vector that maximizes the minimum value of each membership function. Since in this thesis we are only concerned with linear methods for generating weights, we focused solely on membership functions of the linear variety for all computations using the FPP method. The FPP's nonlinear membership functions have some appeal, but they lead to nonlinear optimization formulations and would require us to explore different, more time-consuming techniques for finding solutions.

The FPP method is nice, concise, and has a formulation that is easy to implement, and it can be understood without knowing much about fuzzy sets. Switching from one problem to another is easily done by changing only a small number of values in the formulation. In addition, after the optimization software is finished with its computations, normalized weights are available with no extra work. Our LP-S method has a longer formulation that requires more time to create, and it requires three separate optimizations. Once solutions are obtained from Stage 2, the weights generated by the LP-S have to be transformed back from logarithmic space and normalized. Though these limitations are not major problems, the LP-S method requires slightly more work to obtain weights than the FPP method.

Aside from the computational standpoint, the two methods differ in one key aspect: the LP-S method prefers solutions close to the geometric mean of the intervals, while the FPP method prefers solutions close to the arithmetic mean. We have already demonstrated that the geometric mean preserves the reciprocal nature of the pairwise comparison matrix. For the example presented in this subsection, we wish to demonstrate the consequences that arise when measuring solutions with respect to the these means.

The AHP relies on the reciprocal nature of pairwise comparisons. Making comparisons to fill either the upper or lower triangular portion of the pairwise comparison matrix is all that is required of the decision maker. No matter which order the alternatives are compared or what comparison information is inserted into the formulation, the weights generated should remain the same. For example, consider two identical decision makers. One DM might label three alternatives 1, 2, and 3, while the other might label the same three alternatives $c$, $b$, and $a$. Provided their comparisons are identical, the weights found by each DM should agree regardless of their different labels. Denoting $w_i$ as a weight obtained by the first DM, and letting $\tilde{w}_i$ represent a weight obtained by the second DM, the two decision makers should find the following equivalences between the weights:

$$w_1 \longleftrightarrow \tilde{w}_c$$

$$w_2 \longleftrightarrow \tilde{w}_b$$

$$w_3 \longleftrightarrow \tilde{w}_a$$

We tested whether the LP-S and FPP methods are independent of problem formulation. To conduct the experiment, we used two pairwise comparison matrices containing exactly the same comparisons with the exception that for the second pairwise comparison matrix, the alternatives were labeled differently. The pairwise comparison matrix that we examined was presented and studied by Mikhailov [26]. We created a complimentary pairwise comparison matrix by reversing the order of the alternatives and permuting the entries in the pairwise comparison to correspond with this reversal. These two matrices are presented below.

$$C = \begin{bmatrix} 1 & [1,2] & c_{13} \\ [\frac{1}{2},1] & 1 & [2,3] \\ \frac{1}{c_{13}} & [\frac{1}{3},\frac{1}{2}] & 1 \end{bmatrix} \qquad \tilde{C} = \begin{bmatrix} 1 & [\frac{1}{3},\frac{1}{2}] & \frac{1}{c_{13}} \\ [2,3] & 1 & [\frac{1}{2},1] \\ c_{13} & [1,2] & 1 \end{bmatrix}$$

The interval entries in $C$ are the same as those from the example in Mikhailov's paper, where different values of $c_{13}$ are used to create a variety of problem instances. $\tilde{C}$ is the complimentary matrix we built from the entries in $C$.

We compared the performance of our LP-S method versus the linear FPP method on seven different intervals for the seven intervals of $c_{13}$ provided by Mikhailov [26]. We provide the details of one of these calculations on the most strongly-inconsistent example provided in [26], where

| | $w_1(\tilde{w}_c)$ | $w_2(\tilde{w}_b)$ | $w_3(\tilde{w}_a)$ | $\sum\sum g_{ij}$ | $\sum\sum z_{ij}$ | $z_{max}$ | $\lambda$ |
|---|---|---|---|---|---|---|---|
| LP-S on $C$ | 0.198 | 0.380 | 0.422 | 1.8971 | 2.9958 | 0.9986 | N/A |
| LP-S on $\tilde{C}$ | 0.422 | 0.380 | 0.198 | 1.8971 | 2.9958 | 0.9986 | N/A |
| FPP on $C$ | 0.263 | 0.434 | 0.303 | N/A | N/A | N/A | 0.828 |
| FPP on $\tilde{C}$ | 0.412 | 0.400 | 0.188 | N/A | N/A | N/A | 0.788 |

Table 4.3: Priority vectors and objective function values for LP-S and FPP

$c_{13} = [0.1, 0.3] \implies \frac{1}{c_{13}} = [\frac{10}{3}, 10]$. Both methods were implemented in LINDO, and the values for these computations are given in Table 4.3.

We see that the LP-S method is successful in preserving the weights regardless of the reordering of the alternatives. The optimal values for Stages 0, 1, and 2 are equal for both $C$ and $\tilde{C}$, and the resulting weights are the same, permuted accordingly. The FPP method, however, was not able to produce consistent sets of weights. It generated two different priority vectors for $C$ and $\tilde{C}$, and the two sets of weights had different membership values.

We repeated these computations for the other six intervals of this problem. The results of these experiments are displayed in Table 4.4 for the LP-S method and Table 4.5 for the FPP method. In every instance, the LP-S method returned the same weights for both $C$ and $\tilde{C}$. Therefore, we have only displayed the solutions and the first, second, and third stage optimal values in Table 4.5 that come from $C$.

Table 4.5 shows that the FPP's assignment of weights is dependent on the comparison information used. In this table, we have highlighted the optimal membership value of the two solutions, indicating the better priority vector for the given $c_{13}$. For several values of $c_{13}$, using the matrix $\tilde{C}$ for the FPP gave a better set of weights than $C$, but sometimes the priority vector for $C$ was associated with a higher membership value. Thus, given any pairwise comparison matrix, it is unclear as to which matrix to use for the computation of the priority vector for the FPP method.

As noted earlier, the different solutions are a direct result of the fact that the FPP membership functions are maximized at the arithmetic midpoint of each interval. The arithmetic mean for corresponding intervals in $C$ and $\tilde{C}$ are not reciprocals of one another. The maximum value of a membership function on an interval in $C$ occurs at a different ratio than from the corresponding interval in $\tilde{C}$. The weights for the FPP, therefore, depend on which set of information is used in

| $c_{13}$ | LP-S $w_1(\tilde{w}_c)$ | $w_2(\tilde{w}_b)$ | $w_3(\tilde{w}_a)$ | $\sum\sum g_{ij}$ | $\sum\sum z_{ij}$ | $z_{max}$ |
|---|---|---|---|---|---|---|
| $[2, 6]$ | 0.5011 | 0.3543 | 0.1446 | 0 | 0 | 0 |
| $[8, 10]$ | 0.6031 | 0.3016 | 0.0954 | 0.2877 | 0.9485 | 0.3465 |
| $[0.1, 0.3]$ | 0.1982 | 0.3804 | 0.4215 | 2.9958 | 1.8971 | 0.9986 |
| $[0.1, 0.9]$ | 0.3214 | 0.3214 | 0.3572 | 0.7985 | 2.4465 | 1.0986 |
| $[0.5, 0.9]$ | 0.3242 | 0.3962 | 0.2796 | 0.7985 | 1.6417 | 0.5472 |
| $[0.5, 1.5]$ | 0.3463 | 0.4228 | 0.2309 | 0.2876 | 1.383 | 0.546 |
| $[1, 1.9]$ | 0.3979 | 0.3979 | 0.2041 | 0.0512 | 0.9215 | 0.3466 |

Table 4.4: LP-S weights and objective function values for Mikhailov's problem

| | FPP Linear ($C$) | | | | | FPP Linear ($\tilde{C}$) | | | |
|---|---|---|---|---|---|---|---|---|---|
| $c_{13}$ | $w_1$ | $w_2$ | $w_3$ | $\lambda$ | $\tilde{c}_{13}$ | $\tilde{w}_c$ | $\tilde{w}_b$ | $\tilde{w}_a$ | $\lambda$ |
| $[2, 6]$ | 0.462 | 0.385 | 0.154 | **1.077** | $[\frac{1}{6}, \frac{1}{2}]$ | 0.433 | 0.400 | 0.167 | 1.033 |
| $[8, 10]$ | 0.625 | 0.292 | 0.083 | 0.958 | $[\frac{1}{10}, \frac{1}{8}]$ | 0.615 | 0.297 | 0.088 | **0.989** |
| $[0.1, 0.3]$ | 0.263 | 0.434 | 0.303 | **0.828** | $[\frac{10}{3}, 10]$ | 0.413 | 0.400 | 0.188 | 0.788 |
| $[0.1, 0.9]$ | 0.324 | 0.419 | 0.256 | 0.906 | $[\frac{10}{9}, 10]$ | 0.321 | 0.400 | 0.279 | **0.921** |
| $[0.5, 0.9]$ | 0.324 | 0.419 | 0.256 | 0.906 | $[\frac{10}{9}, 2]$ | 0.321 | 0.400 | 0.279 | **0.921** |
| $[0.5, 1.5]$ | 0.370 | 0.407 | 0.222 | 0.963 | $[\frac{2}{3}, 2]$ | 0.375 | 0.400 | 0.225 | **0.975** |
| $[1, 1.9]$ | 0.395 | 0.401 | 0.204 | 0.993 | $[\frac{10}{19}, 1]$ | 0.396 | 0.400 | 0.204 | **0.996** |

Table 4.5: Weights and membership values for the FPP solutions for $C$ and $\tilde{C}$

the formulation. A DM might make the same comparisons in a slightly different order and get entirely different weights. The first example we illustrated in this subsection showed that generating different sets of weights may not be the only problem. Sometimes, the inconsistency might be so great that a different rank ordering might occur. For the example when $c_{13} = [0.1, 0.3]$, we notice that using the FPP on $C$ yields $w_2 > w_3 > w_1$, while using the FPP on $\tilde{C}$ yields $\tilde{w}_c > \tilde{w}_b > \tilde{w}_a$. This should not occur for one method using the same interval information.

Our method is not subject to these deficiencies. Our use of the geometric mean preserves the preference structure of the DM and the reciprocal nature of his or her comparisons. In the minimization of the total error with respect to the geometric mean, the same point of reference is used for the measurement of this error no matter which comparisons are used. The weights will be unchanged regardless of the problem structure. We see this consistency as a necessary feature for any interval method.

## 4.4 Concluding Remarks on Interval Methods

The debate over the most effective way to determine a priority vector for interval pairwise comparison matrices will continue, as evidenced by the growing number of methods introduced in recent years to address the interval problem. In this chapter, we have introduced the LP method and presented a natural adaptation to it. The addition of stretch factors is a straightforward extension to the LP model, and the three-stage formulation synthesizes a variety of techniques. This new method, what we call the LP-S method, finds quality priority vectors and is flexible enough to be applied to a wide variety of pairwise comparison matrices.

In this section of the thesis, we have also illustrated that there are a few problems with two alternative methods (the LGP and FPP) used for generating weights for a inconsistent interval pairwise comparison matrix. While these two methods are able to generate solutions, we have documented some of the problems they encounter. We have also demonstrated how our LP-S method avoids these problems. Through more experimentation, we aim to further explore the LP-S method in search of other potential strengths and weaknesses. We believe that it holds much promise in the future study of interval AHP problems.

# Chapter 5

## Conclusions and Ideas for Future Work

In the first part of this thesis, we revisited the greatest sports record problem that was examined by Golden and Wasil [16] and Richardson [29]. Using up-to-date data and the ratings mode of the analytic hierarchy process, we ranked the greatest sports records of all time. We found Byron Nelson's 11 consecutive professional wins, Joe DiMaggio's 56 consecutive-game hitting streak, and Wilt Chamberlain's 50.4 points per game average to be the three greatest single-season sports records. Babe Ruth's 0.670 career slugging average, Wayne Gretzky's 2857 points, and Jack Nicklaus' 18 majors were the three greatest career sports records. Wilt Chamberlain's 100 points scored in a basketball game and Ernie Nevers' 40 point football game were the greatest single-event sports records.

In the future, it will be relatively easy to monitor the rankings with the ratings mode of the AHP. As values and data associated with the records change, we will be able to update the ratings and determine whether or not any records should receive an updated rating. We are also always in search of additional great individual records that could be considered for our study, such as Lance Armstrong's six Tour de France victories or Edwin Moses's 122 consecutive wins in the 400m hurdles. In future studies, we might consider team records such as the 21-game winning streak of the New England Patriots in 2003-04 and the 33-game winning streak of the Los Angeles Lakers in 1971-72. The ratings mode would allow these records to be easily incorporated into our rankings.

We note that this study analyzes the greatest sports *records* of all time, not necessarily the greatest *athletes* of all time. Though the records we considered in this study are remarkable achievements themselves, they are not always indicators of the greatest athletes in a sport. Determining the greatest athletes of all time is an interesting area where we might apply the AHP in future work. Examples of these studies might include determining the greatest football quarterback, the best point guard in basketball, and the best professional golfer of all time. In such focused studies, we would be able to more explicitly capture a variety of different sport-specific performance indicators and better analyze how changes in sports over time have impacted the determination of a great athlete in his or her respective sport.

In the second part of this thesis, we studied solution techniques for generating priority vectors for interval AHP problems. We introduced the LP method, identified its major deficiency, and modified the method to address this deficiency. By including stretch factors into the formulation, we have given the LP method the ability to allow sets of weights that do not satisfy all of the interval pairwise comparisons. This new LP-S method can be applied to all types of pairwise comparison matrices: discrete, interval, and mixed; consistent and inconsistent. We have also presented a case for why we believe the LP-S method is better than other interval techniques, and we have compared the LP-S method to these techniques.

In future work, we plan to analyze the performance of the LP-S on all types of pairwise comparison matrices and continue to compare its performance to other interval methods. We would also like to explore the role our Stage 0 and Stage 1 objective functions play in measuring the quality of solutions. We plan to investigate consistency measures like the geometric consistency index [2] to help us determine acceptable values for total stretching and total error.

# Appendix A

# Greatest Sports Records Appendix

## A.1 Single-Season Records Data

Throughout the process of determining the greatest records in each of the three categories, we gathered a lot of data to support our analyses. In this section, we include all single-season data used for the study that could not be included in the text because of space considerations.

### A.1.1 Recent Challengers to the Single-Season Records

In determining how long each record is expected to stand, we analyzed some of the best athletic performances in each sport over the past ten years (1994-2004). Table A.1a through Table A.1e list some of the most recent challenges to each record.

| **Basketball** | | | | | |
|---|---|---|---|---|---|
| **Player** | **Mark** | **Year** | **Player** | **Mark** | **Year** |
| STOCKTON | | | WILTSAVG | | |
| John Stockton | 14.5 | 1989-90 | Wilt Chamberlain | 50.4 | 1961-62 |
| John Stockton | 12.6 | 1993-94 | Tracy McGrady | 32.1 | 2002-03 |
| John Stockton | 12.3 | 1994-95 | Allen Iverson | 31.4 | 2001-02 |
| Mark Jackson | 11.4 | 1996-97 | Allen Iverson | 31.1 | 2000-01 |
| John Stockton | 11.2 | 1995-96 | Michael Jordan | 30.4 | 1995-96 |
| Andre Miller | 10.9 | 2001-02 | Kobe Bryant | 30.0 | 2002-03 |
| Jason Kidd | 10.8 | 1998-99 | | | |
| WILTREB | | | | | |
| Wilt Chamberlain | 27.2 | 1960-61 | | | |
| Dennis Rodman | 17.3 | 1993-94 | | | |
| Dennis Rodman | 16.8 | 1994-95 | | | |
| Dennis Rodman | 16.1 | 1996-97 | | | |
| Ben Wallace | 15.4 | 2002-03 | | | |
| Dennis Rodman | 15 | 1997-98 | | | |
| Dennis Rodman | 14.9 | 1995-96 | | | |

Table A.1a: Challengers to each single-season basketball record from 1994-2004

| Baseball | | | | | |
|---|---|---|---|---|---|
| **Player** | **Mark** | **Year** | **Player** | **Mark** | **Year** |
| DIMAGGIO | | | HENDRSON | | |
| Joe DiMaggio | 56 | 1941 | Rickey Henderson | 130 | 1982 |
| Luis Castillo | 35 | 2002 | Kenny Lofton | 75 | 1996 |
| Vladimir Guerrero | 31 | 1999 | Brian Hunter | 74 | 1997 |
| Sandy Alomar | 30 | 1997 | Tony Womack | 72 | 1999 |
| Nomar Garciaparra | 30 | 1997 | Roger Cedeno | 66 | 1999 |
| Albert Pujols | 30 | 2003 | Rickey Henderson | 66 | 1998 |
| Eric Davis | 30 | 1998 | Tom Goodwin | 66 | 1996 |
| Luis Gonzalez | 30 | 1999 | | | |
| BONDSHR | | | GIBSON | | |
| Barry Bonds | 73 | 2001 | Bob Gibson | 1.12 | 1968 |
| Mark McGwire | 70 | 1998 | Greg Maddux | 1.56 | 1994 |
| Sammy Sosa | 66 | 1998 | Greg Maddux | 1.63 | 1995 |
| Mark McGwire | 65 | 1999 | Pedro Martinez | 1.74 | 2000 |
| Sammy Sosa | 64 | 2001 | Kevin Brown | 1.89 | 1996 |
| Sammy Sosa | 63 | 1999 | Pedro Martinez | 1.9 | 1997 |
| Luis Gonzalez | 57 | 2001 | Roger Clemens | 2.05 | 1997 |
| Alex Rodriguez | 57 | 2002 | | | |
| BONDSSA | | | RYAN | | |
| Barry Bonds | 0.863 | 2001 | Nolan Ryan | 383 | 1973 |
| Barry Bonds | 0.799 | 2002 | Randy Johnson | 372 | 2001 |
| Mark McGwire | 0.752 | 1998 | Randy Johnson | 364 | 1999 |
| Jeff Bagwell | 0.750 | 1994 | Randy Johnson | 347 | 2000 |
| Barry Bonds | 0.749 | 2003 | Randy Johnson | 334 | 2002 |
| Sammy Sosa | 0.737 | 2001 | Curt Schilling | 319 | 1997 |
| Mark McGwire | 0.730 | 1996 | Curt Schilling | 316 | 2002 |
| HORNSBY | | | WILSON | | |
| Rogers Hornsby | 0.424 | 1924 | Hack Wilson | 191 | 1930 |
| Tony Gwynn | 0.394 | 1994 | Manny Ramirez | 165 | 1999 |
| Larry Walker | 0.379 | 1999 | Sammy Sosa | 160 | 2001 |
| Tony Gwynn | 0.372 | 1997 | Sammy Sosa | 158 | 1998 |
| Nomar Garciaparra | 0.372 | 2000 | Juan Gonzalez | 157 | 1998 |
| Todd Helton | 0.372 | 2000 | Albert Belle | 152 | 1998 |
| Barry Bonds | 0.370 | 2002 | Andres Galarraga | 150 | 1996 |

Table A.1b: Challengers to each single-season baseball record from 1994-2004

| Golf | | |
|---|---|---|
| **Player** | **Mark** | **Year** |
| NELSON | | |
| Byron Nelson | 11 | 1945 |
| Tiger Woods | 6 | 1999-00 |
| Peter Jacobsen | 3 | 1995 |
| David Duval | 3 | 1998 |

Table A.1c: Challengers to the single-season golf record from 1994-2004

| Hockey | | | | | |
|---|---|---|---|---|---|
| **Player** | **Mark** | **Year** | **Player** | **Mark** | **Year** |
| GRETZKYG | | | GRETZKYP | | |
| Wayne Gretzky | 92 | 1981-82 | Wayne Gretzky | 215 | 1985-86 |
| Mario Lemieux | 69 | 1995-96 | Mario Lemieux | 161 | 1995-96 |
| Pavel Bure | 59 | 2000-01 | Jaromir Jagr | 149 | 1995-96 |
| Pavel Bure | 58 | 1999-2000 | Jaromir Jagr | 127 | 1998-99 |
| Alexander Mogilny | 55 | 1995-96 | Mario Lemieux | 122 | 1996-97 |
| Joe Sakic | 54 | 2000-01 | Jaromir Jagr | 121 | 2000-01 |
| | | | Joe Sakic | 118 | 2000-01 |
| GRETZKYA | | | | | |
| Wayne Gretzky | 163 | 1985-86 | | | |
| Mario Lemieux | 92 | 1995-96 | | | |
| Ron Francis | 92 | 1995-96 | | | |
| Jaromir Jagr | 87 | 1995-96 | | | |
| Peter Forsberg | 86 | 1995-96 | | | |
| Jaromir Jagr | 83 | 1998-99 | | | |
| Peter Forsberg | 77 | 2002-03 | | | |

Table A.1d: Challengers to each single-season hockey record from 1994-2004

| Football | | | | | |
|---|---|---|---|---|---|
| **Player** | **Mark** | **Year** | **Player** | **Mark** | **Year** |
| DICKRSON | | | HORNUNG | | |
| Eric Dickerson | 2105 | 1984 | Paul Hornung | 176 | 1960 |
| Jamal Lewis | 2066 | 2003 | Gary Anderson | 164 | 1998 |
| Barry Sanders | 2053 | 1997 | Jeff Wilkins | 163 | 2003 |
| Terrell Davis | 2008 | 1998 | Priest Holmes | 162 | 2003 |
| Barry Sanders | 1883 | 1994 | Marshall Faulk | 160 | 2000 |
| Ahman Green | 1883 | 2003 | Mike Vanderjagt | 157 | 2003 |
| Ricky Williams | 1853 | 2002 | Emmitt Smith | 150 | 1995 |
| MARINO | | | HOLMES | | |
| Dan Marino | 48 | 1984 | Priest Holmes | 27 | 2003 |
| Kurt Warner | 41 | 1999 | Marshall Faulk | 26 | 2000 |
| Brett Favre | 39 | 1996 | Emmitt Smith | 25 | 1995 |
| Brett Favre | 38 | 1995 | Priest Holmes | 24 | 2002 |
| Steve Young | 36 | 1998 | Terrell Davis | 23 | 1998 |
| Steve Beuerlein | 36 | 1999 | Emmitt Smith | 22 | 1994 |

Table A.1e: Challengers to each single-season football record from 1994-2004

### A.1.2 Single-Season Ratings Pairwise Comparison Matrices

Each subfactor is sorted into three to five ratings, with each record being assigned one of these ratings. The ratings are compared relative to each other, and the priority vector generated from these comparisons is used to assign the local weights to each record. The matrices below contain the pairwise comparisons for the single-season ratings and the priority vectors associated with them.

Years Record Has Stood

| | | | | | Priority |
|---|---|---|---|---|---|
| Over 50 years | 1 | 2 | 5 | 9 | 1.000 |
| 25 to 50 years | | 1 | 3 | 7 | 0.581 |
| 10 to 25 years | | | 1 | 3 | 0.216 |
| Under 10 years | | | | 1 | 0.088 |

Years Expected to Stand

| | | | | Priority |
|---|---|---|---|---|
| Over 20 years | 1 | 2 | 6 | 1.000 |
| 10 to 20 years | | 1 | 3 | 0.500 |
| Under 10 years | | | 1 | 0.167 |

Percent Over Previous Record

| | | | | | | Priority |
|---|---|---|---|---|---|---|
| Over 50% | 1 | 3 | 4 | 7 | 9 | 1.000 |
| 30% to 50% | | 1 | 2 | 3.5 | 7 | 0.461 |
| 15% to 30% | | | 1 | 2 | 4 | 0.262 |
| 5% to 15% | | | | 1 | 1.5 | 0.130 |
| Less than 5% | | | | | 1 | 0.081 |

Percent Over Contemporary Mark

| | | | | | Priority |
|---|---|---|---|---|---|
| Over 55% | 1 | 3 | 5 | 8 | 1.000 |
| 40% to 55% | | 1 | 2 | 5 | 0.408 |
| 20% to 40% | | | 1 | 2 | 0.200 |
| Less than 20% | | | | 1 | 0.101 |

Glamour

| | | | | | Priority |
|---|---|---|---|---|---|
| Well known | 1 | 2 | 5 | 8 | 1.000 |
| Known | | 1 | 3 | 6 | 0.578 |
| Not always known | | | 1 | 2 | 0.201 |
| Not known | | | | 1 | 0.110 |

Purity

| | | | | Priority |
|---|---|---|---|---|
| Not aided | 1 | 2 | 6 | 1.000 |
| Slightly aided | | 1 | 3 | 0.500 |
| Greatly aided | | | 1 | 0.167 |

## A.2 Career Sports Records Data

This section contains the data used to assign a rating to each career record for the Years Record is Expected to Stand subcriterion. We also give the pairwise comparison matrices and their respective priority vector for the five career ratings.

### A.2.1 Current Challengers to Career Records

For estimating the length of time we expect each career record to stand, we researched active players that either are (a) relatively close to passing the current active career record or (b) young in their careers but could challenge the career record in the future. In Tables A.2a through Table A.2c, we give the data we gathered.

| **Challengers** | **Career Total** | **Average per Season** | **2001** | **2002** | **2003** | **Years Needed** | **Age** |
|---|---|---|---|---|---|---|---|
| | | | **Baseball** | | | | |
| AARON | | | | | | | |
| Hank Aaron | 755 | | | | | | |
| Barry Bonds | 658 | 37 | 73 | 46 | 45 | 3 | 39 |
| Sammy Sosa | 539 | 36 | 64 | 49 | 40 | 6 | 35 |
| Alex Rodriguez | 345 | 35 | 52 | 57 | 47 | 12 | 28 |
| COBB | | | | | | | |
| Ty Cobb | 0.370 | | | | | | |
| Todd Helton | 0.337 | | 0.336 | 0.329 | 0.358 | | 30 |
| Nomar Garciaparra | 0.323 | | 0.289 | 0.310 | 0.301 | | 30 |
| Vladimir Guerrero | 0.323 | | 0.307 | 0.336 | 0.330 | | 27 |
| HENDRSON | | | | | | | |
| Ricky Henderson | 1406 | | | | | | |
| Kenny Lofton | 538 | 41 | 16 | 29 | 30 | 22 | |
| Barry Bonds | 500 | 28 | 13 | 9 | 7 | 33 | 39 |
| Roberto Alomar | 474 | 30 | 30 | 16 | 12 | 31 | 35 |
| RIPKEN | | | | | | | |
| Cal Ripken | 2632 | | | | | | |
| Miguel Tejada | 564 | | | | | 13 | 27 |
| ROSE | | | | | | | |
| Pete Rose | 4256 | | | | | | |
| Rickey Henderson | 3055 | 122 | 86 | 40 | 15 | 10 | 45 |
| Rafael Palmeiro | 2780 | 154 | 164 | 149 | 146 | 10 | 39 |
| Roberto Alomar | 2679 | 105 | 193 | 157 | 133 | 15 | 35 |
| RUTH | | | | | | | |
| Babe Ruth | 0.690 | | | | | | |
| Todd Helton | 0.616 | | 0.685 | 0.577 | 0.630 | | 30 |
| Barry Bonds | 0.602 | | 0.863 | 0.799 | 0.749 | | 39 |
| Manny Ramirez | 0.598 | | 0.609 | 0.647 | 0.587 | | 31 |
| RYAN | | | | | | | |
| Nolan Ryan | 5714 | | | | | | |
| Roger Clemens | 4099 | 205 | 213 | 192 | 190 | 8 | |
| Randy Johnson | 3871 | 242 | 372 | 334 | 125 | 8 | 40 |
| Pedro Martinez | 2426 | 202 | 163 | 239 | 206 | 17 | 32 |

Table A.2a: Active challengers to the baseball career records

| Challengers | Career Total | Average per Season | 2001 | 2002 | 2003 | Years Needed | Age |
|---|---|---|---|---|---|---|---|
| **Basketball** | | | | | | | |
| JABBAR | | | | | | | |
| Kareem Abdul-Jabbar | 38387 | | | | | | |
| Karl Malone | 36928 | 1944 | 1788 | 1667 | 554 | 1 | 40 |
| Shaquille O'Neal | 20475 | 1861 | 2125 | 1822 | 1841 | 10 | 31 |
| Tracy McGrady | 10420 | 1489 | 1948 | 2407 | 1878 | 19 | 25 |
| JORDWILT | | | | | | | |
| Michael Jordan | 30.1 | | | | | | |
| Shaquille O'Neal | 27.1 | | 27.2 | 27.5 | 21.5 | | 31 |
| Allen Iverson | 27.0 | | 31.4 | 27.6 | 26.4 | | 28 |
| Karl Malone | 25.0 | | 22.4 | 20.6 | 13.2 | | 40 |
| WILTREB | | | | | | | |
| Wilt Chamberlain | 23924 | | | | | | |
| Karl Malone | 14968 | 788 | 686 | 628 | 367 | 12 | 40 |
| Kevin Garnett | 7493 | 833 | 981 | 1102 | 1139 | 20 | 28 |
| Tim Duncan | 6407 | 915 | 1043 | 1043 | 859 | 20 | 27 |
| STOCKTON | | | | | | | |
| John Stockton | 15806 | | | | | | |
| Gary Payton | 8039 | 574 | 737 | 663 | 449 | 14 | 35 |
| Jason Kidd | 6738 | 674 | 808 | 711 | 618 | 14 | 31 |
| Stephon Marbury | 4830 | 604 | 666 | 654 | 719 | 19 | 27 |
| **Football** | | | | | | | |
| ANDERSON | | | | | | | |
| Gary Anderson | 2346 | | | | | | |
| Morten Andersen | 2259 | 103 | 98 | 117 | 106 | 1 | 43 |
| John Carney | 1433 | 110 | 113 | 130 | 102 | 9 | 39 |
| Matt Stover | 1364 | 97 | 115 | 96 | 134 | 11 | 35 |
| MARINO | | | | | | | |
| Dan Marino | 61243 | | | | | | |
| Brett Favre | 45646 | 3804 | 3921 | 3658 | 3361 | 5 | 34 |
| Drew Bledsoe | 36876 | 3352 | 400 | 4359 | 2860 | 8 | 31 |
| Peyton Manning | 24885 | 4148 | 4131 | 4200 | 4267 | 9 | 27 |
| SMITH | | | | | | | |
| Emmitt Smith | 17418 | | | | | | |
| Jerome Bettis | 12353 | 1123 | 1072 | 666 | 811 | 5 | 31 |
| Curtis Martin | 11669 | 1297 | 1513 | 1094 | 1308 | 5 | 30 |
| Eddie George | 10009 | 1251 | 939 | 1165 | 1031 | 6 | 30 |
| Jamal Lewis | 4757 | 1586 | — | 1327 | 2066 | 8 | 24 |
| UNITAS | | | | | | | |
| Johnny Unitas | 47 | | | | | | |
| Brett Favre | 25 | | | | | | 34 |

Table A.2b: Active challengers to the basketball and football career records

| Challengers | Career Total | Average per Season | 2001 | 2002 | 2003 | Years Needed | Age |
|---|---|---|---|---|---|---|---|
| **Hockey** | | | | | | | |
| GRETZKY | | | | | | | |
| Wayne Gretzky | 2857 | | | | | | |
| Mark Messier | 1887 | 75 | 23 | 40 | 43 | 14 | 43 |
| Joe Sakic | 1402 | 88 | 79 | 58 | 87 | 18 | 34 |
| Jaromir Jagr | 1309 | 94 | 79 | 77 | 74 | 17 | 32 |
| **Golf** | | | | | | | |
| NICKLAUS | | | | | | | |
| Jack Nicklaus | 18 | | | | | | |
| Tiger Woods | 8 | | | | | | 28 |
| Nick Faldo | 6 | | | | | | |
| Ernie Els | 3 | | | | | | 34 |
| BERG | | | | | | | |
| Patty Berg | 15 | | | | | | |
| Juli Inkster | 7 | | | | | | 43 |
| Annika Sorenstam | 6 | | | | | | 33 |
| Se Ri Pak | 4 | | | | | | 26 |
| **Tennis** | | | | | | | |
| SAMPRAS | | | | | | | |
| Pete Sampras | 14 | | | | | | |
| Andre Agassi | 8 | | | | | | 33 |
| Gustavo Keurten | 3 | | | | | | 27 |
| Lleyton Hewitt | 2 | | | | | | 22 |
| COURT | | | | | | | |
| Margaret Smith Court | 24 | | | | | | |
| Monica Seles | 9 | | | | | | 30 |
| Serena Williams | 6 | | | | | | 22 |
| Venus Williams | 4 | | | | | | 23 |

Table A.2c: Active challengers to the hockey, golf, and tennis career records

### A.2.2 Pairwise Comparisons for Career Ratings

Like the single-season records, we pairwise compared the ratings and generated weights for these ratings using Expert Choice. These pairwise comparison matrices are contained below.

Years Record has Stood

| | | | | | Priority |
|---|---|---|---|---|---|
| Over 50 years | 1 | 2 | 5 | 9 | 1.000 |
| 25 to 50 years | | 1 | 3 | 7 | 0.581 |
| 10 to 25 years | | | 1 | 3 | 0.216 |
| Less than 10 years | | | | 1 | 0.088 |

Years Expected to Stand

| | | | | Priority |
|---|---|---|---|---|
| Over 20 years | 1 | 3 | 7 | 1.000 |
| 10 to 20 years | | 1 | 2 | 0.317 |
| Less than 10 years | | | 1 | 0.150 |

Percent over Second Best

| | | | | | Priority |
|---|---|---|---|---|---|
| Over 40% | 1 | 3 | 5 | 8 | 1.000 |
| 15% to 40% | | 1 | 2 | 5 | 0.408 |
| 5% to 15% | | | 1 | 2 | 0.200 |
| Less than 5% | | | | 1 | 0.101 |

Glamour

| | | | | | Priority |
|---|---|---|---|---|---|
| Well known | 1 | 2 | 5 | 8 | 1.000 |
| Known | | 1 | 3 | 6 | 0.578 |
| Not always known | | | 1 | 2 | 0.201 |
| Not known | | | | 1 | 0.110 |

Purity

| | | | | Priority |
|---|---|---|---|---|
| Not aided | 1 | 2 | 6 | 1.000 |
| Slightly aided | | 1 | 3 | 0.500 |
| Greatly aided | | | 1 | 0.167 |

## A.3 Single-Event Records Data

### A.3.1 Recent Challenges to the Single-Event Records

To make estimations as to how long we believe each single-event record will last, we collected information regarding recent performances that came close to challenging each record. Table A.3 contains data regarding the best single-event performances from 1994-2004.

| **Sport** | **Identifier** | **Player** | **Mark** | **Date** |
|---|---|---|---|---|
| Basketball | | | | |
| | WILTPTS | Wilt Chamberlain | 100 | |
| | | David Robinson | 71 | 04/24/94 |
| | | Tracy McGrady | 62 | 03/10/04 |
| | | Shaquille O'Neal | 61 | 03/06/00 |
| | WILTREB | Wilt Chamberlain | 55 | |
| | | Dennis Rodman | 32 | 01/22/94 |
| Football | | | | |
| | NEVERS | Ernie Nevers | 40 | |
| | | Clinton Portis | 30 | 12/07/03 |
| | | Shaun Alexander | 30 | 09/29/02 |
| | | James Stewart | 30 | 10/12/97 |
| | | Corey Dillon | 24 | 12/04/97 |
| | LEWIS | Jamal Lewis | 295 | |
| | | Corey Dillon | 278 | 10/22/00 |
| | | Shaun Alexander | 266 | 11/11/01 |
| | | Mike Anderson | 251 | 12/03/00 |
| | | Corey Dillon | 246 | 12/04/97 |
| | VBRCKLN | Norm Van Brocklin | 554 | |
| | | Boomer Esiason | 522 | 11/10/96 |
| | | Elvis Grbac | 504 | 11/05/00 |
| | | Vinny Testaverde | 481 | 12/24/00 |
| Track & Field | | | | |
| | POWELL | Mike Powell | 8.95 | |
| | | Erick Walder | 8.74 | 04/01/94 |
| | | Ivan Pedroso | 8.71 | 07/18/95 |
| | | Melvin Lister | 8.49 | 05/13/00 |
| | | Jai Taurima | 8.49 | 09/28/00 |

Table A.3: Recent challengers to the single-event records

A.3.2 Pairwise Comparisons Matrices

In contrast to the single-season and career records, the single-event weights were found using the standard AHP. With respect to each subcriterion, we made pairwise comparisons between each pair of records. The resulting pairwise comparison matrices and the priority vectors associated with each matrix are contained in Table A.4 through Table A.8.

- Years a Record has Stood

| | | | | | | | **Priority** |
|---|---|---|---|---|---|---|---|
| NEVERS | 1 | 1.42 | 1.72 | 1.80 | 6.17 | 9 | 0.307 |
| VBRKLN | | 1 | 1.21 | 1.27 | 4.33 | 9 | 0.224 |
| WILTREB | | | 1 | 1.05 | 3.58 | 9 | 0.191 |
| WILTPTS | | | | 1 | 3.42 | 9 | 0.183 |
| POWELL | | | | | 1 | 9 | 0.074 |
| LEWIS | | | | | | 1 | 0.021 |

Table A.4: Pairwise comparisons for Years Stood

- Years Expected to Stand

| | | | | | | | **Priority** |
|---|---|---|---|---|---|---|---|
| WILTPTS | 1 | 1.2 | 1.5 | 3 | 4 | 7 | 0.311 |
| NEVERS | | 1 | 1.2 | 2 | 3 | 5 | 0.237 |
| WILTREB | | | 1 | 2 | 2 | 5 | 0.199 |
| POWELL | | | | 1 | 1.5 | 5 | 0.112 |
| VBRKLN | | | | | 1 | 3 | 0.104 |
| LEWIS | | | | | | 1 | 0.037 |

Table A.5: Pairwise comparisons for Years Expected to Stand

- Percent over Previous Mark

| | | | | | | | **Priority** |
|---|---|---|---|---|---|---|---|
| WILTPTS | 1 | 1.23 | 2.22 | 5.21 | 6.67 | 9 | 0.325 |
| NEVERS | | 1 | 1.81 | 4.25 | 5.45 | 9 | 0.274 |
| VBRKLN | | | 1 | 2.34 | 3 | 9 | 0.172 |
| WILTREB | | | | 1 | 1.28 | 9 | 0.100 |
| LEWIS | | | | | 1 | 9 | 0.089 |
| POWELL | | | | | | 1 | 0.023 |

Table A.6: Pairwise comparisons for Percent over Previous

- Glamour

| | | | | | | | Priority |
|---|---|---|---|---|---|---|---|
| WILTPTS | 1 | 2.0 | 3.5 | 3.5 | 8.0 | 8.0 | 0.430 |
| POWELL | | 1 | 1.8 | 1.8 | 4.0 | 4.0 | 0.217 |
| WILTREB | | | 1 | 1.0 | 2.3 | 2.3 | 0.123 |
| LEWIS | | | | 1 | 2.3 | 2.3 | 0.123 |
| NEVERS | | | | | 1 | 1.0 | 0.054 |
| VBRKLN | | | | | | 1 | 0.054 |

Table A.7: Pairwise comparisons for Glamour

- Purity

| | | | | | | | Priority |
|---|---|---|---|---|---|---|---|
| POWELL | 1 | 2.0 | 4.0 | 4.0 | 5.0 | 7.0 | 0.426 |
| LEWIS | | 1 | 2.0 | 2.0 | 2.5 | 3.5 | 0.213 |
| WILTPTS | | | 1 | 1.0 | 1.3 | 1.8 | 0.108 |
| VBRKLN | | | | 1 | 1.3 | 1.8 | 0.108 |
| NEVERS | | | | | 1 | 1.6 | 0.086 |
| WILTREB | | | | | | 1 | 0.059 |

Table A.8: Pairwise comparisons for Purity

# Appendix B

# Interval AHP Appendix

## B.1 LP-S Method on Example I

This section contains the formulation for the three stages of the LP-S method applied to the sample problem presented in Subsection 4.2. The pairwise comparison matrix used for Example I is found in Equation 4.28. The subsections that follow demonstrate and how we used the entries in this matrix to create the Stage 0, Stage 1, and Stage 2 formulations for this example.

### B.1.1 Stage 0 Formulation and Output

Stage 0 finds the minimum stretching required to remove the inconsistency among the intervals. The objective function sums the stretch factors in the upper triangular entries of the pariwise comparison matrix. To create the remaining formulation, we computed the natural logarithm of the geometric mean of each interval in the upper triangular portion of the pairwise comparison matrix. These values were used in the set of equality constraints. For the soft interval inequalities, we computed the natural logarithm of both the upper and lower bound for these same intervals. The remaining constraints are problem independent.

LINDO was used to optimize the linear program. Following the formulation, the output of the solver is presented. We note that all values returned by LINDO are in the natural logarithm space. They must be exponentiated to be returned to the normal weight space.

- Formulation

$$
\begin{aligned}
\text{Min} \quad & \sum_{i=1}^{3} \sum_{j=i+1}^{4} g_{ij} \\
\text{s.t.} \quad & x_1 - x_2 - y_{12} = 1.1513 & x_1 - x_2 + g_{12} \geq 0.6931 \\
& x_2 - x_1 - y_{21} = -1.1513 & x_1 - x_2 - g_{12} \leq 1.6094 \\
& x_1 - x_3 - y_{13} = 1.0397 & x_1 - x_3 + g_{13} \geq 0.6931 \\
& x_3 - x_1 - y_{31} = -1.0397 & x_1 - x_3 - g_{13} \leq 1.3863 \\
& x_1 - x_4 - y_{14} = 0.34657 & x_1 - x_4 + g_{14} \geq 0.0 \\
& x_4 - x_1 - y_{41} = -0.34657 & x_1 - x_4 - g_{14} \leq 0.6931 \\
& x_2 - x_3 - y_{23} = 1.0075 & x_2 - x_3 + g_{23} \geq 0.9163 \\
& x_3 - x_2 - y_{32} = -1.0075 & x_2 - x_3 - g_{23} \leq 1.0986 \\
& x_2 - x_4 - y_{24} = 0.2028 & x_2 - x_4 + g_{24} \geq 0.0 \\
& x_4 - x_2 - y_{42} = -0.2028 & x_2 - x_4 - g_{24} \leq 0.4055 \\
& x_3 - x_4 - y_{34} = -0.34657 & x_3 - x_4 + g_{34} \geq -0.6931 \\
& x_4 - x_3 - y_{43} = 0.34657 & x_3 - x_4 - g_{34} \leq 0.0 \\
& z_{12} - y_{12} \geq 0 & x_1 = 0 \\
& z_{12} - y_{21} \geq 0 & z_{ij} \geq 0 \quad \forall i, j \\
& z_{13} - y_{13} \geq 0 & g_{ij} \geq 0 \quad \forall i, j \\
& z_{13} - y_{31} \geq 0 \\
& z_{14} - y_{14} \geq 0 \\
& z_{14} - y_{41} \geq 0 \\
& z_{23} - y_{23} \geq 0 \\
& z_{23} - y_{32} \geq 0 \\
& z_{24} - y_{24} \geq 0 \\
& z_{24} - y_{42} \geq 0 \\
& z_{34} - y_{34} \geq 0 \\
& z_{34} - y_{43} \geq 0
\end{aligned}
$$

- LINDO Output

```
LP OPTIMUM FOUND AT STEP     40

       OBJECTIVE FUNCTION VALUE

       1)     0.2232000

 VARIABLE        VALUE          REDUCED COST
      G12         0.000000          0.000000
      G13         0.000000          1.000000
      G14         0.000000          0.000000
      G23         0.223200          0.000000
      G24         0.000000          0.000000
      G34         0.000000          0.000000
       X1         0.000000          0.000000
       X2        -0.693100          0.000000
      Y12        -0.458200          0.000000
      Y21         0.458200          0.000000
       X3        -1.386200          0.000000
      Y13         0.346500          0.000000
      Y31        -0.346500          0.000000
       X4        -0.693100          0.000000
      Y14         0.346530          0.000000
      Y41        -0.345443          0.000000
      Y23        -0.314400          0.000000
      Y32         0.314400          0.000000
      Y24        -0.202800          0.000000
      Y42         0.202800          0.000000
      Y34        -0.346530          0.000000
      Y43         0.346530          0.000000
      Z12         0.458200          0.000000
      Z13         0.458200          0.000000
      Z14         0.458200          0.000000
      Z23         0.458200          0.000000
      Z24         0.458200          0.000000
      Z34         0.346530          0.000000
```

### B.1.2 Stage 1 Formulation and Output

The Stage 1 formulation is exactly the same as Stage 0 with only one addition. The Stage 0 optimal solution is saved and entered as a constraint into the Stage 1 formulation. Only solutions that satisfy the minimum stretching needed to remove the inconsistency are feasible in Stage 1. This formulation is optimized to find the minimum total error.

- Formulation

$$\text{Min} \quad \sum_{i=1}^{3} \sum_{j=i+1}^{4} z_{ij}$$

$$\text{s.t.} \quad \sum_{i=1}^{3} \sum_{j=i+1}^{4} g_{ij} = 0.2232$$

$$\begin{aligned}
x_1 - x_2 - y_{12} &= 1.1513 & x_1 - x_2 + g_{12} &\geq 0.6931 \\
x_2 - x_1 - y_{21} &= -1.1513 & x_1 - x_2 - g_{12} &\leq 1.6094 \\
x_1 - x_3 - y_{13} &= 1.0397 & x_1 - x_3 + g_{13} &\geq 0.6931 \\
x_3 - x_1 - y_{31} &= -1.0397 & x_1 - x_3 - g_{13} &\leq 1.3863 \\
x_1 - x_4 - y_{14} &= 0.34657 & x_1 - x_4 + g_{14} &\geq 0.0 \\
x_4 - x_1 - y_{41} &= -0.34657 & x_1 - x_4 - g_{14} &\leq 0.6931 \\
x_2 - x_3 - y_{23} &= 1.0075 & x_2 - x_3 + g_{23} &\geq 0.9163 \\
x_3 - x_2 - y_{32} &= -1.0075 & x_2 - x_3 - g_{23} &\leq 1.0986 \\
x_2 - x_4 - y_{24} &= 0.2028 & x_2 - x_4 + g_{24} &\geq 0.0 \\
x_4 - x_2 - y_{42} &= -0.2028 & x_2 - x_4 - g_{24} &\leq 0.4055 \\
x_3 - x_4 - y_{34} &= -0.34657 & x_3 - x_4 + g_{34} &\geq -0.6931 \\
x_4 - x_3 - y_{43} &= 0.34657 & x_3 - x_4 - g_{34} &\leq 0.0 \\
z_{12} - y_{12} &\geq 0 & x_1 &= 0 \\
z_{12} - y_{21} &\geq 0 & z_{ij} &\geq 0 \quad \forall i, j \\
z_{13} - y_{13} &\geq 0 & g_{ij} &\geq 0 \quad \forall i, j \\
z_{13} - y_{31} &\geq 0 \\
z_{14} - y_{14} &\geq 0 \\
z_{14} - y_{41} &\geq 0 \\
z_{23} - y_{23} &\geq 0 \\
z_{23} - y_{32} &\geq 0 \\
z_{24} - y_{24} &\geq 0 \\
z_{24} - y_{42} &\geq 0 \\
z_{34} - y_{34} &\geq 0 \\
z_{34} - y_{43} &\geq 0
\end{aligned}$$

- LINDO Output

```
LP OPTIMUM FOUND AT STEP     31

       OBJECTIVE FUNCTION VALUE

       1)      1.812160

 VARIABLE        VALUE          REDUCED COST
      Z12         0.661000          0.000000
      Z13         0.346500          0.000000
      Z14         0.346530          0.000000
      Z23         0.111600          0.000000
      Z24         0.000000          1.000000
      Z34         0.346530          0.000000
      G12         0.202800          0.000000
      G13         0.000000          1.000000
      G14         0.000000          1.000000
      G23         0.020400          0.000000
      G24         0.000000          1.000000
      G34         0.000000          1.000000
       X1         0.000000          0.000000
       X2        -0.490300          0.000000
      Y12        -0.661000          0.000000
      Y21         0.661000          0.000000
       X3        -1.386200          0.000000
      Y13         0.346500          0.000000
      Y31        -0.346500          0.000000
       X4        -0.693100          0.000000
      Y14         0.346530          0.000000
      Y41        -0.345443          0.000000
      Y23        -0.111600          0.000000
      Y32         0.111600          0.000000
      Y24         0.000000          0.000000
      Y42         0.000000          0.000000
      Y34        -0.346530          0.000000
      Y43         0.346530          0.000000
```

We note that this solution is different from the solution obtained in Stage 0. The value for the total stretching is the same for these two stages (it has to be), while the variables adjust to find the solution with the smallest total error.

### B.1.3 Stage 2 Formulation and Output

For the final stage, both Stage 0 and Stage 1 optimal objective values were entered as constraints in Stage 2. As before, the Stage 2 formulation has the same form as Stages 0, with only these two constraints and conditions for $z_{\max}$ added.

- Formulation

$$
\begin{aligned}
\text{Min} \quad & z_{\max} \\
\text{s.t.} \quad & \sum_{i=1}^{3}\sum_{j=i+1}^{4} g_{ij} = 0.2232 & \sum_{i=1}^{3}\sum_{j=i+1}^{4} z_{ij} = 1.81216 \\
& x_1 - x_2 - y_{12} = 1.1513 & x_1 - x_2 + g_{12} \geq 0.6931 \\
& x_2 - x_1 - y_{21} = -1.1513 & x_1 - x_2 - g_{12} \leq 1.6094 \\
& x_1 - x_3 - y_{13} = 1.0397 & x_1 - x_3 + g_{13} \geq 0.6931 \\
& x_3 - x_1 - y_{31} = -1.0397 & x_1 - x_3 - g_{13} \leq 1.3863 \\
& x_1 - x_4 - y_{14} = 0.34657 & x_1 - x_4 + g_{14} \geq 0.0 \\
& x_4 - x_1 - y_{41} = -0.34657 & x_1 - x_4 - g_{14} \leq 0.6931 \\
& x_2 - x_3 - y_{23} = 1.0075 & x_2 - x_3 + g_{23} \geq 0.9163 \\
& x_3 - x_2 - y_{32} = -1.0075 & x_2 - x_3 - g_{23} \leq 1.0986 \\
& x_2 - x_4 - y_{24} = 0.2028 & x_2 - x_4 + g_{24} \geq 0.0 \\
& x_4 - x_2 - y_{42} = -0.2028 & x_2 - x_4 - g_{24} \leq 0.4055 \\
& x_3 - x_4 - y_{34} = -0.34657 & x_3 - x_4 + g_{34} \geq -0.6931 \\
& x_4 - x_3 - y_{43} = 0.34657 & x_3 - x_4 - g_{34} \leq 0.0 \\
& z_{12} - y_{12} \geq 0 & z_{\max} - z_{12} \geq 0 \\
& z_{12} - y_{21} \geq 0 & z_{\max} - z_{13} \geq 0 \\
& z_{13} - y_{13} \geq 0 & z_{\max} - z_{14} \geq 0 \\
& z_{13} - y_{31} \geq 0 & z_{\max} - z_{23} \geq 0 \\
& z_{14} - y_{14} \geq 0 & z_{\max} - z_{24} \geq 0 \\
& z_{14} - y_{41} \geq 0 & z_{\max} - z_{34} \geq 0 \\
& z_{23} - y_{23} \geq 0 & x_1 = 0 \\
& z_{23} - y_{32} \geq 0 & z_{ij} \geq 0 \quad \forall i,j \\
& z_{24} - y_{24} \geq 0 & g_{ij} \geq 0 \quad \forall i,j \\
& z_{24} - y_{42} \geq 0 & \\
& z_{34} - y_{34} \geq 0 & \\
& z_{34} - y_{43} \geq 0 &
\end{aligned}
$$

- LINDO Output

```
LP OPTIMUM FOUND AT STEP     47

       OBJECTIVE FUNCTION VALUE

       1)      0.6610000

 VARIABLE        VALUE          REDUCED COST
     ZMAX         0.661000          0.000000
      G12         0.202800          0.000000
      G13         0.000000          1.000000
      G14         0.000000          0.000000
      G23         0.020400          0.000000
      G24         0.000000          1.000000
      G34         0.000000          1.000000
      Z12         0.661000          0.000000
      Z13         0.346500          0.000000
      Z14         0.346530          0.000000
      Z23         0.111600          0.000000
      Z24         0.000000          0.000000
      Z34         0.346530          0.000000
       X1         0.000000          0.000000
       X2        -0.490300          0.000000
      Y12        -0.661000          0.000000
      Y21         0.661000          0.000000
       X3        -1.386200          0.000000
      Y13         0.346500          0.000000
      Y31        -0.346500          0.000000
       X4        -0.693100          0.000000
      Y14         0.346530          0.000000
      Y41        -0.345443          0.000000
      Y23        -0.111600          0.000000
      Y32         0.111600          0.000000
      Y24         0.000000          0.000000
      Y42         0.000000          0.000000
      Y34        -0.346530          0.000000
      Y43         0.346530          0.000000
```

The output from Stage 2 show that the solution to Stage 1 also minimized the maximum error for this example (it is not always the case). The overall priority vector associated with the pairwise comparison matrix can be found by exponentiating $x_1, x_2, x_3$, and $x_4$ and then normalizing the weights. These weights can be found in the text.

## B.2 Formulations for Computational Comparison I

The first computational comparison in the text presented the interval pairwise comparison matrix found in Equation 4.29. For that matrix, we found a priority vector using five different methods: LP-S, LGP, FPP, EM, and LLS. To illstrate the the differences in the linear programs used by three of these methods, we have shown the formulations for the LP-S, LGP, and FPP methods. and found a priority vector using five different methods. In this section, we have included the formulations for the linear programming methods used to generate three of these solutions: the LP method with stretching, Lexicographic Goal Programming, and Fuzzy Preference Programming.

### B.2.1 LP-S Stage 2 Formulation - Comparison I

We ran Stage 0 and Stage 1 for the LP-S method to find the minimum stretching and minimum error for the pairwise comparison matrix used for this comparison. To save space, we have only included the formulation for the third and final stage. The optimal values from these two stages were entered as constraints in Stage 2, and the optimal priority vector was obtained using LINDO. This priority vector is given in Table 4.1.

$$
\begin{array}{lll}
\text{Max} & z_{\max} & \\
\text{s.t.} & \displaystyle\sum_{i=1}^{3}\sum_{j=i+1}^{4} g_{ij} = 1.7918 & \displaystyle\sum_{i=1}^{3}\sum_{j=i+1}^{4} z_{ij} = 2.7403 \\
 & x_1 - x_2 - y_{12} = 0.34657 & x_1 - x_2 + g_{12} \geq 0 \\
 & x_2 - x_1 - y_{21} = -0.34657 & x_1 - x_2 - g_{12} \leq 0.6931 \\
 & x_1 - x_3 - y_{13} = 0.34657 & x_1 - x_3 + g_{13} \geq 0 \\
 & x_3 - x_1 - y_{31} = -0.34657 & x_1 - x_3 - g_{13} \leq 0.6931 \\
 & x_1 - x_4 - y_{14} = 0.89588 & x_1 - x_4 + g_{14} \geq 0.6931 \\
 & x_4 - x_1 - y_{41} = -0.89588 & x_1 - x_4 - g_{14} \leq 1.0986 \\
 & x_2 - x_3 - y_{23} = 1.3540 & x_2 - x_3 + g_{23} \geq 1.0986 \\
 & x_3 - x_2 - y_{32} = -1.3540 & x_2 - x_3 - g_{23} \leq 1.6094 \\
 & x_2 - x_4 - y_{24} = 1.4979 & x_2 - x_4 + g_{24} \geq 1.3863 \\
 & x_4 - x_2 - y_{42} = -1.4979 & x_2 - x_4 - g_{24} \leq 1.6094 \\
 & x_3 - x_4 - y_{34} = 1.9356 & x_3 - x_4 + g_{34} \geq 1.7918 \\
 & x_4 - x_3 - y_{43} = -1.9356 & x_3 - x_4 - g_{34} \leq 2.0794 \\
 & z_{12} - y_{12} \geq 0 & z_{\max} - z_{12} \geq 0 \\
 & z_{12} - y_{21} \geq 0 & z_{\max} - z_{13} \geq 0 \\
 & z_{13} - y_{13} \geq 0 & z_{\max} - z_{14} \geq 0 \\
 & z_{13} - y_{31} \geq 0 & z_{\max} - z_{23} \geq 0 \\
 & z_{14} - y_{14} \geq 0 & z_{\max} - z_{24} \geq 0 \\
 & z_{14} - y_{41} \geq 0 & z_{\max} - z_{34} \geq 0 \\
 & z_{23} - y_{23} \geq 0 & x_1 = 0 \\
 & z_{23} - y_{32} \geq 0 & z_{ij} \geq 0 \quad \forall i,j \\
 & z_{24} - y_{24} \geq 0 & g_{ij} \geq 0 \quad \forall i,j \\
 & z_{24} - y_{42} \geq 0 & \\
 & z_{34} - y_{34} \geq 0 & \\
 & z_{34} - y_{43} \geq 0 &
\end{array}
$$

### B.2.2 LGP Formulation - Comparison I

This subsection gives the specific formulation of the Lexicographic Goal Programming for the pairwise comparison matrix from the first comparison. The weights that are returned by this method are given in Table 4.1.

$$
\begin{aligned}
\text{Min} \quad & \sum_{i=1}^{3} \sum_{j=i+1}^{4} p_{ij} + p'_{ij} \\
\text{s.t.} \quad & -w_1 + 1w_2 + n_{12} - p_{12} = 0 \\
& w_1 - 2w_2 + n'_{12} - p'_{12} = 0 \\
& -w_1 + 1w_3 + n_{13} - p_{13} = 0 \\
& w_1 - 2w_3 + n'_{13} - p'_{13} = 0 \\
& -w_1 + 2w_4 + n_{14} - p_{14} = 0 \\
& w_1 - 3w_4 + n'_{14} - p'_{14} = 0 \\
& -w_2 + 3w_3 + n_{23} - p_{23} = 0 \\
& w_2 - 5w_3 + n'_{23} - p'_{23} = 0 \\
& -w_2 + 4w_4 + n_{24} - p_{24} = 0 \\
& w_2 - 5w_4 + n'_{24} - p'_{24} = 0 \\
& -w_3 + 6w_4 + n_{34} - p_{34} = 0 \\
& w_3 - 8w_4 + n'_{34} - p'_{34} = 0 \\
& p_{44} + n_{44} = 0 \\
& \sum_{i=1}^{n} w_i + n_{44} - p_{44} = 1 \\
& w_i \geq 0 \quad i = 1, 2, 3, 4 \\
& n_{ij}, p_{ij}, n'_{ij}, p'_{ij} \geq 0 \quad \forall i, j
\end{aligned}
$$

We observe that the goal of the objective function of the LGP is very similar to the objective function used by Stage 0 of the LP-S. The $p_{ij}$'s and $p'_{ij}$'s serve a similar purpose as our stretch factors by allowing solutions to violate the interval comparisons. Whenever $p_{ij}$ or $p'_{ij}$ is positive, the lower or upper interval endpoint for $\frac{w_i}{w_j}$ will be violated, respectively. Though the stretching for the LGP is additive in nature, the magnitude of the minimum total stretching required to remove the inconsistencies will be equal for the LGP and the LP-S.

### B.2.3 FPP Formulation - Comparison I

We also provide the formulation for the Fuzzy Preference Programming on the pairwise comparison matrix in the first computational comparison. We employed linear membership functions for the formulation, allowing us to use linear optimization techniques for generating a priority vector. This formulation is given below.

$$
\begin{aligned}
\text{Max} \quad & \lambda \\
\text{s.t.} \quad & \lambda + w_1 - 2w_2 \leq 1 \\
& \lambda - w_1 + 1w_2 \leq 1 \\
& \lambda + w_1 - 2w_3 \leq 1 \\
& \lambda - w_1 + 1w_3 \leq 1 \\
& \lambda + w_1 - 3w_4 \leq 1 \\
& \lambda - w_1 + 2w_4 \leq 1 \\
& \lambda + w_2 - 5w_3 \leq 1 \\
& \lambda - w_2 + 3w_3 \leq 1 \\
& \lambda + w_2 - 5w_4 \leq 1 \\
& \lambda - w_2 + 4w_4 \leq 1 \\
& \lambda + w_3 - 8w_4 \leq 1 \\
& \lambda - w_3 + 6w_4 \leq 1 \\
& \sum_{i=1}^{4} w_i = 1 \\
& w_i \geq 0 \quad i = 1, 2, 3, 4
\end{aligned}
$$

Each pair of constraints correspond to the two pieces of the linear membership function on each interval. The $\lambda$ that is maximized by the FPP is the membership value of the weight ratio for each fuzzy set (interval comparison). The membership functions have been created so that they assign membership values for weight ratios both inside and outside the interval comparisons. Weight ratios inside the intervals are assigned membership values greater than one. Weight ratios outside the intervals but inside the deviation parameters have membership values between zero and one. Negative membership values are assigned to weight ratios far outside the interval comparisons. The priority vector generated for this formulation can be found in Table 4.1.

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