ABSTRACT

Title of dissertation:	Hydrodynamic Limits of the Boltzmann Equation
	Chunting Lu, Doctor of Philosophy, 2018
Dissertation directed by:	Professor Charles David Levermore Department of Mathematics

This dissertation studies two problems that are related to the question of how solutions of the Boltzmann equation behave in various fluid dynamic regimes. The Boltzmann equation models so-called rarefied gases of identical particles, for which all but binary collisions between particles can be neglected. When the mean free path of gas particles is small comparing to the macroscopic length scale, one can derive fluid equations from the Boltzmann equations.

The first problem is to establish the acoustic limit for a family of appropriately scaled DiPerna-Lions solutions with finite zeroth to second moments over $\mathbb{R}^{\mathrm{D}}$. Every initial data with finite zeroth to second moments has a unique nonhomogeneous global Maxwellian associated with it by matching values of conserved quantities. The fluid fluctuations converge to a unique limit governed by the solution of an acoustic system with variable coefficients. This differs from the acoustic system with constant coefficient obtained by scaling the Boltzmann equation around a homogeneous Maxwellian ([6], [24]). Moreover, unlike the regimes around the homogeneous Maxwellian, there is no higher order Navier-Stokes correction in the regime around

the nonhomogeneous Maxwellian.

The second problem is the approximation of solutions to the linearized Boltzmann equation by solutions of the linearized compressible Navier-Stokes system and by solutions of the weakly dissipative linearized compressible Navier-Stokes system over a periodic domain. We show that if the initial data of the linearized Boltzmann equation is smooth enough and lies within the fluid regime, then fluid moments of its solutions are close to the associated linearized compressible Navier-Stokes system in $L^2(\mathbb{T}^{\mathrm{D}})$ uniformly for $t > 0$. We also show that solutions of the weakly dissipative linearized compressible Navier-Stokes systems approximate solutions of the linearized compressible Navier-Stokes system uniformly for $t > 0$ in $L^2(\mathbb{T}^{\mathrm{D}})$. Therefore, we justified weakly dissipative linearized compressible Navier-Stokes approximation to the linearized Boltzmann equation. Our work differs from that of Ellis and Pinsky [17] in that (1) we consider a periodic domain instead of $\mathbb{R}^{\mathrm{D}}$, and (2) the collision kernels we consider include those arising from inverse power potentials, as well as the hard sphere case considered in [17].

Hydrodynamic Limits of the Boltzmann Equation

by

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Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2018

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Dedication

To my family.

Acknowledgments

I thank Professor Levermore, who guided me through the thesis. I greatly appreciate the efforts and the time that Professor Levermore has put on various stages of my graduate study. I would also like to thank Lucie, for her incredible warmth and kindness in our brief encounters.

I thank Professor Cameron, Professor Drake, Professor Grillakis and Professor Jabin for kindly agreeing to be my committee members and take their precious time to read my thesis and attend the defense.

I thank Jingting Zhou, Elif Kuz and Kanna Nakamura and other graduate students and postdocs in UMD for their invaluable friendships. I thank the staff of UMD and IPAM for their support.

I thank my grandparents, Xiuzhen Zhu and Jinlong Lu for their constant moral support in my education, although they passed away during my graduate study. I thank my parents, Jianping Lu and Zhiming Zhao, for being there for me. I thank my husband Wujun Zhang for his support and the delicious food he cooked. Finally, I thank my 101-year-old grandma, Yundi Li, who shows me the possibility of staying optimistic in tough times.

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Chapter 1: Introduction

1.1 Overview

This dissertation studies how solutions of the Boltzmann equation behave in various fluid dynamic regimes. The Boltzmann equation was introduced by Maxwell [44] and Boltzmann [8] to model so-called rarefied gases of identical particles, for which all but binary collisions between particles can be neglected. It describes the gas by a single-particle phase-space density $F(t, x, v)$ rather than the density $\rho(t, x)$, bulk velocity $u(t, x)$, and temperature $\theta(t, x)$ used in fluid dynamics. As such, it provides a bridge between the fluid dynamic description and an atomic description.

Fluid dynamic regimes are characterized by the smallness of a nondimensional parameter ϵ called the Knudsen number, which is the ratio of the mean free path to a macroscopic length scale:

$$\epsilon = \mathrm{Kn} = \frac{\text{mean free path}}{\text{macroscopic length scale}} .$$

The mean free path is a length scale typical of how far particles travel between collisions. Regimes in which the Knudsen number is small are regimes in which the binary collisions play a dominant role in the dynamics. It was first observed by Maxwell [44] that in these regimes the binary collisions will drive $F(t, x, v)$ towards the local Maxwellian whose form $\mathcal{M}_{(\rho,u,\theta)}$ is given by

$$\mathcal{M}_{(\rho,u,\theta)} = \frac{\rho(t,x)}{(2\pi\theta(t,x))^{\mathrm{D}/2}} \exp\left(-\frac{|v-u(t,x)|^2}{2\theta(t,x)}\right) \tag{1.1}$$

where $\rho(t,x)$, $u(t,x)$, and $\theta(t,x)$ are governed by a system of fluid dynamic equations. To leading order he derived the system of gas dynamics that is now often called the compressible Euler system

$$\begin{aligned} \partial_t\rho + \nabla_x \cdot (\rho u) &= 0\,, \\ \partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x(\rho\theta) &= 0\,, \\ \partial_t(\rho(\tfrac{1}{2}|u|^2 + \tfrac{\mathrm{D}}{2}\theta)) + \nabla_x \cdot (\rho u(\tfrac{1}{2}|u|^2 + \tfrac{\mathrm{D}+2}{2}\theta)) &= 0\,. \end{aligned} \tag{1.2}$$

This is an extension of the original system of gas dynamics that Euler studied, which had no temperature equation. Maxwell [44] also first derived the correction to this system that is now often called the compressible Navier-Stokes system

$$\begin{aligned} \partial_t\rho_\epsilon + \nabla_x \cdot (\rho_\epsilon u_\epsilon) &= 0\,, \\ \partial_t(\rho_\epsilon u_\epsilon) + \nabla_x\cdot (\rho_\epsilon u_\epsilon \otimes u_\epsilon + \rho_\epsilon\theta_\epsilon I) &= \epsilon\mu\nabla_x\cdot \sigma(u_\epsilon)\,, \\ \partial_t(\rho_\epsilon(\tfrac{1}{2}|u_\epsilon|^2 + \tfrac{\mathrm{D}}{2}\theta_\epsilon)) + \nabla_x\cdot (\rho_\epsilon u_\epsilon(\tfrac{1}{2}|u_\epsilon|^2 + \tfrac{\mathrm{D}+2}{2}\theta_\epsilon)) &= \\ &\epsilon\nabla_x\cdot\, (\kappa\nabla_x\theta_\epsilon + \mu\nabla_x\cdot (\sigma(u_\epsilon)\cdot u_\epsilon))\,. \end{aligned} \tag{1.3}$$

Here $\sigma(u)$ is the deformation rate tensor

$$\sigma(u) := \nabla_x u + (\nabla_x u)^T - \tfrac{2}{\mathrm{D}} I \nabla_x\cdot u\,. \tag{1.4}$$

The viscosity μ and thermal conductivity κ depend upon the temperature θ.

To identify additional fluid regimes, we introduce a global Maxwellian. A global Maxwellian is a local Maxwellian that solves the Boltzmann equation. One standard choice of global Maxwellian is the homogeneous Maxwellian, i.e. ρ, u, θ are all constants. By a choice of reference frame and units, we may take $(\rho, u, \theta) = (1, 0, 1)$. We call $\mathcal{M}_{(1,0,1)}$ the unit Maxwellian

$$M(v) \equiv \frac{1}{(2\pi)^{(D/2)}} \exp\left(-\frac{1}{2}|v|^2\right) . \tag{1.5}$$

One can scale the solution F around the global Maxwellian

$$F = \mathcal{M}(1 + \delta_\epsilon g_\epsilon) , \tag{1.6}$$

and find the fluid equations satisfied by fluid moments of the fluctuations g_ϵ when $\delta_\epsilon \to 0$. Various fluid regimes can be identified by the different rate in which $\delta_\epsilon \to 0$ as $\epsilon \to 0$.

When $\epsilon \to 0$ and $\delta_\epsilon \to 0$, and assuming that $g_\epsilon \to g$ formally, we will see that g has the form

$$g = \rho(t,x) + u(t,x) \cdot v + (\tfrac{1}{2}|v|^2 - \tfrac{\mathrm{D}}{2})\theta(t,x) ,$$

and if $\mathcal{M} = \mathcal{M}_{(1,0,1)}$, the fluid fluctuations (ρ, u, θ) satisfies the acoustic equation

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot u = 0 , \qquad & \rho(x,0) = \rho^{\mathrm{in}}(x) , \\ \partial_t u + \nabla_x(\rho + \theta) = 0 , \qquad & u(x,0) = u^{\mathrm{in}}(x) , \\ \tfrac{\mathrm{D}}{2}\partial_t \theta + \nabla_x \cdot u = 0 , \qquad & \theta(x,0) = \theta^{\mathrm{in}}(x) , \end{aligned} \tag{1.7}$$

which is the linearization of compressible Euler system (1.2) around $(\rho, u, \theta) = (1, 0, 1)$. We denote the fluid moments by U

$$U := \begin{pmatrix} \rho \\ u \\ \theta \end{pmatrix} , \tag{1.8}$$

and the acoustic operator by $\mathcal{A}$

$$\mathcal{A}U := \begin{pmatrix} \nabla_x \cdot u \\ \nabla_x(\rho+\theta) \\ \nabla_x \cdot u \end{pmatrix} . \tag{1.9}$$

A shorthand notation for (1.7) is

$$\partial_t U + \mathcal{A}U \;=\; 0 . \tag{1.10}$$

We define the inner product (U_1, U_2) between $U_1 = (\rho_1, u_1, \theta_1)^T$ and $U_2 = (\rho_2, u_2, \theta_2)^T$ as:

$$(U_1, U_2) := \int_{\mathbb{T}^{\mathrm{D}}} \rho\rho_1 + uu_1 + \tfrac{\mathrm{D}}{2}\theta\theta_1 \,\mathrm{d}x . \tag{1.11}$$

The acoustic operator $\mathcal{A}$ is skew-adjoint in the Hilbert space

$$\mathbb{H} = \left\{ \widetilde{V} \in L^2(\mathrm{d}x; \mathbb{R}^{D+2}) : \int_{\mathbb{T}^{\mathrm{D}}} \widetilde{V} \,\mathrm{d}x = 0 \right\} \tag{1.12}$$

equipped with the inner product (1.11). It was shown in [35] that (1.11) is a *natural* inner product implied by the entropy structure.

Because $\mathcal{A}$ is skew-adjoint in the Hilbert space $\mathbb{H}$, it follows that $\mathrm{Range}(\mathcal{A}) = \mathrm{Null}(\mathcal{A})^\perp$, where $\mathrm{Null}(\mathcal{A})^\perp$ is the orthogonal complement of $\mathrm{Null}(\mathcal{A})$ with respect to the natural inner product given by (1.11). The null space of the acoustic operator $\mathrm{Null}(\mathcal{A})$ contains the incompressiblity and Boussinesq relations

$$\nabla_x \cdot u \;=\; 0 , \quad \rho + \theta \;=\; 0 . \tag{1.13}$$

We call it the *incompressible mode*, and its orthogonal complement space $\mathrm{Null}(\mathcal{A})^\perp$ the *acoustic mode*.

Note that the acoustic equation has a large class of stationary solutions, so we need a longer time scale to see the evolution. Consider the evolution over the time $[0, \frac{t}{\tau_\epsilon}]$. Over this time scale, if $\epsilon \ll \delta_\epsilon = \tau_\epsilon$, the Boltzmann equation converges to the incompressible Euler equation. If $\delta_\epsilon = \tau_\epsilon = \epsilon$, the Boltzmann equation converges to the incompressible Navier-Stokes equations. If $\delta_\epsilon \ll \epsilon = \tau_\epsilon$, the Boltzmann equation converges to the incompressible Stokes equation. Note that the initial data of the Boltzmann equation has to be "well-prepared" to get the incompressible limits, i.e., the corresponding fluid fluctuations have to satisfy the incompressibility and Boussinesq relations. The projection to the acoustic mode is zero for solutions of incompressible equations.

Weakly nonlinear-dissipative approximation to the compressible Navier-Stokes system (1.3) emerges when we don't wish to suppress the acoustic modes. Weakly nonlinear-dissipative approximation govern regimes which are close to a global equilibrium and the dissipation coeficients are small. Assuming that $\mu > 0$ and $\kappa > 0$, the only global equilibria of the compressible Navier-Stokes system (1.3) over a periodic domain are the constant states [35]. For simplicity, we write the constant equilibria as $(1, 0, 1)$. To see long-term behavior of perturbations, we introduce two times: a slow time of order 1 and a fast time of order $\frac{1}{\tau_\epsilon}$. Averaging over the fast time scale, we see that weakly nonlinear-dissipative approximation governs perturbations W of the constant equilibrium by the system

$$\partial_t W_\epsilon + \mathcal{A} W_\epsilon + \epsilon \bar{\mathcal{N}}(W_\epsilon, W_\epsilon) \,=\, \epsilon \bar{\mathcal{D}} W_\epsilon \,, \tag{1.14}$$

where

$$
\begin{aligned}
\bar{\mathcal{D}}W &:= \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{t\mathcal{A}}\mathcal{D}e^{-t\mathcal{A}}W \,\mathrm{d}t\,, \\
\bar{\mathcal{N}}(W,W) &:= \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{t\mathcal{A}}\mathcal{N}(e^{-t\mathcal{A}}W, e^{-t\mathcal{A}}W) \,\mathrm{d}t\,,
\end{aligned}
\tag{1.15}
$$

in which the linear dissipative term $\mathcal{D}W$ is

$$
\mathcal{D}W = \mathcal{D}\begin{pmatrix} \rho \\ u \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ \mu\nabla_x\cdot\sigma(u) \\ \kappa\frac{2}{D}\nabla_x\cdot\nabla_x\theta \end{pmatrix}\,, \tag{1.16}
$$

and the quadratic term $\mathcal{N}(W,W)$ is

$$
\mathcal{N}(W,W) = \begin{pmatrix} 0 \\ \nabla_x\cdot(u\otimes u) - \frac{1}{D}\nabla_x|u|^2 I \\ \frac{D+2}{D}\nabla_x\cdot(u\theta) \end{pmatrix}\,. \tag{1.17}
$$

Formal derivation of (1.14) can be found in [36]. For details of the formal derivation, we refer the readers to Chapter 3.

We call (1.14) the *weakly compressible Navier-Stokes system* as a shorthand notation for the weakly nonlinear-dissipative approximation to the compressible Navier-Stokes system (1.3). It is shown in [35] that the weakly compressible Navier-Stokes system has global weak solutions for all initial data in a natural Hilbert space.

On the other hand, linearizing the Boltzmann equation around the unit Maxwellian (1.5) yields the linearized Boltzmann equation. We may establish linearized compressible Navier-Stokes approximation, incompressible Stokes and linearized weakly compressible Navier-Stokes approximation in the same fashion as in the nonlinear setting. In this thesis, we established a linearized compressible Navier-Stokes approximation, and showed that the difference between solution of linearized compressible Navier-Stokes equation and linearized weakly compressible Navier-Stokes equation is “small” uniformly in time.

The following diagram describes some of the fluid equations that arise as limits or approximations of the Boltzmann equation. All the limits and approximations hold formally; we will defer the question of rigorous justification to later discussion.

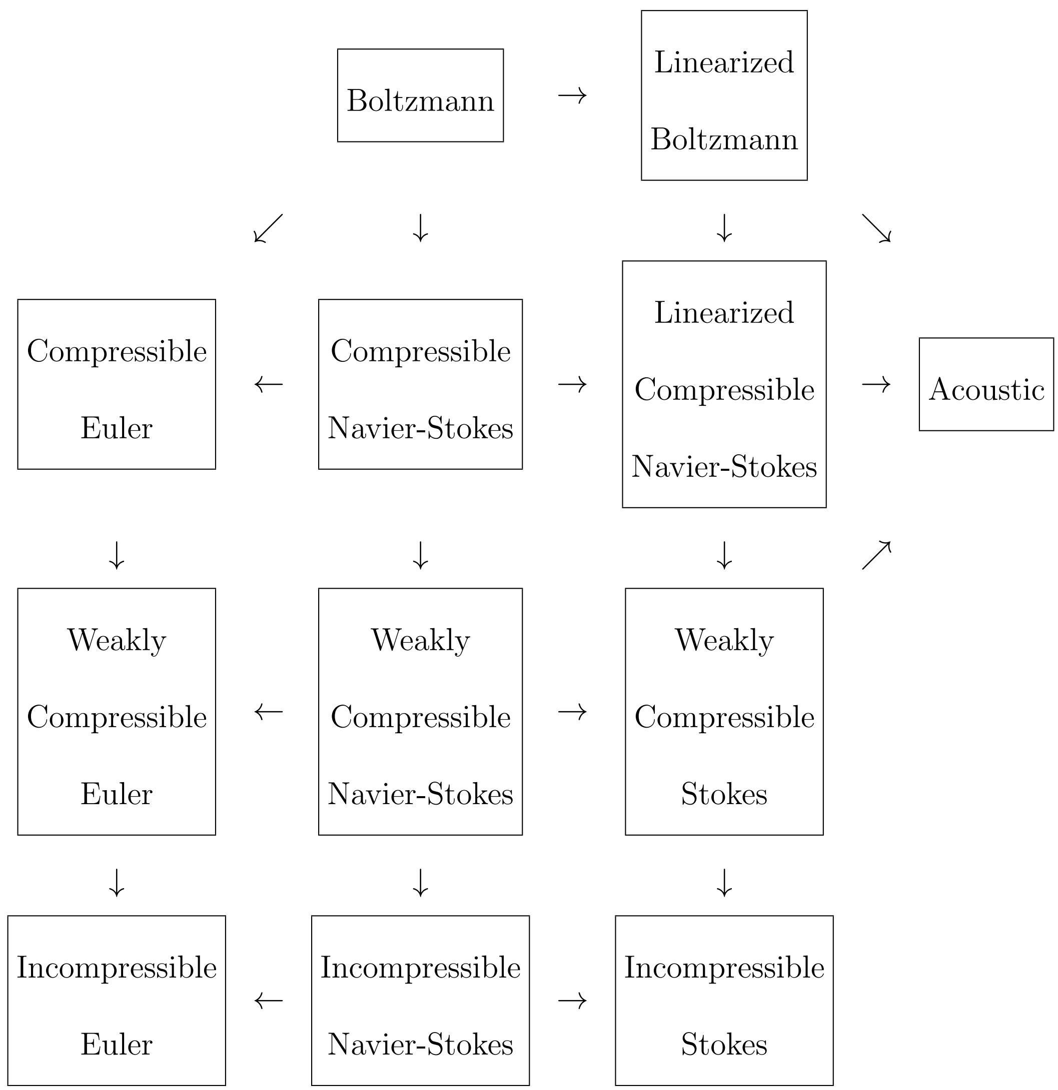

Diagram 1: Various Fluid Limits and Approximations of the Boltzmann Equation

Setting the dissipation coefficients in the second column (the Navier-Stokes systems) to be zero yields the corresponding Euler systems in the first column. Also note that the right half of the diagram is a linearized counterpart of the left half of the diagram. For example, the linearized compressible Navier-Stokes system is the linearization ofthe compressible Naiver-Stokes system around $(1, 0, 1)$, and the weakly compressible Stokes system and the incompressible Stokes system are derived by linearizing the weakly compressible Navier-Stokes system and the incompressible Navier-Stokes system around $(0, 0, 0)$.

For rigorous justification, we restricted the spatial domain in the periodic box $\mathbb{T}^{\mathrm{D}}$. DiPerna and Lions [16] proved the global existence of a type of weak solution to the Boltzmann equation over the whole space $\mathbb{R}^{\mathrm{D}}$ for any initial data satisfying natural physical bounds. After the construction of the DiPerna-Lions renormalized solutions, there was a program initiated by Bardos, Golse and Levermore [4] to derive incompressible models from the Boltzmann equation. The program was completed in a series of papers ([3], [4], [5], [6], [23], [24], [40], [47]) that appeared over two decades by Bardos, Golse, Levermore, Masmoudi and Saint-Raymond. Various scalings have been considered by these authors, leading to equations of acoustic waves [6], the incompressible Stokes [24], incompressible Navier-Stokes [23], [40] and incompressible Euler [47]. Compressible limits and approximations of the Boltzmann equation in the DiPerna-Lions framework is largely an open question, partially because the mathematical understanding of compressible Euler and compressible Navier-Stokes systems is not satisfactory. For example, in the compressible Navier-Stokes system derived from the Boltzmann equation, dissipation terms are small and of the same order as ϵ. Therefore these dissipation terms vanish in the hydrodynamic limit. Since almost nothing is known about the uniformity of the solutions of the compressible Navier-Stokes system in the vanishing viscosity regime, the compressible Navier-Stokes is not a realistic target for rigorous hydrodynamic limit. Moreover, solutions of all the fluid limits and approximations sit in a natural L^2 space except for those of compressible Euler and compressible Navier-Stokes.

The structure of the weakly compressible Navier-Stokes system ensures that it has global weak solutions [35], [36]. Because we work on a periodic domain $\mathbb{T}^{\mathrm{D}}$, we get strict dissipation, although the dissipation coefficients are small. Note that projection of the weakly compressible Navier-Stokes system on the null space of acoustic operator yields incompressible Navier-Stokes, which means the slow incompressible modes are completely decoupled from the fast acoustic mode. The projection on $\mathrm{Null}(\mathcal{A})^{\perp}$, however, is a nonlocal quadratic system which is coupled with the projection on incompressible mode and describes how the fast acoustic waves propagate. This is the reason we call the weakly nonlinear-dissipative approximation of the compressible Navier-Stokes system the *weakly compressible Navier-Stokes system.*

On the other hand, several results have been obtained in the framework of the classical solutions of Boltzmann equation. Kaniel-Shinbrot [38] and Hamdache [29] constructed constructed mild solutions to the Boltzmann equation over the whole space; Guo [27] constructed global-in-time classical solutions near Maxwellian in a periodic box. Recently, Bardos, Gamba, Golse and Levermore [2] modified the arguments in [29] and constructed positive mild solutions near global Maxwellians.

Since the mathematics of linearized Boltzmann and linearizd fluid equations are well-established, in the second part of this thesis, we work on the linear counterpart of the weakly compressible and compressible Navier-Stokes limit of Boltzmann equation. Several works have been published in this direction, notably Ellis and Pinsky [17], who worked on whole space $\mathbb{R}^{\mathrm{D}}$ and showed the difference between the solution of linearized Boltzmann equation and the weakly compressible Navier-Stokes approximation is $O(\epsilon)$ for sufficiently smooth initial data. On the fluid regime, Hoff and Zumbrun [31], [32] showed the Cauchy problem for compressible Navier-Stokes on whole space is asymptotically given by the solution of weakly compressible Navier-Stokes. The domain we are working on is $\mathbb{T}^{\mathrm{D}}$, so the gas is confined and we expect to see dissipation instead of dispersion over the whole space (in which case the acoustic waves will run away to infinity). In Chapter 4, we get a uniform-in-time estimate for the difference of solutions of linearized compressible Navier-Stokes on $\mathbb{T}^{\mathrm{D}}$ and establish a linearized compressible Navier-Stokes approximation of the linearized Boltzmann equation. For a statement of the main result of the second part of the thesis, please refer to Section 1.6.

In this chapter, we give an introduction to the two problems studied in this dissertation. The proof of main results will be presented in later chapters.

1.2 Boltzmann Equation Preliminaries

In this section, we review some preliminary results of the Boltzmann equation. All of the materials of this section are well-known and standard. We follow mostly the presentation in [11, 21, 22].

In kinetic theory, the state of a (rarefied) gas is described by the distribution of molecules in phase space, $F = F(t, x, v)$, which is the density of particles located at the position $x \in \Omega$ with the velocity $v \in \mathbb{R}^{\mathrm{D}}$ at time $t \geq 0$.

The Boltzmann equation governs the evolution of the distribution of molecules in rarefied gases. The following assumptions are made:

- particles interact via binary collisions
- collision are elastic
- collisions involve only uncorrelated particles

The Boltzmann equation reads therefore

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F), \tag{1.18}$$

where $\mathcal{B}(F, F)$ is the Boltzmann collision integral

The quadratic collision kernel in the Boltzmann equation (1.18) is

$$\mathcal{B}(F, F) = \iint_{\mathbb{S}^{\mathrm{D}-1} \times \mathbb{R}^{\mathrm{D}}} (F(v_1')F(v') - F(v_1)F(v))b(v_1 - v, \omega)\, \mathrm{d}\omega \mathrm{d}v_1, \tag{1.19}$$

where the post-collision velocities v' and v_1' are defined in terms of the pre-collision velocities v, v_1 and the unit vector ω by the formulas

$$\begin{aligned} v' &= v'(v, v_1, \omega) = v - (v - v_1) \cdot \omega\omega, \\ v_1' &= v_1'(v, v_1, \omega) = v_1 + (v - v_1) \cdot \omega\omega. \end{aligned} \tag{1.20}$$

One designates $F(t, x, v_1), F(t, x, v')$ and $F(t, x, v_1')$ respectively by F_1, F' and F_1'.

1.2.1 The Boltzmann Collision Kernel

In the Boltzmann collision kernel $b(v_1 - v, \omega)$, the unit vector ω is perpendicular to the reflection plane with $\mathrm{d}\omega$ being the rotationally invariant unit measure for $\mathbb{S}^{\mathrm{D}-1}$. In fact, (v, v_1) and (v', v_1') conserve both momentum and energy because we only consider elastic collisions:

$$\begin{aligned} v' + v_1' &= v + v_1, \\ |v'|^2 + |v_1'|^2 &= |v|^2 + |v_1|^2. \end{aligned} \tag{1.21}$$

From the mechanical viewpoint, the origin $\frac{1}{2}(v+v_1)$ is the velocity of the center of mass for any pair of molecules with velocities v and v_1; in (1.20), the first equality is the conservation of momentum for any pair of colliding molecules with velocities v, v_1 after collision, and v', v_1' before collision.

We now give the explicit forms of some classical collision kernel b. The collision kernel for hard spheres of mass m and radius r_0 has the form

$$b(\omega, v_1 - v) = |\omega \cdot (v_1 - v)| \frac{(2r_0)^{\mathrm{D}-1}}{2m} \,. \tag{1.22}$$

The classical collision kernels that derive from a repulsive intermolecular potential of the form c/r^k with $k > 2\frac{\mathrm{D}-1}{\mathrm{D}+1}$ have the form

$$b(\omega, v_1 - v) = \hat{b}\left(\omega \cdot \frac{v_1 - v}{|v_1 - v|}\right) |v_1 - v|^{\beta} \quad \text{with}\, \beta = 1 - 2\frac{\mathrm{D}-1}{k}\,, \tag{1.23}$$

where $\hat{b}(\omega \cdot \frac{v_1 - v}{|v_1 - v|})$ is positive everywhere, and has even symmetry in ω. Note that $\hat{b}$ has a singularity at $\omega \cdot \frac{v_1 - v}{|v_1 - v|}$. This corresponds to the grazing collisions; the colliding molecules are deflected only slightly. Because of the singularity, $\hat{b}$ is not integrable with respect to $\mathrm{d}\omega$. This means the Boltzmann collision integral will not make sense. To mitigate this issue, we impose a weak small deflection cutoff assumption

$$\int_{\mathbb{S}^{\mathrm{D}-1}} \hat{b}(\omega \cdot \frac{v_1 - v}{|v_1 - v|})\mathrm{d}\omega < \infty \,. \tag{1.24}$$

The cases $\beta > 0$, $\beta = 0$, $\beta < 0$ correspond respectively to the so-called hard, Maxwell and soft potential cases.

1.2.2 Conservation Laws

Momentum and kinetic energy, together with the number of gas molecules, are the only natural conserved quantities at the microscopic level. The most important properties of the Boltzmann equation, described in the following two sections, are straight foreword consequences of the structure of the collision integral, and more specifically of the conservation laws at the microscopic level.

Maxwell first showed that quantities $1, v$ and $|v|^2$ are conserved by $\mathcal{B}$. This means for every $F = F(v)$ satisfying certain growth conditions,

$$\int_{\mathbb{R}^{\mathrm{D}}} \mathcal{B}(F,F)\mathrm{d}v = 0, \qquad \int_{\mathbb{R}^{\mathrm{D}}} v\mathcal{B}(F,F)\mathrm{d}v = 0, \qquad \int_{\mathbb{R}^{\mathrm{D}}} |v|^2\mathcal{B}(F,F)\mathrm{d}v = 0\,.$$

Moreover, every quantity conserved by $\mathcal{B}$ is a linear combination of these. More specifically, given any $\xi = \xi(v)$, the following are equivalent:

$$\text{(i)} \quad \int_{\mathbb{R}^{\mathrm{D}}} \xi(v)\, \mathcal{B}(F,F)\, \mathrm{d}v = 0\,,$$

$$\text{(ii)} \quad \xi \in \mathrm{Span}\{1, v_1, ..., v_{\mathrm{D}}, |v|^2\}\,.$$

Any solution F of the Boltzmann equation formally satisfies the local conservation law

$$\partial_t \int_{\mathbb{R}^{\mathrm{D}}} \xi\, F\, \mathrm{d}v + \nabla_x \cdot \int_{\mathbb{R}^{\mathrm{D}}} v\, \xi\, F\, \mathrm{d}v = 0\,,$$

when $\xi \in \mathrm{Span}\{1, v_1, v_2, \cdots, v_{\mathrm{D}}, |v|^2\}$ and

$$\partial_t \xi + v \cdot \nabla_x \xi = 0\,.$$

It has been known essentially since Boltzmann [8] [10], who worked out the case $\mathrm{D} = 3$, that the only such quantities ξ are linear combinations of the $4 + 2\mathrm{D} + \frac{\mathrm{D}(\mathrm{D}-1)}{2}$ quantities

$$1\ ,\quad v\ ,\quad x - vt\ ,\quad \tfrac{1}{2}|v|^2\ ,\quad v \wedge x \quad v\cdot(x-vt)\ ,\quad \tfrac{1}{2}|x-vt|^2\,, \tag{1.25}$$

where $v \wedge x = v\,x^T - x\,v^T$ is the skew tensor product. By integrating the corresponding local conservation laws over space and time, we formally obtain the global conservation laws

$$\iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} \begin{pmatrix} 1 \\ v \\ x - vt \\ \frac{1}{2}|v|^2 \\ v \wedge x \\ v\cdot(x-vt) \\ \frac{1}{2}|x-vt|^2 \end{pmatrix} F(v,x,t)\,\mathrm{d}v\,\mathrm{d}x = \iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} \begin{pmatrix} 1 \\ v \\ x \\ \frac{1}{2}|v|^2 \\ v \wedge x \\ v\cdot x \\ \frac{1}{2}|x|^2 \end{pmatrix} F^{\mathrm{in}}(v,x)\,\mathrm{d}v\,\mathrm{d}x\,, \tag{1.26}$$

where the right-hand sides above exist for every initial data F^{in} that satisfies the bound

$$\iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} \left(1 + |v|^2 + |x|^2\right) F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x < \infty\,. \tag{1.27}$$

These conserved quantities are associated respectively with the conservation laws of mass, momentum, initial center of mass, energy, angular momentum, scalar momentum moment, and scalar inertial moment.

1.2.3 Boltzmann's $H-$Theorem and Entropy Dissipation Laws

In this subsection, we discuss Boltzmann's H-Theorem and its implication for control of entropy and entropy dissipation. Using results from the H-Theorem and conservation laws, one can also write down the form of a general class of global Maxwellians – the nonhomogeneous global Maxwellian. Boltzmann's H-Theorem states

$$\int_{\mathbb{R}^{\mathrm{D}}} \log(F)\, \mathcal{B}(F,F)\,\mathrm{d}v = -\frac{1}{4}\iiint (F'F_1' - FF_1)\log\left(\frac{F'F_1'}{FF_1}\right) b\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v \leq 0\,. \quad (1.28)$$

Moreover, the following conditions are equivalent:

$$\begin{aligned} &\text{(i)} \quad \int_{\mathbb{R}^{\mathrm{D}}} \log(F)\, \mathcal{B}(F,F)\,\mathrm{d}v = 0\,, \\ &\text{(ii)} \quad \mathcal{B}(F,F) = 0\,, \\ &\text{(iii)} \quad F \text{ is a Maxwellian density, } i.e., F = \frac{\rho}{(2\pi\theta)^{\frac{\mathrm{D}}{2}}}\exp\left(-\frac{|v-u|^2}{2\theta}\right), \end{aligned} \quad (1.29)$$

for some $(\rho, u, \theta) \in \mathbb{R}_+ \times \mathbb{R}^{\mathrm{D}} \times \mathbb{R}_+$.

The inequality (1.28) implies the estimate

$$\begin{aligned} &\partial_t \int F\log(F)\,\mathrm{d}v + \nabla_x \cdot \int vF\log(F) \\ &= -\frac{1}{\epsilon}\int_0^t \iiint \frac{1}{4}\log(\frac{F_1'F'}{F_1F})(F_1'F' - F_1F)b(\omega, v_1 - v)M_1M\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v \leq 0\,. \end{aligned} \quad (1.30)$$

Integrating over the space, we get the global entropy equality

$$H(F(t)) + \frac{1}{\epsilon}\int_0^t R(F(s))\,\mathrm{d}s \;=\; H(F^{\mathrm{in}})\,, \quad (1.31)$$

where $H(F)$ is the entropy functional

$$H(F) = \iint F \log(F) \, \mathrm{d}v \mathrm{d}x \,, \tag{1.32}$$

and $R(F)$ is the entropy dissipation rate functional

$$R(F) = \iiiint \frac{1}{4} \log(\frac{F_1' F'}{F_1 F})(F_1'F' - F_1 F) b(\omega, v_1 - v) M_1 M \, \mathrm{d}\omega \mathrm{d}v_1 \mathrm{d}v \mathrm{d}x \,. \tag{1.33}$$

We can now derive the explicit form of nonhomogeneous global Maxwellian. Recall that a local Maxwellian has the form

$$\mathcal{M} = \frac{\varrho(x,t)}{\big(2\pi\vartheta(t)\big)^{\frac{\mathrm{D}}{2}}} \exp\left(-\frac{|v - U(x,t)|^2}{2\vartheta(t)}\right) \,, \tag{1.34}$$

and a local Maxwellian that satisfies the Boltzmann equation is a global Maxwellian.

To derive the explicit form of the global nonhomogeneous Maxwellian with finite mass, zero net momentum, and center of mass at the origin, we first note that $\mathcal{B}(\mathbb{M},\mathbb{M}) = 0$, so $(\partial_t + v \cdot \nabla_x) \log(\mathcal{M}) = 0$. By the equivalence of (ii) and (iii) in (1.29), $\log(\mathcal{M})$ must satisfy

$$\log(\mathcal{M}) \in \mathrm{Span}\{1, v_1, ... v_{\mathrm{D}}, |v|^2\} \,.$$

Therefore by section 1.2.2, we see that $\log(\mathcal{M})$ must be a linear combination of the quantities in (1.25) . The form

$$\mathcal{M} = \frac{m}{(2\pi)^{\mathrm{D}}} \sqrt{\det(Q)} \exp\left(-\frac{1}{2} \begin{pmatrix} v \\ x - vt \end{pmatrix}^T \begin{pmatrix} cI & bI + B \\ bI - B & aI \end{pmatrix} \begin{pmatrix} v \\ x - vt \end{pmatrix} \right) . \tag{1.35}$$

then comes from the requirements that $\mathcal{M}$ have finite mass, zero net momentum, and center of mass at the origin. By completing the square in its exponent we can write $\mathcal{M}$ in the local Maxwellian form

$$\mathcal{M}_{(\varrho,U,\vartheta)} = \frac{\varrho(t,x)}{(2\pi\vartheta(t,x))^{\mathrm{D}/2}} \exp\left(-\frac{|v-U(t,x)|^2}{2\vartheta(t,x)}\right), \tag{1.36}$$

where the temperature $\vartheta(t)$, bulk velocity $U(x,t)$, and mass density $\varrho(x,t)$ are given by

$$\begin{aligned} \vartheta(t) &= \frac{1}{at^2 - 2bt + c}, \\ U(x,t) &= \vartheta(t)\big(axt - bx - Bx\big), \\ \varrho(x,t) &= m\left(\frac{\vartheta(t)}{2\pi}\right)^{\frac{\mathrm{D}}{2}} \sqrt{\det(Q)}\, \exp\Big(-\frac{\vartheta(t)}{2}\, x^T Q x\Big). \end{aligned} \tag{1.37}$$

with $m > 0$, $Q = (ac - b^2)I + B^2$,

$(a,b,c,B) \in \Omega = \big\{(a,b,c,B) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{\mathrm{D}\wedge\mathrm{D}} \,:\, Q > 0\big\}$. Because $Q > 0$, we see that $\varrho(x,t)$ is integrable over $\mathbb{R}^{\mathrm{D}}$.

Levermore [41] showed that every Cauchy problem

$$\partial_t F + v\cdot\nabla_x F = \mathcal{B}(F,F), \qquad F|_{t=0} = F^{\mathrm{in}} \tag{1.38}$$

with initial data $F^{\mathrm{in}}(v,x)$ that satisfies the bounds (1.27) can be associated with a unique global Maxwellian determined by matching the values of the conserved quantities computed from F^{in}, i.e. quantities on the right-hand side of (1.26) in section 1.2.2. This is the global Maxwellian that we will scale the Boltzmann equation around in the first result.

We can get results similar to the local entropy dissipation law (1.30) and the global entropy equality (1.31) for Boltzmann equation scaled around the global Maxwellian that satisfies

$$\int \mathcal{M}\mathrm{d}v = \int F^{\mathrm{in}}\mathrm{d}v, \qquad \int v\mathcal{M}\mathrm{d}v = \int vF^{\mathrm{in}}\mathrm{d}v, \qquad \int |v|^2\mathcal{M}\mathrm{d}v = \int |v|^2 F^{\mathrm{in}}\mathrm{d}v\,. \tag{1.39}$$

Consider a family of formal solutions F_ϵ to the scaled Boltzmann initial-value problem

$$\partial_t F_\epsilon + v\cdot\nabla_x F_\epsilon = \frac{1}{\epsilon}\mathcal{B}(F_\epsilon, F_\epsilon)\,, \qquad F_\epsilon\big|_{t=0} = F^{\mathrm{in}}\,, \tag{1.40}$$

Scaling the densities around global Maxwellian that satisfies (1.39) , we get relative densities G_ϵ, defined by $F_\epsilon = \mathcal{M}G_\epsilon$. Recasting the initial-value problem (1.40) yields

$$\partial_t G_\epsilon + v\cdot\nabla_x G_\epsilon = \frac{1}{\epsilon}\mathcal{Q}(G_\epsilon, G_\epsilon)\,, \qquad G_\epsilon\big|_{t=0} = G^{\mathrm{in}}\,, \tag{1.41}$$

where

$$\mathcal{Q}(G,G) = \iint (G_1'G' - G_1G)b(v_1 - v,\omega)\mathrm{d}\omega M_1 \mathrm{d}v_1\,. \tag{1.42}$$

If G solves the Boltzmann equation (1.41), then G satisfies local conservation laws of mass, momentum, and energy as well. If G solves the Boltzmann equation (1.41), then G satisfies the local entropy dissipation law

$$\begin{aligned}&\partial_t \int (G\log(G) - G + 1)\mathcal{M}\,\mathrm{d}v + \nabla_x\cdot \int v(G\log(G) - G + 1)\mathcal{M}\,\mathrm{d}v\\ &= -\frac{1}{\epsilon}\iiint \frac{1}{4}\log(\frac{G_1'G}{G_1G})(G_1'G' - G_1G)b(\omega, v_1 - v)\mathcal{M}_1\mathcal{M}\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v \le 0\,.\end{aligned} \tag{1.43}$$

Integrating this over space and time gives the global entropy equality

$$H(G(t)) + \frac{1}{\epsilon}\int_0^t R(G(s))\,\mathrm{d}s \;=\; H(G^{\mathrm{in}})\,, \tag{1.44}$$

where $H(G)$ is the relative entropy functional

$$H(G) \;=\; \int_{\mathbb{R}^{\mathrm{D}}}\int_{\mathbb{R}^{\mathrm{D}}} (G\log(G) - G + 1)\mathcal{M}\,\mathrm{d}v\mathrm{d}x\,, \tag{1.45}$$

and $R(G)$ is the entropy dissipation rate functional

$$R(G) \;=\; \iiiint \frac{1}{4}\log(\frac{G_1'G}{G_1G})(G_1'G' - G_1G)b(\omega, v_1 - v)\mathcal{M}_1\mathcal{M}\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v\mathrm{d}x\,. \tag{1.46}$$

1.2.4 Linearized Collision Operator

Linearized collision operator is the linearization of the Boltzmann collision operator at a local Maxwellian $\mathcal{M}_{(\rho,u,\theta)}$

$$\mathcal{L}_{\mathcal{M}_{(\rho,u,\theta)}}\phi = \iint (\phi + \phi_1 - \phi' - \phi'_1) b(v_1 - v, \omega) \mathcal{M}_{(\rho,u,\theta)} \mathrm{d}v_1 \mathrm{d}\omega \,, \tag{1.47}$$

Using the translation and scaling invariance of the collision kernel, we can actually restrict our discussion to the case where $M = \mathcal{M}_{(1,0,1)}$. The corresponding linearized collision operator is denoted by $\mathcal{L}$.

The next theorem establishes the main property of the linearized collision operator $\mathcal{L}_{\mathcal{M}}$, i.e., that it satisfies the Fredholm alternative in some weighted L^2 space. We define the attenuation coefficient $a(v)$ by

$$a(v) = \iint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^{\mathrm{D}}} b(v - v_1, \omega) M(v_1) \mathrm{d}\omega \mathrm{d}v_1 \,. \tag{1.48}$$

Theorem 1.2.1 *(Hilbert [30]) The linear operator $\mathcal{L}$ is a nonnegative unbounded self-adjoint Fredholm operator on $L^2(M\mathrm{d}v)$. Its null space is the space of collision invariants:*

$$\mathrm{Null}(\mathcal{L}) \;=\; \mathrm{Span}\{1, v_1, \ldots v_D, |v|^2\} \,. \tag{1.49}$$

Moreover the following coercivity estimate on $\mathrm{Null}(\mathcal{L})^{\perp}$ holds: there exists $C > 0$ such that, for each $\phi \in L^2(aM\mathrm{d}v)$, one has

$$\int \phi \mathcal{L}\phi(v) M(v) \mathrm{d}v \geq C \int_{\mathbb{R}^{\mathrm{D}}} (\phi - \Pi\phi)^2 a M \mathrm{d}v \,, \tag{1.50}$$

where Π is the $L^2(aM\mathrm{d}v)$-orthogonal projection on $\mathrm{Null}(\mathcal{L})$.

An important consequence of Theorem 1.2.1 is that the integral equation

$$\mathcal{L}\phi = \psi, \quad \psi \in L^2(M(\mathrm{d}v)) \tag{1.51}$$

has a Fredholm alternative:

- Either $\psi \perp \mathrm{Null}(\mathcal{L})$, in which case (1.51) has a unique solution

  $$\phi_0 \in L^2(aM\mathrm{d}v) \cap \mathrm{Null}(\mathcal{L})^{\perp}; \tag{1.52}$$

  then any solution to (1.51) is of the form

  $$\phi = \phi_0 + \phi_1, \quad \text{where}\, \phi_1 \text{ is an arbitrary element of}\, \mathrm{Null}(\mathcal{L}); \tag{1.53}$$

- Or $\psi \notin \mathrm{Null}(\mathcal{L})$, in which case (1.51) has no solution.

1.3 DiPerna-Lions Solutions

In this section discuss the analytic settings of the problem. DiPerna and Lions [16] proved the global existence of a type of weak solution to the Boltzmann equation over the whole space $\mathbb{R}^{\mathrm{D}}$ for any initial data satisfying natural physical bounds. The DiPerna-Lions theory does not yield solutions that are known to solve the Boltzmann equation in the usual weak sense. Rather, it gives the existence of a global weak solution to a class of formally equivalent initial-value problems that are obtained by dividing the Boltzmann equation in (2.27) by normalizing functions $N = N(G) > 0$:

$$(\partial_t + v \cdot \nabla_x)\Gamma(G) = \frac{1}{\epsilon}\frac{Q(G,G)}{N(G)}\,, \qquad G(v,x,0) \,=\, G^{\mathrm{in}}(v,x) \geq 0\,, \tag{1.54}$$

where $\Gamma'(Z) = \frac{1}{N}(Z)$. Here each normalizing function N is a positive-valued, continuous function over $[0, \infty)$ that for some constant $C_N < \infty$ satisfies the bound

$$\frac{1}{N(Z)} \leq \frac{C_N}{1+Z} \qquad \text{for every } Z \geq 0 \,. \tag{1.55}$$

$G \geq 0$ is a weak solution of (1.54) provided that it is initially equal to G^{in}, and that it satisfies (1.54) in the following sense that for every $\chi \in C_0^1(\mathbb{R}^{\mathrm{D}})$ and $[t_1, t_2] \subset [0, \infty)$ it satisfies

$$\begin{aligned} &\int_{\mathbb{R}^{\mathrm{D}}} \int_{\mathbb{R}^{\mathrm{D}}} \chi \Gamma(t_2) \mathcal{M} \,\mathrm{d}v \mathrm{d}x - \int_{\mathbb{R}^{\mathrm{D}}} \int_{\mathbb{R}^{\mathrm{D}}} \chi \Gamma(t_1) \mathcal{M} \,\mathrm{d}v \mathrm{d}x \\ &\quad = \int_{t_1}^{t_2} \int_{\mathbb{R}^{\mathrm{D}}} \int_{\mathbb{R}^{\mathrm{D}}} \Gamma(G) v \cdot \nabla_x \chi \mathcal{M} \,\mathrm{d}v \mathrm{d}x \mathrm{d}t + \frac{1}{\epsilon} \int_{t_1}^{t_2} \int_{\mathbb{R}^{\mathrm{D}}} \int_{\mathbb{R}^{\mathrm{D}}} \frac{Q(G,G)}{N(G)} \chi \,\mathrm{d}v \mathrm{d}x \mathrm{d}t \,. \end{aligned} \tag{1.56}$$

They show that if G is a weak solution of (1.54) for one such N, and if G satisfies certain bounds, then it is a weak solution of (1.54) for every such N. They call such solutions *renormalized solutions* of the Boltzmann equation. Specifically, cast in our setting, their theory yields the following:

Proposition 1.3.1 *Let b satisfy*

$$b \in L^1_{loc}(\mathbb{R}^{\mathrm{D}} \times \mathbb{S}^{\mathrm{D}-1}), \quad \text{and} \quad \lim_{|v| \to \infty} \frac{1}{1+|v|^2} \iint_{\mathbb{S}^{\mathrm{D}-1} \times K} b(\omega, v_1 - v) \mathrm{d}\omega \mathrm{d}v_1 = 0 \tag{1.57}$$

for every compact set $K \subset \mathbb{R}^{\mathrm{D}}$. Given any initial data G^{in} which satisfies

$$H(G^{\text{in}}) < \infty \,, \qquad G^{\text{in}} \geq 0 \,, \tag{1.58}$$

there exists at least one $G \geq 0$ that is a weak solution of (1.54) in the sense of (1.56) for every normalizing function N. Moreover, G satisfies the global entropy inequality

$$H(G(t)) + \frac{1}{\epsilon} \int_0^t R(G(s)) ds \leq H(G^{\text{in}}) \,, \tag{1.59}$$

a weak form of the local conservation law of mass

$$\partial_t \int G\mathcal{M}\,\mathrm{d}v + \nabla_x \cdot \int vG\mathcal{M}\,\mathrm{d}v \,=\, 0\,, \tag{1.60}$$

the global conservation law of momentum

$$\iint vG(t)\mathcal{M}\,\mathrm{d}v\mathrm{d}x \,=\, \iint vG^{\mathrm{in}}\mathcal{M}\mathrm{d}v\mathrm{d}x\,, \tag{1.61}$$

and, finally, the global energy inequality

$$\iint \frac{1}{2}|v|^2 G(t)\mathcal{M}\,\mathrm{d}v\mathrm{d}x \,\leq\, \iint \frac{1}{2}|v|^2 G^{\mathrm{in}}\mathcal{M}\,\mathrm{d}v\mathrm{d}x\,. \tag{1.62}$$

Note that the assumption on collision kernel b implies that the measure $b(\omega, v_1 - v)\mathrm{d}\omega M_1 \mathrm{d}v_1 M \mathrm{d}v$ is finite. DiPerna-Lions renormalized solutions are not known to satisfy many properties that one would formally expect to be satisfied by solutions of the Boltzmann equation. In particular, the theory does not assert the local conservation of momentum without defect, the global conservation of energy without defect, and the local entropy equality; nor does it assert the uniqueness of the solution. Nevertheless, it provides enough control to establish the acoustic limit here.

1.4 Statement of the First Result

In this section, we state the first result – establishing acoustic limit around a global nonhomogeneous Maxwellian.

We consider a family of formal solutions F_ϵ to the scaled Boltzmann initial-value problem

$$\partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon}\mathcal{B}(F_\epsilon, F_\epsilon)\,, \qquad F_\epsilon\big|_{t=0} = F_\epsilon^{\mathrm{in}}\,.$$

We scale the densities F_ϵ around the unique Maxwellian associated with F_ϵ^{in}, discussed in Section 1.2.3 (cf. the paragraph after (1.37)). We introduce relative densities G_ϵ, defined by $F_\epsilon = \mathcal{M} G_\epsilon$ and consider the fluctuations g_ϵ, defined by

$$G_\epsilon = 1 + \delta_\epsilon g_\epsilon \, .$$

Assuming that fluctuations g_ϵ^{in} and g_ϵ are bounded. $\delta_\epsilon > 0$ satisfies

$$\delta_\epsilon \to 0 \; as \; \epsilon \to 0.$$

Assume that g_ϵ converges formally to g, our goal is to find the limiting function g.

The first step is to determine the form of the limiting function g. Observe that by the fluctuations g_ϵ satisfy

$$\partial_t g_\epsilon + v \cdot \nabla_x g_\epsilon + \frac{1}{\epsilon} \mathcal{L}_{\mathcal{M}} g_\epsilon = \frac{\delta_\epsilon}{\epsilon} \mathcal{Q}_{\mathcal{M}}(g_\epsilon, g_\epsilon), \tag{1.63}$$

where $\mathcal{L}_{\mathcal{M}}$ and $\mathcal{Q}_{\mathcal{M}}$ are formally defined by

$$\begin{aligned} \mathcal{L}_{\mathcal{M}} g_\epsilon &= \iint_{\mathbb{S}^{D-1} \times \mathbb{R}^D} (g_\epsilon + g_{\epsilon_1} - g'_\epsilon - g'_{\epsilon_1}) b \mathcal{M}_1 dw dv_1. \\ \mathcal{Q}_{\mathcal{M}}(g_\epsilon, g_\epsilon) &= \iint_{\mathbb{S}^{D-1} \times \mathbb{R}^D} (g'_\epsilon g'_{\epsilon_1} - g_\epsilon g_{\epsilon_1}) b \mathcal{M}_1 dw dv_1. \end{aligned} \tag{1.64}$$

Assuming $\delta_\epsilon \to 0$ and multiplying both sides by ϵ, one finds that $\mathcal{L}_{\mathcal{M}}\, g = 0$. It is known that the null space of $\mathcal{L}_{\mathcal{M}}$ is given by Null($\mathcal{L}_{\mathcal{M}}$)=Span $\{1, v_1, ..., v_D, |v|^2\}$. We conclude that g has the form of a so-called infinitesimal Maxwellian, namely,

$$g = \rho(t,x) + u(t,x) \cdot (v - U) + \tfrac{1}{2} \theta(t,x) (|v - U|^2 - D\vartheta) \, , \tag{1.65}$$

where $U(x,t), \vartheta(x,t)$ are given by (1.37). Acoustic system can be formally derived upon plugging in the form of g in the local conservation laws:

$$\begin{aligned}\partial_t\,[\varrho\rho]\,+\,\nabla_x\cdot\,[U\varrho\,+\,u\varrho\vartheta\rho]\,&=0\,,\\ \partial_t\,[\varrho\vartheta u]\,+\nabla_x\cdot\,\left[\varrho\vartheta(u\otimes U)\,+\,(\varrho\vartheta\rho+\varrho\vartheta^2\theta)Id\right]\,&=0\,,\\ \partial_t\left[\tfrac{\mathrm{D}}{2}\varrho\vartheta^2\theta\right]\,+\nabla_x\cdot\,\left[\varrho\vartheta^2u+\tfrac{D}{2}U\vartheta^2\varrho\theta\right]\,&=\,0\,.\end{aligned}\tag{1.66}$$

For rigorous justification, we work in the setting of DiPerna-Lions renormalized solutions. Besides the assumption of entropy bound $H(G_\epsilon^{\mathrm{in}})\leq C^{\mathrm{in}}\delta_\epsilon^2$ for some $C^{\mathrm{in}}<\infty$, we need to assume furthermore that

$$\delta_\epsilon\,\to\,0\quad\text{and}\quad\frac{\delta_\epsilon}{\epsilon^{\frac{1}{2}}}|\log(\delta_\epsilon)|^{\beta/2}\,\to\,0\quad\text{as }\epsilon\,\to\,0\,,$$

where β arises from the assumption on the collision kernel:

$$\int_{\mathbb{S}^{\mathrm{D}-1}}b(\omega,v_1-v)\mathrm{d}\omega\leq C_b\big(1+\frac{1}{2}|v_1-v|^2\big)^\beta\ \text{ almost everywhere,}$$

for some $C_b<\infty$ and $\beta\in[0,1)$. This assumption is used to control the conservation defect.

We now state our main result for the acoustic limit.

Theorem 1.4.1 *Let b be a collision kernel that satisfies the assumption of DiPerna-Lions (4.110). In addition, assume that there exists constants $C_b\in(0,\infty)$ and $\beta\in[0,1]$ such that b satisfies*

$$\int_{\mathbb{S}^{\mathrm{D}-1}}b(\omega,v)\mathrm{d}\omega\leq C_b(1+\tfrac{1}{2}|v|^2)^\beta\quad\textit{almost everywhere}.\tag{1.67}$$

Let $G_\epsilon^{\mathrm{in}}\geq0$ be a family such that $\iint G_\epsilon^{\mathrm{in}}\,\mathrm{d}v\mathrm{d}x<\infty$ and satisfies the entropy bound $H(G_\epsilon^{\mathrm{in}})\leq C^{\mathrm{in}}\delta_\epsilon^2$ for some $C^{\mathrm{in}}<\infty$ and $\delta_\epsilon>0$ that satisfies

$$\delta_\epsilon\,\to\,0\quad\textit{and}\quad\frac{\delta_\epsilon}{\epsilon^{\frac{1}{2}}}|\log(\delta_\epsilon)|^{\beta/2}\,\to\,0\quad\textit{as }\epsilon\,\to\,0\tag{1.68}$$

for the β that arises in condition (1.67).

Assume, moreover, that for some

$$(\rho^{\rm in}\varrho, u^{\rm in}\varrho\vartheta, \theta^{\rm in}\varrho\vartheta^2) \in \Big(L^\infty(dt; L^2(\frac{1}{\varrho}dvdx)), L^\infty(dt; L^2(\frac{1}{\varrho\vartheta}dvdx)),$$
$$L^\infty(dt; L^2(\frac{1}{\varrho\vartheta^2}dvdx))\Big)$$

the family of fluctuations $g_\epsilon^{\rm in}$ satisfies

$$\Big(\rho^{\rm in}\varrho^{\rm in}, \rho^{\rm in}U\varrho^{\rm in} + u^{\rm in}\varrho^{\rm in}\vartheta^{\rm in}, \tfrac{D\vartheta^{\rm in}}{2}\rho^{\rm in}\varrho^{\rm in} + \tfrac{1}{2}|U|^2\rho^{\rm in}\varrho^{\rm in} + (\sum_{i=1}^{\rm D} U_i u_i^{\rm in})\varrho^{\rm in}\vartheta^{\rm in} + \tfrac{\rm D}{2}\theta^{\rm in}\varrho^{\rm in}{\vartheta^{\rm in}}^2\Big)$$
$$= \lim_{\epsilon\to 0}\Big(\int g_\epsilon^{\rm in}\mathcal{M}^{\rm in}\,{\rm d}v, \int v g_\epsilon^{\rm in}\mathcal{M}^{\rm in}\,{\rm d}v, \int \tfrac{1}{2}|v|^2 g_\epsilon^{\rm in}\mathcal{M}^{\rm in}\,{\rm d}v\Big) \tag{1.69}$$

in the sense of distributions. Let G_ϵ be any family of DiPerna-Lions renormalized solutions of the Boltzmann equation (1.41) that have $G_\epsilon^{\rm in}$ as initial values. Then, as $\epsilon \to 0$, the family of fluctuations $g_\epsilon^{\rm in}$ satisfies

$$\sigma\mathcal{M}g_\epsilon \to \sigma\mathcal{M}(\rho + u\cdot(v-U) + \tfrac{1}{2}\theta(|v-U|^2 - D\vartheta))\,in\,w\text{-}(L^1_{loc}dt; w\text{-}(L^1(dxdv))), \tag{1.70}$$

where

$$(\rho\varrho, u\varrho\vartheta, \theta\varrho\vartheta^2) \in C\Big([0,\infty); L^2(\tfrac{1}{\varrho}dvdx)), L^\infty(dt; L^2(\tfrac{1}{\varrho\vartheta}dvdx)), L^\infty(dt; L^2(\tfrac{1}{\varrho\vartheta^2}dvdx))\Big)$$

is the unique solution of the acoustic system (1.66) with initial data $(\rho^{\rm in}\varrho, u^{\rm in}\varrho\vartheta, \theta^{\rm in}\varrho\vartheta^2)$.

In addition, one has that

$$\int g_\epsilon\mathcal{M}\,{\rm d}v \;\to\; \rho\varrho, \qquad \int (v-U)g_\epsilon\mathcal{M}\,{\rm d}v \;\to (\rho U + u\vartheta)\varrho,$$
$$\int \tfrac{1}{2}|v|^2 g_\epsilon\mathcal{M}\,{\rm d}v - \tfrac{{\rm D}\vartheta}{2}\int g_\epsilon\mathcal{M}\,{\rm d}v - \tfrac{1}{2}\sum_{i=1}^{\rm D} U_i \int v_i g_\epsilon\mathcal{M}\,{\rm d}v$$
$$\to \tfrac{D\vartheta}{2}\rho\varrho + \tfrac{1}{2}|U|^2\rho\varrho + (\sum_{i=1}^{\rm D} U_i u_i)\varrho\vartheta + \tfrac{\rm D}{2}\theta\varrho\vartheta^2$$

in $C([0,\infty); w\text{-}L^1(dx))$.

We now recall some of the previous results. All the previous results we mentioned here scaled the Boltzmann equation around a homogeneous global Maxwellian and work on a bounded spatial domain. The acoustic limit around homogeneous global Maxwellian was first established in the setting of DiPerna-Lions solutions in [6] over a periodic domain for bounded collision kernels. It is assumed that $\frac{\delta_\epsilon}{\epsilon}|\log(\delta_\epsilon)| \to 0$ as $\epsilon \to 0$. In [24], the assumption on collision kernels is relaxed:

$$\int_{\mathbb{S}^{\mathrm{D}-1}} b(\omega, v_1 - v) \mathrm{d}\omega \leq C_b \big(1 + \frac{1}{2}|v_1 - v|^2\big)^\beta \text{ almost everywhere,}$$

for some $C_b < \infty$ and $\beta \in [0, 1)$. This includes all classical kernels that are derived from Maxwell or hard potentials and that satisfy a weak small deflection cutoff. In addition, it is assumed that $\frac{\delta_\epsilon}{\epsilon^{1/2}}|\log(\delta_\epsilon)|^{\beta/2} \to 0$ as $\epsilon \to 0$. This is the assumption we used for the collision kernels in the first result.

1.5 Linearized Boltzmann Equation: Formal Navier-Stokes Approximations

In the second part of the thesis, we work on compressible and weakly compressible Navier-Stokes approximations of the linearized Boltamann equation satisfied by the density fluctuation g_ϵ

$$\partial_t g_\epsilon + v \cdot \nabla_x g_\epsilon + \frac{1}{\epsilon}\mathcal{L} g_\epsilon = 0 \,. \tag{1.71}$$

$\mathcal{L}$ is the linearized collision operator with M being the unit Maxwellian

$$\mathcal{L} g_\epsilon = \iint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^{\mathrm{D}}} (g_\epsilon + g_{\epsilon_1} - g'_\epsilon - g'_{\epsilon_1}) b M_1 dw dv_1 \,. \tag{1.72}$$

The collision kernel b is normalized so that

$$\iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^{D}\times\mathbb{R}^{D}} b\mathrm{d}\omega M_1 \mathrm{d}v_1 M\mathrm{d}v \,=\, 1\,.$$

We use the following notation:

$$\langle\, f\rangle \,=\, \int f\, M\mathrm{d}v\,.$$

The projection of the linearized Boltzmann equation (1.71) onto

Null($\mathcal{L}$) yields the system of conservation laws

$$\begin{aligned}\partial_t\langle g_\epsilon\rangle + \nabla_x\cdot\langle v\, g_\epsilon\rangle \,&= 0\,,\\ \partial_t\langle v\, g_\epsilon\rangle \,+ \nabla_x\cdot\langle v\otimes v\, g_\epsilon\rangle \,&= 0\,,\\ \partial_t\langle(\tfrac{1}{2}|v|^2-\tfrac{D}{2})g_\epsilon\rangle + \nabla_x\cdot\langle v(\tfrac{1}{2}|v|^2-\tfrac{D}{2})g_\epsilon\rangle \,&= 0\,.\end{aligned}\tag{1.73}$$

Define the fluid variables

$$\rho_\epsilon = \langle g_\epsilon\rangle\,,\qquad u_\epsilon = \langle v g_\epsilon\rangle\,,\qquad \theta_\epsilon = \tfrac{2}{\mathrm{D}}\langle\big(\tfrac{1}{2}|v|^2-\tfrac{\mathrm{D}}{2}\big)\, g_\epsilon\rangle\,.\tag{1.74}$$

Recast the conservation laws (1.73) as

$$\begin{aligned}\partial_t\rho_\epsilon + \nabla_x\cdot u_\epsilon \,&=\, 0\,,\\ \partial_t u_\epsilon + \nabla_x(\rho_\epsilon+\theta_\epsilon) + \nabla_x\cdot\langle A(v)g_\epsilon\rangle \,&=\, 0\,,\\ \tfrac{D}{2}\partial_t\theta_\epsilon + \nabla_x\cdot u_\epsilon + \nabla_x\cdot\langle B(v)g_\epsilon\rangle \,&=\, 0\,,\end{aligned}\tag{1.75}$$

where

$$A(v) = v\otimes v - \tfrac{1}{D}|v|^2 I,\quad B(v) = \tfrac{1}{2}v(|v|^2 - \tfrac{D+2}{2})\,.$$

Clearly

$$A_{jk}\perp \mathrm{Null}(\mathcal{L}),\quad B_l\perp\mathrm{Null}(\mathcal{L}),\quad A_{jk}\perp B_l,\quad j,k,l = 1,2,...D\,.\tag{1.76}$$

Denote the projection of g_ϵ onto Null($\mathcal{L}$) by Π, we have

$$\Pi g_\epsilon = \rho_\epsilon + v \cdot u_\epsilon + (\tfrac{1}{2}|v|^2 - \tfrac{\mathrm{D}}{2})\theta_\epsilon \,.$$

Then g_ϵ can be decomposed into its infinitesimal Maxwellian Πg_ϵ and its deviation $\Pi^\perp g_\epsilon$ as

$$g_\epsilon \,=\, \Pi g_\epsilon + \Pi^\perp g_\epsilon \,.$$

We use $\tilde{g}_\epsilon := \Pi^\perp g_\epsilon$ as a shorthand notation. We see that the system of conservation laws becomes

$$\begin{aligned} \partial_t \rho_\epsilon + \nabla_x \cdot u_\epsilon \,&=\, 0 \,, \\ \partial_t u_\epsilon + \nabla_x(\rho_\epsilon + \theta_\epsilon) + \nabla_x \cdot \langle A(v)\tilde{g}_\epsilon \rangle \,&=\, 0 \,, \\ \tfrac{D}{2}\partial_t \theta_\epsilon + \nabla_x \cdot u_\epsilon + \nabla_x \cdot \langle B(v)\tilde{g}_\epsilon \rangle \,&=\, 0 \,. \end{aligned} \tag{1.77}$$

To close the system we must express $\langle A(v)\tilde{g}_\epsilon \rangle$ and $\langle B(v)\tilde{g}_\epsilon \rangle$ in terms of the fluid variables $\rho_\epsilon, u_\epsilon, \theta_\epsilon$. Note that if we simply set $\langle A(v)\tilde{g}_\epsilon \rangle = 0$ and $\langle B(v)\tilde{g}_\epsilon \rangle = 0$, (1.75) becomes the acoustic system. This is the acoustic approximation to the linearized Boltzmann equation. Solutions of acoustic system do not decay like solutions of the linearized Boltzmann equation, which is problematic. To get higher order approximations for $\tilde{g}_\epsilon$, we project the linearized Boltzmann equation (1.71) onto Null($\mathcal{L}$)$^\perp$

$$\partial_t \tilde{g}_\epsilon + \Pi^\perp v \cdot \nabla_x \tilde{g}_\epsilon + \frac{1}{\epsilon}\mathcal{L}\tilde{g}_\epsilon \,=\, -\Pi^\perp v \cdot \nabla_x \Pi g_\epsilon \,. \tag{1.78}$$

By orthogonality relations (1.76),

$$\Pi^\perp v \cdot \nabla_x \Pi g_\epsilon \,=\, A : \nabla_x u_\epsilon + B \cdot \nabla_x \theta_\epsilon \,. \tag{1.79}$$

The Fredholm alternative implies that there exist pseudo-inverse of $\mathcal{L}$, denoted by $\mathcal{L}^{-1}$. Multiplying the deviation equation (1.78) by $\epsilon\mathcal{L}^{-1}$ and using (1.79) yields

$$\epsilon\mathcal{L}^{-1}(\partial_t + v\cdot\nabla_x)\tilde{g}_\epsilon + \tilde{g}_\epsilon \;=\; -\epsilon\mathcal{L}^{-1}(A:\nabla_x u_\epsilon + B\cdot\nabla_x\theta_\epsilon)\,. \tag{1.80}$$

Therefore, to the leading order

$$\tilde{g}_\epsilon^{[1]} \;=\; -\epsilon\mathcal{L}^{-1}(A:\nabla_x u_\epsilon + B\cdot\nabla_x\theta_\epsilon)\,. \tag{1.81}$$

Acting the pseudo-inverse on $A(v)$ and $B(v)$, we have

$$\widehat{A}(v) = L^{-1}(A(v)), \quad \widehat{B}(v) = L^{-1}(B(v)).$$

By rotational symmetry

$$\widehat{A}(v) = \tau^A(v)A(v), \qquad \widehat{B}(v) = \tau^B(v)B(v)\,,$$

where $\tau^A(v)$ and $\tau^B(v)$ are both functions of $|v|$ only. Plugging $\tilde{g}_\epsilon = \tilde{g}_\epsilon^{[1]}$ in (1.77), we obtain the linearized compressible Navier-Stokes approximation:

$$\partial_t U_\epsilon + \mathcal{A}U_\epsilon = \epsilon\mathcal{D}U_\epsilon\,, \tag{1.82}$$

where U_ϵ are the fluid moments

$$U_\epsilon := \begin{pmatrix} \rho_\epsilon \\ u_\epsilon \\ \theta_\epsilon \end{pmatrix},$$

$\mathcal{A}$ is the acoustic operator and $\mathcal{D}$ is a dissipation operator

$$\mathcal{A}U := \begin{pmatrix} \nabla_x\cdot u \\ \nabla_x(\rho+\theta) \\ \nabla_x\cdot u \end{pmatrix}, \quad \mathcal{D}U := \mathcal{D}\begin{pmatrix} \rho \\ u \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ \mu\nabla_x\cdot\sigma(u) \\ \kappa\frac{2}{D}\nabla_x\cdot\nabla_x\theta \end{pmatrix},$$

$\sigma(u)$ being $\nabla_x u + (\nabla_x u)^T - \frac{2}{D}\nabla_x \cdot uI$, and the dissipation coefficients μ, κ are

$$\mu = \frac{1}{(\mathrm{D}-1)(\mathrm{D}+2)}\langle A : \widehat{A}\rangle\,, \quad \kappa = \frac{2}{\mathrm{D}(\mathrm{D}+2)}\langle B \cdot \widehat{B}\rangle\,.$$

Note that there are two time scales present: the fast acoustic mode and the slow incompressible mode. Averaging over the fast time in the linearized compressible Navier-Stokes system, we get the linearized weakly compressible Navier-Stokes system

$$\partial_t U_\epsilon + \mathcal{A} U_\epsilon = \epsilon \bar{\mathcal{D}} U_\epsilon\,, \tag{1.83}$$

where

$$\bar{\mathcal{D}} U := \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{t\mathcal{A}}\mathcal{D}e^{-t\mathcal{A}} U \,\mathrm{d}t\,.$$

For the computation of $\bar{\mathcal{D}}$, see Chapter 3.

We note furthermore that we can expand the deviation equation (1.80) systematically:

$$\tilde{g}_\epsilon = -\epsilon\mathcal{L}^{-1}\sum_{j=0}^{\infty}\left(-\epsilon(\partial_t + v\cdot\nabla_x)\mathcal{L}^{-1}\right)^j (A : \nabla_x u_\epsilon + B \cdot \nabla_x \theta_\epsilon)\,. \tag{1.84}$$

This is the Chapman-Enskog expansion. It gives a formal expansion of the deviation $\tilde{g}_\epsilon$ in terms of derivatives of the fluid variables $\rho_\epsilon, u_\epsilon$ and θ_ϵ. We define the n^{th}-order Chapman-Enskog approximation to $\tilde{g}_\epsilon$ by

$$\tilde{g}_\epsilon{}^{[n]} = -\epsilon\mathcal{L}^{-1}\sum_{j=0}^{n-1}\left(-\epsilon(\partial_t + v\cdot\nabla_x)\mathcal{L}^{-1}\right)^j (A : \nabla_x u_\epsilon + B \cdot \nabla_x \theta_\epsilon)\,. \tag{1.85}$$

In particular, we will use the second-order Chapman-Enskog approximation in the second result.

1.6 Statement of the Second Result

For the second part of the thesis, we show that solutions of the linearized weakly compressible Navier-Stokes systems approximate solutions of the linearized compressible Navier-Stokes system uniformly for $t > 0$ in $L^2(\mathbb{T}^{\mathrm{D}})$. Furthermore, establish a linearized compressible Navier-Stokes approximation of the linearized Boltzmann equation.

In Chapter 4, we show that the difference between the solution of linearized compressible Navier-Stokes (1.82) and the solution of linearized weakly compressible Navier-Stokes (1.83) is $\mathrm{O}(\sqrt{(\epsilon)})$ uniform in time.

Theorem 1.6.1 *Let U, V be solutions of linearized compressible Navier-Stokes equation and weakly compressible Stokes equation with the same initial data $U^{\mathrm{in}} \in H^1(\mathbb{T}^{\mathrm{D}})$. Then*

$$||U(t) - V(t)||_{L^\infty(dt;L^2(\mathbb{T}^{\mathrm{D}}))} \leq C\sqrt{\epsilon}\,.$$

The constant C depends on dimension D and transportation coefficients only.

Moreover, we establish the linearized compressible Navier-Stokes approximation of the linearized Boltzmann equation.

Theorem 1.6.2 *Let g_ϵ be the solution of the linearized Boltzmann equation. Let*

$$g_\epsilon^{[2]} := \rho_\epsilon + v \cdot u_\epsilon + (\tfrac{1}{2}|v|^2 - \tfrac{\mathrm{D}}{2})\theta_\epsilon - \epsilon\mathcal{L}^{-1}(A : \nabla_x u_\epsilon + B \cdot \nabla_x \theta_\epsilon) + \epsilon^2\mathcal{L}^{-1}(\partial_t + v \cdot \nabla_x)\mathcal{L}^{-1}(A : \nabla_x u_\epsilon + B \cdot \nabla_x \theta_\epsilon)\,,$$

where $U_\epsilon := (\rho_\epsilon,\, u_\epsilon,\, \theta_\epsilon)^T$ are solutions of the associated Cauchy problem of the linearized compressible Navier-Stokes approximation. Denote the fluid moments of g_ϵ

by $U^B_\epsilon := (\rho^B_\epsilon, u^B_\epsilon, \theta^B_\epsilon)^T$*. Assume* $\langle g^{[2]\text{in}}_\epsilon - g^{\text{in}}_\epsilon \rangle$, $\left\langle (|v| g^{[2]\text{in}}_\epsilon - g^{\text{in}}_\epsilon) \right\rangle$, $\left\langle |v|^2 (g^{[2]\text{in}}_\epsilon - g^{\text{in}}_\epsilon) \right\rangle$ *are bounded by* η *in* $L^2(\mathbb{T}^\text{D})$ *and* $U^{\text{in}}_\epsilon \in H^5(\mathbb{T}^\text{D})$*. Then*

$$||U_B - U_\epsilon||_{L^2(\mathbb{T}^\text{D})} \le C \max\{\sqrt{\epsilon}||U^{\text{in}}_\epsilon||_{H^5(\mathbb{T}^\text{D})}, \eta\}$$

uniformly for $t > 0$*.*

1.7 Organization of the Dissertation

We now lay out the organization of this dissertation. Chapter 2 establishes an acoustic limit of Boltzmann equation around a global Maxwellian on the whole space $\mathbb{R}^\text{D}$ that matches with the initial data of the Boltzmann equation on all conserved quantities. Chapter 3 is the formal derivation of weakly compressible fluid limits of the Boltzmann equation for the general initial data, i.e. those that not satisfy the incompressible and Boussinesq relations and use two-time scale to derive the corresponding averaged systems. Chapter 4 gives provides a uniform in time estimate for the $L^2(\mathbb{T}^\text{D})$ norm of the difference between the solution of linearized compressible Navier-Stokes and linearized weakly compressible Navier-Stokes system. We also showed the linearized compressible Navier-Stokes approximation to the linearized Boltzmann equation is uniform in time.

Chapter 2: Acoustic Limit for the Boltzmann Equation around a Global Maxwellian

In this chapter, we establish the acoustic limit for the classical Boltzmann equation considered over the spatial domain $\mathbb{R}^{\mathrm{D}}$. We do so in the physical setting of DiPerna-Lions renormalized solutions (cf. Chapter 1, Section 1.3). We scale the Boltzmann equation around a nonhomogeneous global Maxwellian (cf. Chapter 1, 1.37). Recall the the global Maxwellian has the form

$$\mathcal{M}_{(\varrho,U,\vartheta)} = \frac{\varrho(t,x)}{(2\pi\vartheta(t,x))^{\mathrm{D}/2}} \exp\left(-\frac{|v-U(t,x)|^2}{2\vartheta(t,x)}\right), \tag{2.1}$$

where the temperature $\vartheta(t)$, bulk velocity $U(x,t)$, and mass density $\varrho(x,t)$ are given by

$$\begin{aligned}
\vartheta(t) &= \frac{1}{at^2 - 2bt + c}, \\
U(x,t) &= \vartheta(t)\big(axt - bx - Bx\big), \\
\varrho(x,t) &= m\left(\frac{\vartheta(t)}{2\pi}\right)^{\frac{\mathrm{D}}{2}} \sqrt{\det(Q)}\, \exp\!\left(-\frac{\vartheta(t)}{2}\, x^T Q x\right).
\end{aligned} \tag{2.2}$$

with $m > 0$, $Q = (ac - b^2)I + B^2$,

$(a,b,c,B) \in \Omega = \big\{(a,b,c,B) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{\mathrm{D}\wedge\mathrm{D}} \,:\, Q > 0\big\}$.

Every Cauchy problem

$$\partial_t F_\epsilon + v\cdot\nabla_x F_\epsilon = \frac{1}{\epsilon}\mathcal{B}(F_\epsilon, F_\epsilon), \qquad F_\epsilon\big|_{t=0} = F^{\mathrm{in}} \tag{2.3}$$

with initial data $F^{\rm in}(v,x)$ that satisfies the bounds (1.27):

$$\iint_{\mathbb{R}^{\rm D}\times\mathbb{R}^{\rm D}} \left(1+|v|^2+|x|^2\right) F^{\rm in}\,{\rm d}v\,{\rm d}x < \infty \tag{2.4}$$

can be associated with a unique global Maxwellian determined by matching the values of the conserved quantities computed from $F^{\rm in}$, i.e. quantities on the right-hand side of (1.26) in Section 1.2.2 ([41]). This is the global Maxwellian that we will scale the Boltzmann equation around.

Let $F_\epsilon = \mathcal{M}G_\epsilon$. The fluid density fluctuations g_ϵ are defined by

$$G_\epsilon = 1 + \delta_\epsilon g_\epsilon\,.$$

We will show in Section 2.2 that g_ϵ is weakly relatively compact in L^1 topology. We will then show that every limit point g is governed by the acoustic system

$$\begin{aligned} \partial_t\,[\rho\varrho] \,+\, \nabla_x\cdot\,[\rho U\varrho \,+\, u\varrho\vartheta] &= 0\,,\\ \partial_t\,[u\varrho\vartheta] \,+ \nabla_x\cdot\,\left[(u\otimes U)\varrho\vartheta \,+\, (\rho\varrho\vartheta+\theta\varrho\vartheta^2)Id\right] &= 0\,,\\ \partial_t\left[\tfrac{\rm D}{2}\theta\varrho\vartheta^2\right] \,+ \nabla_x\cdot\,\left[\varrho\vartheta^2 u+\tfrac{D}{2}U\vartheta^2\varrho\theta\right] &= 0\,, \end{aligned} \tag{2.5}$$

with initial conditions

$$\rho\varrho(x,0)=\rho^{\rm in}\varrho^{\rm in},\qquad u\varrho\vartheta(x,0)=u^{\rm in}\varrho^{\rm in}\vartheta^{\rm in},\qquad \tfrac{\rm D}{2}\theta\varrho\vartheta^2(x,0)=\tfrac{\rm D}{2}\theta^{\rm in}\varrho^{\rm in}{\vartheta^{\rm in}}^2\,.$$

Here ϱ, U, ϑ are functions of (x,t) associated with the global Maxwellian. This is a linear hyperbolic system, and is therefore well-posed over $\mathbb{R}^{\rm D}\times[0,T]$.

We impose an initial condition that $F_\epsilon^{\rm in}$ is close to the global Maxwellian $\mathcal{M}$. Specifically, recall the entropy equality (1.44) in Chapter 1, Section 1.2.3:

$$H(G_\epsilon(t)) + \frac{1}{\epsilon}\int_0^t R(G_\epsilon(s))\,{\rm d}s \;=\; H(G_\epsilon^{\rm in})\,, \tag{2.6}$$

where $H(G)$ is the relative entropy functional

$$H(G) = \int_{\mathbb{R}^D} \int_{\mathbb{R}^D} (G \log(G) - G + 1) \mathcal{M} \, \mathrm{d}v \mathrm{d}x \,, \tag{2.7}$$

and $R(G)$ is the entropy dissipation rate functional

$$R(G) = \iiiint \frac{1}{4} \log(\frac{G_1' G}{G_1 G})(G_1' G' - G_1 G) b(\omega, v_1 - v) \mathcal{M}_1 \mathcal{M} \, \mathrm{d}\omega \mathrm{d}v_1 \mathrm{d}v \mathrm{d}x \,. \tag{2.8}$$

We assume the initial data F_ϵ^{in} is close to the global Maxwellian $\mathcal{M}$ in the sense that $H(G_\epsilon^{\mathrm{in}}) \leq C {\delta_\epsilon}^2$. (2.6) suggests the possibility of controlling G_ϵ in terms of G_ϵ^{in}. We prove $(1+|v|^2+|x|^2) g_\epsilon \mathcal{M}$ is relatively compact in w-$(L^1_{loc} dt; w\text{-}(L^1(dxdv)))$. This is a critical element in the proof of the weak acoustic limit theorem, stated in 2.3. We show that the the fluctuations converge weakly to a unique limit governed by a solution of the acoustic system in a weighted L^2 space. Note that recently it is shown in [2] that $\mathcal{M}$ will not be attracting for all close initial data with the same values for the conserved quantities, so the weak limit theorem is nontrivial.

In contrast, in earlier works of Bardos, Golse and Levermore [6] [24], acoustic limit was established for the Boltzmann equation considered over a spatial domain $\mathbb{T}^D$. They chose a scaling in which the density F is close to a spatially homogeneous Maxwellian $M = M(v)$ that has the same total mass, momentum, and energy as the initial data F_ϵ^{in}. By an appropriate choice of a Galilean frame and of mass and velocity units, it can be assumed that this so-called unit Maxwellian M has the form

$$M(v) \equiv \frac{1}{(2\pi)^{(D/2)}} \exp\left(-\frac{1}{2}|v|^2 \right) . \tag{2.9}$$

This corresponds to the spatially homogeneous fluid state with density and temperature equal to 1 and bulk velocity equal to 0.

It was shown in [6] [24] that the fluctuations around the absolute Maxwellian globally in time converge weakly to a unique limit governed by a solution of the acoustic systems

$$
\begin{aligned}
\partial_t \rho + \nabla_x \cdot u = 0\,, \qquad & \rho(x,0) = \rho^{\mathrm{in}}(x)\,, \\
\partial_t u + \nabla_x(\rho + \theta) = 0\,, \qquad & u(x,0) = u^{\mathrm{in}}(x)\,, \\
\tfrac{\mathrm{D}}{2}\partial_t \theta + \nabla_x \cdot u = 0\,, \qquad & \theta(x,0) = \theta^{\mathrm{in}}(x).
\end{aligned}
\tag{2.10}
$$

provided that the fluid moments of their initial fluctuations converge to appropriate L^2 initial data of the acoustic equations. They also worked in the physical setting of the DiPerna-Lions renormalized solutions. However, the fluid regime that we work on is different from theirs; in particular, it's worth noting that there's no higher order correction to the Eulerian regime in our setting, whereas in the regime chosen by [6] and [24], higher-order Navier-Stokes limit can be established.

2.1 Formal Scalings and Derivations

The acoustic system (2.25) can be formally derived from the Boltzmann equation through a scaling in which the fluctuations of the kinetic density F about the global Maxwellian $\mathcal{M}$ are scaled to be on the order of δ_ϵ. Specifically, we consider a family of formal solutions F_ϵ to the scaled Boltzmann initial-value problem

$$
\partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon}\mathcal{B}(F_\epsilon, F_\epsilon)\,, \qquad F_\epsilon\big|_{t=0} = F_\epsilon^{\mathrm{in}}\,, \tag{2.11}
$$

and the fluctuations g_ϵ, defined by

$$
F_\epsilon = \mathcal{M}(1 + \delta_\epsilon g_\epsilon)\,. \tag{2.12}
$$

The fluctuations g_ϵ^{in} and g_ϵ are bounded while $\delta_\epsilon > 0$ satisfies

$$\delta_\epsilon \to 0 \ as \ \epsilon \to 0. \tag{2.13}$$

In this formal derivation we assume that g_ϵ converges formally to g, our goal is to find the limiting function g.

The first step is to determine the form of the limiting function g. Observe that by (1.41) the fluctuations g_ϵ satisfy

$$\partial_t g_\epsilon + v \cdot \nabla_x g_\epsilon + \frac{1}{\epsilon}\mathcal{L}_{\mathcal{M}} g_\epsilon = \frac{\delta_\epsilon}{\epsilon}\mathcal{Q}_{\mathcal{M}}(g_\epsilon, g_\epsilon), \tag{2.14}$$

where $\mathcal{L}_{\mathcal{M}}$ and $\mathcal{Q}_{\mathcal{M}}$ are defined by

$$\begin{aligned} \mathcal{L}_{\mathcal{M}} g_\epsilon &= \iint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^{\mathrm{D}}} (g_\epsilon + g_{\epsilon_1} - g'_\epsilon - g'_{\epsilon_1}) b \mathcal{M}_1 dw dv_1. \\ \mathcal{Q}_{\mathcal{M}}(g_\epsilon, g_\epsilon) &= \iint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^{\mathrm{D}}} (g'_\epsilon g'_{\epsilon_1} - g_\epsilon g_{\epsilon_1}) b \mathcal{M}_1 dw dv_1. \end{aligned} \tag{2.15}$$

Assuming $\delta_\epsilon \to 0$ and multiplying both sides by ϵ, one finds that $\mathcal{L}_{\mathcal{M}}\, g = 0$. It is known that the null space of $\mathcal{L}_{\mathcal{M}}$ is given by $\text{Null}(\mathcal{L}_{\mathcal{M}}) = \text{Span}\{1, v_1, ..., v_D, |v|^2\}$. We conclude that g has the form of a so-called infinitesimal Maxwellian, namely,

$$g = \rho(t,x) + u(t,x)\cdot(v-U) + \frac{1}{2}\theta(t,x)(|v-U|^2 - D\vartheta). \tag{2.16}$$

The second step shows that the evolution of (ρ, u, θ) is governed by the system (2.25). Observe that the fluctuations g_ϵ formally satisfy the local conservation laws

$$\begin{aligned} &\text{(i)} \quad \partial_t \int_{\mathbb{R}^{\mathrm{D}}} g_\epsilon\, \mathcal{M}\, \mathrm{d}v + \nabla_x \cdot \int_{\mathbb{R}^{\mathrm{D}}} v\, g_\epsilon\, \mathcal{M}\, \mathrm{d}v = 0\,, \\ &\text{(ii)} \quad \partial_t \int_{\mathbb{R}^{\mathrm{D}}} v\, g_\epsilon\, \mathcal{M}\, \mathrm{d}v + \nabla_x \cdot \int_{\mathbb{R}^{\mathrm{D}}} v \otimes v\, g_\epsilon\, \mathcal{M}\, \mathrm{d}v = 0\,, \\ &\text{(iii)} \quad \partial_t \int_{\mathbb{R}^{\mathrm{D}}} \frac{1}{2}|v|^2 g_\epsilon\, \mathcal{M}\, \mathrm{d}v + \nabla_x \cdot \int_{\mathbb{R}^{\mathrm{D}}} v\, \frac{1}{2}|v|^2 g_\epsilon\, \mathcal{M}\, \mathrm{d}v = 0\,. \end{aligned} \tag{2.17}$$

By letting $\epsilon \to 0$ in these equations and using the infinitesimal Maxwellian form of g given by (3.16), one finds that (ρ, u, θ) solves the system (2.25). Indeed, the terms in the local conservation laws can be calculated as follows:

$$\int_{\mathbb{R}^{\mathrm{D}}} g\,\mathcal{M}\,\mathrm{d}v \,= \rho\varrho\,. \tag{2.18}$$

$$\int_{\mathbb{R}^{\mathrm{D}}} v\,g\,\mathcal{M}\,\mathrm{d}v \,= \rho U\varrho \,+\, u\varrho\vartheta\,. \tag{2.19}$$

As for $\int_{\mathbb{R}^{\mathrm{D}}} v\otimes v\,g\,\mathcal{M}\,\mathrm{d}v$, when $i \neq j$,

$$\int_{\mathbb{R}^{\mathrm{D}}} v_i\,v_j\,g\,\mathcal{M}\,\mathrm{d}v \,= \rho U_i U_j\varrho \,+\, (U_i u_j + U_j u_i)\varrho\vartheta\,. \tag{2.20}$$

when $i = j$,

$$\int_{\mathbb{R}^{\mathrm{D}}} v_i^2\,g\,\mathcal{M}\,\mathrm{d}v \,= \rho\varrho\vartheta \,+ U_i^2\rho\varrho \,+\, 2u_i U_i\varrho\vartheta \,+\, \theta\varrho\vartheta^2\,. \tag{2.21}$$

Therefore,

$$\int_{\mathbb{R}^{\mathrm{D}}} v\otimes v\,g\,\mathcal{M}\,\mathrm{d}v \,= U\otimes U\rho\varrho \,+\, (U\otimes u + u\otimes U)\varrho\vartheta \,+\, (\rho\varrho\vartheta + \theta\varrho\vartheta^2)Id\,. \tag{2.22}$$

$$\int_{\mathbb{R}^{\mathrm{D}}} \frac{1}{2}|v|^2\,g\,\mathcal{M}\,\mathrm{d}v \,=\, \frac{\mathrm{D}\vartheta}{2}\rho\varrho \,+\, \frac{1}{2}|U|^2\rho\varrho \,+\, (\sum_{i=1}^{\mathrm{D}} U_i u_i)\varrho\vartheta \,+\, \frac{\mathrm{D}}{2}\theta U\vartheta^2\,. \tag{2.23}$$

$$\begin{aligned}\int_{\mathbb{R}^{\mathrm{D}}} v\frac{1}{2}|v|^2 g\mathcal{M}\,\mathrm{d}v = &\frac{1}{2}\rho\varrho U\,[(\mathrm{D}+2)\vartheta + |U|^2] \\ &+ \frac{1}{2}\varrho\vartheta\,[(D+2)u\vartheta + |U|^2 u + 2U\,(\sum_{i=1}^{\mathrm{D}} U_i u_i)] \,+\, \frac{\mathrm{D}+2}{2}U\vartheta^2\varrho\theta\,.\end{aligned} \tag{2.24}$$

Plugging all the terms above in the local conservation laws (2.17), we get

$$
\begin{aligned}
&\text{(i)}\quad \partial_t\,[\rho\varrho]\,+\,\nabla_x\cdot\,[\rho U\,\varrho\,+\,u\varrho\vartheta]\,=0\,,\\
&\text{(ii)}\quad \partial_t[\rho U\,\varrho\,+\,u\varrho\vartheta]+\\
&\nabla_x\cdot\,[U\otimes U\rho\varrho\,+\,(U\otimes u+u\otimes U)\varrho\vartheta\,+\,(\rho\varrho\vartheta+\theta\varrho\vartheta^2)Id]\,=0\,,\\
&\text{(iii)}\quad \partial_t[\tfrac{\mathrm{D}\vartheta}{2}\rho\varrho\,+\,\tfrac{1}{2}|U|^2\rho\varrho\,+\,(\sum_{i=1}^{\mathrm{D}}U_iu_i)\varrho\vartheta\,+\,\tfrac{\mathrm{D}}{2}\theta(t,x)\varrho\vartheta^2]+\\
&\nabla_x\cdot\,\{\tfrac{1}{2}\rho\varrho U[(\mathrm{D}+2)\vartheta+|U|^2]\,+\,\tfrac{1}{2}\varrho\vartheta[(D+2)u\vartheta\\
&+|U|^2u+2U(\sum_{i=1}^{\mathrm{D}}U_iu_i)]\,+\,\tfrac{\mathrm{D}+2}{2}U\vartheta^2\varrho\theta\}\,=\,0\,.
\end{aligned}
\tag{2.25}
$$

One finds that

$$
\begin{aligned}
&\Big(\int g_\epsilon^{\mathrm{in}}\mathcal{M}^{\mathrm{in}}\,dv,\int vg_\epsilon^{\mathrm{in}}\mathcal{M}^{\mathrm{in}}\,dv,\int\tfrac{1}{2}|v|^2g_\epsilon^{\mathrm{in}}\mathcal{M}^{\mathrm{in}}\,dv\Big)\\
\to&\Big(\rho^{\mathrm{in}}\varrho^{\mathrm{in}},\rho^{\mathrm{in}}U\varrho^{\mathrm{in}}+u^{\mathrm{in}}\varrho^{\mathrm{in}}\vartheta^{\mathrm{in}},\\
&\qquad\tfrac{D\vartheta^{\mathrm{in}}}{2}\rho^{\mathrm{in}}\varrho^{\mathrm{in}}+\tfrac{1}{2}|U|^2\rho^{\mathrm{in}}\varrho^{\mathrm{in}}+(\sum_{i=1}^{\mathrm{D}}U_iu_i^{\mathrm{in}})\varrho^{\mathrm{in}}\vartheta^{\mathrm{in}}+\tfrac{\mathrm{D}}{2}\theta^{\mathrm{in}}\varrho^{\mathrm{in}}(\vartheta^{\mathrm{in}})^2\Big)
\end{aligned}
\tag{2.26}
$$

as $\epsilon\to 0$ provided we assume that the limits exist in the sense of distributions.

The above formal derivation can be stated more precisely as follows:

Theorem 2.1.1 *(Formal Limit Theorem) Let G_ϵ be a family of distribution solutions of the scaled Boltzmann initial-value problem (*1.41*). Let G_ϵ^{in} and G_ϵ have fluctuations g_ϵ^{in} and g_ϵ given by (3.12) that are bounded families for some $\delta_\epsilon>0$ that vanishes with ϵ as in (*3.13*). Also:*

*(i) Assume that the local conservation laws (*2.17*) are also satisfied in the sense of distributions for every g_ϵ.*

(ii) Assume that the family g_ϵ converges in the sense of distributions as $\epsilon\to 0$ to g.

Assume furthermore that $\mathcal{L}_\mathcal{M} g_\epsilon \to \mathcal{L}_\mathcal{M} g$*, that the moments*

$$\int g_\epsilon \mathcal{M}\,dv, \int v g_\epsilon \mathcal{M}\,dv, \int v\otimes v g_\epsilon \mathcal{M}\,dv, \int v|v|^2 g_\epsilon \mathcal{M}\,dv,$$

converge to the corresponding moments

$$\int g \mathcal{M}\,dv, \int v g \mathcal{M}\,dv, \int v\otimes v g \mathcal{M}\,dv, \int v|v|^2 g \mathcal{M}\,dv,$$

and that every formally small terms vanishes, all in the sense of distributions as $\epsilon \to 0$*.*

(iii) Assume that for some $(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}})$ *the family* g_ϵ^{in} *satisfies (*2.26*) in the sense of distributions.*

Then g *is the unique local infinitesimal Maxwellian (*3.16*) determined by the solution* (ρ, u, θ) *of the system (*2.25*) with the initial data* $(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}})$ *obtained from (*2.26*).*

2.2 Analytic Setting

In order to mathematically justify the acoustic limit, we must make precise: (1) the notion of solution for the Boltzmann equation, and (2) the notion of solution for the acoustic system (2.25). We therefore work in the setting of DiPernaLions renormalized solutions for the Boltzmann equation, as discussed in Chapter 1, Section 1.3. We will show that the acoustic system (2.25) is hyperbolic. We then prove $(1 + |v|^2 + |x|^2) g_\epsilon \mathcal{M}$ is relatively compact in w-$(L^1_{loc} dt; w$-$(L^1(dxdv)))$. The scaled Boltzmann initial-value problems for the acoutstic limits can be put into the general form

$$\partial_t G_\epsilon + v\cdot\nabla_x G_\epsilon = \frac{1}{\epsilon} Q(G_\epsilon, G_\epsilon), \qquad G_\epsilon(v,x,0) = G_\epsilon^{\mathrm{in}}(v,x)\,. \tag{2.27}$$

2.2.1 The Acoustic System

In this section, we show that the acoustic system (2.25) is hyperbolic. Consider the Cauchy problem

$$\begin{aligned} \partial_t\, [\rho\varrho] \,+\, \nabla_x \cdot\, [\rho U \varrho \,+\, u\varrho\vartheta] &= 0\,, \\ \partial_t\, [u\varrho\vartheta] \,+ \nabla_x \cdot\, \big[(u\otimes U)\varrho\vartheta \,+\, (\rho\varrho\vartheta + \theta\varrho\vartheta^2) Id\big] &= 0\,, \\ \partial_t\, \big[\tfrac{\mathrm{D}}{2}\theta\varrho\vartheta^2\big] \,+ \nabla_x \cdot\, \big[\varrho\vartheta^2 u + \tfrac{D}{2} U\vartheta^2\varrho\theta\big] &= 0 \end{aligned} \tag{2.28}$$

with initial data $V^{\text{in}} = (\rho^{\text{in}}\varrho, u^{\text{in}}\varrho\vartheta, \theta^{\text{in}}\varrho\vartheta^2)$. In order to show the hyperbolicity, it suffices to find a pair of entropy $\Phi(V)$ and entropy flux $\Psi(V)$ associated with the system. Here $V = (\rho\varrho, u\varrho\vartheta, \theta\varrho\vartheta^2)$. Indeed, we may take

$$\Phi = \int g^2 \mathcal{M} \mathrm{d}v = \rho^2\varrho + |u|^2\varrho\vartheta + \frac{1}{2}\theta^2\varrho\vartheta^2\,.$$

A direct calculation shows that its Hessian is positive definite. The existence of entropy flux Ψ can be proved by looking at $\partial_{VV}\Phi(V)\partial_V F(V)\cdot\eta$, which turns out to be symmetric for all possible choice of V and η. Therefore, there exists an entropy flux Ψ which associates with the entropy Φ. Hence the system is indeed hyperbolic. In particular, we have

Proposition 2.2.1 *For any*

$$V^{\text{in}} = (\rho^{\text{in}}\varrho, u^{\text{in}}\varrho\vartheta, \theta^{\text{in}}\varrho\vartheta^2) \in \Big(L^\infty(dt; L^2(\tfrac{1}{\varrho} dvdx)), L^\infty(dt; L^2(\tfrac{1}{\varrho\vartheta} dvdx)), L^\infty(dt; L^2(\tfrac{1}{\varrho\vartheta^2} dvdx))\Big)\,,$$

there is a unique solution

$$V = (\rho\varrho, u\varrho\vartheta, \theta\varrho\vartheta^2) \in C\Big([0,\infty); L^2(\tfrac{1}{\varrho} dvdx)), L^\infty(dt; L^2(\tfrac{1}{\varrho\vartheta} dvdx)), L^\infty(dt; L^2(\tfrac{1}{\varrho\vartheta^2} dvdx))\Big)$$

to the Cauchy problem (2.28).

2.2.2 Fluctuations

We now show the fluctuations are relatively compact in a weak-L^1 space. Later in this chapter we will show the limit point g is governed by the acoustic system $(I-10)$. We start from the entropy inequality

$$H(G_\epsilon(t)) + \frac{1}{\epsilon}\int_0^t R(G_\epsilon(s))\mathrm{d}s \leq H(G_\epsilon^{\mathrm{in}}) \leq C^{\mathrm{in}}\delta_\epsilon^2, \tag{2.29}$$

where

$$H(G) = \iint (G\log G - G + 1)\mathcal{M}\,\mathrm{d}v\mathrm{d}x.$$

and

$$R(G) = \iiiint \frac{1}{4}\log\big(\frac{G_1'G'}{G_1G}\big)(G_1'G' - G_1G)b(\omega, v_1 - v)\mathcal{M}_1\mathcal{M}\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v\mathrm{d}x.$$

If we rewrite this in terms of g_ϵ and q_ϵ, where $q_\epsilon = \frac{1}{\epsilon^{\frac{1}{2}}\delta_\epsilon}(G_1'G' - G_1G)$, we get

$$\begin{aligned}&\iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} h(\delta_\epsilon g_\epsilon)\mathcal{M}\,\mathrm{d}v\mathrm{d}x \\ &\qquad + \frac{1}{\epsilon}\int_0^t \iiiint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} \frac{1}{4} r\big(\frac{q_\epsilon}{G_{\epsilon 1}G_\epsilon}\epsilon^{\frac{1}{2}}\delta_\epsilon\big)G_{\epsilon 1}G_\epsilon b\mathrm{d}\omega\mathcal{M}_1\mathcal{M}\mathrm{d}v_1\mathrm{d}v\mathrm{d}x\mathrm{d}s \\ &\qquad \leq H(G_\epsilon^{\mathrm{in}}) \leq C^{\mathrm{in}}\delta_\epsilon^2,\end{aligned} \tag{2.30}$$

i.e.

$$\begin{aligned}&\iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} \frac{1}{\delta_\epsilon^2} h(\delta_\epsilon g_\epsilon)\mathcal{M}\,\mathrm{d}v\mathrm{d}x \\ &+ \frac{1}{\epsilon\delta_\epsilon^2}\int_0^t \iiiint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} \frac{1}{4} r\big(\frac{q_\epsilon}{G_{\epsilon 1}G_\epsilon}\epsilon^{\frac{1}{2}}\delta_\epsilon\big)G_{\epsilon 1}G_\epsilon b\mathcal{M}_1\mathcal{M}\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v\mathrm{d}x\mathrm{d}s \leq C^{\mathrm{in}}.\end{aligned}$$

Here

$$h(z) = (1+z)\log(1+z) - z, \quad z = \delta_\epsilon g_\epsilon. \tag{2.31}$$

$$r(\tilde{z}) = \tilde{z}\log(1+\tilde{z}), \quad \tilde{z} = \frac{G'_{\epsilon_1} G'_\epsilon}{G_{\epsilon_1} G_\epsilon} - 1. \tag{2.32}$$

Entropy Controls

Proposition 2.2.2 *Assume that there exists C^{in}, such that*

$$\iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} \frac{1}{\delta_\epsilon^2} h(\delta_\epsilon g_\epsilon)\mathcal{M}\,\mathrm{d}v\mathrm{d}x \le C^{\mathrm{in}} \qquad \textit{for all} \quad t\,.$$

Then

(i) $\displaystyle\iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} (1+|v|^2+|x|^2) g_\epsilon \mathcal{M}\,\mathrm{d}v\mathrm{d}x \in L^\infty(dt)$ *and* $(1+|v|^2+|x|^2)g_\epsilon\mathcal{M}$ *is relatively compact in* w-$(L^1_{loc}dt; w$-$(L^1(dxdv)))$.

(ii) If $(1+|v|^2+|x|^2)g\mathcal{M}$ *is a* w-$(L^1_{loc}dt; w$-$(L^1(dxdv)))$ *limit point of* $(1+|v|^2+|x|^2)g_\epsilon\mathcal{M}$*, then* $g^2\mathcal{M} \in L^\infty(dt; L^1(dvdx))$. *Moreover,* $\displaystyle\frac{1}{2}\iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} g^2\mathcal{M}\,\mathrm{d}v\mathrm{d}x \le C^{\mathrm{in}}$.

Proof. (i) Notice that h is a convex function, hence by Young's inequality

$$yz \le h(y) + h^*(z), \qquad \text{for} \quad y > -1. \tag{2.33}$$

Here $h^*(z)$ is the Legendre dual of h, and can be given explicitly by $e^z - 1 - z$. $h^*(z)$ also satisfies the superquadratic property and the reflection property:

$$h^*(az) \le a^2 h^*(z), \qquad \text{for} \qquad 0 \le a \le 1, z > 0, \tag{2.34}$$

$$h^*(|z|) \le h(z). \tag{2.35}$$

Applying the properties above,

$$(1+|v|^2+|x|^2)|g_\epsilon|\mathcal{M} \ \leq \frac{3}{\alpha}\big[\frac{\sigma}{\delta_\epsilon^2}h(\delta_\epsilon g_\epsilon)+\frac{1}{\sigma}h^*\big(\frac{\alpha(1+|v|^2+|x|^2)}{3}\big)\big]\mathcal{M}, \tag{2.36}$$

for all $\frac{\delta_\epsilon}{\sigma}<1$, where σ is to be chosen later.

We may now choose $\alpha>0$ such that

$$\iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} h^*\big(\frac{\zeta(1+|v|^2+|x|^2)}{3}\big)\mathcal{M}\,\mathrm{d}v\mathrm{d}x \in L^\infty(dt)\,.$$

The possibility of such choice is guaranteed by the following observation: The global Maxwellian can be written as

$$\mathcal{M}=\frac{m}{(2\pi)^{\mathrm{D}}}\sqrt{\det(Q)}\,\exp\left(-\frac{1}{2}\begin{pmatrix} v \\ x-vt\end{pmatrix}^T\begin{pmatrix} cI & bI+B \\ bI-B & aI\end{pmatrix}\begin{pmatrix} v \\ x-vt\end{pmatrix}\right). \tag{2.37}$$

Note that $\begin{pmatrix} cI & bI+B \\ bI-B & aI\end{pmatrix}$ is positive definite. Therefore, there exists $\lambda_0>0$, λ_0 depending on (a,b,c,B), such that

$$\begin{aligned}
&-\frac{1}{2}\begin{pmatrix} v \\ x-vt\end{pmatrix}^T\begin{pmatrix} cI & bI+B \\ bI-B & aI\end{pmatrix}\begin{pmatrix} v \\ x-vt\end{pmatrix}\\
&\leq -\frac{1}{2}\begin{pmatrix} v \\ x-vt\end{pmatrix}^T\begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0\end{pmatrix}\begin{pmatrix} v \\ x-vt\end{pmatrix}\\
&= -\frac{1}{2}\lambda_0\begin{pmatrix} v \\ x\end{pmatrix}^T\begin{pmatrix} 1+t^2 & -t \\ -t & 1\end{pmatrix}\begin{pmatrix} v \\ x\end{pmatrix}.
\end{aligned}$$

The smaller eigenvalue of $\begin{pmatrix} 1+t^2 & -t \\ -t & 1 \end{pmatrix}$, $\dfrac{2+t^2-\sqrt{t^4+4t^2}}{2}$, decreases monotonically. Hence $\mathcal{M}$ is bounded by

$$\frac{m}{(2\pi)^{\mathrm{D}}}\sqrt{\det(Q)}\exp\big(-\frac{\lambda_0}{2}(\frac{2+T^2-\sqrt{T^4+4T^2}}{2})(|v|^2+|x|^2)\big)$$

for $t\in[0,T]$. Observing that $h^*(z)$ asymptotically behaves like e^z, we may simply choose α, such that

$$\frac{\alpha}{3}-\frac{\lambda_0}{2}(\frac{2+T^2-\sqrt{T^4+4T^2}}{2})<0\,.$$

Therefore $\iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} h^*\big(\frac{\zeta(1+|v|^2+|x|^2)}{3}\big)\mathcal{M}\,\mathrm{d}v\mathrm{d}x \in L^\infty(dt)$ upon such choice. We then apply the Dunford-Pettis criterion to prove (i).

For any $\eta>0$, choose $\sigma=\dfrac{\alpha\eta}{8TC^{\mathrm{in}}}$, then pick $\xi>0$, such that $|\Omega|<\xi$ implies

$$\iiint_\Omega h^*\big(\frac{\zeta(1+|v|^2+|x|^2)}{3}\big)\mathcal{M}\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\le\frac{1}{8}\sigma\alpha\eta.$$

Hence for any Ω, such that $|\Omega|<\xi$,

$$\iiint_\Omega(1+|v|^2+|x|^2)|g_\epsilon|\mathcal{M}\,\mathrm{d}v\mathrm{d}x\mathrm{d}t<\eta.$$

(ii) Let $(1+|v|^2+|x|^2)g\mathcal{M}$ be the weak limit of any convergent subsequence of $(1+|v|^2+|x|^2)g_\epsilon\mathcal{M}$. The convexity of h yields the inequality

$$\frac{1}{\delta_\epsilon^2}h(\delta_\epsilon g)+\frac{1}{\delta_\epsilon}h'(\delta_\epsilon g)(g_\epsilon-g)\le\frac{1}{\delta_\epsilon^2}h(\delta_\epsilon g_\epsilon). \tag{2.38}$$

Fix $\lambda>0$ and multiply this inequality by the global Maxwellian $\mathcal{M}$ and the characteristic function $1_{\{|g|<\lambda\}}$; the non-negativity of h then implies

$$\frac{1}{\delta_\epsilon^2}h(\delta_\epsilon g)1_{\{|g|<\lambda\}}\mathcal{M}+\frac{1}{\delta_\epsilon}h'(\delta_\epsilon g)(g_\epsilon-g)1_{\{|g|<\lambda\}}\mathcal{M}\le\frac{1}{\delta_\epsilon^2}h(\delta_\epsilon g_\epsilon)\mathcal{M}. \tag{2.39}$$

Average this over $[t_1,t_2]\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}$ for an arbitrary time interval $[t_1,t_2]$ and then consider its limit as $\epsilon\to 0$. Use the strong L^∞ limits $\frac{1}{\delta_\epsilon^2}h(\delta_\epsilon g)1_{\{|g|<\lambda\}}\mathcal{M}\to\frac{1}{2}g^2 1_{\{|g|<\lambda\}}\mathcal{M}$, $\frac{1}{\delta_\epsilon}h'(\delta_\epsilon g)1_{\{|g|<\lambda\}}\mathcal{M}\to g1_{\{|g|<\lambda\}}\mathcal{M}$ and the weak-L^1 limit $\mathcal{M}(g_\epsilon-g)\rightharpoonup 0$, we then have

$$\begin{aligned}&\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}}\frac{1}{2}g^2 1_{\{|g|<\lambda\}}\mathcal{M}\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\\ &\le\liminf_{\epsilon\to 0}\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}}\frac{1}{\delta_\epsilon^2}h(\delta_\epsilon g_\epsilon)\mathcal{M}\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\le C^{\mathrm{in}}.\end{aligned}\tag{2.40}$$

Taking $\lambda\to+\infty$ and use the arbitrariness of the interval $[t_1,t_2]$, we proved

$$\iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} g^2\mathcal{M}\,\mathrm{d}v\mathrm{d}x\in L^\infty(dt),\ \text{for almost any}\ t\in[0,\infty).$$

Before we state the next proposition, several notations need to be introduced. We define $N_\epsilon:=1+\frac{1}{3}\delta_\epsilon g_\epsilon$, hence g_ϵ can be composed as $\frac{g_\epsilon}{N_\epsilon}+\frac{1}{3}\delta_\epsilon\frac{g_\epsilon^2}{N_\epsilon}$. Also, we define $\gamma_\epsilon:=\frac{1}{\delta_\epsilon}t(\delta_\epsilon g_\epsilon)$, where $t(z)=3\log(1+\frac{1}{3}z)$.

Corollary 2.2.1

(i) $\mathcal{M}(g_\epsilon-\gamma_\epsilon)=o(\delta_\epsilon)$ *in* $L^\infty(dt;L^1(dvdx))$;

(ii) $\frac{\mathcal{M}g_\epsilon^2}{N_\epsilon}$ *is bounded in* $L^\infty(dt;L^1(dvdx))$.

Proof. We will prove (i) first. Set $z=\delta_\epsilon g_\epsilon$ into the elementary inequality

$$\frac{z}{1+\frac{1}{3}z}\le t(z)\le z,$$

thus

$$0\le g_\epsilon-\gamma_\epsilon\le g_\epsilon-\frac{g_\epsilon}{1+\frac{1}{3}\delta_\epsilon g_\epsilon}=\frac{1}{3}\delta_\epsilon\frac{g_\epsilon^2}{N_\epsilon}.$$

If (ii) holds, then (i) also holds. In order to prove (ii), we introduce the function $s(z)=\frac{1}{2}\frac{z^2}{1+\frac{1}{3}z}$. Notice that $s(z)\le h(z)$ for $z\ge -1$; indeed $s''(z)=(1+\frac{1}{3}z)^{-3}\le$

$(1+z)^{-1} = h''(z)$ and $h(z) - s(z) = o(z^4)$ as $z \to 0$. Therefore $\frac{1}{2}\frac{g_\epsilon^2}{N_\epsilon} = \frac{1}{\delta_\epsilon^2} s(\delta_\epsilon g_\epsilon) \leq \frac{1}{\delta_\epsilon^2} h(\delta_\epsilon g_\epsilon)$, whence (ii) follows from the entropy bound.

Proposition 2.2.3 *As $\epsilon \to 0$, $\mathcal{M}(|v|^2 + |x|^2)\frac{g_\epsilon^2}{N_\epsilon} = o\big(\log(\delta_\epsilon)\big)$ in $L^\infty(dt; L^1(dvdx))$.*

Proof. The proof is almost identical to that of proposition 3.4 of [5], except for replacing $|v|^2$ by $\alpha(|v|^2 + |x|^2)$, and observing that $\iint \exp\big(\frac{\alpha}{3}(|v|^2 + |x|^2)\mathcal{M}\big)\, \mathrm{d}v\mathrm{d}x$ is uniformly bounded for fixed T.

Dissipation Controls

Proposition 2.2.4 *Let g_ϵ, q_ϵ be sequences of functions satisfying the entropy inequality and bound (2.30), then*

(i) $\frac{\mathcal{M}_1\mathcal{M}}{N_\epsilon}(1 + |v|^2 + |x|^2)q_\epsilon$ *is relatively compact in* $w\text{-}L^1_{loc}(dt; w\text{-}L^1(bd\omega dv dv_1 dx))$.

(ii) If $\mathcal{M}_1\mathcal{M}q$ *is* $w\text{-}L^1_{loc}(dt; w\text{-}L^1(bd\omega dv dv_1 dx))$ *limit of any converging subsequence of* $\frac{\mathcal{M}_1\mathcal{M}q_\epsilon}{N_\epsilon}$, $\sqrt{\mathcal{M}_1\mathcal{M}}q \in L^2(dt; L^2(bd\omega dv_1 dv dx))$ *and for a.e.* $t \in [0, \infty)$, *it satisfies*

$$\iint_{\mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}}} \frac{1}{2} g^2(t)\mathcal{M} dv dx + \frac{1}{4}\int_0^t \iiiint_{\mathbb{S}^{\mathrm{D}-1} \times \mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}}} q^2 \mathcal{M}_1\mathcal{M} bd\omega dv dv_1 dx ds \leq C^{\mathrm{in}}.$$

Proof. (i) Choose α such that

$$\frac{\alpha}{3} - \frac{\lambda_0}{2}\Big(\frac{2 + T^2 - \sqrt{T^4 + 4T^2}}{2}\Big) < 0.$$

Take $\eta > 0$ arbitrarily small. Choose $\sigma = \frac{\alpha\eta}{32C^{\mathrm{in}}}$.

Applying Young's inequality and using superquadratic property of $r^*(y)$, we get

$$\begin{aligned} &(1 + |v|^2 + |x|^2)\frac{q_\epsilon}{N_\epsilon} \\ &\leq \frac{3}{\alpha} G_{\epsilon_1} G_\epsilon \Big[\frac{\sigma}{\delta_\epsilon^2 \epsilon} r\Big(\frac{q_\epsilon \delta_\epsilon \epsilon^{\frac{1}{2}}}{G_{\epsilon_1} G_\epsilon}\Big) + \frac{1}{\sigma} r^*\Big(\frac{\alpha}{3}(1 + |v|^2 + |x|^2)\Big)\frac{1}{N_\epsilon^2}\Big], \ \forall \frac{\delta_\epsilon \epsilon}{N_\epsilon \sigma} < 1. \end{aligned} \tag{2.41}$$

Since $N_\epsilon \geq \frac{2}{3}$, $\dfrac{\delta_\epsilon \epsilon}{N_\epsilon \sigma} < 1$ holds for $\delta_\epsilon \epsilon < \frac{2}{3}\sigma$.

We know that $r^*(y) = O(\exp\{y\})$ as $y \to +\infty$. Therefore,

$$r^*\big(\frac{\alpha}{3}(1+|v|^2+|x|^2)\big) \sim \exp\big(\frac{\alpha}{3}(1+|v|^2+|x|^2)\big)$$

for large $|x|$, $|v|$. Note that $\dfrac{G_\epsilon}{N_\epsilon^2} \leq \dfrac{9}{8}$ and converges to 0 in measure and $\mathcal{M}_1 G_{\epsilon_1}$ is relatively compact in w-$L^1_{loc}(dt; w\text{-}L^1(dv_1 dx))$. This yields the relative compactness of $r^*(\dfrac{\alpha}{3}(1+|v|^2+|x|^2))G_{\epsilon_1}\mathcal{M}_1\mathcal{M}$ in w-$L^1_{loc}(dt; w\text{-}L^1(bd\omega dv_1 dv dx))$.

Then by Product Limit Theorem, $r^*(\dfrac{\alpha}{3}(1+|v|^2+|x|^2))\dfrac{G_{\epsilon_1}G_\epsilon}{N_\epsilon^2}\mathcal{M}_1\mathcal{M}$ is relatively compact in w-$L^1_{loc}(dt; w\text{-}L^1(bd\omega dv_1 dv dx))$.

Observe that

$$\int_0^t \iiiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D\times\mathbb{R}^D} \frac{1}{\epsilon\delta_\epsilon^2}\frac{1}{4} r\big(\frac{q_\epsilon}{G_{\epsilon_1}G_\epsilon}\epsilon^{\frac{1}{2}}\delta_\epsilon\big) G_{\epsilon_1}G_\epsilon b\mathcal{M}_1\mathcal{M}\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v\mathrm{d}x\mathrm{d}s \leq C^{\mathrm{in}}\,.$$

Applying the same argument as was in Proposition 2.2.2 gives assertion (i). To show (ii), we use (ii) of Proposition 2.2.2:

$$\begin{aligned} &\iint_{\mathbb{R}^D\times\mathbb{R}^D} \frac{1}{2} g^2(t)\mathcal{M} dv dx + \\ \liminf_{\epsilon\to 0} &\int_0^t \iiiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D\times\mathbb{R}^D} \frac{1}{\epsilon\delta_\epsilon^2}\frac{1}{4}\big(\frac{q_\epsilon}{G_{\epsilon_1}G_\epsilon}\epsilon^{\frac{1}{2}}\delta_\epsilon\big) G_{\epsilon_1}G_\epsilon b dw \mathcal{M}_1 dv_1 \mathcal{M} dv dx ds \leq C^{\mathrm{in}}. \end{aligned} \tag{2.42}$$

For the second term on the left-hand side, notice that convexity of r yields

$$\frac{1}{\epsilon\delta_\epsilon^2} r(\epsilon^{\frac{1}{2}}\delta_\epsilon q) + \frac{1}{\epsilon^{\frac{1}{2}}\delta_\epsilon} r'(\epsilon^{\frac{1}{2}}\delta_\epsilon q)\big(\frac{q_\epsilon}{G_{\epsilon_1}G_\epsilon} - q\big) \leq \frac{1}{\epsilon\delta_\epsilon^2} r\big(\frac{\epsilon^{\frac{1}{2}}\delta_\epsilon q_\epsilon}{G_{\epsilon_1}G_\epsilon}\big). \tag{2.43}$$

Fix $\lambda > 0$ and multiply this inequality by $1_{|q|<\lambda}$ times $G_{\epsilon_1}G_\epsilon$ over the normalization $N_\epsilon^{abs} := 1 + \frac{1}{3}\delta_\epsilon |g_\epsilon|$, the non-negativity of r then implies

$$\frac{1}{\epsilon\delta_\epsilon^2} r(\epsilon^{\frac{1}{2}}\delta_\epsilon q)\frac{G_{\epsilon_1}G_\epsilon}{N_\epsilon^{abs}} 1_{|q|<\lambda} + \frac{1}{\epsilon^{\frac{1}{2}}\delta_\epsilon} r'(\epsilon^{\frac{1}{2}}\delta_\epsilon q)\big(\frac{q_\epsilon}{G_{\epsilon_1}G_\epsilon} - q\big)\frac{G_{\epsilon_1}G_\epsilon}{N_\epsilon^{abs}} 1_{|q|<\lambda} \leq \frac{1}{\epsilon\delta_\epsilon^2} r\big(\frac{\epsilon^{\frac{1}{2}}\delta_\epsilon q_\epsilon}{G_{\epsilon_1}G_\epsilon}\big) G_{\epsilon_1}G_\epsilon. \tag{2.44}$$

Note that

$$\frac{\mathcal{M}_1\mathcal{M}G_{\epsilon_1}G_\epsilon}{N_\epsilon^{abs}} \to \mathcal{M}_1\mathcal{M} \text{ in } L^1_{loc}(dt; L^1(bd\omega dv_1 dv dx))\,, \tag{2.45}$$

and

$$\frac{\mathcal{M}_1\mathcal{M}q_\epsilon}{N_\epsilon^{abs}} = \mathcal{M}_1\mathcal{M}\frac{q_\epsilon}{N_\epsilon}\frac{N_\epsilon}{N_\epsilon^{abs}} \to \mathcal{M}_1\mathcal{M}q \text{ in } w\text{-}L^1_{loc}(dt; w\text{-}L^1(bd\omega dv_1 dv dx)) \tag{2.46}$$

by Product Limit Theorem.

The w-L^1 limit of the left side of inequality (2.44) can be evaluated using the strong L^∞ limits

$$\frac{\mathcal{M}_1\mathcal{M}}{\epsilon\delta_\epsilon^2} r(\epsilon^{\frac{1}{2}}\delta_\epsilon q)\frac{G_{\epsilon_1}G_\epsilon}{N_\epsilon^{abs}}1_{|q|<\lambda} \to \mathcal{M}_1\mathcal{M}q^2 1_{|q|<\lambda},$$

$$\mathcal{M}_1\mathcal{M}\frac{1}{\epsilon^{\frac{1}{2}}\delta_\epsilon} r'(\epsilon^{\frac{1}{2}}\delta_\epsilon q)\Big(\frac{q_\epsilon}{G_{\epsilon_1}G_\epsilon} - q\Big)\frac{G_{\epsilon_1}G_\epsilon}{N_\epsilon^{abs}}1_{|q|<\lambda} \to 2\mathcal{M}_1\mathcal{M}q 1_{|q|<\lambda},$$

and the w-L^1 limit

$$\mathcal{M}_1\mathcal{M}\Big(\frac{q_\epsilon}{G_{\epsilon_1}G_\epsilon} - q\Big)\frac{G_{\epsilon_1}G_\epsilon}{N_\epsilon^{abs}}1_{|q|<\lambda} = \mathcal{M}_1\mathcal{M}\Big(\frac{q_\epsilon}{N_\epsilon^{abs}} - q\frac{G_{\epsilon_1}G_\epsilon}{N_\epsilon^{abs}}\Big)1_{|q|<\lambda} \to 0\,.$$

Taking $\lambda \to +\infty$ then provides the estimate needed in (2.42) to complete the proof of assertion (ii).

Proposition 2.2.5 *$\mathcal{M}(1+|v|^2+|x|^2)\frac{1}{\delta_\epsilon\epsilon^{\frac{1}{2}}}\frac{Q(G_\epsilon, G_\epsilon)}{N_\epsilon}$ is relatively compact in w-$L^1_{loc}(dt; w\text{-}L^1(dv_1 dx))$, where*

$$Q(G,G) = \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} (G_1'G' - G_1 G)b(\omega, v_1 - v)\mathrm{d}\omega\mathcal{M}_1\mathrm{d}v_1\,.$$

Proof. Observe that for any $\chi \in L^\infty_{loc}(dt; L^\infty(dvdx))$,

$$\begin{aligned}&\int_{t_1}^{t_2}\iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} \chi(1+|v|^2+|x|^2)\mathcal{M}\frac{1}{\delta_\epsilon^2\epsilon^{\frac{1}{2}}}\frac{Q(G_\epsilon,G_\epsilon)}{N_\epsilon}\,\mathrm{d}\omega\mathrm{d}x\mathrm{d}t\\ &= \int_{t_1}^{t_2}\iiiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D\times\mathbb{R}^D} \chi(1+|v|^2+|x|^2)\frac{q_\epsilon}{N_\epsilon}\mathcal{M}_1\mathcal{M}b\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{2.47}$$

We get the desired property by applying Proposition 2.2.4 to the right side.

The Infinitesimal Maxwellian

Proposition 2.2.6 *Let $\mathcal{M}g$ be the limit of a convergent sequence of $\mathcal{M}g_\epsilon$ in w-$(L^1_{loc}dt; w$-$(L^1(dxdt)))$. Assume that g_ϵ satisfies the entropy inequality and bound (2.30). Then, for almost every (t,x), g is of the form*

$$g = \rho(t,x) + u(t,x)\cdot(v-U) + \tfrac{1}{2}\theta(t,x)(|v-U|^2 - D\vartheta)\,.$$

Moreover,

$$(\rho\varrho, u\varrho\vartheta, \theta\varrho\vartheta^2) \in (L^\infty(dt; L^2(\tfrac{1}{\varrho}dvdx)), L^\infty(dt; L^2(\tfrac{1}{\varrho\vartheta}dvdx)), L^\infty(dt; L^2(\tfrac{1}{\varrho\vartheta^2}dvdx)))\,.$$

Proof. We start with an observation on the linearized collision kernel $\mathcal{L}_\mathcal{M}g_\epsilon$.

$$\begin{aligned}\mathcal{M}\mathcal{L}_\mathcal{M}g_\epsilon &= \mathcal{M}\iint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^\mathrm{D}} (g_\epsilon + g_{\epsilon_1} - g'_\epsilon - g'_{\epsilon_1})b(v_1-v)\mathcal{M}_1\,\mathrm{d}\omega\mathrm{d}v_1 \\ &= \mathcal{M}\delta_\epsilon Q(g_\epsilon,g_\epsilon) - \mathcal{M}\frac{1}{\delta_\epsilon}Q(G_\epsilon,G_\epsilon).\end{aligned} \tag{2.48}$$

Recall that

$$\begin{aligned}Q(g_\epsilon,g_\epsilon) &= \iint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^\mathrm{D}} (g'_{\epsilon_1}g'_\epsilon - g_{\epsilon_1}g_\epsilon)b(\omega,v_1-v)\,\mathrm{d}\omega\mathcal{M}_1\mathrm{d}v_1,\\ Q(G_\epsilon,G_\epsilon) &= \iint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^\mathrm{D}} (G'_{\epsilon_1}G'_\epsilon - G'_{\epsilon_1}G_\epsilon)b(\omega,v_1-v)\,\mathrm{d}\omega\mathcal{M}_1\mathrm{d}v_1.\end{aligned}$$

The idea is to show weak L^1 convergence of the left side to $\mathcal{M}\mathcal{L}_\mathcal{M}g$, as well as weak L^1 convergence of the right side to 0. To prove weak convergence, we use a modified form of (2.48):

$$\begin{aligned}&\frac{\mathcal{M}}{N^{abs}_\epsilon}\iint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^\mathrm{D}} \frac{1}{N^{abs}_{\epsilon_1}}(g_\epsilon + g_{\epsilon_1} - g'_\epsilon - g'_{\epsilon_1})b(\omega,v_1-v)\mathcal{M}_1\,\mathrm{d}\omega\mathrm{d}v_1\\ =&\frac{\mathcal{M}\delta_\epsilon}{N^{abs}_\epsilon}\iint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^\mathrm{D}} \frac{1}{N^{abs}_{\epsilon_1}}(g'_{\epsilon_1}g'_\epsilon - g_{\epsilon_1}g_\epsilon)b(\omega,v_1-v)\mathcal{M}_1\,\mathrm{d}\omega\mathrm{d}v_1\\ &-\frac{\mathcal{M}}{N^{abs}_\epsilon\delta_\epsilon}\iint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^\mathrm{D}} \frac{1}{N^{abs}_{\epsilon_1}}(G'_{\epsilon_1}G'_\epsilon - G_{\epsilon_1}G_\epsilon)b(\omega,v_1-v)\mathcal{M}_1\,\mathrm{d}\omega\mathrm{d}v_1.\end{aligned} \tag{2.49}$$

Lemma 2.2.1 *Let $\mathcal{M}g$ be the limit of a convergent sequence of $\mathcal{M}g_\epsilon$ in w-$(L^1_{loc}dt; w\text{-}(L^1(dxdt)))$, then the left side of (2.49) converges weakly to $\mathcal{M}\mathcal{L}_{\mathcal{M}}g$ in $L^1_{loc}(dt; L^1(dvdx))$.*

Proof. The bound on the Boltzmann collision kernel implies

$$\int_{\mathbb{S}^{D-1}} bdw \leq C(1+|v|^2)^\beta(1+|v_1|^2)^\beta \leq C(1+|v|^2)(1+|v_1|^2). \tag{2.50}$$

Weak convergence of the first part $\frac{\mathcal{M}}{N^{abs}_\epsilon}\iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D}\frac{1}{N^{abs}_{\epsilon_1}}g_\epsilon b(v_1-v)\mathcal{M}_1\,\mathrm{d}\omega\mathrm{d}v_1$ then follows from product limit theorem, since

$$\frac{1}{N^{abs}_\epsilon}\frac{1}{1+|v|^2}\iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D}\frac{1}{N^{abs}_{\epsilon_1}}\mathcal{M}_1\,b\,\mathrm{d}\omega dv \in L^\infty_{loc}(dt; L^\infty(dvdx))$$

converges almost everywhere, and $(1+|v|^2)\mathcal{M}g_\epsilon$ converges in w-$(L^1_{loc}dt; w\text{-}(L^1(dxdv)))$. We can take care of the other terms, using the invariance of $b\,d\omega dv_1 dv$ under the changes $(v,v_1)\leftrightarrow(v',v_1')$, $(v,v')\leftrightarrow(v_1,v_1')$.

For the right side of (2.49), we observe that the first term

$$\frac{\mathcal{M}\delta_\epsilon}{N^{abs}_\epsilon}\iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D}(g'_{\epsilon_1}g'_\epsilon - g_{\epsilon_1}g_\epsilon)\frac{\mathcal{M}_1}{N^{abs}_{\epsilon_1}}b\,\mathrm{d}\omega\mathrm{d}v_1$$

converges weakly to 0 in $L^1_{loc}(dt; L^1(dvdx))$ by another application of the Product Limit Theorem. In fact, $(1+|v|^2)g_\epsilon\frac{\mathcal{M}}{N^{abs}_\epsilon}$ converges weakly in L^1 and $\iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D}\frac{\delta_\epsilon g_{\epsilon_1}}{N^{abs}_{\epsilon_1}}\frac{\mathcal{M}_1}{(1+|v|^2)}b\,\mathrm{d}\omega\mathrm{d}v_1$ converges to 0 in measure.

It remains to verify the weak converges of the second term

$$\frac{\mathcal{M}}{N^{abs}_\epsilon\delta_\epsilon}\iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D}\frac{\mathcal{M}_1}{N^{abs}_{\epsilon_1}}(G'_{\epsilon_1}G'_\epsilon - G_{\epsilon_1}G_\epsilon)b(\omega, v_1-v)\,\mathrm{d}\omega\mathrm{d}v_1\,.$$

A modification of Proposition 2.2.5 gives

$$\frac{\mathcal{M}}{\delta_\epsilon}\iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D}\frac{\mathcal{M}_1}{N^{abs}_{\epsilon_1}}(G'_{\epsilon_1}G'_\epsilon - G_{\epsilon_1}G_\epsilon)b(\omega, v_1-v)\,\mathrm{d}\omega\mathrm{d}v_1 = O(\epsilon)\,\mathrm{in}\,L^1_{loc}(dt; L^1(dvdx))\,.$$

Therefore, the second term converges to 0 in $L^1_{loc}(dt; L^1(dvdx))$ since $\frac{1}{N_\epsilon^{abs}}$ is bounded. Assertion (ii) of Proposition 2.2.2 states that $\int g^2 \mathcal{M}\, dvdx \in L^\infty(dt)$, so that

$$(\rho\varrho, u\varrho\vartheta, \theta\varrho\vartheta^2) \in (L^\infty(dt; L^2(\frac{1}{\varrho}dvdx)), L^\infty(dt; L^2(\frac{1}{\varrho\vartheta}dvdx)), L^\infty(dt; L^2(\frac{1}{\varrho\vartheta^2}dvdx)))\,.$$

We then proved Proposition 2.2.6.

2.3 The Weak Acoustic Limit Theorem

We now state our main result for the acoustic limit.

Theorem 2.3.1 *Let b be a collision kernel that satisfies the assumption of DiPerna-Lions (4.110). In addition, assume that there exists constants $C_b \in (0,\infty)$ and $\beta \in [0,1]$ such that b satisfies*

$$\int_{\mathbb{S}^{\mathrm{D}-1}} b(\omega, v)\mathrm{d}\omega \le C_b(1 + \tfrac{1}{2}|v|^2)^\beta \quad \textit{almost everywhere}. \tag{2.51}$$

Let $G_\epsilon^{\mathrm{in}} \ge 0$ be a family such that $\iint G_\epsilon^{\mathrm{in}}\,\mathrm{d}v\mathrm{d}x < \infty$ and satisfies the entropy bound $H(G_\epsilon^{\mathrm{in}}) \le C^{\mathrm{in}}\delta_\epsilon^2$ for some $C^{\mathrm{in}} < \infty$ and $\delta_\epsilon > 0$ that satisfies

$$\delta_\epsilon \to 0 \quad \textit{and} \quad \frac{\delta_\epsilon}{\epsilon^{\frac{1}{2}}}|\log(\delta_\epsilon)|^{\beta/2} \to 0 \quad \textit{as } \epsilon \to 0 \tag{2.52}$$

for the β that arises in condition (2.51).

Assume, moreover, that for some

$$\left(\rho^{\mathrm{in}}\varrho, u^{\mathrm{in}}\varrho\vartheta, \theta^{\mathrm{in}}\varrho\vartheta^2\right) \in \Big(L^\infty(dt; L^2(\tfrac{1}{\varrho}dvdx)), L^\infty(dt; L^2(\tfrac{1}{\varrho\vartheta}dvdx)),$$
$$L^\infty(dt; L^2(\tfrac{1}{\varrho\vartheta^2}dvdx))\Big)\,,$$

the family of fluctuations g_ϵ^{in} satisfies

$$\Big(\rho^{\mathrm{in}}\varrho^{\mathrm{in}}, \rho^{\mathrm{in}}U\varrho^{\mathrm{in}} + u^{\mathrm{in}}\varrho^{\mathrm{in}}\vartheta^{\mathrm{in}}, \tfrac{D\vartheta^{\mathrm{in}}}{2}\rho^{\mathrm{in}}\varrho^{\mathrm{in}} + \tfrac{1}{2}|U|^2\rho^{\mathrm{in}}\varrho^{\mathrm{in}} + (\sum_{i=1}^{\mathrm{D}} U_i u_i^{\mathrm{in}})\varrho^{\mathrm{in}}\vartheta^{\mathrm{in}} + \tfrac{\mathrm{D}}{2}\theta^{\mathrm{in}}\varrho^{\mathrm{in}}{\vartheta^{\mathrm{in}}}^2\Big)$$
$$= \lim_{\epsilon\to 0}\Big(\int g_\epsilon^{\mathrm{in}}\mathcal{M}^{\mathrm{in}}\,\mathrm{d}v, \int v g_\epsilon^{\mathrm{in}}\mathcal{M}^{\mathrm{in}}\,\mathrm{d}v, \int \tfrac{1}{2}|v|^2 g_\epsilon^{\mathrm{in}}\mathcal{M}^{\mathrm{in}}\,\mathrm{d}v)\Big) \tag{2.53}$$

*in the sense of distributions. Let G_ϵ be any family of DiPerna-Lions renormalized solutions of the Boltzmann equation (*1.41*) that have G_ϵ^{in} as initial values. Then, as $\epsilon \to 0$, the family of fluctuations g_ϵ^{in} satisfies*

$$\sigma\mathcal{M}g_\epsilon \to \sigma\mathcal{M}(\rho + u\cdot(v-U) + \tfrac{1}{2}\theta(|v-U|^2 - D\vartheta))\,in\,w\text{-}(L^1_{loc}dt; w\text{-}(L^1(dxdv))), \tag{2.54}$$

*where $(\rho\varrho, u\varrho\vartheta, \theta\varrho\vartheta^2) \in C([0,\infty); L^2(\frac{1}{\varrho}dvdx)), L^\infty(dt; L^2(\frac{1}{\varrho\vartheta}dvdx)),$ $L^\infty(dt; L^2(\frac{1}{\varrho\vartheta^2}dvdx)))$ is the unique solution of the acoustic system (*2.25*) with initial data $(\rho^{\mathrm{in}}\varrho, u^{\mathrm{in}}\varrho\vartheta, \theta^{\mathrm{in}}\varrho\vartheta^2)$. In addition, one has that*

$$\begin{aligned}&\Big(\int g_\epsilon\mathcal{M}\,\mathrm{d}v, \int v g_\epsilon\mathcal{M}\,\mathrm{d}v - U\int g_\epsilon\mathcal{M}\,\mathrm{d}v,\\ &\int \tfrac{1}{2}|v|^2 g_\epsilon\mathcal{M}\,\mathrm{d}v - \tfrac{\mathrm{D}\vartheta}{2}\int g_\epsilon\mathcal{M}^{\mathrm{in}}\,\mathrm{d}v - \tfrac{1}{2}\sum_{i=1}^{\mathrm{D}} U_i\int v_i g_\epsilon\mathcal{M}^{\mathrm{in}}\,\mathrm{d}v\Big)\\ \to&\Big(\rho\varrho, \rho U\varrho + u\varrho\vartheta, \tfrac{D\vartheta}{2}\rho\varrho + \tfrac{1}{2}|U|^2\rho\varrho + (\sum_{i=1}^{\mathrm{D}} U_i u_i)\varrho\vartheta + \tfrac{\mathrm{D}}{2}\theta\varrho\vartheta^2\Big)\end{aligned} \tag{2.55}$$

in $C([0,\infty); w\text{-}L^1(dx))$.

2.4 Proof of the Weak Acoustic Limit Theorem

2.4.1 Approximate Local Conservation Laws

All that remains to be done to establish (2.54) is to show that $(\rho\varrho, u\varrho\vartheta, \theta\varrho\vartheta^2)$ is the aforementioned weak solution of the acoustic system by passing to the limit in approximate local conservation laws built from the renormalized Boltzmann equation (1.54). We choose to use the normalization of that equation given by

$$\Gamma(Z) = 3\log(\frac{2}{3} + \frac{1}{3}Z), \quad N(Z) = \frac{2}{3} + \frac{1}{3}Z. \tag{2.56}$$

Dividing equation by δ_ϵ, we get

$$\partial_t \gamma_\epsilon + v \cdot \nabla_x \gamma_\epsilon = \frac{1}{\epsilon^{\frac{1}{2}}} \iint \frac{q_\epsilon}{N_\epsilon} \mathcal{M}_1 b(\omega, v_1 - v)\, \mathrm{d}\omega \mathrm{d}v_1 \tag{2.57}$$

where

$$\gamma_\epsilon = \frac{3}{\delta_\epsilon} \log(1 + \frac{1}{3}\delta_\epsilon g_\epsilon), \quad N_\epsilon = 1 + \frac{1}{3}\delta_\epsilon g_\epsilon. \tag{2.58}$$

When the moment of the renormalized Boltzmann equation (2.57) is formally taken with respect to any $\zeta \in \mathrm{Span}\{1, v_1, ...v_D, |v|^2\}$, one obtains

$$\partial_t \int \zeta \gamma_\epsilon \mathcal{M}\, \mathrm{d}v + \nabla_x \cdot \int v \zeta \gamma_\epsilon \mathcal{M}\, \mathrm{d}v = \frac{1}{\epsilon^{\frac{1}{2}}} \iiint \zeta \frac{q_\epsilon}{N_\epsilon} \mathcal{M}_1 \mathcal{M} b\, \mathrm{d}\omega \mathrm{d}v_1 \mathrm{d}v. \tag{2.59}$$

It can be shown from (1.56) that every DiPerna-Lions solution of (2.57) satisfies (2.59) in the sense that for all $\chi \in C_0^1(\mathbb{R}^{\mathrm{D}})$ and $[t_1, t_2] \subset [0, \infty)$ it satisfies

$$\begin{aligned} &\int \chi \int \zeta \gamma_\epsilon(t_2) \mathcal{M}\, \mathrm{d}v \mathrm{d}x - \int \chi \int \zeta \gamma_\epsilon(t_1) \mathcal{M}\, \mathrm{d}v \mathrm{d}x \\ &= \int_{t_1}^{t_2} \int \nabla_x \chi \cdot \int v \zeta \gamma_\epsilon \mathcal{M}\, \mathrm{d}v \mathrm{d}x \mathrm{d}t \\ &+ \int_{t_1}^{t_2} \chi \frac{1}{\epsilon^{\frac{1}{2}}} \iiint \zeta \frac{q_\epsilon}{N_\epsilon} \mathcal{M}\mathcal{M}_1 b\, \mathrm{d}\omega \mathrm{d}v_1 \mathrm{d}v \mathrm{d}x \mathrm{d}t. \end{aligned} \tag{2.60}$$

We observe that (2.60) still holds if we replace $\zeta \in \mathrm{Span}\{1, v_1, ...v_D, |v|^2\}$ by

$$\tilde{\zeta} \in \mathrm{Span}\{1, x_1, ...x_D, |x|^2\}.$$

2.4.2 Removal of the Conservation Defect

The fact that the conservation defect term on the right-hand side of (2.60) vanishes as $\epsilon \to 0$ follows from the scaling assumption (2.52), the fact χ is bounded, the fact ζ is a collision invariant, and the estimate

$$\frac{1}{\epsilon^{\frac{1}{2}}}\iiint \zeta \frac{q_\epsilon}{N_\epsilon}\mathcal{M}_1\mathcal{M}b\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v \;=\; O(\frac{\delta_\epsilon|\log(\delta_\epsilon)|^{\beta/2}}{\epsilon^{\frac{1}{2}}}) + O(\delta_\epsilon|\log(\delta_\epsilon)|) \qquad (2.61)$$

in $L^1_{loc}(dt; L^1(dx))$ as $\epsilon \to 0$. Given this estimate, the argument is as follows: The scaling assumption (2.52) directly implies that the first term on the right-hand side of (2.61) vanishes as $\epsilon \to 0$. The second term also vanishes as $\epsilon \to 0$. All that remains is to establish the estimate (2.61), but this follows from Theorem 2.5.1. The whole argument still holds if ζ is replaced by $\tilde{\zeta}$.

2.4.3 Control of the Flux

The flux term on the right-hand side of (2.60) contains the sequence $\int v\zeta\gamma_\epsilon\mathcal{M}\,dv$. To control this term, first observe that when one sets $z = \frac{1}{3}\delta_\epsilon g_\epsilon$ in the elementary inequality

$$(\log(1+z))^2 \le \frac{z^2}{1+z}, \quad z > -1,$$

one obtains $\gamma_\epsilon^2 \le \dfrac{g_\epsilon^2}{N_\epsilon}$. The nonlinear bound in Proposition (2.2.3) then shows that

$$\iint \gamma_\epsilon^2\mathcal{M}\,\mathrm{d}v\mathrm{d}x \le \iint \frac{g_\epsilon^2}{N_\epsilon}\mathcal{M}\,\mathrm{d}v\mathrm{d}x \le 2C^{\mathrm{in}}, \quad \text{for all } t \ge 0.$$

Note that $\int v^2\zeta^2\mathcal{M}\,\mathrm{d}v \le C$ uniformly in t, $\int v^2\tilde{\zeta}^2\mathcal{M}\,\mathrm{d}v \in L^\infty_{loc}(dt)$. Then, $\int v\zeta\gamma_\epsilon\mathcal{M}dv$ is relatively compact in w-$L^1_{loc}(dt; w\text{-}L^2(dx))$. In fact, for any $X \in L^2(dx)$,

$$\begin{aligned}
&\int X(\int v\zeta\gamma_\epsilon\mathcal{M}\,\mathrm{d}v)\,\mathrm{d}x\\
\le&(\int X^2\mathrm{d}x)^{\frac{1}{2}}(\int(\int v\zeta\gamma_\epsilon\mathcal{M}\,\mathrm{d}v)^2\,\mathrm{d}x)^{\frac{1}{2}}\\
\le&(\int X^2\mathrm{d}x)^{\frac{1}{2}}(\iint v^2\zeta^2\mathcal{M}\,\mathrm{d}v)(\int \gamma_\epsilon^2\mathcal{M}\,\mathrm{d}v)\,\mathrm{d}x)^{\frac{1}{2}}\\
\le&C(\int X^2\mathrm{d}x)^{\frac{1}{2}}\iint \gamma_\epsilon^2\mathcal{M}\,\mathrm{d}v\mathrm{d}x \le (2C^{\mathrm{in}})^{\frac{1}{2}}C(\int X^2\mathrm{d}x)^{\frac{1}{2}}.
\end{aligned}$$

Here C is the bound for $(\iint v^2\zeta^2\mathcal{M}\,\mathrm{d}v)^{\frac{1}{2}}$, which is independent of t. Hence we the proof is complete by applying Dunford-Pettis Criterion.

Moreover, $\int v\tilde{\zeta}\gamma_\epsilon\mathcal{M}dv$ is also relatively compact in w-$L^1_{loc}(dt; w\text{-}L^2(dx))$. To this end, we observe that for any $X \in L^2(dx)$,

$$\int X(\int v\tilde{\zeta}\gamma_\epsilon\mathcal{M}\,\mathrm{d}v)\,\mathrm{d}x \le (\iint X^2\tilde{\zeta}^2\mathcal{M}\,\mathrm{d}v\mathrm{d}x)^{\frac{1}{2}}(\iint v^2\gamma_\epsilon^2\mathcal{M}\,\mathrm{d}v\mathrm{d}x)^{\frac{1}{2}}\,.$$

But $\int\tilde{\zeta}^2\mathcal{M}\,\mathrm{d}v \in L^\infty_{loc}(dt)$, hence we get relative compactness by the same argument as above.

2.4.4 Control of the Density Terms

The density terms on the left-hand side of (2.60) contain the sequence $\int\zeta\gamma_\epsilon\mathcal{M}\,dv$. We use the Arzela-Ascoli theorem to establish that this sequence is relatively compact in $C([0,\infty); w\text{-}L^2(dx))$. First, for any $Y \in C^1_0(\mathbb{R}^{\mathrm{D}}), \int Y\int\zeta\gamma_\epsilon(t)\mathcal{M}\,dvdx$ is bounded in $C([0,\infty)$ by the same argument in the previous subsection. For equicontinuity, the left-hand side of (2.60) is controlled by

$$\left|\int_{t_1}^{t_2}\int\nabla_x Y\cdot\int v\zeta\gamma_\epsilon\mathcal{M}\,dvdxdt\right| + \left|\int_{t_1}^{t_2}\int Y\frac{1}{\epsilon^{\frac{1}{2}}}\iiint\zeta\frac{q_\epsilon}{N_\epsilon}\mathcal{M}\mathcal{M}_1 b\,dwdv_1dvdxdt\right|\,.$$

The first term can be controlled by $C||\nabla_x Y||_{L^\infty}|t_2-t_1|$, when the second term can be controlled by $C|t_2-t_1|$ following from the arguments in "removal of conservation defects". Hence we get equicontinuity. By a density argument, we see that

$$\int\zeta\gamma_\epsilon\mathcal{M}\,\mathrm{d}v\,\text{is relatively compact in}\,C([0,\infty); w\text{-}L^2(dx)). \tag{2.62}$$

Following the same argument, we also have

$$\int\zeta\gamma_\epsilon\mathcal{M}\,\mathrm{d}v\,\text{is relatively compact in}\,C([0,\infty); w\text{-}L^2(dx)). \tag{2.63}$$

2.4.5 Passing to the Limit

Proposition 2.2.2 (i) allows us to pass to a subsequence of the sequence $\mathcal{M}g_\epsilon$, still abusively denoted $\mathcal{M}g_\epsilon$, such that as $\epsilon \to 0$

$$\sigma\mathcal{M}g_\epsilon \to \sigma\mathcal{M}g \,\text{in}\, w\text{-}L^1_{loc}(dt; w\text{-}L^1(dvdx)) \,\text{as}\, \epsilon \to 0. \tag{2.64}$$

Now observe that when one sets $z = \frac{1}{3}\delta_\epsilon g_\epsilon$ in the elementary inequalities

$$0 \le z - \log(1+z) \le \frac{z^2}{1+z}. \,\text{for all}\, z > -1,$$

one obtains

$$0 \le g_\epsilon - \gamma_\epsilon \le \frac{1}{3}\delta_\epsilon \frac{g_\epsilon^2}{N_\epsilon}.$$

Since $\sigma\mathcal{M}\frac{g_\epsilon^2}{N_\epsilon} = O(|\log(\delta_\epsilon)|)$ in $L^\infty(dt; L^1(dvdx))$ as $\epsilon \to 0$, we have $\sigma g_\epsilon \mathcal{M} - \sigma\gamma_\epsilon\mathcal{M} \to 0$ in $L^\infty(dt; L^1(dvdx))$ as $\epsilon \to 0$. From Proposition 2.2.2 (ii), we have

$$\frac{1}{2}\iint g(t)^2\mathcal{M}\, dvdx \le \liminf_{\epsilon\to 0} \frac{1}{\delta_\epsilon^2} H(G_\epsilon(t)).$$

Therefore,

$$\iint (\gamma_\epsilon - g)^2 \mathcal{M}\, dvdx \in L^\infty(dt).$$

By the same argument as in 'Control of the flux', we have

$$\iiint |Y(\gamma_\epsilon - g)\mathcal{M}|\, \mathrm{d}v\mathrm{d}x\mathrm{d}t \to 0, \,\text{for all}\, Y \,\text{such that}\, Y^2\mathcal{M} \in L^\infty_{loc}(dt; L^1(dvdx))\,. \tag{2.65}$$

Moreover, if the Y in (2.65) is independent of t, then

$$\iint |Y(\gamma_\epsilon - g)\mathcal{M}|\, \mathrm{d}v\mathrm{d}x \to 0 \,\text{in}\, L^\infty(dt) \,\text{for all}\, Y \,\text{such that}\, Y^2\mathcal{M} \in L^\infty_{loc}(dt; L^1(dvdx)). \tag{2.66}$$

Now we will be able to show that

$$\iint X v \zeta (\gamma_\epsilon - g) \mathcal{M} \, \mathrm{d}v \mathrm{d}x \to 0, \text{ for all } X \in L^\infty_{loc}(dt; L^2(dx)). \tag{2.67}$$

Choose $Y = Xv\zeta$ in (2.65) for any $X \in C_0^1(\mathbb{R}^{\mathrm{D}})$ and by a density argument, we obtain

$$\int v \zeta \gamma_\epsilon \mathcal{M} \, \mathrm{d}v \rightharpoonup \int v \zeta g \mathcal{M} \, \mathrm{d}v \, \mathrm{in} \, w\text{-}L^1_{loc}(dt; w\text{-}L^2(dx)). \tag{2.68}$$

On the other hand, we can show that

$$\sup_{t>0} \left| \iint X \zeta (\gamma_\epsilon - g) \mathcal{M} \, dv dx \right| \to 0, \text{ for all } X \in L^2(dx) \tag{2.69}$$

by choosing $Y = X\zeta$ in (2.65) for some $X \in C_0^1(\mathbb{R}^{\mathrm{D}})$ and a standard density argument. Hence

$$\int \zeta \gamma_\epsilon \mathcal{M} \, \mathrm{d}v \to \int \zeta g \mathcal{M} \, \mathrm{d}v \, \mathrm{in} \, C([0,\infty); w\text{-}L^2(dx)). \tag{2.70}$$

Both (2.68, 2.70) still hold if ζ is replaced by $\tilde{\zeta}$. Moreover, because the initial fluctuations g_ϵ^{in} satisfy

$$(\rho^{\mathrm{in}} \varrho^{\mathrm{in}}, \rho^{\mathrm{in}} U \varrho^{\mathrm{in}} + u^{\mathrm{in}} \varrho^{\mathrm{in}} \vartheta^{\mathrm{in}}, \tfrac{\mathrm{D}\vartheta^{\mathrm{in}}}{2} \rho^{\mathrm{in}} \varrho^{\mathrm{in}} + \tfrac{1}{2}|U|^2 \rho^{\mathrm{in}} \varrho^{\mathrm{in}} + (\sum_{i=1}^{\mathrm{D}} U_i u_i^{\mathrm{in}}) \varrho^{\mathrm{in}} \vartheta^{\mathrm{in}} + \tfrac{\mathrm{D}}{2} \theta^{\mathrm{in}} \varrho^{\mathrm{in}} {\vartheta^{\mathrm{in}}}^2)$$

$$= \lim_{\epsilon \to 0} \Big(\int g_\epsilon^{\mathrm{in}} \mathcal{M}^{\mathrm{in}} \, \mathrm{d}v, \int v g_\epsilon^{\mathrm{in}} \mathcal{M}^{\mathrm{in}} \, \mathrm{d}v, \int \tfrac{1}{2}|v|^2 g_\epsilon^{\mathrm{in}} \mathcal{M}^{\mathrm{in}} \, \mathrm{d}v \Big),$$

one sees from

$$\sigma g_\epsilon \mathcal{M} - \sigma \gamma_\epsilon \mathcal{M} \to 0 \, \mathrm{in} \, L^1(dvdx) \, \mathrm{as} \, \epsilon \to 0, \text{ for any } t$$

and

$$\int \sigma (g_\epsilon^{\mathrm{in}} - \gamma^{\mathrm{in}}) \mathcal{M} \, \mathrm{d}v \text{ is bounded in } L^2(dx) \text{ for any } t,$$

that

$$\Big(\int \gamma_\epsilon^{\text{in}}\mathcal{M}^{\text{in}}\,\mathrm{d}v, \int v\gamma_\epsilon^{\text{in}}\mathcal{M}^{\text{in}}\,\mathrm{d}v, \int \tfrac{1}{2}|v|^2\gamma_\epsilon^{\text{in}}\mathcal{M}^{\text{in}}\,\mathrm{d}v\Big) \to \Big(\rho^{\text{in}}\varrho^{\text{in}},$$
$$\rho^{\text{in}}U\varrho^{\text{in}} + u^{\text{in}}\varrho^{\text{in}}\vartheta^{\text{in}}, \tfrac{D\vartheta^{\text{in}}}{2}\rho^{\text{in}}\varrho^{\text{in}} + \tfrac{1}{2}|U|^2\rho^{\text{in}}\varrho^{\text{in}} + (\sum_{i=1}^{\mathrm{D}} U_i u_i^{\text{in}})\varrho^{\text{in}}\vartheta^{\text{in}} + \tfrac{\mathrm{D}}{2}\theta^{\text{in}}\varrho^{\text{in}}{\vartheta^{\text{in}}}^2\Big) \tag{2.71}$$

in w-$L^2(dx)$ as $\epsilon \to 0$, where we define $\gamma_\epsilon^{\text{in}} = \gamma_\epsilon(0)$. Furthermore,

$$\Big(\int \gamma_\epsilon^{\text{in}}\mathcal{M}^{\text{in}}\,\mathrm{d}v, \int v\gamma_\epsilon^{\text{in}}\mathcal{M}^{\text{in}}\,\mathrm{d}v - U\int \gamma_\epsilon^{\text{in}}\mathcal{M}^{\text{in}}\,\mathrm{d}v,$$
$$\int \tfrac{1}{2}|v|^2\gamma_\epsilon^{\text{in}}\mathcal{M}^{\text{in}}\,\mathrm{d}v - \tfrac{\mathrm{D}\vartheta}{2}\int \gamma_\epsilon^{\text{in}}\mathcal{M}^{\text{in}}\ dv - \tfrac{1}{2}\sum_{i=1}^{\mathrm{D}} U_i\int v_i\gamma_\epsilon^{\text{in}}\mathcal{M}^{\text{in}}\,\mathrm{d}v\Big) \tag{2.72}$$
$$\to \Big(\rho^{\text{in}}\varrho^{\text{in}}, u^{\text{in}}\varrho^{\text{in}}\vartheta^{\text{in}}, \tfrac{\mathrm{D}}{2}\theta^{\text{in}}\varrho^{\text{in}}{\vartheta^{\text{in}}}^2\Big).$$

Taking limits in (2.60) as $\epsilon \to 0$ leads to

$$\begin{aligned}&\int \chi\int \zeta g(t_2)\mathcal{M}\,\mathrm{d}v\mathrm{d}x - \int \chi\int \zeta g(t_1)\mathcal{M}\,\mathrm{d}v\mathrm{d}x\\ &= \int_{t_1}^{t_2}\int \nabla_x\chi\cdot\int v\zeta g\mathcal{M}\,\mathrm{d}v\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{2.73}$$

which is the weak form of the local conservation law

$$\partial_t\int \zeta g\mathcal{M}\,\mathrm{d}v + \nabla_x\cdot\int v\zeta g\mathcal{M}\,\mathrm{d}v = 0.$$

Setting $\zeta = 1, v_1, ...v_D$ and $\frac{1}{2}|v|^2$ into this equation and using the infinitesimal Maxwellian form of g gives the resulting system. We then set $\zeta = 1, v_1, ...v_D$ and $\frac{1}{2}|v|^2$ into

$$\int \zeta\gamma_\epsilon\mathcal{M}\,\mathrm{d}v \to \int \zeta g\mathcal{M}\,\mathrm{d}v\,\mathrm{in}\,C([0,\infty); w\text{-}L^1(dx))$$

and combine this with the fact that

$$\int \zeta\gamma_\epsilon\mathcal{M}\,\mathrm{d}v \to \int \tilde{\zeta}g\mathcal{M}\,\mathrm{d}v\,\mathrm{in}\,C([0,\infty); w\text{-}L^1(dx)), \quad \mathrm{for}\,\tilde{\zeta}\in \mathrm{Span}\{1, x_1, ...x_D, |x|^2\},$$

and

$$\sigma\gamma_\epsilon\mathcal{M} - \sigma g_\epsilon\mathcal{M} \to 0 \,\mathrm{in}\, L^1(dvdx) \,\mathrm{as}\, \epsilon \to 0.$$

Therefore,

$$\begin{aligned} &\Big(\int g_\epsilon\mathcal{M}\,\mathrm{d}v, \int v g_\epsilon\mathcal{M}\,\mathrm{d}v - U\int g_\epsilon\mathcal{M}\,\mathrm{d}v, \\ &\int \tfrac{1}{2}|v|^2 g_\epsilon\mathcal{M}\,\mathrm{d}v - \tfrac{\mathrm{D}\vartheta}{2}\int g_\epsilon\mathcal{M}^{\mathrm{in}}\,\mathrm{d}v - \tfrac{1}{2}\sum_{i=1}^{\mathrm{D}} U_i \int v_i g_\epsilon\mathcal{M}^{\mathrm{in}}\,\mathrm{d}v\Big) \\ \to &\Big(\rho\varrho, \rho U\varrho + u\varrho\vartheta, \tfrac{D\vartheta}{2}\rho\varrho + \tfrac{1}{2}|U|^2\rho\varrho + (\sum_{i=1}^{\mathrm{D}} U_i u_i)\varrho\vartheta + \tfrac{\mathrm{D}}{2}\theta\varrho\vartheta^2\Big) \end{aligned} \tag{2.74}$$

in $C([0,\infty); w\text{-}L^1(dx))$. This concludes the proof of the Weak Acoustic Limit Theorem.

2.5 Control of the Conservation Defects

In this section we derive the conservation defect bounds (2.61). We prove the following

Theorem 2.5.1 *Let the collision kernel b satisfy the bound (2.51) for some $\beta \in [0,1]$. Let $\mathcal{M}G_\epsilon$ be a family of functions in $C([0,\infty); w\text{-}L^1(\mathrm{d}v\mathrm{d}x))$ that satisfies the entropy bound (2.29). Let $\zeta \in \mathrm{Span}\{1, v_1, ..., v_D, |v|^2, x_1, ..., x_D, |x|^2\}$. Let $\delta_\epsilon > 0$ vanish as $\epsilon \to 0$. Then*

$$\iiint \zeta \frac{q_\epsilon}{N_\epsilon}\mathcal{M}_1\mathcal{M} b\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v \;=\; O(\delta_\epsilon|\log(\delta_\epsilon)|^{\beta/2}) \;+\; O(\epsilon^{\frac{1}{2}}\delta_\epsilon|\log(\delta_\epsilon)|) \tag{2.75}$$

in $L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}x))$ as $\epsilon \to 0$.

2.5.1 Proof of the Conservation Defect Theorem

We exploit the symmetries of $\mathcal{M}_1\mathcal{M}b\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v$ and the fact that ζ is a collision invariant to decompose the defect into three parts, each of which is then shown to vanish as $\epsilon \to 0$. We use the following decomposition

$$\begin{aligned}&\iiint \zeta\frac{q_\epsilon}{N_\epsilon}b\mathcal{M}_1\mathcal{M}\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v\\ &\quad= \iiint \zeta(1-\frac{1}{N_{\epsilon_1}})\frac{q_\epsilon}{N_\epsilon}b\mathcal{M}_1\mathcal{M}\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v \,+\, \iiint \zeta\frac{q_\epsilon}{N_{\epsilon_1}N_\epsilon}b\mathcal{M}_1\mathcal{M}\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v.\end{aligned} \tag{2.76}$$

Then

$$\begin{aligned}&\iiint \zeta\frac{q_\epsilon}{N_{\epsilon_1}N_\epsilon}b\mathcal{M}_1\mathcal{M}\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v\\ &=\frac{1}{2}\iiint(\zeta_1\,+\,\zeta)\frac{q_\epsilon}{N_{\epsilon_1}N_\epsilon}\mathcal{M}_1\mathcal{M}b\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v\\ &=\frac{1}{4}\iiint(\zeta_1\,+\,\zeta)(\frac{1}{N_{\epsilon_1}N_\epsilon}-\frac{1}{N'_{\epsilon_1}N'_\epsilon})q_\epsilon\mathcal{M}_1\mathcal{M}b\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v\\ &=\frac{1}{4}\iiint(\zeta_1\,+\,\zeta)\frac{N'_{\epsilon_1}N'_\epsilon-N_{\epsilon_1}N_\epsilon}{N_{\epsilon_1}N_\epsilon N'_{\epsilon_1}N'_\epsilon}q_\epsilon\mathcal{M}_1\mathcal{M}b\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v\,.\end{aligned} \tag{2.77}$$

Observe that

$$\begin{aligned}N'_{\epsilon_1}N'_\epsilon-N_{\epsilon_1}N_\epsilon &= \frac{2}{9}\delta_\epsilon(g'_{\epsilon_1}+g'_\epsilon-g_{\epsilon_1}-g_\epsilon)+\frac{1}{9}(G'_{\epsilon_1}G'_\epsilon-G_{\epsilon_1}G_\epsilon)\\ &= -\frac{2}{9}{\delta_\epsilon}^2(g'_{\epsilon_1}g'_\epsilon-g_{\epsilon_1}g_\epsilon)+\frac{1}{3}\epsilon^{\frac{1}{2}}\delta_\epsilon q_\epsilon\,,\end{aligned} \tag{2.78}$$

therefore (2.77) decomposes as

$$\begin{aligned}&\iiint \zeta\frac{q_\epsilon}{N_{\epsilon_1}N_\epsilon}\mathcal{M}_1\mathcal{M}b\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v\\ &=-\frac{1}{18}{\delta_\epsilon}^2\iiint(\zeta_1+\zeta)\frac{g'_{\epsilon_1}g'_\epsilon-g_{\epsilon_1}g_\epsilon}{N'_{\epsilon_1}N'_\epsilon N_{\epsilon_1}N_\epsilon}q_\epsilon\mathcal{M}_1\mathcal{M}b\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v\\ &\quad+\frac{1}{12}\epsilon^{\frac{1}{2}}\delta_\epsilon\iiint(\zeta_1+\zeta)\frac{q_\epsilon^2}{N_{\epsilon_1}N_\epsilon N'_{\epsilon_1}N'_\epsilon}\mathcal{M}_1\mathcal{M}b\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v\,.\end{aligned} \tag{2.79}$$

By the symmetries and the fact that ζ is a collision invariant, the right hand side of (2.79) becomes

$$\begin{aligned}
&\iiint (\zeta_1+\zeta)\frac{g'_{\epsilon_1}g'_\epsilon - g_{\epsilon_1}g_\epsilon}{N'_{\epsilon_1}N'_\epsilon N_{\epsilon_1}N_\epsilon} q_\epsilon \mathcal{M}_1\mathcal{M} b \,\mathrm{d}\omega \mathrm{d}v_1 \mathrm{d}v \\
=&\, 2\iiint (\zeta'_1+\zeta')\frac{g'_{\epsilon_1}g'_\epsilon}{N'_{\epsilon_1}N'_\epsilon N_{\epsilon_1}N_\epsilon} q_\epsilon \mathcal{M}_1\mathcal{M} b \,\mathrm{d}\omega \mathrm{d}v_1 \mathrm{d}v \\
=&\, 4\iiint \frac{\zeta' g'_{\epsilon_1}g'_\epsilon}{N'_{\epsilon_1}N'_\epsilon N_{\epsilon_1}N_\epsilon} q_\epsilon \mathcal{M}_1\mathcal{M} b \,\mathrm{d}\omega \mathrm{d}v_1 \mathrm{d}v, \\
&\iiint (\zeta_1+\zeta)\frac{q_\epsilon^2}{N'_{\epsilon_1}N'_\epsilon N_{\epsilon_1}N_\epsilon} \mathcal{M}_1\mathcal{M} b \,\mathrm{d}\omega \mathrm{d}v_1 \mathrm{d}v \\
=&\, 2\iiint \frac{\zeta q_\epsilon^2}{N'_{\epsilon_1}N'_\epsilon N_{\epsilon_1}N_\epsilon} \mathcal{M}_1\mathcal{M} b \,\mathrm{d}\omega \mathrm{d}v_1 \mathrm{d}v.
\end{aligned} \tag{2.80}$$

Now we arrive at the decomposition

$$\begin{aligned}
&\iiint \zeta \frac{q_\epsilon}{N_\epsilon} b \mathcal{M}_1\mathcal{M} \mathrm{d}\omega \mathrm{d}v_1 \mathrm{d}v \\
=&\, \frac{1}{3}\iiint \zeta \frac{\delta_\epsilon g_{\epsilon_1}}{N_{\epsilon_1}N_\epsilon} q_\epsilon \mathcal{M}_1\mathcal{M} b \,\mathrm{d}\omega \mathrm{d}v_1 \mathrm{d}v \\
&- \frac{2}{9}\iiint \zeta' \frac{\delta_\epsilon^2 g'_{\epsilon_1}g'_\epsilon}{N'_{\epsilon_1}N'_\epsilon N_{\epsilon_1}N_\epsilon} q_\epsilon \mathcal{M}_1\mathcal{M} b \,\mathrm{d}\omega \mathrm{d}v_1 \mathrm{d}v \\
&+ \frac{1}{6}\iiint \zeta \frac{\epsilon^{\frac{1}{2}}\delta_\epsilon^2 q_\epsilon^2}{N'_{\epsilon_1}N'_\epsilon N_{\epsilon_1}N_\epsilon} \mathcal{M}_1\mathcal{M} b \,\mathrm{d}\omega \mathrm{d}v_1 \mathrm{d}v.
\end{aligned} \tag{2.81}$$

The above argument still holds if we replace $\zeta \in \mathrm{Span}\{1, v_1, ..., v_D, |v|^2\}$ by $\tilde{\zeta} \in \mathrm{Span}$ $\{1, x_1, ..., x_D, |x|^2\}$. Because for every ζ or $\tilde{\zeta}$ there exists a constant $C < \infty$ such that $|\zeta| \le C\sigma$, $|\tilde{\zeta}| \le C\sigma$, where $\sigma(v) := 1 + |v|^2 + |x|^2$, the result will follow upon establishing the bounds

$$\sigma \frac{\delta_\epsilon g_{\epsilon_1}}{N_{\epsilon_1}N_\epsilon} q_\epsilon \mathcal{M}_1\mathcal{M} \;=\; O(\delta_\epsilon |\log(\delta_\epsilon)|^{\beta/2}), \tag{2.82}$$

$$\sigma' \frac{{\delta_\epsilon}^2 g'_{\epsilon_1}g'_\epsilon}{N'_{\epsilon_1}N'_\epsilon N_{\epsilon_1}N_\epsilon} q_\epsilon \mathcal{M}_1\mathcal{M} \;=\; O(\delta_\epsilon |\log(\delta_\epsilon)|^{\beta/2}), \tag{2.83}$$

$$\sigma \frac{\epsilon^{\frac{1}{2}} \delta_\epsilon {q_\epsilon}^2}{N'_{\epsilon_1} N'_\epsilon N_{\epsilon_1} N_\epsilon} \mathcal{M}_1 \mathcal{M} \;=\; O(\epsilon^{\frac{1}{2}} \delta_\epsilon |\log(\epsilon^{\frac{1}{2}} \delta_\epsilon)|), \tag{2.84}$$

in $L^1_{loc}(\mathrm{d}t; L^1(b\mathrm{d}\omega \mathrm{d}v_1 \mathrm{d}v \mathrm{d}x))$ as $\epsilon \to 0$ and then observing that

$$\begin{aligned} \epsilon^{\frac{1}{2}} \delta_\epsilon |\log(\epsilon^{\frac{1}{2}} \delta_\epsilon)| \;&\leq\; \epsilon^{\frac{1}{2}} \delta_\epsilon |\log(\epsilon^{\frac{1}{2}})| + \epsilon^{\frac{1}{2}} \delta_\epsilon |\log(\delta_\epsilon)| \\ &=\; O(\delta_\epsilon |\log(\delta_\epsilon)|^{\beta/2}) + O(\epsilon^{\frac{1}{2}} \delta_\epsilon |\log(\delta_\epsilon)|). \end{aligned}$$

The bounds (2.82)-(2.84) follow directly from Lemmas 2.5.1, 2.5.2 and 2.5.3 respectively.

2.5.2 Dissipation Rate Control Lemmas

The proofs of the following lemmas rely on the following control for dissipation rate R, implied from the entropy inequality (1.59)

$$\frac{1}{\epsilon \delta_\epsilon^2} \int_0^\infty R(G_\epsilon)\,\mathrm{d}t \;\leq\; C^{\mathrm{in}}\,.$$

the bound can be recast in terms of R and q_ϵ as

$$\frac{1}{\epsilon \delta_\epsilon^2} \int_0^t \iiiint_{\mathbb{S}^{\mathrm{D}-1} \times \mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}}} \frac{1}{4} r\Big(\frac{q_\epsilon}{G_{\epsilon_1} G_\epsilon} \epsilon^{\frac{1}{2}} \delta_\epsilon \Big) G_{\epsilon_1} G_\epsilon b \mathcal{M}_1 \mathcal{M} \,\mathrm{d}\omega \mathrm{d}v_1 \mathrm{d}v \mathrm{d}x \mathrm{d}s \leq C^{\mathrm{in}}. \tag{2.85}$$

The proof of the next two lemmas are based on the classical Young inequality

$$pz \;\leq\; r^\star(p) + r(z) \;\text{ for every}\, p \in \mathbb{R} \,\text{and}\, z > -1.$$

Upon choosing

$$p \;=\; \frac{\epsilon^{\frac{1}{2}} \delta_\epsilon y}{\gamma} \quad \text{and} \quad z = \frac{\epsilon^{\frac{1}{2}} \delta_\epsilon |q_\epsilon|}{G_{\epsilon_1} G_\epsilon}$$

and noticing that $r(|z|) \leq r(z)$ for every $z > -1$, for every positive γ and y one obtains

$$y|q_\epsilon| \leq \frac{\gamma}{\epsilon\delta_\epsilon^2} r^\star \left(\frac{\epsilon^{\frac{1}{2}}\delta_\epsilon y}{\gamma} \right) G_{\epsilon_1} G_\epsilon + \frac{\gamma}{\epsilon\delta_\epsilon^2} r \left(\frac{\epsilon^{\frac{1}{2}}\delta_\epsilon q_\epsilon}{G_{\epsilon_1} G_\epsilon} \right) G_{\epsilon_1} G_\epsilon \,. \tag{2.86}$$

This inequality will be the starting point for the proofs of Lemmas 2.5.1 and 2.5.2.

Lemma 2.5.1 *Let $\beta, \delta_\epsilon, \epsilon^{\frac{1}{2}}, g_\epsilon$ and N_ϵ be as in Theorem 2.5.1. Then*

$$\sigma \frac{\delta_\epsilon g_{\epsilon_1}}{N_{\epsilon_1} N_\epsilon} q_\epsilon \mathcal{M}_1 \mathcal{M} = O(\delta_\epsilon |\log(\delta_\epsilon)|^{\beta/2}) \; in \, L^1_{loc}(\mathrm{d}t; L^1(b\mathrm{d}\omega \mathrm{d}v_1 \mathrm{d}v \mathrm{d}x)) \; as \quad \epsilon \to 0.$$

Proof. We first set

$$y = \frac{\gamma\sigma}{\delta_\epsilon N_\epsilon} \left| 1 - \frac{1}{N_{\epsilon_1}} \right| = \frac{\gamma\sigma |g_{\epsilon_1}|}{3 N_{\epsilon_1} N_\epsilon} \,, \tag{2.87}$$

and apply the superquadratic property

$$r^\star(\lambda p) \leq \lambda^2 r^\star(p) \quad \text{for every}\, p > 0 \,\text{and}\, \lambda \in [0,1] \,. \tag{2.88}$$

with

$$\lambda = \frac{\epsilon^{\frac{1}{2}}\delta_\epsilon |g_{\epsilon_1}|}{\gamma N_{\epsilon_1} N_\epsilon} \,\text{and}\, p = \frac{\alpha\sigma}{3} \,, \tag{2.89}$$

where $\lambda \leq 1$ whenever $\epsilon^{\frac{1}{2}} \leq \frac{2}{9}\gamma$. This leads to

$$\begin{aligned} &\frac{1}{\delta_\epsilon} \frac{\sigma}{N_\epsilon} \left| 1 - \frac{1}{N_{\epsilon_1}} \right| |q_\epsilon| \\ &\leq \frac{1}{\gamma} \left[\frac{1}{\gamma} \frac{g_{\epsilon_1}^2}{N_{\epsilon_1}^2 N_\epsilon^2} r^\star (\frac{\alpha\sigma}{3} G_{\epsilon_1} G_\epsilon) + \frac{\gamma}{\epsilon\delta_\epsilon^2} r \left(\frac{\epsilon^{\frac{1}{2}}\delta_\epsilon q_\epsilon}{G_{\epsilon_1} G_\epsilon} \right) G_{\epsilon_1} G_\epsilon \right] \\ &\leq \frac{1}{\gamma} \left[\frac{3^3}{2\gamma} \frac{g_{\epsilon_1}^2}{N_{\epsilon_1}} r^\star (\frac{\alpha\sigma}{3} G_{\epsilon_1} G_\epsilon) + \frac{4\gamma}{\epsilon\delta_\epsilon^2} \frac{1}{4} r \left(\frac{\epsilon^{\frac{1}{2}}\delta_\epsilon q_\epsilon}{G_{\epsilon_1} G_\epsilon} \right) G_{\epsilon_1} G_\epsilon \right] . \end{aligned} \tag{2.90}$$

By the assumption on the collision kernel, we have

$$\int b(\omega, v_1 - v) \, \mathrm{d}\omega \leq C_b (1 + |v_1|^2)^\beta (1 + |v|^2)^\beta \,. \tag{2.91}$$

Then

$$\begin{aligned} &\frac{1}{\delta_\epsilon}\int_0^T \iiiint \frac{\sigma}{N_\epsilon}\left|1-\frac{1}{N_{\epsilon_1}}\right| |q_\epsilon|\mathcal{M}_1\mathcal{M}b\,\mathrm{d}\omega \mathrm{d}v_1\mathrm{d}v\mathrm{d}x\mathrm{d}t \\ \le&\frac{1}{\alpha\sigma}\frac{3^3}{2}C_b\int_0^T \iiiint \frac{g_{\epsilon_1}^2}{N_{\epsilon_1}}r^\star(\frac{\alpha\sigma}{3})\sigma_1^\beta\sigma^\beta\mathcal{M}_1\mathcal{M}b\mathrm{d}\omega \mathrm{d}v_1\mathrm{d}v\mathrm{d}x\mathrm{d}t+\frac{4\gamma}{\alpha}C^{\mathrm{in}}\,. \end{aligned} \tag{2.92}$$

Interpolating between the entropy estimates in Corollary 2.2.1(ii) and Proposition 2.2.3, we get

$$\iint \frac{\sigma_1^\beta\mathcal{M}_1 g_{\epsilon_1}^2}{N_{\epsilon_1}}\,\mathrm{d}v_1\mathrm{d}x \;=\; O(|\log(\delta_\epsilon)|^\beta) \tag{2.93}$$

in $L^\infty(\mathrm{d}t)$, while

$$\int (1+|v|^2)^\beta r^\star(\frac{\alpha\sigma}{3})\mathcal{M}\,\mathrm{d}v \quad \text{is uniformly bounded in}\, x\,. \tag{2.94}$$

Hence we got the lemma by optimizing over α and multiplying the result by δ_ϵ.

Lemma 2.5.2 *Let $\beta, \delta_\epsilon, \epsilon^{\frac12}, g_\epsilon$ and N_ϵ be as in Theorem* 2.5.1*. Then*

$$\sigma\frac{\delta_\epsilon g_{\epsilon_1}}{N_{\epsilon_1}N_\epsilon}q_\epsilon\mathcal{M}_1\mathcal{M} \;=\; O(\delta_\epsilon|\log(\delta_\epsilon)|^{\beta/2}) \quad in \quad L^1_{loc}(\mathrm{d}t;L^1(b\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v\mathrm{d}x))\;as \quad \epsilon\to 0.$$

Proof. Set

$$y=\frac{1}{9}\frac{\delta_\epsilon\sigma'|g'_{\epsilon_1}||g'_\epsilon|}{N'_{\epsilon_1}N'_\epsilon N_{\epsilon_1}N_\epsilon} \tag{2.95}$$

in Young's inequality (2.86). Then apply the superquadratic property with

$$\lambda=\frac{1}{3}\frac{\epsilon^{\frac12}\delta_\epsilon^2|g'_{\epsilon_1}||g'_\epsilon|}{\gamma N'_{\epsilon_1}N'_\epsilon N_{\epsilon_1}N_\epsilon} \quad \text{and}\, p=\frac{\sigma'\alpha}{3}. \tag{2.96}$$

Note that $\lambda\le 1$ whenever $\epsilon^{\frac12}\le\frac{4}{27}\sigma$.

Therefore

$$\begin{aligned} &\frac{1}{9}\frac{\delta_\epsilon\sigma'|g'_{\epsilon_1}||g'_\epsilon|}{N'_{\epsilon_1}N'_\epsilon N_{\epsilon_1}N_\epsilon}|q_\epsilon| \\ \le&\frac{1}{\alpha}\left[\frac{1}{3^2\gamma}\frac{\delta_\epsilon^2 g'^2_{\epsilon_1}g'^2_\epsilon}{(N'_{\epsilon_1}N'_\epsilon N_{\epsilon_1}N_\epsilon)^2}r^\star(\frac{\alpha\sigma'}{3})G_{\epsilon_1}G_\epsilon+\frac{\sigma}{\epsilon\delta_\epsilon^2}r\left(\frac{\epsilon^{\frac12}\delta_\epsilon g_\epsilon}{G_{\epsilon_1}G_\epsilon}\right)G_{\epsilon_1}G_\epsilon\right] \\ \le&\frac{9^2}{8^2\alpha\sigma}\frac{g'^2_{\epsilon_1}}{N_{\epsilon_1}}r^\star(\frac{\alpha\sigma}{3})+\frac{4\gamma}{\alpha\epsilon\delta_\epsilon^2}\frac{1}{4}r\left(\frac{\epsilon\delta_\epsilon g_\epsilon}{G_{\epsilon_1}G_\epsilon}\right)G_{\epsilon_1}G_\epsilon\,. \end{aligned} \tag{2.97}$$

Multiplying the inequality by $\mathcal{M}_1\mathcal{M}$ and integrating both sides with respect to $b\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v\mathrm{d}x\mathrm{d}t$, we get

$$\begin{aligned}&\frac{1}{9}\int_0^T\iiiint \frac{\delta_\epsilon\sigma'|q'_{\epsilon_1}||q'_\epsilon|}{N'_{\epsilon_1}N'_\epsilon N_{\epsilon_1}N_\epsilon}|q_\epsilon|\mathcal{M}_1\mathcal{M}b\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v\mathrm{d}x\mathrm{d}t\\ \leq&\frac{3^5}{2^7\alpha\sigma}C_0\int_0^T\iiint\frac{(1+|v'_1|^2)^\beta(1+|v'|^2)^\beta g'^\beta_{\epsilon_1}}{N'_{\epsilon_1}}r^\star(\frac{\alpha\sigma'}{3})\mathcal{M}_1\mathcal{M}b\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v\mathrm{d}x\mathrm{d}t+\frac{4\gamma}{\alpha}C^{\mathrm{in}}.\end{aligned} \tag{2.98}$$

The first term of the right hand side is bounded because

$$\int(1+|v|^2)^\beta r^\star(\frac{\alpha\sigma}{3})\mathcal{M}\,\mathrm{d}v\quad\text{is uniformly bounded in}\,x\,,$$

and

$$\iiint\frac{(1+|v_1|^2)^\beta g_\epsilon^2}{N_{\epsilon_1}}\,\mathrm{d}v_1\mathrm{d}x\mathrm{d}t=O(|\log(\delta_\epsilon)|^\beta)\,.$$

We then get Lemma 2.5.2 by optimizing over α and multiplying the result by δ_ϵ.

Lemma 2.5.3 *Let $\beta,\delta_\epsilon,\epsilon^{\frac{1}{2}},g_\epsilon$ and N_ϵ be as in Theorem* 2.5.1*. Then*

$$(\sigma+\sigma_1)\frac{{q_\epsilon}^2}{N'_{\epsilon_1}N'_\epsilon N_{\epsilon_1}N_\epsilon}\mathcal{M}_1\mathcal{M}\;=\;O(|\log(\epsilon^{\frac{1}{2}}\delta_\epsilon)|)\,in\,L^1_{loc}(\mathrm{d}t;L^1(b\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v\mathrm{d}x))\,as\,\epsilon\to0.$$

Proof. The argument is similar to Proposition 2.2.3. Notice that h and r satisfy the elementary inequality

$$h(z)\leq r(z)\quad\text{for every}\,z>-1\,, \tag{2.99}$$

the dissipation control (2.85) implies that

$$\frac{1}{\epsilon\delta_\epsilon^2}\int_0^t\iiiint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}}h\Big(\frac{q_\epsilon}{G_{\epsilon_1}G_\epsilon}\epsilon^{\frac{1}{2}}\delta_\epsilon\Big)G_{\epsilon_1}G_\epsilon b\mathcal{M}_1\mathcal{M}\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v\mathrm{d}x\mathrm{d}s\leq4C^{\mathrm{in}}\,. \tag{2.100}$$

Applying the argument in the proof of Proposition 3.3 of [5], we get the Young-type inequality

$$\Lambda(\epsilon^{\frac{1}{2}}\delta_\epsilon)\frac{\alpha}{8}(\sigma+\sigma_1)\frac{1}{\epsilon\delta_\epsilon^2}s\left(\frac{\epsilon^{\frac{1}{2}}\delta_\epsilon q_\epsilon}{G_{\epsilon_1}G_\epsilon}\right) \leq \frac{1}{\epsilon\delta_\epsilon^2}h\left(\frac{\epsilon^{\frac{1}{2}}\delta_\epsilon q_\epsilon}{G_{\epsilon_1}G_\epsilon}\right)+C\exp\left(\frac{\alpha}{3}(\sigma+\sigma_1)\right)\,, \quad (2.101)$$

where C is a positive constant, $s(z)$ is defined by

$$s(z) \;=\; \frac{\frac{1}{2}z^2}{1+\frac{1}{3}z}\,, \quad (2.102)$$

and $0<\Lambda(y)<1$ is defined implicitly for every $y\in(0,1)$ by

$$1-\Lambda\log(\Lambda)-(1-\Lambda)\log(1-\Lambda)+\Lambda\log(y) \;=\; 0\,. \quad (2.103)$$

After some asymptotic analysis, it follows from this definition that

$$\frac{1}{\Lambda(y)} \;=\; O(|\log(y)|) \quad as\, y\to 0\,. \quad (2.104)$$

Let $T\in[0,\infty)$ and integrate both sides of the inequality (2.101) over the set $\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times[0,T]$ with respect to the measure

$$\frac{G_{\epsilon_1}\mathcal{M}_1}{N_{\epsilon_1}}\frac{G_\epsilon\mathcal{M}}{N_\epsilon}b\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v\mathrm{d}x\mathrm{d}t\,.$$

By using the bound (2.91), the fact

$$\iiint(1+|v|^2)^\beta\exp(\frac{\alpha\sigma}{3})(1+|v_1|^2)^\beta\exp(\frac{\alpha\sigma_1}{3})\mathcal{M}_1\mathcal{M}\,\mathrm{d}v_1\mathrm{d}v\mathrm{d}x \;<\; \infty\,,$$

and (2.104), one obtains

$$\int_0^T\iiiint(\sigma+\sigma_1)\frac{q_\epsilon^2}{G_{\epsilon_1}G_\epsilon+G'_{\epsilon_1}G'_\epsilon}\frac{\mathcal{M}_1\mathcal{M}}{N_{\epsilon_1}N_\epsilon}b\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v\mathrm{d}x\mathrm{d}t \;=\; O(|\log(\epsilon^{\frac{1}{2}}\delta_\epsilon)|) \quad (2.105)$$

as $\epsilon\to 0$. Using the symmetric property of the collision integrand and the elementary inequality

$$G_{\epsilon_1}G_\epsilon+G'_{\epsilon_1}G'_\epsilon \;\leq\; 3^2(N_{\epsilon_1}N_\epsilon+N'_{\epsilon_1}N'_\epsilon)\,,$$

we see that the left-hand side of (2.105) satisfies

$$
\begin{aligned}
&2\int_0^T \iiiint (\sigma+\sigma_1)\frac{q_\epsilon^2}{G_{\epsilon_1}G_\epsilon + G'_{\epsilon_1}G'_\epsilon}\frac{\mathcal{M}_1\mathcal{M}}{N_{\epsilon_1}N_\epsilon} b\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v\mathrm{d}x\mathrm{d}t \\
&= \int_0^T \iiiint (\sigma+\sigma_1)\frac{q_\epsilon^2}{G_{\epsilon_1}G_\epsilon + G'_{\epsilon_1}G'_\epsilon}\left(\frac{1}{N_{\epsilon_1}N_\epsilon} + \frac{1}{N'_{\epsilon_1}N'_\epsilon}\right)\mathcal{M}_1\mathcal{M} b\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v\mathrm{d}x\mathrm{d}t \\
&= \int_0^T \iiiint (\sigma+\sigma_1)\frac{q_\epsilon^2}{G_{\epsilon_1}G_\epsilon + G'_{\epsilon_1}G'_\epsilon}\frac{N'_{\epsilon_1}N'_\epsilon + N_{\epsilon_1}N_\epsilon}{N_{\epsilon_1}N_\epsilon N'_{\epsilon_1}N'_\epsilon}\mathcal{M}_1\mathcal{M} b\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v\mathrm{d}x\mathrm{d}t \\
&\geq \frac{1}{3^2}\int_0^T \iiiint (\sigma+\sigma_1)\frac{q_\epsilon^2}{N_{\epsilon_1}N_\epsilon N'_{\epsilon_1}N'_\epsilon}\mathcal{M}_1\mathcal{M} b\,\mathrm{d}\omega\mathrm{d}v_1\mathrm{d}v\mathrm{d}x\mathrm{d}t\,.
\end{aligned}
$$

The estimates then follows from (2.105).

Chapter 3: Acoustic Limit and Compressible Navier-Stokes Approximation of the Boltzmann Equation: Formal Scalings and Derivations

This chapter lays the groundwork for the next by presenting formal derivation of the acoustic limit and the compressible Navier-Stokes Approximation of the Boltzmann equation scaling around a unit Maxwellian $\mathcal{M}_{(1,0,1)}$ on spatial domain $\mathbb{T}^{\mathrm{D}}$. It follows Jiang's presentation in [34]. We first use moment-based methods [4–6] to formally derive the acoustic approximation to the Boltzmann equation (cf. Chapter 1, (1.40)) scaled around the unit global Maxwellian. Next, we refine the approximation for the Boltzmann equation, and formally derive the weakly compressible Navier Stokes system (cf. Chapter 1, (1.14)) by asymptotic expansion and averaging [49].

In Section 3.1, we present moment-based formal derivations of the acoustic system from the Boltzmann equation.

In Section 3.2, we state the formal derivations of the weakly nonlinear hydrodynamic limits for the general initial data, i.e., the initial data not necessarily satisfy the incompressibility and Boussenesq relations. It is observed that there exists a fast time scale (the fast acoustic waves), and a slow time scale (the incompressible mode). Averaging method is used to formally derive that asymptotically, the fluid behavior of the Boltzmann equation is governed by linear or weakly nonlinear models, such as weakly compressible Stokes and weakly compressible Navier-Stokes system. The projections of these weakly nonlinear fluid systems on the slow modes are consistent with the formal limits results with well-prepared initial data. When the initial data are not well-prepared, averaging method is used to describe the propagagtions of the fast waves. Section 3.2 mostly follows Jiang's presentation in [34].

The weakly compressible Stokes (linearized weakly compressible Navier-Stokes) system and weakly nonlinear Navier-Stokes system can be formally derived from the Boltzmann equation through a scaling in which the density F is close to the unit global Maxwellian M. Specifically, we consider families of solutions parametrized by the Knudsen number ϵ (Knudsen number characterized the ratio of mean free path and macroscopic length scales, so a small Knudsen number indicates fluid dynamics regime) that have the form (1.40)

$$\partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon} \mathcal{B}(F_\epsilon, F_\epsilon) \,, \qquad F_\epsilon \big|_{t=0} = F_\epsilon^{\mathrm{in}} \,, \tag{3.1}$$

around the unit Maxwellian, homogeneous in space and time:

$$M(v) = \frac{1}{(2\pi)^{\frac{\mathrm{D}}{2}}} \exp(-\frac{1}{2}|v|^2) \,. \tag{3.2}$$

We introduce relative densities G_ϵ, defined by $F_\epsilon = M G_\epsilon$. Recasting the initial-value problem (3.1) yields (1.41):

$$\partial_t G_\epsilon + v \cdot \nabla_x G_\epsilon = \frac{1}{\epsilon} \mathcal{Q}(G_\epsilon, G_\epsilon) \,, \qquad G_\epsilon \big|_{t=0} = G_\epsilon^{\mathrm{in}} \,, \tag{3.3}$$

where the collision operator is given by Chapter 1, (1.42):

$$\mathcal{Q}(G, G) = \iint_{\mathbb{S}^{\mathrm{D}-1} \times \mathbb{R}^{\mathrm{D}}} (G_1' G' - G_1 G) b(\omega, v_1 - v) M_1 \, \mathrm{d}\omega \mathrm{d}v_1 . \tag{3.4}$$

Here G_1', G', G_1 and $M_1 = M(v_1)$ follow the same notation in (1.20). We assume formally that the fluctuations g_ϵ^{in} and g_ϵ are bounded while $\delta_\epsilon > 0$ satisfies

$$\delta_\epsilon \to 0 \quad \text{as} \quad \epsilon \to 0\,. \tag{3.5}$$

We also assume the normalizations of the collision kernel b

$$\iiint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} b(\omega, v_1 - v)\,\mathrm{d}\omega M_1 \mathrm{d}v_1 M \mathrm{d}v = 1\,, \tag{3.6}$$

the measures on $\mathbb{S}^{\mathrm{D}-1}, \mathbb{R}^{\mathrm{D}}, \mathbb{T}^{\mathrm{D}}$

$$\int_{\mathbb{S}^{\mathrm{D}-1}} \mathrm{d}\omega = 1, \quad \int_{\mathbb{R}^{\mathrm{D}}} M\mathrm{d}v = 1, \quad \int_{\mathbb{T}^{\mathrm{D}}} \mathrm{d}x = 1\,, \tag{3.7}$$

and of the initial data

$$\iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{T}^{\mathrm{D}}} G_\epsilon^{\text{in}} M\mathrm{d}v\mathrm{d}x = 1, \iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{T}^{\mathrm{D}}} vG_\epsilon^{\text{in}} M\mathrm{d}v\mathrm{d}x = 0, \iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{T}^{\mathrm{D}}} \tfrac{1}{2}|v|^2 G_\epsilon^{\text{in}} M\mathrm{d}v\mathrm{d}x = \tfrac{\mathrm{D}}{2}\,. \tag{3.8}$$

In these derivations we assume that g_ϵ converges formally to g, where the limiting function g is in $L^\infty(\mathrm{d}t; L^2(M\mathrm{d}v\mathrm{d}x))$, and that all formally small terms vanish.

3.1 Acoustic Limit

Before we formally derive the weakly compressible Stokes system and weakly nonlinear Navier-Stokes system, we derive the acoustic system. All the results in this section belong to Bardos-Golse-Levermore [6, 24]. The acoustic system is the linearization about the homogeneous state of the compressible Euler system. After a suitable choice of units, the fluid fluctuations (ρ, u, θ) satisfy

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot u = 0, \quad & \rho(0,x) = \rho^{\text{in}}(x)\,, \\ \partial_t u + \nabla_x (\rho + \theta) = 0, \quad & u(0,x) = u^{\text{in}}(x)\,, \\ \tfrac{\mathrm{D}}{2} \partial_t \theta + \nabla_x \cdot u = 0, \quad & \theta(0,x) = \theta^{\text{in}}(x)\,. \end{aligned} \tag{3.9}$$

In 3.1.1, we give the formal derivation of acoustic system. We study some further properties of the acoustic operator in 3.1.2, as it will be useful in the derivation of weakly compressible approximations.

3.1.1 Formal derivation of the Acoustic Limit

We consider a family of formal solutions G_ϵ to the scaled Boltzmann initial-value problem

$$\partial_t G_\epsilon + v \cdot \nabla_x G_\epsilon = \frac{1}{\epsilon} \mathcal{Q}(G_\epsilon, G_\epsilon), \quad G_\epsilon(0,x,v) = G_\epsilon^{\text{in}}(x,v)\,. \tag{3.10}$$

G_ϵ satisfies local conservation laws of mass, momentum, and energy:

$$\begin{aligned} \partial_t \int G_\epsilon M \mathrm{d}v + \nabla_x \cdot \int v G_\epsilon M \, dv \;&=\; 0, \\ \partial_t \int v G_\epsilon M \mathrm{d}v + \nabla_x \cdot \int v \otimes v G_\epsilon M \, dv \;&=\; 0, \\ \partial_t \int \frac{1}{2}|v|^2 G_\epsilon M \mathrm{d}v + \nabla_x \cdot \int v \frac{1}{2}|v|^2 G_\epsilon M \, dv \;&=\; 0. \end{aligned} \tag{3.11}$$

We now consider the fluctuations g_ϵ, defined by

$$G_\epsilon = 1 + \delta_\epsilon g_\epsilon, \tag{3.12}$$

where the fluctuations g_ϵ^{in} and g_ϵ are bounded while $\delta_\epsilon > 0$ satisfies

$$\delta_\epsilon \to 0 \;\; as \;\; \epsilon \to 0. \tag{3.13}$$

In this formal derivation we assume that g_ϵ converges formally to g. The goal is to find the limiting function g.

The first step is to determine the form of the limiting function g. Observe that by (3.3) the fluctuations g_ϵ satisfy

$$\partial_t g_\epsilon + v \cdot \nabla_x g_\epsilon + \frac{1}{\epsilon}\mathcal{L} g_\epsilon = \frac{\delta_\epsilon}{\epsilon}\mathcal{Q}(g_\epsilon, g_\epsilon), \tag{3.14}$$

where the linearized collision operator $\mathcal{L}$ is formally defined by

$$\mathcal{L} g_\epsilon = \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} (g_\epsilon + g_{\epsilon_1} - g'_\epsilon - g'_{\epsilon_1}) b(v_1 - v) M_1 \mathrm{d}\omega \mathrm{d}v_1. \tag{3.15}$$

Assuming $\delta_\epsilon \to 0$ and multiplying both sides by ϵ, one finds that $\mathcal{L}\, g = 0$. The null space of $\mathcal{L}$ is given by $\mathrm{Null}(\mathcal{L}) = \mathrm{Span}\{1, v_1, ..., v_D, |v|^2\}$ according to Theorem (1.2.1). We conclude that g has the form of an infinitesimal Maxwellian, namely,

$$g = \rho(t,x) + u(t,x) \cdot v + \tfrac{1}{2}(|v|^2 - D)\theta(t,x). \tag{3.16}$$

The second step shows that the evolution of (ρ, u, θ) is governed by the *acoustic system*. Observe that the fluctuations g_ϵ satisfy the local conservation laws

$$\begin{aligned}
&\text{(i)} \quad \partial_t \langle g_\epsilon \rangle + \nabla_x \cdot \langle v\, g_\epsilon \rangle = 0\,,\\
&\text{(ii)} \quad \partial_t \langle v\, g_\epsilon \rangle + \nabla_x \cdot \langle v \otimes v\, g_\epsilon \rangle = 0\,,\\
&\text{(iii)} \quad \partial_t \langle (\tfrac{1}{2}|v|^2 - \tfrac{D}{2}) g_\epsilon \rangle + \nabla_x \cdot \langle v (\tfrac{1}{2}|v|^2 - \tfrac{D}{2}) g_\epsilon \rangle = 0\,,
\end{aligned} \tag{3.17}$$

and $g_\epsilon \to g$ formally, we have

$$\begin{aligned}
&\text{(i)} \quad \partial_t \langle g \rangle + \nabla_x \cdot \langle v\, g \rangle = 0\,,\\
&\text{(ii)} \quad \partial_t \langle v\, g \rangle + \nabla_x \cdot \langle v \otimes v\, g \rangle = 0\,,\\
&\text{(iii)} \quad \partial_t \langle (\tfrac{1}{2}|v|^2 - \tfrac{D}{2}) g \rangle + \nabla_x \cdot \langle v (\tfrac{1}{2}|v|^2 - \tfrac{D}{2}) g \rangle = 0\,,
\end{aligned} \tag{3.18}$$

here we use the angle-bracket notation

$$\langle h \rangle \,=\, \int_{\mathbb{T}^{\mathrm{D}}} h(v) M \mathrm{d}v \,. \tag{3.19}$$

Recall that

$$A(v) \,=\, v \otimes v - \tfrac{1}{D}|v|^2 I, \quad B(v) \,=\, \tfrac{1}{2}|v|^2 v - \tfrac{D+2}{2} v \,, \tag{3.20}$$

the local conservation laws are recast as

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot u \,&=\, 0 \,, \\ \partial_t u + \nabla_x (\rho + \theta) + \nabla_x \cdot \langle A(v) g \rangle \,&=\, 0 \,, \\ \tfrac{D}{2} \partial_t \theta + \nabla_x \cdot u + \nabla_x \cdot \langle B(v) g \rangle \,&=\, 0 \,. \end{aligned} \tag{3.21}$$

Since $A(v) \perp \mathrm{Null}(\mathcal{L})$, $B(v) \perp \mathrm{Null}(\mathcal{L})$ and $g \in \mathrm{Null}(\mathcal{L})$, we have

$$\langle A(v) g \rangle \,=\, 0, \quad \langle B(v) g \rangle \,=\, 0 \,, \tag{3.22}$$

then we obtain the *acoustic system*

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot u \,&=\, 0 \,, \\ \partial_t u + \nabla_x (\rho + \theta) \,&=\, 0 \,, \\ \tfrac{\mathrm{D}}{2} \partial_t \theta + \nabla_x \cdot u \,&=\, 0 \,. \end{aligned} \tag{3.23}$$

We denote the fluid moments by U:

$$U := \begin{pmatrix} \rho \\ u \\ \theta \end{pmatrix} \tag{3.24}$$

and (3.23) becomes

$$\partial_t U + \mathcal{A} U \,=\, 0 \,, \tag{3.25}$$

where

$$\mathcal{A}U := \begin{pmatrix} \nabla_x \cdot u \\ \nabla_x(\rho+\theta) \\ \nabla_x \cdot u \end{pmatrix} \tag{3.26}$$

is called the acoustic operator. Note that solutions of the acoustic system (3.23) satisfy

$$\partial_t(\tfrac{1}{2}\rho^2 + \tfrac{1}{2}|u|^2 + \tfrac{D}{4}\theta^2) + \nabla_x \cdot ((\rho+\theta)u) \,=\, 0\,. \tag{3.27}$$

For periodic domain $\mathbb{T}^{\mathrm{D}} = \mathbb{R}^{\mathrm{D}}/\mathbb{Z}^{\mathrm{D}}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int (\tfrac{1}{2}\rho^2 + \tfrac{1}{2}|u|^2 + \tfrac{D}{4}\theta^2)\,\mathrm{d}x \,=\, 0\,, \tag{3.28}$$

which shows solutions of the acoustic system do not decay like solutions of the linearized Boltzmann equation. Equally problematic is the fact that the acoustic system has a large class of nontrivial stationary solutions while the linear Boltzmann equation does not. Specifically, (ρ, u, θ) is a stationary solution of the acoustic system if and only if

$$\nabla_x \cdot u \,=\, 0\,, \quad \nabla_x(\rho+\theta) \,=\, 0\,. \tag{3.29}$$

On the other hand, g is a stationary solution of the linearized Boltzmann equation over $\mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}}$ if and only if

$$g \,=\, \rho + v\cdot u + (\tfrac{1}{2}|v|^2 - \tfrac{D}{2})\theta + v\cdot \Omega x\,, \tag{3.30}$$

where $(\rho, u, \theta, \Omega) \in \mathbb{R} \times \mathbb{R}^{\mathrm{D}} \times \mathbb{R} \times \mathbb{R}^{D\times D}$ with $\Omega^T = -\Omega$.

3.1.2 Properties of the Acoustic Operator

Because the operator $\mathcal{A}$ is the linearization of the compressible Euler system about a constant state, the only dynamics associated with it is that of sound waves. It is thereby called the *acoustic* operator. Solutions of the weakly compressible Navier-Stokes system can be decomposed into a so-called incompressible component that lies in Null($\mathcal{A}$), and an acoustic component that lies in Range($\mathcal{A}$). We characterize the spectral decomposition of $\mathcal{A}$ in this subsection. For convenience, part of the calculation will be carried out in the Fourier space. First, note the Fourier transform of the acoustic operator is $i\widehat{\mathcal{A}}$, where

$$\widehat{\mathcal{A}} := \begin{pmatrix} 0 & \xi^T & 0 \\ \xi & \mathbf{0} & \xi \\ 0 & \frac{2}{\mathrm{D}}\xi^T & 0 \end{pmatrix} . \tag{3.31}$$

$\widehat{\mathcal{A}}$ has a symmetrizer; in fact, $H\widehat{\mathcal{A}}$ is a symmetric matrix, with

$$H := \begin{pmatrix} \mathbf{I}_{\mathrm{D}+1\times\mathrm{D}+1} & 0 \\ 0 & \frac{\mathrm{D}}{2} \end{pmatrix} . \tag{3.32}$$

Hence, we may define the inner product (U_1, U_2) between $U_1 = (\rho_1, u_1, \theta_1)^T$ and $U_2 = (\rho_2, u_2, \theta_2)^T$ as:

$$(U_1, U_2) := \int_{\mathbb{T}^{\mathrm{D}}} \rho\rho_1 + uu_1 + \tfrac{\mathrm{D}}{2}\theta\theta_1 \, \mathrm{d}x \, . \tag{3.33}$$

So the operator $\mathcal{A}$ is skew-adjoint in the Hilbert space

$$\mathbb{H} = \left\{ \widetilde{V} \in L^2(\mathrm{d}x; \mathbb{R}^{\mathrm{D}+2}) : \int_{\mathbb{T}^{\mathrm{D}}} \widetilde{V} \, \mathrm{d}x = 0 \right\} \tag{3.34}$$

equipped with the inner product (3.33). It was shown in [35] that (3.33) is a *natural* inner product implied by the entropy structure.

Because $\mathcal{A}$ is skew-adjoint in its domain–the Hilbert space $\mathbb{H}$, it follows that $\text{Range}(\mathcal{A}) = \text{Null}(\mathcal{A})^{\perp}$, where $\text{Null}(\mathcal{A})^{\perp}$ is the orthogonal complement of $\text{Null}(\mathcal{A})$ with respect to the natural inner product given by (3.33).

It is clear from (3.26) that the range and null space of $\mathcal{A}$ are given by

$$\begin{aligned} \text{Range}(\mathcal{A}) &= \left\{ \begin{pmatrix} \beta \\ \nabla_x \phi \\ \beta \end{pmatrix} : \beta \in L_0^2(\mathrm{d}x;\mathbb{R}), \phi \in H^1(\mathrm{d}x;\mathbb{R}) \right\}, \\ \text{Null}(\mathcal{A}) &= \left\{ \begin{pmatrix} \gamma \\ \omega \\ -\gamma \end{pmatrix} : \gamma \in L_0^2(\mathrm{d}x;\mathbb{R}), \omega \in L_0^2(\mathrm{d}x;\mathbb{R}^{\mathrm{D}}), \nabla_x \cdot \omega = 0 \right\}, \end{aligned} \tag{3.35}$$

where $L_0^2(\mathrm{d}x)$ denotes L^2 functions with mean zero.

The spectral decomposition of $\mathcal{A}$ can be characterized in terms of eigenvectors of $\widehat{\mathcal{A}}$. We observe that $\widehat{\mathcal{A}}$ has $D+2$ independent eigenvectors; moreover, they are orthogonal under the inner product (3.33). More specifically, for $\xi \neq \mathbf{0}$, the eigenvalues and their corresponding eigenvectors are

$$\lambda^{(1)} = \sqrt{1+\tfrac{2}{\mathrm{D}}}||\xi||, \quad \phi^{(\mathbf{1})} = \frac{1}{\sqrt{2\frac{\mathrm{D}+2}{\mathrm{D}}}} \begin{pmatrix} 1 \\ \sqrt{1+\frac{2}{\mathrm{D}}}\frac{\xi}{||\xi||} \\ \frac{2}{\mathrm{D}} \end{pmatrix}$$

$$\lambda^{(2)} = -\sqrt{1+\tfrac{2}{\mathrm{D}}}||\xi||, \quad \phi^{(\mathbf{2})} = \frac{1}{\sqrt{2\frac{\mathrm{D}+2}{\mathrm{D}}}} \begin{pmatrix} 1 \\ -\sqrt{1+\frac{2}{\mathrm{D}}}\frac{\xi}{||\xi||} \\ \frac{2}{\mathrm{D}} \end{pmatrix}$$

$\lambda^{(a)} = 0$, for $a = \{3, ..., \mathrm{D}+2\}$, and $\phi^{(\mathbf{a})}$ are D-dimensional basis of solutions to

$$\xi \cdot \mathbf{y} = 0, x + z = 0. \tag{3.36}$$

Here (x, y^T, z) denotes an eigenvector, where $x, z \in \mathbb{R}, y \in \mathbb{R}^{\mathrm{D}}$. In particular, we may choose

$$\phi^{(\mathbf{3})} = \frac{1}{\sqrt{1+\frac{\mathrm{D}}{2}}} \begin{pmatrix} 1 \\ \mathbf{0} \\ -1 \end{pmatrix}, \phi^{(\mathbf{a})} = \begin{pmatrix} 0 \\ \mathbf{y} \\ 0 \end{pmatrix} \text{ for } a = \{4, ...\mathrm{D}{+}2\}, \text{where } \xi{\cdot}\mathbf{y} = 0, ||y||_{\mathbb{R}^{\mathrm{D}}} = 1\,. \tag{3.37}$$

Note that for the $\mathrm{D}-1$ independent solutions to $\xi \cdot \mathbf{y}$, we could always make them orthogonal under the regular inner product on $\mathbb{R}^{\mathrm{D}}$. It's straightforward to check that $\phi^{(\mathbf{a})}, a \in \{1, ...\mathrm{D}+2\}$ are orthonormal under the new inner product defined in (3.33). Note also that $\phi^{(1)}, \phi^{(2)}$ span Range($\widehat{\mathcal{A}}$), and $\phi^{(k)}$, $k = 3, ..., D+2$ span Null($\widehat{\mathcal{A}}$).

Every $U \in \mathbb{H}$ has the unique decomposition

$$U = \mathcal{P}U + \mathcal{P}^{\perp}U\,, \tag{3.38}$$

where $\mathcal{P}$ and $\mathcal{P}^{\perp}$ are projections onto $\mathrm{Null}(\mathcal{A})$ and $\mathrm{Null}(\mathcal{A})^{\perp}$ with

$$\mathcal{P}:\mathbb{H}\longrightarrow \mathrm{Null}(\mathcal{A}), \qquad \mathcal{P}^{\perp}:\mathbb{H}\longrightarrow \mathrm{Range}(\mathcal{A})=\mathrm{Null}(\mathcal{A})^{\perp}, \tag{3.39}$$

defined by

$$\mathcal{P}\begin{pmatrix}\rho\\ u\\ \theta\end{pmatrix}=\begin{pmatrix}\frac{2}{D+2}\rho-\frac{D}{D+2}\theta\\ \Pi u\\ -\frac{2}{D+2}\rho+\frac{D}{D+2}\theta\end{pmatrix} \qquad \mathcal{P}^{\perp}\begin{pmatrix}\rho\\ u\\ \theta\end{pmatrix}=\begin{pmatrix}\frac{D}{D+2}(\rho+\theta)\\ (I-\Pi)u\\ \frac{2}{D+2}(\rho+\theta)\end{pmatrix}, \tag{3.40}$$

where Π is the usual Leray projection onto the space of divergence-free vector fields defined by

$$\Pi = I-\nabla_x\Delta^{-1}\nabla_x\cdot\ . \tag{3.41}$$

We define that

$$\vartheta=-\tfrac{2}{D+2}\rho+\tfrac{D}{D+2}\theta, \quad w=\Pi u\,, \tag{3.42}$$

and

$$\pi=\tfrac{D}{D+2}(\rho+\theta), \quad v=(I-\Pi)u\,. \tag{3.43}$$

Then we have the following orthogonal decomposition: for every $U\in\mathbb{H}$,

$$U=\mathcal{P}U+\mathcal{P}^{\perp}U=\begin{pmatrix}-\vartheta\\ w\\ \vartheta\end{pmatrix}+\begin{pmatrix}\pi\\ v\\ \frac{2}{D}\pi\end{pmatrix}. \tag{3.44}$$

$\widehat{\mathcal{P}U}(\xi)$ can be represented by $\phi^{(k)}$, $k=3,\dots,D+2$:

$$\widehat{\mathcal{P}U}(\xi)=\sum_{k=3,\dots,D+2}(U,\phi^{(k)})\phi^{(k)}:=\sum_{k=3,\dots,D+2}U^{(k)}\phi^{(k)}\,, \tag{3.45}$$

and $\widehat{\mathcal{P}^{\perp}U}$ can be represented by $\phi^{(1)},\phi^{(2)}$:

$$\widehat{\mathcal{P}^{\perp}U}(\xi)=\sum_{k=1,2}(U,\phi^{(k)})\phi^{(k)}:=\sum_{k=1,2}U^{(k)}\phi^{(k)}\,. \tag{3.46}$$

We denote the coefficient of $\phi^{(k)}$ in the above representation as

$$U^{(k)} = (U, \phi^{(k)}) . \tag{3.47}$$

3.2 Formal Derivation of the Weakly Compressible Navier-Stokes System

A natural question to ask is whether one can refine the acoustic approximations. It is clear that the time scale at which the acoustic system is derived is not long enough to see the evolution of these solutions. By considering the Boltzmann equation over a longer time scale, one can give formal derivations of these incompressible fluid dynamics, depending on the limiting behavior of the ratio $\frac{\delta_\epsilon}{\epsilon}$ as $\epsilon \to 0$.

In this section, we state the formal derivations of the weakly nonlinear hydrodynamic limits for the general initial data (Jiang [34]), i.e., the initial data are not necessary to satisfy the incompressibility ($\nabla_x \cdot u = 0$) and Boussenesq relations ($\nabla_x(\rho + \theta) = 0$). We refer readers to [4, 5, 24] for the derivations of the incompressible fluid models with well-prepared initial data. In the case of general initial data, the fast acoustic waves occur. Averaging method is used to formally derive that asymptotically, the fluid behavior of the Boltzmann equation is governed by linear or weakly nonlinear models, such as weakly compressible Stokes and weakly compressible Navier-Stokes system. The projections of these weakly nonlinear fluids systems on the incompressible modes are incompressible Stokes and Navier-Stokes systems, which are consistent with the formal limits results before. The approach in this section is slightly different than that of [34]. We introduce a fast time scale t and a slow time scale $\tau = \epsilon t$. We then use Hilbert expansion [30] to get systematic expansion of

$$\partial_t g_\epsilon + v \cdot \nabla_x g_\epsilon + \frac{1}{\epsilon}\mathcal{L} g_\epsilon = \mathcal{Q}(g_\epsilon, g_\epsilon), \tag{3.48}$$

with two time scales.

After taking fluid moments at the leading order, it turns out that the projection of fluid fluctuations on the incompressible mode (Null($\mathcal{A}$)) can be decoupled from the acoustic mode (Range($\mathcal{A}$) = Null$(\mathcal{A})^\perp$). We then average over the fast time to get propagation of the fast acoustic waves.

Throughout the section, we set $\delta_\epsilon = \epsilon$ unless otherwise noted. Under this assumption we get the weakly compressible Navier-Stokes approximation. It will be clear in the derivation that if

$$\frac{\delta_\epsilon}{\epsilon} \to 0, \tag{3.49}$$

then all the nonlinear terms will vanish, and thus weakly compressible Stokes approximation is derived. For this reason, we will not give a separate description of the weakly compressible Stokes derivation.

3.2.1 Asymptotic Expansion

Hilbert's expansion is historically the older and goes back to Hilbert's fundamental paper [30] on the kinetic theory of gases. Writing the fluctuations of the scaled Boltzmann equation as formal power series in ϵ

$$g_\epsilon(t, x, v) = \sum_{n \geq 0} \epsilon^n g_n(t, x, v), \tag{3.50}$$

the leading order approximation g_0 is expected to be the limiting hydrodynamic distribution function, while the successive corrections g_n account for the finite Knudsen effects. Note that there are two time scales in (3.48), so we introduce a slow time $\tau = \epsilon t$ additionally, i.e. recast $g_n(t,x,v)$ as $g_n(t,\tau,x,v)$. Therefore

$$\partial_t \to \partial_t + \epsilon\partial_\tau \tag{3.51}$$

in the scaled Boltzmann equation (3.48). These coefficients g_n are found by plugging ansatz (3.50) in the scaled equation (3.48), and balancing the resulting coefficients of the successive powers of ϵ on each side of (3.48):

Order ϵ^{-1}:

$$\mathcal{L}(g_0) \;=\; 0\,, \tag{3.52}$$

Order ϵ^0:

$$\partial_t g_0 + v\cdot\nabla_x g_0 + \mathcal{L}g_1 \;=\; \mathcal{Q}(g_0,g_0)\,, \tag{3.53}$$

Order ϵ:

$$\partial_t g_1 + \partial_\tau g_0 + v\cdot g_1 + \mathcal{L}g_2 \;=\; \mathcal{Q}(g_0,g_1)\,, \tag{3.54}$$

....

Order ϵ^n:

$$\partial_t g_n + v\cdot\nabla_x g_n + \mathcal{L}(g_{n+1}) \;=\; \sum_{\substack{i+j=n\\1\le i,j\le n}} \mathcal{Q}(g_i,g_j)\,. \tag{3.55}$$

Solving for equation at order ϵ^{-1}, we get the leading order term is of the form of a infinitesimal Maxwellian:

$$g_0 \;=\; \rho_0 + v\cdot u_0 + (\tfrac{1}{2}|v|^2 - \tfrac{\mathrm{D}}{2})\theta_0. \tag{3.56}$$

To determine the coefficients ρ_0, u_0 and θ_0, we go to the next order

$$\partial_t g_0 + v \cdot \nabla_x g_0 + \mathcal{L} g_1 \;=\; \mathcal{Q}(g_0, g_0)\,. \tag{3.57}$$

Note that $\mathcal{L}$ satisfies the Fredholm alternative (Chapter 2), the compatibility condition at order 0 is therefore

$$\partial_t g_0 + v \cdot \nabla_x g_0 - \mathcal{Q}(g_0, g_0) \perp \mathrm{Null}(\mathcal{L})\,, \tag{3.58}$$

For each $f \in \mathrm{Null}(\mathcal{L})$, we have

$$\mathcal{Q}(f, f) \;=\; \tfrac{1}{2}\mathcal{L}(f^2)\,. \tag{3.59}$$

To prove the above identity, we take the second derivative of the relation

$$\mathcal{B}(\mathcal{M}_{(\rho,u,\theta)}, \mathcal{M}_{(\rho,u,\theta)}) \;=\; 0 \tag{3.60}$$

with respect to the parameters ρ, u, θ and evaluate it at $(1, 0, 1)$. See [4] for a complete argument. Taking fluid moments at order 0 and let

$$U_0 := \begin{pmatrix} \rho_0 \\ u_0 \\ \theta_0 \end{pmatrix}, \tag{3.61}$$

we then have

$$\partial_t U_0 + \mathcal{A} U_0 \;= 0\,, \tag{3.62}$$

where i.e. ρ_0, u_0, θ_0 satisfies the acoustic system (3.23). We write the solution of the acoustic system as

$$U_0 \;=\; e^{-t\mathcal{A}} V_\tau(\tau)\,. \tag{3.63}$$

To find V_τ we go to higher orders. The compatibility condition at order 1 is:

$$\partial_t g_1 + \partial_\tau g_0 + v \cdot \nabla_x g_1 - 2\mathcal{Q}(g_0, g_1) \perp \mathrm{Null}(\mathcal{L})\,, \tag{3.64}$$

i.e.

$$\begin{aligned} \partial_t \langle g_1 \rangle + \partial_\tau \langle g_0 \rangle + \nabla_x \cdot \langle v g_1 \rangle \;&=\; 0\,, \\ \partial_t \langle v g_1 \rangle + \partial_\tau \langle v g_0 \rangle + \nabla_x \cdot \langle v \otimes v g_1 \rangle \;&=\; 0\,, \\ \partial_t \langle (\frac{1}{2}|v|^2 - \frac{\mathrm{D}}{2}) g_1 \rangle + \partial_\tau \langle (\frac{1}{2}|v|^2 - \frac{\mathrm{D}}{2}) g_0 \rangle + \nabla_x \cdot \langle v(\frac{1}{2}|v|^2 - \frac{\mathrm{D}}{2}) g_1 \rangle \;&=\; 0\,. \end{aligned} \tag{3.65}$$

Let

$$U_1 := \begin{pmatrix} \rho_1 \\ u_1 \\ \theta_1 \end{pmatrix} := \begin{pmatrix} \langle g_1 \rangle \\ \langle v g_1 \rangle \\ \frac{2}{D} \langle (\frac{1}{2}|v|^2 - \frac{\mathrm{D}}{2}) g_1 \rangle \end{pmatrix}\,, \tag{3.66}$$

then (3.65) are recast as

$$\begin{aligned} \partial_t \rho_1 + \partial_\tau \rho_0 + \nabla_x \cdot u_1 \;&=\; 0\,, \\ \partial_t u_1 + \partial_\tau u_0 + \nabla_x(\rho_1 + \theta_1) + \nabla_x \cdot \langle A(v) g_1 \rangle \;&=\; 0\,, \\ \tfrac{D}{2} \partial_t \theta_1 + \tfrac{D}{2} \partial_\tau \theta_0 + \nabla_x \cdot u_1 + \nabla_x \cdot \langle B(v) g_1 \rangle \;&=\; 0\,. \end{aligned} \tag{3.67}$$

By (3.53) and (3.105), we have

$$\begin{aligned} g_1 \;&=\; \mathcal{P} g_1 + \mathcal{L}^{-1}(\mathcal{L}(g_0^2) - (\partial_t + v \cdot \nabla_x) g_0) \\ &=\; \mathcal{P} g_1 + \mathcal{P}^{\perp}(g_0^2) - \mathcal{L}^{-1}(v \cdot \nabla_x g_0)\,, \end{aligned} \tag{3.68}$$

where $\mathcal{P} g_1$ is the orthogonal projection of g_1 onto $\mathrm{Null}(\mathcal{L})$:

$$\mathcal{P} g \;=\; \langle g \rangle + v \cdot \langle v g \rangle + \tfrac{2}{\mathrm{D}} (\tfrac{1}{2}|v|^2 - \tfrac{2}{\mathrm{D}}) \langle (\tfrac{1}{2}|v|^2 - \tfrac{\mathrm{D}}{2}) g \rangle\,. \tag{3.69}$$

We showed in Chapter 2 there exists $\widehat{A}(v), \widehat{B}(v) \in \mathrm{Null}(\mathcal{L})^{\perp}$ such that

$$\mathcal{L}\widehat{A} \;=\; A\,, \qquad \mathcal{L}\widehat{B} \;=\; B\,. \tag{3.70}$$

and two scalar functions a and b such that

$$\widehat{A}(v) \;=\; a(|v|)A(v), \quad \widehat{B}(v) \;=\; b(v)B(v)\,. \tag{3.71}$$

Applying the self-adjoint property of the linearized collision operator $\mathcal{L}$, the terms $\langle A(v)g_1\rangle, \langle B(v)g_1\rangle$ in (3.67) are

$$\begin{aligned}\langle A(v)g_1\rangle &= \langle A(v)\mathcal{P}^{\perp}(g_0^2)\rangle - \langle \widehat{A}(v)(v\cdot\nabla_x g_0)\rangle,\\ \langle B(v)g_1\rangle &= \langle B(v)\mathcal{P}^{\perp}(g_0^2)\rangle - \langle \widehat{B}(v)(v\cdot\nabla_x g_0)\rangle\,,\end{aligned} \tag{3.72}$$

The terms $\langle A(v)\mathcal{P}^{\perp}(g_0^2)\rangle, \langle B(v)\mathcal{P}^{\perp}(g_0^2)\rangle, \langle \widehat{A}(v)(v\cdot\nabla_x g_0)\rangle, \langle \widehat{B}(v)(v\cdot\nabla_x g_0)\rangle$ can be calculated explicitly, as shown in the following lemmas:

Lemma 3.2.1

$$\begin{aligned}\langle \widehat{A}(v)v\cdot\nabla_x g_0\rangle &= \mu(\nabla_x u_0 + \nabla_x u_0^T - \tfrac{2}{D}\nabla_x\cdot u_0)\,,\\ \langle \widehat{B}(v)v\cdot\nabla_x g_0\rangle &= \kappa\nabla_x\theta_0\,,\end{aligned} \tag{3.73}$$

where

$$\mu = \tfrac{1}{(\mathrm{D}-1)(\mathrm{D}+2)}\langle A : \widehat{A}\rangle\,, \qquad \kappa = \tfrac{2}{\mathrm{D}(\mathrm{D}+2)}\langle B\cdot\widehat{B}\rangle\,. \tag{3.74}$$

Proof: After simple calculations, we obtain

$$\begin{aligned}v\cdot\nabla_x(\mathcal{P}g_0) \;&= A(v):\nabla_x u_0 + B(v)\cdot\nabla_x\theta_0\\ &\quad + v\cdot\nabla_x(\rho_0+\theta_0) + \tfrac{1}{D}|v|^2\nabla_x\cdot u_0\,.\end{aligned} \tag{3.75}$$

Let $\xi(v)$ denote $A(v)$ or $B(v)$, then $\widehat{\xi}(v) \in \mathrm{Null}(\mathcal{L})^{\perp}$. Thus the inner product of $\widehat{\xi}(v)$ with the last two terms in (3.75) vanish because they are in the null space of $\mathcal{L}$. Then

$$\langle\widehat{\xi}(v)v\cdot\nabla_x\mathcal{P}g_0\rangle \;=\; \langle\widehat{\xi}A\rangle : \nabla_x u_0 + \langle\widehat{\xi}B\rangle\cdot\nabla_x\theta_0\,. \tag{3.76}$$

Notice that $\widehat{A}(v)$ is even in v and $\widehat{B}(v)$ is odd in v, we obtain

$$\langle \widehat{A} B \rangle = 0, \quad \langle \widehat{B} A \rangle = 0 \,. \tag{3.77}$$

Thus

$$\langle \widehat{A}(v) v \cdot \nabla_x \mathcal{P} g_0 \rangle \; = \; \langle \widehat{A} \otimes \mathcal{A} \rangle : \nabla_x u_0 \,, \tag{3.78}$$

and

$$\langle \widehat{B}(v) v \cdot \nabla_x \mathcal{P} g_0 \rangle \; = \; \langle \widehat{B} \otimes B \rangle \cdot \nabla_x \theta_0 \,. \tag{3.79}$$

To finish the proof of Lemma (3.2.1), we state the following lemma which was proved in [5] (Lemma 4.4) .

Lemma 3.2.2

$$\begin{aligned} \langle \widehat{A}_{ij}, A_{kl} \rangle &= \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{2}{\mathrm{D}}\delta_{ij}\delta_{kl}), \\ \langle \widehat{B}_i, B_j \rangle &= \kappa \delta_{ij} \,, \end{aligned} \tag{3.80}$$

Applying Lemma (3.2.2) to (3.78) and (3.79), we finish the proof of Lemma (3.2.1). The derivation of the convection terms which are stated in the following lemma are more difficult.

Lemma 3.2.3

$$\begin{aligned} \langle A(v) \mathcal{P}^{\perp}(g_0^2) \rangle &= u_0 \otimes u_o - \tfrac{1}{D}|u_0|^2 I \,, \\ \langle B(v) \mathcal{P}^{\perp}(g_0^2) \rangle &= \tfrac{D+2}{2} u_0 \theta_0 \,. \end{aligned} \tag{3.81}$$

g_0 is given by

$$\begin{aligned} g_0^2 \; &= \rho_0^2 + 2\rho_0 u_0 \cdot v + 2\rho_0\theta_0(\tfrac{1}{2}|v|^2 \tfrac{\mathrm{D}}{2}) + \theta_0^2(\tfrac{\mathrm{D}}{2}|v|^2 + \tfrac{\mathrm{D}^2}{4}) \\ &+ (u_0 \cdot v)^2 + \theta_0^2(\tfrac{1}{4}|v|^4) + \theta_0 u_0 \cdot v(|v|^2 - \mathrm{D}) \,. \end{aligned} \tag{3.82}$$

The first four terms above are in the null space of $\mathcal{L}$, so their inner products with either A or B vanish. Furthermore, the last term is odd in v, and $A(v)$ is even in v, so their inner product is zero. Thus

$$\langle A_{ij}(v)(g_0)^2\rangle \;=\; \langle A_{ij}(v)(u_0\cdot v)^2\rangle + \tfrac{1}{4}\langle |v|^4 A_{ij}(v)\rangle\theta_0^2\,. \tag{3.83}$$

For a fixed pair (i,j), if $i \neq j$,

$$\langle A_{ij}(v)(u\cdot v)^2\rangle \;=\; 2\langle v_i^2 v_j^2\rangle u_i u_j \;=\; 2(u\otimes u)_{ij}\,. \tag{3.84}$$

If $i = j$,

$$\begin{aligned}\langle v_i^2(u\cdot v)^2\rangle &= \langle v_i^4\rangle|u_i|^2 + \sum j \neq i\langle v_i^2 v_j^2\rangle|u_j|^2\\ &= 3|u_i^2| + \sum j\neq i|u_j|^2 = \;|u|^2 + 2|u_i|^2\,.\end{aligned} \tag{3.85}$$

Thus

$$\begin{aligned}\langle A_{ij}(v)(u\cdot v)^2\rangle &= \langle v_i^2(u\cdot v)^2\rangle - \tfrac{1}{\mathrm{D}}\langle|v|^2(u\cdot v)^2\rangle\\ &= |u|^2 + 2|u_i|^2 - \tfrac{1}{\mathrm{D}}\sum_{j=1}^{\mathrm{D}}\langle v_j^2(u\cdot v)^2\rangle\\ &= |u|^2 + 2|u_i|^2 - \tfrac{1}{\mathrm{D}}(\mathrm{D}|u|^2 + 2|u|^2) = \;2(1-\tfrac{1}{\mathrm{D}})|u|^2\,.\end{aligned} \tag{3.86}$$

Then we proved

$$\tfrac{1}{2}\langle A_{ij}(v)(u_0\cdot v)^2\rangle \;=\; (u_0\otimes u_0)_{ij} - \tfrac{1}{\mathrm{D}}|u_0|^2\delta_{ij}\,. \tag{3.87}$$

Observe that

$$\langle\tfrac{1}{4}|v|^4 A_{ij}(v)\rangle \;=\; \tfrac{1}{4}\langle v_i v_j|v|^4\rangle - \tfrac{1}{4\mathrm{D}}\langle|v|^6\rangle\delta_{ij}\,. \tag{3.88}$$

If $i\neq j$, then $\langle v_i v_j|v|^4\rangle = 0$, so $\frac{1}{4}|v|^4 A_{ij}(v)\rangle = 0$.

If $i = j$, then $\frac{1}{4}\langle v_i^2|v|^4\rangle = \frac{1}{4\mathrm{D}}\langle|v|^6\rangle$, we also obtain $\langle\frac{1}{4}|v|^4 A_{ij}(v)\rangle = 0$. Combine with

(3.83), we proved the first identity in (3.81). Notice that $B(v)$ is in $\mathrm{Null}^{\perp}(\mathcal{L})$ and is odd in v, after taking inner product with (3.82), what is left is

$$\langle B_i(v)(g_0)^2\rangle \;=\; \langle B_i(v)v_j(|v|^2-\mathrm{D})\rangle u_{0j}\theta_0\,. \tag{3.89}$$

The coefficient $\langle B_i(v)v_j(|v|^2-\mathrm{D})\rangle$ is

$$\langle B_i(v)v_j(|v|^2-\mathrm{D})\rangle \;=\; \tfrac{1}{2}\langle v_iv_j|v|^4\rangle-(\mathrm{D}+1)\langle v_iv_j|v|^2\rangle+\tfrac{\mathrm{D}(\mathrm{D}+2)}{2}\delta_{ij}\,. \tag{3.90}$$

After some simple calculations, we get

$$\begin{aligned}\tfrac{1}{2}\langle v_iv_j|v|^4\rangle &=\tfrac{1}{2}[15+(\mathrm{D}-1)(\mathrm{D}+7)]\delta_{ij}\,,\\ (\mathrm{D}+1)\langle v_iv_j|v|^2\rangle &=(\mathrm{D}+1)(\mathrm{D}+2)\delta_{ij}\,.\end{aligned} \tag{3.91}$$

Then

$$\tfrac{1}{2}\langle B_i(v)v_j(|v|^2-\mathrm{D})\rangle = \tfrac{\mathrm{D}+2}{2}\,. \tag{3.92}$$

Thus we proved the second identity in (3.81).

Combining the above lemmas, the compatability condition (3.65) has the form of

$$\partial_t U_1+\partial_\tau U_0+\mathcal{A}U_1+\mathcal{N}(U_0,U_0) \;=\; \mathcal{D}U_0\,, \tag{3.93}$$

in which the linear term $\mathcal{D}(U)$ is (1.16):

$$\mathcal{D}U \;=\; \mathcal{D}\begin{pmatrix}\rho\\ u\\ \theta\end{pmatrix} = \begin{pmatrix}0\\ \mu\nabla_x\cdot\sigma(u)\\ \kappa\frac{2}{D}\nabla_x\cdot\nabla_x\theta\end{pmatrix}, \tag{3.94}$$

where $\sigma(u) \;=\; \nabla_x u+(\nabla_x u)^T-\frac{2}{D}\nabla_x\cdot uI$ and the quadratic term $\mathcal{N}(U,U)$ is (1.17):

$$\mathcal{N}(U,U) \;=\; \begin{pmatrix}0\\ \nabla_x\cdot(u\otimes u)-\frac{1}{D}\nabla_x|u|^2I\\ \frac{D+2}{D}\nabla_x\cdot(u\theta)\end{pmatrix}. \tag{3.95}$$

The linear operator $\mathcal{A}$ is skew-symmetric under the inner product

$$\langle U, V\rangle = \int_\Omega (\rho\tilde{\rho} + u\cdot\tilde{u} + \tfrac{D}{2}\theta\tilde{\theta})\mathrm{d}x \tag{3.96}$$

for $U = (\rho, u, \theta)$ and $V = (\tilde{\rho}, \tilde{u}, \tilde{\theta})$, i.e.,

$$\langle \mathcal{A}U, V\rangle = -\langle U, \mathcal{A}V\rangle. \tag{3.97}$$

Then the semi-group $e^{t\mathcal{A}}$ preserves the norm defined by this inner product, i.e.,

$$||e^{t\mathcal{A}}U|| = ||U||\,, \tag{3.98}$$

where $||U|| = \langle U, U\rangle$.

Applying the semi-group $e^{t\mathcal{A}}U$ to the identity (3.93), we obtain

$$\partial_t(e^{t\mathcal{A}}U_1) + \partial_\tau V_\tau + e^{t\mathcal{A}}\mathcal{N}(e^{-t\mathcal{A}}V_\tau, e^{-t\mathcal{A}}V_\tau) = e^{t\mathcal{A}}\mathcal{D}e^{-t\mathcal{A}}V_\tau\,. \tag{3.99}$$

here we use the identity (3.63): $U_0 = e^{-t\mathcal{A}}V_\tau$.

We now introduce some basic properties of almost-periodic functions, which were introduced by Bohr [7] in the case of complex functions and then extended to Banach spaces by Bochner and others. We also refer to [1] for the case of almost periodic functions in Banach spaces. A classic definition is given as follows:

Definition 3.2.1 *Lef $F \in C(\mathbb{R}, \mathbf{B})$, where $\mathbf{B}$ is a Banach space. F is said to be almost-periodic if and only if, given an $\epsilon > 0$, there exists a length L such that each interval of $\mathbb{R}$ of length L contains an almost-period p associated to ϵ, namely,*

$$\sup_{\tau\in\mathbb{R}} ||f(\tau + p) - f(\tau)||_{\mathbf{B}} \leq \epsilon\,. \tag{3.100}$$

We then denote by $AP(\mathbb{R}, \mathbf{B})$ the set of all such functions F.

We will use the following proposition in the sequel, which could have been given as an equivalent definition:

Proposition 3.2.1 *Let $F \in C(\mathbb{R}, \mathbf{B})$, F is almost-periodic if and only if it can be approximated uniformly by trigonometric polynomials*

$$\begin{aligned} \forall \alpha > 0, \exists N, a_n \in \mathbf{B}, w_n \in \mathbb{R}, 0 \le n \le N, suchthat \\ \left\| F - \sum_{n=0}^{N} a_n e^{i w_n \tau} \right\|_{L^\infty(\mathbb{R},\mathbf{B})} \le \alpha. \end{aligned} \tag{3.101}$$

The lemma stated below has a seen wide application in multiple time scales problems.

Lemma 3.2.4 *Let $F \in AP(\mathbb{R}, \mathbf{B})$ with $\mathbf{B} = L^\infty([0,T], H^s)$. Let $\tau = \epsilon t$. Then*

$$F(\frac{\tau}{\epsilon}, \tau) \rightharpoonup \bar{F}(\tau) \quad \textit{in weak-star sense in } \mathbf{B}, \tag{3.102}$$

where

$$\bar{F}(\tau) = \lim_{t \to \infty} \frac{1}{t} \int_0^t F(s, \tau)\, \mathrm{d}s\,. \tag{3.103}$$

The existence of $\bar{F}$ is a classical consequence of the definition and is called the mean value of F (see [1]).

Applying the characterization of the almost-periodic function, see Proposition (3.2.1), it is easy to see $e^{t\mathcal{A}}\mathcal{D}e^{-t\mathcal{A}}V_\tau$ and $e^{t\mathcal{A}}\mathcal{Q}(e^{-t\mathcal{A}}V_\tau, e^{-t\mathcal{A}}V_\tau)$ are almost-periodic in t.

Therefore, if we assume $V_\tau \to U$ as $\epsilon \to 0$, we obtain

$$\begin{aligned} &\lim_{\epsilon \to 0} e^{\frac{\tau}{\epsilon}\mathcal{A}}\mathcal{D}e^{-\frac{\tau}{\epsilon}\mathcal{A}}V_\tau = \bar{\mathcal{D}}V\,, \\ &\lim_{\epsilon \to 0} e^{\frac{\tau}{\epsilon}\mathcal{A}}\mathcal{N}(e^{-\frac{\tau}{\epsilon}\mathcal{A}}V_\tau, e^{-\frac{\tau}{\epsilon}\mathcal{A}}V_\tau) = \bar{\mathcal{N}}(V, V)\,. \end{aligned} \tag{3.104}$$

After time averaging of (3.105), we have

$$\begin{aligned}&\lim_{T\to\infty}\frac{1}{T}\int_0^T\{\partial_t(e^{t\mathcal{A}}U_1)+\partial_\tau V+e^{t\mathcal{A}}\mathcal{N}(e^{-t\mathcal{A}}V,e^{-t\mathcal{A}}V)\}\,\mathrm{d}t\\ =&\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{t\mathcal{A}}\mathcal{D}e^{-t\mathcal{A}}V\,\mathrm{d}t\,.\end{aligned}\tag{3.105}$$

Assuming U_1 is bounded, and note that $e^{t\mathcal{A}}$ is norm-preserving, we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\{\partial_t(e^{t\mathcal{A}}U_1)\}\,\mathrm{d}t=0\,,\tag{3.106}$$

so (3.105) becomes

$$\partial_\tau V+\bar{\mathcal{N}}(V,V)\;=\bar{\mathcal{D}}V\,,\tag{3.107}$$

where

$$\begin{aligned}\bar{\mathcal{D}}V&:=\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{t\mathcal{A}}\mathcal{D}e^{-t\mathcal{A}}V\,\mathrm{d}t\,,\\ \bar{\mathcal{N}}(V,V)&:=\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{t\mathcal{A}}\mathcal{N}(e^{-t\mathcal{A}}V,e^{-t\mathcal{A}}V)\,\mathrm{d}t\,.\end{aligned}\tag{3.108}$$

We now proceed to calculate $\bar{\mathcal{D}}$ and $\bar{\mathcal{N}}$.

3.2.2 Averaged Dissipation Operator

We decompose $\bar{\mathcal{D}}V = \mathcal{P}V+\mathcal{P}^\perp V$ and calculate the projection of $\bar{\mathcal{D}}$ onto Null($\mathcal{A}$) and Null$(\mathcal{A})^\perp$. Recall that $\widehat{\mathcal{P}V}(\xi)$ can be represented by:

$$\widehat{\mathcal{P}V}(\xi)=\sum_{k=3,\dots,D+2}(V,\phi^{(k)})\phi^{(k)}:=\sum_{k=3,\dots,D+2}V_k\phi^{(k)}\,,\tag{3.109}$$

where $\phi^{(k)}$, $k=3,...,D+2$ are eigenvectors of $\widehat{\mathcal{A}}$ with eigenvalue 0.

Let η be any eigenvector of $\mathcal{A}$ associated with the eigenvalue 0. The exponential operator $e^{t\mathcal{A}}$ does not affect Null($\mathcal{A}$), i.e. $e^{t\mathcal{A}}\eta=\eta$. The inner product of $\mathcal{P}\bar{\mathcal{D}}$ with

η is :

$$
\begin{aligned}
(\bar{\mathcal{D}}V,\eta) &= \lim_{T\to\infty}\frac{1}{T}\int_0^T (\mathcal{D}e^{-t\mathcal{A}}V, e^{-t\mathcal{A}}\eta)\,\mathrm{d}t \\
&= \lim_{T\to\infty}\frac{1}{T}\int_0^T (\mathcal{D}e^{-t\mathcal{A}}V, \eta)\,\mathrm{d}t \\
&= \lim_{T\to\infty}\frac{1}{T}\int_0^T (\mathcal{D}e^{-t\mathcal{A}}\mathcal{P}V, \eta)\,\mathrm{d}t + \lim_{T\to\infty}\frac{1}{T}\int_0^T (\mathcal{D}e^{-t\mathcal{A}}\mathcal{P}V, \eta)\,\mathrm{d}t \\
&= \lim_{T\to\infty}\frac{1}{T}\int_0^T (\mathcal{P}\mathcal{D}\mathcal{P}V, \eta)\,\mathrm{d}t + \lim_{T\to\infty}\frac{1}{T}\int_0^T (\mathcal{D}e^{-t\mathcal{A}}\mathcal{P}^{\perp}V, \eta)\,\mathrm{d}t\,.
\end{aligned}
\tag{3.110}
$$

The first term is the resonant term which is independent of t, so it is not affected by time averaging. The second term is non-resonant, which is filtered by time averaging. The following Riemann-Lebesgue lemma, the proof of which can be found in [?], guarantees that this second term vanishes. Thus we have

$$(\bar{\mathcal{D}}V,\eta) = (\mathcal{P}\mathcal{D}\mathcal{P}V,\eta)\,. \tag{3.111}$$

Lemma 3.2.5 *In the time averaging, the oscillatory integral*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{isA(k)}\phi(s)\,\mathrm{d}x \tag{3.112}$$

for any integrable function $\phi(t)$ vanishes when $A(k)\neq 0$. The only nonzero contributions that survive the averaging process are the resonance $A(k)=0$. Here $A(k)$ is any polynomial of k so that (3.112) is integrable.

Applying the above lemma, we deduce that the projection of the averaged dissipation operator $\bar{\mathcal{D}}$ on Null($\mathcal{A}$) is

$$\mathcal{P}\bar{\mathcal{D}}V = \mathcal{P}\mathcal{D}\mathcal{P}V = \begin{pmatrix} -\frac{2}{D+2}\kappa\Delta\vartheta \\ \mu\Pi\Delta\Pi u \\ \frac{2}{D+2}\kappa\Delta\vartheta \end{pmatrix}, \tag{3.113}$$

where

$$\vartheta = -\tfrac{2}{D+2}\rho + \tfrac{D}{D+2}\theta\,, \quad \Pi = I - \nabla_x \Delta^{-1} \nabla_x \cdot\ . \tag{3.114}$$

The projection on $\mathrm{Null}(\widehat{\mathcal{A}})^{\perp}$ is :

$$\widehat{\mathcal{P}^{\perp}\bar{\mathcal{D}}V}(\xi) = \sum_{k=1,2} (\bar{\mathcal{D}}V, \phi^{(k)})\phi^{(k)}\,, \tag{3.115}$$

where

$$\begin{aligned}
(\bar{\mathcal{D}}V, \phi^{(k)}) &= \lim_{T\to\infty} \frac{1}{T} \int_0^T (\widehat{\mathcal{D}} e^{-i\widehat{\mathcal{A}}s} V, e^{-i\widehat{\mathcal{A}}s}\phi^{(k)}) \\
&= \lim_{T\to\infty} \frac{1}{T} \int_0^T (\widehat{\mathcal{D}} \sum_l e^{-i\lambda_l s} V_l \phi^{(l)}, e^{-i\lambda_k s}\phi^{(k)}) \\
&= \sum_l \lim_{T\to\infty} \frac{1}{T} \int_0^T V_l e^{-is(\lambda_l - \lambda_k)} (\widehat{\mathcal{D}}\phi^{(l)}, \phi^{(k)})\, \mathrm{d}x\,.
\end{aligned} \tag{3.116}$$

The Riemann-Lebesgue lemma imply that the integral above is nonzero only when $\lambda_k = \lambda_l$, for $k = 1, 2$, this means $k = l$. Simple calculations show that

$$(\widehat{\mathcal{D}}\phi^{(k)}, \phi^{(k)}) = -\bar{\mu}|\xi|^2\,, \tag{3.117}$$

where

$$\bar{\mu} = [\mu\tfrac{D-1}{D} + \kappa\tfrac{2}{D(D+2)}]\,, \tag{3.118}$$

so

$$(\bar{\mathcal{D}}V, \phi^{(k)}) = -\bar{\mu}|\xi|^2 V_k\,, \tag{3.119}$$

and

$$\widehat{\mathcal{P}^{\perp}\bar{\mathcal{D}}V}(\xi) = \sum_{k=1,2} (\bar{\mathcal{D}}V, \phi^{(k)})\phi^{(k)} = -\sum_{k=1,2} \bar{\mu}|\xi|^2 V_k \phi^{(k)}\,, \tag{3.120}$$

i.e.

$$\mathcal{P}^{\perp}\bar{\mathcal{D}}V = \bar{\mu}\Delta\mathcal{P}^{\perp}V\,. \tag{3.121}$$

Simple calculations shows that the averaged diffusion term is strictly dissipated, in other words,

$$
\begin{aligned}
-(\bar{\mathcal{D}}V, V) &= -(\mathcal{P}\bar{\mathcal{D}}V, \mathcal{P}V) - (\mathcal{P}^{\perp}\bar{\mathcal{D}}V, \mathcal{P}^{\perp}V) \\
&\geq || - \sqrt{\tfrac{2}{D+2}\kappa}\nabla_x\vartheta, \sqrt{\mu}\nabla_x w, \sqrt{\tfrac{2}{D+2}\kappa}\nabla_x\vartheta||_{\mathbb{H}}^2 + \bar{\mu}||\mathcal{P}^{\perp}U||_{\mathbb{H}}^2 \\
&\geq \delta_0||V||_{\mathbb{H}}^2\,,
\end{aligned}
\tag{3.122}
$$

for some $\delta_0 > 0$. Furthermore, $(\bar{\mathcal{D}}V, V) = 0$ if and only if $V = 0$.

Remark 3.2.1 *The original dissipation operator $\mathcal{D}$ is only partially dissipative. That is one of the difficulties for the equations of compressible model because the equation of continuity is just a transport equation and does not have dissipation. According to the derivation, after taking time averaging, the diffusion term in the averaged system is strictly dissipative. This averaged dissipation operator appeared in the work of Hoff and Zumbrun [31, 32]. They called it an "artificial viscosity" term [31, 32], applied to the isentropic gas without energy equation. So the averaged system discussed in this chapter is a natural generalization of the Hoff-Zumbrun's so-called "effective artificial viscosity system". Actually, one of the main motivation of Hoff-Zumbrun's consideration is to modify the dissipative operator so that it has strict parabolicity.*

3.2.3 Averaged Quadratic Operator

By a similar approach, we can compute $\bar{\mathcal{N}}(V, V)$. We outline the calculation and refer interested readers to [36].

For any $\eta \in \mathrm{Null}(\mathcal{A})$,

$$
\begin{aligned}
(\bar{\mathcal{N}}(V,V),\eta) &= \lim_{T\to\infty}\frac{1}{T}\int_0^T (\bar{\mathcal{N}}(e^{-s\mathcal{A}}V, e^{-s\mathcal{A}}V),\eta)\,\mathrm{d}s \\
&= \lim_{T\to\infty}\frac{1}{T}\int_0^T (I_1+I_2+I_3,\eta)\,\mathrm{d}s\,,
\end{aligned}
\tag{3.123}
$$

where

$$
\begin{aligned}
&I_1 = \mathcal{N}(\mathcal{P}V,\mathcal{P}V),\\
&I_2 = \mathcal{N}(\mathcal{P}V, e^{-s\mathcal{A}}\mathcal{P}^\perp V) + \mathcal{N}(e^{-s\mathcal{A}}\mathcal{P}^\perp V, \mathcal{P}V)\,,\\
&I_3 = \mathcal{N}(e^{-s\mathcal{A}}\mathcal{P}^\perp V, e^{-s\mathcal{A}}\mathcal{P}^\perp V)\,.
\end{aligned}
\tag{3.124}
$$

We claim that

$$
\lim_{T\to\infty}\frac{1}{T}\int_0^T (I_2,\eta)\,\mathrm{d}s = 0\,,\quad \lim_{T\to\infty}\frac{1}{T}\int_0^T (I_3,\eta)\,\mathrm{d}s = 0\,.
\tag{3.125}
$$

Both can be proved using Riemann-Lebesgue lemma. Thus $\mathcal{P}\bar{\mathcal{N}}(V,V) = \mathcal{P}\mathcal{N}(\mathcal{P}V,\mathcal{P}V)$.

A direct calculation yields

$$
\mathcal{P}\mathcal{N}(\mathcal{P}V,\mathcal{P}V) = \begin{pmatrix} -w\cdot\nabla_x\vartheta \\ w\cdot\nabla_x w \\ w\cdot\nabla_x\vartheta \end{pmatrix} = \begin{pmatrix} w\cdot\nabla_x(\frac{2}{D+2}\rho - \frac{D}{D+2}\theta) \\ w\cdot\nabla_x w \\ w\cdot\nabla_x(-\frac{2}{D+2})\rho + \frac{D}{D+2}\theta \end{pmatrix}.
\tag{3.126}
$$

Now we take a look at the projection on the acoustic mode $\mathcal{P}^\perp\bar{\mathcal{Q}}(V,V)$. Let γ_k be unit eigenvectors of $\mathcal{A}$ that span $\mathrm{Null}^\perp(\mathcal{A})$, then

$$
\begin{aligned}
\mathcal{P}^\perp\bar{\mathcal{N}}(V,V) &= \sum_k (\bar{\mathcal{N}}(V,V),\gamma_k)\gamma_k \\
&= \lim_{T\to\infty}\frac{1}{T}\int_0^T (I_1+I_2+I_3, e^{-s\mathcal{A}}\gamma_k)\gamma_k\,\mathrm{d}s \\
&= \mathcal{N}_{2r}(\mathcal{P}V,\mathcal{P}^\perp V) + \mathcal{N}_{3r}(\mathcal{P}^\perp V,\mathcal{P}^\perp V)\,.
\end{aligned}
\tag{3.127}
$$

The term $(I_1, e^{-s\mathcal{A}}\gamma_k)$ contains only nonresonant terms and will vanish under time averaging. $\mathcal{N}_{2r}$ and $\mathcal{N}_{3r}$ denote the averaged quadratic operator over the two-wave and three-wave resonant sets respectively. Note that $\mathcal{N}_{2r}$ depends on both the incompressible and acoustic modes, while $\mathcal{N}_{3r}$ depends only on the acoustic modes. For complete forms of $\mathcal{N}_{2r}$ and $\mathcal{N}_{3r}$, see [36] .

Now we state a theorem on the formal derivation of the weakly nonlinear approximation of the Boltzmann equation with the general initial data.

Theorem 3.2.1 *(The Formal Weakly Compressible Approximation Theorem) Let G_ϵ be a family of distribution solutions to the scaled Boltzmann initial-value problem (3.3) with initial data G_ϵ^{in} that satisfy the normalizations (3.6), (3.7), (3.8). Let $G_\epsilon^{\text{in}} = 1 + \delta_\epsilon g_\epsilon^{\text{in}}$ and $G_\epsilon = 1 + \delta_\epsilon g_\epsilon$ where $\delta_\epsilon \to 0$ as $\epsilon \to 0$, and the fluctuations g_ϵ^{in} and g_ϵ are bounded in $L^\infty(\mathrm{d}t; L^2(M\mathrm{d}v\mathrm{d}x))$. Moreover:*

1. Assume that in the sense of distributions the family g_ϵ^{in} satisfies

$$\lim_{\epsilon\to 0}(\langle g_\epsilon^{\text{in}}\rangle, \langle v g_\epsilon^{\text{in}}\rangle, \langle(\tfrac{1}{\mathrm{D}}|v|^2 - 1)g_\epsilon^{\text{in}}\rangle) \tag{3.128}$$

for some $(\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}}) \in L^2(\mathrm{d}x; \mathbb{T}^{\mathrm{D}+2})$;

2. Assume that the local conservation laws (3.17) are also satisfied in the sense of distributions for every g_ϵ;

3. For the family of the fluctuations g_ϵ, assume that

$$\mathcal{P}^\perp g_\epsilon = (I - \mathcal{P})g_\epsilon \to 0, \quad as\,\epsilon \to 0 \tag{3.129}$$

and the following moments with $\widehat{\xi} = \widehat{A}$ or $\widehat{B}$

$$\langle\widehat{\xi}(v) v\cdot\nabla_x \mathcal{P}^\perp g_\epsilon\rangle, \langle\widehat{\xi}(v)\mathcal{Q}(\mathcal{P}^\perp g_\epsilon, \mathcal{P}g_\epsilon + \mathcal{P}^\perp g_\epsilon)\rangle \tag{3.130}$$

go to zero as $\epsilon \to 0$*; and*

$$\epsilon\langle\widehat{\xi}\partial_t g_\epsilon\rangle \to 0, \quad as\, \epsilon \to 0; \tag{3.131}$$

Then the family of the moments

$$U_\epsilon = (\langle g_\epsilon\rangle, \langle v g_\epsilon\rangle, \langle \tfrac{1}{\mathrm{D}}|v|^2 g_\epsilon\rangle) \tag{3.132}$$

satisfy the asymptotics

$$U_\epsilon - \mathcal{P}V - e^{-t\mathcal{A}}(\mathcal{P}^\perp V) \to 0, \quad as \epsilon \to 0\,, \tag{3.133}$$

where $\mathcal{P}V$ *and* $\mathcal{P}^\perp V$ *satisfy the equations: 1. when* $\frac{\delta_\epsilon}{\epsilon} \to 0$*,* $\mathcal{P}V$ *satisfies the incompressible Stokes system*

$$\begin{aligned} \nabla_x \cdot u \; &= 0\,, \\ \nabla_x(\rho+\theta) \; &= 0\,, \\ \partial_t u + \nabla_x p \; &= \mu \Delta_x u, \\ \tfrac{\mathrm{D}+2}{2}\partial_t \theta \; &= \kappa \Delta_x \theta, \end{aligned} \tag{3.134}$$

with initial data $\mathcal{P}U^{\mathrm{in}}$*; and* $\mathcal{P}^V$ *satisfies the averaged equation*

$$\begin{aligned} \partial_t \mathcal{P}^\perp V \; &= \bar{\mu}\Delta_x \mathcal{P}^\perp V\,, \\ \mathcal{P}^\perp V(0,x) \; &= \mathcal{P}^\perp U^{\mathrm{in}}(x); \end{aligned} \tag{3.135}$$

with

$$\overline{\mu} = \mu \tfrac{\mathrm{D}-1}{\mathrm{D}} + \kappa \tfrac{2}{\mathrm{D}(\mathrm{D}+2)}\,. \tag{3.136}$$

2. when $\frac{\delta_\epsilon}{\epsilon} \to 1$*,* $\mathcal{P}V$ *satisfies the incompressible Navier-Stokes system*

$$\begin{aligned} \nabla_x \cdot u &= 0\,, \\ \nabla_x(\rho + \theta) &= 0\,, \\ \partial_t u + u \cdot \nabla_x u + \nabla_x p &= \mu \Delta_x u, \\ \tfrac{\mathrm{D}+2}{2}(\partial_t \theta + u \cdot \nabla_x \theta) &= \kappa \Delta_x \theta, \end{aligned} \tag{3.137}$$

with initial data $\mathcal{P}U^{\mathrm{in}}$*; and* $\mathcal{P}^\perp U$ *satisfies the averaged equation*

$$\begin{aligned} \partial_t \mathcal{P}^\perp V + \mathcal{N}_{2r}(\mathcal{P}V, \mathcal{P}^\perp V) + \mathcal{N}_{3r}(\mathcal{P}^\perp V, \mathcal{P}^\perp V) &= \bar{\mu} \Delta_x \mathcal{P}^\perp V\,, \\ \mathcal{P}^\perp V(0,x) &= \mathcal{P}^\perp U^{\mathrm{in}}(x)\,. \end{aligned} \tag{3.138}$$

Remark 3.2.2 *When the initial data are well-prepared, i.e.,* $\mathcal{P}^\perp U^{\mathrm{in}} = 0$*, the above theorem exactly matches with Bardos-Golse-Levermore's theorem on the formal incompressible limits. For the Stokes dynamics, the averaged equation is completely decoupled from the projection on the incompressible regime. For the Navier-Stokes dynamics, the averaged equations are coupled with the corresponding incompressible regime.*

Chapter 4: Linearized Compressible and Weakly Compressible Navier-Stokes System

In this chapter, we showed the solutions of the linearized compressible compressible Navier-Stokes system over $\mathbb{T}^{\mathrm{D}}$

$$\begin{aligned} \partial_t \rho_\epsilon + \nabla_x \cdot u_\epsilon &= 0 \,, \\ \partial_t u_\epsilon + \nabla_x(\rho_\epsilon + \theta_\epsilon) &= \epsilon \nabla_x \cdot \mu \left[\nabla_x u_\epsilon + (\nabla_x u_\epsilon)^T - \tfrac{2}{\mathrm{D}} \nabla_x \cdot u_\epsilon I \right] \,, \\ \partial_t \theta_\epsilon + \tfrac{2}{\mathrm{D}} \nabla_x \cdot u_\epsilon &= \epsilon \tfrac{2}{\mathrm{D}} \nabla_x \cdot (\kappa \nabla_x \theta_\epsilon) \,, \end{aligned} \tag{4.1}$$

and the weakly compressible linearized Navier-Stokes system

$$\begin{aligned} \partial_t \rho_\epsilon + \nabla_x \cdot u_\epsilon &= \epsilon \left((\tfrac{2}{\mathrm{D}+2})^2 \kappa + \tfrac{\mathrm{D}}{\mathrm{D}+2} \bar{\mu} \right) \rho_\epsilon + \epsilon \tfrac{\mathrm{D}}{\mathrm{D}+2} \left(\bar{\mu} - \tfrac{2}{\mathrm{D}+2} \kappa \right) \theta_\epsilon \,, \\ \partial_t u_\epsilon + \nabla_x(\rho_\epsilon + \theta_\epsilon) &= \epsilon \mu \Delta u_\epsilon + \epsilon (\bar{\mu} - \mu) \nabla_x \nabla_x \cdot u_\epsilon \,, \\ \partial_t \theta_\epsilon + \tfrac{2}{\mathrm{D}} \nabla_x \cdot u_\epsilon &= \epsilon \tfrac{2}{\mathrm{D}+2} \left(-\tfrac{2}{\mathrm{D}+2} \kappa + \bar{\mu} \right) \rho_\epsilon + \epsilon \tfrac{2}{\mathrm{D}+2} \left(\tfrac{\mathrm{D}}{\mathrm{D}+2} \kappa + \bar{\mu} \right) \theta_\epsilon \,, \end{aligned} \tag{4.2}$$

approximates the fluid moments of the linearized Boltzmann equation

$$\partial_t g_\epsilon + v \cdot \nabla_x g_\epsilon + \frac{1}{\epsilon} \mathcal{L} g_\epsilon \;=\; 0 \tag{4.3}$$

in $L^2(\mathbb{T}^{\mathrm{D}})$ uniformly for $t > 0$. Recall that $\bar{\mu}$ in (4.2) is (cf. Chapter 3, (3.118))

$$\bar{\mu} = [\mu \tfrac{D-1}{D} + \kappa \tfrac{2}{D(D+2)}] \,.$$

We denote the linearized compressible Navier-Stokes system (4.1) by

$$\partial_t U_\epsilon + \mathcal{A} U_\epsilon \;=\; \epsilon \mathcal{D} U_\epsilon \,, \tag{4.4}$$

and the linearized weakly compressible Navier-Stokes system (4.2) by

$$\partial_t U_\epsilon + \mathcal{A} U_\epsilon = \epsilon \bar{\mathcal{D}} U_\epsilon \,. \tag{4.5}$$

The linearized Navier-Stokes system is not strictly dissipative. By arguments in Kawashima [39], no nonconstant eigenfunction of $\mathcal{A}$ is in the null space of $\mathcal{D}$, so the averaged dissipation operator $\bar{\mathcal{D}}$ is strictly dissipative. We first consider a special case of (4.1), in which the ratio of μ and κ is equal to a specific constant. In this case, we can write the solution of (4.1) explicitly. We then compute the decay rate of the solution of (4.1) and show the difference between the solution of the linearized compressible Navier-Stokes system (4.4) and the linearized weakly compressible Navier-Stokes system (4.5) is "small" and the estimate is uniform in time. This argument can be generalized for all $\mu > 0$ and $\kappa > 0$. Finally, we use Chapman-Enskog expansion (cf. Chapter 1, Section 1.5) to show that if the initial fluctuations g_ϵ of the linearized Boltzmann equation (4.3) is in the fluid regime, then the moments of fluid fluctuations g_ϵ can be approximated by solutions of the linearized compressible Navier-Stokes system, and the estimate is also uniform in time.

Several works have been published in this direction, notably Ellis and Pinsky [17], who worked on whole space $\mathbb{R}^{\mathrm{D}}$ and showed the difference between the solution of linearized Boltzmann equation and the weakly compressible Navier-Stokes approximation is $O(\epsilon)$ for sufficiently smooth initial data. On the fluid regime, Hoff and Zumbrun [31], [32] showed the Cauchy problem for compressible Navier-Stokes on whole space is asymptotically given by the solution of weakly compressible

Navier-Stokes. The domain we are working on is $\mathbb{T}^{\mathrm{D}}$, so the gas is confined and we expect to see dissipation instead of dispersion over the whole space (in which case the acoustic waves will run away to infinity). Additionally, we treat a larger class of collision kernel b (hard sphere and all the inverse power kernels) than [17], in which only hard sphere case is considered.

4.1 Decay Estimate for the Linearized Compressible Navier-Stokes System

In this section, we give the estimate of decay rate of $\widehat{U}(\xi, t)$ for all wave numbers: larger ξ (all roots are real for the corresponding characteristic polynomial of $\mathbf{Q}$) and smaller ξ (for which the characteristic polynomial has a conjugate pair of roots). To start with, we discuss the structure of the linearized Navier-Stokes system (4.1) in more details. As a first attempt, we resolve the velocity field u into the sum of a solenoidal vector field ω_ϵ and $\nabla_x \phi_\epsilon$, where ϕ_ϵ is a scalar. In other words,

$$\begin{aligned} u_\epsilon &= \nabla_x \phi_\epsilon + \omega_\epsilon, \\ \nabla_x \cdot u_\epsilon &= \Delta \phi_\epsilon, \\ \nabla_x \cdot \omega_\epsilon &= 0\,. \end{aligned} \tag{4.6}$$

We now show that the the linearized Navier-Stokes system (4.1) can be decoupled under the above decomposition of velocity field.

In fact, the first and third equation of the linearized Navier-Stokes system (4.1) immediately become

$$\partial_t \rho_\epsilon + \Delta \phi_\epsilon = 0 \tag{4.7}$$

and

$$\partial_t \theta_\epsilon + \tfrac{2}{\mathrm{D}} \Delta \phi_\epsilon \; = \; \epsilon \tfrac{2}{\mathrm{D}} \kappa \Delta \theta_\epsilon . \tag{4.8}$$

Moreover, after projecting the second equation of the linearized Navier-Stokes system (4.1) into the divergence free vector field, we get

$$\partial_t \omega_\epsilon \; = \; \epsilon \mu \Delta \omega_\epsilon , \tag{4.9}$$

and

$$\begin{aligned} \partial_t (\nabla_x \phi_\epsilon) + \nabla_x (\rho_\epsilon + \theta_\epsilon) \; &= \; \epsilon \mu [\nabla_x \nabla_x \cdot \nabla_x \phi_\epsilon + \Delta \nabla_x \phi] - \epsilon \mu \tfrac{2}{\mathrm{D}} \nabla_x \nabla_x \cdot \nabla_x \phi \\ &= \; \epsilon \mu \tfrac{\mathrm{D}-2}{\mathrm{D}} \nabla_x \nabla_x \cdot \nabla_x \phi_\epsilon + \epsilon \mu \Delta \nabla_x \phi_\epsilon \, , \end{aligned} \tag{4.10}$$

which, after integration on $\mathbb{T}^{\mathrm{D}}$, becomes

$$\partial_t \phi_\epsilon + (\rho_\epsilon + \theta_\epsilon) \; = \; \epsilon \mu \tfrac{\mathrm{D}-2}{D} \Delta \phi_\epsilon + \epsilon \mu \Delta \phi \; = \; \epsilon \mu \tfrac{2\mathrm{D}-2}{\mathrm{D}} \Delta \phi . \tag{4.11}$$

Therefore, the linearized Navier-Stokes system (4.1) are decomposed into

$$\partial_t \omega_\epsilon \; = \; \epsilon \mu \Delta \omega_\epsilon , \tag{4.12}$$

and

$$\begin{aligned} \partial_t \rho_\epsilon + \Delta \phi_\epsilon \; &= \; 0, \\ \partial_t \phi_\epsilon + (\rho_\epsilon + \theta_\epsilon) \; &= \; \epsilon \mu \tfrac{2\mathrm{D}-2}{\mathrm{D}} \Delta \phi, \\ \partial_t \theta_\epsilon + \tfrac{2}{\mathrm{D}} \Delta \phi_\epsilon \; &= \; \epsilon \tfrac{2}{\mathrm{D}} \kappa \Delta \theta_\epsilon . \end{aligned} \tag{4.13}$$

We see the divergence-free vector field ω_ϵ decays like a heat kernel, whereas the behavior of $\rho_\epsilon, \phi_\epsilon$ and θ_ϵ is less clear. Thus, as an alternative approach, we proceed to analyze the linearized Navier-Stokes system (4.1) in the Fourier space.

The Fourier transform of the linearized Navier-Stokes system (4.1) is

$$\partial_t \begin{pmatrix} \widehat{\rho} \\ \widehat{u} \\ \widehat{\theta} \end{pmatrix} + \begin{pmatrix} 0 & i\xi^T & 0 \\ i\xi & \epsilon\mu(I|\xi|^2 + \frac{\mathrm{D}-2}{\mathrm{D}}\xi\xi^T) & i\xi \\ 0 & i\frac{2}{\mathrm{D}}\xi^T & \epsilon\kappa\frac{2}{\mathrm{D}}|\xi|^2 \end{pmatrix} \begin{pmatrix} \widehat{\rho} \\ \widehat{u} \\ \widehat{\theta} \end{pmatrix} = 0, \tag{4.14}$$

and will be denoted by

$$\frac{d}{dt}\widehat{U} = \mathbf{Q}\widehat{U}, \tag{4.15}$$

with

$$\mathbf{Q} = \begin{pmatrix} 0 & -i\xi^T & 0 \\ -i\xi & -\epsilon\mu(I|\xi|^2 + \frac{\mathrm{D}-2}{\mathrm{D}}\xi\xi^T) & -i\xi \\ 0 & -i\frac{2}{\mathrm{D}}\xi^T & -\epsilon\kappa\frac{2}{\mathrm{D}}|\xi|^2 \end{pmatrix}, \quad \widehat{U} = \begin{pmatrix} \widehat{\rho} \\ \widehat{u} \\ \widehat{\theta} \end{pmatrix}. \tag{4.16}$$

in the further narrative.

The eigenspace of $-\mathbf{Q}$ can be further decomposed into two subspaces. To see this, we decompose $\mathbb{R}^{\mathrm{D}}$ into the $\mathrm{D}-1$ dimensional subspace $V_1 := \{y \in \mathbb{C}^{\mathrm{D}} | \xi^T \cdot y = 0\}$ and the 3D subspace $V_2 := \{(a, b\frac{\xi}{|\xi|}, c)^T | a, b, c \in \mathbb{C}^{\mathrm{D}}\}$. Clearly every vector in V_1 is an eigenvector of $-\mathbf{Q}$ with corresponding eigenvalue $\epsilon\mu|\xi|^2$. Moreover, note that solving for

$$-\mathbf{Q} \begin{pmatrix} a \\ b\frac{\xi}{|\xi|} \\ c \end{pmatrix} = \begin{pmatrix} ib|\xi| \\ i\xi(a+c) + \epsilon\mu\frac{2\mathrm{D}-2}{\mathrm{D}}|\xi|\xi \\ i\frac{2}{\mathrm{D}}b|\xi| + \epsilon\kappa\frac{2}{\mathrm{D}}c|\xi|^2 \end{pmatrix} = \lambda \begin{pmatrix} a \\ b\frac{\xi}{|\xi|} \\ c \end{pmatrix} \tag{4.17}$$

is equivalent to solving for

$$\begin{pmatrix} 0 & i|\xi| & 0 \\ i|\xi| & \epsilon\mu\frac{2\mathrm{D}-2}{\mathrm{D}}|\xi|^2 & i|\xi| \\ 0 & i\frac{2}{\mathrm{D}}|\xi| & \epsilon\kappa\frac{2}{\mathrm{D}}|\xi|^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \tag{4.18}$$

so the characteristic polynomial of $-\mathbf{Q}$ is

$$
\begin{aligned}
P_{-\mathbf{Q}(z)} \;=\; & (z+\epsilon\mu|\xi|^2)^{\mathrm{D}-1} \\
& \left[z^3 - (\epsilon\mu\frac{2\mathrm{D}-2}{\mathrm{D}}|\xi|^2 + \epsilon\kappa\frac{2}{\mathrm{D}}|\xi|^2)z^2 + (\epsilon^2\mu\kappa\frac{2(2\mathrm{D}-2)}{\mathrm{D}^2}|\xi|^4 + \frac{2+\mathrm{D}}{\mathrm{D}}|\xi|^2 z) - \epsilon\kappa\frac{2}{\mathrm{D}}|\xi|^4 \right].
\end{aligned}
\tag{4.19}
$$

Remark 4.1.1 *The decomposition of* $\mathbb{R}^{\mathrm{D}}$ *reflects exactly the decoupling of system by resolving velocity field into* $\nabla_x\phi + \omega$. *In fact,*

$$
\begin{aligned}
\widehat{\omega} &= (I - \frac{\xi\xi^T}{|\xi|^2})\widehat{u}, \\
\widehat{\nabla_x\phi} &= i\xi\widehat{\phi} = \frac{\xi\xi^T}{|\xi|^2}\widehat{u},
\end{aligned}
\tag{4.20}
$$

and

$$
\begin{aligned}
V_1 &= \{(I - \frac{\xi\xi^T}{|\xi|^2})\widehat{u} | \widehat{u} \in \mathbb{C}^{\mathrm{D}}\}, \\
V_2 &= \{(\widehat{\rho}, \frac{\xi\xi^T}{|\xi|^2}\widehat{u}, \widehat{\theta})^T | \widehat{\rho}, \widehat{\theta} \in \mathbb{C}, \widehat{u} \in \mathbb{C}^{\mathrm{D}}\}.
\end{aligned}
\tag{4.21}
$$

By the above decomposition, we see the solutions of $\frac{d}{dt}\widehat{U} = \mathbf{Q}\widehat{U}$ has decay rate $e^{-\epsilon\mu|\xi|^2 t}$ in at least $\mathrm{D}-1$ directions; whether the solutions at the other directions decay is less clear at this point. There is also the possibility that not all the roots of characteristic polynomial are simple, so the matrix $\mathbf{Q}$ is not necessarily diagonalizable everywhere. We will discuss these issues later in this chapter.

At this point one may ask whether any "qualitative" estimates are available for $\widehat{U}$. We've stated earlier that the first order symbol

$$
\widehat{A}(\xi) = \begin{pmatrix} 0 & i\xi^T & 0 \\ i\xi & 0 & i\xi \\ 0 & i\frac{2}{\mathrm{D}}\xi^T & 0 \end{pmatrix}
\tag{4.22}
$$

and the second order symbol

$$\widehat{B}(\xi) = \begin{pmatrix} 0 & \mathbf{0} & 0 \\ \mathbf{0} & \epsilon\mu(I|\xi|^2 + \frac{\mathrm{D}-2}{\mathrm{D}}\xi\xi^T) & \mathbf{0} \\ 0 & \mathbf{0} & \epsilon\kappa\frac{2}{\mathrm{D}}|\xi|^2 \end{pmatrix} \tag{4.23}$$

don't commute, so we cannot get dissipative properties directly. However, according to Kawashima [39], if none of the null vectors of $\widehat{B}$ lie in eigenspace of $\widehat{A}$ (a condition that is satisfied by the linearized Navier-Stokes system), then the interaction between convective and diffusive effects actually result in a weak smoothing of the density. In fact,

Lemma 4.1.1 *[39] The solution of the Fourier transform of the linearized Navier-Stokes system (*4.1*) satisfies*

$$|\widehat{U}(t,\xi)|^2 \le Me^{-\frac{\epsilon c}{2}\frac{|\xi|^2}{1+|\xi|^2}t}|\widehat{U}(0,\xi)|^2, \tag{4.24}$$

where constant c depends on dimension D *and μ, κ only.*

Note, though, the decay rate is less than ideal for large ξ in the above expression. We compute $e^{\mathbf{Q}t}\widehat{U}^{\mathrm{in}}$ explicitly in the rest of the chapter.

We now use natural fundamental set of solutions to derive an explicit expression of the matrix exponential $e^{\mathbf{Q}t}$. Our method to compute the matrix exponential $e^{\mathbf{Q}t}$ has three step

1. Find a polynomial $p(z)$ that annihilates $\mathbf{Q}$. Let m denote its degree.

2. Compute $N_0(t)$, $N_1(t)$,...,$N_{m-1}(t)$, the natural fundamental set of solutions associated with the mth-order differential operator $p(\mathbf{D})$ and the initial time 0.

3. Compute the matrix exponential $e^{\mathbf{Q}t}$ by the formula

$$e^{\mathbf{Q}t} = N_0(t)\mathbf{I} + N_1(t)\mathbf{Q} + ... + N_{m\text{-}1}(t)\mathbf{Q}^{m\text{-}1}. \tag{4.25}$$

By the Cayley-Hamilton Theorem, we can always find a polynomial of degree $\mathrm{D}+2$ that annihilates $\mathbf{Q}$ namely, the characteristic polynomial of $\mathbf{Q}$. So, we will have to compute $\mathbf{Q}^{\mathrm{D}+1}$ in step 3. We would like to find a polynomial of smaller degree that also annihilates $\mathbf{Q}$.

In fact, the characteristic polynomial of $\mathbf{Q}$ is

$$\begin{aligned} P_{\mathbf{Q}}(z) &= (z\epsilon\mu|\xi|^2)^{\mathrm{D}-1} \\ &\quad \left[z^3 + (\epsilon\mu\tfrac{2(\mathrm{D}-1)}{\mathrm{D}}|\xi|^2 + \epsilon\kappa\tfrac{2}{\mathrm{D}}|\xi|^2)z^2 + (\epsilon^2\mu\kappa\tfrac{4(\mathrm{D}-1)}{\mathrm{D}^2}|\xi|^4 + \tfrac{\mathrm{D}+2}{\mathrm{D}}|\xi|^2)z + \epsilon\kappa\tfrac{2}{\mathrm{D}}|\xi|^4\right]. \end{aligned} \tag{4.26}$$

Clearly $-\epsilon\mu|\xi|^2$ is a root of multiplicity $\mathrm{D}-1$; moreover, $\{y\,|\xi^T \cdot y = 0, y \in \mathbb{R}^{\mathrm{D}}\}$ forms a $\mathrm{D}-1$-dimensional eigenspace corresponding to the eigenvalue $-\epsilon\mu|\xi|^2$. Thus if

$$p_4(z) = (z+\epsilon\mu|\xi|^2)(z^3+(\epsilon\mu\tfrac{2(\mathrm{D}-1)}{\mathrm{D}}|\xi|^2+\epsilon\kappa\tfrac{2}{\mathrm{D}}|\xi|^2)z^2+(\epsilon^2\mu\kappa\tfrac{4(\mathrm{D}-1)}{\mathrm{D}^2}|\xi|^4+\tfrac{\mathrm{D}+2}{\mathrm{D}}|\xi|^2)z+\epsilon\kappa\tfrac{2}{\mathrm{D}}|\xi|^4)$$

has four distinct roots, $\mathbf{Q}$ is diagonalizable, and $p_4(\mathbf{Q}) = 0$.

To see what values of ξ would yield double roots of $p_4(z)$, we now write the exact forms of the cubic roots of $p_3(z)$. First, let $z = -\epsilon\kappa|\xi|^2 y$ in the cubic polynomial $p_3(z) = z^3 + (\epsilon\mu\frac{2(\mathrm{D}-1)}{\mathrm{D}}|\xi|^2 + \epsilon\kappa\frac{2}{\mathrm{D}}|\xi|^2)z^2 + (\epsilon^2\mu\kappa\frac{4(\mathrm{D}-1)}{\mathrm{D}^2}|\xi|^4 + \frac{\mathrm{D}+2}{\mathrm{D}}|\xi|^2)z + \epsilon\kappa\frac{2}{\mathrm{D}}|\xi|^4$, and $p_3(z)$ becomes

$$-\epsilon^3\kappa^3|\xi|^6\left(y^3 - (\tfrac{\mu}{\kappa}\tfrac{2(\mathrm{D}-1)}{\mathrm{D}} + \tfrac{2}{\mathrm{D}})y^2 + (\tfrac{\mu}{\kappa}\tfrac{4(\mathrm{D}-1)}{\mathrm{D}^2} + \tfrac{1}{\epsilon^2\kappa^2|\xi|^2}\tfrac{\mathrm{D}+2}{\mathrm{D}})y - \tfrac{2}{\mathrm{D}}\tfrac{1}{\epsilon^2\kappa^2|\xi|^2}\right).$$

To find the roots of

$$y^3 - (\frac{\mu}{\kappa}\frac{2(\mathrm{D}-1)}{\mathrm{D}} + \frac{2}{\mathrm{D}})y^2 + (\frac{\mu}{\kappa}\frac{4(\mathrm{D}-1)}{\mathrm{D}^2} + \frac{1}{\epsilon^2\kappa^2|\xi|^2}\frac{\mathrm{D}+2}{\mathrm{D}})y - \frac{2}{\mathrm{D}}\frac{1}{\epsilon^2\kappa^2|\xi|^2}\,,$$

it suffices to find points of intersection of $y(y^2 - (\frac{\mu}{\kappa}\frac{2(\mathrm{D}-1)}{\mathrm{D}} + \frac{2}{\mathrm{D}})y + \frac{\mu}{\kappa}\frac{4(\mathrm{D}-1)}{\mathrm{D}^2})$ (with x-intercepts 0, $\frac{2}{\mathrm{D}}$, $\frac{\mu}{\kappa}\frac{2(\mathrm{D}-1)}{\mathrm{D}}$)and the line $l = -\frac{1}{\epsilon^2\kappa^2|\xi|^2}\frac{\mathrm{D}+2}{\mathrm{D}}y + \frac{2}{\mathrm{D}}\frac{1}{\epsilon^2\kappa^2|\xi|^2}$(with x-intercept $\frac{2}{\mathrm{D}+2}$).

We now consider a special case in which explicit estimate can be done for e^Qt. Notice that when

$$\mu \;=\; \tfrac{\mathrm{D}}{(\mathrm{D}-1)(\mathrm{D}+2)}\kappa, \tag{4.27}$$

$\frac{\mu}{\kappa}\frac{2(\mathrm{D}-1)}{\mathrm{D}} = \frac{2}{\mathrm{D}+2}$, so $\frac{2}{\mathrm{D}+2}$ is one point of intersection, therefore $-\frac{2}{\mathrm{D}+2}\epsilon\kappa|\xi|^2$ is one root of the cubic polynomial $p_3(z)$.

Under the relation (4.27),

$$p_3(z) = z^3 + \epsilon\kappa|\xi|^2\tfrac{4(\mathrm{D}+1)}{(\mathrm{D}+2)\mathrm{D}}z^2 + \left[\tfrac{4\epsilon^2\kappa^2|\xi|^4}{(\mathrm{D}+2)\mathrm{D}} + \tfrac{\mathrm{D}+2}{\mathrm{D}}|\xi|^2\right]z + \tfrac{2}{\mathrm{D}}\epsilon\kappa|\xi|^4\,,$$

and the two other roots of $p_3(z)$ are solutions of

$$z^2 + \frac{2\epsilon\kappa|\xi|^2}{\mathrm{D}}z + \tfrac{\mathrm{D}+2}{\mathrm{D}}|\xi|^2 = 0, \tag{4.28}$$

for which

$$\Delta \;=\; \frac{4(\epsilon\kappa|\xi|^2)^2}{\mathrm{D}^2} - \frac{4(\mathrm{D}+2)}{\mathrm{D}}|\xi|^2.$$

We thus split ξ into three groups accordingly:

Case 1: $|\xi|^2 \;<\; \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}$, therefore $\Delta < 0$, and (4.28) has a complex conjugate pair of roots: $-\frac{1}{\mathrm{D}}\epsilon\kappa|\xi|^2 \pm \frac{i}{\mathrm{D}}\sqrt{-\epsilon^2\kappa^2|\xi|^4 + \mathrm{D}(\mathrm{D}+2)|\xi|^2}$; $p_3(z)$ has three distinct roots.

Case 2: $|\xi|^2 = \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}$, therefore $\Delta = 0$; $p_3(z)$ has double roots $-\frac{1}{\mathrm{D}}\epsilon\kappa|\xi|^2$.

Case 3: $|\xi|^2 > \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}$, therefore $\Delta > 0$, and Eq (4.28) has distinct real roots:

$-\frac{1}{\mathrm{D}}\epsilon\kappa|\xi|^2 \pm \frac{1}{\mathrm{D}}\sqrt{\epsilon^2\kappa^2|\xi|^4 - \mathrm{D}(\mathrm{D}+2)|\xi|^2}$.

To see if $p_4(z)$ has any double roots, it remains to check whether any of the following terms are equal to one another:

$$\begin{aligned} a &= -\frac{\epsilon\kappa\mathrm{D}|\xi|^2}{(\mathrm{D}-1)(\mathrm{D}+2)}, \\ b &= -\frac{2\epsilon\kappa|\xi|^2}{\mathrm{D}+2}, \\ c &= -\frac{1}{\mathrm{D}}\epsilon\kappa|\xi|^2 + \frac{1}{\mathrm{D}}\sqrt{\epsilon^2\kappa^2|\xi|^4 - \mathrm{D}(\mathrm{D}+2)|\xi|^2} \\ d &= -\frac{1}{\mathrm{D}}\epsilon\kappa|\xi|^2 - \frac{1}{\mathrm{D}}\sqrt{\epsilon^2\kappa^2|\xi|^4 - \mathrm{D}(\mathrm{D}+2)|\xi|^2}\,. \end{aligned} \tag{4.29}$$

These are not always distinct roots. In fact,

$a = b$ iff $\mathrm{D} = 2$;

$a = c$ iff

$-\frac{\epsilon\kappa\mathrm{D}|\xi|^2}{(\mathrm{D}-1)(\mathrm{D}+2)} = -\frac{1}{\mathrm{D}}\epsilon\kappa|\xi|^2 + \frac{1}{\mathrm{D}}\sqrt{\epsilon^2\kappa^2|\xi|^4 - \mathrm{D}(\mathrm{D}+2)|\xi|^2}$

iff $\epsilon\kappa\frac{\mathrm{D}-2}{(\mathrm{D}-1)(\mathrm{D}+2)}|\xi|^2 = \frac{1}{\mathrm{D}}\sqrt{\epsilon^2\kappa^2|\xi|^4 - \mathrm{D}(\mathrm{D}+2)|\xi|^2}$

iff $\mathrm{D}(\mathrm{D}+2)|\xi|^2 == \epsilon^2\kappa^2|\xi|^4[1 - \frac{(\mathrm{D}-2)^2}{(\mathrm{D}-1)^2(\mathrm{D}+2)^2}]$

iff

$$|\xi|^2 = \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2[1 - \frac{(\mathrm{D}-2)^2}{(\mathrm{D}-1)^2(\mathrm{D}+2)^2}]} \tag{4.30}$$

For $\mathrm{D} > 2$, $\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2[1-\frac{(\mathrm{D}-2)^2}{(\mathrm{D}-1)^2(\mathrm{D}+2)^2}]} > \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}$.

$b = d$ iff

$-\frac{2\epsilon\kappa|\xi|^2}{\mathrm{D}+2} = -\frac{1}{\mathrm{D}}\epsilon\kappa|\xi|^2 - \frac{1}{\mathrm{D}}\sqrt{\epsilon^2\kappa^2|\xi|^4 - \mathrm{D}(\mathrm{D}+2)|\xi|^2}$

iff $\frac{1}{\mathrm{D}}\sqrt{\epsilon^2\kappa^2|\xi|^4 - \mathrm{D}(\mathrm{D}+2)|\xi|^2} = \frac{\epsilon\kappa(\mathrm{D}-2)|\xi|^2}{\mathrm{D}(\mathrm{D}+2)}$

iff $[1-\frac{(\mathrm{D}-2)^2}{(\mathrm{D}+2)^2}]\epsilon^2\kappa^2|\xi|^4 = \mathrm{D}(\mathrm{D}+2)|\xi|^2$

iff

$$|\xi|^2 = \frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2} \tag{4.31}$$

Notice that $\frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2} > \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}$ for $\mathrm{D} > 2$. Further calculation shows that $a = d$, $b = c$ iff $\mathrm{D} = 2$ and $|\xi|^2 = \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}$. In fact, $a = d$ iff $\frac{1}{\mathrm{D}}\sqrt{\epsilon^2\kappa^2|\xi|^4 - \mathrm{D}(\mathrm{D}+2)|\xi|^2} = \frac{-\epsilon\kappa(\mathrm{D}-2)|\xi|^2}{\mathrm{D}(\mathrm{D}-1)(\mathrm{D}+2)}$.
$b = c$ iff $\frac{1}{\mathrm{D}}\sqrt{\epsilon^2\kappa^2|\xi|^4 - \mathrm{D}(\mathrm{D}+2)|\xi|^2} = \frac{-\epsilon\kappa(\mathrm{D}-2)|\xi|^2}{\mathrm{D}(\mathrm{D}+2)}$.

The two equalities hold iff $\mathrm{D} = 2$ and $|\xi|^2 = \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}$.

We already know that $c = d$ when $|\xi|^2 = \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}$.

In the following discussion, we only consider $\mathrm{D} > 2$. In this case, $p_4(z)$ has double roots when $|\xi|^2 = \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}, \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2[1-\frac{(\mathrm{D}-2)^2}{(\mathrm{D}-1)^2(\mathrm{D}+2)^2}]}, \frac{\mathrm{D}(\mathrm{D}+2)}{8\epsilon^2\kappa^2}$. Thus we split $|\xi|^2 > \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}$ furthermore into three subintervals:

(I) $\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2} < |\xi|^2 < \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2[1-\frac{(\mathrm{D}-2)^2}{(\mathrm{D}-1)^2(\mathrm{D}+2)^2}]}$,
(II) $\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2[1-\frac{(\mathrm{D}-2)^2}{(\mathrm{D}-1)^2(\mathrm{D}+2)^2}]} < |\xi|^2 < \frac{\mathrm{D}(\mathrm{D}+2)}{8\epsilon^2\kappa^2}$,
(III) $|\xi|^2 > \frac{\mathrm{D}(\mathrm{D}+2)}{8\epsilon^2\kappa^2}$.

So $\mathbf{Q}$ is diagonalizable over (I), (II), (III) as well as $|\xi|^2 < \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}$. Further calculation shows that $p_4(\mathbf{Q}) \neq 0$ when $|\xi|^2 = \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}, \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2[1-\frac{(\mathrm{D}-2)^2}{(\mathrm{D}-1)^2(\mathrm{D}+2)^2}]}, \frac{\mathrm{D}(\mathrm{D}+2)}{8\epsilon^2\kappa^2}$. Except when ξ taking those values, the annihilating polynomial of $\mathbf{Q}$ is of degree 4, therefore by (4.25), we only need to compute $\mathbf{Q}^2$ and $\mathbf{Q}^3$ for such ξ.

Here we give each entry of $\mathbf{Q}^2$ and $\mathbf{Q}^3$ under the relation (4.27) as follows:

$$\mathbf{Q} = \begin{pmatrix} 0 & i\xi^T & 0 \\ i\xi & \epsilon\frac{\mathrm{D}}{(\mathrm{D}-1)(\mathrm{D}+2)}\kappa(I|\xi|^2 + \frac{\mathrm{D}-2}{\mathrm{D}}\xi\xi^T) & i\xi \\ 0 & i\frac{2}{\mathrm{D}}\xi^T & \epsilon\kappa\frac{2}{\mathrm{D}}|\xi|^2 \end{pmatrix}$$

$$\mathbf{Q}^2 = \begin{pmatrix} -|\xi|^2 & i\epsilon\kappa\frac{2}{\mathrm{D}+2}|\xi|^2\xi^T & -|\xi|^2 \\ i\epsilon\kappa\frac{2}{\mathrm{D}+2}|\xi|^2\xi & \substack{-\frac{\mathrm{D}+2}{\mathrm{D}}\xi\xi^T + \frac{\epsilon^2\kappa^2\mathrm{D}^2}{(\mathrm{D}-1)^2(\mathrm{D}+2)^2} * \\ [|\xi|^4\mathbf{I} + 2\frac{\mathrm{D}-2}{\mathrm{D}}\xi\xi^T|\xi|^2 + \left(\frac{\mathrm{D}-2}{\mathrm{D}}\right)^2\xi\xi^T|\xi|^2]} & i4\epsilon\kappa\frac{\mathrm{D}+1}{(\mathrm{D}+2)\mathrm{D}} \\ -\frac{2}{\mathrm{D}}|\xi|^2 & i8\epsilon\kappa\frac{\mathrm{D}+1}{(\mathrm{D}+2)\mathrm{D}^2} & -\frac{2}{\mathrm{D}}|\xi|^2 + \epsilon^2\kappa^2\left(\frac{2}{\mathrm{D}}\right)^2|\xi|^4 \end{pmatrix}$$

$$\mathbf{Q}^3 = \begin{pmatrix} Q_{11}^{(3)} & Q_{12}^{(3)} & Q_{13}^{(3)} \\ Q_{21}^{(3)} & Q_{22}^{(3)} & Q_{23}^{(3)} \\ Q_{31}^{(3)} & Q_{32}^{(3)} & Q_{33}^{(3)} \end{pmatrix}, \tag{4.32}$$

where

$$\begin{aligned}
Q_{11}^{(3)} &= \epsilon\kappa\tfrac{2}{\mathrm{D}+2}|\xi|^4, \\
Q_{12}^{(3)} &= i\tfrac{\mathrm{D}+2}{\mathrm{D}}|\xi|^2\xi^T - i\epsilon^2\kappa^2\left(\tfrac{2}{\mathrm{D}+2}\right)^2|\xi|^4\xi^T \\
Q_{13}^{(3)} &= 4\epsilon\kappa|\xi|^4\tfrac{\mathrm{D}+1}{\mathrm{D}(\mathrm{D}+2)} \\
Q_{21}^{(3)} &= i\tfrac{\mathrm{D}+2}{\mathrm{D}}|\xi|^2\xi - i\epsilon^2\kappa^2\left(\tfrac{2}{\mathrm{D}+2}\right)^2|\xi|^4\xi \\
Q_{22}^{(3)} &= \epsilon\kappa\tfrac{4(\mathrm{D}+1)}{\mathrm{D}^2}|\xi|^2\xi\xi^T - \tfrac{\epsilon^3\kappa^3\mathrm{D}^3}{(\mathrm{D}-1)^3(\mathrm{D}+2)^3}\left[|\xi|^6\mathbf{I} + [3\tfrac{\mathrm{D}-2}{\mathrm{D}} + 3\left(\tfrac{\mathrm{D}-2}{\mathrm{D}}\right)^2 + \left(\tfrac{\mathrm{D}-2}{\mathrm{D}}\right)^3]|\xi|^4\xi\xi^T\right] \\
Q_{23}^{(3)} &= i\tfrac{\mathrm{D}+2}{\mathrm{D}}|\xi|^2\xi - i4\epsilon^2\kappa^2\left[\tfrac{1}{(\mathrm{D}+2)^2} + \tfrac{1}{(\mathrm{D}+2)\mathrm{D}} + \tfrac{1}{\mathrm{D}^2}\right]|\xi|^4\xi \\
Q_{31}^{(3)} &= \epsilon\kappa\tfrac{8(\mathrm{D}+1)}{\mathrm{D}^2(\mathrm{D}+2)}|\xi|^4 \\
Q_{32}^{(3)} &= i\tfrac{2(\mathrm{D}+2)}{\mathrm{D}^2}|\xi|^2\xi^T - i\epsilon^2\kappa^2\tfrac{8}{\mathrm{D}}\left[\tfrac{1}{(\mathrm{D}+2)^2} + \tfrac{1}{(\mathrm{D}+2)\mathrm{D}} + \tfrac{1}{\mathrm{D}^2}\right]|\xi|^4\xi^T \\
Q_{33}^{(3)} &= \epsilon\kappa\tfrac{4(3\mathrm{D}+4)}{(\mathrm{D}+2)\mathrm{D}^2}|\xi|^4 - \epsilon^3\kappa^3(\tfrac{2}{\mathrm{D}})^3|\xi|^6.
\end{aligned} \tag{4.33}$$

In the next two sections, we compute the corresponding natural fundamental set of solutions $N_0(t)$, $N_1(t)$,...,$N_{m-1}(t)$ for each group of ξ, hence getting explicit expression of the matrix exponential $e^{\mathbf{Q}t}$.

4.1.1 Decay Estimate for Large Wave Number

We see from the previous discussion that for $|\xi|^2 > \frac{D(D+2)}{\epsilon^2\kappa^2}$, all roots of the cubic polynomial $p_3(z) = z^3 + \epsilon\kappa|\xi|^2\frac{4(D+1)}{(D+2)D}z^2 + \left[\frac{4\epsilon^2\kappa^2|\xi|^4}{(D+2)D} + \frac{D+2}{D}|\xi|^2\right]z + \frac{2}{D}\epsilon\kappa|\xi|^4$ are real, and are asymptotically of order $-\epsilon\kappa|\xi|^2$ or $-\frac{1}{\epsilon\kappa}$, suggesting a decay rate faster that of Kawashima-type($\sim e^{-\epsilon t}$) might be obtained.

Note, however, that the characteristic polynomial of $\mathbf{Q}$ has double roots when $|\xi|^2 = \frac{D(D+2)}{\epsilon^2\kappa^2}, \frac{D(D+2)}{\epsilon^2\kappa^2[1-\frac{(D-2)^2}{(D-1)^2(D+2)^2}]}, \frac{D(D+2)}{8\epsilon^2\kappa^2}$, so it is unclear whether entries of $\mathbf{e}^{\mathbf{Q}t}$ would decay like $e^{-\frac{1}{\epsilon\kappa}}$ or $e^{-\epsilon\kappa|\xi|^2}$ when $|\xi|^2$ take values which are close to $\frac{D(D+2)}{\epsilon^2\kappa^2}$, $\frac{D(D+2)}{\epsilon^2\kappa^2[1-\frac{(D-2)^2}{(D-1)^2(D+2)^2}]}$, $\frac{D(D+2)}{8\epsilon^2\kappa^2}$. In this section, we 'zoom in' around these special values and examine the corresponding entries of $e^{\mathbf{Q}t}$. We also derive general estimates of entries of $e^{\mathbf{Q}t}$ for other ξ in the domain $|\xi|^2 > \frac{D(D+2)}{\epsilon^2\kappa^2}$. We conclude that for $t \geq 1$, all entries of $e^{\mathbf{Q}t}$ are controlled by $C(D)e^{-\frac{C(D)t}{\epsilon\kappa}}$.

To start with, we give the explicit form of the matrix exponential $e^{\mathbf{Q}t}$ for $|\xi|^2 > \frac{D(D+2)}{\epsilon^2\kappa^2}$. Recall that the annihilating polynomial is

$$p(z) = (z+\frac{\epsilon\kappa D|\xi|^2}{(D-1)(D+2)})(z^3+\epsilon\kappa|\xi|^2\frac{4(D+1)}{(D+2)D}|\xi|^2z^2+\left[\frac{4\epsilon^2\kappa^2|\xi|^4}{(D+2)D} + \frac{D+2}{D}|\xi|^2\right]z+\epsilon\kappa\frac{2}{D}|\xi|^4). \tag{4.34}$$

To compute the natural fundamental set of solutions associated with $p(\mathbf{D})$, we write a general solution associated with $p(\mathbf{D})$:

$$X(t) = c_1e^{at} + c_2e^{bt} + c_3e^{ct} + c_4e^{dt}, \tag{4.35}$$

with a, b, c, d defined by (4.29), therefore we write

$$N_i(t) = c_1^ie^{at} + c_2^ie^{bt} + c_3^ie^{ct} + c_4^ie^{dt}. \tag{4.36}$$

To find $N_0(t)$, $N_1(t)$,...,$N_3(t)$, it suffices to find $c_1^0, ..., c_4^0$,..., $c_1^3, ..., c_4^3$ in (4.35), such that

$$N_i^{(j)}(0) \;=\; \delta_{ij}, \;\text{for}\, i = 0, 1, ..., 3. \tag{4.37}$$

Note that

$$\begin{pmatrix} X(0) \\ X'(0) \\ X''(0) \\ X'''(0) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}. \tag{4.38}$$

Therefore,

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{pmatrix} \begin{pmatrix} c_1^0 & c_1^1 & c_1^2 & c_1^3 \\ c_2^0 & c_2^1 & c_2^2 & c_2^3 \\ c_3^0 & c_3^1 & c_3^2 & c_3^3 \\ c_4^0 & c_4^1 & c_4^2 & c_4^3 \end{pmatrix} = \mathbf{I}\,. \tag{4.39}$$

$$\begin{pmatrix} c_1^0 & c_1^1 & c_1^2 & c_1^3 \\ c_2^0 & c_2^1 & c_2^2 & c_2^3 \\ c_3^0 & c_3^1 & c_3^2 & c_3^3 \\ c_4^0 & c_4^1 & c_4^2 & c_4^3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{bcd}{(-a+b)(-a+c)(-a+d)} & -\frac{cd+bc+bd}{(-a+b)(-a+c)(-a+d)} & \frac{b+c+d}{(-a+b)(-a+c)(-a+d)} & -\frac{1}{(-a+b)(-a+c)(-a+d)} \\ \frac{acd}{(a-b)(b-c)(b-d)} & -\frac{cd+ac+ad}{(a-b)(b-c)(b-d)} & \frac{a+c+d}{(a-b)(b-c)(b-d)} & -\frac{1}{(a-b)(b-c)(b-d)} \\ \frac{abd}{(a-c)(-b+c)(c-d)} & -\frac{bd+ab+ad}{(a-c)(-b+c)(c-d)} & \frac{a+b+d}{(a-c)(-b+c)(c-d)} & -\frac{1}{(a-c)(-b+c)(c-d)} \\ \frac{abc}{(a-d)(-b+d)(-c+d)} & -\frac{bc+ab+ac}{(a-d)(-b+d)(-c+d)} & \frac{a+b+c}{(a-d)(-b+d)(-c+d)} & -\frac{1}{(a-d)(-b+d)(-c+d)} \end{pmatrix}. \tag{4.40}$$

We now illustrate the procedure for computing the $(1,1)$th entry of $e^{\mathbf{Q}t}$, and generalize it to all entries. Since the first entries of $\mathbf{Q}$, $\mathbf{Q}^2$ and $\mathbf{Q}^3$ are 0, $-|\xi|^2$ and $\epsilon\kappa\frac{2}{\mathrm{D}+2}|\xi|^4$, we may write the $(1,1)$th entry as

$$c_1e^{at}+c_2e^{bt}+c_3e^{ct}+c_4e^{dt}, \tag{4.41}$$

where a,b,c,d are defined by (4.29), and

$$\begin{aligned}
c_1 &= \frac{bcd-|\xi|^2(b+c+d)-\frac{2\epsilon\kappa}{\mathrm{D}+2}|\xi|^4}{(-a+b)(-a+c)(-a+d)},\\
c_2 &= \frac{acd-|\xi|^2(a+c+d)-\frac{2\epsilon\kappa}{\mathrm{D}+2}|\xi|^4}{(a-b)(b-c)(b-d)},\\
c_3 &= \frac{abd-|\xi|^2(a+b+d)-\frac{2\epsilon\kappa}{\mathrm{D}+2}|\xi|^4}{(a-c)(-b+c)(c-d)},\\
c_4 &= \frac{abc-|\xi|^2(a+b+c)-\frac{2\epsilon\kappa}{\mathrm{D}+2}|\xi|^4}{(a-d)(-b+d)(-c+d)}.
\end{aligned} \tag{4.42}$$

. From this point, we denote any positive value depending on dimension D only by $C(\mathrm{D})$. For $|\xi|^2\sim\frac{C(\mathrm{D})}{\epsilon^2\kappa^2}$ (which includes all ξ in Group(I), (II), and those in Group(III) whose values are close to $\dfrac{\mathrm{D}(\mathrm{D}+2)}{8\epsilon^2\kappa^2}$, $\epsilon\kappa|\xi|^2\sim\frac{C(\mathrm{D})}{\epsilon\kappa}$. Therefore, $a,b,d\sim\frac{-C(\mathrm{D})}{\epsilon\kappa}$; as for c, we notice that

$$\begin{aligned}
c &= -\frac{1}{\mathrm{D}}\epsilon\kappa|\xi|^2+\frac{1}{\mathrm{D}}\sqrt{\epsilon^2\kappa^2|\xi|^4-\mathrm{D}(\mathrm{D}+2)|\xi|^2}\\
&= \frac{\frac{\epsilon^2\kappa^2|\xi|^4}{\mathrm{D}^2}-\frac{1}{\mathrm{D}^2}(\epsilon^2\kappa^2|\xi|^4-\mathrm{D}(\mathrm{D}+2)|\xi|^2)}{-\frac{1}{\mathrm{D}}\epsilon\kappa|\xi|^2-\frac{1}{\mathrm{D}}\sqrt{\epsilon^2\kappa^2|\xi|^4-\mathrm{D}(\mathrm{D}+2)|\xi|^2}}\\
&= \frac{\frac{\mathrm{D}+2}{\mathrm{D}}|\xi|^2}{d}\sim\frac{-C(\mathrm{D})}{\epsilon\kappa}.
\end{aligned} \tag{4.43}$$

Moreover, by (4.42), we see that as long as $a-b,a-c,a-d,b-c,b-d,c-d$ are "not too close", say, for example, that they are all equivalent to $\frac{1}{\epsilon\kappa}$, then c_1,c_2,c_3,c_4 have constant(depending on dimension D only) bounds.

On the other hand, c_1,c_2,c_3,c_4 will become unbounded when $a-b,a-c,a-d,b-c,b-d,c-d$ are close to zero, i.e. around those ξ for which $|\xi|^2=\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}$, $|\xi|^2=$

$\frac{\mathrm{D(D+2)}}{\epsilon^2\kappa^2[1-\frac{(\mathrm{D}-2)^2}{(\mathrm{D}-1)^2(\mathrm{D}+2)^2}]}$,$|\xi|^2 = \frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2}$. To elaborate, we now zoom in around one of these values, $|\xi| = \frac{(\mathrm{D}+2)^{\frac{3}{2}}}{2\sqrt{2}\epsilon\kappa}$. The goal is to show that (4.41) remain bounded even for $|\xi|$ close to $\frac{(\mathrm{D}+2)^{\frac{3}{2}}}{2\sqrt{2}\epsilon\kappa}$(for which the value of $b-d$ is close to zero). To this end, we make the following change of variables:

Step 1: Switching to polar coordinates: $(x_1, ..., x_D) \to (r, \theta_1, ..., \theta_{\mathrm{D}-1})$.

Step 2: Rescaling around r around $r = \frac{(\mathrm{D}+2)^{\frac{3}{2}}}{2\sqrt{2}\epsilon\kappa}$, i.e. let $\tilde{r} = \frac{2\sqrt{2}\epsilon\kappa}{(\mathrm{D}+2)^{\frac{3}{2}}} r$.

Step 3: Recentering $\tilde{r}$ around 0: let $\gamma = \tilde{r} - 1$. Since $b = d$ when $\gamma = 0$, it would be of interest to represent b, d and c_2, c_4 in terms of γ.

$$
\begin{aligned}
b - d &= -\frac{2\epsilon\kappa|\xi|^2}{\mathrm{D}+2} + \frac{1}{\mathrm{D}}\epsilon\kappa|\xi|^2 + \frac{1}{\mathrm{D}}\sqrt{\epsilon^2\kappa^2|\xi|^4 - \mathrm{D}(\mathrm{D}+2)|\xi|^2} \\
&= \frac{\epsilon\kappa|\xi|^2}{\mathrm{D}}\left[\frac{-(\mathrm{D}-2)}{\mathrm{D}+2} + \sqrt{1 - \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2|\xi|^2}}\right] \\
&= \frac{\epsilon\kappa|\xi|^2}{\mathrm{D}} \frac{1 - \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2|\xi|^2} - \frac{(\mathrm{D}-2)^2}{(\mathrm{D}+2)^2}}{\frac{\mathrm{D}-2}{\mathrm{D}+2} + \sqrt{1 - \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2|\xi|^2}}}.
\end{aligned}
\tag{4.44}
$$

Since $|\xi|^2 = \frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2}(1+\gamma)^2$, $1 - \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2|\xi|^2} - \frac{(\mathrm{D}-2)^2}{(\mathrm{D}+2)^2} = \frac{8\mathrm{D}}{(\mathrm{D}+2)^2}[1 - \frac{1}{(1+\gamma)^2}]$. In particular, we have $\delta := d - b \sim -C(\mathrm{D})\epsilon\kappa|\xi|^2[1 - \frac{1}{(1+\gamma)^2}]$. We then proceed to express c_2, c_4 in terms of δ as well:

$$
\begin{aligned}
c_2 &= \frac{acb - |\xi|^2(a+c+b) - \frac{2\epsilon\kappa|\xi|^4}{\mathrm{D}+2} - \delta|\xi|^2 + ac\delta}{-(a-b)(b-c)\delta}, \\
c_4 &= \frac{abc - |\xi|^2(a+b+c) - \frac{2\epsilon\kappa|\xi|^4}{\mathrm{D}+2}}{(a-b-\delta)(b-c-\delta)\delta} \\
&= \frac{[abc - |\xi|^2(a+b+c) - \frac{2\epsilon\kappa|\xi|^4}{\mathrm{D}+2}](1 + \frac{\delta}{a-b} + \frac{\delta}{b-c} + O(\frac{\delta}{a-b} + \frac{\delta}{b-c})^2)}{(a-b)(b-c)\delta},
\end{aligned}
\tag{4.45}
$$

and

$$\begin{aligned}\Delta :=& c_2 + c_4\\ =&\frac{|\xi|^2\delta - ac\delta + (1+\frac{\delta}{a-b}+\frac{\delta}{b-c}+O(\frac{\delta}{a-b}+\frac{\delta}{b-c})^2)[abc - |\xi|^2(a+b+c) - \frac{2\epsilon\kappa|\xi|^4}{\mathrm{D}+2}]}{(a-b)(b-c)\delta}\\ =&\frac{|\xi|^2 - ac}{(a-b)(b-c)} + \frac{(1+\frac{\delta}{a-b}+\frac{\delta}{b-c}+O(\frac{\delta}{a-b}+\frac{\delta}{b-c})^2)[abc - |\xi|^2(a+b+c) - \frac{2\epsilon\kappa|\xi|^4}{\mathrm{D}+2}]}{(a-b)(b-c)\delta}.\end{aligned} \tag{4.46}$$

Note that we have to pick γ such that $|1-\frac{1}{(1+\gamma)^2}| \leq \eta_1'$, where $\eta_1' \sim o(1)$, so that $\frac{\delta}{a-b}, \frac{\delta}{b-c} \sim o(1)$. *For such* γ, $|\left(\frac{\mathrm{D}+2}{\mathrm{D}-2}\right)^2(2\gamma+\gamma^2)| < 1$ *always hold.* Since $\epsilon\kappa|\xi|^2 \sim \frac{1}{\epsilon\kappa}$, and

$$\frac{abc - |\xi|^2(a+b+c) - \frac{2\epsilon\kappa|\xi|^4}{\mathrm{D}+2}}{(a-b)^2(b-c)} \sim C(\mathrm{D}),$$
$$\frac{abc - |\xi|^2(a+b+c) - \frac{2\epsilon\kappa|\xi|^4}{\mathrm{D}+2}}{(a-b)(b-c)^2} \sim C(\mathrm{D}),$$

combining this with (4.46), we conclude that $|\Delta| \leq C(\mathrm{D})$. Therefore,

$$\begin{aligned}c_2e^{bt} + c_4e^{dt} =& c_2e^{bt} + (-c_2+\Delta)e^{(b+\delta)t}\\ =& c_2e^{bt} + (-c_2+\Delta)e^{bt} + (c_2-\Delta)e^{bt} - (c_2-\Delta)e^{bt}e^{\delta t}\\ =& \Delta e^{bt} - c_4e^{bt}(1-e^{\delta t}).\end{aligned} \tag{4.47}$$

Clearly the first term in the last equality is bounded by $C(\mathrm{D})e^{-\frac{C(\mathrm{D})}{\epsilon\kappa}t}$. The second term:

$$\begin{aligned}|c_4(1-e^{\delta t})e^{bt}| \leq& C(\mathrm{D})\left|\frac{1}{1-\frac{1}{(1+\gamma)^2}}\left[1-e^{C(\mathrm{D})\epsilon\kappa|\xi|^2(1-\frac{1}{(1+\gamma)^2})t}\right]\right|e^{-C(\mathrm{D})\epsilon\kappa|\xi|^2t}\\ \leq& C(\mathrm{D})\epsilon\kappa|\xi|^2te^{-C(\mathrm{D})\epsilon\kappa|\xi|^2t} \sim \frac{C(\mathrm{D})t}{\epsilon\kappa}e^{-\frac{C(\mathrm{D})t}{\epsilon\kappa}}\end{aligned} \tag{4.48}$$

($C(\mathrm{D})$ may be different from each other) by expansion of $e^{\delta t}$, for $|1-\frac{1}{(1+\gamma)^2}| \leq \frac{\epsilon\kappa}{C(\mathrm{D})t}$. Hence $|c_2e^{bt}+c_4e^{dt}| \leq \frac{C(\mathrm{D})t}{\epsilon\kappa}e^{-\frac{C(\mathrm{D})t}{\epsilon\kappa}}$ for γ such that

$$\left|1-\frac{1}{(1+\gamma)^2}\right| < \min\left\{\eta_1', \frac{\epsilon\kappa}{C(\mathrm{D})t}\right\}.$$

We may assume that $\frac{1}{(1+\gamma)^2} < 2$, in which case the inequality above holds for all γ satisfying $|\gamma^2 + 2\gamma| < \eta_1(t)$, where $\eta_1(t) = \frac{1}{2}\min\left\{\eta_1', \frac{\epsilon\kappa}{C(\mathrm{D})t}\right\}$, i.e.

$$-\eta_1(t) + 1 < (1+\gamma)^2 < \eta_1(t) + 1, \quad \eta_1(t) = \frac{1}{2}\min\left\{\eta_1', \frac{\epsilon\kappa}{C(\mathrm{D})t}\right\} \tag{4.49}$$

Recall that both c_1e^{at} and c_3e^{ct} are bounded by $C(\mathrm{D})e^{-\frac{C(\mathrm{D})t}{\epsilon\kappa}}$ by the previous arguments. Therefore the $(1,1)$th entry is bounded by $\frac{C(\mathrm{D})t}{\epsilon\kappa}e^{-\frac{C(\mathrm{D})t}{\epsilon\kappa}}$ for all γ satisfying (4.49), which may be translated back in terms of $|\xi|$ as

$$\frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2}(1-\eta_1(t)) < |\xi|^2 < \frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2}(1+\eta_1(t)). \tag{4.50}$$

Next, we zoom in around $|\xi|^2 = \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2[1-\frac{(\mathrm{D}-2)^2}{(\mathrm{D}-1)^2(\mathrm{D}+2)^2}]}$ and show that $c_1e^{at} + c_3e^{ct}$ is bounded in a neighborhood of $|\xi| = \sqrt{\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2[1-\frac{(\mathrm{D}-2)^2}{(\mathrm{D}-1)^2(\mathrm{D}+2)^2}]}}$. The argument is identical; nevertheless we include the details for the record.

First, we change variables as follows:

Step1: Switching to polar coordinates: $(x_1, ..., x_D) \to (r, \theta_1, ..., \theta_{\mathrm{D}-1})$.

Step 2: Rescaling around $r = \sqrt{\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2[1-\frac{(\mathrm{D}-2)^2}{(\mathrm{D}-1)^2(\mathrm{D}+2)^2}]}}$, i.e. let $\tilde{r} = \sqrt{\frac{\epsilon^2\kappa^2[1-\frac{(\mathrm{D}-2)^2}{(\mathrm{D}-1)^2(\mathrm{D}+2)^2}]}{\mathrm{D}(\mathrm{D}+2)}}r$.

Step 3: Recentering $\tilde{r}$ around 0: let $\gamma = \tilde{r} - 1$.

For simplicity we denote $\frac{\mathrm{D}(\mathrm{D}+2)}{1-\frac{(\mathrm{D}-2)^2}{(\mathrm{D}-1)^2(\mathrm{D}+2)^2}}$ by $\widetilde{C}$, thereby $|\xi|^2 = \frac{(\gamma+1)^2\widetilde{C}}{\epsilon^2\kappa^2}$.

Recall that $a = -\frac{\epsilon\kappa \mathrm{D}|\xi|^2}{(\mathrm{D}-1)(\mathrm{D}+2)}$, $c = -\frac{1}{\mathrm{D}}\epsilon\kappa|\xi|^2 + \frac{1}{\mathrm{D}}\sqrt{\epsilon^2\kappa^2|\xi|^4 - \mathrm{D}(\mathrm{D}+2)|\xi|^2}$,

$$\begin{aligned} a - c &= -\tfrac{\epsilon\kappa \mathrm{D}|\xi|^2}{(\mathrm{D}-1)(\mathrm{D}+2)} + \tfrac{1}{\mathrm{D}}\epsilon\kappa|\xi|^2 - \tfrac{1}{\mathrm{D}}\sqrt{\epsilon^2\kappa^2|\xi|^4 - \mathrm{D}(\mathrm{D}+2)|\xi|^2} \\ &= \epsilon\kappa|\xi|^2\left[\tfrac{-\mathrm{D}^2+(\mathrm{D}-1)(\mathrm{D}+2)}{\mathrm{D}(\mathrm{D}-1)(\mathrm{D}+2)}\right] - \tfrac{1}{\mathrm{D}}\sqrt{\epsilon^2\kappa^2|\xi|^4 - \mathrm{D}(\mathrm{D}+2)|\xi|^2} \\ &= \epsilon\kappa|\xi|^2\tfrac{(\mathrm{D}-2)}{\mathrm{D}(\mathrm{D}-1)(\mathrm{D}+2)} - \tfrac{1}{\epsilon\kappa \mathrm{D}}\sqrt{(\gamma+1)^4\widetilde{C}^2 - (\gamma+1)^2\widetilde{C}\mathrm{D}(\mathrm{D}+2)} \\ &= \frac{\epsilon\kappa(\gamma+1)^2\widetilde{C}(\mathrm{D}-2)}{\epsilon^2\kappa^2\mathrm{D}(\mathrm{D}-1)(\mathrm{D}+2)} - \tfrac{(\gamma+1)\sqrt{\widetilde{C}(\widetilde{C}-\mathrm{D}(\mathrm{D}+2))}}{\epsilon\kappa \mathrm{D}}\sqrt{1 + \tfrac{\widetilde{C}(\gamma^2+2\gamma)}{\widetilde{C}-\mathrm{D}(\mathrm{D}+2)}} \\ &= \tfrac{\widetilde{C}}{\epsilon\kappa \mathrm{D}}\left(\tfrac{(\mathrm{D}-2)}{(\mathrm{D}-1)(\mathrm{D}+2)} - \sqrt{1 - \tfrac{\mathrm{D}(\mathrm{D}+2)}{\widetilde{C}}}\right) + \tfrac{C(\mathrm{D})\gamma + O(\gamma^2)}{\epsilon\kappa}, \\ &= \tfrac{C(\mathrm{D})\gamma + O(\gamma^2)}{\epsilon\kappa}, \end{aligned} \tag{4.51}$$

since

$$\sqrt{1 - \frac{\mathrm{D}(\mathrm{D}+2)}{\widetilde{C}}} = \sqrt{\frac{(\mathrm{D}-2)^2}{(\mathrm{D}-1)^2(\mathrm{D}+2)^2}}.$$

Therefore the $(1,1)$th entry is bounded by $\frac{C(\mathrm{D})t}{\epsilon\kappa}e^{-\frac{C(\mathrm{D})t}{\epsilon\kappa}}$ for all γ such that $1-\eta_2(t) < (1+\gamma)^2 < 1+\eta_2(t)$, $(\eta_2 \sim o(1), \eta_2(t) = \min\{\eta_2', \frac{\epsilon\kappa}{C(\mathrm{D})t}\})$ which may be translated back in terms of $|\xi|$ as

$$\frac{\widetilde{C}}{\epsilon^2\kappa^2}(1-\eta_2(t)) < |\xi|^2 < \frac{\widetilde{C}}{\epsilon^2\kappa^2}(1+\eta_2(t)). \tag{4.52}$$

For the neighborhood of $|\xi|^2 = \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}$, despite the fact that $\delta := c - d \sim \frac{\sqrt{\gamma}}{\epsilon\kappa}$ instead of $\frac{\gamma}{\epsilon\kappa}$ as in the previous two cases, the argument is more or less the same: let

$$1 + \gamma = \frac{\epsilon\kappa}{\sqrt{\mathrm{D}(\mathrm{D}+2)}}|\xi|, \tag{4.53}$$

then

$$\begin{aligned} c - d &= \tfrac{2}{\mathrm{D}}\sqrt{\epsilon^2\kappa^2|\xi|^4 - \mathrm{D}(\mathrm{D}+2)|\xi|^2} \\ &= \tfrac{2|\xi|}{\mathrm{D}}\sqrt{(1+\gamma)\mathrm{D}(\mathrm{D}+2) - \mathrm{D}(\mathrm{D}+2)} \\ &= \tfrac{2(\mathrm{D}+2)}{\epsilon\kappa}\sqrt{\gamma}(1+\gamma), \end{aligned} \tag{4.54}$$

and

$$c_3 = \frac{abd - |\xi|^2(a+b+d) - \frac{2\epsilon\kappa}{\mathrm{D}+2}|\xi|^4}{(a-d-\delta)(-b+d+\delta)\delta},$$
$$c_4 = \frac{ab(d+\delta) - |\xi|^2(a+b+d+\delta) - \frac{2\epsilon\kappa}{\mathrm{D}+2}|\xi|^4}{-(a-d)(-b+d)\delta},$$

$$c_3 + c_4 = \frac{-ab\delta - |\xi|^2\delta + (abd - |\xi|^2(a+b+d) - \frac{2\epsilon\kappa}{\mathrm{D}+2}|\xi|^4)(1 + \frac{\delta}{a-b} + \frac{\delta}{-b+d} + O((\frac{\delta}{a-b} + \frac{\delta}{-b+d})^2))}{(a-d)(-b+d)\delta} \tag{4.55}$$

for γ such that $|\sqrt{\gamma}(1+\gamma)| < \eta_3{}'$, $\eta_3{}' \sim o(1)$.

Therefore $|\Delta := c_3 + c_4| \le C(\mathrm{D})$ still holds; hence the first term of $c_3 e^{ct} + c_4 e^{dt} = \Delta e^{dt} - c_3 e^{dt}(1 - e^{\delta t})$ is bounded by $C(\mathrm{D}) e^{-\frac{C(\mathrm{D})}{\epsilon\kappa}t}$. The second term in that expression is bounded by $\frac{C(\mathrm{D})}{\sqrt{\gamma}(1+\gamma)}\left[1 - e^{\frac{\sqrt{\gamma}(1+\gamma)}{\epsilon\kappa}C(\mathrm{D})t}\right] e^{-\frac{C(\mathrm{D})}{\epsilon\kappa}t}$, thus further bounded by $\frac{C(\mathrm{D})t}{\epsilon\kappa} e^{-\frac{C(\mathrm{D})t}{\epsilon\kappa}}$ if $\sqrt{\gamma}|1+\gamma| < \frac{\epsilon\kappa}{C(\mathrm{D})t}$; assuming that $|1+\gamma| < 2$, we see this inequality is satisfied when

$$\sqrt{\gamma} < \eta_3(t), \quad \eta_3(t) = \tfrac{1}{2}\min\left\{\eta_3, \tfrac{\epsilon\kappa}{C(\mathrm{D})t}\right\}. \tag{4.56}$$

Moreover, the $(1,1)$th entry is bounded by $\frac{C(\mathrm{D})t}{\epsilon\kappa} e^{-\frac{C(\mathrm{D})t}{\epsilon\kappa}}$ for all γ satisfying (4.56) which may be translated back in terms of $|\xi|$ as

$$\tfrac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2} < |\xi|^2 < \tfrac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}(1 + \eta_3^2(t)). \tag{4.57}$$

So far we've shown that the $(1,1)$th entry $c_1 e^{at} + c_2 e^{bt} + c_3 e^{ct} + c_4 e^{dt}$ remain bounded around those $|\xi|$ which yield double roots of the annihilating polynomial, even if c_1, c_2, c_3, c_4 may become unbounded. We then check the same bound $\frac{C(\mathrm{D})t}{\epsilon\kappa} e^{-\frac{C(\mathrm{D})t}{\epsilon\kappa}}$ holds for $|c_1 e^{at} + c_2 e^{bt} + c_3 e^{ct} + c_4 e^{dt}|$, when ξ take values away from those singular points.

In fact, for ξ such that $\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}(1 + \eta_3^2(t)) < |\xi|^2 < \frac{\widetilde{C}}{\epsilon^2\kappa^2}(1 - \eta_2(t))$ or $\frac{\widetilde{C}}{\epsilon^2\kappa^2}(1 + \eta_2(t)) <$

$|\xi|^2 < \frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2}(1-\eta_1(t))$, we let

$$\gamma + 1 = \sqrt{\frac{\epsilon^2\kappa^2[1-\frac{(\mathrm{D}-2)^2}{(\mathrm{D}-1)^2(\mathrm{D}+2)^2}]}{\mathrm{D}(\mathrm{D}+2)}}|\xi|, \tag{4.58}$$

by (4.51), we see that $|a-c| \geq C(\mathrm{D})\epsilon\kappa|\xi|^2\eta_2(t)$, therefore

$$|c_1e^{at} + c_2e^{bt} + c_3e^{ct} + c_4e^{dt}| \leq \frac{C(\mathrm{D})}{\eta_2(t)}e^{-\frac{C(\mathrm{D})}{\epsilon\kappa}t}. \tag{4.59}$$

For $|\xi|^2 > \frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2}(1+\eta_1(t))$, c_1, c_3 are bounded by $C(\mathrm{D})$. As for c_2, c_4, recall that(c.f. *case* "b=d")

$$b - d \;=\; \frac{\epsilon\kappa|\xi|^2}{\mathrm{D}}\frac{\frac{8\mathrm{D}}{(\mathrm{D}+2)^2}[1-\frac{1}{(1+\gamma)^2}]}{\frac{\mathrm{D}-2}{\mathrm{D}+2}+\sqrt{1-\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2|\xi|^2}}}. \tag{4.60}$$

Since $|\xi|^2 = \frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2}(1+\gamma)^2$, $|b-d| \geq C(\mathrm{D})\epsilon\kappa|\xi|^2\eta_1(t)$. So c_2, c_4 are bounded by $\frac{C(\mathrm{D})}{\eta_1(t)}$. On the other hand, $a, b, d \sim -\epsilon\kappa|\xi|^2$, whereas $c = \frac{\frac{\mathrm{D}+2}{\mathrm{D}}|\xi|^2}{d} \sim -\frac{1}{\epsilon\kappa}$.

Therefore,

$$\begin{aligned}
&|c_1e^{at}| \leq C(\mathrm{D})e^{-C(\mathrm{D})\epsilon\kappa|\xi|^2t},\\
&|c_2e^{bt}| \leq \frac{C(\mathrm{D})}{\eta_1(t)}e^{-C(\mathrm{D})\epsilon\kappa|\xi|^2t},\\
&|c_3e^{ct}| \leq C(\mathrm{D})e^{-\frac{C(\mathrm{D})}{\epsilon\kappa}t},\\
&|c_4e^{dt}| \leq \frac{C(\mathrm{D})}{\eta_1(t)}e^{-C(\mathrm{D})\epsilon\kappa|\xi|^2t}.
\end{aligned} \tag{4.61}$$

So

$$|c_1e^{at} + c_2e^{bt} + c_3e^{ct} + c_4e^{dt}| \leq \frac{C(\mathrm{D})}{\eta_1(t)}e^{-C(\mathrm{D})\epsilon\kappa|\xi|^2t} + C(\mathrm{D})e^{-\frac{C(\mathrm{D})}{\epsilon\kappa}t}. \tag{4.62}$$

To conclude this section, we collect all the bounds obtained for (1,1)th entry so far as follows:

$$|\mathbf{e}^{\mathbf{Q}t}_{(1,1)}| \leq \begin{cases} \frac{C(\mathrm{D})t}{\epsilon\kappa}e^{-\frac{C(\mathrm{D})t}{\epsilon\kappa}}, & \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2} < |\xi|^2 < \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}(1+\eta_3^2) \\ \frac{C(\mathrm{D})}{\eta_2}e^{-\frac{C(\mathrm{D})t}{\epsilon\kappa}}, & \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}(1+\eta_3^2) < |\xi|^2 < \frac{\widetilde{C}}{\epsilon^2\kappa^2}(1-\eta_2) \\ \frac{C(\mathrm{D})t}{\epsilon\kappa}e^{-\frac{C(\mathrm{D})t}{\epsilon\kappa}}, & \frac{\widetilde{C}}{\epsilon^2\kappa^2}(1-\eta_2) < |\xi|^2 < \frac{\widetilde{C}}{\epsilon^2\kappa^2}(1+\eta_2) \\ \frac{C(\mathrm{D})}{\eta_2}e^{-\frac{C(\mathrm{D})t}{\epsilon\kappa}}, & \frac{\widetilde{C}}{\epsilon^2\kappa^2}(1+\eta_2) < |\xi|^2 < \frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2}(1-\eta_1) \\ \frac{C(\mathrm{D})t}{\epsilon\kappa}e^{-\frac{C(\mathrm{D})t}{\epsilon\kappa}}, & \frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2}(1-\eta_1) < |\xi|^2 < \frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2}(1+\eta_1) \\ \frac{C(\mathrm{D})}{\eta_1}e^{-C(\mathrm{D})\epsilon\kappa|\xi|^2 t} + C(\mathrm{D})e^{-\frac{C(\mathrm{D})}{\epsilon\kappa}t}, & |\xi|^2 > \frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2}(1+\eta_1) \end{cases},$$

where $\eta_1(t) = \min\left\{\eta_1', \frac{\epsilon\kappa}{C(\mathrm{D})t}\right\}, \eta_2(t) = \min\left\{\eta_2', \frac{\epsilon\kappa}{C(\mathrm{D})t}\right\}, \eta_3(t) = \frac{1}{2}\min\left\{\eta_3', \frac{\epsilon\kappa}{C(\mathrm{D})t}\right\}$, $\eta_1', \eta_2', \eta_3' \sim o(1)$, and $\widetilde{C} = \frac{\mathrm{D}(\mathrm{D}+2)}{1-\frac{(\mathrm{D}-2)^2}{(\mathrm{D}-1)^2(\mathrm{D}+2)^2}}$.

We now generalize the argument to all entries. It suffices to investigate how c_1, c_2, c_3, c_4 change across the entries.

$$\begin{aligned} c_1 &= \frac{bcd + \mathbf{Q}^2_{(i,j)}(b+c+d) - \mathbf{Q}^3_{(i,j)}}{(-a+b)(-a+c)(-a+d)}, \\ c_2 &= \frac{acd + \mathbf{Q}^2_{(i,j)}(a+c+d) - \mathbf{Q}^3_{(i,j)}}{(a-b)(b-c)(b-d)}, \\ c_3 &= \frac{abd + \mathbf{Q}^2_{(i,j)}(a+b+d) - \mathbf{Q}^3_{(i,j)}}{(a-c)(-b+c)(c-d)}, \\ c_4 &= \frac{abc + \mathbf{Q}^2_{(i,j)}(a+b+c) - \mathbf{Q}^3_{(i,j)}}{(a-d)(-b+d)(-c+d)}, \end{aligned} \tag{4.63}$$

where $\mathbf{Q}^2_{(i,j)}, \mathbf{Q}^3_{(i,j)}$ are the (i,j)th entries of $\mathbf{Q}^2, \mathbf{Q}^3$.

In (4.42), we used the expression above for $\mathbf{Q}^2_{(1,1)} = -|\xi|^2$ and $\mathbf{Q}^3_{(1,1)} = \frac{2\epsilon\kappa}{\mathrm{D}-2}|\xi|^4$. Observe further more that $|\mathbf{Q}^2_{(i,j)}| \leq C(\mathrm{D})\epsilon^2\kappa^2|\xi|^4$, $|\mathbf{Q}^3_{(i,j)}| \leq C(\mathrm{D})\epsilon^3\kappa^3|\xi|^6$. For $\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2} < |\xi|^2 < \frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2}(1+\eta_1(t))$,

$$-|\xi|^2 \sim \epsilon^2\kappa^2|\xi|^4\,, \qquad \frac{2\epsilon\kappa}{\mathrm{D}-2}|\xi|^4 \sim \epsilon^3\kappa^3|\xi|^6\,,$$

so all the bounds obtained previously remain the same. For $|\xi|^2 > \frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2}(1+\eta_1(t))$,

$$\begin{aligned}
|c_1| &= \left|\frac{bcd+\mathbf{Q}^2_{(i,j)}(b+c+d)-\mathbf{Q}^3_{(i,j)}}{(-a+b)(-a+c)(-a+d)}\right| \le C(\mathrm{D})\frac{(\epsilon\kappa|\xi|^2)^3}{(\epsilon\kappa|\xi|^2)^3} = C(\mathrm{D}),\\
|c_2| &= \left|\frac{acd+\mathbf{Q}^2_{(i,j)}(a+c+d)-\mathbf{Q}^3_{(i,j)}}{(a-b)(b-c)(b-d)}\right| \le C(\mathrm{D})\frac{(\epsilon\kappa|\xi|^2)^3}{(\epsilon\kappa|\xi|^2)^2\eta_1\epsilon\kappa|\xi|^2} = \frac{C(\mathrm{D})}{\eta_1(t)},\\
|c_3| &= \left|\frac{abd+\mathbf{Q}^2_{(i,j)}(a+b+d)-\mathbf{Q}^3_{(i,j)}}{(a-c)(-b+c)(c-d)}\right| \le C(\mathrm{D})\frac{(\epsilon\kappa|\xi|^2)^3}{(\epsilon\kappa|\xi|^2)^3} = C(\mathrm{D}),\\
|c_4| &= \left|\frac{abc+\mathbf{Q}^2_{(i,j)}(a+b+c)-\mathbf{Q}^3_{(i,j)}}{(a-d)(-b+d)(-c+d)}\right| \le C(\mathrm{D})\frac{(\epsilon\kappa|\xi|^2)^3}{(\epsilon\kappa|\xi|^2)^2\eta_1\epsilon\kappa|\xi|^2} = \frac{C(\mathrm{D})}{\eta_1(t)},
\end{aligned} \tag{4.64}$$

which are the same bounds that we obtained for the $(1,1)$th entry. We thus conclude that for entries of $\mathbf{e}^{\mathbf{Q}t}$,

$$|\mathbf{e}^{\mathbf{Q}t}_{(i,j)}| \le \begin{cases}
\frac{C(\mathrm{D})t}{\epsilon\kappa}e^{-\frac{C(\mathrm{D})t}{\epsilon\kappa}}, & \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2} < |\xi|^2 < \frac{\widetilde{C}}{\epsilon^2\kappa^2}(1+\eta_3)^2\\
\frac{C(\mathrm{D})}{\eta_2}e^{-\frac{C(\mathrm{D})t}{\epsilon\kappa}}, & \frac{\widetilde{C}}{\epsilon^2\kappa^2}(1+\eta_3)^2 < |\xi|^2 < \frac{\widetilde{C}}{\epsilon^2\kappa^2}(1-\eta_2)\\
\frac{C(\mathrm{D})t}{\epsilon\kappa}e^{-\frac{C(\mathrm{D})t}{\epsilon\kappa}}, & \frac{\widetilde{C}}{\epsilon^2\kappa^2}(1-\eta_2) < |\xi|^2 < \frac{\widetilde{C}}{\epsilon^2\kappa^2}(1+\eta_2)\\
\frac{C(\mathrm{D})}{\eta_2}e^{-\frac{C(\mathrm{D})t}{\epsilon\kappa}}, & \frac{\widetilde{C}}{\epsilon^2\kappa^2}(1+\eta_2) < |\xi|^2 < \frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2}(1-\eta_1)\\
\frac{C(\mathrm{D})t}{\epsilon\kappa}e^{-\frac{C(\mathrm{D})t}{\epsilon\kappa}}, & \frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2}(1-\eta_1) < |\xi|^2 < \frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2}(1+\eta_1)\\
\frac{C(\mathrm{D})}{\eta_1}e^{-C(\mathrm{D})\epsilon\kappa|\xi|^2t} + C(\mathrm{D})e^{-\frac{C(\mathrm{D})}{\epsilon\kappa}t}, & |\xi|^2 > \frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2}(1+\eta_1)
\end{cases},$$

where $\eta_1(t) = \frac{1}{2}\min\{\eta_1', \frac{\epsilon\kappa}{C(\mathrm{D})t}\}, \eta_2(t) = \frac{1}{2}\min\{\eta_2', \frac{\epsilon\kappa}{C(\mathrm{D})t}\}, \eta_3(t) = \frac{1}{2}\min\{\eta_3', \frac{\epsilon\kappa}{C(\mathrm{D})t}\}$, $\eta_1', \eta_2', \eta_3' \sim o(1)$, and $\widetilde{C} = \frac{\mathrm{D}(\mathrm{D}+2)}{1-\frac{(\mathrm{D}-2)^2}{(\mathrm{D}-1)^2(\mathrm{D}+2)^2}}$.

4.1.2 Decay Estimate for Small Wave Number

In this section, we compute the explicit form of $\mathbf{e}^{\mathbf{Q}t}$ for smaller ξ, i.e. those satisfying $|\xi|^2 < \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}$. For ξ in this range, the characteristic polynomial of $\mathbf{Q}$ has one conjugate pair of roots. Moreover, the real parts of all roots are of order $\epsilon\kappa|\xi|^2$.

By expansion, we give a detailed description of coefficients associated with each of these real roots. For ξ such that $\epsilon^2\kappa^2|\xi|^2 \sim o(1)$, we will match their coefficients with those of the approximate system in Chapter 6, and show their difference is "small" in L^2 norm in Chapter 7. For those ξ for which $\epsilon^2\kappa^2|\xi|^2 \sim o(1)$ is closer to order 1, by exactly the same expansion, we show that entries of $\mathbf{e}^{\mathbf{Q}t}$ can be controlled by $C(D)(1 + O(\epsilon^2\kappa^2|\xi|^2))e^{-C(D)\epsilon\kappa|\xi|^2 t}$.

We now estimate $e^{\mathbf{Q}t}$ entry by entry for $|\xi|^2 < \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}$. The annihilating polynomial

$$p(z) = (z + \frac{\epsilon\kappa\mathrm{D}|\xi|^2}{(\mathrm{D}-1)(\mathrm{D}+2)})(z^3 + \epsilon\kappa|\xi|^2\tfrac{4(\mathrm{D}+1)}{(\mathrm{D}+2)\mathrm{D}}|\xi|^2 z^2 + \left[\tfrac{4\epsilon^2\kappa^2|\xi|^4}{(\mathrm{D}+2)\mathrm{D}} + \tfrac{\mathrm{D}+2}{\mathrm{D}}|\xi|^2\right] z + \epsilon\kappa\tfrac{2}{\mathrm{D}}|\xi|^4) \tag{4.65}$$

has two real roots a, b and a complex conjugate pair $p \pm iq$, where

$$\begin{aligned} a &= -\tfrac{\epsilon\kappa\mathrm{D}|\xi|^2}{(\mathrm{D}-1)(\mathrm{D}+2)}, \\ b &= -\tfrac{2\epsilon\kappa|\xi|^2}{\mathrm{D}+2}, \\ p &= -\tfrac{1}{\mathrm{D}}\epsilon\kappa|\xi|^2 \\ q &= \tfrac{1}{\mathrm{D}}\sqrt{-\epsilon^2\kappa^2|\xi|^4 + \mathrm{D}(\mathrm{D}+2)|\xi|^2}. \end{aligned} \tag{4.66}$$

We then write the general solution associated with $p(D)$:

$$X(t) = c_1 e^{at} + c_2 e^{bt} + c_3 e^{pt}\cos(qt) + c_4 e^{pt}\sin(qt), \tag{4.67}$$

thus

$$N_i(t) = c_1^i e^{at} + c_2^i e^{bt} + c_3^i e^{pt}\cos(qt) + c_4^i e^{pt}\sin(qt). \tag{4.68}$$

Notice that

$$\begin{pmatrix} X(0) \\ X'(0) \\ X''(0) \\ X'''(0) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ a & b & p & q \\ a^2 & b^2 & p^2-q^2 & 2pq \\ a^3 & b^3 & p^3-3pq^2 & 3p^2q-q^3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}. \tag{4.69}$$

Therefore,

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ a & b & p & q \\ a^2 & b^2 & p^2-q^2 & 2pq \\ a^3 & b^3 & p^3-3pq^2 & 3p^2q-q^3 \end{pmatrix} \begin{pmatrix} c_1^0 & c_1^1 & c_1^2 & c_1^3 \\ c_2^0 & c_2^1 & c_2^2 & c_2^3 \\ c_3^0 & c_3^1 & c_3^2 & c_3^3 \\ c_4^0 & c_4^1 & c_4^2 & c_4^3 \end{pmatrix} = \mathbf{I} \tag{4.70}$$

.

$$\begin{pmatrix} c_1^0 & c_1^1 & c_1^2 & c_1^3 \\ c_2^0 & c_2^1 & c_2^2 & c_2^3 \\ c_3^0 & c_3^1 & c_3^2 & c_3^3 \\ c_4^0 & c_4^1 & c_4^2 & c_4^3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ a & b & p & q \\ a^2 & b^2 & p^2-q^2 & 2pq \\ a^3 & b^3 & p^3-3pq^2 & 3p^2q-q^3 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{pmatrix} \tag{4.71}$$

where

$$
\begin{aligned}
c_{11} &= \tfrac{-(p^2+q^2)b}{((a-p)^2+q^2)(a-b)}\,, & c_{31} &= \tfrac{ab(3p^2-q^2-2pb-2ap+ab)}{((a-p)^2+q^2)((b-p)^2+q^2)} \\
c_{12} &= \tfrac{p^2+q^2+2pb}{((a-p)^2+q^2)(a-b)}\,, & c_{32} &= \tfrac{2a^2p+(a+b)(q^2-3p^2+2pb)}{((a-p)^2+q^2)((b-p)^2+q^2)}\,, \\
c_{13} &= \tfrac{-(2p+b)}{((a-p)^2+q^2)(a-b)}\,, & c_{33} &= -\tfrac{a^2-3p^2+q^2+ab+b^2}{((a-p)^2+q^2)((b-p)^2+q^2)}\,, \\
c_{14} &= \tfrac{1}{((a-p)^2+q^2)(a-b)}\,, & c_{34} &= \tfrac{a+b-2p}{((a-p)^2+q^2)((b-p)^2+q^2)}\,, \\
c_{21} &= \tfrac{a(p^2+q^2)}{((b-p)^2+q^2)(a-b)}\,, & c_{41} &= \tfrac{-ab[(q^2-p^2)(a+b)+p^3-3pq^2+abp]}{q((a-p)^2+q^2)((b-p)^2+q^2)}\,, \\
c_{22} &= -\tfrac{2ap+p^2+q^2}{((b-p)^2+q^2)(a-b)}\,, & c_{42} &= \tfrac{(a+b)(p^3-3pq^2-p^2b+q^2b)+a^2(q^2-p^2+b^2)}{q((a-p)^2+q^2)((b-p)^2+q^2)}\,, \\
c_{23} &= \tfrac{a+2p}{((b-p)^2+q^2)(a-b)}\,, & c_{43} &= \tfrac{a(a+b)(p-b)+p(-p^2+3q^2+b^2)}{q((a-p)^2+q^2)((b-p)^2+q^2)}\,, \\
c_{24} &= -\tfrac{1}{((b-p)^2+q^2)(a-b)}\,, & c_{44} &= -\tfrac{q^2+(a-p)(p-b)}{q((a-p)^2+q^2)((b-p)^2+q^2)}\,.
\end{aligned}
$$

As usual, we illustrate the procedure for the $(1,1)$th entry, and we generalize it to all the entries later.

Since the first entries of $\mathbf{Q}$, $\mathbf{Q}^2$ and $\mathbf{Q}^3$ are 0, $-|\xi|^2$ and $\epsilon\kappa\frac{2}{\mathrm{D}+2}|\xi|^4$, we may write the $(1,1)$th entry as

$$c_1e^{at} + c_2e^{bt} + c_3e^{pt}\cos(qt) + c_4e^{pt}\sin(qt), \tag{4.72}$$

where a, b, p, q are defined by (4.90), and

$$
\begin{aligned}
c_1 &= \frac{-(p^2+q^2)b+|\xi|^2(2p+b)+\epsilon\kappa\frac{2}{\mathrm{D}+2}|\xi|^4}{((b-p)^2+q^2)(a-b)}, \\
c_2 &= \frac{a(p^2+q^2)-|\xi|^2(a+2p)+\epsilon\kappa\frac{2}{\mathrm{D}+2}|\xi|^4}{((b-p)^2+q^2)(a-b)}, \\
c_3 &= \frac{ab(3p^2-q^2-2pb-2ap+ab)+|\xi|^2(a^2-3p^2+q^2+ab+b^2)+\epsilon\kappa\frac{2}{\mathrm{D}+2}|\xi|^4(a+b-2p)}{((a-p)^2+q^2)((b-p)^2+q^2)}, \\
c_4 &= \frac{-ab[(q^2-p^2)(a+b)+p^3-3pq^2+abp]-|\xi|^2(a(a+b)(p-b)+p(-p^2+3q^2+b^2))-\epsilon\kappa\frac{2}{\mathrm{D}+2}|\xi|^4(q^2+(a-p)(p-b))}{q((a-p)^2+q^2)((b-p)^2+q^2)}.
\end{aligned} \tag{4.73}
$$

. Further calculation shows that

$$
\begin{aligned}
c_1 &= O(\epsilon^2\kappa^2|\xi|^2), \qquad c_2 = \tfrac{2}{\mathrm{D}+2}[1+O(\epsilon^2\kappa^2|\xi|^2)], \\
c_3 &= \tfrac{\mathrm{D}}{\mathrm{D}+2}[1+O(\epsilon^2\kappa^2|\xi|^2)], \qquad c_4 = O(\epsilon^2\kappa^2|\xi|^2).
\end{aligned} \tag{4.74}
$$

We may estimate all the other entries in the same fashion. In fact, for the $(1,2),...(1,\mathrm{D}+1)$th entry:

$$c_1 = O(\epsilon^2\kappa^2|\xi|^2), \qquad c_2 = O(\epsilon^2\kappa^2|\xi|^2),$$

$$c_3 = O(\epsilon^2\kappa^2|\xi|^2), \qquad c_4 = -i\sqrt{\tfrac{\mathrm{D}}{\mathrm{D}+2}}\tfrac{\xi^T}{|\xi|}[1+O(\epsilon^2\kappa^2|\xi|^2)];$$

for the $(1,\mathrm{D}+2)$th entry:

$$c_1 = O(\epsilon^2\kappa^2|\xi|^2), \qquad c_2 = -\tfrac{\mathrm{D}}{\mathrm{D}+2}[1+O(\epsilon^2\kappa^2|\xi|^2)],$$

$$c_3 = \tfrac{\mathrm{D}}{\mathrm{D}+2}[1+O(\epsilon^2\kappa^2|\xi|^2)], \qquad c_4 = O(\epsilon^2\kappa^2|\xi|^2);$$

for the $(2,1)...(\mathrm{D}+1)$th entry:

$$c_1 = O(\epsilon^2\kappa^2|\xi|^2), \qquad c_2 = O(\epsilon^2\kappa^2|\xi|^2),$$

$$c_3 = O(\epsilon^2\kappa^2|\xi|^2), \qquad c_4 = -i\sqrt{\tfrac{\mathrm{D}}{\mathrm{D}+2}}\tfrac{\xi^T}{|\xi|}[1+O(\epsilon^2\kappa^2|\xi|^2)];$$

for the $(2,\mathrm{D}+2),...(\mathrm{D}+1,\mathrm{D}+2)$th entry:

$$c_1 = O(\epsilon^2\kappa^2|\xi|^2), \qquad c_2 = O(\epsilon^2\kappa^2|\xi|^2),$$

$$c_3 = O(\epsilon^2\kappa^2|\xi|^2), \qquad c_4 = -i\sqrt{\tfrac{\mathrm{D}}{\mathrm{D}+2}}\tfrac{\xi^T}{|\xi|}[1+O(\epsilon^2\kappa^2|\xi|^2)];$$

for the $(\mathrm{D}+2,1)$th entry:

$$c_1 = O(\epsilon^2\kappa^2|\xi|^2), \qquad c_2 = -\tfrac{2}{\mathrm{D}+2}[1+O(\epsilon^2\kappa^2|\xi|^2)],$$

$$c_3 = \tfrac{2}{\mathrm{D}+2}[1+O(\epsilon^2\kappa^2|\xi|^2)], \qquad c_4 = O(\epsilon^2\kappa^2|\xi|^2)$$

for the $(\mathrm{D}+2,2)...(\mathrm{D}+2,\mathrm{D}+1)$th entry:

$$c_1 = O(\epsilon^2\kappa^2|\xi|^2), \qquad c_2 = O(\epsilon^2\kappa^2|\xi|^2),$$

$$c_3 = O(\epsilon^2\kappa^2|\xi|^2), \qquad c_4 = -2i\sqrt{\tfrac{1}{(\mathrm{D}+2)\mathrm{D}}}\tfrac{\xi^T}{|\xi|}[1+O(\epsilon^2\kappa^2|\xi|^2)];$$

for the $(\mathrm{D}+2,\mathrm{D}+2)$th entry:

$$c_1 = O(\epsilon^2\kappa^2|\xi|^2), \qquad c_2 = \tfrac{\mathrm{D}}{\mathrm{D}+2}[1+O(\epsilon^2\kappa^2|\xi|^2)],$$

$$c_3 = \tfrac{2}{\mathrm{D}+2}[1+O(\epsilon^2\kappa^2|\xi|^2)], \qquad c_4 = O(\epsilon^2\kappa^2|\xi|^2).$$

. We now describe the estimate D × D matrix in the middle in more details:

$$
\begin{aligned}
c_1 = & \Big[\frac{-b}{a-b}\mathbf{I} + \frac{-\frac{\epsilon\kappa \mathrm{D}}{(\mathrm{D}-1)(\mathrm{D}+2)}|\xi|^2\mathbf{I} - \frac{\epsilon\kappa(\mathrm{D}-2)}{(\mathrm{D}-1)(\mathrm{D}+2)}\xi\xi^T}{a-b} + \frac{(2p+b)(-\frac{\mathrm{D}+2}{\mathrm{D}}\xi\xi^T)}{q^2(-a+b)} \\
& + \frac{\epsilon\kappa\frac{4(\mathrm{D}+1)}{\mathrm{D}^2}|\xi|^2\xi\xi^T}{q^2(a-b)} + O(\epsilon^2\kappa^2|\xi|^2)\Big]\Big[1 + O(\epsilon^2\kappa^2|\xi|^2)\Big] \\
= & \Big(\mathbf{I} - \frac{\xi\xi^T}{|\xi|^2}\Big)\Big[1 + O(\epsilon^2\kappa^2|\xi|^2)\Big], \\
c_2 = & \Big[\frac{a}{a-b}\mathbf{I} + \frac{\frac{\epsilon\kappa \mathrm{D}}{(\mathrm{D}-1)(\mathrm{D}+2)}|\xi|^2\mathbf{I} + \frac{\epsilon\kappa(\mathrm{D}-2)}{(\mathrm{D}-1)(\mathrm{D}+2)}\xi\xi^T}{a-b} + \frac{a+2p}{\frac{\mathrm{D}+2}{\mathrm{D}}|\xi|^2(a-b)}(-\frac{\mathrm{D}+2}{\mathrm{D}}\xi\xi^T) \\
& - \frac{\epsilon\kappa\frac{4(\mathrm{D}+1)}{\mathrm{D}^2}|\xi|^2\xi\xi^T}{\frac{\mathrm{D}+2}{\mathrm{D}}|\xi|^2(a-b)} + O(\epsilon^2\kappa^2|\xi|^2)\Big]\Big[1 + O(\epsilon^2\kappa^2|\xi|^2)\Big] = O(\epsilon^2\kappa^2|\xi|^2), \\
c_3 = & \frac{(1+\frac{2}{\mathrm{D}})\xi\xi^T}{\frac{\mathrm{D}+2}{\mathrm{D}}|\xi|^2}\Big[1 + O(\epsilon^2\kappa^2|\xi|^2)\Big] = \frac{\xi\xi^T}{|\xi|^2}\Big[1 + O(\epsilon^2\kappa^2|\xi|^2)\Big], \\
c_4 = & O(\epsilon^2\kappa^2|\xi|^2).
\end{aligned}
$$

We have got estimate of the exact solution for all ξ.

4.2 Linearized Weakly Compressible Navier-Stokes Approximation to the Linearized Compressible Navier-Stokes system

In this section, we show that the difference between the solution of linearized compressible Navier-Stokes (i.e., compressible Stokes) and the solution of linearized weakly compressible Navier-Stokes (A.37) is $\mathrm{O}(\sqrt(\epsilon))$ uniform in time:

Theorem 4.2.1 *Let U be the solution of compressible Stokes equation (*4.91*) and V be solution of weakly compressible Stokes equation (A.37) with the same initial data $U^{\mathrm{in}} \in H^1(\mathbb{T}^{\mathrm{D}})$. Then*

$$||U(t) - V(t)||_{L^\infty(dt;L^2(\mathbb{T}^{\mathrm{D}}))} \leq C\sqrt{\epsilon}\,.$$

Here C depends on dimension D and transportation coefficients only.

4.2.1 Estimate for Small Wave Number

In this section, we give $L^\infty(\mathrm{d}t; L^2(\mathrm{d}\xi))$ estimate for $|\xi|^2 < \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}$. For simplicity, we denote $e^{\mathbf{Q}t}_{i,j}$ by $a_{i,j}$, and the (i,j)th entry of the approximation matrix by $b_{i,j}$. We estimate the difference $\sum_{|\xi|^2<\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}} (a_{1,1} - b_{1,1})^2|\widehat{U}(\xi)|^2$ first, and then generalize the argument to all entries.

Since the (1,1)th entry in the approximation matrix (A.36) is

$$b_{1,1} = \tfrac{2}{\mathrm{D}+2}e^{bt} + \tfrac{\mathrm{D}}{\mathrm{D}+2}e^{pt}\cos\big(\sqrt{\tfrac{\mathrm{D}+2}{\mathrm{D}}}|\xi|t\big), \tag{4.75}$$

and we've shown in the discussion of exact solution that

$$\begin{aligned} a_{1,1} =& O(\epsilon^2\kappa^2|\xi|^2)e^{at} + \tfrac{2}{\mathrm{D}+2}[1 + O(\epsilon^2\kappa^2|\xi|^2)]e^{bt} \\ &+ \tfrac{\mathrm{D}}{\mathrm{D}+2}[1 + O(\epsilon^2\kappa^2|\xi|^2)]e^{pt}\cos(qt) + O(\epsilon^2\kappa^2|\xi|^2)e^{pt}\sin(qt) \end{aligned} \tag{4.76}$$

Hence for $\epsilon^2\kappa^2|\xi|^2 < \mathrm{D}(\mathrm{D}+2)$, we may split the ξ into three groups:

Group ①: $|\xi|^2 < \eta_4\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}$,

Group ②: $\eta_4\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2} < |\xi|^2 < (1-\eta_4)\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}$,

Group ③: $(1-\eta_4)\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2} < |\xi|^2 < \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}$. We take $\eta_4 = (\epsilon\kappa)^\alpha$. For ξ in group ①, $\sum_{|\xi|^2<\eta_4\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}} (a_{1,1} - b_{1,1})^2|\widehat{U}(\xi)|^2$ can be split into two types of series:

$$(I) \qquad \sum_{|\xi|^2<\eta_4\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}} O(\epsilon^4\kappa^4|\xi|^4)e^{2at}|\widehat{U}(\xi)|^2,$$

$$\sum_{|\xi|^2<\eta_4\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}} O(\epsilon^4\kappa^4|\xi|^4)e^{2bt}|\widehat{U}(\xi)|^2;$$

$$(II) \quad C(\mathrm{D}) \sum_{|\xi|^2<\eta_4\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}} \big(O(\epsilon^2\kappa^2|\xi|^2)e^{pt}\sin(qt)\big)^2\,|\widehat{U}(\xi)|^2\,,$$

$$C(\mathrm{D}) \sum_{|\xi|^2<\eta_4\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}} \left(\tfrac{\mathrm{D}}{\mathrm{D}+2}[1 + O(\epsilon^2\kappa^2|\xi|^2)]e^{pt}\cos(qt) - \tfrac{\mathrm{D}}{\mathrm{D}+2}e^{pt}\cos\big(\sqrt{\tfrac{\mathrm{D}+2}{\mathrm{D}}}|\xi|t\big)\right)^2|\widehat{U}(\xi)|^2\,.$$

For type (I) integral, notice that $\sum |\xi|^2 |\widehat{U}(\xi)|^2$ is bounded, therefore

$$\sum_{|\xi|^2 < \eta_4 \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}} O(\epsilon^4\kappa^4|\xi|^4)|\widehat{U}(\xi)|^2 \leq O(\epsilon^2\kappa^2) \sum C(\mathrm{D})\eta_4 |\xi|^2 |\widehat{U}(\xi)^2| \leq (\epsilon\kappa)^{2+\alpha}$$

For type (II) integral, notice that

$$\begin{aligned}
q &= \tfrac{1}{\mathrm{D}}\sqrt{-\epsilon^2\kappa^2|\xi|^4 + \mathrm{D}(\mathrm{D}+2)|\xi|^2} \\
&= \tfrac{1}{\mathrm{D}}\sqrt{\mathrm{D}(\mathrm{D}+2)|\xi|^2\left(1 - \tfrac{\epsilon^2\kappa^2|\xi|^2}{\mathrm{D}(\mathrm{D}+2)}\right)} \\
&= \sqrt{\tfrac{\mathrm{D}+2}{\mathrm{D}}}|\xi|\left(1 - \tfrac{\epsilon^2\kappa^2|\xi|^2}{2\mathrm{D}(\mathrm{D}+2)} + O(\epsilon^4\kappa^4|\xi|^4)\right), \\
\cos(qt) &= \cos(\sqrt{\tfrac{\mathrm{D}+2}{\mathrm{D}}}|\xi|t) + \tfrac{1}{2}\sin(\xi_c t)\sqrt{\tfrac{\mathrm{D}+2}{\mathrm{D}}}|\xi|t\left(\tfrac{\epsilon^2\kappa^2|\xi|^2}{\mathrm{D}(\mathrm{D}+2)} + O(\epsilon^4\kappa^4|\xi|^4)\right), \\
\sin(qt) &= \sin(\sqrt{\tfrac{\mathrm{D}+2}{\mathrm{D}}}|\xi|t) - \tfrac{1}{2}\cos(\xi_c t)\sqrt{\tfrac{\mathrm{D}+2}{\mathrm{D}}}|\xi|t\left(\tfrac{\epsilon^2\kappa^2|\xi|^2}{\mathrm{D}(\mathrm{D}+2)} + O(\epsilon^4\kappa^4|\xi|^4)\right),
\end{aligned}$$

where $\xi_c \in (q, \sqrt{\frac{\mathrm{D}+2}{\mathrm{D}}}|\xi|)$. Therefore

$$\begin{aligned}
|II| \leq &\sum_{|\xi|^2 < \eta_4 \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}} C(\mathrm{D})O(\epsilon^4\kappa^4|\xi|^4)e^{2pt}|\widehat{U}(\xi)|^2 \\
&+ \sum_{|\xi|^2 < \eta_4 \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}} C(\mathrm{D})|\xi|^2 t^2 \epsilon^4\kappa^4|\xi|^4 e^{2pt}|\widehat{U}(\xi)|^2.
\end{aligned}$$

The first term is another type (I) series; for the second term, notice that

$$C(\mathrm{D})|\xi|^2 t^2 e^{2pt} = C(\mathrm{D})|\xi|^2 t^2 e^{-C(\mathrm{D})\epsilon\kappa|\xi|^2 t} \leq \frac{C(\mathrm{D})}{\epsilon^2\kappa^2|\xi|^2},$$

so

$$\sum_{|\xi|^2 < \eta_4 \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}} C(\mathrm{D})|\xi|^2 t^2 \epsilon^4\kappa^4|\xi|^4 e^{2pt}|\widehat{U}(\xi)|^2 \leq C(\mathrm{D}) \sum \epsilon^2\kappa^2|\xi|^2|\widehat{U}(\xi)|^2 \leq C(\mathrm{D})(\epsilon\kappa)^2,$$

By the same argument, we have

$$\sum_{|\xi|^2 < \eta_4 \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}} (a_{i,j} - b_{i,j})^2 |\widehat{U}(\xi)|^2 \leq C(\mathrm{D})(\epsilon\kappa)^2$$

for all (i,j). We now estimate $\sum |(a_{i,j}-b_{i,j})^2||\widehat{U}(xi)|^2$ for ξ in group ②, ③. To do so, we give separate estimates for $\sum |a_{i,j}^2||\widehat{U}(\xi)|^2$ and $\sum b_{i,j}^2|\widehat{U}(\xi)|^2$. In the next subsection we estimate $\sum_{\eta_4\frac{D(D+2)}{\epsilon^2\kappa^2}<|\xi|^2<\frac{D(D+2)}{\epsilon^2\kappa^2}} |a_{i,j}^2||\widehat{U}(\xi)|^2$; estimate for

$$\sum_{\eta_4\frac{D(D+2)}{\epsilon^2\kappa^2}<|\xi|^2<\frac{D(D+2)}{\epsilon^2\kappa^2}} |b_{i,j}^2||\widehat{U}(\xi)|^2$$

will be given in the last section as part of of the estimate for $\sum_{|\xi|^2>\eta_4\frac{D(D+2)}{\epsilon^2\kappa^2}} |b_{i,j}^2||\widehat{U}(\xi)|^2$.

4.2.2 Estimate for Large Wave Number

Case 1: $\sum_{\eta_4\frac{D(D+2)}{\epsilon^2\kappa^2}<|\xi|^2<\frac{D(D+2)}{\epsilon^2\kappa^2}} |a_{i,j}^2||\widehat{U}(\xi)|^2$

In order to estimate c_1, c_2, c_3, c_4 or ξ in group ②, we use the following change of variable:

$$|\xi| = \tfrac{(1-\gamma)\sqrt{D(D+2)}}{\epsilon\kappa},$$

and express $C(D)\epsilon\kappa|\xi|^2$(since $a, b, p \sim -C(D)\epsilon\kappa|\xi|^2$) and q as follows:

$$\begin{aligned} C(D)\epsilon\kappa|\xi|^2 &= \tfrac{(1-\gamma)^2 C(D)}{\epsilon\kappa}, \\ q &= \sqrt{D(D+2)|\xi|^2\left(1-\tfrac{\epsilon^2\kappa^2|\xi|^2}{D(D+2)}\right)} \\ &= \sqrt{D(D+2)}|\xi|\sqrt{1-(1-\gamma)^2} \\ &= \tfrac{D(D+2)(1-\gamma)\sqrt{1-(1-\gamma)^2}}{\epsilon\kappa}. \end{aligned} \tag{4.77}$$

Since $\eta_4\frac{D(D+2)}{\epsilon^2\kappa^2} < |\xi|^2 < (1-\eta_4)\frac{D(D+2)}{\epsilon^2\kappa^2}$, we then have

$$\begin{aligned} &\eta_4 < (1-\gamma)^2 < 1-\eta_4, \\ &\sqrt{\eta_4(1-\eta_4)} < (1-\gamma)\sqrt{1-(1-\gamma)^2} < \frac{1}{2}. \end{aligned} \tag{4.78}$$

Hence for all entries,

$$\begin{aligned} |c_1| &\le \tfrac{C(\mathrm{D})|\xi|^3}{\eta_4|\xi|^2\sqrt{\eta_4}|\xi|} = \tfrac{C(\mathrm{D})}{\eta_4^{\frac{3}{2}}}, & |c_2| &\le \tfrac{C(\mathrm{D})|\xi|^3}{\eta_4|\xi|^2\sqrt{\eta_4}|\xi|} = \tfrac{C(\mathrm{D})}{\eta_4^{\frac{3}{2}}}, \\ |c_3| &\le \tfrac{C(\mathrm{D})|\xi|^4}{\eta_4^2|\xi|^4} = \tfrac{C(\mathrm{D})}{\eta_4^2}, & |c_4| &\le \tfrac{C(\mathrm{D})|\xi|^5}{\eta_4^{\frac{5}{2}}|\xi|^5} = \tfrac{C(\mathrm{D})}{\eta_4^{\frac{5}{2}}}. \end{aligned} \tag{4.79}$$

Therefore for ξ in group ②,

$$|c_1e^{at} + c_2e^{bt} + c_3e^{pt}\cos(qt) + c_4e^{pt}\sin(qt)| \le \tfrac{C(\mathrm{D})}{\eta_4^{\frac{5}{2}}}e^{-C(\mathrm{D})\epsilon\kappa|\xi|^2t} \tag{4.80}$$

and

$$\begin{aligned} &\sum_{\eta_4\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}<|\xi|^2<(1-\eta_4)\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}} |c_1e^{at} + c_2e^{bt} + c_3e^{pt}\cos(qt) + c_4e^{pt}\sin(qt)|^2|\widehat{U}(\xi)|^2 \\ &\le C(\mathrm{D})(\epsilon\kappa)^{2-\frac{7}{2}\alpha}\sum |\xi|^2|\widehat{U}(\xi)|^2 \end{aligned} \tag{4.81}$$

For ξ in group ③, we see that c_1, c_2, c_3 are all bounded by $C(\mathrm{D})$ by observing that $a, b, p \sim -C(\mathrm{D})\epsilon\kappa|\xi|^2$ and q is "close to zero". As for c_4, we have $|c_4| \le \frac{C(\mathrm{D})\epsilon\kappa|\xi|^2}{q}$. Therefore,

$$\begin{aligned} |c_4e^{pt}\sin(qt)| &\le \frac{C(\mathrm{D})\epsilon\kappa|\xi|^2t}{qt}e^{pt}|\sin(qt)| \\ &\le |C(\mathrm{D})\epsilon\kappa|\xi|^2t\,e^{pt}| \le C(\mathrm{D})\epsilon\kappa|\xi|^2te^{-C(\mathrm{D})\epsilon\kappa|\xi|^2t}, \end{aligned}$$

and

$$\begin{aligned} &\sum_{(1-\eta_4)\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}<|\xi|^2<\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}} |c_1e^{at} + c_2e^{bt} + c_3e^{pt}\cos(qt) + c_4e^{pt}\sin(qt)|^2|\widehat{U}(\xi)|^2 \\ &\le C(\mathrm{D})(\epsilon\kappa)^2 \end{aligned} \tag{4.82}$$

Case 2: Estimate for $\sum_{|\xi|^2>\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}}(a_{i,j})^2|\widehat{U}(\xi)|^2$

According to the previous discussion, we may split ξ furthermore into six groups:

Group ①: $\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2} < |\xi|^2 < \frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}(1+\eta_3^2(t))$, Group ②: $\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}(1+\eta_3^2(t)) < |\xi|^2 < \frac{\widetilde{C}}{\epsilon^2\kappa^2}(1-\eta_2(t))$,

Group ③: $\frac{\widetilde{C}}{\epsilon^2\kappa^2}(1-\eta_2(t)) < |\xi|^2 < \frac{\widetilde{C}}{\epsilon^2\kappa^2}(1+\eta_2(t))$,

Group ④: $\frac{\widetilde{C}}{\epsilon^2\kappa^2}(1+\eta_2(t)) < |\xi|^2 < \frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2}(1-\eta_1(t))$,

Group ⑤: $\frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2}(1-\eta_1(t)) < |\xi|^2 < \frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2}(1+\eta_1(t))$,

Group ⑥: $|\xi|^2 > \frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2}(1+\eta_1(t))$.

Also, we assume ${\eta_1}' \sim {\eta_2}' \sim {\eta_3}'$ without loss of generality. We then split the time variable t into several groups:

Case (a) $\eta_i(t) = \frac{1}{2}{\eta_i}'$, i.e. $t \le \frac{\epsilon\kappa}{{\eta_i}' C(\mathrm{D})} = \frac{\epsilon\kappa}{\eta C(\mathrm{D})}$, where $\eta := \min\{\eta_i\}$, $i = 1, 2, 3$. Therefore $\frac{C(\mathrm{D})t}{\epsilon\kappa} \le \frac{C(\mathrm{D})}{{\eta_2}'}$ for ξ in Group ①,③,⑤.

Hence

$$\begin{aligned}\sum_{\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}<|\xi|^2<\frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2}(1+\frac{1}{2}{\eta_1}')} |e^{\mathbf{Q}t}|^2|\widehat{U}(\xi)|^2 \le& \left(\frac{C(\mathrm{D})}{{\eta_2}'}e^{-\frac{C(\mathrm{D})t}{\epsilon\kappa}}\right)^2(\epsilon\kappa)^2\sum|\xi|^2|\widehat{U}(\xi)|^2 \\ \le& C(\mathrm{D})(\frac{\epsilon\kappa}{{\eta_2}'})^2.\end{aligned} \tag{4.83}$$

As for ξ in Group ⑥,

$$\begin{aligned}\sum_{|\xi|^2>\frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2}(1+\frac{1}{2}{\eta_1}')} |e^{\mathbf{Q}t}|^2|\widehat{U}(\xi)|^2 \le& \left(\frac{C(\mathrm{D})}{{\eta_1}'}e^{-\frac{C(\mathrm{D})t}{\epsilon\kappa}}\right)^2(\epsilon\kappa)^2\sum|\xi|^2|\widehat{U}(\xi)|^2 \\ \le& C(\mathrm{D})(\frac{\epsilon\kappa}{{\eta_1}'})^2.\end{aligned} \tag{4.84}$$

So

$$\sum_{|\xi|^2>\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}} |e^{\mathbf{Q}t}|^2|\widehat{U}(\xi)|^2 \le C(\mathrm{D})[(\frac{\epsilon\kappa}{{\eta_2}'})^2 + (\frac{\epsilon\kappa}{{\eta_1}'})^2]. \tag{4.85}$$

Case (b) $\eta_i(t) = \frac{1}{2}\frac{\epsilon\kappa}{C(\mathrm{D})t}$, i.e. $t \ge \frac{\epsilon\kappa}{{\eta_i}' C(\mathrm{D})}$, $i = 1, 2, 3$. In particular, $\eta_2(t) = \frac{1}{2}\frac{\epsilon\kappa}{C(\mathrm{D})t}$.

Hence

$$\begin{aligned}\sum_{\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}<|\xi|^2<\frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2}(1+\eta_1(t))} |e^{\mathbf{Q}t}|^2|\widehat{U}(\xi)|^2 \le& \left(\frac{C(\mathrm{D})t}{\epsilon\kappa}e^{-\frac{C(\mathrm{D})t}{\epsilon\kappa}}\right)^2(\epsilon\kappa)^2\sum|\xi|^2|\widehat{U}(\xi)|^2 \\ \le& C(\mathrm{D})(\epsilon\kappa)^2,\end{aligned} \tag{4.86}$$

and

$$\sum_{|\xi|^2>\frac{(\mathrm{D}+2)^3}{8\epsilon^2\kappa^2}(1+\eta_1(t))} |e^{\mathbf{Q}t}|^2|\widehat{U}(\xi)|^2 \leq \left(\frac{C(\mathrm{D})t}{\epsilon\kappa}e^{-\frac{C(\mathrm{D})t}{\epsilon\kappa}}\right)^2(\epsilon\kappa)^2|\widehat{U}(\xi)|^2 \leq C(\mathrm{D})(\epsilon\kappa)^2. \tag{4.87}$$

So

$$\sum_{|\xi|^2>\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}} |e^{\mathbf{Q}t}|^2|\widehat{U}(\xi)|^2 \leq C(\mathrm{D})(\epsilon\kappa)^2\,. \tag{4.88}$$

Case (c) All the other values of t:

In this case, $\frac{\epsilon\kappa}{C(\mathrm{D})t} \sim \eta_i{}'$, thus reduce Case(c) to Case(a) or (b). (They are equivalent under the assumption of Case(c)). Therefore,

$$\sum_{|\xi|^2>\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}} (a_{i,j})^2|\widehat{U}(\xi)|^2 \leq C(\mathrm{D})[(\frac{\epsilon\kappa}{\eta_2{}'})^2+(\frac{\epsilon\kappa}{\eta_1{}'})^2]. \tag{4.89}$$

Case 3: Estimate for $\sum_{|\xi|^2>\eta_4\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}} |b_{i,j}^2||\widehat{U}(\xi)|^2$(approximation solution)

We now estimate $P := \Phi e^{-i\Lambda t}e^{M^{diag}\tau}\Phi^T H\widehat{U}^{\mathrm{in}}$ in $L^2(\mathrm{d}\xi)$ for $|\xi|^2 > \eta_4\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}$.

$$\sum_{|\xi|^2>\eta_4\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}} |P_{(i,j)}|^2|\widehat{U}(\xi)|^2 \leq \sum_{|\xi|^2>\eta_4\frac{\mathrm{D}(\mathrm{D}+2)}{\epsilon^2\kappa^2}} e^-C(\mathrm{D})\epsilon\kappa|\xi|^2\frac{1}{|\xi|^2} \leq C(\mathrm{D})(\epsilon\kappa)^{2-\alpha}e^{-C(\mathrm{D})\frac{t}{\epsilon\kappa}} \leq\ C(\mathrm{D})(\epsilon\kappa)^{2-\alpha} \tag{4.90}$$

Thus we've collected estimates for all ξ and proved Theorem 4.2.1.

4.3 Linearized Compressible Navier-Stokes Approximation to the Linearized Boltzmann Equation

In this section, we establish a linearized compressible Navier-Stokes approximation

$$
\begin{aligned}
\partial_t \rho_\epsilon + \operatorname{div} u_\epsilon \;&=\; 0\,, \\
\partial_t u_\epsilon + \nabla_x(\rho_\epsilon + \theta_\epsilon) \;&=\; \epsilon \operatorname{div}(\mu D_x u_\epsilon)\,, \\
\partial_t \theta_\epsilon + \tfrac{2}{\mathrm{D}} \operatorname{div} u_\epsilon \;&=\; \epsilon \tfrac{2}{\mathrm{D}} \operatorname{div}(\kappa \nabla_x \theta_\epsilon)\,,
\end{aligned}
\tag{4.91}
$$

to the linearized Boltzmann equation

$$
\partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon + \frac{1}{\epsilon} \mathcal{L} F_\epsilon \;=\; 0\,. \tag{4.92}
$$

We show that the solutions of 4.91 approximates the fluid moments of the linearized Boltzmann equation uniformly in time if the initial data of the linearized Boltzmann equation is in the fluid regime.

First, we construct an approximate solution $g_\epsilon^{[2]}$ from the solution of compressible Stokes equation. $g_\epsilon^{[2]}$ is the Chapman-Enskog expansion described in Chapter 1, Section 1.5 up to second order. $g_\epsilon^{[2]}$ has the same fluid moment with the solution to the linearized Boltzmann equation g_ϵ, and formally approximates the solution of the linearized Boltzmann equation up to $O(\epsilon^2)$.

Lemma 4.3.1 *Let*

$$
\begin{aligned}
g_\epsilon^{[2]} = \rho_\epsilon + v \cdot u_\epsilon + (\tfrac{1}{2}|v|^2 - \tfrac{\mathrm{D}}{2})\theta_\epsilon - \epsilon \mathcal{L}^{-1}(A : \nabla_x u_\epsilon + B \cdot \nabla_x \theta_\epsilon) \\
+ \epsilon^2 \mathcal{L}^{-1}(\partial_t + v \cdot \nabla_x)\mathcal{L}^{-1}(A : \nabla_x u_\epsilon + B \cdot \nabla_x \theta_\epsilon)\,,
\end{aligned}
\tag{4.93}
$$

where ρ_ϵ, u_ϵ and θ_ϵ are solutions of the compressible Stokes system (4.91)*. Then $g_\epsilon^{[2]}$ satisfies*

$$\partial_t g_\epsilon^{[2]} + v\cdot\nabla_x g_\epsilon^{[2]} + \tfrac{1}{\epsilon}\mathcal{L} g_\epsilon^{[2]} \;=\; \epsilon^2\left((\partial_t + v\cdot\nabla_x)\mathcal{L}^{-1}\right)^2(A:\nabla_x u_\epsilon + B\cdot\nabla_x\theta_\epsilon)\,. \tag{4.94}$$

Proof: Note that

$$\begin{aligned}
&(\partial_t + v\cdot\nabla_x) g_\epsilon^{[2]}\\
&= (\partial_t + v\cdot\nabla_x)(\rho_\epsilon + v\cdot u_\epsilon + (\tfrac12|v|^2 - \tfrac{\mathrm{D}}{2})\theta_\epsilon) - \epsilon(\partial_t + v\cdot\nabla_x)\mathcal{L}^{-1}(A:\nabla_x u_\epsilon + B\cdot\nabla_x\theta_\epsilon)\\
&\quad + \epsilon^2(\partial_t + v\cdot\nabla_x)\mathcal{L}^{-1}(\partial_t + v\cdot\nabla_x)\mathcal{L}^{-1}(A:\nabla_x u_\epsilon + B\cdot\nabla_x\theta_\epsilon)\,,
\end{aligned} \tag{4.95}$$

and

$$\begin{aligned}
\tfrac{1}{\epsilon}\mathcal{L} g_\epsilon^{[2]} \;&=\; -\,(A:\nabla_x u_\epsilon + B\cdot\nabla_x\theta_\epsilon)\\
&\quad + \epsilon\mathcal{P}^\perp\left((\partial_t + v\cdot\nabla_x)\mathcal{L}^{-1}(A:\nabla_x u_\epsilon + B\cdot\nabla_x\theta_\epsilon)\right)\,,
\end{aligned} \tag{4.96}$$

so

$$\begin{aligned}
&(\partial_t + v\cdot\nabla_x) g_\epsilon^{[2]} + \tfrac{1}{\epsilon}\mathcal{L} g_\epsilon^{[2]}\\
&= (\partial_t + v\cdot\nabla_x)(\rho_\epsilon + v\cdot u_\epsilon + (\tfrac12|v|^2 - \tfrac{\mathrm{D}}{2})\theta_\epsilon) - \epsilon\mathcal{P}[(\partial_t + v\cdot\nabla_x)\mathcal{L}^{-1}(A:\nabla_x u_\epsilon + B\cdot\nabla_x\theta_\epsilon)]\\
&\quad + \epsilon^2\left((\partial_t + v\cdot\nabla_x)\mathcal{L}^{-1}\right)^2(A:\nabla_x u_\epsilon + B\cdot\nabla_x\theta_\epsilon) - (A:\nabla_x u_\epsilon + B\cdot\nabla_x\theta_\epsilon)\,.
\end{aligned} \tag{4.97}$$

Plugging

$$\begin{aligned}
\partial_t\rho_\epsilon \;&=\; -\operatorname{div} u_\epsilon\,,\\
v\cdot(\partial_t u_\epsilon) \;&=\; -\,v\cdot\nabla_x(\rho_\epsilon + \theta_\epsilon) + \epsilon v\cdot\operatorname{div}(\mu D_x u_\epsilon)\,,\\
(\tfrac12|v|^2 - \tfrac{\mathrm{D}}{2})\partial_t\theta_\epsilon \;&=\; -\,(\tfrac12|v|^2 - \tfrac{\mathrm{D}}{2})\tfrac{2}{\mathrm{D}}\operatorname{div} u_\epsilon + \epsilon\tfrac{2}{\mathrm{D}}(\tfrac12|v|^2 - \tfrac{\mathrm{D}}{2})\operatorname{div}(\kappa\nabla_x\theta_\epsilon)\,,
\end{aligned} \tag{4.98}$$

into $(\partial_t + v\cdot\nabla_x)(\rho_\epsilon + v\cdot u_\epsilon + (\frac{1}{2}|v|^2 - \frac{\mathrm{D}}{2})\theta_\epsilon)$, we have

$$
\begin{aligned}
&(\partial_t + v\cdot\nabla_x)(\rho_\epsilon + v\cdot u_\epsilon + (\tfrac{1}{2}|v|^2 - \tfrac{\mathrm{D}}{2})\theta_\epsilon) \\
=\,& v\cdot\nabla_x(v\cdot u_\epsilon) - \tfrac{1}{\mathrm{D}}|v|^2 \operatorname{div} u_\epsilon + v\cdot\nabla_x(\tfrac{1}{2}|v|^2 - \tfrac{\mathrm{D}+2}{2})\theta_\epsilon \\
&+ \epsilon v\cdot \operatorname{div}(\mu D_x u_\epsilon) + \epsilon\tfrac{2}{\mathrm{D}}(\tfrac{1}{2}|v|^2 - \tfrac{\mathrm{D}}{2})\operatorname{div}(\kappa\nabla_x\theta_\epsilon) \\
=\,& A(v) : \nabla_x u_\epsilon + B(v)\cdot\nabla_x\theta_\epsilon \\
&+ \epsilon v\cdot \operatorname{div}(\mu D_x u_\epsilon) + \epsilon\tfrac{2}{\mathrm{D}}(\tfrac{1}{2}|v|^2 - \tfrac{\mathrm{D}}{2})\operatorname{div}(\kappa\nabla_x\theta_\epsilon)\,.
\end{aligned}
\tag{4.99}
$$

It remains to show

$$
v\cdot \operatorname{div}(\mu D_x u_\epsilon) + \tfrac{2}{\mathrm{D}}(\tfrac{1}{2}|v|^2 - \tfrac{\mathrm{D}}{2})\operatorname{div}(\kappa\nabla_x\theta_\epsilon) \;=\; \mathcal{P}[(\partial_t + v\cdot\nabla_x)\mathcal{L}^{-1}(A:\nabla_x u_\epsilon + B\cdot\nabla_x\theta_\epsilon)]\,.
\tag{4.100}
$$

Note that

$$
\mathcal{P}[(\partial_t + v\cdot\nabla_x)\mathcal{L}^{-1}(A:\nabla_x u_\epsilon + B\cdot\nabla_x\theta_\epsilon)] \;=\; \mathcal{P}[(v\cdot\nabla_x)(\widehat{A}:\nabla_x u_\epsilon + \widehat{B}\cdot\nabla_x\theta_\epsilon)]\,,
\tag{4.101}
$$

the lemma will be proved once we show

$$
\mathcal{P}[(v\cdot\nabla_x)(\widehat{A}:\nabla_x u_\epsilon)] \;=\; v\cdot \operatorname{div}(\mu D_x u_\epsilon)
\tag{4.102}
$$

and

$$
\mathcal{P}[(v\cdot\nabla_x)(\widehat{B}\cdot\nabla_x\theta_\epsilon)] \;=\; \tfrac{2}{\mathrm{D}}(\tfrac{1}{2}|v|^2 - \tfrac{\mathrm{D}}{2})\operatorname{div}(\kappa\nabla_x\theta_\epsilon)\,.
\tag{4.103}
$$

here we have used the notations and results in Chapter 3. Applying the orthogonal relations (1.76) in Chapter 1, we see that

$$
\begin{aligned}
\mathcal{P}[(v\cdot\nabla_x)(\widehat{A}:\nabla_x u_\epsilon)] &= \langle (v\cdot\nabla_x)(\widehat{A}:\nabla_x u_\epsilon)\rangle + v\cdot\langle v(v\cdot\nabla_x)(\widehat{A}:\nabla_x u_\epsilon)\rangle \\
&\quad + \tfrac{2}{\mathrm{D}}(\tfrac{1}{2}|v|^2-\tfrac{\mathrm{D}}{2})\langle(\tfrac{1}{2}|v|^2-\tfrac{\mathrm{D}+2}{2}+1)(v\cdot\nabla_x)(\widehat{A}:\nabla_x u_\epsilon)\rangle \\
&= v\cdot\langle v(v\cdot\nabla_x)(\widehat{A}:\nabla_x u_\epsilon)\rangle \\
&= v\cdot\operatorname{div}[\langle A(v)\otimes\widehat{A}(v)\rangle:\nabla_x u_\epsilon] = \, v\cdot\operatorname{div}(\mu D_x u_\epsilon)\,.
\end{aligned}
\tag{4.104}
$$

Similarly,

$$
\begin{aligned}
\mathcal{P}[(v\cdot\nabla_x)(\widehat{B}\cdot\nabla_x\theta_\epsilon)] &= \tfrac{2}{\mathrm{D}}(\tfrac{1}{2}|v|^2-\tfrac{\mathrm{D}}{2})\langle B(v)\cdot\nabla_x(\widehat{B}\cdot\nabla_x\theta_\epsilon)\rangle \\
&= \tfrac{2}{\mathrm{D}}(\tfrac{1}{2}|v|^2-\tfrac{\mathrm{D}}{2})\operatorname{div}(\langle B(v)\otimes\widehat{B}(v)\rangle\nabla_x\theta_\epsilon) \\
&= \tfrac{2}{\mathrm{D}}(\tfrac{1}{2}|v|^2-\tfrac{\mathrm{D}}{2})\operatorname{div}(\kappa\nabla_x\theta_\epsilon)\,.
\end{aligned}
\tag{4.105}
$$

Here we have used

$$
\begin{aligned}
&\langle A(v)\otimes\widehat{A}(v)\rangle:\nabla u_\epsilon = \mu(\nabla u_\epsilon+\nabla u_\epsilon^T-\tfrac{2}{\mathrm{D}}\operatorname{div}u_\epsilon)\,, \\
&\langle B(v)\otimes\widehat{B}(v)\rangle\nabla_x\theta_\epsilon = \kappa\nabla_x\theta_\epsilon\,.
\end{aligned}
\tag{4.106}
$$

in the last steps of (4.104) and (4.105). We have proved Lemma 4.3.1. □

We now estimate the difference between $g_\epsilon^{[2]}$ (cf. Lemma 7.1.1) and g_ϵ, solution of the linearized Boltzmann equation

$$
\partial_t g_\epsilon + v\cdot\nabla_x g_\epsilon + \tfrac{1}{\epsilon}\mathcal{L}g_\epsilon \,=\, 0\,. \tag{4.107}
$$

Before we state the main theorem of this chapter, recall that for the collision kernel b in the linearized collision operator

$$
\mathcal{L}g_\epsilon = \iint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^{\mathrm{D}}}(g_\epsilon+g_{\epsilon_1}-g'_\epsilon-g'_{\epsilon_1})bM_1\,dw\,dv_1\,. \tag{4.108}
$$

the attenuation coefficient is defined by

$$a(v) := \int\int_{\mathbb{S}^{D-1}\times\mathbb{R}^D} b(\omega, v_1 - v)\, \mathrm{d}\omega M_1 \mathrm{d}v_1. \tag{4.109}$$

We now give assumptions regarding the collision kernal b and collect some properties satisfied by linearized collision operators with b satisfying these assumptions. These assumptions are satisfied by many classical collision kernels, including those discussed in Section 1.2.1. Most of the presentation in this subsection is from [40], we refer the readers to [40] for detailed discussion of the assumptions and the proof of properties satisfied by the linearized collision operators. The first assumption is that the collision kernel b satisfies the requirement of the DiPerna-Lions theory (cf. Chapter 1, (4.110)): for every compact set $K \subset \mathbb{R}^D$,

$$b \in L^1_{loc}(\mathbb{R}^D \times \mathbb{S}^{D-1}), \quad \text{and} \quad \lim_{|v|\to\infty} \frac{1}{1+|v|^2} \iint_{\mathbb{S}^{D-1}\times K} b(\omega, v_1 - v)\mathrm{d}\omega \mathrm{d}v_1 = 0\,. \tag{4.110}$$

The second assumption is that the attenuation coefficient

$$a(v) = \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} b(v - v_1, \omega) M(v_1)\mathrm{d}\omega \mathrm{d}v_1\,. \tag{4.111}$$

is bounded below as

$$C_a(1+|v|)^\alpha \le a(v), \tag{4.112}$$

for some constants $C_a > 0$ and $\alpha \in \mathbb{R}$. The third assumption is there exists $s \in (1, \infty]$ and $C_b \in (0, \infty)$ such that

$$\left(\int_{\mathbb{R}^D} \left|\frac{\bar{b}(v_1 - v)}{a(v_1)a(v)}\right|^s a(v_1) M_1 \mathrm{d}v_1\right)^{\frac{1}{s}} \le C_b\,, \tag{4.113}$$

where

$$\bar{b}(v_1 - v) := \int_{\mathbb{S}^{D-1}} b(\omega, v_1 - v)\mathrm{d}\omega\,. \tag{4.114}$$

The fourth assumption is that

$$\mathcal{K}^+ : L^2(aM\mathrm{d}v) \to L^2(aMdv) \quad \text{is compact}\,, \tag{4.115}$$

where

$$\mathcal{K}^+(g) := \frac{1}{2a} \iint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^{\mathrm{D}}} (g' + g_1')b(\omega, v_1 - v)\mathrm{d}\omega M_1 \mathrm{d}v_1\,. \tag{4.116}$$

With these assumptions, we have

$$\frac{1}{a}\mathcal{L} : L^p(aM\mathrm{d}v) \to L^p(aM\mathrm{d}v) \quad \text{is Fredholm for every}\, p \in (0,\infty)\,. \tag{4.117}$$

Moreover,

$$\begin{aligned} \mathcal{L} : L^p(aM\mathrm{d}v) &\to L^p(a^{1-p}M\mathrm{d}v) \quad \text{is bounded}\,, \\ \mathcal{L}^{-1} : L^p(a^{1-p}M\mathrm{d}v) &\to L^p(aM\mathrm{d}v) \quad \text{is bounded}\,. \end{aligned} \tag{4.118}$$

The next theorem establishes the main property of the linearized collision operator $\mathcal{L}_{\mathcal{M}}$, i.e., that it satisfies the Fredholm alternative in some weighted L^2 space. We call

$$a(v) = \iint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^{\mathrm{D}}} b(v - v_1, \omega)M(v_1)\mathrm{d}\omega\mathrm{d}v_1 \tag{4.119}$$

the attenuation coefficient.

Theorem 4.3.1 *Let g_ϵ be the solution of the linearized Boltzmann equation*

$$\partial_t g_\epsilon + v\cdot\nabla_x g_\epsilon + \tfrac{1}{\epsilon}\mathcal{L}g_\epsilon \,=\, 0. \tag{4.120}$$

Assume the collision operator satisfies the assumptions (4.110)*-*(4.115) *above. Let*

$$\begin{aligned} g_\epsilon^{[2]} := \rho_\epsilon + v\cdot u_\epsilon + (\tfrac{1}{2}|v|^2 - \tfrac{\mathrm{D}}{2})\theta_\epsilon - \epsilon\mathcal{L}^{-1}(A:\nabla_x u_\epsilon + B\cdot\nabla_x\theta_\epsilon) \\ + \epsilon^2\mathcal{L}^{-1}(\partial_t + v\cdot\nabla_x)\mathcal{L}^{-1}(A:\nabla_x u_\epsilon + B\cdot\nabla_x\theta_\epsilon)\,, \end{aligned}$$

where $U_\epsilon := (\rho_\epsilon,\, u_\epsilon,\, \theta_\epsilon)^T$ *are solutions of the associated Cauchy problem of the linearized compressible Navier-Stokes approximation. Denote the fluid moments of* g_ϵ *by* $U_\epsilon^B := (\rho_\epsilon^B, u_\epsilon^B, \theta_\epsilon^B)^T$*. Assume* $\langle g_\epsilon^{[2]\mathrm{in}} - g_\epsilon^{\mathrm{in}}\rangle$, $\left\langle (|v| g_\epsilon^{[2]\mathrm{in}} - g_\epsilon^{\mathrm{in}})\right\rangle$, $\left\langle |v|^2 (g_\epsilon^{[2]\mathrm{in}} - g_\epsilon^{\mathrm{in}})\right\rangle$ *are bounded by* η *in* $L^2(\mathbb{T}^{\mathrm{D}})$ *and* $U_\epsilon^{\mathrm{in}} \in H^5(\mathbb{T}^{\mathrm{D}})$*. Then*

$$||U_B - U_\epsilon||_{L^2(\mathbb{T}^{\mathrm{D}})} \leq C \max\{\sqrt{\epsilon}||U_\epsilon^{\mathrm{in}}||_{H^5(\mathbb{T}^{\mathrm{D}})}, \eta\}\,.$$

uniformly for $t > 0$.

Proof: According to Lemma 7.1.1.

$$\partial_t g_\epsilon^{[2]} + v\cdot\nabla_x g_\epsilon^{[2]} + \tfrac{1}{\epsilon}\mathcal{L} g_\epsilon^{[2]} \;=\; \epsilon^2 \left((\partial_t + v\cdot\nabla_x)\mathcal{L}^{-1}\right)^2 (A : \nabla_x u_\epsilon + B\cdot\nabla_x\theta_\epsilon)\,, \quad (4.121)$$

so

$$\partial_t (g_\epsilon^{[2]} - g_\epsilon) + v\cdot\nabla_x (g_\epsilon^{[2]} - g_\epsilon) + \tfrac{1}{\epsilon}\mathcal{L}(g_\epsilon^{[2]} - g_\epsilon) \;=\; \epsilon^2 \left((\partial_t + v\cdot\nabla_x)\mathcal{L}^{-1}\right)^2 (A : \nabla_x u_\epsilon + B\cdot\nabla_x\theta_\epsilon)\,. \quad (4.122)$$

Taking inner product of the above equation with $(g_\epsilon^{[2]} - g_\epsilon)$ and integrating with respect to $M\mathrm{d}v\mathrm{d}x$ yields

$$\partial_t \int \langle (g_\epsilon^{[2]} - g_\epsilon)^2\rangle \mathrm{d}x + \tfrac{1}{\epsilon}\int \langle \mathcal{L}(g_\epsilon^{[2]} - g_\epsilon), (g_\epsilon^{[2]} - g_\epsilon)\rangle \mathrm{d}x$$
$$= \epsilon^2 \int \langle \left((\partial_t + v\cdot\nabla_x)\mathcal{L}^{-1}\right)^2 (A : \nabla_x u_\epsilon + B\cdot\nabla_x\theta_\epsilon), (g_\epsilon^{[2]} - g_\epsilon)\rangle\, \mathrm{d}x \quad (4.123)$$

Let

$$a(t) := \left(\int \langle (g_\epsilon^{[2]} - g_\epsilon)^2\rangle \mathrm{d}x\,(t)\right)^{\frac{1}{2}}\,,$$

$$f(s) := ||\left((\partial_t + v\cdot\nabla_x)\mathcal{L}^{-1}\right)^2 (A : \nabla_x u_\epsilon + B\cdot\nabla_x\theta_\epsilon)||_{L^2(M\mathrm{d}v\mathrm{d}x)}\,,$$

$$\Phi(t) := \epsilon^2 \int_0^t f(s)\, a^2(s)\, \mathrm{d}s\,.$$

We get

$$a^2(t) \le \Phi(t) + a^2(0)$$
$$\le \epsilon^2 \int_0^t f(s)\sqrt{\Phi(s)+a^2(0)}\mathrm{d}s + a^2(0)\,.$$

Since $\Phi(s) \le \Phi(t)$ for $s \in [0,t]$, we have

$$\Phi(t) + a^2(0) \le \Theta\sqrt{\Phi(t)+a^2(0)} + a^2(0)$$

where Θ is the upper bound for $\epsilon^2 \int_0^t f(s)\mathrm{d}s$. Therefore

$$\sqrt{\Phi(t)+a^2(0)} \le C\max\{\Theta, a^2(0)\}$$

So $a(t)$ is bounded by $\max\{\Theta, a^2(0)\}$. By Minkowski inequality and Cauchy-Schwartz inequality,

$$\left(\int_{\mathbb{T}^{\mathrm{D}}}\left(\int_{\mathbb{R}^{\mathrm{D}}}(g_\epsilon^{[2]}-g_\epsilon)M\mathrm{d}v\right)^2\mathrm{d}x\right)^{\frac{1}{2}} \le C\left(\int_{\mathbb{T}^{\mathrm{D}}}\int_{\mathbb{R}^{\mathrm{D}}}(g_\epsilon^{[2]}-g_\epsilon)^2M\mathrm{d}v\mathrm{d}x\right)^{\frac{1}{2}}$$

So $||\rho_\epsilon - \langle g_\epsilon\rangle||_{L^2(\mathbb{T}^{\mathrm{D}})} \le C\max\{\Theta, a^2(0)\}$. It suffices to estimate

$$f(s) = ||\left((\partial_t + v\cdot\nabla_x)\mathcal{L}^{-1}\right)^2(A:\nabla_x u_\epsilon + B\cdot\nabla_x\theta_\epsilon)||_{L^2(M\mathrm{d}v\mathrm{d}x)}\,.$$

We now show the estimate for $||\left((v\cdot\nabla_x)\mathcal{L}^{-1}\right)^2(A:\nabla_x u_\epsilon)||_{L^2(M\mathrm{d}v\mathrm{d}x)}$. Note that

$$\begin{aligned}
&||\left((v\cdot\nabla_x)\mathcal{L}^{-1}\right)^2(A:\nabla_x u_\epsilon)||^2_{L^2(M\mathrm{d}v\mathrm{d}x)}\\
\le&C\sum_{i,j,m,n}\int(v_i\mathcal{L}^{-1}(v_j\mathcal{L}^{-1}(A_{mn})))^2M\mathrm{d}v\int_{\mathbb{T}^{\mathrm{D}}}(D_x^3u_\epsilon)^2\,\mathrm{d}x \qquad (4.124)\\
\le&Ce^{-\epsilon c_1 t}||D_x^3U_\epsilon^{\mathrm{in}}||^2\sum_{i,j,m,n}\int(v_i\mathcal{L}^{-1}(v_j\mathcal{L}^{-1}(A_{mn})))^2M\mathrm{d}v,
\end{aligned}$$

we will show that $\int(v_i\mathcal{L}^{-1}(v_j\mathcal{L}^{-1}(A_{mn})))^2M\mathrm{d}v$ is bounded by a constant. In fact,

$$\begin{aligned}
&\int(v_i\mathcal{L}^{-1}(v_j\mathcal{L}^{-1}(A_{mn})))^2M\mathrm{d}v\\
\le&\left(\int(v_i)^4M\mathrm{d}v\right)^{\frac{1}{2}}\left(\int(\mathcal{L}^{-1}v_j\mathcal{L}^{-1}(A_{mn}))^4M\mathrm{d}v\right)^{\frac{1}{2}} \qquad (4.125)\\
\le&C\left(\int(\mathcal{L}^{-1}v_j\mathcal{L}^{-1}(A_{mn}))^4M\mathrm{d}v\right)^{\frac{1}{2}},
\end{aligned}$$

and

$$\begin{aligned}
&\int (\mathcal{L}^{-1}(v_j\mathcal{L}^{-1}(A_{mn}))^4 M\mathrm{d}v \\
\leq& \left(\int (\mathcal{L}^{-1}(v_j\mathcal{L}^{-1}(A_{mn}))^4)^2 aM\mathrm{d}v\right)^{\frac{1}{2}} \left(\int (a^{-1}M\mathrm{d}v\right)^{\frac{1}{2}} .
\end{aligned} \tag{4.126}$$

By the attenuation assumptions, $\int a^{-1}M\mathrm{d}v \leq C$. Also, for every $p \in (1,\infty)$, $\mathcal{L}^{-1} : L^p(a^{1-p}M\mathrm{d}v) \to L^p(aM\mathrm{d}v)$ is bounded (cf. [40]), therefore

$$\begin{aligned}
&\left(\int (\mathcal{L}^{-1}(v_j\mathcal{L}^{-1}(A_{mn}))^8) aM\mathrm{d}v\right)^{\frac{1}{2}} \\
\leq& C\left(\int (v_j\mathcal{L}^{-1}(A_{mn}))^8 a^{-1}M\mathrm{d}v\right)^{\frac{1}{2}} \\
\leq& C\left(\int (\mathcal{L}^{-1}(A_{mn}))^{16} aM\mathrm{d}v\right)^{\frac{1}{4}} \left(v_j^2(a^{-3})M\mathrm{d}v\right)^{\frac{1}{4}} \\
\leq& C\left(\int (A_{mn})^{16} a^{-15}M\mathrm{d}v\right)^{\frac{1}{4}} \leq C.
\end{aligned} \tag{4.127}$$

Therefore,

$$||\left((v\cdot\nabla_x)\mathcal{L}^{-1}\right)^2 (A:\nabla_x u_\epsilon)||_{L^2(M\mathrm{d}v\mathrm{d}x)} \leq Ce^{-\epsilon c_1 t}||D_x^3 U_\epsilon^{\mathrm{in}}||^2. \tag{4.128}$$

Applying the same estimate for other terms in $||\left((\partial_t + v\cdot\nabla_x)\mathcal{L}^{-1}\right)^2 (A:\nabla_x u_\epsilon + B\cdot\nabla_x\theta_\epsilon)||_{L^2(M\mathrm{d}v\mathrm{d}x)}$, we then have

$$||\left((\partial_t + v\cdot\nabla_x)\mathcal{L}^{-1}\right)^2 (A:\nabla_x u_\epsilon + B\cdot\nabla_x\theta_\epsilon)||_{L^2(M\mathrm{d}v\mathrm{d}x)} \leq Ce^{-\epsilon c_1 t}||D_x^5 U_\epsilon^{\mathrm{in}}||^2 . \tag{4.129}$$

So $\epsilon^2\int_0^t f(s)\mathrm{d}s$ has upper bound $\epsilon||D_x^5 U_\epsilon^{\mathrm{in}}||^2$ and

$$||\rho_\epsilon - \langle g_\epsilon\rangle||_{L^2(\mathbb{T}^{\mathrm{D}})} \leq C\max\{\sqrt{\epsilon}||U_\epsilon^{\mathrm{in}}||_{H^5(\mathbb{T}^{\mathrm{D}})}, \eta\} .$$

The proof can be generalized to higher fluid moments. This completes the proof of Theorem 4.3.1.

Appendix A: Formal Derivation of the Linearized Weakly Compressible Navier-Stokes System from the Linearized Compressible Navier-Stokes Equation in Matrix Form

In this appendix, we derive the first-order approximation of the Fourier transform of the linearized compressible Navier-Stokes equations. The linearized compressible Navier-Stokes equations are:

$$\begin{aligned} \partial_t \rho_\epsilon + \nabla_x \cdot u_\epsilon &= 0 \\ \partial_t u_\epsilon + \nabla_x(\rho_\epsilon + \theta_\epsilon) &= \epsilon \nabla_x \cdot \mu[\nabla_x u_\epsilon + (\nabla_x u_\epsilon)^T - \tfrac{2}{\mathrm{D}} \nabla_x \cdot u_\epsilon I] \\ \partial_t \theta_\epsilon + \tfrac{2}{\mathrm{D}} \nabla_x \cdot u_\epsilon &= \epsilon \tfrac{2}{\mathrm{D}} \nabla_x \cdot (\kappa \nabla_x \theta_\epsilon), \end{aligned} \tag{A.1}$$

Applying Fourier transform to the system above:

$$\partial_t \begin{pmatrix} \widehat{\rho} \\ \widehat{u} \\ \widehat{\theta} \end{pmatrix} + \begin{pmatrix} 0 & i\xi^T & 0 \\ i\xi & \epsilon\mu(I|\xi|^2 + \frac{\mathrm{D}-2}{\mathrm{D}}\xi\xi^T) & i\xi \\ 0 & i\frac{2}{\mathrm{D}}\xi^T & \epsilon\kappa\frac{2}{\mathrm{D}}|\xi|^2 \end{pmatrix} \begin{pmatrix} \widehat{\rho} \\ \widehat{u} \\ \widehat{\theta} \end{pmatrix} = 0 \tag{A.2}$$

We denote (4.14) by

$$\partial_t \widehat{U} + i\widehat{\mathcal{A}}\widehat{U} = \epsilon\widehat{\mathcal{B}}\widehat{U}, \tag{A.3}$$

where

$$\widehat{\mathcal{A}} = \begin{pmatrix} 0 & i\xi^T & 0 \\ i\xi & 0 & i\xi \\ 0 & i\frac{2}{\mathrm{D}}\xi^T & 0 \end{pmatrix}, \quad \widehat{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\mu(I|\xi|^2 + \frac{\mathrm{D}-2}{\mathrm{D}}\xi\xi^T) & 0 \\ 0 & 0 & -\kappa\frac{2}{\mathrm{D}}|\xi|^2 \end{pmatrix}. \tag{A.4}$$

We now describe the approximation scheme.

Formally, we expand $\widehat{U}(t,\tau)$ as:

$$\widehat{U} = \widehat{U}^{[0]} + \epsilon\widehat{U}^{[1]} + \epsilon^2\widehat{U}^{[2]} + ... \tag{A.5}$$

Plugging the above expression to $(A.3)$ and truncating at $O(1)$:

$$\partial_t\widehat{U}^{[0]} + i\widehat{\mathcal{A}}\widehat{U}^{[0]} = 0, \tag{A.6}$$

i.e.

$$\partial_t(e^{i\widehat{\mathcal{A}}t}\widehat{U}^{[0]}) = 0, \tag{A.7}$$

thus

$$\widehat{U}^{[0]}(t,\tau) = e^{-i\widehat{\mathcal{A}}t}\widehat{U}^{(\tau)}, \text{ where } \widehat{U}^{(\tau)} = \widehat{U}(0,\tau). \tag{A.8}$$

At $O(\epsilon)$: (recall that $\partial_t \to \partial_t + \epsilon\partial_\tau$ under the two-time scaling)

$$\partial_t\widehat{U}^{[1]} + \partial_\tau\widehat{U}^{[0]} + i\widehat{\mathcal{A}}\widehat{U}^{[1]} = \widehat{\mathcal{B}}\widehat{U}^{[0]}, \tag{A.9}$$

i.e.

$$\partial_t(e^{i\widehat{\mathcal{A}}t}\widehat{U}^{[1]}) + \partial_\tau(e^{i\widehat{\mathcal{A}}t}\widehat{U}^{[0]}) = e^{i\widehat{\mathcal{A}}t}\widehat{\mathcal{B}}\widehat{U}^{[0]}. \tag{A.10}$$

By $(A.8)$, $e^{i\widehat{\mathcal{A}}t}\widehat{U}^{[0]} = \widehat{U}^{(\tau)}$, so $\partial_\tau(e^{i\widehat{\mathcal{A}}t}\widehat{U}^{[0]}) = \frac{\mathrm{d}\widehat{U}^{(\tau)}}{\mathrm{d}\tau}$.

Since $e^{i\widehat{\mathcal{A}}t}$ is unitary, we may expect

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \partial_s(e^{i\widehat{\mathcal{A}}s}\widehat{U}^{[1]})\,\mathrm{d}s = 0. \tag{A.11}$$

Therefore, averaging $(A.10)$ over time would lead to

$$\frac{\mathrm{d}\widehat{U}^{(\tau)}}{\mathrm{d}\tau} = \lim_{t\to\infty}\frac{1}{t}\int_0^t e^{i\widehat{\mathcal{A}}s}\widehat{\mathcal{B}}\widehat{U}^{[0]}\,\mathrm{d}s = \lim_{t\to\infty}\frac{1}{t}\int_0^t e^{i\widehat{\mathcal{A}}s}\widehat{\mathcal{B}}e^{-i\widehat{\mathcal{A}}s}\widehat{U}^{(\tau)}\,\mathrm{d}s. \tag{A.12}$$

So

$$\widehat{U}^{[0]}(t,\tau) = e^{-i\widehat{\mathcal{A}}t}\widehat{U}^{(\tau)}, \tag{A.13}$$

with $\widehat{U}^{(\tau)}$ being the solution of $(A.20)$.

It remains to compute $\widehat{U}^{(\tau)}$.

A.1 First-order Approximation up to a Unitary Transform

In this section, we compute $\widehat{U}^{(\tau)}$. Since $\widehat{U}^{[0]}(t,\tau) = e^{-i\widehat{\mathcal{A}}t}\widehat{U}^{(\tau)}$, $\widehat{U}^{(\tau)}$ is essentially the first-order approximation up to a unitary transform. All the calculation are carried on over the orthonormal basis formed by eigenvectors of $\mathcal{A}$, and an inner product induced by $\mathcal{A}$. For this purpose, we observe that $\widehat{\mathcal{A}}$ has a symmetrizer; in fact, $H\widehat{\mathcal{A}}$ is a symmetric matrix, where

$$H := \begin{pmatrix} \mathbf{I}_{\mathrm{D}-1\times\mathrm{D}-1} & 0 \\ 0 & \frac{\mathrm{D}}{2} \end{pmatrix} \tag{A.14}$$

Hence, we may define the inner product $U_1 \diamond U_2$ between $U_1 = (\rho_1, u_1, \theta_1)^T$ and $U_2 = (\rho_2, u_2, \theta_2)^T$ as:

$$U_1 \diamond U_2 := \rho_1\rho_2 + u_1u_2 + \tfrac{\mathrm{D}}{2}\theta_1\theta_2 \tag{A.15}$$

We observe that $\widehat{\mathcal{A}}$ has $D+2$ independent eigenvectors; moreover, they are orthogonal under the new inner product. More specifically, for $\xi \neq \mathbf{0}$,

$$\lambda_1 = \sqrt{1+\tfrac{2}{\mathrm{D}}}||\xi||, \quad \phi^{(\mathbf{1})} = \frac{1}{\sqrt{2\frac{\mathrm{D}+2}{\mathrm{D}}}} \begin{pmatrix} 1 \\ \sqrt{1+\frac{2}{\mathrm{D}}}\frac{\xi}{||\xi||} \\ \frac{2}{\mathrm{D}} \end{pmatrix}$$

$$\lambda_2 = -\sqrt{1+\tfrac{2}{\mathrm{D}}}||\xi||, \quad \phi^{(\mathbf{2})} = \frac{1}{\sqrt{2\frac{\mathrm{D}+2}{\mathrm{D}}}} \begin{pmatrix} 1 \\ -\sqrt{1+\frac{2}{\mathrm{D}}}\frac{\xi}{||\xi||} \\ \frac{2}{\mathrm{D}} \end{pmatrix}$$

$$\lambda_a = 0 \text{ for } a \in \{3, ..., \mathrm{D}+2\}, \text{and } \phi^{(\mathbf{a})} \text{ are D-dimensional basis of solutions to}$$
$$\xi \cdot \mathbf{y} = 0, x + z = 0\,. \tag{A.16}$$

Here $(x, \mathbf{y}^T, z)$ denotes an eigenvector, where $x, z \in \mathbb{R}, \mathbf{y} \in \mathbb{R}^{\mathrm{D}}$. In particular, we may choose

$$\phi^{(\mathbf{3})} = \frac{1}{\sqrt{1+\frac{\mathrm{D}}{2}}} \begin{pmatrix} 1 \\ \mathbf{0} \\ -1 \end{pmatrix}, \phi^{(\mathbf{a})} = \begin{pmatrix} 0 \\ \mathbf{y} \\ 0 \end{pmatrix} \text{ for } a \in \{4, ..., \mathrm{D}+2\}, \text{where } \xi \cdot \mathbf{y} = 0, ||\mathbf{y}||_{\mathbb{R}^{\mathrm{D}}} = 1\,. \tag{A.17}$$

Note that for the $\mathrm{D}-1$ independent solutions $\mathbf{y}_1, ..., \mathbf{y}_{\mathrm{D}-1}$ to $\xi^T\mathbf{y} = 0$, we could always make them orthogonal under the regular inner product on $\mathbb{R}^{\mathrm{D}}$. It's straightforward to check that $\phi^{(\mathbf{a})}, a \in \{1, ..., \mathrm{D}+2\}$ are orthonormal under the new inner product defined in $(A.15)$, i.e.

$$\Phi\Phi^T H = \mathbf{I}, \quad \text{where } \Phi = \begin{pmatrix} \phi^{(1)} & \phi^{(2)} & ... & \phi^{(\mathrm{D}+2)} \end{pmatrix}. \tag{A.18}$$

We now expand $\widehat{U}^{(\tau)}$ in terms of the orthonormal basis $\phi^{(i)}$:

$$\widehat{U}^{(\tau)} = \sum_{i=1}^{\mathrm{D}+2} c_i \phi^{(i)} = \Phi \mathbf{c}(\tau), \quad \text{where } \mathbf{c} = \begin{pmatrix} c_{(1)} \\ \vdots \\ c_{(\mathrm{D}+2)} \end{pmatrix}. \tag{A.19}$$

Recall that

$$\frac{\mathrm{d}\widehat{U}^{(\tau)}}{\mathrm{d}\tau} = \lim_{t\to\infty} \frac{1}{t} \int_0^t e^{i\widehat{\mathcal{A}}s} \widehat{\mathcal{B}} \widehat{U}^{[0]} \, \mathrm{d}s = \lim_{t\to\infty} \frac{1}{t} \int_0^t e^{i\widehat{\mathcal{A}}s} \widehat{\mathcal{B}} e^{-i\widehat{\mathcal{A}}s} \widehat{U}^{(\tau)} \, \mathrm{d}s, \tag{A.20}$$

i.e.

$$\Phi \frac{\mathrm{d}\mathbf{c}}{\mathrm{d}\tau} = \lim_{t\to\infty} \frac{1}{t} \int_0^t e^{i\widehat{\mathcal{A}}s} \widehat{\mathcal{B}} e^{-i\widehat{\mathcal{A}}s} \Phi \mathbf{c} \; \mathrm{d}s. \tag{A.21}$$

Notice that

$$e^{-i\widehat{\mathcal{A}}s} \Phi = \Phi \begin{bmatrix} e^{-i\lambda_1 s} & & 0 & \\ & e^{-i\lambda_2 s} & & \\ & & \ddots & \\ & 0 & & e^{-i\lambda_{\mathrm{D}+2} s} \end{bmatrix}, \tag{A.22}$$

and write

$$e^{-i\Lambda t} = \begin{bmatrix} e^{-i\lambda_1 s} & & 0 & \\ & e^{-i\lambda_2 s} & & \\ & & \ddots & \\ & 0 & & e^{-i\lambda_{\mathrm{D}+2} s} \end{bmatrix}. \tag{A.23}$$

Therefore,

$$\widehat{B}(\xi) e^{-i\widehat{\mathcal{A}}s} \Phi = \widehat{B}(\xi) \Phi e^{-i\Lambda t}. \tag{A.24}$$

We then write

$$\widehat{\mathcal{B}}(\xi) \phi_{(a)} = M_{(b,a)} \phi_{(b)}, \tag{A.25}$$

i.e.,

$$\widehat{\mathcal{B}}(\xi)\Phi = \Phi \begin{pmatrix} M_{(1,1)} & M_{(1,2)} & \dots & M_{(1,\mathrm{D}+2)} \\ \vdots & \vdots & \ddots & \vdots \\ M_{(\mathrm{D}+2,1)} & M_{(\mathrm{D}+2,2)} & \dots & M_{(\mathrm{D}+2,\mathrm{D}+2)} \end{pmatrix}. \tag{A.26}$$

In particular,

$$M = \Phi^{-1}\widehat{\mathcal{B}}(\xi)\Phi = \Phi^T H \widehat{\mathcal{B}}(\xi)\Phi. \tag{A.27}$$

Therefore

$$e^{i\widehat{\mathcal{A}}s}\widehat{\mathcal{B}}(\xi)e^{-i\widehat{\mathcal{A}}s}\Phi = e^{i\widehat{\mathcal{A}}s}\Phi M e^{-i\Lambda s} = \Phi e^{i\Lambda s} M e^{-i\Lambda s}. \tag{A.28}$$

So

$$\begin{aligned} &\lim_{t\to\infty}\frac{1}{t}\int_0^t e^{i\widehat{\mathcal{A}}s}\widehat{\mathcal{B}}e^{-i\widehat{\mathcal{A}}s}\Phi\mathbf{c}\ \mathrm{d}s. \\ =&\lim_{t\to\infty}\frac{1}{t}\int_0^t \Phi e^{i\Lambda s}Me^{-i\Lambda s}\mathbf{c}\ \mathrm{d}s. \\ =&\Phi\lim_{t\to\infty}\frac{1}{t}\int_0^t e^{i\Lambda s}Me^{-i\Lambda s}\ \mathrm{d}s\ \mathbf{c}. \end{aligned} \tag{A.29}$$

Since

$$\begin{aligned} &\lim_{t\to\infty}\frac{1}{t}\int_0^t e^{i\Lambda s}Me^{-i\Lambda s}\ \mathrm{d}s \\ =&\begin{bmatrix} M_{(1,1)} & 0 & 0 & \dots \\ 0 & M_{(2,2)} & \dots & \dots \\ 0 & \vdots & \ddots & 0 \\ 0 & 0 & \dots & M_{(\mathrm{D}+2,\mathrm{D}+2)} \end{bmatrix} := M^{diag}, \end{aligned} \tag{A.30}$$

$(A.21)$ becomes

$$\Phi\frac{\mathrm{d}\mathbf{c}(\tau)}{\mathrm{d}\tau} = \Phi M^{diag}\mathbf{c}(\tau). \tag{A.31}$$

We then have

$$\widehat{U}^{\tau} = \Phi e^{M^{diag}\tau}\Phi^T H\widehat{U}^{\mathrm{in}}, \quad \text{where } \widehat{U}^{\mathrm{in}} = \widehat{U}(0), \tag{A.32}$$

since

$$\Phi\Phi^T H = \mathbf{I}.$$

A.2 The Approximation Matrix

We now compute $\Phi e^{M^{diag}\tau}\Phi^T H$. For this purpose, notice that

$$M_{(i,i)} = {\phi^{(i)}}^T H \phi^{(i)}. \tag{A.33}$$

Specifically, we have

$$\begin{aligned} M_{(1,1)} &= M_{(2,2)} = -\left[\mu\frac{\mathrm{D}-1}{\mathrm{D}}|\xi|^2 + \kappa\frac{2}{\mathrm{D}(\mathrm{D}+2)}|\xi|^2\right] := u, \\ M_{(3,3)} &= -\kappa\frac{2}{\mathrm{D}+2}|\xi|^2 := v, \\ M_{(j,j)} &= -\mu|\xi|^2 := w,\ j = 4, ..., \mathrm{D}+2. \end{aligned} \tag{A.34}$$

We thus write each entry of $\Phi e^{M^{diag}\tau}\Phi^T H$:

$$\begin{pmatrix} \frac{\mathrm{D}}{\mathrm{D}+2}e^{u\tau} + \frac{2}{\mathrm{D}+2}e^{v\tau} & \mathbf{0} & \frac{\mathrm{D}}{\mathrm{D}+2}e^{u\tau} - \frac{\mathrm{D}}{\mathrm{D}+2}e^{v\tau} \\ \mathbf{0} & \frac{\xi\xi^T}{|\xi|^2}e^{u\tau} + (I - \frac{\xi\xi^T}{|\xi|^2})e^{w\tau} & \mathbf{0} \\ \frac{2}{\mathrm{D}+2}e^{u\tau} & \mathbf{0} & \frac{2}{\mathrm{D}+2}e^{u\tau} + \frac{\mathrm{D}}{\mathrm{D}+2}e^{v\tau} \end{pmatrix}.$$

Note that

$$\begin{aligned} \widehat{U}^{[0]}(t,\tau) &= e^{-i\widehat{A}t}\widehat{U}^{(\tau)} \\ &= e^{-i\widehat{A}t}\Phi e^{M^{diag}\tau}\Phi^T H \widehat{U}^{\mathrm{in}} \\ &= \Phi e^{-i\Lambda t} e^{M^{diag}\tau}\Phi^T H \widehat{U}^{\mathrm{in}}. \end{aligned} \tag{A.35}$$

We thus compute entries of $P := \Phi e^{-i\Lambda t} e^{M^{diag}\tau} \Phi^T H$:

$$\begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix}. \tag{A.36}$$

where

$$\begin{aligned}
P_{11} &= \frac{\mathrm{D}}{\mathrm{D}+2} e^{u\tau} \cos(\sqrt{\frac{\mathrm{D}+2}{\mathrm{D}}}|\xi|t) + \frac{2}{\mathrm{D}+2} e^{v\tau}\,, \\
P_{12} &= -i\sqrt{\frac{\mathrm{D}}{\mathrm{D}+2}} \frac{\xi^T}{|\xi|} e^{u\tau} \sin(\sqrt{\frac{\mathrm{D}+2}{\mathrm{D}}}|\xi|t)\,, \\
P_{13} &= \frac{\mathrm{D}}{\mathrm{D}+2} e^{u\tau} \cos(\sqrt{\frac{\mathrm{D}+2}{\mathrm{D}}}|\xi|t) - \frac{\mathrm{D}}{\mathrm{D}+2} e^{v\tau}\,, \\
P_{21} &= -i\sqrt{\frac{\mathrm{D}}{\mathrm{D}+2}} \frac{\xi}{|\xi|} e^{u\tau} \sin(\sqrt{\frac{\mathrm{D}+2}{\mathrm{D}}}|\xi|t)\,, \\
P_{22} &= \frac{\xi\xi^T}{|\xi|^2} e^{u\tau} \cos(\sqrt{\frac{\mathrm{D}+2}{\mathrm{D}}}|\xi|t) + e^{w\tau}(I - \frac{\xi\xi^T}{|\xi|^2})\,, \\
P_{23} &= -i\sqrt{\frac{\mathrm{D}}{\mathrm{D}+2}} \frac{\xi}{|\xi|} e^{u\tau} \sin(\sqrt{\frac{\mathrm{D}+2}{\mathrm{D}}}|\xi|t)\,, \\
P_{31} &= \frac{2}{\mathrm{D}+2} e^{u\tau} \cos(\sqrt{\frac{\mathrm{D}+2}{\mathrm{D}}}|\xi|t) - \frac{2}{\mathrm{D}+2} e^{v\tau}\,, \\
P_{32} &= -2i\frac{\xi^T}{|\xi|} \sqrt{\frac{1}{(\mathrm{D}+2)\mathrm{D}}} e^{u\tau} \sin(\sqrt{\frac{\mathrm{D}+2}{\mathrm{D}}}|\xi|t)\,, \\
P_{33} &= \frac{2}{\mathrm{D}+2} e^{u\tau} \cos(\sqrt{\frac{\mathrm{D}+2}{\mathrm{D}}}|\xi|t) + \frac{\mathrm{D}}{\mathrm{D}+2} e^{v\tau}\,.
\end{aligned}$$

Therefore, the Fourier transform of solution of linearized weakly compressible Navier-Stokes with initial data $\widehat{U}^{\mathrm{in}}(\xi)$ can be expressed as

$$\widehat{V}(\xi) = P\widehat{U}^{\mathrm{in}}\,. \tag{A.37}$$

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