ABSTRACT

Title of dissertation: NEW EXAMPLES OF S-UNIMODAL MAPS WITH A SIGMA-FINITE ABSOLUTELY CONTINUOUS INVARIANT MEASURE

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We construct new types of examples of S-unimodal maps φ on an interval I that do not have a finite absolutely continuous invariant measure but that do have a σ-finite one. These examples satisfy two important properties. The first property is topological, namely, the forward orbit of the critical point c is dense, i.e., $\omega_\varphi(c) = I$. On the other hand, the second property is metric, we are able to conclude that this measure is infinite on every non-trivial interval. In the process, we show that we have the following dichotomy. Every absolutely continuous invariant measure, in our setting, is either σ-finite, or else it is infinite on every set of positive Lebesgue measure. Our method of construction is based on the method of inducing a power map defined piecewise on a countable collection of non-overlapping intervals that partition I modulo a Cantor set of Lebesgue measure zero. The power map then satisfies what is known as the

Folklore Theorem and therefore has a finite a.c.i.m. that is pulled back to define our φ-invariant measure on I, with the above stated properties.

NEW EXAMPLES OF S-UNIMODAL MAPS WITH A SIGMA-FINITE ABSOLUTELY CONTINUOUS INVARIANT MEASURE

by

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DEDICATION

To my mother and in memory of my father

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CHAPTER 1

INTRODUCTION

Section 1.1. Overview

1.1.1. Let us first recollect some definitions. If

$$\varphi : I \to I$$

is a C^3 map on the closed interval I then the *Schwarzian derivative* at x is given by

$$\mathcal{S}\varphi(x) = \frac{\varphi'''(x)}{\varphi'(x)} - \frac{3}{2}\left(\frac{\varphi''(x)}{\varphi'(x)}\right)^2$$

provided of course that x is not a critical point, i.e., $\varphi'(x) \neq 0$. We say that φ has *negative Schwarzian derivative* when $\mathcal{S}\varphi(x) < 0$ for all non-critical points x, and we write $\mathcal{S}\varphi < 0$. The map φ is called *unimodal* when the graph of φ has one and only one turning point $w = \varphi(c)$, which we will refer to as the critical value and c as the critical point. A unimodal map with negative Schwarzian derivative is called $\mathcal{S}$-unimodal.

A unimodal map φ is said to be *renormalizable* if there exists a closed proper sub-interval $J \subset I$ that contains the critical point c and a positive number n such that

$$\varphi^n(J) \subset J$$

and the restriction

$$\varphi^n|J: J \to J$$

is a unimodal map. A unimodal map φ is *infinitely renormalizable* if for every interval J and iterate φ^n as above, the restriction

$$\varphi^n|J: J \to J$$

is in turn renormalizable.

Remark 1.1.2. It is known, see [G-Jo], that if φ is infinitely renormalizable, then we obtain a Sinai-Bowen-Ruelle (SBR) measure μ supported on the closure of the forward orbit of the critical point which is an attracting Cantor set of zero Lebesgue measure. Clearly, in this case, φ has no absolutely continuous invariant measure, (abbreviated by a.c.i.m.).

The examples we give, are by construction, non-renormalizable. The topological behavior of such maps is easily described. The iterate of every point, except the endpoints of I, eventually falls inside the interval I' bounded by the critical value $\varphi(c)$ and its image $\varphi^2(c)$, and φ restricted to this interval is topologically mixing. In addition, $\omega_\varphi(x)$ coincides with I' for x belonging to a residual subset B of I'. As is customary, $\omega_\varphi(x)$ denotes the omega limit set of points of the forward iterates $\varphi^n(x)$.

Much work has been done on the question of whether B has full measure in I', and it was discovered that this is related to the limit behavior of the iterates of the Lebesgue measure m; (See Remark 1.2.3 at the end of this chapter). We often simply write $|\cdot|$ for $m(\cdot)$. Note that for every φ-invariant measure μ, the support of μ is necessarily contained in the set of non-wandering points of φ. So, if the set of non-wandering points has zero Lebesgue measure, then φ has no a.c.i.m. For example, if φ has an attracting periodic k-cycle

$$\{x_1, x_2, \ldots, x_k\} \qquad (\varphi(x_i) = x_{i+1},\, \varphi^k(x_1) = x_1;\; i = 1, 2, \ldots, k-1)$$

then every invariant measure is singular, and the forward iterates of Lebesgue measure $\varphi_*^n dm$ converge weakly to the discrete invariant measure supported on the above cycle.

Several interesting unsolved questions in the theory are related to metric properties of maps with *sensitive dependence on initial conditions*. These maps have the property that iterates of points belonging to a set of positive Lebesgue measure diverge exponentially fast away from the iterates of nearby points. F. Ledrappier [L] has shown that when $\varphi: I \to I$ admits a finite a.c.i.m. with positive entropy then there exists a subset $A \subset I$ of positive Lebesgue measure such that if $x \in A$ then the Lyapunov exponent

$$\lim_{n\to\infty} \frac{1}{n} \log |D\varphi^n(x)|$$

exists and is a positive number. This implies that the map φ exhibits sensitive dependence on initial conditions. However, sensitive dependence in itself is not sufficient for the existence of a finite a.c.i.m. It was S. D. Johnson who first showed the existence of such non-renormalizable maps with *no finite* a.c.i.m. [Jo]. Then F. Hofbauer and G. Keller constructed examples where the iterates of the Lebesgue measure do not converge [H-K1]. In [H-K1], [K1] [G-J] a question about whether such maps have a σ-finite a.c.i.m. was formulated, (this corresponds to having an almost everywhere finite density).

It turns out that for S-unimodal maps φ with sensitive dependence on initial conditions, the omega limit set $\omega_\varphi(x)$ is the same for Lebesgue almost every point x, in which case we refer to this set as the *attractor*, see [H-K2]. In [H-K2], Hofbauer and Keller, then proposed the following classification for maps with sensitive dependence (recall that c denotes the critical point):

Type I The attractor is a Cantor set of zero Lebesgue measure and coincides with $\omega(c)$.

Type II The attractor is a finite collection of intervals but $\omega(c)$ is still a Cantor set.

Type III The attractor is a finite collection of intervals which coincide with $\omega(c)$.

Clearly Type I maps have no a.c.i.m. Among the results in [H-K2] is that maps of Type II always have σ-finite a.c.i.m. and that there are examples of Type II maps with no finite a.c.i.m.

1.1.3. In this dissertation, we construct new examples of Type III maps defined on the interval I that admit no finite a.c.i.m. and for which $\omega(c)$ is actually the whole interval $I' = [\varphi^2 c, \varphi(c)]$. We describe a method of construction that enables us to control the forward orbit of the critical point and at the same time lets us establish the existence of a σ-finite a.c.i.m. μ. We will show that if the measure μ, which we define explicitly, is not finite, then no finite a.c.i.m. exists, (Theorem 2.1.4). Furthermore, either μ is σ-finite or $\mu(B) = \infty$ for every set B of positive Lebesgue measure, (Theorem 2.2.2.). Our main result, which is proved in a purely constructive manner, then concludes:

Main Theorem. *There are Type III maps that admit no finite a.c.i.m. but that have a σ-finite a.c.i.m. μ. Moreover, the μ-measure of every non-trivial interval is infinite.*

Our work uses the technique of induced hyperbolicity introduced by M. Jakobson [J1] in which he proved that for one parameter families of interval maps $\varphi_t(x) = t\varphi(x)$, e.g., the Quadratic family $\varphi_t(x) = tx(1-x)$, there exists a set of parameter values t of positive Lebesgue measure such that φ_t admits a unique ergodic absolutely continuous invariant probability measure μ_t.

A main feature of this technique is that the critical point is, so to speak, induced away in the following sense. One constructs a new transformation T, called the power map, which piecewise coincides with some iterate of φ_t. Then the transformation T, thus constructed, satisfies the hypothesis of the *Folklore Theorem*, as formulated in Chapter 5, and therefore has a finite a.c.i.m. ν that is then pulled back to obtain a σ-finite a.c.i.m. μ for φ_t. Simultaneously, we employ the technique of Johnson's boxes [Jo], to ensure that μ is not finite. The essential feature of our methods is the ability to control where the invariant measure has infinite mass, so for example we can construct examples of maps with σ-finite a.c.i.m. such that every interval has infinite Lebesgue measure. A different method was used by Bruin, [Br]. He consecutively chooses the parameter values so that the graphs of the central branches h_n are almost tangent to the diagonal line $y = x$, exhibiting almost saddle-node bifurcations.

For more detailed topological properties of S-unimodal maps we advise the reader to consult [G] or [M]. For basic definitions in topological dynamics and ergodic theory see for example, the book "Introduction to Dynamical Systems" by M. Brin and G. Stuck [B-S].

Section 1.2. The Quadratic Family and Associated Power Map

1.2.1. For ease of exposition we will construct our examples from the one-parameter family $\{\varphi_t : t \in [0,4]\}$ of quadratic maps $x \mapsto tx(1-x)$ on the unit interval $[0,1]$. We will describe how to select the parameter t so that the corresponding map φ_t is of the desired type and admits a σ-finite a.c.i.m. μ. Our procedures generalize to any *full* family of S-unimodal maps $\varphi_t(x) = t\varphi(x)$, i.e., the topological entropy of φ_t depends continuously on the parameter t in the C^1 topology, and varies between 0 and $\log 2$. Note that the critical point $x = c$ is fixed for all parameter values t. In the quadratic model given above we have $c = 1/2$ for all t and the fact that these maps are symmetric about the line $x = c = 1/2$ merely simplifies the notation as this allows us to represent φ_t as an even function.

For any map φ belonging to the family $\{\varphi_t\}$ we do the induced construction described in [J-S]. We start by constructing the *first return map* $G: I \to I$ on the interval $I := [q^{-1}, q]$ bounded by the fixed point $q \in [1/2, 1]$ of φ and its second preimage $q^{-1} \in [0, 1/2]$, i.e., $\varphi(q^{-1}) = q$. When t is close to 4, G has many monotone branches G_i and a central parabolic (folding) branch h, (see Section 2 of Chapter 3). The domains of these branches form a partition $\tilde{\xi}_0$ of I, which we then refine to a partition ξ_0 of I with desired properties. Then we pull back ξ_0 onto the domain of h and obtain new monotone and folding branches and a refined partition ξ_1, and so on *ad infinitum* in accordance with [J-S]. All our examples, in the terminology of [J-S], are *expansion inducing*. This means that there exists a partition ξ of I into a countable union of non-overlapping intervals Δ_i and a complementary Cantor set of Lebesgue measure zero such that every Δ_i is mapped diffeomorphically onto I by some iterate G^{N_i}. Moreover, the *power* map T defined almost everywhere on I by

$$T|\Delta_i = G^{N_i}$$

satisfies the conditions of the *Folklore Theorem* [A] and therefore has a unique ergodic invariant probability measure ν, which is absolutely continuous with respect to Lebesgue measure $|\cdot|$, and has a density bounded away from zero and infinity. So that, ν is actually equivalent to Lebesgue measure, ($\nu \equiv |\cdot|$).

Since ν is ergodic, Birkhoff's theorem yields

$$\frac{1}{n}\sum_{i=0}^{n-1} \delta_{T^i x} \longrightarrow \nu \quad \text{a.e.}$$

and for every continuous function ψ

$$\frac{1}{n}\sum_{i=0}^{n-1} \psi(T^i x) \longrightarrow \int \psi d\nu \quad \text{a.e.}$$

and hence ν is an SBR measure.

Remark 1.2.2. We remind that an invariant probability measure ν is called SBR if for x belonging to a set of positive Lebesgue measure the time average $\frac{1}{n}\sum_{i=0}^{n-1}\psi(T^i x)$ converges to the space average $\int \psi d\nu$. Thus every ergodic absolutely continuous invariant probability measure is SBR.

Remark 1.2.3. Applying the Birkhoff theorem to the Indicator function on the attractor, we see that for Lebesgue almost every point

$$\omega_T(x) = I \quad \text{and} \quad \omega_{\varphi_t}(x) = [\varphi^2(c), \varphi(c)]$$

so that the topological and measure theoretic notions of the attractor coincide, (See [BKNS] for a thorough treatment of this subject).

CHAPTER 2

TOWER CONSTRUCTION AND SIGMA-FINITE MEASURES

Section 2.1. Tower Construction

2.1.1. We follow the tower construction of [J1]. A similar construction adjusted for "Hofbauer's tower" was used in [H-K1, K1] and also by Bruin in [Br].

First, recall that the power map T was defined piecewise by

$$T_i = T|\Delta_i = G^{N_i}$$

where each T_i is a monotone branch and G is the first return map induced by φ_t on I. We construct the so-called tower as follows:

Let

$$A_{ij} = G^j(\Delta_i) \qquad (i = 0, 1, \dots ; j = 0, 1, \dots, N_i - 1)$$

and

$$\mathcal{H} = \bigcup_{i=0}^{\infty} \bigcup_{j=0}^{N_i-1} A_{ij} \quad \text{(disjoint)}$$

We call the sets A_{ij} (i fixed; j varies) the tower over Δ_i. Define the projection $\pi\colon \mathcal{H} \to I$ by assigning to any $u \in A_{ij}$ its image under the natural inclusion $\pi u \in G^j \Delta_i \subset I$. Let

$$\mathcal{G}(u) = \begin{cases} \pi^{-1} \circ G \circ \pi\,(u) \cap A_{i,j+1} & \text{if } u \in A_{ij}\ (j = 0, 1, \dots, N_i - 2); \\ \pi^{-1} \circ G \circ \pi\,(u) \cap (\cup_i A_{i,0}) & \text{if } u \in A_{i,N_i-1}. \end{cases}$$

By construction, $G \circ \pi = \pi \circ \mathcal{G}$. Define a measure ρ on $\mathcal{H}$ by

$$\rho(A) = \nu(\mathcal{G}^{-j}(A)) \quad \text{when} \quad A \subset A_{ij}.$$

Since ν is T-invariant, ρ is $\mathcal{G}$-invariant. Notice that if we view I as the base of the tower $\mathcal{H}$ then T is the *first return map* and $\nu = \rho|I$.

Put $\mu = \pi_* \rho$. Evidently μ is G-invariant and absolutely continuous with respect to Lebesgue measure. In fact, it turns out that the two measures are equivalent.

2.1.2. Our aim is to determine when the measure μ is σ-finite (and not finite). An interesting fact is that this measure is the only possible candidate, in the sense that if μ is not finite then *no finite* a.c.i.m. exists. First we prove a summability criteria for the existence of a finite a.c.i.m.

Theorem 2.1.3. *φ has a finite a.c.i.m. iff*

$$\sum_i N_i |\Delta_i| < \infty. \tag{1}$$

Proof.

(i) The convergence of the sum in (1) above is sufficient:

Suppose that $\mu = \pi_* \rho$ is given as above, then

$$\begin{aligned}\mu(I) &= \rho(\mathcal{H}) \\ &= \sum_i \sum_{j=0}^{N_i - 1} \rho(A_{ij}) \\ &= \sum_i N_i \cdot \nu(\Delta_i) \\ &< \infty\end{aligned}$$

where the last inequality follows from (1) because ν has a bounded density. Thus φ admits a finite a.c.i.m.

(ii) The convergence of the sum in (1) above is necessary:

Assume there exits an absolutely continuous probability G-invariant measure α on I. Then, by a theorem of G. Keller [K2] it lifts to a finite a.c.i.m., say β, on the tower $\mathcal{H}$ such that

$$\pi\beta = \alpha.$$

Let β_0 denote the measure β restricted to the zero level $\bigcup_i A_{i0}$ of the tower $\mathcal{H}$. By construction

$$I = \pi\big(\bigcup_i A_{i0}\big)$$

Since β is $\mathcal{G}$-invariant, we have

$$\beta(A_{ij}) = \beta_0(A_{i0}) \qquad (j = 0, 1, \ldots N_i - 1)$$

As β is finite this implies

$$\sum_i N_i \beta_0(A_{i0}) < \infty.$$

On the other hand, the tower construction implies that β is invariant under the power map T. But, we know from the Folklore Theorem that the absolutely continuous T-invariant probability measure ν on I is unique, therefore $\pi\beta = \text{const} \cdot \nu$, and since $\Delta_i = \pi(A_{i0})$, we obtain

$$\sum_i N_i \nu(\Delta_i) = \text{const} \cdot \sum_i N_i \beta_0(A_{i0}) < \infty.$$

Now using that ν has a density bounded away from zero yields

$$\sum_i N_i |(\Delta_i)| < \infty$$

This concludes the proof of (ii) and the theorem as well. ∎

As the measure on the tower is determined by its restriction to the zero level and that restriction is unique by the Folklore Theorem, we get

Theorem 2.1.4. *Up to a multiplicative constant, μ is the only possible finite absolutely continuous G-invariant measure on I.*

Section 2.2. A property of σ-finite a.c.i.m.

2.2.1. Consider the T-invariant measure ν (equivalent to Lebesgue measure) and the measure ρ, which is defined on the tower as indicated above, with $\mu = \pi\rho$. Let m denote normalized Lebesgue measure.

2.2.2. *Theorem. Either μ is σ-finite or else $\mu B = \infty \ \forall B$ with $mB > 0$.*

Proof. Assume $\mu \ll m$ is not σ-finite and let $\mu(B) > 0$. Then $m(B) > 0$. As μ is G-invariant

$$\mu(B) = \mu(G^{-1}B) = \mu(G^{-2}B) = \cdots$$

Let $B_0 = B$ and

$$B_n = G^{-n}(B) - \left(\bigcup_{i=0}^{n-1} B_i\right) \qquad (n = 1, 2, \ldots)$$

Now, consider the set

$$\begin{aligned} \mathcal{A} &= \bigcup_{n=0}^{\infty} G^{-n}(B) \\ &= \bigcup_{n=0}^{\infty} B_n \end{aligned} \tag{2}$$

Clearly $m(\mathcal{A}) > 0$. Now, if $m(\mathcal{A}) = 1$ and $\mu B < \infty$, then equality (2) gives us a decomposition of I into a countable union of disjoint sets B_n of finite μ measure, contradicting that μ is not σ-finite. On the other hand, if $0 < m(\mathcal{A}) < 1$, then

$$G^{-1}(\mathcal{A}) \subset (\mathcal{A})$$

implies that

$$T^{-1}(\mathcal{A}) \subset (\mathcal{A})$$

However, this contradicts that T is ergodic with respect to the invariant measure ν equivalent to m. ∎

CHAPTER 3

Basic Structure and First Step of Induction.

Section 3.1. Introducing Some Properties of S-unimodal Maps.

3.1.1. The *Schwarzian derivative* $S\varphi(x)$ of a C^3 map $\varphi(x)$ is defined for all non-critical points x as:

$$S\varphi(x) = \frac{\varphi'''(x)}{\varphi'(x)} - \frac{3}{2}\left(\frac{\varphi''(x)}{\varphi'(x)}\right)^2.$$

When $S\varphi(x) < 0$ for all non-critical points x, then we write $S\varphi < 0$ and say that the map φ has *negative Schwarzian derivative*. These maps have the following useful properties that will be used throughout.

(i) Compositions of maps with negative Schwarzian derivative have negative Schwarzian derivative. This follows from the chain rule for derivatives, which yields

$$S(\psi \circ \varphi)(x) = S\psi(\varphi(x))|\varphi'(x)|^2 + S\varphi(x).$$

Furthermore, albeit with some calculation, we arrive at the formula for the Schwarzian derivative of the iterates of φ:

$$S\varphi^n(x) = \sum_{i=0}^{n-1} S\varphi(\varphi^i(x))|\mathrm{D}\varphi^i(x)|^2.$$

(ii) *Koebe distortion constant*

Diffeomorphisms with negative Schwarzian derivative have bounded distortion in the following sense: Let $J, I, \hat{I}$ be intervals, with

$$\hat{I} = L \cup I \cup R,$$

where L is the interval adjacent to the left of I and R to the right. Note that L and R form a *collar* around I. Suppose

$$\min\left\{\frac{|L|}{|I|}, \frac{|R|}{|I|}\right\} > \tau$$

Let

$$F \colon \hat{J} \to \hat{I}$$

be a diffeomorphism with negative Schwarzian derivative and set $J = F^{-1}(I)$. Then there exists a constant $c = c(\tau) > 1$, independent of F, such that

$$1/c < \left|\frac{F'(x)}{F'(y)}\right| < c$$

for all $x, y \in F^{-1}(I)$.

We refer to c as the *Koebe distortion constant*, and say that a map has *small distortion*, whenever $c = 1 + \varepsilon$, for very small ε. The important point here is that the Koebe distortion constant $c = c(\tau)$ *only* depends on the range of the map and is completely independent of the map F itself and its domain.

A very powerful and useful tool indeed. In addition, another fact about such maps F is that if I contains the middle point of $\hat{I}$ and the ratio $r(I,\hat{I}) = |I|/|\hat{I}|$ is very small then F has small distortion. More precisely, the uniform distortion constant $c = c(\tau)$ depends on the ratio $r = r(I,\hat{I})$ as follows

$$c(\tau) < \exp\bigl(r\kappa(\tau)\bigr)$$

with

$$\kappa(\tau) \longrightarrow 4 \quad \text{as } r \longrightarrow 0$$

see [J-S].

(iii) *The Minimum principle* If $F\colon J \to \mathbf{R}$ is an interval map with no critical points in J then $|F'(x)|$ has no positive minimum in the interior of the interval J. The proof of this statement is elementary. Indeed, if $|F'(x)|$ has a minimum at a point x_0, then x_0 must be a critical point for F'', i.e., $F''(x_0) = 0$. Therefore,

$$SF(x_0) = \frac{F'''(x_0)}{F'(x_0)} < 0$$

implies that $F'''(x_0)$ and $F'(x_0)$ have opposite signs, which is impossible. Because, either $F'(x_0) > 0$ is a local minimum in which case $F'''(x_0) > 0$, or $F'(x_0) < 0$ is a local maximum and $F'''(x_0) < 0$.

3.1.5. For the one parameter family $\{\,\varphi_t : t \in [0,4]\,\}$ of quadratic maps

$$x \mapsto tx(1-x)$$

on $[0,1]$, one clearly has $S\varphi_t < 0$.

Section 3.2. The First Return Map

3.2.1 For any fixed t close to 4, the quadratic map

$$\varphi_t\colon [0,1] \to [0,1]$$

has two repelling fixed points 0 and $q_t^+ = 1 - 1/t$. Let $q_t^- = 1/t$ denote the second preimage of q_t^+ and consider the first return map G_t induced by φ_t on the interval $I := [q_t^-, q_t^+]$, then G_t has $2K$ monotone branches (diffeomorphisms) and one central parabolic branch. Since $t \approx 4$, K is very large.

In general, we suppress the parameter t denote the monotone branches by

$$f_i\colon \Delta_i^{\pm} \to I,$$

where Δ_i^- denotes the domain to the left of the critical point $1/2$ and Δ_i^+ denotes the symmetrical one to the right of $1/2$ that has the same return time $i = 2, 3, \ldots K+1$. The central parabolic branch is denoted by

$$h_0\colon \delta_0 \to I$$

which has return time $K+2$. We denote the two boundary intervals of I by Δ_l (l for left) and Δ_r (r for right) which have return time equal to 2. The next pair of intervals $\Delta_3^{\pm}$, adjacent to Δ_l and Δ_r respectively, have return time equal 3, and so on. More specifically, if we let $\underline{\varphi} = \varphi_t|[0,q]$, $\varphi_0 = \varphi_t|I$, and $\overline{\varphi} = \varphi_t|[q,1]$, then $G: I \to I$ is given by:

$$
\begin{aligned}
f_l &= \overline{\varphi} \circ \varphi_0 | \Delta_l \\
f_r &= \overline{\varphi} \circ \varphi_0 | \Delta_r \\
f_i^{\pm} &= \underline{\varphi}^{i-2} \circ \overline{\varphi} \circ \varphi_0 | \Delta_i^{\pm} \qquad (i = 3, 4, \ldots K+1) \\
h_0 &= \underline{\varphi}^{K} \circ \overline{\varphi} \circ \varphi_0 | \delta_0 \qquad (3)
\end{aligned}
$$

Denote the resulting partition of I by $\tilde{\xi}_0$.

3.2.2 In the sections that follow, we construct by induction a sequence of partitions ξ_n of I such that at each step n of induction, we obtain a transformation $T_{(n)}$, which piecewise coincides with an iterate of G, defined on a the partition ξ_n of I modulo a Cantor set of zero Lebesgue measure. The elements of ξ_n consist of one central domain δ_n of a (folding) parabolic branch denoted by h_n with a countable collection δ_i^{-k} of diffeomorphic preimages $G^{-k}(\delta_i)$ for $i = 0, 1, \ldots, n$, and a countable collection of non-overlapping domains of (strictly) monotone branches denoted by f with lower indices such that f_i maps Δ_i onto I. The central branch and preimages of central branches are called holes and are continuously filled in by a new central branch, by preimages of central branches and by monotone branches. One of the primary charactarestics of the construction is that monotone domains created at previous steps are never altered at later steps and so they belong to ξ_n for all later inductive steps n.

Section 3.3. Pullback.

3.3.1. In this section we describe procedures which we use throughout our construction. Let us first explain some of the notations we use. If f_i and f_j are monotone maps then we write

$$f_{ij} = f_j \circ f_i$$

to denote the composition restricted to the domain

$$\Delta_{ij} = \Delta_i \cap f_i^{-1} \Delta_j.$$

This produces a new monotone branch

$$f_{ij} \colon \Delta_{ij} \to I.$$

Similarly for the monotone maps

$$f_{i_1 i_2 \cdots i_k} \colon \Delta_{i_1 i_2 \cdots i_k} \to I.$$

for every k.

3.3.2. *Monotone Pullback*

Suppose

$$f_0\colon \Delta_0 \to I$$

is a monotone branch and let ξ denote a partition of I. Then we refer to $f_0^{-1}(\xi)$ as the monotone pullback of the partition ξ onto Δ_0. This creates a partition of Δ_0 into domains of various types. For every domain J of the partition ξ we have the corresponding domain $f_0^{-1}(J) \subset \Delta_0$. For example, if $\Delta_i \in \xi$ is a monotone domain then

$$f_{0i} = f_i \circ f_0\colon \Delta_{0i} \to I$$

is a monotone branch where Δ_{0i} as indicated above is the monotone domain $\Delta_0 \cap f_0^{-1}(\Delta_i)$.

3.3.3. *Critical Pullback.*

If

$$h\colon \delta \to I$$

is a central branch and Δ_0 is contained in the image of h, but does not contain the critical value $h(1/2)$ then $h^{-1}(\Delta_0)$ consists of two new monotone domains inside the central domain δ, with branches

$$F_{-} = f_0 \circ h_0|(0, 1/2)$$

and

$$F_+ = f_0 \circ h_0 | (1/2, 1).$$

That is, the domain of F_- is located in $(0, 1/2)$ while $\Delta(F_+) \subset (1/2, 1)$, but both lie inside δ.

Section 3.4. Uniform Extendibility

3.4.1. Recall that monotone branches are diffeomorphisms

$$f \colon \Delta \to I.$$

The monotone domain Δ is *extendible* when we can enlarge Δ to a domain $\hat{\Delta}$ that is mapped diffeomorphically onto a larger interval $\hat{I}$, containing I, by a map which we denote by $\hat{f}$. In our construction $I = [q^{-1}, q]$ is extended to the interval $\hat{I} := [a^-, a^+]$ where $a^- \in (0, q^{-1})$, and $a^+ \in (q, 1)$ are specified in Lemma 3.5.2. For an extendible $f \colon \Delta \to I$ we use the notation

$$\hat{f} \colon \hat{\Delta} \to \hat{I}$$

where

$$\hat{\Delta} = \Delta_L \cup \Delta \cup \Delta_R$$

and $\hat{I} := [a^-, a^+]$ such that

$$\hat{f} \colon \Delta_L \to [a^-, q^{-1}], \qquad \hat{f} \colon \Delta_R \to [q, a^+].$$

When the collar $\hat{I} - I$ remains the same for all branches, then we refer to these extensions as *uniform* and the collar is said to be a *Uniform Extendibility Collar*.

From the construction of the first return map, all extensions $\hat{\Delta}$ are contained in I except for the extensions of the two boundary domains of I, Δ_l and Δ_r. Since the fixed point q is repelling it follows that the extensions of Δ_l and Δ_r are both contained in $\hat{I}$.

3.4.2. Now, suppose

$$f_i \colon \Delta_i \to I$$

and

$$f_j \colon \Delta_i \to I$$

are two monotone branches then

$$f_{ij} \colon \Delta_{ij} \to I$$

is an extendible monotone branch

$$f_j \circ f_i \colon \Delta_{ij} \to I$$

where

$$\Delta_{ij} := \Delta_i \cap f^{-1}(\Delta_j)$$

and its extension $\hat{\Delta}_{ij}$ lies within Δ_i except for the two boundary intervals of $f_i^{-1}(\Delta_{l(r)})$ whose extensions lie inside $\hat{\Delta}_i$. Similarly for all compositions

$$f_{j_1 j_2 \cdots j_k}$$

provided that each of the maps

$$f_{j_1}, f_{j_2}, \cdots f_{j_k}$$

is an extendible monotone domain.

3.4.3. A critical branch has the form $h = F \circ Q$, where F is a monotone branch and Q is the restriction of the initial quadratic map

$$\varphi_t = tx(1-x)$$

to a small interval δ around the critical point 1/2. A central branch h is said to be *extendible* if F is extendible. In which case the extension $\hat{h} = \hat{f} \circ Q$ is a critical branch defined on $\hat{\delta} \supset \delta$ whose image contains either $[a^-, q^{-1}]$ or $[q, a^+]$. In particular, the initial critical branch

$$h_0 \colon \delta_0 \to I$$

of the first return map is extendible and its extension $\hat{h}_0$ is given by equation (3) in 3.2.1. Moreover, the image of the extension

$$\hat{h}_0 \colon \hat{\delta} \to \hat{I}$$

contains $[q, a^+]$.

Preimages of central domains δ^{-k} are mapped by diffeomorphisms

$$\chi\colon \delta^{-k} \to \delta$$

which are iterates $G^k|\delta^{-k}$ and are said to be extendible whenever they extend up to the preimage of the extension $\hat{\delta}$ of the central domain δ.

Remark 3.4.4. As discussed above, all extensions $\hat{\Delta}$ are contained in I except for the two boundary intervals of I, Δ_r and Δ_l whose extension lie inside $\hat{I}$. A critical pullback of an extendible domain Δ is extendible if that domain is not too close to the critical value (this will be made more precise later) and its extension lies within δ, except when Δ is either one of of the preimages of the boundary intervals $\Delta_{l(r)}$, which are contained in $\hat{\delta}$.

Section 3.5. Preliminary Construction

3.5.1. We shall refine the initial partition $\tilde{\xi}_0$ induced by the first return map G into a partition of the form:

$$\xi_0 \colon I = (\cup_i \Delta_i) \cup (\cup_k \delta_0^{-k}) \cup \delta_0$$

where Δ_i denotes domains of uniformly extendible monotone branches, δ_0^{-k} denotes preimages of δ_0 by extendible diffeomorphisms

$$\chi = G^k | \delta_0^{-k}$$

and δ_0 is the domain of an extendible parabolic branch h_0.

Lemma 3.5.2. *For every $\varepsilon > 0$ we can construct the partition ξ_0 to have the following properties:*

(i) Each monotone domain has length less than ε.

(ii) The aggregate sum of lengths of the "holes" δ_0^{-k} is less than ε.

(iii) Extendibility Collar does not depend on ε.

Proof. When t approaches 4 the initial map φ_t approaches the map $4x(1-x)$ which by a linear change of coordinates is the Chebyshev polynomial $1-2x^2$ and we say that φ_t approaches the Chebyshev polynomial. Let us consider the relationship between parameter space and phase space. The parameter interval consists of a sequence of adjacent subintervals $[t_j, t_j + 1]$ such that when $t \in (t_j, t_{j+1})$, the first return map G_t has $2j$ monotone branches and a central parabolic branch δ_0. Then as t moves across each subinterval, t crosses over the boundary point t_{j+1}, the central domain δ_0 splits into two (boundary) monotone domains and a new central domain is born. One can check that when $t = 4$, the size of each step of the staircase as we approach zero decreases as $1/4^n$. So for parameters t very close to 4, the parameter interval $[t_j, t_{j+1}]$ is of the order $1/4^j$. At the limiting case, when $t = 4$, we have $\varphi_4(1/2) = 1$ and $\varphi_4^2(1/2) = 0$ is a fixed point. So $j \to \infty$ as $t \to 4$, and G_4 has a countably infinite number of monotone branches that converge toward the middle point $1/2$ and has no central parabolic branch. On the other hand,

$$\left.\frac{\partial \varphi_t(x)}{\partial x}\right|_{x=0} = t$$

It follows that there exists a constant c_0, such that, for every j

$$|\Delta_j| < \frac{c_0}{2^j}$$

when the parameter interval is very close to 4.

Let us now suppose that G_t has $2K$ monotone branches and one central branch parabolic branch, where K is extremely large. Then choose a large index $j_0 \ll K$ such that

$$\frac{c_0}{2^{j_0}} < \varepsilon \tag{4}$$

and consider the initial partition $\tilde{\xi}_0$ described in Section 2. Then for every $j \geq j_0$, we have

$$|\Delta_j| < \frac{c_0}{2^j} < \varepsilon \tag{5}$$

Derivatives of all monotone branches f_j besides possibly the two branches $f_K^{\pm}$ next to the middle central branch satisfy

$$\left| \frac{df_j}{dx} \right| > c_1 2^j \tag{6}$$

In our construction, we can choose the position of the critical value $h_0(1/2)$ is close to $1/2$, so that the derivatives of $f_K^{\pm}$ will also satisfy (6).

Next, we pullback the partition $\tilde{\xi}_0$ onto all monotone domains with indices smaller than j_0, thus creating inside each of these monotone domains a copy of the partition $\tilde{\xi}_0$. Newly created domains have lengths either less than ε — for example the domain $\Delta_{j_0-1,j_0} \subset \Delta_{j_0-1}$ one has

$$|\Delta_{j_0-1,j_0}| < \frac{1}{2^{j_0-1}} \frac{1}{2^{j_0}} c_1$$

—or else they are still larger than ε so we continue pulling back $\tilde{\xi}_0$ onto such domains.

A straightforward calculation shows that, for t sufficiently close to 4 the expansion of all monotone branches f_j for $j < K$ is larger than 3. Therefore the pullback of monotone domains Δ_j, where $K > j \geq j_0$ must satisfy the inequality (5). Finally, for the pullbacks of the domains with indices $j < j_0$ onto domains also with indices $j < j_0$, we use the fact that the absolute value of the derivatives of the monotone branches f_j of the first return map G_t, even the leftmost and the rightmost with return time 2, exceed 3. Then after $k_0 := \lceil \log_3 2^{j_0} \rceil + 1$ steps the absolute value of derivatives will be larger than 2^{j_0}. Thus inequality (5) holds for these pullbacks as well. In addition pullbacks δ_0^{-j} of the central domain δ_0, (recall that $|\delta_0| \approx c_0/2^K$), are created and using bounded distortion the total sum of their lengths is estimated by assuming that we pullback a maximal number of times k_0:

$$
\begin{aligned}
\text{(total lengths of pullbacks } \delta_0^{-j}) &< 1 - (1 - c_1|\delta_0|)^{k_0} \\
&< k_0 \cdot \left(\frac{c_2}{2^K}\right) \quad \text{by the binomial expansion} \\
&< j_0 \cdot \left(\frac{c_2}{2^K}\right) \quad \text{because } k_0 < j_0 \\
&< \frac{1}{2^{j_0}} \quad \text{by the choice of } j_0 \ll K \\
&< \varepsilon \quad \text{by definition}
\end{aligned}
$$

where c_1 and c_2 are constants that arise due to the approximation to linear maps when we have bounded distortion. Finally, using that the quadratic map φ_t for t close to the Chebyshev value 4 is mapping $[0, 1]$ almost onto $[0, 1]$, we have that all monotone branches f_j with the possible exception of the two monotone branches adjacent to the central branch δ_0 extend up to the interval $\tilde{I}$ bounded by the two preimages of q_t^{-1}, i.e.

$$\tilde{I} = [\tilde{a}^-, \tilde{a}^+]$$

where

$$\varphi_t(\tilde{a}^-) = \varphi_t(\tilde{a}^+) = q_t^{-1} \qquad \tilde{a}^- \in [0, q_t^{-1}],\ \tilde{a}^+ \in [q_t, 1].$$

It turns out that for future purposes we need the extendibility collar to be smaller. Since q_t is a repelling fixed point, we take a sequence of preimages

$$\tilde{a}_+^{-k} := \varphi^{-k}(\tilde{a}^+) \in [1/2, 1]$$

that spiral towards q_t, we then fix a preimage

$$a^+ := \tilde{a}_+^{-k} \in [q_t, 1]$$

when k is large enough. Then we set

$$a^- := {\varphi_t}^{-2}(a^+) \in [0, q_t^{-1}]$$

and $I := [q_t^{-1}, q_t]$ extends uniformly to

$$\hat{I} := [a^-, q_t^{-1}] \cup [q_t^{-1}, q_t] \cup [q_t, a^+]$$

Then all branches of the first return map will be extendible up to $\hat{I} = [a^-, a^+]$ for these choices of a^- and a^+. Further, by construction, the compositions

$f_{i_1 i_2 \ldots i_k}$ extend uniformly up to the interval $\hat{I}$, the central domain δ_0 of the first return map is extendible, via equation (3) given in section 2, and we denote its extension by $\hat{\delta}_0$. In addition, all the preimages δ_0^{-k} are extendible to the preimages of the extension $\hat{\delta}_0$, which are denoted by $\hat{\delta}_0^{-k}$.

We thus obtain the partition:

$$\xi_0 \colon I = (\cup_i \Delta_i) \cup (\cup_k \delta_0^{-k}) \cup \delta_0 \tag{7}$$

This partition satisfies the properties (i), (ii), (iii) of the Lemma. ∎

Remark 3.5.3. By the choice of a^+, (resp. a^-), close to q (resp. q^{-1}) the right(left) extension of a monotone domain of the first return map is contained within the respective adjacent domain. Thus, extensions of all elements of the partition ξ_0 are contained inside I, except for the extensions of the boundary domains Δ_r and Δ_l which are contained in $\hat{I}$.

Remark 3.5.4. As mentioned above branches of the first return map are expanding with derivative growing exponentially with the number of iterates, except for the central branch h_0 and possibly the two branches adjacent to h_0. Since the monotone branches decrease in size the closer they are to the central domain, the size of these two monotone domain is very small compared with the whole interval I as the number of monotone branches $2K$ becomes very large. By choice of parameter, we may assume that that the image of $h_0(\delta_0)$

covers a fixed interval larger than some constant c. Then the extension of these two branches covers $[q^{-1}, q]$ and an additional fixed interval surrounding $[q^{-1}, q]$ that does not contain the critical value. Therefore, by the mean value theorem the derivative at some point must be large and it follows by bounded distortion that the derivative at any other point can be made larger than A_0 for any given $A_0 > 1$, provided we take K sufficiently large. The same argument holds for monotone branches

$$f\colon \Delta \to I$$

constructed at subsequent steps because their domains are created inside the central domain δ_0 or in the preimages δ_0^{-k} and since δ_0 is small the monotone domains inside them must also be small. In essence, as uniform extendibility ensures uniformly bounded distortion by the Koebe distortion principle, the expansion of these monotone domains grows as the central branch and its preimages decrease in size when K increases.

Section 3.6. Basis of induction

3.6.1. Let us first recall the preliminary partition ξ_0 of I that was constructed in Lemma 3.5.2. That partition has the form

$$\xi_0: I = (\cup_i \Delta_i) \cup (\cup_k \delta_0^{-k}) \cup \delta_0$$

where Δ_i denotes domains of uniformly extendible monotone branches, δ_0^{-k} denotes preimages of δ_0 by extendible diffeomorphisms

$$\chi^k = G^k|\delta_0^{-k}: \delta_0^{-k} \to \delta_0$$

and δ_0 is the domain of an extendible parabolic branch h_0. Then this partition has the following geometrical properties.

Every monotone domain is adjacent to another monotone domain or to a preimage of δ_0 which at the next step of induction would be refined and this refinement always creates two monotone domains that lie at the boundary of this preimage. In other words, every monotone domain is adjacent to another monotone domain or will be adjacent to another monotone domain at the next step. This includes the two monotone domains adjacent to the central domain δ_0. The next property is that no two preimages of δ_0 are adjacent, there are always many monotone domains in between.

The construction of our examples proceeds by induction, namely given the partition ξ_{n-1} with certain properties we show how to construct the next partition ξ_n with the desired properties so that in the limit we have a partition of I modulo a set of Lebesgue measure zero that consists of monotone domains and satisfies the hypothesis of the Folklore Theorem of Adler as formulated in Chapter 5 Theorem 5.2.3.

3.6.2. *The Johnson Box*

Let h_0 be the folding parabolic branch of the initial first return map G_t. Note that the first return map reverses orientation and consequently h_0 has a minimum at the critical point. By choice of parameter interval Λ_0, we can arrange that for $t \in \Lambda_0$, $h_0(1/2) \in \delta_0$ with $h_0(1/2) < 1/2$ in order that the graph of h_0 crosses the diagonal. In view of the fact that the boundary endpoints of the interval δ_0 are mapped by h_0 onto q^+, we see that the image of h_0 contains all the domains of ξ_0 that are located to the right of δ_0. Since the graph of h_0 crosses the diagonal, we can define the *Johnson's box* with bottom B_0 bounded by the points q_0, q_0^{-1} where q_0 is one of the two fixed point of h_0— the one farther away from 1/2— and q_0^{-1} is its second preimage. Since we choose our maps non-renormalizable, we place the critical value outside of $[q_0^{-1}, q_0]$. We call the part of the graph outside this box the *hat* and denote its base by H_0.

3.6.3. *Constructing the First Step of the Staircase*

Define

$$r_0 = \min\{\, r : h_0^r(1/2) \notin \delta_0 \,\}.$$

The first critical pullback $h_0^{-1}(\xi_0)$ creates inside δ_0 a preimage of the right half of the partition ξ_0. That is, we create a pair of preimages of every element of ξ_0 that lies to the right side of δ_0. This is because $h''(1/2) < 0$. Since h_0 is a two to one map we create a pair of preimages to every element of ξ_0 that is to the right of $1/2$. We call this collection the first step of the staircase, and we denote it by

$$\mathcal{S}_{1,\text{left}} \quad \text{and} \quad \mathcal{S}_{1,\text{right}}$$

3.6.4. *The Infinite Staircase Construction*

We proceed by constructing the *infinite staircase* $\mathcal{S} = \cup_{j\geq 1}\mathcal{S}_j$ where each $\mathcal{S}_j$ consists of two components $\mathcal{S}_{j,\text{left}}$ and $\mathcal{S}_{j,\text{right}}$, symmetric about $1/2$, each of which contains the preimages

$$h_0^{-j}(\Delta),\ h_0^{-j}(\delta_{n-1}^{-p}) \quad \text{where} \quad \Delta, \delta_{n-1}^{-p} \in \xi_0$$

All these preimages are outside Johnson's box, in fact $\mathcal{S} = \delta_0 - B_0$.

3.6.5. *The Partition of δ_0*

Next, by the definition of r_0, the critical value belongs to $\mathcal{S}_{r_0}$ so we fill the base of the hat H_0 by critical pullback

$$h_0^{-1}\left(\bigcup_{j=r_0}^{\infty} \mathcal{S}_{j,\text{left}}\right)$$

thus creating new monotone branches inside H_0 in addition to the critical branch

$$h_1 := f_0^* \circ h_0^{r_0}$$

Here f_0^* is the monotone branch whose domain $\Delta_0^* \in \mathcal{S}_1$ contains the iterate $h_0^{r_0}(1/2)$ of the critical point. Restricting h_0 to the two symmetric intervals of $B_0 - H_0$, we obtain two monotone maps g_1, g_2. Since g_1, g_2 and all their iterates are uniformly extendible branches of an $\mathcal{S}$-unimodal map, they have uniformly bounded distortion. This implies that almost every point of $B_0 - H_0$ under the iterations of g_1 and g_2 eventually 'escapes' the box through H_0. However, for example, the set of endpoints of the interval H_0 and their preimages under the compositions of g_1, g_2 get eventually mapped onto q_0 and are thus trapped inside the box. This set is countable and therefore has Lebesgue measure zero. In fact, we will show in section 3.6.8 - and 3.6.10 that the set of all points that do not belong to the interior of the preimages of H_0 under all compositions $g_{j_1 j_2 \cdots j_k}$ of the two branches g_1 and g_2, form a Cantor set of Lebesgue measure zero. Therefore, this partition of B_0 (mod 0), adjoined with that of the staircase $\mathcal{S}$ constitute the desired partition η_0 of δ_0:

$$\eta_0 : \delta_0 = (\cup\Delta) \cup (\cup\delta_0^{-p}) \cup (\cup\delta_1^{-p}) \qquad \text{(mod 0)}.$$

3.6.6. *The refinement ξ_1 of ξ_0*

Now ξ_1 is obtained by filling in the preimages δ_0^{-P} of δ_0 by pulling back the partition of δ_0 we just created.

That is,

$$\xi_1 : I = (\cup_i \Delta_i) \cup (\cup_{i=0,1} \cup_k \delta_i^{-k}) \cup \delta_0$$

3.6.7. *The Need for Boundary Refinement*

In general, when we do the critical pullback we may encounter the following situation. If the critical value falls inside the extension of a monotone domain Δ then the critical pullback of Δ is not extendible. In this case we need to subdivide Δ using the boundary refinement procedure which we shall describe later on. However, we can avoid the need for this boundary refinement procedure at the *first* step of our construction, if the extendibility collar $\hat{I} - I$ is small enough. By choice of parameter, (which is explained in detail in the next section), we arrange that the critical value falls in the middle of a monotone domain Δ_0. Then when we consider h_0 and related staircases we obtain that the preimages of all these elements are extendible.

3.6.8. *Cantor Sets*

It is customary terminology to call perfect, nowhere dense subsets *Cantor sets.* The standard Cantor middle third set $C = \bigcap_j C_j$ is constructed by removing a ratio of 1/3 at each stage so that

$$|C_j| = (2/3)|C_{j-1}|$$

In general, we can remove a fixed ratio $\beta = 1 - \alpha$ to obtain the recursion

$$|C_j| = \alpha \cdot |C_{j-1}| = \cdots = \alpha^j \cdot |C_0|$$

Thus $|C_j| \longrightarrow 0$ as $j \longrightarrow \infty$, and since $C \subset C_j$ for all j it follows that $|C| = 0$. This construction can be further generalized by removing at each stage a variable ratio β_j as long as β_j is bounded away from zero. In this case $\alpha_j = 1 - \beta_j \leq \alpha$ for some uniform $\alpha < 1$. Hence

$$\begin{aligned} |C_j| &= \alpha_j \cdot |C_{j-1}| \\ &\leq \alpha \cdot |C_j| \leq \cdots \leq \alpha^j \cdot |C_0| \longrightarrow 0 \quad \text{as} \quad j \longrightarrow \infty \end{aligned}$$

So

$$\begin{aligned} |C| &= |\bigcap_{j=0}^{\infty} C_j| \\ &= \lim_{j \to \infty} |C_j| = 0 \end{aligned}$$

as before.

Remark 3.6.9. The condition that β_j is bounded away from zero ensures that the Cantor set has zero Lebesgue measure. However, in the the case when we remove a diminishing amount $\beta_j \to 0$ we can obtain a Cantor set of positive Lebesgue measure, and actually this is the type of set that we will use in order to construct maps that admit a σ-finite a.c.i.m.

3.6.10. Now returning to the branches g_1 and g_2. We note that these monotone maps are uniformly extendible, although the extendibility collar may be very small when the hat H_0 is small relative to the base B_0. However, this uniform extendibility, implies that the distortion of all the compositions of g_1 and g_2 will remain uniformly bounded. Now, writing $B = B_k$, $H = H_k$, and using the above notation we have

$$C_0 = B - H$$

$$C_1 = C_0 - H^{-1}$$

$$\vdots$$

$$C_j = C_{j-1} - H^{-j}.$$

In view of bounded distortion, as we discussed above, the ratio $|H^{-j}|/|C_{j-1}|$ is bounded away from zero by some uniform constant $\beta = 1 - \alpha$. Therefore

$$|C_j| \leq \alpha \cdot |C_{j-1}|$$

and so

$$\lim_{j\to\infty} |C_j| \leq \lim_{j\to\infty} \alpha^j \cdot |C_0| = 0.$$

Hence, the discussion in 3.6.8 applied to $\bigcup_{j=0}^{\infty} H_k^{-j}$ shows that it has full measure in B_k as desired.

CHAPTER 4

Construction of Partitions.

Section 4.1. Basic Step

4.1.1. The construction proceeds inductively through refining ξ_0 by creating a new central domain δ_1 and filling in the preimages $\{\delta_0^{-k}\}_k$ with monotone domains and preimages $\{\delta_1^{-k}\}_k$. The main point is that monotone domains created at a previous step remain unchanged, thus we only partition the central domain and then its preimages accordingly. So, we assume by induction that after step $n-1$ we have the following partition of I

$$\xi_{n-1} : I = (\cup \Delta) \cup (\cup_i \cup_k \delta_i^{-k}) \cup \delta_{n-1} \cup C_{n-1}$$

where the collection $\{\Delta\}$ are monotone domains mapped onto I by uniformly extendible diffeomorphisms, δ_{n-1} is the domain of the extendible central parabolic branch and each δ_i^{-k} is a preimage of some δ_i $(i = 0, 1, \ldots, n-1)$ by an extendible diffeomorphism χ. Here C_{n-1} denotes a Cantor-like set with zero Lebesgue measure and $C_0 = \emptyset$. Recall that all maps consist of iterates of the initial quadratic map φ_t and t belongs to a parameter interval Λ_{n-1}

Let us now consider the location of the critical value $h_{n-1}(1/2)$. In accordance with [J-S], if

$$h_{n-1}(1/2) \in \Delta^*_{n-1}, \quad (\Delta_{n-1} \in \xi_{n-1})$$

then we refer to this as a *Basic step* and we proceed with the construction of the partition ξ_n using the procedures described below.

(1) *Grow up procedure*:

If the image of the central branch $(h_{n-1}(\delta_{n-1}))$ is contained in the rightmost or leftmost boundary domain of the initial partition ξ_0, which we denote by Δ_l and Δ_r respectively then no refinement is done. Rather, we perform iterations of f_l if the critical value falls in Δ_l, and if the critical value falls in Δ_r then we need to apply f_r once and then, if necessary, follow it by the required iterations of f_l (because at this point the central branch becomes concave down) image of the central branch contains more than the boundary interval. More specifically, we choose the smallest m such that the image

$$f_l^m \circ h_{n-1}(\delta_{n-1})$$

or

$$f_l^{m-1} \circ f_r \circ h_{n-1}(\delta_{n-1})$$

covers more than just the boundary interval, in which case we replace the central branch

$$f_l \circ h_{n-1} \quad \text{or} \quad f_r \circ h_{n-1}$$

respectively, by

$$f_l^m \circ h_{n-1} \quad \text{or} \quad f_l^{m-1} \circ f_r \circ h_{n-1}$$

respectively, which is again redenoted by h_{n-1}. In our construction, we will have $m \leq 2^{n-1}$, (see 5.1.1 in Chapter 5).

(2) Extra Pullback Procedure:

In our estimates on the measure of holes in Chapter 5 we use that ratio $|\delta_n|/|\delta_{n-1}|$ is small. In Lemma 3.5.2 we showed that given any ε we are able to arrange that all elements belonging to the preliminary partition are of length less than ε. If the image of the central branch h_{n-1} covers more than half the length of I then

$$\frac{|\delta_n|}{|\delta_{n-1}|} \leq c\sqrt{\frac{2|\Delta^*_{n-1}|}{|I|}}$$

is small, (here c denotes the Koebe distortion constant). However, if the image of h_{n-1} does not cover that much, then the length of Δ^*_{n-1} may be comparable to the height of that image. This may happen, for example, if Δ^*_{n-1} is the domain adjacent to one of the boundary domains $\Delta_{l(r)}$. Therefore, we introduce the following rule of "Extra Pullback".

If

$$|Im(h_{n-1})| < \frac{1}{2}|I|$$

then we do one extra monotone pullback of ξ_0 onto Δ^*_{n-1} which ensures that after critical pullback the ratio

$$\frac{|\delta_n|}{|\delta_{n-1}|} \leq c_1\sqrt{\varepsilon} \tag{8}$$

is arbitrarily small, depending on our choice of ε. Here c_1 is a uniform constant

(3) *Boundary Refinement Procedure*:

Suppose $F\colon \Delta \to I$ is an extendable monotone branch, where $\Delta \in \xi_{n-1}$, $\Delta \subset h_{n-1}(\delta_{n-1})$, and $h_{n-1}(1/2) \notin \Delta$. If Δ is too close to $h_{n-1}(1/2)$ then when we do critical pullback onto δ_{n-1}, we may create monotone domains $h^{-1}_{n-1}(\Delta)$ which are not extendible. In which case, we perform the *boundary refinement procedure* as follows.

Consider the initial partition

$$\xi_0 : I = (\cup_i \Delta_i) \cup (\cup \delta_0^{-k}) \cup \delta_0.$$

It contains the boundary rightmost monotone branch $f_r\colon \Delta_r \to I$ which has a repelling fixed point $q^+ = 1 - 1/t$. We refine Δ_r by *monotone pullback*, thus creating the partition $f_r^{-1}(\xi_0)$. Then

$$q \in f_r^{-1}(\Delta_r) \stackrel{\text{def}}{=} \Delta_{rr}.$$

We refine Δ_{rr} by *monotone pullback* of ξ_0 by f_r^{-2} and so on. The k^{th} step refinement creates a copy of ξ_0 on

$$\Delta_{\underbrace{rr\ldots r}_{k}}$$

contracted approximately by $|f_r'(q)|^{-k}$. We call the resulting partition the k^{th} *right boundary refinement of* ξ_0 and is denoted by $\xi_{0,k}$. After constructing such a partition on Δ_r, we pull back $\xi_{0,k-1}$ by f_l onto the leftmost boundary interval Δ_l of ξ_0 to create the k^{th} *left boundary refinement of* ξ_0 denoted by $\xi_{k,0}$. Both refinements have the following properties:

(i) The sizes of the elements of $\xi_{0,k}$, (resp. $\xi_{k,0}$) in

$$\Delta_{\underbrace{rr\ldots r}_{p}} \quad (\text{resp. } \Delta_{\underbrace{lrr\ldots r}_{p}}) \qquad (p \leq k)$$

are up to a uniform constant, $|f_r'(q)|^{-p}$ times smaller their images in ξ_0.

(ii) For any element $\mathcal{A}$ of the partition $\xi_{0,k}$ ($\xi_{k,0}$) except for the rightmost (leftmost) boundary interval

$$\Delta_{\underbrace{rr\ldots r}_{k}} \quad (\text{resp. } \Delta_{\underbrace{lrr\ldots r}_{k}})$$

the ratio

$$\frac{|\mathcal{A}|}{dist(\mathcal{A},q)} \quad \left(\text{resp. } \frac{|\mathcal{A}|}{dist(q^-,\mathcal{A})}\right)$$

is uniformly bounded.

(iii) Monotone branches from elements $\Delta \in \xi_{0,k}$ (reps). $\xi_{k,0}$) onto I are uniformly extendible up to the interval $\hat{I} = I_L \cup I \cup I_R$. Also, diffeomorphisms $\chi\colon \delta_0^{-m} \to \delta_0$ are uniformly extendable up to $\hat{\delta}_0 = \delta_{0l} \cup \delta_0 \cup \delta_{0r}$.

(iv) All extended elements

$$\hat{\Delta} \mapsto \hat{I}$$

and

$$\hat{\delta}_0^{-m} \mapsto \hat{\delta}_0$$

are contained in I except the for the extensions of the leftmost and rightmost boundary domains which are contained in $\hat{I}$ because q is a repelling fixed point.

Then, the Koebe distortion property implies

Lemma 4.1.2. *Let $f_i\colon \Delta_i \to I$ be any extendible monotone branch, with $\hat{f}_i\colon \hat{\Delta}_i \to \hat{I}$ its extension. Denote the pullback $f_i^{-1}(\Delta_r)$ by $\Delta_{ir} \subset \Delta_i$ and its extension $\hat{f}_{ir}\colon \hat{\Delta}_{ir} \to \hat{I}$. Then*

$$\frac{|\Delta_{ir}|}{|\Delta_i|} < c_1 \cdot |Df_r(q)|^{-1},$$

where c_1 is a uniform (distortion) constant independent of i. Since

$$\Delta_{irr} = f_i^{-1}(\Delta_{rr}) \subset \Delta_{ir}, \ldots$$

we have

$$|\Delta_{i\underbrace{rr\ldots r}_{k}}|/|\Delta_i| < c_2 \cdot |Df_r(q)|^{-k},$$

for some other uniform constant c_2.

Remarks 4.1.3:

(i) We may pullback onto $\Delta_{rr\ldots r}$ any partition ξ_n created at later stages of our construction. Properties (i) through (iv) above will still hold for these versions of refinement and consequently the above corollary as well.

(ii) The above corollary applies to the *left boundary refinement procedure* so that

$$|\Delta_{i\underbrace{lrr\ldots r}_{k}}|/|\Delta_i| < c_2' \cdot |Df_r(q)|^{-k}.$$

Now coming back to $\Delta \subset \mathrm{Im}(h_{n-1})$ which is too close to the critical value $h_{n-1}(1/2)$, and assume that the right boundary point of Δ is closest to that critical value so that the reason for the non-extendibility of $h_{n-1}^{-1}(\Delta)$ is that the extension of Δ to the right is not contained in $\mathrm{Im}(h_{n-1})$, (the case when the problem is with the left-side extension we us the *left boundary refinement*), then we pullback onto Δ the refinement $\xi_{0,k}$ with k sufficiently large so that the preimage of

$$\Delta_{\underbrace{rr\ldots r}_{k}}$$

which we denote by

$$\Delta^{-1}_{\underbrace{rr\ldots r}_{k}}$$

and is closest to $h_{n-1}(1/2)$ is contained in $\mathrm{Im}(h_{n-1})$ together with its extension

$$\hat{\Delta}^{-1}_{\underbrace{rr\ldots r}_{k}}.$$

Then the critical pullback by h_{n-1}^{-1} of that refinement of Δ creates a partition on $h_{n-1}^{-1}(\Delta)$ with uniformly extendable elements. It's important to realize that Δ remains unchanged its refinement was done so as the critical pullback onto $h_{n-1}(\Delta) \subset \delta_{n-1}$ creates uniformly extendable monotone domains.

Remark 4.1.4. By the choice of parameter, as we will show later on, the critical value $h_{n-1}(1/2)$ always belongs to a monotone domain and moreover, if a preimage δ_j^{-k} is contained in the image of the central critical branch h_{n-1}, then the whole extension $\hat{\delta}_j^{-k}$ of that preimage is also contained in the preimage of that critical branch. This means that the pullbacks onto these preimages will not need any boundary refinement.

(4) *Critical Pullback*:

Let us denote by ξ'_{n-1} the new partition in which all elements that needed boundary refinement were refined. We then induce on δ_{n-1} the partition

$h_{n-1}^{-1}(\xi'_{n-1})$ thus creating preimages of all the elements of ξ_{n-1} that are contained in the image of h_{n-1}. This gives us domains inside δ_{n-1} of extendible branches of the following type:

- Two new monotone branches $f \circ h_{n-1}$ for each monotone domain $\Delta(f)$ which lies wholly inside $\mathrm{Im}(h_{n-1})$. In view of the grow up procedure this includes either $f_l \circ h_{n-1}$, or $f_r \circ h_{n-1}$.
- A central parabolic branch $h_n := f_n^* \circ h_{n-1}$, where $f_n^*\colon \Delta_n^* \to I$ is the monotone branch containing the critical value $h_{n-1}(1/2)$.
- We also obtain the diffeomorphisms $\chi \circ h_{n-1}$ from the corresponding diffeomorphisms $\chi\colon \delta_i^{-k} \to \delta_i$ of ξ_{n-1}. Observe that if $\mathrm{Im}(h_{n-1})$ contains the central domain δ_{n-1}, then we have two primary preimages of δ_{n-1}.

Denote the resulting partition of δ_{n-1} by

$$\eta_{n-1} : \delta_{n-1} = \delta_n \cup (\cup \Delta) \cup (\cup_i \cup_p \delta_i^{-p}) \qquad (\text{mod } 0).$$

Notice that by construction at every step $i = 0, 1, \ldots, n-1$, similar partitions ξ_{i-1} and η_{i-1} are defined. This is reflected by the appearance of preimages of central domains δ_i^{-p} in the partition η_{n-1} above.

(5) *Filling-in* :

We fill each preimage

$$\delta_j^{-k} = \chi^{-1}(\delta_j) \quad j = 0, 1, \ldots, n-1$$

with the pullback $\chi^{-1}(\eta_j)$. In this way we get a 'copy' of the elements of η_j inside each δ_j^{-k}. This includes the new partition of δ_{n-1} created at this step. Thus, ξ_n now has the form

$$\xi_n = (\bigcup \Delta) \cup (\bigcup_{j \leq n} \bigcup_{p > 0} \delta_j^{-p}) \cup \delta_n \tag{10}$$

where all the monotone domains Δ are uniformly extendible, as well as, the preimages δ_j^{-k} for all $j \leq n$.

Section 4.2. Enlargements

4.2.1 When constructing the partitions ξ_n we emphasized that the critical value $h_n(1/2)$ falls in a monotone domain. So, clearly that excludes $h_n(1/2)$ from being inside a hole δ_n^{-k}. However, we will require slightly more freedom, namely, we will add the assumption that the critical value does not belong to an enlargement of δ_n^{-k} which we will define below.

The procedures we follow in Chapter 4 to construct the sequence of partitions ξ_n have the property that holes are never adjacent, but are separated by monotone domains. At each step n of induction, we have the central branch

$$h_{n-1} : \delta_{n-1} \to I.$$

Then after critical pullback we create the next central domain δ_n. In the Johnson box situation, we create multiple preimages of δ_n inside the box. After one step of filling in a primary domain, say δ_i^{-p}, that domain breaks up into several disjoint primary domains.

At the start, when we construct the partition ξ_0, we choose a^- and a^+ to define the extension $\hat{I} := [a^-, a^+]$ of the interval $I = [q^{-1}, q]$ so that all monotone branches

$$f_i \colon \Delta_i \to I$$

where the domain $\Delta_i \in \xi_0$, can be extended as a diffeomorphism over a larger domain $\hat{\Delta}_i$. That is, we have the extended diffeomorphism

$$\hat{f}_i\colon \hat{\Delta}_i \to \hat{I}.$$

As for the central domain δ_0 the extension is contained in the union

$$\Delta_K^- \cup \delta_0 \cup \Delta_K$$

where $\Delta_K^\pm$ are the monotone domains that are adjacent to δ_0, with Δ_K^- is to the left of δ_0 while Δ_K^+ is located to the right. However, to ensures that all diffeomorphisms

$$\xi\colon \delta_0^{-k} \to \delta_0$$

have uniformly bounded small distortion we need to enlarge this extension further. to contain many initial adjacent domains. Let us define the set W as

$$W = \bigcup_{m=2j_0}^{K} \left(\Delta_m^\pm \cup \delta_0\right)$$

where $\Delta_K^\pm$ are the monotone domains that are adjacent to δ_0, with Δ_K^- is to the left of δ_0 while Δ_K^+ is located to the right, and $\Delta_{K-1}^\pm$ is adjacent to $\Delta_K^\pm$ and so on, for $\Delta_m^\pm$ $(m = 2j_0, 2j_0 + 1, \ldots, K)$, where j_0 is defined by the inequality (4) of Lemma 3.5.2. We now adopt the following notation.

Notation 4.2.2. Let us first define the *enlargement* $\tilde{\delta}_0$ of the central domain δ_0 as the set W given above. Then, for all subsequent central domains we define the *enlargement* of δ_i to be $\tilde{\delta}_i = \delta_{i-1}$. When we apply the critical pullback procedure, we make sure that the critical value does not belong to the union of enlargements

$$\bigcup \tilde{\delta}_i.$$

In this way, we have the property that for any hole δ_i^{-k} its enlargement $\tilde{\delta}_i^{-k}$ is well defined. Also, if δ_i^{-k} is obtained by filling in of δ_j^{-p} then $\tilde{\delta}_i^{-k} \subset \delta_j^{-p}$. So, by construction, the enlargement of every δ_i^{-k} is defined and denoted by $\tilde{\delta}_i^{-k}$. That is,

$$\tilde{\delta}_i^{-k} := \delta_{i-1}^{-k}.$$

4.2.3. Now, when we construct a new central domain δ_n, its extension is the critical pullback of the extension $\hat{\Delta}^*$ of the monotone domain Δ^* which contains the critical value

$$h_n(1/2) \in \Delta^*.$$

In view of the grow up procedure $\hat{\Delta}^* \subset I$ and therefore

$$h_n^{-1}(\Delta^*) \subset \delta_{n-1}.$$

As a result

$$\hat{\delta}_n \subset \delta_{n-1} = \tilde{\delta}_n$$

which can be formulated simply as: *Extension is a subset of enlargement.*

For all monotone domains constructed inside δ_{n-1} at step n, their enlargements are inside δ_{n-1} as well, except for the boundary elements $h_{n-1}^{-1}(\Delta_l)$ and $h_{n-1}^{-1}(\Delta_r)$. In particular, all enlargements $\tilde{\delta}_i^{-k}$ of δ_i^{-k} belong to $\delta_{n-1} = \tilde{\delta}_n$. Clearly, similarly to holes, enlargements are either disjoint or one of them contains the other.

Since we know that the total measure of holes $|\mathcal{H}_n|$ decreases at least by a constant factor $\theta < 1$, the same clearly applies to their enlargements. After all, enlargements were previously holes at one step back in our inductive construction of the sequence of partitions ξ_n.

4.2.4. In view of equation (8) in section 4.1, we have

$$\frac{|\delta_{n+1}|}{|\delta_n|} \leq \varepsilon_1$$

for some small ε_1 determined by our construction of the preliminary partition ξ_0. As all maps

$$\delta_i^{-k} \mapsto \delta_i$$

are extendible up to

$$\delta_{i-1}^{-k} \mapsto \delta_{i-1}$$

we obtain from the Koebe distortion property that all diffeomorphisms

$$\chi\colon \delta_i^{-k} \to \delta_i$$

have small distortion. Recall that we say that a map has small distortion when the Koebe distortion constant is close to 1, see property (ii) in 3.1.1.

Section 4.3. Delayed Basic Step.

4.3.1. Now consider the case $h_{n-1}(1/2) \in \delta_{n-1}$, such that $1/2$ belongs to $\mathrm{Im}(h_{n-1})$. In this case, we stipulate that an iterate of $h_{n-1}(1/2)$ falls in a monotone domain belonging to the partition ξ_{n-1}. This situation is best described as a *delayed basic step*, because even though the critical value falls inside the central domain which leads to the box construction, it is still essentially a basic step since after the trajectory of the critical point is *delayed* in Johnson's box, the critical value eventually "escapes and falls in a preimage of a *monotone* domain on the staircase and not in a preimage of a central domain, as is the case in the so called *box situation* described in [J-S]. For historical reasons, we also refer to this step as a Johnson step and proceed with the construction as follows.

4.3.2. For each delayed basic step $n = n_k$, we define

$$r_k = \min \{ r : h^r_{n-1}(1/2) \notin \delta_{n-1} \}.$$

Then, as in the first step of our construction in 3.6.3 and 3.6.4, we partition δ_{n-1} using the so-called staircases. Recall that the graph of h_{n-1} crosses the diagonal, so we can define *Johnson's box* with bottom B_k bounded by the points q_k, q_k^{-1} where q_k is one of the two fixed point of h_k — the one farther away from 1/2 — and q_k^{-1} is its second preimage where with out loss of generality we assume for definitiveness that $h'_{n-1}(q_k) > 0$, so that $q_k \in [0, 1/2]$. We call the part of the graph outside this box the *hat* and denote its base by H_{n-1}. Then we continue as in the first step by constructing an *infinite staircase* $S = \cup_{j\geq 1} S_j$ where each $\mathcal{S}_j$ consists of two components $\mathcal{S}_{j,\text{left}}$ and $\mathcal{S}_{j,\text{right}}$, symmetric about 1/2, each of which contains the preimages

$$h_{n-1}^{-j}(\Delta),\ h_{n-1}^{-j}(\delta_{n-1}^{-p}) \text{ where } \Delta, \delta_{n-1}^{-p} \in \xi_{n-1}$$

Then we fill the base of the hat H_k by critical pullback, thus creating new monotone branches inside H_k in addition to the critical branch

$$h_n := f_n^* \circ h_{n-1}^{r_{n-1}}$$

here f_n^* is the monotone branch whose domain $\Delta_n^* \in \mathcal{S}_{r_k}$ contains the critical value $h_{n-1}(1/2)$. Restricting h_{n-1} to the two symmetric intervals of $B_k - H_k$, we obtain two monotone maps g_1 and g_2. So, as before almost every point of $B_k - H_k$ under the iterations of g_1 and g_2 eventually 'escapes' the box through H_k. The preimages of the partition of H_k under the two monotone branches g_1 and g_2 generates a partition of $B_k - H_k$ (modulo a Cantor set of zero Lebesgue measure). This partition of B_k adjoined with that of the staircase $\mathcal{S}$ constitute the desired partition η_{n-1} of δ_{n-1}:

$$\eta_{n-1} : \delta_{n-1} = \delta_n \cup (\cup\Delta) \cup (\cup_j \cup_p \delta_j^{-p}) \qquad \text{(mod 0).}$$

Finally, the partition

$$\xi_n : I = (\bigcup \Delta) \cup (\bigcup_{j \leq n} \bigcup_{k>0} \delta_j^{-k}) \cup \delta_n$$

having the same form as (10) in section 4.1 is obtained by filling in the preimages of δ_{n-1} as in the basic situation.

Section 4.4. The Limit Partition

4.4.1 Let

$$\mathcal{H}_{n-1} = \bigcup_{j<n;p\geq 0} \delta_j^{-p}$$

denote the collection of holes. Then we observe that at each step of induction we construct domains of monotone branches which are not changed anymore, domains δ_i^{-k} which are filled-in at the next steps and Cantor sets of zero measure. This implies

Proposition 4.4.2. *Suppose that at each step n of our construction the relative measure of $\mathcal{H}_{n-1}$ within δ_n is less than a uniform constant $\theta < 1$. Then as $n \to \infty$ we obtain a limiting partition $\xi = \xi_\infty$ of I consisting of a countably infinite number of uniformly extendible domains Δ_i of monotone branches $f_i \colon \Delta_i \to I$ and a Cantor set of Lebesgue measure zero.*

Further, since by construction, all maps f_i are extendible then in view of Remark 3.5.4 in chapter 3 we have

Corollary 4.4.3. *Under the conditions of proposition 4.1 above, we obtain that all monotone branches f_i are expanding and have uniformly bounded distortion.*

CHAPTER 5

PROOF OF THE MAIN THEOREM

Section 5.1. Preliminary Definitions and Terminology

5.1.1 In the course of our construction we need to keep track of certain quantities associated with the successive partitions ξ_n. Recall that after completing step n of our inductive construction, we obtain the partition

$$\xi_n = (\bigcup \Delta) \cup (\bigcup_{j \leq n} \bigcup_{p>0} \delta_j^{-p}) \cup \delta_n$$

as described previously in chapter 4.

Associated with each partition ξ_n we have the following:

(i) *Holes*

Let

$$\mathcal{H}_n := \bigcup_{j \leq n} \bigcup_{p>0} \delta_j^{-p}$$

denote the union of holes δ_j^{-p}, consisting of preimages of central domains δ_j, where $j = 0, 1, 2, \ldots, n$. We then let $\alpha_n = |\mathcal{H}_n|$ denote the total Lebesgue measure of these holes.

(ii) If Δ is a proper subset of the image of the central domain and is too close to the critical value $w = h_n(1/2)$, then the critical pullback $h_{n-1}(\Delta)$ consisting of the two preimages $\{\, \Delta^-, \Delta^+ \,\}$ of Δ may create non-extendible domains. In which case, we will need to do the boundary refinement procedure described in Chapter 4. For future estimates, in this situation, we define $R_n(\Delta)$ to be the number of boundary refinements needed so that all the intervals created inside both intervals $\Delta^{\pm}$ are extendible. Clearly, $R_n(\Delta)$ depends on the length $|\Delta|$ and the distance from the critical value to nearest endpoint of the interval Δ. We will prove below that $R_n(\Delta)$ is bounded as Δ ranges over all the domains except for the domain Δ_{adj}, which lies inside the image of the central branch and is adjacent to the monotone domain Δ_n^* that contains the critical value w. In this case, $R_n(\Delta)$ depends on the location of w within Δ_n^* and tends to infinity as w approaches the boundary of Δ_{adj}. (See the boundary refinement Lemma later on in this chapter).

(iii) If $n = n_k$ is a delayed basic step and we have the box B_k and the base of the hat H_k we will have the ratio

$$|H_k|/|B_k| \leq \beta_k.$$

where β_k, to be specified later, is chosen in advance to be small enough in order that the a.c.i.m. μ is on the one hand infinite and on the other hand σ-finite. This an important aspect of our proof, namely, the ratio β_k being small has the dual effect of ensuring both that the measure μ is infinite and yet, it is also σ-finite.

(iv) If $h_n(1/2)$ falls in one of the boundary intervals of I, namely Δ_l or Δ_r, but the image of h_n contains no other intervals, then we need to perform the Grow-up procedure described in Chapter 4. More explicitly, the central branch

$$h_n = f^*_{n-1} \circ h_{n-1}$$

is replaced by either

$$f_l^m \circ h_n$$

or

$$f_l^{m-1} \circ f_r \circ h_n$$

until it's image covers more than just a boundary interval. We then re-denote the central domain by h_n keeping in mind that the power $N(h_n)$ is increased by m iterates of h_n. However, by using the parameter choice Lemma 5.3.2 below, we can arrange that the position of the critical value $h_n(1/2)$ at step n such that m does not exceed 2^n.

Section 5.2. Strategy of The Construction

5.2.1. The examples we give are constructed by giving a decreasing sequence of nested parameter intervals Λ_n such that for all $t \in \Lambda_n$ the map φ_t admits the partition ξ_n as described previously in Chapter 4. In addition, we will arrange that ξ_n satisfies certain conditions specified below, so that for $t = \cap_n \Lambda_n$, φ_t is of type III and has a non-integrable invariant density. The main topological ingredient of the inductive construction of Chapter 4 is that at each step either $h_n(1/2)$ falls in a monotone domain created at a previous step, denoted by Δ_n^* and belongs to ξ_n (*Basic Case*); Or $h_n(1/2)$ is "delayed" in δ_n and falls instead in a preimage of the monotone domain Δ_n^* belonging to ξ_n, so that $h_n^{r_n}(1/2) \in \Delta_n^*$, (*delayed basic case*). Notice that in the latter case $h_n(1/2)$ still falls in a monotone domain, except that this monotone domain is created at the current step, that is, it belongs to the partition ξ_{n+1}.

Thus, in either situation, the critical value falls in a domain which is mapped *onto* I by a monotone branch. It follows from the monotonicity of the kneading invariant, (see Graczyk and Swiatek [G-S]), that if the critical value enters a certain domain $\Delta = [a_1, a_2]$, say through a_1 when the parameter $t = t_1$, then it remains inside Δ until the parameter reaches $t = t_2$ when it then leaves Δ through a_2. Therefore, by varying the parameter, we can arrange that the new critical value

$$h_{n+1}(1/2) = f_n^* \circ h_n(1/2)$$

at a basic step, or

$$h_{n+1}(1/2) = f_n^* \circ h_n^{r_n}(1/2)$$

at a delayed basic step, is mapped anywhere in I. In this way, we can ensure that the forward G-orbit of the critical point is dense, i.e., $\omega_G(1/2) = I$ and hence $\omega_{\varphi_t}(1/2) = [\varphi_t^2(1/2), \varphi_t(1/2)]$, so that φ_t is of Type III.

Moreover, every time the critical value $h_n(1/2)$ is delayed in the box, the level of the staircase, r_n, which contains $h_n(1/2)$, and as a consequence, the size of the hat can be chosen independent of the topological requirements on the critical orbit because each level of the infinite staircase consists of the preimage of the previous level.

5.2.2. Let us recall some definitions. Every monotone branch

$$f_i : \Delta_i \to I$$

is by construction a composition of iterates of the first return map G. Accordingly $f_i = G^{N_i}|\Delta_i$ and we call N_i the *power* of f_i. Every critical branch h_n can be factored into $h_n = F_n \circ h_0$, where F_n is a composition of monotone branches and h_0 is the central parabolic branch of the first return map G restricted to a small neighborhood of the critical point $x = 1/2$. In this case, we define the power of h_n as 1 plus the sum of powers of each of the monotone branches in the composition F_n. Notice that in this sense, the power of all branches of the first return map G is 1, and all monotone branches can be factored into compositions of branches of G. In terms of the *Tower Construction* of Chapter 2, N_i corresponds to the number of domains in the tower over Δ_i and thus may be referred to as the *height* of Δ_i. In fact, $N(\Delta_i) = N_i$ defines a step function $N\colon \cup_i \Delta_i \to \mathbf{N}$ by

$$N(x) = \sum_i N_i \, \mathbf{I}_{\Delta_i}(x).$$

Here $\mathbf{I}$ denotes the usual characteristic or indicator function.

In accordance with [J-S], a map T is called expansion inducing when there exists a partition ξ of I (mod 0) into a countable union of non-overlapping intervals $\{\Delta_i\}_i$, where each Δ_i is the domain of the monotone branch

$$f_i := G^{N_i}|\Delta_i \colon \Delta_i \to I$$

and the transformation $T\colon I \to I$ defined piecewise by

$$T|\Delta_i = f_i : \Delta_i \to I$$

is expanding with uniformly bounded distortion for all branches f_i. In which case, T will satisfy the hypothesis of the so-called Folklore Theorem as formulated below, see [A].

The Folklore Theorem 5.2.3. *Assume we are given a C^2 map T defined on a countable collection $\{\Delta_i\}$ of disjoint open intervals with the following properties:*

1. *The Lebesgue measure of $I - \cup_i \Delta_i$ is zero.*
2. *Every interval Δ_i is the domain of an expanding diffeomorphism $f_i \colon \Delta_i \to I$ such that $|f_i'| > L > 1$ for a uniform constant L independent of i.*
3. *There exists a uniform upper bound M such that for every branch f_i,*

$$\frac{|f_i''(x)|}{|f_i'(x)|^2} < M \quad \textit{for all} \quad x \in \Delta_i$$

Then T has an ergodic a.c.i.m. ν with a density function that is continuous and bounded away from zero and infinity.

5.2.4. Using the Tower Construction from chapter 2, we obtain the G-invariant measure μ on I given by the formula

$$\mu(E) = \sum_i \sum_{j=0}^{N_i - 1} \nu(\Delta_i \cap G^{-j} E)$$

for every measurable set $E \subset I$. Since G is a smooth map we have $\mu \ll \nu$. By transitivity, μ is absolutely continuous:

$$\mu \ll \nu,\, \nu \ll |\cdot| \Rightarrow \mu \ll |\cdot|$$

Since ν has a bounded density, μE is finite if and only if

$$\sum_i \sum_{j=0}^{N_i-1} \left|(\Delta_i \cap G^{-j}E)\right| \tag{11}$$

converges, and μ is finite if and only if

$$\sum_i N_i|\Delta_i| < \infty. \tag{12}$$

Our aim is to construct the map T in such a way so that:

(A) The convergence of the sum in (12) does not hold.

(B) There exists a set E with positive Lebesgue measure for which the sum in (11) converges.

We will also have

(C) The μ-measure of every interval is infinite.

Theorem 2.1.3 of Chapter 2, implies by property (B) that the measure μ is σ-finite.

Section 5.3. Parameter Choice Lemma

5.3.1. Let us assume, as in Lemma 3.5.2 of Chapter 3, that the first return map

$$G: I \to I$$

induced by φ_t has $2K$ monotone branches for all parameter values t inside a fixed interval denoted by (t_K, t_{K+1}). Then when $t = t_{K+1}$ the critical branch splits into two new monotone branches and a new critical branch is born in between. This follows from the implicit function theorem, see for example [J2].

So, our first parameter interval is given by $\Lambda_0 = [t_K, t_{K+1}]$ and for all parameter values in the interior of Λ_0, a partition

$$\xi_0 : I = (\cup\Delta) \cup (\delta_0^{-k}) \cup \delta_0$$

is defined and its elements vary continuously with t. In the course of our construction we determine a nested sequence of closed parameter intervals $\Lambda_n \subset \Lambda_{n-1}$ such that for all parameter values $t \in \Lambda_n$, φ_t induces the partition ξ_n with desired properties. Then for $\tau = \cap_i^\infty \Lambda_i$ we obtain the limit partition ξ_∞ and the expansion inducing map φ_τ. In order to do this, we will use what we refer to as the

Parameter Choice Lemma 5.3.2. *At each step n, there exists a parameter interval $\Lambda_n \subset \Lambda_{n-1}$, such that as t varies in the interior of Λ_n, the following two properties hold:*

$(i)_n$ *All intervals of the partition ξ_n vary continuously, in particular none of them disappear and no new ones appear.*

$(ii)_n$ *The critical value $h_n(1/2)$ moves continuously across the whole interval I.*

Proof. As we pointed out above, we start with the parameter interval Λ_0 that the first return map G induced by φ_t for $t \in \Lambda_0$ has exactly $2K + 1$ branches. Then as t varies in the interior of Λ_0, the lengths of these branches varies continuously with t and the critical value $h_0(1/2)$ moves all the way across the whole interval I. Assume by induction that the two properties $(i)_j$ and $(ii)_j$ hold for all $j \leq n$. Then by monotonicity of the kneading invariant ([G-S]), we get that given a prescribed monotone domain Δ, there exists a parameter subinterval $\Lambda_{n+1} \subset \Lambda_n$ such that when $t \in \Lambda_{n+1}$, $h_n(1/2)$ moves all the way through Δ without leaving Δ. According to our inductive construction of Chapter 4, the next central branch is

$$h_{n+1} = F_n \circ h_n,$$

where $F_n = f_n^*$ at a basic step, and $F_n = f_n^* \circ h_n^{r_n}$ at a delayed basic step. Since, in both cases, F_n maps Δ *onto* the whole interval I, it follows that

$h_{n+1}(1/2)$ satisfies $(ii)_{n+1}$ as $h_n(1/2)$ moves across the interval Δ. Next, since $h_n(1/2)$ depends continuously on the parameter t and stays inside the domain Δ when $t \in \Lambda_{n+1}$, the new partition of δ_n which we had denoted by η_n in our construction in chapter 4 will satisfy $(i)_{n+1}$. Moreover, the new branches of the partition ξ_{n+1} constructed outside δ_n are compositions of branches of ξ_n with those branches inside δ_n. As both vary continuously, all new branches satisfy (i)$_{n+1}$. ∎

Section 5.4. Generating Partitions

5.4.1. In this section, we define an additional sequence of partitions which allows us to ensure that the forward orbit of the critical point is dense in I. Using the sequence of partitions constructed in Chapter 4, we define a subordinate partition $\mathcal{P}_n$ as follows:

Let $\mathcal{P}_0 = \xi_0$ — the preliminary partition constructed in Chapter 4. Construct the refinement $\mathcal{P}_1$ as follows: $\mathcal{P}_1$ agrees with ξ_1 on the partitions of $\delta_0^{-k} \in \xi_0$, $(k \geq 0)$. On the other hand, each $\Delta_i \in \mathcal{P}_0$ is mapped 1-1 and onto I by the monotone branch f_i, so we can pull back the partition ξ_1 thus subdividing each $\Delta_i \in \xi_0$ into *good* intervals Δ_{ij} which are mapped onto I by $f_{ij} = f_j \circ f_i$, and "holes" $f_i^{-1}(\delta_0^{-k})$, $(k \geq 0)$. Continuing in the same manner and using the partition ξ_n, we can construct the refinement $\mathcal{P}_n \prec \mathcal{P}_{n-1}$ as follows. First, we consider the holes

$$\{\, \delta_m^{-k} : m = 0, 1, \ldots, n-1 \,\} \quad (k \geq 0)$$

coming from the partition ξ_{n-1}. These are refined by the filling in procedure of the general construction of the partition ξ_n. Consequently, this creates new monotone domains and holes all of which are elements of ξ_n, and as such, are refined later when constructing $\mathcal{P}_{n+1}$. So, we have that the refinement $\mathcal{P}_n$ coincides with ξ_n on the holes

$$\delta_m^k \in \xi_{n-1} \quad (m = 0, 1, \ldots, n-1).$$

However every monotone domain of ξ_n that has length greater than 3^{-n} is refined by monotone pullback of the partition ξ_n. In this way we create new good intervals of the form Δ_{ij} and holes $f_i^{-1}(\delta_m^{-k})$ where $\Delta_i \in \xi_{n-1} - \xi_{n-2}$, $\Delta_j \in \xi_n$ and $m = 0, 1, \ldots, n-1$. Since every good interval Δ of $\mathcal{P}_{n-1}$ has the representation $\Delta_{ij\cdots k}$ and is mapped onto I by an expanding diffeomorphism $F := f_{ij\cdots k}$ we can continue in the same manner by performing the monotone pullback $F^{-1}(\xi_n)$ until every good interval Δ has length at most 3^{-n}. Indeed, we have shown that under the conditions of Proposition 4.4.2 in Chapter 4,

$$\frac{\text{mes}\,(\textit{holes in } \xi_n)}{|I|} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

and so after each monotone pullback the ratio

$$\frac{\text{mes}\,(\textit{holes in }\Delta)}{|\Delta|}$$

decreases in view of the fact that the monotone map F is not only expanding but has bounded distortion as well. Therefore we obtain

Lemma 5.4.2. *The sum of lengths of all the holes $|\delta_m^{-k}|$ belonging to the refinement $\mathcal{P}_n$ tends to zero as $n \to \infty$, and every good interval that belongs to $\mathcal{P}_n$ has length at most 3^{-n}.*

As a result we have

Proposition 5.4.3. *The collection of good intervals, as defined above, form a basis for the standard topology on the set of Real numbers $\mathbf{R}$ restricted to the interval I.*

Proof. Given any open set $U \subset I$ there exists an n such that the partition $\mathcal{P}_n$ contains a good interval $\Delta \subset U$. It follows immediately that the countable collection of all good intervals form a basis and therefore generates the standard topology of $\mathbf{R}$ restricted to I. ∎

Section 5.5. Positioning the critical value at Johnson's step

5.5.1. In this section we describe how to determine the parameter intervals Λ_n to achieve the following two properties.

(i) We want the trajectory of the critical point to be dense, which involves visiting a certain number of good intervals between two consecutive delayed basic steps.

(ii) Second, given a sequence of numbers γ_k we will want to arrange that at each delayed basic step $n = n_k$ the hat is small enough in order that the ratio $|H_k|/|B_k| < \gamma_k$.

We may start the construction of Chapter 4 with a delayed basic step, that is

$$h_0(1/2) \in \delta_0, \ldots, h_0^{r_0-1}(1/2) \in \delta_0$$

and

$$h_0^{r_0}(1/2) \in I - \delta_0.$$

Thus, $h_0(1/2) \in \mathcal{S}_{r_0}$ — the r_0^{th} level of the staircase $\mathcal{S}$, and $h_0^{r_0-1}(1/2)$ falls in the first level $\mathcal{S}_1$. Let $\Delta_0^* \in \xi_0$ denote the monotone domain that contains $h_0^{r_0}(1/2)$.

The idea is to look ahead. Since $h_0^{r_0}(1/2)$ falls in a monotone domain Δ_0^* that is mapped *onto* the whole interval I, the location of $h_0(1/2)$ may be chosen (it actually is *a priori* determined) so as for every finite collection of good intervals Δ of $\mathcal{P}_0$ there corresponds a basic step such that $h_j(1/2) \in \Delta$. This determines a sequence of basic steps $j = 1, 2, \ldots, n_1 - 1$ and then the following step is delayed basic: $h_{n_1}(1/2) \in \delta_{n_1}$. For each of these basic steps we let f_j^* denote the monotone branch whose domain Δ_j^* contains the critical value $h_j(1/2)$. Then $h_{j+1} = f_j^* \circ h_j$, and it follows that

$$h_{n_1} = f_{n_1-1}^* \circ f_{n_1-2}^* \circ \cdots \circ f_0^* \circ h_0^{r_0}.$$

Therefore the above requirement on the critical value for steps $n = 1, 2, \ldots, n_1$ is that the collection of domains

$$\Delta_1^*, \Delta_2^*, \ldots, \Delta_{n_1-1}^*$$

includes a given collection of good intervals of $\mathcal{P}_0$. Notice that this requirement is independent of the value of r_0 which is chosen to be sufficiently large in order that $|H_0|/|B_0| < \gamma_0$ for any prescribed γ_0. Using the Parameter Choice Lemma for each of the steps $n = 1, 2, \ldots, n_1$, we obtain a sequence of parameter intervals

$$\Lambda_0 \supset \Lambda_1 \supset \cdots \supset \Lambda_{n_1}$$

such that for $t \in \Lambda_{n_1}$, φ_t induces the partition ξ_{n_1} with the properties described above. Observe that Λ_{n_1} contains a subinterval such that when the parameter runs through this subinterval, $h_{n_1}(1/2)$ moves across the staircase $\mathcal{S} = \cup_j \mathcal{S}_j$

belonging to δ_{n_1}. Now, $h_{n_1}^{r_1}(1/2)$ falls in a monotone domain $\Delta_{n_1}^* \in \xi_{n_1}$, and so the location of the critical value $h_{n_1}(1/2)$ may be chosen so that for the next series of basic steps, the critical value $h_j(1/2)$, $(j = n_1+1, n_1+2, \ldots, n_2-1)$, falls in a prescribed collection of good intervals in $\mathcal{P}_1$. Then follows a delayed basic step: $h_{n_2}(1/2) \in \delta_{n_2}$ and $h_{n_2}^{r_2}(1/2)$ is the smallest iterate of $h_{n_2}(1/2)$ outside the central domain δ_{n_2}. This requirement on the critical orbit at step $n = n_1 + 1$ determines the location of $h_{n_1}^{r_1}(1/2)$ within $\Delta_{n_1}^*$, however, the level of the staircase $\mathcal{S}_{r_1}$ which contains the critical value $h_{n_1}(1/2)$ is completely independent of the aforementioned requirement, and so the choice of r_1 can be done *after* determining the critical orbits for the steps $n = n_1+2, n_1+3, \ldots, n_2$. We arrange that r_1 is sufficiently large so as the ratio $|H_1|/|B_1| < \gamma_1$. Again, we use the parameter choice lemma successively to obtain a sequence of parameter intervals

$$\Lambda_{n_1} \supset \cdots \supset \Lambda_{n_2}$$

for which φ_t induces the construction above. We continue in the same fashion, h_{n_2} is a delayed basic step with $|H_2|/|B_2| < \gamma_2$, and for $j = n_2+1, n_2+2, \ldots, n_3-1$, the critical value $h_j(1/2)$ visits a prescribed collection of good intervals of $\mathcal{P}_2$, then h_{n_3} is delayed basic with $|H_3|/|B_3| < \gamma_3$, and so on. In this way, we may select a sequence of nested parameter intervals Λ_{n_k} such that for $t = \cap \Lambda_{n_k}$, the orbit of $1/2$ under G_t is ε_k-dense, where $\varepsilon_k \downarrow 0$. It follows that for $t \in \cap_k \Lambda_{n_k} (= \cap_n \Lambda_n)$, φ_t has $\omega_G(1/2) = I$. Moreover, φ_t induces the sequence of partitions ξ_n, and in the construction of ξ_n, the critical value $h_n(1/2)$ falls in a monotone domain belonging to ξ_n when $n \neq n_k$ — basic step, and in a preimage of such a domain when $n = n_k$ — delayed basic step. In the latter situation, we have the box B_k and hat H_k with the ratio $|H_k|/|B_k| < \gamma_k$.

Section 5.6. The Proof that the a.c.i.m. μ is infinite

5.6.1. We will show below that if the sequence of partitions ξ_n includes an infinite number of partitions ξ_{n_k} such that the critical value is "delayed" inside the Johnson box B_k, then the invariant measure μ will accumulate enough mass forcing the divergence of the sum

$$\sum_i N_i |\Delta_i|.$$

Using the same notation as in the end of the previous section, we assume that at step $n = n_k$, the central branch

$$h_n \colon \delta_n \to I$$

falls in the delayed basic situation and we construct the box $B = B_k$ with hat $H = H_k$. Set

$$H^{-j} = h_n^{-j}(H),$$

i.e., if g_1 and g_2 denote the two monotone branches of $h_n|(B-H)$ then H^{-j} consists of the collection of 2^j intervals that are mapped onto H by the compositions $g_{i_1\cdots i_j}$ of g_1 and g_2 for all possible arrangements $i_1\cdots i_j$ of the numbers 1 and 2. These intervals are called preimages of the hat of *order* j. Let s be the smallest integer larger than $1/|B|$. The next lemma says that if the base hat is small enough, then the intervals of order larger than s constitute more than half of the box (cf. [Jo]):

Lemma 5.6.2. *There exists* $a = a(k)$ *such that if* $|H|/|B| < a$ *then*

$$|H| + \cdots + |H^{-s}| < \frac{1}{2}|B|$$

Proof. Obvious by continuity. ∎

This leads to

Proposition 5.6.3. *Assume that in the construction of* ξ *there are infinitely many delayed basic steps* $n = n_k$ *such that* $|H_k|/|B_k| < a_k$*, where the* a_k *are given by the above Lemma, then the measure* μ *is infinite.*

Proof. We only need to show that the sum

$$\Sigma = \sum_i N_i|\Delta_i|$$

diverges. For each step $n = n_k$, we have by the previous Lemma

$$\left|\left\{ x \in B_k : N(x) \geq \frac{1}{|B_k|} \right\}\right| > \frac{1}{2}|B_k|.$$

Therefore the contribution to the sum Σ at each step $n = n_k$ exceeds

$$\frac{1}{|B_k|} \cdot \left| \left\{ x \in B_k : N(x) \geq \frac{1}{|B_k|} \right\} \right| > \frac{1}{2}.$$

Since we have infinitely many such steps $n = n_k$, the sum Σ in diverges and μ is not finite. ∎

This satisfies condition (A) given in section 5.1.1.

Section 5.7. Construction of the set E

5.7.1. Recall from Section 5.1.1. that we wish to construct a set E with non-zero Lebesgue measure for which the sum in (11) converges. From the previous section we see that we need to exclude the intervals that go back and forth within Johnson's box. With this in mind, we construct the set E by defining a sequence of open sets U_k, such that their union $U = \bigcup_k U_k$ does not have full measure in I, then $E := I - U$ has strictly positive measure. Take

$$U_0 = \bigcup_{k \geq 0} \delta_0^{-k}$$

We define U_k inductively by using the partition ξ_{n_k} as follows. At each delayed basic step $n = n_k$ we have $h_n(1/2) \in \delta_n$ and the box construction with related staircases is preformed. Let N denote the power of h_n, and let

$$R = h_n^{-1}(\delta_n).$$

We define U_k as the union of the collection of N iterates

$$G(R), G^2(R), \dots, G^N(R) = \delta_n \cap \mathrm{Im}(h_n).$$

We wish to emphasize that U_k consists of the whole G-orbit of $h_{n_k}^{-1}(\delta_{n_k})$ all the way up to $\delta_{n_k} \cap \mathrm{Im}(h_{n_k})$. Again let $B = B_k$ denote the associated Johnson box with hat $H = H_k$. We will show below that if at each delayed basic step the hat is sufficiently small then we have $|U| < I$ and so $|E| > 0$.

Proposition 5.7.2. *There exists a sequence b_k such that if at each delayed basic case $|H_k| < b_k$ then $|E| > 0$.*

Proof. Let $n = n_k$ and $m = n_{k-1}$ be two consecutive delayed basic steps. In our construction we will have many basic steps in between. So

$$h_n = f_{n-1}^* \circ f_{n-2}^* \circ \cdots \circ f_m^* \circ h_m^{r_m},$$

where the branches f_i^* for $i = m, m+1, \dots, n-1$ are chosen in order to visit prescribed intervals of the generating partition $\mathcal{P}_m$, thus ensuring that orbit of the critical point is everywhere dense. By construction $h_m^{r_m}(1/2)$ is the first iterate of $h_m(1/2)$ that falls outside δ_m, i.e., $h_m(1/2) \in \mathcal{S}_{r_m}(\delta_m)$ — the r_m^{th} level of the staircase construction belonging to δ_m. Let

$$R = h_n^{-1}(\delta_n)$$

and let N_n be the power of h_n. Set $S = h_m^{r_m - 1}(R) \in \mathcal{S}_1(\delta_m)$ — the first level of the staircase belonging to δ_m. Then we decompose the orbit

$$U_k = R \cup G(R) \cup \cdots \cup (G^{N_n}(R) = \delta_n \cap \operatorname{Im} h_n)$$

into two blocks

$$\mathcal{B}_1 = R \cup G(R) \cup \cdots \cup S$$

$$\mathcal{B}_2 = G(S) \cup G2(S) \cup \cdots \cup (\delta_n \cap \operatorname{Im} h_n).$$

Clearly $R \subset \ h_m^{-1}\delta_m$ since $\delta_n \subset \delta_m$ because $n > m$. Consequently, $\mathcal{B}_1 \subset U_{k-1}$ and

$$|U_k - U_{k-1}| \leq |\mathcal{B}_2|.$$

The key point now, is that the number of iterates of S which make up the union in the second block $\mathcal{B}_2$ is **independent** of r_m, (remember that by construction $h_m^{r_m}(S) \subset \Delta_m^*$, irrespective of r_m). So if M denotes the power of h_m, then $\mathcal{B}_2$ consists of a union of

$$M + N(\Delta_m^*) + N(\Delta_{m+1}^*) + \cdots + N(\Delta_{n-1}^*)$$

G-iterates of S. It follows, by continuity that $\mathcal{B}_2$ can be made arbitrarily small provided $\delta_n \subset H_{k-1}$ is small enough, which in turn can be arranged by choosing r_m sufficiently large. Therefore at each delayed basic step, we can determine in advance the level of the staircase because the series of basic steps and the following Johnson box depends only on the location of $h_m^{r_m}(1/2)$ within Δ_m^* and not on r_m. Consequently, there exists a sequence b_k such that if $|H_k| < b_k$ then $|U| < |I|$ and $|E| > 0$. ∎

Let

$$\gamma_n = \min\{a_k, b_k\}$$

be the sequence of numbers for which to carry out the construction as specified in Section 5.5. Then

$$|H_k| < |H_k|/|B_k| < \gamma_k,$$

so that the hypotheses of Propositions 5.6.2 and 5.7.2 are both satisfied.

Our next step is to show that μE is finite to conclude that μ is σ-finite.

5.7.3 *The power of a branch through E*

In Chapter 4 we have shown, following [J-S], that if at every step n our map is basic or delayed basic, then φ is expansion inducing. That means that there exists a partition of I into a countable union of disjoint open intervals $\{\Delta_i\}_i$ such that the map f defined piecewise by $f_i = G^{N_i}|\Delta_i$ which maps Δ_i diffeomorphically onto I and satisfies the conditions of the *Folklore Theorem*. That is, f_i is expanding and has bounded distortion. Consequently f has an a.c.i.m. ν with density bounded away from zero and infinity. Using the *Tower construction* in Chapter 3 we obtain the G-invariant measure μ given by the formula

$$\mu(E) = \sum_i \sum_{j=0}^{N_i-1} \nu(\Delta_i \cap G^{-j}E)$$

Evidently, μ is absolutely continuous w.r.t. *Lebesgue Measure*. Since ν has a bounded density, $\mu(E)$ is finite iff

$$\Sigma := \sum_i \sum_{j=0}^{N_i-1} \left|\Delta_i \cap G^{-j}E\right| \tag{13}$$

converges, and μ is finite iff

$$\sum_i N_i|\Delta_i| < \infty. \tag{14}$$

We have already shown above that the series in (14) diverges so that the measure μ is infinite satisfying condition (A) from Section 5.2.4. Next we need to show that the second condition (B) from 5.2.4 is satisfied to ensure that μ is σ-finite. Recalling that condition , we will now prove:

(B) There exists a set E with positive Lebesgue Measure for which the sum in (13) converges.

Then by Theorem 2.2.2 in Chapter 2, property (B) implies the measure μ is σ-finite.

5.7.4. At each step n of induction when we construct ξ_n this involves creating new monotone domains inside the central domain δ_{n-1} and its preimages $\cup\delta_{n-1}^{-k}$, while the previous monotone domains remain unchanged. This means that we can calculate the contribution of the monotone domains to the sum in (11) as they are created at each step n. Now the contribution of a monotone domain Δ_i is at most $N_i|\Delta_i|$. By construction, $N_i = N(\Delta_i)$ is the number of intervals in the tower over Δ_i and for $k = 1, 2, \ldots, N_i - 1$ each iterate $G^k(\Delta_i)$ falls in some monotone domain created at a previous stage. In order to estimate the sum in (11) we calculate the contribution of each monotone domain Δ_i to the sum Σ given in formula (11). The maximum contribution of each monotone domain Δ_i is:

$$N(\Delta_i)|\Delta_i|$$

Note that, since we performed the Tower construction, using the first return map G we have that the power associated with one iterate of G is 1. That is, if

$$G|\Delta_i\colon \Delta_i \to I \qquad \text{where} \quad G|\Delta_i = G^{N_i}$$

Then $N(\Delta_i) = N_i$.

5.7.5. From the definition of the *exceptional* set E in 5.8.1, it follows that if we only count the intervals $G^k(\Delta_i)$ that intersect E and denote their number by $N_E(\Delta_i)$ we get that the sum Σ given by the formula (13) is majorized by

$$\sum_i N_E(\Delta_i)|\Delta_i| \tag{15}$$

Terminology: Recall that in terms of the tower construction, $N_E(\Delta_i)$ denotes the number of levels of the tower over Δ_i *that intersect the set* E. Let us consider the preliminary partition ξ_0. Since this partition consists of a finite number of intervals we can set

$$N_0 = \max \{ N(J) : J \in \xi_0 \}$$

In this sense then, we define the *power through* E *of* h_n as

$$N(h_n) := 1 + N_0^* + N_1^* + \cdots + N_{n-1}^* \qquad (N(h_0) = 1)$$

where $N_i^* = N_E(\Delta_i^*)$ is the *height through* E of the monotone domain that contains the critical value $h_i(1/2)$. Recall that after each basic step i of induction the new central branch

$$h_{i+1}\colon \delta_{i+1} \to I$$

is given by the composition

$$h_{i+1} = f_i^* \circ h_i$$

and in the delayed basic situation

$$h_{i+1} = f_i^* \circ h_i^{r_i}.$$

5.7.6. The essential feature of our construction, is that all new monotone domains at step $n+1$ are created inside the central domain δ_n and all (disjoint) preimages

$$\{\delta_i^{-p}\} \quad \text{where} \quad (i = 0, 1, \ldots, n+1) \quad \text{and} \quad (p \geq 0)$$

Then the additional contribution at step $n+1$ cannot exceed

$$N(\xi_n)\Big|\bigcup_{i,p} \delta_i^{-p}\Big|$$

where $N_E(\xi_n)$ is given by

$$\begin{aligned} N_E(\xi_n) &= \max_{\xi_n}\{\, N_E(J)\,\} \\ &= \max\{\, N_E(J) : J \in \xi_n \,\} \end{aligned}.$$

In the next section we determine the effect of boundary refinement on the growth of quantity $N_E(\xi_n)$.

Remark 5.7.7. The maximum height $N_E(J)$ is taken over all domains of $J \in \xi_n$, so in particular it applies to $J = \delta_n$. This means that

$$N_E(\delta_n) = N_E(h_n) \leq N_E(\xi_n)$$

Section 5.8. Properties of Boundary Refinement

5.8.1. As indicated in Section 5.3 we use the Parameter Choice Lemma 5.3.2 to enable us to control the trajectory of the critical point. By construction, all delayed basic steps lead to the construction of the Johnson box and related staircases. So assume for now, that we are dealing with two consecutive Johnson steps $n = n_k$ with box B_k and hat H_k and $m = n_{k+1}$ with box B_{k+1} and hat H_{k+1}. Remember that by choice of parameter, we can ensure that no boundary refinement is needed at Johnson steps, however we will usually have many basic steps

$$j = n + 1, n + 2, \ldots, m - 1.$$

in between that may create non-extendible monotone domains and so will require boundary refinement.

Let us first notice that it is still possible to carry out the construction of the main partitions ξ_n (and associated refinements used), even if we add the restriction that the maximum number of boundary refinements needed to make all elements of ξ_n extendible does not exceed, say, 2^n. Let us call the two intervals $\Delta_*^{(n)}$ and $\Delta_*^{(m)}$ that contain the critical values $h_n(1/2)$ and $h_m(1/2)$ as the initial landing element and the final landing element respectively. To achieve the bound 2^n we will use the following argument.

Assume at step n according to our itinerary we must visit certain domains, but it involves more than $M > 2^n$ refinements. Then we interrupt our itinerary and just pullback ξ_0 consecutively without doing any boundary refinement. More explicitly, The first monotone pullback of ξ_0 is onto $\Delta_*^{(m)}$, then the following pullback is onto the preimage of the element of ξ_0 inside the landing element $\Delta_*^{(n)}$ that contains the critical value. By parameter choice, we make sure that the critical value belongs to a preimage of a monotone domain. We do this procedure k-times, where $k > \log_2 m$. After that we return to our original predetermined itinerary that ends with the critical value in the final landing element $\Delta_*^{(n)}$.

5.8.2. The following argument shows that any choice for a position of the critical value at step n that is compatible with our rules for the Parameter Choice Lemma (see section 3), will still alow us to impose the bound 2^n on the maximal number of boundary refinements needed for any monotone domain belonging to ξ_n. Let us first define the following notations.

Notation 5.8.3.

(i) Suppose Δ, $\Delta_0 \in \xi_n$ are monotone domains and Δ_0 contains the critical value $x_0 = h_n(1/2)$. Then we let $R_n(\Delta, \Delta_0, x_0)$ denote the minimum number of boundary refinements needed for Δ in order that the monotone domains $h_n^{-1}(\Delta) \in \xi_{n+1}$ are uniformly extendible.

(ii) Let

$$R_n(\Delta_0, x_0) = \max_{\Delta \in \xi_n} \{ R_n(\Delta, \Delta_0, x_0) \}$$

(iii) Let

$$\xi_0 \prec \xi_1 \prec \cdots \prec \xi_\infty$$

be the partitions constructed in the course of our induction. Then

$$\xi_\infty = \lim_{n \to \infty} \xi_n$$

denotes the limit partition.

Remarks 5.8.4.

(i) If F is a diffeomorphism then

$$R_n(F^{-1}\Delta, F^{-1}\Delta_0, F^{-1}x_0) = R_n(\Delta, \Delta_0, x_0)$$

since extensions of preimages are preimages of extensions.

(ii) By the construction of enlargements in section 5, each hole δ^{-k} belonging to a given partition ξ has an enlargement $\tilde{\delta}^{-k}$ such that for all elements constructed as a result of filling in δ^{-k}, their extensions are totally inside $\tilde{\delta}^{-k}$. In other words, every preimage $\delta^{-k} \in \xi$ has an extension $\hat{\delta}^{-k}$ that is mapped onto the extension of the central domain $\hat{\delta}$ by the same diffeomorphism $\chi\colon \delta^{-k} \to \delta$. Furthermore, extensions of all elements obtained by the filling in of δ^{-k} are properly contained inside $\tilde{\delta}^{-k}$.

(iii) As a consequence of remark 2 above we obtain that for any fixed $x_0 \in \Delta_0$, where

$$\Delta_0 \subset (I - \bigcup \tilde{\delta}^{-k})$$

we have, for all elements $\Delta \subset \cup \delta^{-k}$, $R_n(\Delta, \Delta_0, x_0) = 0$ for all n.

(iv) Because of the choice of parameter, we have the property that for all i and k, $\tilde{\delta}_i^{-k} \cap \Delta_0 = \emptyset$. This means that the interval Δ_0 that contains the critical value is never adjacent to a preimage δ^{-k}. Rather, all such monotone domains are adjacent to other monotone domains Δ. In fact, if Δ_0 is adjacent to the enlargement $\tilde{\delta}_i^{-k}$ then Δ_0 must be adjacent to either Δ_r^{-k} or Δ_l^{-k} which are the preimages of the two boundary intervals of δ_{i-1}. Note that when $i = 0$ then Δ_0 would be in this case adjacent to one of the preimages of the two monotone domains $\Delta_K^{\pm}$ surrounding δ_0.

5.8.5. We now prove what we refer to as the *Boundary Refinement Lemma*. To simplify notation, we may assume $I = [0, 1]$.

The Boundary Refinement Lemma 5.8.6. *Suppose $f\colon [a, b] \to [0, 1]$ is an extendible monotone branch with $f(b) = 1$ and let $J = [b, d]$ be an interval that is adjacent to $[a, b]$. Let us consider the refinements of $[a, b]$ and let ζ_k be the boundary interval of the k^{th} refinement which is adjacent to b. Then* *there exists $k_0 = k_0(|J|)$ such that the extension of the boundary interval ζ_{k_0} is contained in J.*

Proof. By construction, since $f|[a,b]$ is extendible, there exists an extension $[b,c]$ such that $f|([a,b] \cup [b,c] = [0, 1+\alpha]$ where α is a fixed uniform constant. Indeed, $f([b,c]) = [1, 1+\alpha]$ and refinements of $[a,b]$ are refinements of $[0,1]$ by previously created partitions which are pulled back. Let Δ_k denote the boundary interval of the k^{th} partition which we pulled back, then its extension has the form $(1, 1+x_k]$ and their lengths decrease exponentially with k as we showed when we explained the boundary refinement procedure in chapter 4. So the intervals ζ_k are the preimages $f^{-1}(\Delta_k)$. Let ζ_k^u denote the upper extension of ζ_k then $\zeta_k^u = f^{-1}([1, x_k])$. Now, if the extension of $[a,b]$ is actually contained in J then we can set $k_0 = 0$. Otherwise we have two possibilities:

(1) If $f(J) \supset [1, 1+\frac{\alpha}{2}]$, then $f(J) \supset [1, x_{k_1}]$ for some fixed k_1. So, we can set $k_0 = k_1$ in this case.

(2) If $f(J) \subset [1, 1+\frac{\alpha}{2}]$, then $f|([a,b] \cup J)$ is uniformly extendible since the image of the extension of $f|[a,b]$ contains $[1, 1+\alpha]$. Consequently, $f|([a,b] \cup J)$ has uniformly bounded distortion. As $f([a,b]) = [0,1]$, there exists some constant $c_0 = c_0(|J|)$ such that $|f(J)| > c_0$. Therefore, $f(J) \supset [1, x_{k_2}]$ for some fixed $k_2 = k_2(|J|)$ and we can set $k_0 = k_2$ in this case.

This concludes the proof of the Lemma. ∎

Suppose ξ is a partition with the critical value $x_0 = h(1/2)$ contained in $\Delta_0 \in \xi$. Also assume

$$\Delta_a \subset \text{ Image}\,(h)$$

is the monotone domain adjacent to Δ_0. Then, using the boundary refinement lemma we get the following corollary:

Corollary 5.8.7. *If $\Delta \neq \Delta_a$ belongs to ξ and requires boundary refinement, then we will need no more than $k_0|\Delta_a|$ steps of boundary refinements.*

In view of Remark (iv) in 5.8.4 we have

Lemma 5.8.8. *If $\Delta_0 \in \xi_n$ and $x_0 \in \Delta_0$ as above then*

$$\max_{\Delta \in \xi_\infty} \{\, R_n(\Delta, \Delta_0, x_0) \,\} = \max_{\Delta \in \xi_{n+1}} \{\, R_n(\Delta, \Delta_0, x_0) \,\}$$

Proof. It suffices to observe that all monotone domains created after step n are inside the holes of ξ_{n+1}. ∎

Using that

$$|\bigcup_{\Delta \in \xi_\infty} \Delta| = 1$$

we can now prove

Proposition 5.8.9. *Let $m > 0$ be fixed. Then*

$$\lim_{n} |\{ x_0 \in \Delta_0 : R_m(\Delta_0, x_0) < n \}| = 1.$$

Proof. For a given $\Delta_0 \in \xi_m$ we have

$$R_m(\Delta, \Delta_0, x_0) < k_0(\Delta_a)$$

for all Δ non-adjacent to Δ_0. As for the adjacent interval Δ_a the number of boundary refinements is finite for any fixed x_0 inside the interior of Δ_0 and goes to ∞ as x_0 approaches the common boundary between Δ_0 and Δ_a. However,

$$\lim_{n \to \infty} \frac{|\{ x_0 : R_m(\Delta_a, \Delta_0, x_0) > n \}}{|\Delta_0|} = 0$$

Hence, for every finite union U of intervals Δ_0 and every union V of open subintervals of Δ_0 that is separated from the boundary points of Δ_0 and has relative measure (in U) close to 1, there exists an n such that

$$\max_{\Delta_0, x_0 \in V} R_m(\Delta_0, x_0) < n,$$

proving the proposition. ∎

Section 5.9. Growth of the N_E

5.9.1. We remind the reader that after the construction of the limit partition $\xi = \xi_\infty$ of $I = \cup\Delta_i$ (mod 0) where each Δ_i is the monotone domain for the branch

$$f_i = G^{N_i}|\Delta_i$$

mapping Δ_i *onto* I, we showed in a previous section that hypothesis of the Folklore Theorem were satisfied, which produced an invariant measure ν for the power map T. Then the desired G-invariant measure μ was defined by the formula

$$\sum_i \sum_{j=0}^{N_i-1} \nu(\Delta_i \cap G^{-j}E)$$

for any measurable set E. However, as we showed in Theorem 2.2.2, the measure μ is σ-finite if there is a particular set E with nonzero measure that has **finite** μ-measure. Since the density of ν is bounded away from zero and infinity, the sum above representing the μ measure of E is finite iff

$$\sum_i \sum_{j=0}^{N_i-1} \left|(\Delta_i \cap G^{-j}E)\right| < \infty \tag{16}$$

The fact that $|E| > 0$ was shown previously in Proposition 5.7.2, so in this section we show that μE is finite by proving that the condition in (16) holds.

5.9.2. Let us start with the preliminary partition constructed at step zero. Recall that we used the partition induced by the *first return map* G on I. We chose a parameter interval very close to the Chebychev value 4 which ensured that δ_0 was as small as desired. Then we performed monotone pullback on each monotone domain until all sizes became less than an arbitrary given ε. This finite refinement was denoted by ξ_0:

$$\xi_0 : I = (\cup_i \Delta_i) \cup (\cup_k \delta_0^{-k}) \cup \delta_0$$

where all elements have sizes less than an arbitrary given ε, in fact, the whole union of preimages $|\cup_k \delta_0^{-k}| < \varepsilon$. So, we can assume that the measure of $\cup_k \delta_0^{-k}$ is less than $c|\delta_0|$ for some constant c independent of δ_0. At a basic step, we first construct by critical pullback the partition of δ_{n-1} with the new central domain δ_n located in the middle of δ_{n-1}.

5.9.3. At a delayed basic step $n = n_k$, we have the infinite staircase construction and the Johnson box. If we only count the holes that intersect the set E, we need to consider just the intervals inside the first step of the staircase $\mathcal{S}_1$. All other iterates are in "deleted" intervals, whose union was denoted by U_k in 5.7.1, and $E \subset (I - U_k)$.

Let us recall the sum in (16) converges provided

$$\sum_{\Delta \in \xi_n} N_E(\Delta)|\Delta| < \infty \tag{17}$$

Since the refinements ξ_n have the property that once a uniformly extendible monotone domain is created it is never changed, it follows that all new monotone domains come from the critical pullback into the central domain and then from the subsequent filling in procedure. So to calculate the sum in (17) we will estimate the contribution at each step n due to the these procedures.

5.9.4. Let

$$N_E(\xi_n) = \max_{J \in \xi_n} N_E(J)$$

As we previously noted in Remark 5.8.7, the maximum is taken over all elements J of $\$\xi_n$ including δ_n so

$$N_E(\mathcal{H}_n) := N_E(\delta_n) \leq N_E(\xi_n)$$

Since the preliminary partition ξ_0 is a finite partition, we set

$$N_0 = N_E(\xi_0)$$

(see lemma 3.5.2 in chapter 3). Then

Proposition 5.9.5.

$(a)_n$ $$N_E(\xi_n) < N_0 5^n$$

Proof. Clearly $(a)_0$ holds. Now, assume by induction that $(a)_n$ and let us consider the partition ξ_{n+1}. As

$$h_{n+1} = f_n^* \circ h_n$$

we have by critical pullback

$$N_E(h_{n+1}) \le N_E(h_n) + N_E(\xi_n)$$
$$< N_0 5^n + N_0 5^n \tag{18}$$

Consequently, for new elements Δ and δ_i^{-p} of the partition ξ_{n+1} created inside δ_n by critical pullback, we have

$$\max_{J \in \delta_n} N_E(J) < 2N_0 5^n + 2^n + N_0 \tag{19}$$

The number 2^n represents the maximum number of boundary refinements needed to make all new elements of ξ_{n+1} extendible, as well as the maximum number of grow-up steps that may be needed if the critical value falls in one of the boundary domains $\Delta_{l(r)}$ of ξ_0 during a basic step, (see 5.8.2). We also added an extra term N_0 in estimate (19) to take into account that during a Johnson step we may need to insert an extra pullback of ξ_0 to ensure that $|\delta_n|/|H_n|$ is small as desired. Similarly, at a basic step, to ensure that the ratio

$$\frac{|\delta_{n+1}|}{|\delta_n|} \le \varepsilon_1$$

is small, we may need to perform one Extra Pullback Procedure, as explained in the basic construction in Section 4.1. Finally, when we do the filling-in procedure, we add one more term $N_E(\xi_n)$ due to the orbit from δ_i^{-p} to δ_i to obtain from (19)

$$N_E(\xi_{n+1}) \leq 3N_0 5^n + 2^{n+1} + N_0$$

$$< N_0 5^{n+1}$$

which proves $(a)_{n+1}$ as required. ∎

Section 5.10. Contribution of Holes at step $\mathbf{n+1}$

5.10.1. As in the previous section we start with the preliminary partition at step zero

$$\xi_0 : I = (\cup_i \Delta_i) \cup (\cup_k \delta_0^{-k}) \cup \delta_0$$

where we know that the total Lebesgue measure of all the holes is at most

$$| \cup_k \delta_0^{-k}| \leq \varepsilon$$

where $\varepsilon > 0$ is chosen at will when we construct the preliminary partition in accordance with lemma 3.5.2 of chapter 3. In addition, clearly

$$|\delta_0 < | \cup_k \delta_0^{-k}| \leq \varepsilon$$

Let us consider the central domain δ_i, for $i = 1, 2, ldots, n$. If we consider δ_i as a hole without any partition, then its contribution to the sum in (17) at step $i+1$ is at most

$$N(\xi_{i+1})|\delta_i| \leq N_0 5^{i+1} |\delta_i| \tag{20}$$

by proposition 5.9.5.

5.10.2. Let us now estimate the contribution to (17) from the elements constructed inside the preimages δ_i^{-p} created by the filling in procedure. Suppose $\Delta, \delta_j^{-k} \subset \delta_i^{-p}$ are elements obtained by filling in δ_i^{-p}. Then we can subdivide the orbit of these elements into two segments. The first segment consists of the trajectory of δ_i^{-p} until they reach δ_i, the second segment then follows the orbit of the elements inside δ_i that are constructed at step $i+1$. Since we accounted for the first segment at step n, because we counted the contribution of the hole δ_i^{-p} without any partition, we need to account for the second segment of that orbit. That contribution does not exceed

$$N_E(\xi_{i+1})\Big(\sum_{\delta_i^{-p}\in\xi_n} |\delta_i^{-p}|\Big) \tag{21}$$

Since $i \leq n$, we get from Proposition 5.9.5

$$N_E(\xi_{i+1}) \leq N_0 5^{n+1}$$

and consequently estimate (21) is at most

$$N_0 5^{n+1}\Big(\sum_{\delta_i^{-p}\in\xi_n} |\delta_i^{-p}|\Big) \tag{22}$$

Now we just note that adding (20) and (22) can be done by simply including δ_i in the enclosed sum of (22). Therefore, the total contribution to the sum in (17) at step $n+1$ due the preimages δ_i^{-k} for $i = 0, 1, 2. \ldots, n$ and $p \geq 0$ does not exceed the estimate in (22). In the next section we will prove

Proposition 5.10.3.

$$\sum_{\delta_i^{-p} \in \xi_n} |\delta_i^{-p}| < a^i b^n s_0 \tag{23}$$

where $a = a(\delta_0)$, $b = b(\delta_0)$ and $s_0 = s_0(\delta_0)$ all tend to zero with δ_0.

Then we obtain

$$\begin{aligned} \sum_{i=0}^{\infty} a^i b^n s_0 &\leq \frac{1}{1-a} s_0 b^n \\ &\leq 6^{-n} \end{aligned} \tag{24}$$

provided a, b and s_0 are sufficiently small. Combining equations (22) and (24) proves the convergence of the sum of the new contributions added and respectively the sum in formula (17).

Section 5.11. Estimating the measure of holes $\cup\delta_{\mathbf{i}}^{-\mathbf{k}}$ inside $\xi_{\mathbf{n}}$

5.11.1. In our general construction of the refinements ξ_n the central branch $h\colon \delta \to I$ can be written as a composition

$$h(x) = F \circ Q(x)$$

where $Q(x)$ is the standard quadratic map and F is a composition of monotone domains with uniformly bounded distortion. For the quadratic map $Q(x)$ we know that, if $J \subset \delta$ are both symmetric intervals containing the critical point, then

$$\frac{|J|}{|\delta|} = \sqrt{\frac{|Q(J)|}{|Q(\delta)|}}.$$

Since F has bounded distortion we obtain for similar intervals J and δ

$$|J| < c|\delta|\sqrt{\frac{|h(J)|}{|h(\delta)|}}$$

Notice that, in view of the grow up procedure, the image of the central branch covers at least a constant length I_0. So we may write

$$|J| < c|\delta|\sqrt{|h(J)|} \tag{25}$$

where c is another uniform constant.

5.11.2 Let the total measure of the holes $\cup\delta_i^{-k}$ that belong to ξ_n be denoted by

$$\alpha_i^{(n)} = \left|\bigcup_{\xi_n} \delta_i^{-k}\right| \tag{26}$$

We note that the measure of this union is the sum of its respective disjoint elements which are created after we complete step n and the partition ξ_n has been constructed. To estimate the relative measure of the holes created inside δ_n as a result of the critical pullback procedure, we assume the worst position of these holes. By that, we mean that we assume that all the holes are contiguous with one end being bounded by the critical value $w = h_n(1/2)$. Let $M_i^{(n+1)}$ denote the measure of the union of all preimages of δ_i created inside δ_n at step $n+1$. For $i < n+1$ by the inequality (25) we have the following estimate

$$\frac{M_i^{(n+1)}}{|\delta_n|} < c\sqrt{\alpha_i^{(n)}} \tag{27}$$

with c being another uniform constant. This obviously gives us the worst (maximum) estimate on their relative measure inside δ_n.

For $i = n+1$ we get in the basic case

$$M_{n+1}^{(n+1)} = \delta_{n+1} < \beta\delta_n \tag{28}$$

where β is a small constant depending on the maximal size of elements in ξ_0.

5.11.3. We get estimates (27), (28) at basic steps. At a Johnson step, the estimate (27) still holds for preimages created along the first step of the staircase. For subsequent preimages we prove

Lemma 5.11.4. *All preimages located inside all other steps of the staircase together with the box, which are all inside δ_n, have total measure less than $c_1|\delta_n|^{3/2}$.*

Proof. Let $h_n = F \circ Q$, where Q is the initial quadratic map. Write

$$J = \delta_n - \mathcal{S}_1 \tag{29}$$

Then by construction $h_n(J) = \delta_n$, and as we argued in Section 5.11.1 we have

$$\frac{|J|}{|\delta_n|} = \sqrt{\frac{|Q(J)|}{|Q(\delta_n)|}} \tag{30}$$

As F has uniformly bounded distortion we obtain that

$$|J| < c|\delta_n|\sqrt{\frac{|h_n(J)|}{|h_n(\delta_n)|}} \tag{31}$$

for a uniform distortion constant c. Using that $h_n(\delta_n)$ covers more than 1/2 the length of I and $h_n(J) = \delta_n$ we obtain

$$|J| < c_1|\delta_n|^{3/2} \tag{32}$$

for some other uniform constant c_2 as desired. ∎

By a similar argument at Johnson step

$$M_{n+1}^{(n+1)} < c_1|\delta_n|^{3/2} \tag{33}$$

because all new preimages of δ_{n+1} are inside the box contained in $\delta_n - \mathcal{S}_1$. In fact, the preimages of δ^{n+1} are all contained in the box and since the tip of the hat can be chosen arbitrarily small,

$$|h_n(B)| = |B|(1+\varepsilon)$$

for arbitrarily small ε. Therefore, taking $J = B$ in (31) we obtain

$$\begin{aligned} |B| &< c|\delta_n|\sqrt{\frac{|h(B)|}{|h(\delta)|}} \\ &< c|\delta_n|\sqrt{|B|(1+\varepsilon)} \end{aligned}$$

which shows that the box is of order $|\delta_n|^2$. Therefore, instead of (33) we have the stronger estimate

$$M_{n+1}^{(n+1)} < c_1|\delta_n|^2 \tag{34}$$

So, in Johnson's case, we get (33) (or even (34)) when $i = n+1$, and for $i \leq n$ the measure of the union of all preimages of δ_i created inside δ_n at step $n+1$ is at most

$$\begin{aligned} M_i^{(n+1)} &< c|\delta_n|\sqrt{\alpha_i^{(n)}} + c_1|\delta_n|^{3/2} \\ &\leq c|\delta_n|\sqrt{\alpha_i^{(n)}}(1 + \frac{c_1}{c}|\delta_n|^{1/2}) \\ &< c_2|\delta_n|\sqrt{\alpha_i^{(n)}} \end{aligned} \tag{35}$$

since

$$(1 + \frac{c_1}{c}|\delta_n|^{1/2}) \leq c_2$$

for some uniform constant c_2 because

$$|\delta_n| \leq \beta^n |\delta_0| < \varepsilon$$

So (27) holds in all cases with some uniform constant. We keep the same notation c for this constant as well.

5.11.5. When doing filling in of a hole δ_j^{-p} we pullback the structure of δ_j that was created by critical pullback at step $j+1$. So we handle this at step $j+1$ as we did above at step $n+1$, i.e., we get

$$M_i^{(j+1)} < c|\delta_j|\sqrt{\alpha_i^{(j)}} \qquad (36).$$

Then we pullback with small distortion onto the preimage δ_j^{-p} and obtain inside each preimage δ_j^{-p} new preimages δ_i^{-k} with measure less than

$$c|\delta_j^{-p}|\sqrt{\alpha_i^{(j)}} \qquad (37)$$

Notice that we consider $j \geq i-1$ because preimages of δ_i can only appear at steps $i, i+1, \ldots, n+1$. Taking the union over all preimages δ_j^{-p} for $j =$

$i-1, i, \ldots, n$ we get that at step $n+1$ the total measure of all preimages δ_i^{-r} appearing after filling in all preimages δ_m^{-k}, $(m = i, i+1, \ldots, n)$, is at most

$$c \sum_{m=i}^{n} \alpha_m^{(n)} \sqrt{\alpha_i^{(m)}} \tag{38}$$

While when $m = i-1$ (which occurs only at a Johnson step) we get from (33) a much smaller estimate than (28) which holds at a basic step. We note that when we pullback δ_{i-1} onto its preimage then

$$\frac{|\delta_i|}{|\delta_{i-1}|} \leq \beta$$

so that the ratio of preimages

$$\frac{|\delta_i^{-k}|}{|\delta_{i-1}^{-k}|} \leq (1+\varepsilon)\beta$$

for a very small ε because the process of enlargements implies that the diffeomorphisms

$$\chi : \delta_{i-1}^{-k} \to \delta_{i-1}$$

have small distortion. So we may use the same factor β for these preimages as well. Therefore, we conclude from (27) (28) and (38) that at step $n+1$, we obtain

$$\alpha_i^{(n+1)} < \beta\alpha_{i-1}^{n} + c_1(|\delta_n|\sqrt{\alpha_i^{(n)}} + \sum_{j=i}^{n} \alpha_j^{(n)} \sqrt{\alpha_i^{(j)}}) \tag{39}$$

for a some small constant β and another uniform constant which we may keep denoting as c_1.

5.11.6. Now, we prove a restatement of Proposition 5.10.3

Proposition 5.11.7. *There exists small positive constants s_0, a and b, such that for all ≥ 0 and all $i \leq n$ we have*

$$\Gamma_{(i,n)} \qquad \alpha_i^{(n)} < a^i b^n s_0$$

Moreover, one can choose s_0, a and b that tend to zero as $|\delta_0| \to 0$.

Proof. We may assume that δ_0 is small enough — to be specified below. Recall that by 4.2.4 we have

$$\frac{|\delta_{i+1}|}{|\delta_i|} < \beta$$

where β is small because its size is determined by the choice of parameter when constructing the preliminary partition, and according to Lemma 3.5.2 the elements of the preliminary partition ξ_0 can be as small as desired. Also, as we noted before in (34), we have

$$|\delta_{i+1}| < c|\delta_i|^2$$

when i is a delayed basic step. We assume $|\delta_0| \ll \beta$, then the estimate $|\delta_{i+1}|/|\delta_i| < \beta$ holds for all $i \leq n$ at every step n of our construction. Consequently, in our estimates below, we use that

$$|\delta_i| \leq \beta^i |\delta_0| \tag{40}$$

and

$$\alpha_i^{(i)} \leq \delta_i$$
$$\leq c\beta^i|\delta_0| \quad (41)$$

Since δ_0 can be chosen as small as desired. We choose s_0 such that $|\delta_0| \ll s_0$, say $|\delta_0| < s_0^2$. In addition , we choose small constants $a = \beta^x$ and $b = \beta^y$ for some positive exponents $x, y < 1/2$ such that $ab > 3\beta$ and $b^3 < a$ Thus, combining all the above, we will use in our estimates below the following inequalities

$$|\delta_0| < s_0^2$$
$$b^3 < a$$
$$\beta < \frac{1}{3}ab, \quad \beta^n < b^{2n}, \quad \beta^i < a^{2i} \quad (42)$$

In order to check the basis of induction we note that the union of preimages δ_0^{-k} contained in the preliminary partition ξ_0 has measure less than $c|\delta_0|$, therfore by (42)

$$\alpha_0^{(0)} < c|\delta_0| < cs_0^2$$

Let us first check when $i = 0$, in this case (39) becomes

$$\alpha_0^{(n+1)} < c_1\Big(|\delta_0|\beta^n\sqrt{s_0}b^{n/2} + \sum_{m=0}^{n-1} s_0a^m b^n\sqrt{s_0}b^{m/2}\Big)$$
$$< c_1\sqrt{s_0}\left[\delta_0 b^{n/2}\beta^n + s_0\big(b^n(1 + ab^{1/2} + a^2b + \ldots)\big)\right] \quad (43)$$

If a and b are small then the sum of geometric progression

$$1 + ab^{1/2} + a^2 b + \dots$$

is close to 1, and we then have

$$\alpha_0^{(n+1)} \le c_1 \sqrt{s_0} \big[\delta_0 b^{n/2} \beta^n + (1+\varepsilon_1) s_0 b^n \big] \tag{44}$$

where c_1 does not depend on δ_0 nor on β. Also, we can arrange that the elements of the initial partition are small enough (as shown in the construction of the preliminary partition) to ensure that

$$c_1 \sqrt{s_0} < b/10 \ll 1 \tag{45}$$

Then, by using $\beta^n \le b^{2n}$ in (42), the estimate in (44) then becomes

$$\begin{aligned} \alpha_0^{(n+1)} &\le \frac{1}{10} \Big(|\delta_0| b^{n/2} b^{2n+1} + (1+\varepsilon_1) s_0 b^{n+1} \Big) \\ &< s_0 b^{n+1}/5 \end{aligned}$$

which proves formula $\Gamma_{(0,n+1)}$.

Now we assume by induction that $\Gamma_{(i,n)}$ holds for all $i \le n$. Then for all $i = 1, 2, \dots, n$ we get from (39), using $\beta < \frac{1}{3} ab$ and $\beta^n < b^{2n}$ from (42), we get

$$\begin{aligned} \alpha_i^{(n+1)} &< \beta a^{i-1} b^n s_0 + c_1 \Big[|\delta_0| \beta^n \sqrt{s_0} a^{i/2} b^{n/2} + \sum_{j=i}^{n} s_0 a^j b^n \sqrt{s_0} a^{i/2} b^{j/2} \Big] \\ &\le \frac{1}{3} s_0 a^i b^{n+1} + s_0 a^{\frac{i}{2}} b^n \Bigg[c_1 s_0 b^{\frac{3n}{2}} + c_1 \sqrt{s_0} \Big(\sum_{j=i}^{n} a^j b^{j/2} \Big) \Bigg] \end{aligned} \tag{46}$$

now let us consider the term in the square brackets, it consists of the sum of

$$c_1 s_0 b^{\frac{3n}{2}} \tag{47}$$

and

$$c_1 a^i b^{\frac{i}{2}} \sqrt{s_0} \left(\frac{1}{1 - ab^{\frac{1}{2}}} \right) \tag{48}$$

For the term in (47), we have

$$c_1 s_0 b^{\frac{3n}{2}} \leq \frac{1}{3} a^{\frac{i}{2}} b \tag{49}$$

for all $i \leq n$ since by (45)

$$c_1 \sqrt{s_0} < b/10$$

and by (42) $b^3 < a$.

As for the term in (48), we again use $c_1 \sqrt{s_0} < b/10$ given in (45) to obtain the estimate

$$\begin{aligned} c_1 a^i b^{\frac{i}{2}} \sqrt{s_0} \left(\frac{1}{1 - ab^{\frac{1}{2}}} \right) &\leq a^{\frac{i}{2}} b \left[\frac{1}{10} \left(\frac{a^{\frac{i}{2}} b^{\frac{i}{2}}}{1 - ab^{\frac{1}{2}}} \right) \right] \\ &\leq \frac{1}{3} a^{\frac{i}{2}} b \end{aligned} \tag{50}$$

because clearly

$$\frac{1}{10} \left(\frac{a^{\frac{i}{2}} b^{\frac{i}{2}}}{1 - ab^{\frac{1}{2}}} \right) \leq \frac{1}{3}$$

Therefore, combining (49) and (50), the estimate in (46) becomes

$$\begin{aligned} \alpha_i^{(n+1)} &\leq \frac{1}{3} s_0 a^i b^{n+1} + \frac{1}{3} s_0 a^i b^{n+1} + \frac{1}{3} s_0 a^i b^{n+1} \\ &\leq s_0 a^i b^{n+1} \end{aligned}$$

proving formula $\Gamma_{(i,n+1)}$ for all $i \leq n+1$. ∎

Remark 5.11.8. When $i = 0$ we have the strongest restrictions on the smallness of s_0 and δ_0.

As shown in (24) at the end of 12.2, this Proposition implies

$$\sum_{i=0}^{\infty} a^i b^n s_0 < \frac{1}{1-a} s_0 b^n < 6^{-n}$$

which when combined with (22) shows that the sum Σ converges and μ is therefore a σ-finite a.c.i.m. Moreover, the density of the orbit of the critical point, forces every interval to have infinite μ-measure:

Main Theorem. *There are maps from among the family φ_t that admit no finite a.c.i.m. but that have a σ-finite a.c.i.m. measure that is infinite on every interval.*

Proof. It remains to show the last assertion. Let J be any interval in I, by construction there is an n such that $h_n(c)$ passes through J and hence there is a k_0 such that the box B_{k_0} passes through J. This implies that the contribution of the intervals created inside B_k to the summation formula corresponding to $\mu(J)$ exceeds $1/2$ for every $k \geq k_0$. So the sum corresponding to $\mu(J)$ diverges and $\mu(J) = \infty$. ∎

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