ABSTRACT

Title of dissertation: ESSAYS ON FINANCIAL CONSTRAINTS, R&D INVESTMENTS, AND COMPETITION

Danmo Lin, Doctor of Philosophy, 2016

Dissertation directed by: Professor Richmond Mathews
Professor Mark Loewenstein
Department of Finance
Robert H. Smith School of Business

This dissertation consists of two chapters of theoretical studies that investigate the effect of financial constraints and market competition on research and development (R&D) investments. In the first chapter, I explore the impact of financial constraints on two different types of R&D investments. In the second chapter, I examine the impact of market competition on the relationship between financial constraints and R&D investments.

In the first chapter, I develop a dynamic monopoly model to study a firm's R&D strategy. Contrary to intuition, I show that a financially constrained firm may invest more aggressively in R&D projects than an unconstrained firm. Financial constraints introduce a risk that a firm may run out of money before its project bears fruit, which leads to involuntary termination on an otherwise positive-NPV project. For a company that relies on cash flow from assets in place to keep its R&D project alive, early success

can be relatively important. I find that when the discovery process can be expedited by heavier investment ("accelerable" projects), a financially constrained company may find it optimal to "over"-invest in order to raise the probability of project survival. The over-investment will not happen if the project is only "scalable" (investment scales up payoffs). The model generates several testable implications regarding over-investment and project values.

In the second chapter, I study the effects of competition on R&D investments in a duopoly framework. Using a homogeneous duopoly model where two unconstrained firms compete head to head in an R&D race, I find that competition has no effect on R&D investment if the project is not accelerable, and the competing firms are not constrained. In a heterogeneous duopoly model where a financially constrained firm competes against an unconstrained firm, I discover interesting strategic interactions that lead to preemption by the constrained firm in equilibrium. The unconstrained competitor responds to its constrained rival's investment in an inverted-U shape fashion. When the constrained competitor has high cash flow risk, it accelerates the innovation in equilibrium, while the unconstrained firm invests less aggressively and waits for its rival to quit the race due to shortage of funds.

ESSAYS ON FINANCIAL CONSTRAINTS, R&D INVESTMENTS, AND COMPETITION

by

Danmo Lin

Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2016

Advisory Committee:
Professor Richmond Mathews, Chair
Professor Mark Loewenstein, Co-chair
Professor Albert S. "Pete" Kyle
Professor Laurent Fresard
Professor Liu Yang
Professor Daniel Vincent

Dedication

To my father, Mingdai Lin, and my mother, Yaming He.

Acknowledgments

I owe my utmost gratitude to my advisors, Professor Richmond Mathews and Professor Mark Loewenstein. This dissertation would not have been possible without their guidance. Their confidence in me and their patience with me from the very beginning were far beyond what I thought I could possibly ask for. Rich gave me all the freedom to explore my research interests, and provided excellent guidance every step of the way in a caring manner. It has been a privilege to learn corporate finance theories rigorously from him, listen to him talk about the fine details of this profession, read his careful notes on several versions of this dissertation and my job market paper, among many other things. Mark has been unbelievably generous with his time. He taught me the importance of thinking hard on the economics for any research project. I have always admired Rich and Mark for their knowledge, their wisdom, and their passion for research. I feel I may only claim to be a true student of theirs by working very hard in the future. The time we spent together talking about my dissertation in the conference room and in their offices will be my very fond memory forever.

I am also indebted to my committee members Professor Albert "Pete" Kyle, Professor Liu Yang, and Professor Laurent Fresard. They offered invaluable advice to my research, gave me timely feedback at different stages of my dissertation, and talked about various research agenda with me. Special thanks go to Professor Daniel Vincent for serving on my dissertation committee as an external member. I appreciate the opportunity to learn industrial organization theories from him at the beginning of my Ph.D.

I am very thankful to Professor Gerard Hoberg and Professor Nagpurnanand Prabhala, who introduced me to the research of corporate finance and have always been generous to share their perspectives and advice. I benefited enormously from discussions with Vojislav Maskimovic, Cecilia Bustamante, Julien Cujean, Michael Faulkender, Russ Wermers, Shri Santosh, Wei Li, Wen Chen, Su Li, Ben Munyun, and Austin Starkweather. I also appreciate the job market advice from Francesco D'Acunto, Alberto Rossi, and William Mullins. In addition, I want to thank our Ph.D. Program Coordinator, Justina Blanco, all my fellow Ph.D. students, Ben Zou, Ceren Yilmaz, and Dustin Scown for their generous support and friendship.

I owe my deepest thanks to my father Mingdai Lin and my mother Yaming He, for always having faith in me and being there for me. I am truly grateful to my loving husband, Jian Feng. Without your encouragement and understanding, I would not be able to get this far. Thank you to Henry, my toddler son, for making the last two years of the Ph.D. journey more joyful and meaningful. Last but not least, I wish to extend my very special thanks to Anne Peterson and Sol Satin for their unconditional love and support throughout my time at Maryland.

Contents

List of Tables

List of Figures

1 CHAPTER 1: A MONOPOLY MODEL OF R&D INVESTMENT AND FINANCIAL CONSTRAINTS

1.1 Introduction

Innovation has long been identified as the key driver of economic growth. The majority of modern innovation is the result of research and development effort carried out through private and public entities. The financial status of those entities can be an important determinant of R&D activities. After decades of research effort, it is thus surprising how little we understand, both theoretically and empirically, the R&D behavior of the value maximizing private sector. In particular, how financial constraints, financial market development, and public policy may alter that strategy.

This chapter sets out to answer the following questions from a theory perspective: How do a firm's financial constraints affect its decision to initiate an R&D project? How do firms choose optimal investment scales for innovative projects? More specifically, ceteris paribus, do financially constrained companies always invest less than unconstrained companies? If not, what are the mechanisms that may cause over-investment from unconstrained firms? How does an R&D project's expected payoff structure matter for this relationship?

This chapter focuses on the comparison between two types of firms: a financially constrained monopoly (FC hereafter) and a financially unconstrained monopoly (UC hereafter). The former type has infinitely high cost of capital from the financial markets, which effectively prevents it from accessing outside capital; the latter can finance its projects costlessly in external capital market. This exogenous difference in firms' external cost of capital implicitly assumes FC firms have innate and persistent asymmetric information problems or agency problems, which raise their effective cost of external financing relative to UC firms. These problems, however, do not play any role beyond the ex ante separation of FC and UC firms.

A continuous time stochastic dynamic model is used to analyze firms' optimal R&D strategies in a risk neutral setting. R&D projects are modeled by a cash flow process, which consists of a continuous cash outflow in each development period and an expected one time cash inflow from a successful discovery at an uncertain time. I assume no inter-period internal capital accumulation. With this assumption, the implication of financial constraints on project development is that internal cash flow from a firm has to cover its R&D effort in every instance. The moment that the cash flow drops below the required outflow, the R&D project needs to be terminated, but the firm could remain in operation. Once an R&D project is started, the firm commits to run the project by investing consistently each period until one of the two things happens: the project pays out; or the firm voluntarily or involuntarily abandons the project. Two types of R&D projects are discussed: scalable projects for which the expected payoff scales up with a larger investment size; and accelerable projects for which the success intensity increases in investment size. I distinguish these two features of R&D projects to get a clear idea of how constraints affect firms' investment decisions.

When making the decision about R&D investment scale, a firm weighs the benefit of a better and/or sooner R&D outcome versus the cost to run a larger project and the loss from a forced project abandonment. In a monopoly setup where only one firm has an R&D opportunity, a UC firm does not face any termination risk and invests at first best level. Financial constraints add an option-like term in the project value. If the R&D project is not accelerable, an FC firm optimally under-invests relative to first best because financial constraints introduce a marginal cost through option expiration. If the R&D project that the FC firm invests in is accelerable, enlarging investment relative to first best still imposes a marginal cost, but it may also provide a marginal benefit of saving the option value through earlier success. Thus, the FC firm may over-invest or under-invest depending on the possibility to retain the project by speeding it up and the characteristics of assets in place cash flows.

Unlike the indefinite effect of financial constraints on investment scale, a higher catastrophe risk on assets in place cash flows always induces an FC firm to invest more aggressively, and it effectively raises the discount rate for future cash flows. The possibility of a catastrophic event makes the consequence of a larger investment on forced project abandonment less severe because the adverse shock causes an instant project termination anyway. It reduces the marginal cost of additional investment more than it reduces the marginal benefit and leads to a higher investment. On the contrary, a larger volatility on assets in place cash flows reduces the option price, and makes it optimal to choose a lower R&D investment.

The idea that financial constraints are associated with over-investment is not in itself new. For example, two recent papers, Bena (2008) and Phillips and Zhdanov (2013), both show that result. It emerges in the context of competition in the first paper. The model in the latter one further adds existence of an active acquisition market. Our model shows the over-investment result without any force of competition among firms.

My proposed mechanism through the tradeoff between different timing risk has not been explored before, and it is the main contribution of the paper.

In addition, I made some efforts to explore the full spectrum of firms, which include those that have access to costly external financing (CEF hereafter). Not surprisingly, CEFs choose an investment level between that of UC and FC firms, and their abandonment thresholds are also between those two polar cases. Consistent with intuition, the abandonment threshold increases with the cost of external finance.

The chapter is organized as follows: Section 1.2 discusses related literature, Section 1.3 sets up the benchmark R&D investment model for constrained and unconstrained firms, Section 1.4 studies the optimal investment strategy in scalable R&D projects, Section 1.5 investigates R&D behavior in accelerable projects. Section 1.6 considers a few extensions for the benchmark model. Section 1.7 concludes.

1.2 Literature Review

This chapter is closely related to the literature on optimal R&D investment policy in an uncertain economic environment. Berk, Green, and Naik (2004), as a representative recent work, studies R&D investment policy and risk premium along the life cycle of innovative projects. In their model, a successful innovative project requires completion across multiple stages, and it is subject to obsolescence risk, technological risk and cash flow risk in each stage. The uncertainty in my model involves jump risk, discovery risk and diffusion risk, which correspond to their model. However, we emphasize the interaction of these risks on the time dimension, and we allow the scale decision to adjust those risks. Our paper simplifies the R&D staging process, and focuses on studying how financial frictions impact firm value and investment decisions.

The real consequences of financing constraints have caught the attention of many scholars. A large empirical literature has been trying to measure investment cash flow sensitivity (Fazzari, Hubbard, and Petersen (1988),Whited (1992), Kaplan and Zingales (1997), Lamont (1997), Rauh (2006), Almeida and Campello (2007) etc). Their implicit assumption is that there is a wedge between internal and external capital cost, and the reaction of investment upon the availability of additional internal capital can approximate the size of the wedge. They interpret financing constraints by such a wedge, and we interpret financial constraints in the same spirit. However, instead of examining the effect of constraints on usual capital investment, I study its effect on an innovative investment project. The effect of capital constraints on investment is more carefully studied in a series of more recent papers (Faulkender and Petersen (2012) etc). Yet, except for Brown, Martinsson, and Petersen (2013) which only offers an aggregated perspective, not much empirical work studies the impact of financing constraints on R&D investment.

More recently, Li (2011) extends Berk, Green, and Naik (2004) to incorporate financial constraints theoretically. Unlike my work, it focuses on implications for the value premium. Different from my model, its financial constraints come in the form of collaterized project financing based on the change of project value. We believe the way we model firms is more suitable for its empirical testing sample, which is US public companies. Our paper brings asset in place and an R&D project into the same framework, and models constraints with the relationship of cash flow from these two components.

A few other new papers also show the main result of my paper, that is, financially constrained firms might invest more than unconstrained firms. However, without exception, they need competition in the model to generate the result. For example, Bena (2008) tackles the question of how financial constraints affect the relationship between competition intensity and innovation on an aggregate level. There, the financial constraint is described as a prevailing economic condition that applies to all firms. For competition at an intermediate level, the wedge between first-best and financially constrained aggregate innovation intensity can be negative. Meng (2008), with a typical patent race model, has the project accelerability feature of my paper, but such models generally show in a patent race model that competition drives firms to invest in R&D more aggressively, compared with a joint monopoly case. The head to head competition on R&D projects is more obvious in a patent race model. For instance, Meng (2008) argues in such a model that competition drives firms to invest in R&D more aggressively, compared with a joint monopoly case. I borrow the accelerability feature of innovative projects from patent race models since it describes a large class of R&D investment. Further, Phillips and Zhdanov (2013) adds beyond competition an active acquisition market, which boosts incentives for mature and unconstrained firms to wait while their smaller and constrained competitors race in R&D, and buy the successfully developed projects later. It also leads to the over-investment result. However, my model is able to generate this key result without competition.

The distinction on scalable versus accelerable projects is not the first effort to study investment behavior on different projects. Early work asks how capital should be allocated among different projects, for example Childs, Ott, and Triantis (1998). Lately, Almeida, Hsu, and Li (2013) differentiates exploitative technology from exploratory technology in terms of how close the project under development utilizes the firm's current technology. Their paper shows the disciplinary benefit of financial constraints, and the mechanism is through mitigating the free cash flow problem by constrained firms refocusing in the specialized field. While firm with financial slack may sub-optimally over invest in innovation, financing constraints force firms to focus on exploitative as opposed to exploratory innovation strategies. Seru (2014) distinguishes R&D projects by their novelty, and then studies the relationship between organizational structure and that novelty. It concludes that conglomerates conduct less novel R&D, and that those conglomerates with more novel R&D tend to operate with decentralized R&D budgets.

This paper is also related to studies using a real option approach to examine general investment policy. Dixit and Pindyck (1994) extensively educates researchers on that topic. Hackbarth, Mathews, and Robinson (2014) extends that line of research by focusing on the implication of optimal organizational structure. Similar to this chapter, the value of growth option under study is related to an assets in place cash flow process.

The main contribution for this chapter is to show that constrained firms may invest more than unconstrained ones even in a competition-free framework, to provide a new explanation of the mechanism, and to point out that it only happens if the project is accelerable.

1.3 Model setup

This is a continuous time model with an infinite horizon. I assume agents are risk neutral. The monopoly firm has two sources of cash flow: one is from its assets in place and the other is from an R&D project. As soon as an R&D project becomes available, the firm decides whether to carry it out or not. This investment opportunity is non-deferrable. Upon starting the project, the firm chooses its investment level in the project, which remains constant until discovery or abandonment. The firm may decide to voluntarily stop the project at some point, but it may also have to abandon the project if it runs out of money. I study the monopoly firm's optimal strategies in this context. More specifically,

- **Cash flow from assets in place** A firm has assets in place (AIP), which generate a cash flow X_t at each instance t. X_t follows a geometric Brownian motion with constant drift μ and volatility σ, i.e.

$$\mathrm{dX_t} = \mu X_t \mathrm{dt} + \sigma X_t \mathrm{dZ_t},$$

  where $\mathrm{dZ_t}$ is a standard Brownian motion.

- **Cash outflow of R&D before discovery** At time 0, an R&D project becomes available, and the firm has an option to start the project. The project availability could be due to exogenous shocks, such as a one time licensing opportunity, or a government subsidy program. I assume the investment decision cannot be postponed. If the firm chooses to start the project, it has to further decide on the investment scale R. Then a cash outflow of Rdt is required in each instance to keep the project going.

- **R&D project's discovery time and payoff** The discovery time τ_d is random, and follows an exponential distribution with parameter λ_d. That is, its density function is $f_{\tau_d}(t) = \lambda_d e^{-\lambda_d t}$, or $P\{\tau_d \in dt\} = \lambda_d e^{-\lambda_d t} dt$. The discovery payoff is random. The features of the discovery payoff and discovery time depend on the type of R&D project. I explicitly model two types of projects. One is a scalable project in which investment scale impacts discovery payoff, and the other is an accelerable project in which investment size affects the discovery time. Table 1.1 shows the characteristics associated with the two kinds of projects. The scalability factor $I(R)$, which determines how R impacts a scalable project's expected payoff, is increasing and concave. So is the accelerability factor $f(R)$, which determines how R impacts an accelerable project's expected discovery time. When a project is both scalable and accelerable, I also impose $I(R) \times f(R)$ to be concave so that the R&D technology is always decreasing returns to scale. More details are provided when we study these two kinds of projects separately in Section 1.4 and Section 1.5.

[insert Table 1.1 here]

- **R&D project termination** If Rdt is not paid at any moment before discovery, the R&D project becomes worthless instantly, i.e. the scrap value of the project is 0. The firm may choose not to pay Rdt even though it has the financial resources, or it may not be able to pay it due to financial constraints. The zero scrap value assumption is to simplify the payoff of the project when it has to end before its discovery time. Such technology of R&D projects fits in industries where the suspension of an R&D process pushes projects back to initial stages, or when there is a large information asymmetry problem for knowledge accumulated in a half-done innovative project to prohibit it from having a good resale price. The development status of this R&D project does not affect the firm's cash flow from AIP.

- **All equity and no cash flow accumulation assumptions** The firm is all equity financed. Also, the firm in my model is not allowed to accumulate internal cash flow, and it is forced to pay out its excess cash flow from investment as a dividend in each period of time, i.e. $d_t = \max\{X_t - R,\ 0\}$. This assumption restricts the firm from precautionary savings or reinvesting in its AIP. I conjecture that precautionary saving will not change my main result.

- **Financial constraints** I take the common understanding of financial constraints from the literature as the gap of the cost of capital between internal and external financing. In the main part of the paper, there are two types of firms: a financially constrained firm (FC) and a financially unconstrained firm (UC). An FC firm has infinitely high cost of external financing. Thus with the assumption of no retained earnings, if the project was carried out by an FC firm, then it has to be abandoned right away once $R > X_t$. A UC firm faces no additional cost of capital from external financing, and it could finance any funding gap $(R - X_t)^+$ by issuing new equity. A monopoly firm's accessibility to external financing market is determined ex ante, and the endogeneity of financial constraints is not a subject of this dissertation.

- With the above model features, the optimal strategy would be $(R, \underline{X})$, where R is the committed level of R&D and $\underline{X}$ is the threshold of AIP cash flow to abandon the R&D project. Due to different access to the financial market, for FC firms, $\underline{X} \geq R$; and for UC firms or CEF firms, $\underline{X} \geq 0$. Denote the time of project abandonment as τ_c, i.e. $\tau_c = \inf\{t : X_t \leq \underline{X}\}$.

- Here is the timeline of events:

t=0	$\longrightarrow$	t=1	$\longrightarrow$	t=2	$\longrightarrow$ ...
$\downarrow$		$\downarrow$		$\downarrow$	
If start R&D, then pick $\{R(X_0), \underline{X}(R)\}$		terminate the project?		terminate?	...

I use backward induction to study the firm's R&D decisions. First, I analyze the optimal abandonment threshold given a fixed project size R. Then I take one step back and figure out what is the optimal investment level R given the abandonment strategy. Finally, given the optimal decisions on investment size and abandonment, I provide the conditions under which the firm would rationally start such a project.

In practice, the majority of R&D projects possess a combination of both scalability and accelerability. I distinguish the two properties in my analysis to highlight how financial constraints affect R&D investment differently.

Later, I study some extensions of the model and show that the model can be used to examine more complicated setups. In Section1.6.2, I study a firm which has access to costly external financing(CEF). For a CEF firm, it can finance the gap of $R - X_t > 0$ at some cost $g(R, X_t)$. $g(R, X_t)$ can be interpreted as a floatation cost, or cash payment to equity holders to issue new equity. g is increasing and convex in the cash flow gap. In

Section 1.6.1, I study an extension of the model which incorporates a downward jump on the cash flow from AIP at a random time τ_j. I model such a jump as the first time of a Poisson arrival process with intensity λ_j, i.e the density function follows $f_{\tau_j}(t) = \lambda_j e^{-\lambda_j t}$. I further assume the assets in place produce no cash flow after this jump happens, i.e. $X_t = 0$, $\forall t > \tau_j$. Such a catastrophic loss of assets in place may be caused by technical obsolescence, successful development of competitive products or an adverse regulatory event affecting existing product sales.

1.4 Scalable Projects

In this section, I study the effect of financial constraints on those projects that are only scalable, but not accelerable. That is, the success payoff increases with the investment level, but the expected success time is independent of R. An example of a scalable project is an automobile company's R&D that aims at improving fuel efficiency and safety features for its new model. Large investments will probably result in cars with more attractive features and generate more profits when the new model is introduced to the market at the scheduled time. In details,

- The R&D project's success time τ_d follows an exponential distribution with parameter λ_d, which is independent of the R&D investment size.

- Once R&D is successful at τ_d, it generates a one time cash flow which depends on investment size, i.e. $Y_t = \tilde{A} f(R)$ with $E(\tilde{A}) = A$. The project is scalable in the sense that the project payoff can be expanded by enlarging the investment level. Assume f is an increasing and concave function, and $f(0) = 0$, $\lim_{R \to 0} f'(R) = \infty$,

$f'(R) > 0$ and $f"(R) < 0$. One can think of this one time cash flow as a discounted value of future cash flow or the sale price for such a successful innovation. A is the expected innovation efficiency or productivity. In the numerical analysis below, I use $f(R) = R^\beta$ with $\beta \in (0, 1)$.

1.4.1 The firm's problem

The monopoly firm chooses its investment and abandonment strategies to maximize its firm value. Since cash flow from AIP is not affected by this R&D project, firm value maximization is equivalent to project value maximization. In detail, firm value $FV(X)$ can be written as

$$
\begin{aligned}
FV(X) &= \sup_{R,\underline{X}} E[\int_0^{\tau_d \wedge \tau_c} e^{-rt}(X_t - R)\mathrm{dt} + e^{-r\tau_d}\tilde{A}f(R)1_{\{\tau_d < \tau_c\}} + \int_{\tau_d \wedge \tau_c}^{\infty} e^{-rt}X_t\mathrm{dt}] \\
&= \underbrace{\sup_{R,\underline{X}} E[\int_0^{\tau_d \wedge \tau_c} e^{-rt}(-R)dt + e^{-r\tau_d}\tilde{A}f(R)1_{\{\tau_d < \tau_c\}}]}_{\text{value of the project}} + \underbrace{E\int_0^{\infty} e^{-rt}X_t\mathrm{dt}}_{=\frac{X_0}{r-\mu},\ \text{value of AIP cash flow}}
\end{aligned}
$$

Denote $V(X)$ as the project value before project discovery or abandonment, and use the law of iterated expectation in the derivation, we have

$$
\begin{aligned}
V(X) &= \sup_{R,\underline{X}} V(X; R(X_0), \underline{X}(R)) \\
&= \sup_{R,\tau_c} E[\int_0^{\tau_d \wedge \tau_c} e^{-rt}(-R)dt + e^{-r\tau_d}\tilde{A}f(R)1_{\{\tau_d<\tau_c\}}] \qquad (1.1) \\
&= \sup_{R,\tau_c} E[\int_0^{\tau_c} e^{-rt}(-R)1_{\{t\leq\tau_d\}}dt] + Ee^{-r\tau_d}\tilde{A}f(R)1_{\{\tau_d<\tau_c\}} \\
&= \sup_{R,\tau_c} E\int_0^{\tau_c} e^{-rt}(-R)E(1_{\{t\leq\tau_d\}}\Big|\tau_c)dt + Af(R)E[e^{-r\tau_d}1_{\{\tau_d<\tau_c\}}] \\
&= \sup_{R,\tau_c} E\int_0^{\tau_c} e^{-rt}(-R)P\{t\leq\tau_d\}dt + Af(R)E(\int_0^{\infty} e^{-r\tau_d}f(\tau_d)1_{\{\tau_d<\tau_c\}}d\tau_d\Big|\tau_c) \\
&= \sup_{R,\tau_c} E\int_0^{\tau_c} e^{-rt}(-R)e^{-\lambda_d t}dt + E(Af(R)\int_0^{\tau_c} e^{-r\tau_d}\lambda_d e^{-\lambda_d\tau_d}d\tau_d) \\
&= \sup_{R,\tau_c} E\int_0^{\tau_c} e^{-(r+\lambda_d)t}(-R + Af(R)\lambda_d)dt \qquad (1.2)
\end{aligned}
$$

Eq(1.1) shows the project value is composed of two parts: the continuous outflow of Rdt until either discovery or abandonment, whichever happens first, and the potential inflow if the discovery occurs earlier than abandonment. Eq(1.2) states that the project value equals the expected per period cash flow $-R + Af(R)\lambda_d$ discounted at $r + \lambda_d$ before abandonment. The reason why the discount rate includes the success intensity is because the discovery effectively ends the project's cash flow. It plays a similar role as the risk of the catastrophic failure ϕ in the discount factor in Eq(13) of Berk, Green, and Naik (2004).

1.4.2 Solution for the project value

The Hamilton-Jacobi-Bellman equation

The HJB equation for the project value before abandonment is

$$rV\mathrm{dt} = E\mathrm{dV} + (-R)\mathrm{dt}, \tag{1.3}$$

where $E\mathrm{dV} = E\mathscr{D}V + \lambda_d(Af(R) - V)\mathrm{dt}$. The second term is the success intensity times the difference between value of the project if the success happens in the next instance and if it doesn't happen. We can use Ito's lemma on $E\mathrm{dV}$ to expand Eq (1.3) and eliminate dt, to get the following expression at the optimum,

$$rV = \mu XV_X + \frac{1}{2}\sigma^2X^2V_{XX} + \lambda_d(Af(R) - V) - R,$$

or equivalently [1]

$$(r + \lambda_d)V = \mu XV_X + \frac{1}{2}\sigma^2X^2V_{XX} + \lambda_dAf(R) - R. \tag{1.4}$$

---

[1] Another way of getting this HJB equation is by the property of martingale on $V(X)$. More specifically, from Eq(1.2), the following expression is a martingale at the optimal strategy before τ_c:

$$V = E(\int_0^t e^{-(r+\lambda_d)s}(-R + \tilde{A}f(R)\lambda_d)ds + e^{-(r+\lambda_d)t}V)$$

Therefore,

$$\begin{aligned} E(\frac{dV}{dt}) &= E(e^{-(r+\lambda_d)t}(-R + Af(R)\lambda_d) + e^{-(r+\lambda_d)t}(-r - \lambda_d)V + e^{-(r+\lambda_d)t}\mathscr{D}V) \\ &= E(e^{-(r+\lambda_d)t}(-R + Af(R)\lambda_d - (r + \lambda_d)V + \mathscr{D}V)) = 0 \\ \Rightarrow (r + \lambda_d)V &= E(\mathscr{D}V) + \lambda_dAf(R) - R \end{aligned}$$

Using the method of reduction to constant coefficient, this Euler-Cauchy equation has the known general solution

$$V(X) = c_1 X^{\alpha_1} + c_2 X^{\alpha_2} + V_p \tag{1.5}$$

where c_1, c_2 are constants to be determined by the boundary conditions, and V_p is a particular solution for the ODE Eq(1.4). α_1,α_2 are the solutions for the quadratic equation $\frac{1}{2}x(x-1)\sigma^2 + x\mu = (r + \lambda_d)$. I use

$$\alpha_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2(\lambda_d + r)}{\sigma^2}} < 0 \tag{1.6}$$

$$\alpha_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2(\lambda_d + r)}{\sigma^2}} > 0 \tag{1.7}$$

$$V_p = \frac{Af(R)\lambda_d - R}{\lambda_d + r} \text{(one particular solution)}$$

For the ease of representation, denote $B \equiv \frac{1}{2} - \frac{\mu}{\sigma^2}$ and $C = (\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2(\lambda_d + r)}{\sigma^2}$. Thus, $C = B^2 + \frac{2(\lambda_d + r)}{\sigma^2}$, $\alpha_1 = B - C^{\frac{1}{2}}$ and $\alpha_2 = B + C^{\frac{1}{2}}$. Obviously, $|C^{-\frac{1}{2}} \times B| < 1$.

Boundary conditions

As cash flow generated from AIP becomes infinitely large, the monopoly will not hit a fixed abandonment threshold $\underline{X}$ before discovery. Therefore, the project value is bounded from above by the value of a project that achieves success. That gives us the first condition for the ODE Eq (1.4). The second boundary condition comes from the fact that project value is zero at the abandonment. Together[2] :

[2]The reason why we cannot use the boundary condition $\lim_{X \to 0} V(X) \neq \infty$ to rule out a non-zero c_1 is because the value function changes as $X \to 0$.

$$\lim_{X\to\infty} V(X) = \frac{Af(R)\lambda_d - R}{\lambda_d + r}, \text{ and} \tag{1.8}$$

$$V(\underline{X}) = 0. \tag{1.9}$$

The first boundary condition implies that $c_2 = 0$, so we can simplify the project value as $V(X) = c_1 X^{\alpha_1} + V_p$. From the second boundary condition, we have

$$c_1 \underline{X}^{\alpha_1} + V_p = 0 \Rightarrow c_1 = \frac{-V_p}{\underline{X}^{\alpha_1}} = -\frac{A\lambda_d f(R) - R}{\lambda_d + r}\underline{X}^{-\alpha_1},$$

which leads to the following proposition.

Proposition 1. *The value of a scalable R&D project, for an investment size R and an abandonment strategy* $\underline{X}$, *is*

$$V(X;\ R(X_0), \underline{X}(R)) = \frac{Af(R)\lambda_d - R}{\lambda_d + r}(1 - (\frac{X}{\underline{X}})^{\alpha_1}) \tag{1.10}$$

where α_1 *is defined in Eq* (1.6).

Project value $V(X)$ is the market value of a security that claims $Af(R)\lambda_d - R$ units of payment before the AIP cash flow X_t hits the abandonment threshold $\underline{X}$. As shown in Eq(1.10), the R&D project value can be decomposed into two parts. The first term $\frac{Af(R)\lambda_d - R}{\lambda_d + r}$ is the discounted future cash flow of a perpetuity, with the effective cost of capital $\lambda_d + r$ and per period expected payoff $Af(R)\lambda_d - R$. This is the project value with no abandonment. Roughly speaking, the second term $(1 - (\frac{X}{\underline{X}})^{\alpha_1})$ is the pricing density of the project's payoff process. The equivalent pricing density is 1 under the risk neutral measure if there is no abandonment. However, it is reduced to $1 - (\frac{X}{\underline{X}})^{\alpha_1}$ given the existence of a potential abandonment of the project. A higher level of cash flow X, or a lower level of the abandonment threshold $\underline{X}$ implies a higher pricing density. Intuition suggests that the farther away is the abandonment time in the future, ceteris paribus, the higher the project value should be. Alternatively, we can interpret $(\frac{X}{\underline{X}})^{\alpha_1}$ as the market value of a security that claims 1 unit of payment at the hitting time of the abandonment threshold $\tau_c = \inf\{t : X_t \leq \underline{X}\}$[3].

We can see from Proposition (1) that the possibility of success, which is measured by the Poisson success intensity λ_d, has three effects on the valuation. First, as the success rate increases, it enlarges the equivalent cash flow $Af(R)\lambda_d$dt generated by the R&D project in each instance. Second, the success rate effectively increases the discount rate for the project value. Intuitively, a higher discount rate is associated with an earlier innovation success which ends the project's cash flow process. The third effect is played out through the pricing density of the project. Higher λ_d reduces the possibility to hit the abandonment threshold before discovery, thus increasing the price density.

[3]From Chap 11 of Duffie (2010): For any given constant $K \in (0, X_0)$ and a geometric Brownian motion $dX_t = uX_t dt + \sigma X_t dB^Q$, the market value of a security that claims one unit of account at the hitting time $\tau_K = inf\{t : X_t \leq K\}$ is, at any time $t < \tau_K$ is

$$E_t^Q[e^{-r(\tau_K - t)}] = (\frac{X_t}{K})^{-\gamma}$$

where $r = \dfrac{m + \sqrt{m^2 + 2r\sigma^2}}{\sigma^2}$ and $m = u - \frac{\sigma^2}{2}$.

1.4.3 The abandonment decision

From proposition (1), the monopoly firm should choose the lowest possible abandonment threshold $\underline{X}$. Formally,

Lemma 1. *Provided that the project is carried out, the optimal abandonment threshold of AIP cash flow is* $\underline{X} = R$ *for a financially constrained monopoly, and* $\underline{X} = 0$ *for a financially unconstrained monopoly.*

This result suggests that both types of firms never voluntarily abandon an R&D project once it was started. As long as the cost of the R&D project can be covered by the assets in place cash flow, the financially constrained company will keep developing the new technology. The unconstrained monopoly never terminates an ongoing project in this benchmark model. Intuitively, the cash flow prospects of the project remain the same as the innovation process unfolds. The project continues to have a positive NPV as long as the firm has the ability to pay for the R&D expenses.

1.4.4 The scale decision

Suppose $Af(R)\lambda_d - R \geq 0$ for $R \in [\underline{R}, \bar{R}]$, with $\underline{R} \geq 0$ and $\bar{R} \neq \infty$. If the investment scale is fixed and not a choice for the monopoly, then it is optimal for the monopoly to launch this R&D project as long as the fixed R is within $[\underline{R}, \bar{R}]$. Now if the firms can flexibly pick the size of the project, what is the optimal investment level for a financially constrained monopoly and an unconstrained monopoly? Do financial constraints always restrain the incentive to invest in scalable R&D projects? To answer these questions, we analyze the optimal scale decisions for two types of firms respectively.

The UC monopoly

For an unconstrained firm, we know from Lemma 1 and Eq (1.10), that for any given level of R, the optimal abandonment strategy leads to its project value of

$$V_{UC}(X_0;\ R) = \frac{Af(R)\lambda_d - R}{\lambda_d + r}. \tag{1.11}$$

The project value of a UC firm is the value of a perpetuity, since the project runs until it reaches discovery. The next step of solving the firm's problem is to choose the investment level R given its AIP cash flow at the project arrival, i.e.

$$R^*_{UC}(X_0) = \text{argmax}_R V_{UC}(X_0)$$

Suppose the expected project payoff $Af(R)$ is differentiable, and an interior solution exists $R^*_{UC} \in [\underline{R}, \bar{R}]$, then the optimal level of R should satisfy the following first order condition [4]

$$R^*_{UC} = \{R:\ A\lambda_d f'(R) = 1\}. \tag{1.12}$$

With assumptions on the payoff function f, the UC monopoly's optimal investment increases with the project's scaling factor ($\frac{\partial R^*_{UC}}{\partial A} > 0$) and the project's expected discovery speed ($\frac{\partial R^*_{UC}}{\partial \lambda_d} > 0$). However, R^*_{UC} is not sensitive to the AIP process ($\frac{\partial R^*_{UC}}{\partial \mu} = 0,\ \frac{\partial R^*_{UC}}{\partial \sigma^2} = 0,\ \frac{\partial R^*_{UC}}{\partial X_0} = 0$) because the firm has free and unlimited access to the financial market to raise any funding. R^*_{UC} does not depend on the risk free discount rate

[4]An alteration for this condition is represented by the expected time of success, $Af'(R) = \frac{1}{\lambda_d}$. Given $\frac{1}{\lambda_d}$ is the expected time spent before success, which is also $\int_0^{\frac{1}{\lambda_d}} \frac{dR}{dR}\text{dt}$.

either ($\frac{\partial R^*_{UC}}{\partial r} = 0$) since it affects the marginal benefit and marginal cost of investment equally.

The FC monopoly

For a constrained firm, Lemma1 and Eq(1.10) together give us

$$V_{FC}(X;\ R(X_0)) = \frac{Af(R)\lambda_d - R}{\lambda_d + r}(1 - (\frac{X}{R})^{\alpha_1}). \tag{1.13}$$

The project value for an FC monopoly has a pricing term, relating to the fact that the firm will have to terminate the project once its assets in place cash flow is not sufficient to cover the R&D expenses. The reciprocal of the second term is the option value of being financially unconstrained.

To compare the R&D investment scales between the two types of firms qualitatively, I summarize the scale decisions for the two types of firms in the next lemma.

Lemma 2. *For a scalable R&D project, the UC monopoly invests at* $R^*_{UC} = argmax_R \frac{Af(R)\lambda_d - R}{\lambda_d + r}$, *and the FC monopoly invests at* $R^*_{FC} = argmax_R \frac{Af(R)\lambda_d - R}{\lambda_d + r}(1-(\frac{X}{R})^{\alpha_1})$, *where* α_1 *is defined in Eq*(1.6).

Notice that R has an additional effect on the project value for the financially constrained firm. While investment size affects the marginal payoff $Af'(R)\lambda_d - 1$ for both UC and FC monopolies, it also changes the price density for the FC monopoly. More specifically,

$$\frac{\partial V_{FC}}{\partial R}\Big|_{X_0} = \underbrace{(1-(\frac{X_0}{R})^{\alpha_1})}_{+}(\overbrace{\frac{Af'(R)\lambda_d - 1}{\lambda_d + r}}^{\text{marginal payoff}}) + (\overbrace{\underbrace{\alpha_1 X_0^{\alpha_1} R^{-\alpha_1 - 1}}_{-}}^{\text{ME via pricing}})\underbrace{\frac{Af(R)\lambda_d - R}{\lambda_d + r}}_{+}. \quad (1.14)$$

The first term in Eq(1.14) shows the marginal effect of investment level R through project payoff on the valuation. It is positive when the monopoly under-invests relative to the UC monopoly ($R < R^*_{UC}$). It is negative when the monopoly over-invests ($R > R^*_{UC}$). The second term represents the marginal effect of R through the pricing kernel on the valuation. This term is negative because a higher level of investment always leads to earlier abandonment, making the project less valuable. Thus, to satisfy the first order condition $\frac{\partial V_{FC}}{\partial R} = 0$, the first term has to be positive, which leads to the following proposition.

Proposition 2. *A financially constrained monopoly always invests less aggressively in a scalable only R&D project than an unconstrained monopoly ($R^*_{FC} < R^*_{UC}$).*

This result proves the conventional wisdom that financial constraints reduce R&D investment in the context of the model. Intuitively, an FC monopoly under-invests relative to a UC monopoly in order to reduce the probability that it will be forced to terminate an ongoing project and end up getting nothing.

1.4.5 Comparative statics

By taking first order derivatives and applying the implicit function theorem, we can get the following comparative statics results regarding investment in scalable projects.

Corollary 1. *Optimal investment of both an FC monopoly and a UC monopoly on scalable R&D projects increases with the project success intensity λ_d and project payoff scale factor A. In addition, an FC monopoly's investment increases with risk free discount rate r, its AIP cash flow's level at the R&D arrival X_0 and its expected growth rate μ, and it decreases with the AIP cash flow volatility σ.*

To assess the effect of financial constraints quantitatively, I next turn to some numerical solutions. I apply the Nelder-Mead simplex method to search for local maximum. I use the parameter values listed in Table 1.2 as the baseline. These values are chosen based on the fact that they are reasonable[5].

[insert Table 1.2 here]

1.4.5.1 Investment scale

In Figure 1.1, I plot how optimal R&D investment changes with project characteristics and the discount rate r, fixing all other parameters. I use dashed lines to denote a UC monopoly's optimal investment scale, and solid lines to represent an FC monopoly's optimal investment scale. Panel (a), Panel (b), and Panel (d) all show sharp increase of investment of a UC monopoly as the project payoff scale factor increases $A \uparrow$, success intensity increases $\lambda_d \uparrow$, and project scalability increases $\beta \uparrow$. In these panels, we also see the same directional changes for an FC monopoly but the changes are milder. Obviously, it is optimal for a firm to invest more in a better project, but financial constraints restrict investments to some extent. Panel (c) shows that a UC monopoly does not react

[5] It is very difficult to do a calibration since I have not yet found reliable stylized facts on contemporary technologies in the literature.

to changes in the risk free discount rate, but an FC monopoly increases its investment as $r\uparrow$. It is because a larger discount rate makes the future cash flow matter less and thus the concern of hitting financial constraints is reduced. In all the plots, the UC monopoly invests more than the FC monopoly, and the difference is larger as $A\uparrow$,$\lambda_d\uparrow$, $\beta\uparrow$, and $r\downarrow$.

[insert Figure 1.1 here]

In Figure (1.2), I plot how optimal R&D investment changes with a monopoly's assets in place cash flow, fixing all other parameters. Panel (a), Panel (c), Panel (d) illustrate a flat investment for a UC monopoly, but the investment of an FC monopoly is higher when the firm has more cash flow to burn $X_0\uparrow$, higher growth rate of future cash flow $\mu\uparrow$, and less cash flow volatility $\sigma\downarrow$. A change of X_0 from 5 to 30 leads to an increase in the FC firm's investment from 1.8 to about 4. A change of μ from -0.3 to 0.3 leads to an increase in the FC firm's investment from 2.5 to about 6. As a matter of fact, as $X_0\uparrow\uparrow$ or $\mu\uparrow\uparrow$, we expect a constrained firm to invest at the level of an unconstrained firm because the constraints will never be binding. In all the plots, the UC monopoly invests more than the FC monopoly, and the difference is larger as $X_0\downarrow$, $\mu\downarrow$, and $\sigma\uparrow$.

[insert Figure 1.2 here]

1.4.5.2 Project value

In Figure (1.3), I plot R&D project values with regard to changes in project characteristics and discount rate r, fixing all other parameters at the baseline. An R&D project always has higher project value if it is carried out by an unconstrained monopoly, as opposed to a constrained one. The financial friction reduces project value from its first best level due to the investment distortion. Similar to the results on investment scales, we see sharp increases in project values of a UC monopoly in Plot (a), Plot (b), and Plot (d) as the project becomes better ($A \uparrow$, $\lambda_d \uparrow$, or $\beta \uparrow$). The same directional changes in an FC monopoly's project values are milder. For example, when the success rate λ_d increases from 0.05 to 0.15, the project value for an FC monopoly increases from about 3 to 20, while it increases for a UC monopoly from about 3 to about 55. In Plot (c), we observe that project value decreases with discount rate $r \downarrow$ for both types of firms because the future cash flow is discounted more heavily.

[insert Figure 1.3 here]

In Figure 1.4, I plot project values when one aspect of assets in place cash flow changes, with all other parameters fixed at the baseline. We will again defer the discussion of Panel (b) to Section 1.6.1. Panel (a), (c), and (d) show that the project value for a UC monopoly is independent of its assets in place, but an FC monopoly's project value is higher and gets closer to the UC monopoly's when its AIP cash flow is higher, the growth is stronger, and the uncertainty is lower ($X_0 \uparrow$, $\mu \uparrow$, $\sigma \downarrow$). For a change of growth rate μ from -0.3 to 0.3, the project value of an FC monopoly increases from about 7.5 to the UC monopoly's level 18.

[insert Figure 1.4 here]

We conclude the findings in many other numerical analysis with the following corollary.

Corollary 2. *For an FC monopoly, a scalable R&D project's value increases with the level and growth rate of its assets in place, but decreases with its cash flow volatility. For a UC monopoly, the project value is independent of its assets in place cash flow process.*

Project values for both type of firms increase with project success rate and project scale factor, and decrease with risk free rate. It is always true that $V_{FC} \leq V_{UC}$.

Furthermore, the effect of the current cash flow on the project value of an FC monopoly can be decomposed into two parts. One is its effect on the net present value of the project, which is the same as a UC monopoly; the other is its effect on the pricing density. The first effect is incorporated in the dependence of optimal R on X_0. Both effects impact project value through the optimal choice of project size positively. Figure 1.5 shows such a decomposition around the baseline. I use "NPV" to denote the first term of project value in Eq(1.13). The effect on pricing density, represented by the red line, becomes more important when the cash flow level is high.

[insert Figure 1.5 here]

1.4.6 The initiation decision

Given the optimal abandonment strategy ($\underline{X}_{FC} = R$, $\underline{X}_{UC} = 0$) and investment strategy (R^* in Section 1.4.4), the monopoly calculates the project value. If the R&D investment has a positive expected value, then the monopoly initiates the project at the arrival of such an opportunity. The traditional NPV rule is followed in the project initiation decision.

1.5 Accelerable Projects

In this section, I examine the effect of financial constraints on investment of accelerable R&D projects, i.e. those can be sped up with more investment. [6] Examples of such projects are ubiquitous. While explorative research projects fit better into the category of scalable projects, development process are more likely to be accelerable in general. For example, By hiring more coding staff and expanding computer power or server capacity, a project aiming at building an internet platform can get to the test and delivery stage sooner. Another example is a pharmaceutical company's search for the best chemical compound for a drug among many candidates. The effort to set up a larger or better equipped-lab and hire more capable technicians helps find the most suitable compound sooner. To distinguish from scalable projects, I focus on accelerable projects which are not scalable. More specifically, this type of project can be described with the following features:

1. The R&D project's payoff is random and does not depend on investment scale. I denote the random discovery payoff as $\tilde{A}$ with $E(\tilde{A}) = A$.

2. The discovery process can be expedited by heavier investment, but the marginal effect of investment on discovery time declines with investment scale. I model this property by assuming $\lambda_d = h(R)$ with $h' > 0$ and $h'' < 0$. Recall λ_d is the parameter in the exponential density function of discovery time τ_d, which is the

[6] The model is general enough to study projects with different combinations of scalability and accelerability. For example, the projects which higher investment scales up the discovery payoff as well speeds up the discovery process, or the project which more investment speeds up the discovery process but reduces the discovery payoff. However, such a comprehensive exploration is beyond the scope of this dissertation.

first arrival time of a Poisson process. The Poisson process is independent of the assets in place cash flow. In the numerical analysis, I further assume the functional form of the success intensity is $\lambda_d = \eta R^\gamma$ where $\gamma \in (0, 1)$.

Such characterization is standard in the patent race literature. However, that literature has not yet provided a satisfactory answer to the effect of financial constraints on R&D investment. With accelerable projects, is it true that financial constraints always dampen the incentive to invest in R&D? The answer is less clear. Financial constraints cause a disruption in the project development, and the marginal effect of investment scale on the expected time to hit constraints is still negative. However, the FC monopoly may be able to obtain an earlier discovery by making a larger investment and yet avoid hitting its constraints. By risking to hit the constraints sooner, the project may have a chance to survive until discovery.

1.5.1 The project value

Similar with scalable projects in Sec(1.4), the monopoly chooses its abandonment and investment strategy to maximize the project value of an accelerable project.

$$\begin{aligned} V(X) &= \sup_{\tau_c, R} E[\int_0^{\tau_c \wedge \tau_d} e^{-rt}(-R)dt + e^{-r\tau_d}\tilde{A}1_{\{\tau_d < \tau_c\}}] \\ &= \sup_{\tau_c, R} E \int_0^{\tau_c} e^{-(r+h(R))t}(-R + Ah(R))\mathrm{dt}, \end{aligned}$$

where τ_c is the abandonment time, i.e $\tau_c = \inf\{t : X_t < \underline{X}\}$, and $\underline{X}$ is the abandonment threshold on assets in place cash flow.

The corresponding HJB equation is

$$rV = \mu X V_X + \frac{1}{2}\sigma^2 X^2 V_{XX} + h(R)(A - V) - R$$

Boundary conditions for the ODE are

$$\begin{aligned} \lim_{X\to\infty} V(X) &= \frac{Ah(R) - R}{h(R) + r} \\ V(\underline{X}) &= 0 \end{aligned}$$

Solving for the ODE gives us

$$V(X;\ R(X_0), \underline{X}(R)) = \frac{Ah(R) - R}{r + h(R)}(1 - (\frac{X}{\underline{X}})^{\alpha_1}),$$

where

$$\alpha_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma})^2 + \frac{2(r + h(R))}{\sigma^2}}. \tag{1.15}$$

1.5.2 The abandonment decision

Following the same argument in Lemma 1, a monopoly should never voluntarily abandon a project that is under development in the model. Thus we have $\underline{X}^* = R$ for an FC monopoly and $\underline{X}^* = 0$ for a UC monopoly, the same as scalable projects. Following this abandonment decision, we can simplify the project values for the two kinds of firms.

Proposition 3. *The value of an accelerable R&D project for a UC monopoly is*

$$V_{UC} = \sup_R \frac{Ah(R) - R}{r + h(R)} \tag{1.16}$$

The value of an accelerable R&D project for an FC monopoly is

$$V_{FC} = \sup_R \frac{Ah(R) - R}{r + h(R)}(1 - (\frac{X}{R})^{\alpha_1}) \tag{1.17}$$

where α_1 *is defined in Eq* (1.15).

1.5.3 The scale decision

The UC monopoly

To find out the optimal R&D size choice, i.e. $R^*_{UC} = \text{argmax}_R V_{UC}(X_0)$, we use the first order condition of Eq (1.16). $\dfrac{\partial V_{UC}}{\partial R} = 0$ implies

$$\frac{\overbrace{(Ah'(R) - 1) \times (r + h(R))}^{\text{marginal effect on instantaneous payoff,}+} - \overbrace{(Ah(R) - R) \times h'(R)}^{\text{marginal effect on discount rate,}+}}{(r + h(R))^2} = 0 \tag{1.18}$$

Apparently, at the unconstrained monopoly's optimal investment level, R has positive effects on both the instantaneous payoff and the discount rate, and the two offset one another. Unlike the results for a UC monopoly in Sec (1.4), the marginal instantaneous payoff for the UC monopoly at R^*_{UC} is positive in an accelerable project[7]. Loosely speaking, the fact that the firm can get the R&D result faster by investing more intensely motivates a UC to invest less than otherwise.

[7]With the assumption that $h(R) = \eta R^\gamma$, and $\gamma \in (0, 1)$, we could derive from Eq(1.18) that $(A\eta h' - 1)r + \eta R h' - \eta h|_{R^*_{UC}} = 0$ and $A\eta h'(R) - 1\Big|_{R^*_{UC}} > 0,\ Rh' - h < 0$.

With the assumption on the discovery rate $\lambda_d = h(R) = \eta R^\gamma$ with $\gamma \in (0,1)$, the Implicit Function Theorem gives us $\frac{\partial R^*_{UC}}{\partial r} > 0$, $\frac{\partial R^*_{UC}}{\partial A} > 0$, and $\frac{\partial R^*_{UC}}{\partial \gamma} > 0$. R^*_{UC} does not change with any other parameters in the model. [8]

The result on riskless rate ($\frac{\partial R^*_{UC}}{\partial r} > 0$) is in contrast with Bena (2008), where the optimal investment decreases with the risk free rate. It was interpreted that heavier discounting over the future payoff destroys the firm's incentive to exert effort. However, it was ignored that a heavier discount also incentivizes the firm to push the project to succeed at an earlier time. The later force dominates the former, when the marginal success rate decreases in R. Alternatively, we can think of an extreme case where the risk free discount rate is close to zero. With a minimal discount on future payoffs, a UC monopoly only cares about having the discovery payoff eventually but is not concerned about the time value of money. Therefore, it makes sense for a UC monopoly to choose a low level of investment to keep the project going and to reap the final payoff while paying a low cost each period.

The FC monopoly

We cannot get the closed form solution for the optimal investment of an FC monopoly, so I use some numerical solutions to demonstrate the key result. However, I will first

[8] The proof is straightforward. Also, when $\beta = \frac{1}{2}$, R^*_{UC} has an analytical solution and $R^*_{UC} = (r - \sqrt{r^2 + Ar})^2$, which is similar to results in Proposition 1 in Bena (2008). Then we have

$$\begin{aligned} \frac{\partial R^*_{UC}}{\partial r} &= A + 4r - \frac{r^2 + 3r(A+r)}{\sqrt{r(A+r)}} > 0 \\ \frac{\partial R^*_{UC}}{\partial A} &= r - \frac{r^2}{\sqrt{r(A+r)}} > 0 \end{aligned}$$

decompose the project value of a constrained firm into two parts to understand the project value better.

$$V_{FC} = \underbrace{\frac{Ah(R) - R}{r + h(R)}}_{\equiv V1} \underbrace{(1 - (\frac{X}{R})^{\alpha_1})}_{\equiv V2}$$

Investment scale R appears four times in the project value. The first two are the same as when the project is undertaken by a UC monopoly: the discount rate and the instantaneous payoff. The third is through its closeness with the cash flow X, and the fourth effect is through α_1.

V_1 follows the same expression as the project value of a UC monopoly. To analyze $V2$, let's apply $\frac{\partial(f(x)^{g(x)})}{\partial x} = f(x)^{g(x)}(g' \ln f + \frac{g}{f}f')$,

$$\begin{aligned}
\frac{\partial V2}{\partial R} &= -(\frac{X}{R})^{\alpha_1}(\frac{\partial \alpha_1}{\partial R}\ln(\frac{X}{R}) + \frac{\alpha_1 R}{X}(-1)XR^{-2}) \\
&= -(\frac{X}{R})^{\alpha_1}[\frac{\partial \alpha_1}{\partial R}\ln(\frac{X}{R}) - \frac{\alpha_1}{R}] \\
&= \underbrace{(\frac{X}{R})^{\alpha_1}(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2(r + h(R))}{\sigma^2})^{-\frac{1}{2}}\frac{h'(R)}{\sigma^2}\ln(\frac{X}{R})}_{\text{marginal effect through } \alpha_1, +} + \underbrace{(\frac{X}{R})^{\alpha_1}\frac{\alpha_1}{R}}_{\text{marginal effect from closeness of R to X,-}} \\
&= \underbrace{(\frac{X}{R})^{\alpha_1}}_{+}[\underbrace{\underbrace{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2(r + h(R))}{\sigma^2})^{-\frac{1}{2}}\frac{h'(R)}{\sigma^2}\ln(\frac{X}{R})}_{+} + \underbrace{\frac{\alpha_1}{R}}_{-}}_{-/+}]
\end{aligned} \tag{1.19}$$

I find that $V2$ may increase or decrease with the investment scale. In other words, the third and fourth effects combined may be positive or negative. If the marginal effect through the risk adjusted factor overweighs the marginal effect through the imminence of hitting the constraints, then the marginal value of an additional investment on the pricing term is positive; otherwise the net effects of the two forces is negative.

The above analysis further suggests the necessity to use numerical solutions to understand an FC firm's over-investment or under-investment behavior in accelerable projects.

1.5.4 Comparative statics

I use the same baseline parameter values for assets in place cash flow and the risk free rate as in scalable projects. See Table 1.2 in the appendix for the details. The only differences from Section 1.4.4 are that I shut down the project's scalability by assuming $\beta = 0$, and I assume the baseline project accelerability $\gamma = 0.7$ and the scale factor $\eta = 0.1$.

The following key result of this chapter is from numerical comparison of investment scales by a UC and an FC monopolies in accelerable projects.

Proposition 4. *Financial constraints do not always reduce R&D investment. When developing accelerable R&D projects, a financially constrained monopoly may optimally invest on a larger scale than an unconstrained monopoly.*

Intuitively, financial constraints impose a termination risk that the firm may run out of money before the R&D project bears fruit. When that happens, the firm will lose the project. Thus, if the firm can expedite the project by investing more heavily, it may find it optimal to do so to increase the likelihood of project survival. This is more likely to happen when the firm's assets in place cash flow declines rapidly and the risk free discount rate is low. I obtain such over-investment results using a wide range of parameter values.

1.5.4.1 Investment scale

In Figure (1.6), I plot optimal investment scales against project characteristics, changing one parameter in each panel. The red dashed lines represent the investment by a UC monopoly, and the blue solid lines represent the investment by an FC monopoly. We can observe the over-investment result from an FC monopoly in all the four panels[9]. Furthermore, the over-investment is more likely to happen when the project final payoff is high enough ($A \uparrow$ in Panel (a)), the accelerability is large ($\gamma \uparrow$ in Panel (c)) , and the scale factor in accelerability (η in Panel (d)) is not too high. These results suggest that the incentive for a constrained firm to over-invest, relative to the first best, relates with several aspects of project characteristics. Among these, a necessary condition is that the project has to be accelerable ($\gamma > 0$). Apart of the evidence from the numerical results here, I have also shown in Proposition 2 on page 22 that there is never over-investment when the project is not accelerable. From Panel (b), the over-investment is more likely to occur when the risk free discount rate is low. This is because a higher discount rate motivates a firm to speed up to capture the time value of money, more so for an unconstrained monopoly than for a constrained one. Notice that the investment difference between the two kinds of firms decreases with regard to r in accelerable projects while

[9] Around this parameterization, the UC and the FC always invest at levels close to each other. In the parameterization I used in my job market paper, the two differ more obviously, with some cases that $R_{UC} < R_{FC}$. I will deter the discussion of the green line regarding an FC monopoly with a jump process in Section 1.6.1 on page 40.

increasing in r in scalable projects. It further highlights the importance to study an R&D project's characteristics more carefully to understand investment incentives.

[insert Figure 1.6 here]

In Figure 1.7, I plot optimal investment scales with regard to AIP cash flow parameters, changing only one parameter in each panel. Over-investment is evident in this set of graphs too, represented by the blue line (for an FC monopoly's investment) being above the red line (for a UC monopoly's investment). The red lines are flat because AIP does not affect a UC firm's investment decision. We see over-investment when the cash flow starts at a high level ($X \uparrow$ in Panel (a)), declines at a high rate($\mu \downarrow$ in Panel (c)), and is less volatile ($\sigma \downarrow$ in Panel (d)). As the cash flow increases further, the FC monopoly decreases its investment and converges to a UC monopoly's level. If cash flow deteriorates at a faster rate, the constrained monopoly is more strongly incentivized to make the discovery happen sooner, and thus invests more aggressively. In addition, Panel (c) shows a non-monotonic relationship between R_{FC} and μ which worth exploring in future works.

[insert Figure 1.7 here]

1.5.4.2 Project value

Next I investigate how the value of an accelerable project changes with different model parameters using numerical solutions.

In Figure 1.8, I show the project value of an accelerable project by a UC monopoly and an FC monopoly, as we change some project characteristics. It is always the case that $V_{FC} < V_{UC}$ because the friction from financial constraints distorts investment. In

Panel (a), both V_{UC} and V_{FC} increase as $A \uparrow$. The difference between V_{UC} and V_{FC} enlarges as $A \uparrow$ since the distortion from financial constraints is more severe as the accelerable project is more profitable. Similarly, in Panel (c), the difference between V_{UC} and V_{FC} increases as the project becomes more readily accelerable ($\gamma \uparrow$). In the same plot, both V_{UC} and V_{FC} follow a U-shape as $\gamma \uparrow$. From Panel (d), both V_{UC} and V_{FC} increase as the scale factor for accelerability becomes larger ($\eta \uparrow$), but the difference between the two follows an inverse U-shape. The non-monotonic relationships highlight the complicated tradeoff related with project characteristics. In Panel (b), both V_{UC} and V_{FC} decrease with discount rate r, and the difference between the two is most striking when $r = 0$ because the UC monopoly only invests at the minimal level to keep the option alive.

[insert Figure 1.8 here]

In Figure 1.9, I plot the value of an accelerable project as we change some aspects of the firm's cash flow. While V_{UC} remains constant in all four graphs, V_{FC} is higher and closer to V_{UC} when cash flow is high ($X \uparrow$ in Panel (a)), declines at a lower rate or increases at a higher rate ($\mu \uparrow$ in Panel (c)), and is less volatile ($\sigma \downarrow$ in Panel (d)).

[insert Figure 1.9 here]

1.5.5 Deterministic cash flow from assets in place

One of the uncertainties a constrained firm faces when developing an R&D project is when it will run out of money. In order to better understand the mechanism of over-investment, let's remove this cash flow risk, and instead assume the cash flow from assets in place is deterministic, e.g. $dX_t = \mu X_t dt$. Now an FC firm knows exactly how long it could keep developing the project. If the termination risk is the source of over-investment, we should expect such model simplification to deliver the key result as well.

With $\sigma = 0$, the project value should satisfy

$$rV = \mu X V_X + h(R)(A - V) - R, \tag{1.20}$$

or

$$(r + h(R))V = \mu X V_X + h(R)A - R. \tag{1.21}$$

From which we can rewrite

$$V_X + (-\frac{r + h(R)}{\mu X})V = \frac{R - h(R)A}{\mu X}. \tag{1.22}$$

This ODE has a solution in the form of

$$V(X) = \frac{\int u(s)g(s)ds + C}{u(X)}, \tag{1.23}$$

where

$$u(X) = \exp(\int p(t)dt) = X^{-\frac{r+h(R)}{\mu}} \tag{1.24}$$

is the integrating factor. Thus

$$V(X) = \frac{h(R)A - R}{r + h(R)} + CX^{\frac{r+h(R)}{\mu}}.$$

The boundary conditions are:

$$V(\underline{X}) = 0, \ \lim_{X \to \infty} V(X) = \frac{Ah(R) - R}{r + h(R)}.$$

We can then get the solution for the ODE:

$$\begin{aligned}
V(\underline{X}) &= \frac{h(R)A - R}{r + h(R)} + C\underline{X}^{\frac{r+h(R)}{\mu}} = 0 \\
\Rightarrow C &= -\frac{h(R)A - R}{r + h(R)} \underline{X}^{-\frac{r+h(R)}{\mu}} \\
\Rightarrow V(X) &= \frac{h(R)A - R}{r + h(R)} (1 - (\frac{\underline{X}}{X})^{-\frac{r+h(R)}{\mu}})
\end{aligned}$$

Notice that if $\mu > 0$, then the FC monopoly never runs out of money and the constraints don't play any role in the investment decision. Thus we stick to the more interesting case where $\mu < 0$.

$$\frac{\partial V(X)}{\partial \underline{X}} = \frac{r + h(R)}{\mu} (\frac{\underline{X}}{X})^{-\frac{r+h(R)}{\mu} - 1} \times \frac{1}{X} < 0 \Rightarrow \underline{X}_{FC} = R, \ \underline{X}_{UC} = 0$$

With no voluntary abandonment, project value of an FC monopoly can be written as

$$V(X) = \frac{h(R)A - R}{r + h(R)} (1 - (\frac{R}{X})^{-\frac{r+h(R)}{\mu}}). \tag{1.25}$$

To check whether we can get the over-investment in this simplified setting, we take the first order derivative for the project value. The sign of it tells us how the value changes with respect to investment.

$$
\begin{aligned}
\frac{\partial V(X)}{\partial R} &= (1-(\frac{R}{X})^{-\frac{r+h(R)}{\mu}}) \times \frac{(Ah'-1)(r+h(R))-(h(R)A-R)h'(R)}{(r+h(R))^2} \\
&+\frac{h(R)A-R}{r+h(R)}(-(\frac{R}{X})^{-\frac{r+h(R)}{\mu}}) \times (-\frac{h'(R)}{\mu}\ln(\frac{R}{X})-\frac{r+h(R)}{\mu}\times\frac{1}{R}) \\
&= (\frac{R}{X})^{-\frac{r+h(R)}{\mu}}[(\frac{h(R)A-R}{r+h(R)}(\frac{h'(R)}{\mu}\ln(\frac{R}{X})+\frac{r+h(R)}{\mu}\times\frac{1}{R})-\frac{Ah'r-r-h+Rh'}{(r+h(R))^2}] \\
&+\frac{Ah'r-r-h+Rh'}{(r+h(R))^2}
\end{aligned} \tag{1.26}
$$

Let's focus on the sign of

$$
m = -\frac{h'(R)}{\mu}\ln(\frac{R}{X})-\frac{r+h(R)}{\mu}\times\frac{1}{R}
$$

If $m|_{R^*_{FC}} > 0$, then the first part in Eq(1.26) has to be positive for the FOC to hold, and because the second half of the first part in Eq(1.26) is decreasing in R, we then have $R^*_{FC} < R^*_{UC}$. If $m|_{R^*_{FC}} < 0$, then $R^*_{FC} > R^*_{UC}$. I can show that $m|_{R^*_{FC}} < 0$ holds for some parameter values, thus sometimes we have $R^*_{FC} > R^*_{UC}$.

Thus, without the cash flow risk, an FC monopoly may still invest more than a UC firm. This is more likely when the cash flow deteriorates quickly. It suggests that the essential ingredients that drive the over-investment result are (1) accelerability of the project; (2) deteriorating cash flow from assets in place. What seem to matter are the relative timing of involuntary project termination and the project discovery, but not the randomness of both.

1.6 Extended models

In this section, I study two variations of the basic model. First, I introduce a jump risk on the cash flow and ask whether the new source of termination risk also motivates investment. If so, does this motivation only work on accelerable projects? The second variation deals with a monopoly that has costly external finance, and asks whether the extent to which firms are financially constrained affects their R&D investment strategies monotonically?

1.6.1 Limited assets in place cash flow process

In this modified setup, I assume there is a random downward jump on the cash flow from assets in place which wipes out all future cash flow. The new cash flow process follows

$$dX_t = \mu X_t dt + \sigma X_t dZ_t - X_t dq_1, \tag{1.27}$$

where $Z = \{Z_t; 0 < t < \infty\}$ is a standard Brownian motion, and dq_1 is the increment of a Poisson process with an exogenous arrival rate λ_j. The jump happens at a random time τ_j. When this jump occurs, an FC monopoly will not be able to pay for its R&D project anymore, and thus abandons the project involuntarily. However, a UC monopoly will be able to continue funding the project through the financial market, and does not have to terminate the project. Examples of the jump process include a recall crisis for an auto manufacturer, or a successful manufacturing of a generic drug from a pharmaceutical firm's competitor after the patent expiration of its branded drug.

The project values in this new setting are as follows, with the proof in the appendix.

Proposition 5. *The value of an R&D project for a financially unconstrained monopoly is*

$$V_{UC}(X) = \sup_R \frac{u\lambda_d - R}{\lambda_d + r} \tag{1.28}$$

The value of an R&D project for a financially constrained monopoly is

$$V_{FC}(X) = \sup_R \frac{u\lambda_d - R}{\lambda_d + \lambda_j + r}(1 - (\frac{X}{R})^{\alpha_1}) \tag{1.29}$$

where $\alpha_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2(\lambda_d+\lambda_j+r)}{\sigma^2}}$, *$u$ is the expected payoff, and λ_d measures the expected discovery speed.*

The project value of a UC monopoly remains the same as in the basic model, since any changes related to the AIP cash flow does not affect project development. The possibility for such a catastrophic event in the future makes an FC firm discount cash flows more heavily, and it also reduces the pricing term because having to involuntarily abandon the project becomes a more urgent concern. Below, I separate the discussion for two kinds of projects. Intuitively, the jump risk motivates an FC monopoly to speed up the discovery if the project is accelerable, but it is less clear how the investment incentive will change for scalable projects.

1.6.1.1 Scalable projects

In the benchmark model, we always have $R^*_{UC} > R^*_{FC}$ for scalable projects. Does this result still hold if the cash flow has a downward jump? When an FC monopoly expects such a jump in the future, it is likely to choose a larger investment because such a catastrophic event destroys the project regardless of how far X_t is from R. However, we can prove that this jump risk alone does not push an FC monopoly to over-invest in scalable projects.

Proposition 6. *If there is an expected downward jump in the AIP cash flow, then an FC monopoly invests more than when such a jump is absent. The increase in investment is positively related with the jump intensity. The optimal investment scale remains the same for a UC monopoly. As in the case with infinite horizon AIP cash flow, if the project is scalable, it always holds that* $R^*_{UC} < R^*_{FC}$.

From Eq (2.16), it is clear that the AIP jump adds an extra term in the discount rate. Panel (b) of Figure 1.2 on page 56 shows that, with the obsolescence risk from AIP's cash flow, the optimal project size for a constrained firm is higher and closer to an unconstrained company. We are equivalently comparing R_{UC} and R_{FC} in the benchmark model at different levels of discount rate r, with a lower r for R_{UC}.

This extra incentive, however, doesn't push an FC firm to ever invest more than a UC firm. Technically, it is because the negative marginal effect of investment on the pricing term forces the marginal effect of R on the payoff to be positive. Thus, with a concave payoff function, it still leads to a lower level of investment comparing with a UC firm. Take the baseline parameters as an example. In Panel (c) of Figure 1.1, R_{FC} is larger at a higher r, but it will never be above R_{UC}. In Panel (b) of Figure 1.1, we can see the investment by an FC monopoly increases with the jump intensity λ_j and gets closer to the investment level by a UC monopoly but never goes over it. In Panel (b) of Figure 1.3, the project value by an FC monopoly is lower when it is subject to a higher jump likelihood. A higher catastrophic risk by no surprise has a larger effect on value distortion.

Accelerable projects

The main result of a downward jump on accelerable projects investment can be summarized as follows.

Lemma 3. *When investing in accelerable projects, an expected fall of cash flow from AIP motivates an FC monopoly to invest more heavily while not affecting a UC monopoly. With this extra incentive, over-investment by an FC monopoly is more severe.*

Thus, for both scalable and accelerable projects, a downward jump motivates investment. While the jump cannot trigger over-investment in scalable projects, it can exacerbate and even cause over-investment in accelerable projects. In Figure 1.6 and Figure 1.7, I use green circled lines to represent the investment by an FC firm subject to a jump at and around the baseline. We see that the jump makes an FC firm more aggressive in its R&D strategy in both figures. In Figure 1.6, this positive effect on investment is stronger ($R_{FC,\lambda=0.05} - R_{FC,\lambda=0}$ is larger) when the project has a higher payoff ($A \uparrow$ in Panel (a)), is more readily accelerable ($\gamma \uparrow$ in Panel (c)), and the scale factor in project accelerability is larger ($\eta \uparrow$ in Panel (d)). In Figure 1.7, this positive effect on investment is stronger when the FC monopoly has more cash flow ($X \uparrow$ in Panel (a)), lower cash flow volatility ($\sigma \downarrow$ in Panel (c)), and higher cash flow growth rate ($\mu \uparrow$ in Panel (c)).

Meanwhile, from Panel (b) and Panel (d) in Figure 1.2 and from Panel (b) and Panel (d) in Figure 1.7, the two kinds of cash flow risk (σ and λ_j) have opposite effects in motivating R&D investment. While a higher cash flow volatility σ leads to lower investment, a higher jump risk leads to higher investment. Intuitively, a higher cash flow volatility tampers investment incentive because the FC firm expects to hit financial constraints sooner, but a higher jump risk makes the financial constraints less of a concern because of the new risk of having all the cash flow evaporated at some point in the near future.

The model also generates several testable implications regarding project values. I illustrate the jump effects on project values in Figure 1.8 and Figure 1.9 by the difference between the green lines ($\lambda_j = 0.05$) and the blue lines ($\lambda_j = 0$). We can see from all panels of the two figures that the jump risk always reduces project value, and more so in situations where the investment is distorted more severely ($A \uparrow, \gamma \downarrow, X \uparrow, \mu \uparrow, \sigma \downarrow$, $\lambda_j \uparrow$, $r \downarrow$).

1.6.2 Costly External Financing

So far, I have focused on firms at the two extreme cases on the spectrum of financial constraints. In reality, most firms can pay some cost to raise funding from the financial market. In this model variation, I examine the R&D strategy for a monopoly which faces an increasing and convex cost of capital (a "CEF firm"), and study the effect of the cost of capital on R&D investment. A natural conjecture is that such a monopoly invests at a level between an FC monopoly and a UC monopoly.

Following Kaplan and Zingales (1997), I measure the degree of financial constraints as the wedge between a firm's internal and external cost of funds. Whenever cash flow X_t falls short of the project's instantaneous investment Rdt, I assume a CEF firm could finance the gap $(R - X_t)^+$ at the cost of $g((R - X_t)^+)$ in the financial market. This cost can be a cash transfer to issue new equity or simply the floatation cost. For the simplicity of the analysis, I assume $g((R - X_t)^+) = \gamma((R - X_t)^+)^2$. Meanwhile, we are back to $\lambda_j \to \infty$ to separate out the effect of costly financing.

A CEF monopoly should follow a threshold strategy of investment and abandonment. For the abandonment, it chooses to keep investing in R&D until $X_t \leq \underline{X}_{CEF}$[10]. Denote τ_f as the time to stop using external financing, i.e. $\tau_f = \inf\{t : X_t \leq \underline{X}_{CEF}\}$. Then the project value by a CEF monopoly becomes

$$\begin{aligned} V_{CEF}(X) &= \sup_{\tau_f, R} E\{\int_0^{\tau_f \wedge \tau_d} [-R - g(X_t)] \times e^{-rt} dt + 1_{\{\tau_s < \tau_f\}} A f(R) e^{-t\tau_d}\} \\ &= \sup_{\tau_f, R} E\{\int_0^{\tau_f} [-R - g(X_t) + A f(R)\lambda_d] e^{-(\lambda_d + r)t} dt\} \end{aligned}$$

In what follows, we focus on the analysis on scalable projects for mathematical tractability.

[10]Another possible threshold strategy is based on the state of accumulative investment, as it is used in Berk, Green, and Naik (2004). However, unlike their model, the knowledge from developing R&D project is not accumulative in our model. It doesn't matter in this setup how long the firm has been investing continuously. Given the success intensity is exogenous, and follows a Poisson distribution, the success possibility in any instance remains constant from the very beginning. Thus, this alternative threshold strategy is eliminated.

HJB equation

The HJB equation for the project value by a CEF monopoly, from either dynamic programming or contingent claim, is:

$$(r+\lambda_d)V = \mu X V_X + \frac{1}{2}\sigma^2 X^2 V_{XX} + \lambda_d A f(R) - R - g(X), \ \forall t < \tau_f. \tag{1.30}$$

This is a linear second order ODE. Similar with Liu and Loewenstein (2002), I follow the method in Boyce and DiPrima (2000) to get its solution[11]. The general solution should be

$$V(X) = c_1 X^{\alpha_1} + c_2 X^{\alpha_2} + V_p(X)$$

The fundamental solutions for the homogeneous equation are X^{α_1} and X^{α_2}, recall from Eq(1.6) and Eq(1.7) from Sec(1.4.2), $\alpha_1, \alpha_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} \pm \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2(\lambda_d + r)}{\sigma^2}}$ with $\alpha_1 < 0$ and $\alpha_2 > 0$. One particular solution for Eq(1.30) is

$$V_p(X) = -X^{\alpha_1}\int_{t_1^*}^{X} \frac{2(R+g(t)-\lambda_d A f(R))}{(\alpha_2-\alpha_1)t^{\alpha_1+1}\sigma^2}dt + X^{\alpha_2}\int_{t_2^*}^{X} \frac{2(R+g(t)-\lambda_d A f(R))}{(\alpha_2-\alpha_1)t^{\alpha_2+1}\sigma^2}dt \tag{1.31}$$

Set both of the lower bounds at a convenient level $\underline{X}_{CEF}$, i.e. $t_1^* = t_2^* = \underline{X}_{CEF}$.

[11] Theorem 3.7.1. in the 12th edition of Boyce and DiPrima (2000) states: If the functions p,q and g are continuous on an open interval I, and if the functions y_1 and y_2 are linearly independent solutions of the homogeneous equation $y" + p(t)y' + q(t)y = 0$ corresponding to the non-homogeneous equation $y" + p(t)y' + q(t)y = g(t)$, then a particular solution of the non-homogeneous equation is

$$Y(t) = -y_1(t)\int_{t_1}^{t} \frac{y_2(s)g(s)}{W(y_1,y_2)(s)}ds + y_2(t)\int_{t_2}^{t} \frac{y_1(s)g(s)}{W(y_1,y_2)(s)}dt$$

where the Wronskian $W = y_1 y_2^{'} - y_1^{'} y_2$, and the general solution is $y = c_1 y_1(t) + c_2 y_2(t) + Y(t)$.

Rewrite $V(X)$ by plugging in the particular solution of Eq(1.31) and substituting t_1^* and t_2^* in the general solution:

$$V(X) = X^{\alpha_1}(c_1 - \int_{\underline{X}}^{X} \frac{2(R + g(t) - \lambda_d A f(R))}{(\alpha_2 - \alpha_1)t^{\alpha_1+1}\sigma^2} dt) + X^{\alpha_2}(c_2 + \int_{\underline{X}}^{X} \frac{2(R + g(t) - \lambda_d A f(R))}{(\alpha_2 - \alpha_1)t^{\alpha_2+1}\sigma^2} dt) \tag{1.32}$$

Boundary conditions

Three boundary conditions for the HJB equation are

$$\begin{aligned} \lim_{X\to\infty} V(X) &= \frac{Af(R)\lambda_d - R}{\lambda_d + r} \quad (1.33) \\ V(\underline{X}_{CEF}) &= 0 \text{ (value matching)} \\ \frac{dV(X)}{dX}|_{X=\underline{X}_{CEF}} &= 0 \text{ (smooth pasting)} \end{aligned}$$

The first boundary condition tells us that the coefficient associated with the term X^{α_2} in Eq(1.32) should be zero as $X \to \infty$ given that $\lim_{X\to\infty} \int_{\underline{X}}^{X} \frac{2(R+g(t)-\lambda_d Af(R))}{(\alpha_2-\alpha_1)t^{\alpha_2+1}\sigma^2}\mathrm{dt}$ has a finite limit[12]. Thus[13],

$$c_2 = -\int_{\underline{X}}^{\infty} \frac{2(R+g(t)-\lambda_d Af(R))}{(\alpha_2-\alpha_1)t^{\alpha_2+1}\sigma^2}\mathrm{dt} \tag{1.34}$$

The second boundary condition yields

$$V(\underline{X}_{CEF}) = c_1\underline{X}_{CEF}^{\alpha_1} + c_2\underline{X}_{CEF}^{\alpha_2} = 0 \Rightarrow c_1 = -c_2\underline{X}_{CEF}^{\alpha_2-\alpha_1}$$

[12]Since g is a polynomial consisting the highest degree of h, a sufficient condition for a finite limit is $\alpha_2 > h$.

[13]By plugging the expression of c_2 into $V(X)$ and take its limit, we can verify

$$\begin{aligned}
\lim_{X\to\infty} V(X) &= \lim_{X\to\infty}\{c_1X^{\alpha_1} - X^{\alpha_2}\int_{\underline{X}_{CEF}}^{\infty} \frac{2(R+g(t)-\lambda_d Af(R))}{(\alpha_2-\alpha_1)t^{\alpha_2+1}\sigma^2}\mathrm{dt} \\
&\quad -X^{\alpha_1}\int_{\underline{X}_{CEF}}^{X} \frac{2(R+g(t)-\lambda_d Af(R))}{(\alpha_2-\alpha_1)t^{\alpha_1+1}\sigma^2}\mathrm{dt} + X^{\alpha_2}\int_{\underline{X}_{CEF}}^{X} \frac{2(R+g(t)-\lambda_d Af(R))}{(\alpha_2-\alpha_1)t^{\alpha_2+1}\sigma^2}\mathrm{dt}\} \\
&= \lim_{X\to\infty}\{-\underbrace{X^{\alpha_1}}_{\to 0}\underbrace{\int_{\underline{X}_{CEF}}^{X} \frac{2(R+g(t)-\lambda_d Af(R))}{(\alpha_2-\alpha_1)t^{\alpha_1+1}\sigma^2}\mathrm{dt}}_{\to\infty \text{ if } \alpha_1+1<0} - \underbrace{X^{\alpha_2}}_{\to\infty}\underbrace{\int_{X}^{\infty} \frac{2(R+g(t)-\lambda_d Af(R))}{(\alpha_2-\alpha_1)t^{\alpha_2+1}\sigma^2}\mathrm{dt}\}}_{\to 0} \\
&= \lim_{X\to\infty}\{-\frac{\frac{2(R+g(X)-\lambda_d Af(R))}{(\alpha_2-\alpha_1)X^{\alpha_1+1}\sigma^2}}{-\alpha_1X^{-\alpha_1-1}} - \frac{-\frac{2(R+g(X)-\lambda_d Af(R))}{(\alpha_2-\alpha_1)X^{\alpha_2+1}\sigma^2}}{-\alpha_2X^{-\alpha_2-1}}\} \text{ (By L'Hopital's Rule)} \\
&= \frac{2(R-\lambda_d Af(R))}{(\alpha_2-\alpha_1)\alpha_1\sigma^2} - \frac{2(R-\lambda_d Af(R))}{(\alpha_2-\alpha_1)\alpha_2\sigma^2} \\
&= \frac{Af(R)\lambda_d - R}{\lambda_d + r} \text{ provided } \alpha_1\alpha_2 = -\frac{2(r+\lambda_d)}{\sigma^2}
\end{aligned}$$

And the third condition[14] leads to

$$
\begin{aligned}
0 &= -c_2\underline{X}_{CEF}^{\alpha_2-\alpha_1}\alpha_1\underline{X}_{CEF}^{\alpha_1-1} + c_2\alpha_2\underline{X}_{CEF}^{\alpha_2-1} - \underline{X}^{\alpha_1}\frac{2(R+g(\underline{X}_{CEF})-\lambda_d Af(R))}{(\alpha_2-\alpha_1)\underline{X}_{CEF}^{\alpha_1+1}\sigma^2} \\
&\quad +\underline{X}_{CEF}^{\alpha_2}\frac{2(R+g(\underline{X}_{CEF})-\lambda_d Af(R))}{(\alpha_2-\alpha_1)\underline{X}_{CEF}^{\alpha_2+1}\sigma^2} \\
&\Rightarrow\ c_2(-\alpha_1+\alpha_2)\underline{X}_{CEF}^{\alpha_2-1} = 0
\end{aligned}
$$

This suggests two solutions, one with $\underline{X}_{CEF} = 0$ and the other with

$$
c_2 = -\int_{\underline{X}_{CEF}}^{\infty} \frac{2(R+g(t)-\lambda_d Af(R))}{(\alpha_2-\alpha_1)t^{\alpha_2+1}\sigma^2}\mathrm{dt} = 0 \tag{1.35}
$$

which indicates the abandonment threshold should be a function of the investment scale, i.e. $\underline{X}_{CEF}(R)$. I take the second solution since it is more sensible.

Since both c_1 and c_2 equal zeros, we can rewrite the project value in Eq (1.32) as

$$
\begin{aligned}
V(X;\underline{X}_{CEF}(R),R) &= X^{\alpha_2}\int_{\underline{X}_{CEF}}^{X}\frac{2(R+g(t)-\lambda_d Af(R))}{(\alpha_2-\alpha_1)t^{\alpha_2+1}\sigma^2}\mathrm{dt} - X^{\alpha_1}\int_{\underline{X}_{CEF}}^{X}\frac{2(R+g(t)-\lambda_d Af(R))}{(\alpha_2-\alpha_1)t^{\alpha_1+1}\sigma^2}\mathrm{dt} \\
&= \frac{2}{(\alpha_2-\alpha_1)\sigma^2}[\int_{\underline{X}_{CEF}^*}^{X}(R+g(t)-\lambda_d Af(R))(\frac{X^{\alpha_2}}{t^{\alpha_2+1}} - \frac{X^{\alpha_1}}{t^{\alpha_1+1}})\mathrm{dt}] \qquad (1.36)
\end{aligned}
$$

To derive $\underline{X}_{CEF}$, notice that Eq(1.35) is equivalent to

$$
\int_{\underline{X}_{CEF}}^{\infty}\frac{(R+g(t)-\lambda_d Af(R))}{t^{\alpha_2+1}}\mathrm{dt} = 0 \tag{1.37}
$$

[14] It is not an optimality condition. To verify $\underline{X}_{CEF}$ is the optimal strategy, we need to check the second order condition: $\frac{\partial^2 V(X,\underline{X}_{CEF};R)}{\partial \underline{X}_{CEF}^2}|_{\underline{X}^*} < 0$.

With the assumption on the cost of capital g:

$$g(X_t) = \begin{cases} \theta(R - X_t)^2 & \underline{X} \leq X_t < R \\ 0 & X_t > R \\ \infty & X_t < \underline{X} \end{cases},$$

Eq(1.35) can then be further simplified as

$$\int_{\underline{X}_{CEF}}^{\infty} \frac{(R + g(t) - \lambda_d Af(R))}{t^{\alpha_2+1}} \mathrm{dt} = 0 \tag{1.38}$$

$$\Rightarrow \int_{\underline{X}_{CEF}}^{\infty} \frac{R - \lambda_d Af(R)}{t^{\alpha_2+1}} \mathrm{dt} + \int_{\underline{X}_{CEF}}^{R} \frac{\theta(R-t)^2}{t^{\alpha_2+1}} \mathrm{dt} = 0 \tag{1.39}$$

$$\Rightarrow [R - \lambda_d Af(R)] \frac{t^{-\alpha_2}}{-\alpha_2}\Big|_{\underline{X}_{CEF}}^{\infty} + \frac{R^2\theta t^{-\alpha_2}}{-\alpha_2}\Big|_{\underline{X}_{CEF}}^{R} - \frac{2R\theta t^{-\alpha_2+1}}{-\alpha_2+1}\Big|_{\underline{X}_{CEF}}^{R} + \frac{\theta t^{-\alpha_2+2}}{-\alpha_2+2}\Big|_{\underline{X}_{CEF}}^{R} = 0$$

$$\Rightarrow [R - \lambda_d Af(R)] \frac{\underline{X}_{CEF}^{-\alpha_2}}{\alpha_2} - \frac{\theta R^2}{\alpha_2}[R^{-\alpha_2} - \underline{X}_{CEF}^{-\alpha_2}] - \frac{2R\theta}{1-\alpha_2}[R^{-\alpha_2+1} - \underline{X}_{CEF}^{-\alpha_2+1}] \tag{1.40}$$

$$+ \frac{\theta}{2-\alpha_2}[R^{-\alpha_2+2} - \underline{X}_{CEF}^{-\alpha_2+2}] = 0$$

The optimal investment scale R should maximize the project value in Eq (1.36), with the solution of $\underline{X}_{CEF}$ from Eq (1.40). Therefore, to solve the problem for a CEF firm, we need to find the abandonment threshold $\underline{X}_{CEF}$ using Eq (1.40) with any given R, and then search for the R^* that maximizes Eq (1.36). Finally, we find the fixed point such that $R = R^*$ so that the abandonment and investment strategies are consistent in optimality.

Obtaining the full solution of the problem depends on the reliability of numerical methods, which is beyond the scope of this dissertation. Instead, I provide some evidence that a CEF firm invests somewhere in between an FC and a UC monopolies: it chooses an abandonment threshold between an FC monopoly's (R) and a UC monopoly's (0). In Figure (1.10), I illustrate the abandonment threshold $\underline{X}^*(R_{CEF})$ for two CEF firms. I use the baseline parameter values in Table 1.2. The blue circled line represents the threshold for a firm that has a cost of capital $0.3((R - X_t)^+)^2 + 1$, and the red line represents the threshold when the cost of capital is $3((R - X_t)^+)^2 + 1$. Both firms take the level of investment R as given.The green dashed line is the 45 degree line. The figure shows that (1) the abandonment threshold is always between 0 and R for a CEF firm. It means a CEF firm abandons later than an FC firm, but earlier than a UC firm. The firm optimally uses some external financing for the project development. (2) A CEF firm that is subject to a higher cost of capital abandons the project earlier than one with lower lost of capital.

[insert Figure 1.10 here]

I solve the threshold numerically using a wide range of parameter values. All those exercises suggest that the abandonment threshold is always lower than their optimal investment scale, and it increases with the degree of financial constraints, and decreases with project scalability. To summarize,

Lemma 4. *When a firm can use the financial market to fund a scalable R&D project by paying a cost, it optimally chooses a threshold on assets in place cash flow below which it abandons the project. This threshold is lower than its optimal investment scale, and it increases with the degree of financial constraints, and decreases with project scalability.*

Furthermore, intuition suggests that a faster decline rate on cash flow from AIP leads to a higher threshold, since it is less likely that the cash flow will come back up to cover the project expense. Moreover, a higher cash flow volatility should induce a CEF firm to choose a lower threshold to allow cash flow to bounce around dramatically without destroying the project. I thus conjecture the following testable implication.

Conjecture 1. *The abandonment threshold in Lemma 4 decreases with the growth rate and volatility of AIP cash flow of a CEF firm.*

1.7 Conclusion

This chapter of my dissertation studies how financial constraints affect a firm's incentive to invest in R&D projects. I build a model to compare R&D investment strategies of a financially constrained monopoly and an unconstrained monopoly. The model generates novel insights by examining project characteristics carefully. I find the effects of financial constraints on R&D investment differ for scalable projects (more investment scales up the expected discovery payoff) vs. accelerable projects (more investment speeds up discovery process in expectation).

Financial constraints always reduce investment on scalable projects. However, with accelerable projects, the termination risk imposed by financial constraints could make a monopoly more aggressive in R&D investments. If a constrained firm can increase the likelihood of project survival by a larger investment, then the over-investment might happen regardless of the resulting higher burn rate of cash flow. This is true even when the firm has a deterministic cash flow from assets in place. It is because the drive for over-investment is the investment strategy's impact on the relative time of project discovery and funding shortage.

A random downward jump on a monopoly's cash flow is studied as another source of termination risk. Unlike financial constraints, such a jump motivates R&D investment for both scalable and accelerable projects. However, it never pushes a constrained firm to over-invest in scalable projects. In addition, I provide some evidence in an extended model that the extent to which a firm is constrained has a monotonic impact on a firm's R&D strategy.

The model generates several testable implications regarding a firm's R&D decision and R&D project values. It also has broader applications by showing the relevance of cross-industry empirical studies to answer the question of how frictions in financial markets affect the real economy. In particular, it suggests that one way to separate industries into different pools is to use the payoff characteristics of new technology development.

Table 1.1 – R&D project characteristics

project type	cash flow before discovery	cash flow at discovery	expected discovery time
scalable	$-Rdt$	$\tilde{u} = \tilde{A}f(R)$	$E(\tau_d) = \eta^{-1}$
accelerable	$-Rdt$	$\tilde{u} = \tilde{A}$	$E(\tau_d) = [\eta I(R)]^{-1}$

Table 1.2 – Baseline parameter values in scalable projects analysis

R&D project parameter	A	λ_d	β	AIP parameter	r	X_0	μ	σ
value	25	0.1	0.7	value	0.05	15	-0.2	0.3

Figure 1.1 – Optimal investment on project characteristics and r- scalable projects

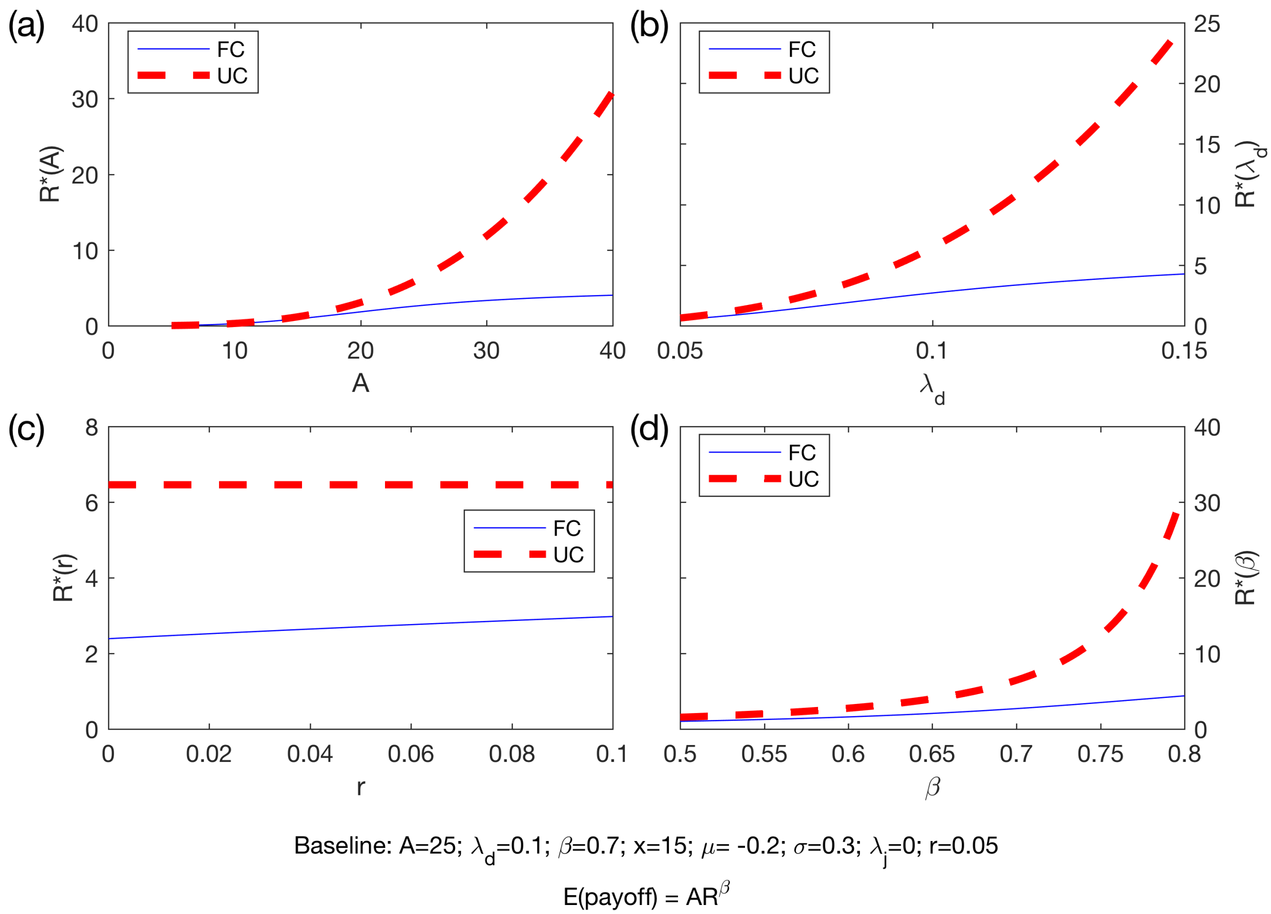


Figure 1.2 – Optimal investment on AIP cash flow parameters - scalable projects

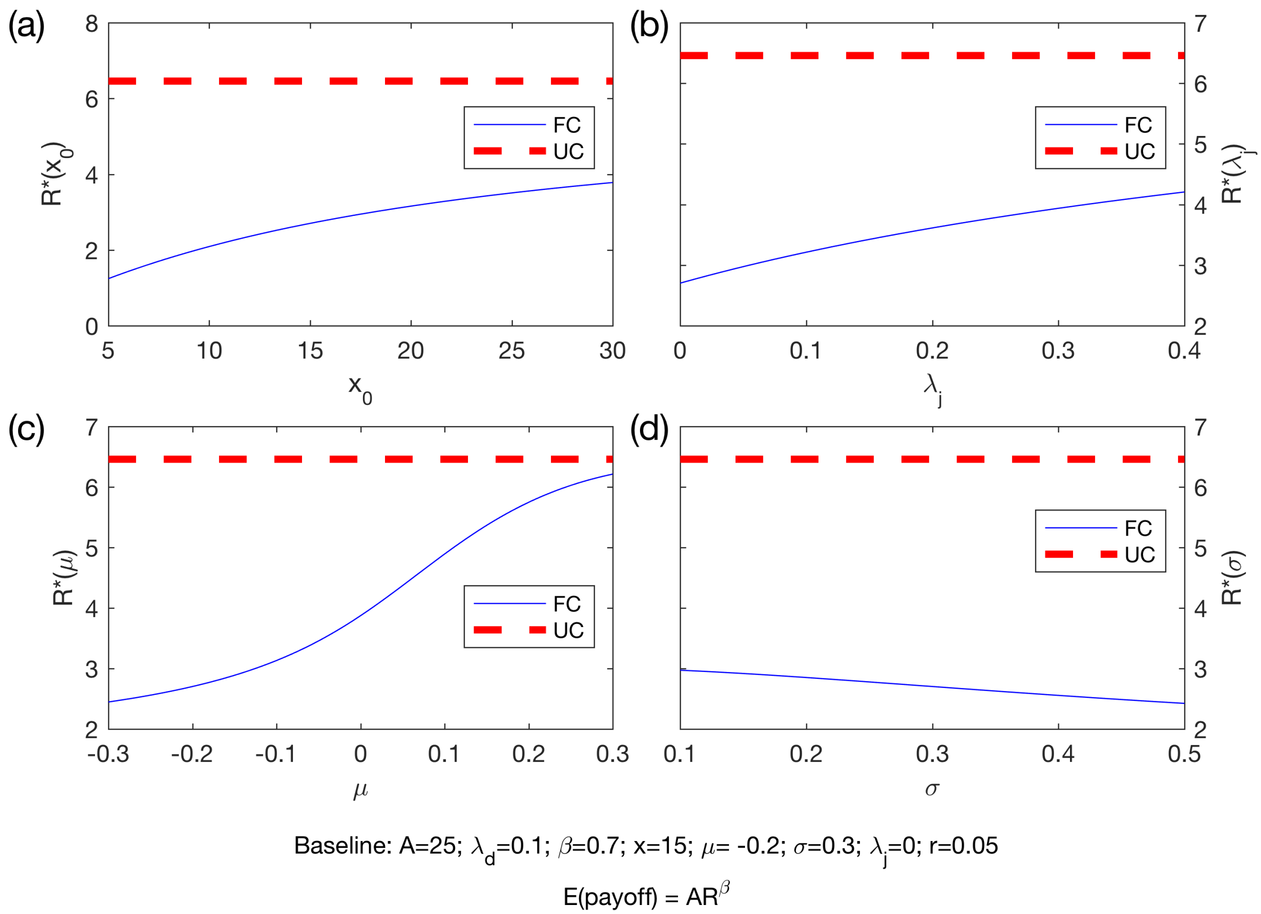
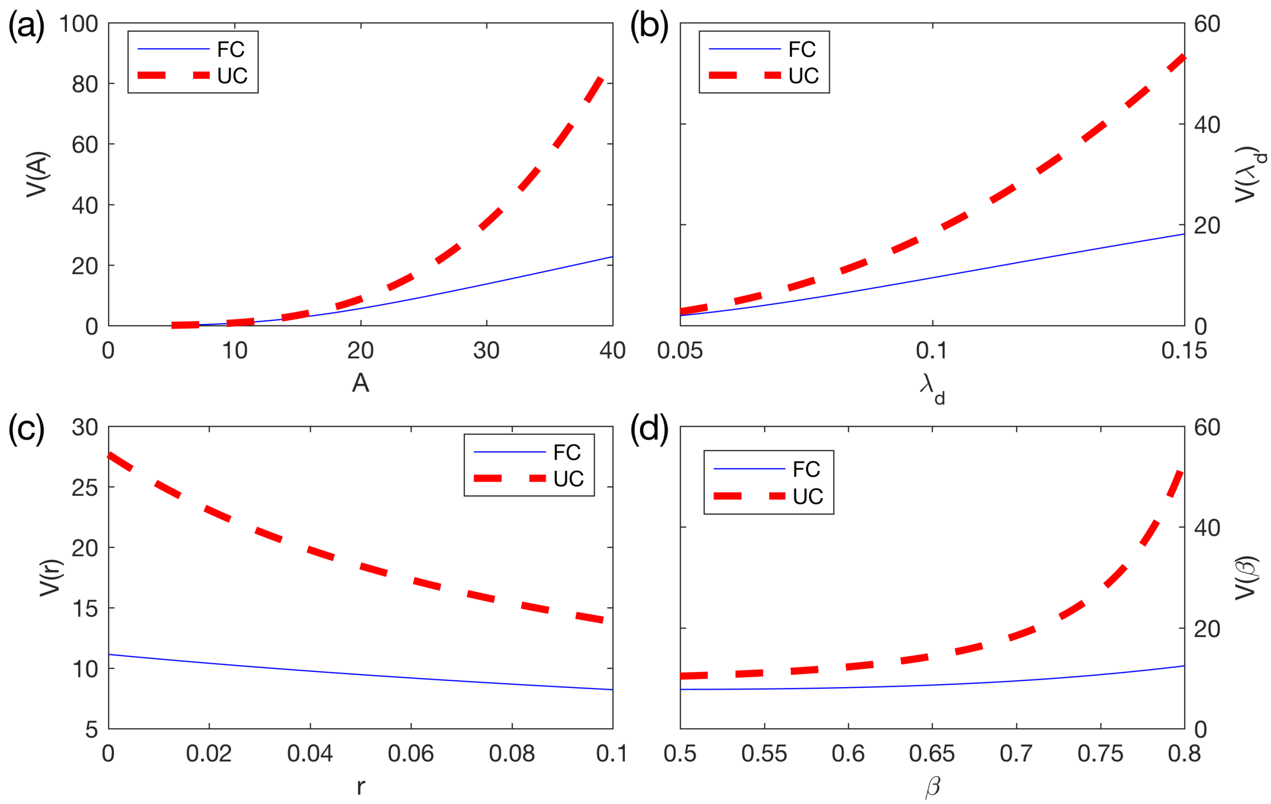


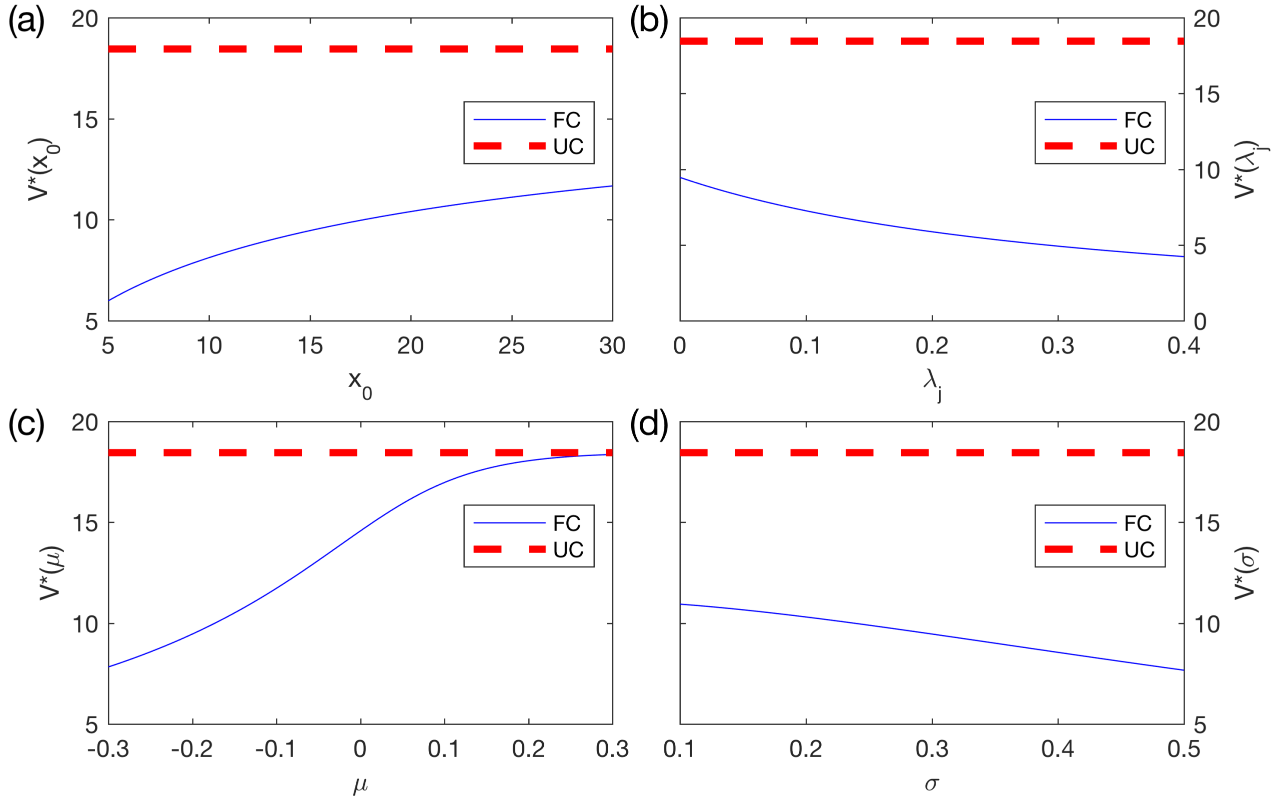
Figure 1.3 – Project value on project characteristics and r- scalable projects



Baseline: A=25; λ_d=0.1; β=0.7; x=15; μ= -0.2; σ=0.3; λ_j=0; r=0.05

E(payoff) = AR^{β}

Figure 1.4 – Project value on AIP cash flow parameters - scalable projects



Baseline: A=25; λ_d=0.1; β=0.7; x=15; μ= -0.2; σ=0.3; λ_j=0; r=0.05

E(payoff) = AR^{β}

Figure 1.5 – Project value decomposition - scalable projects by an FC monopoly

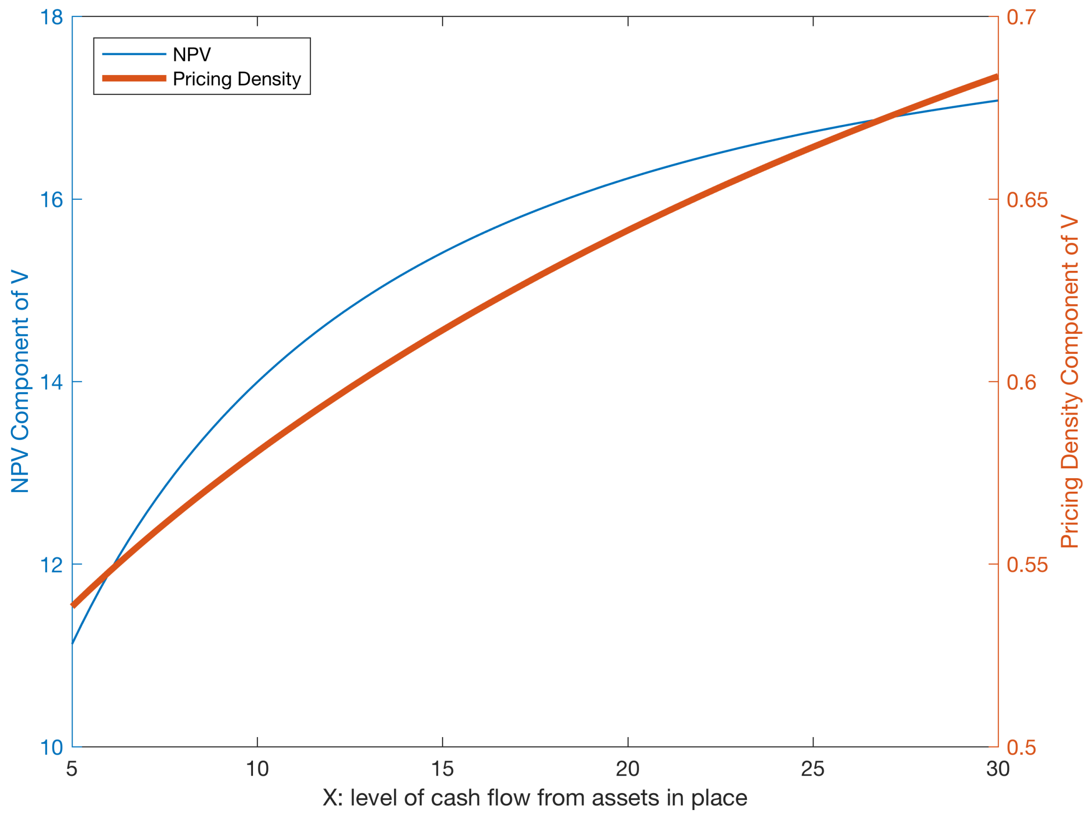
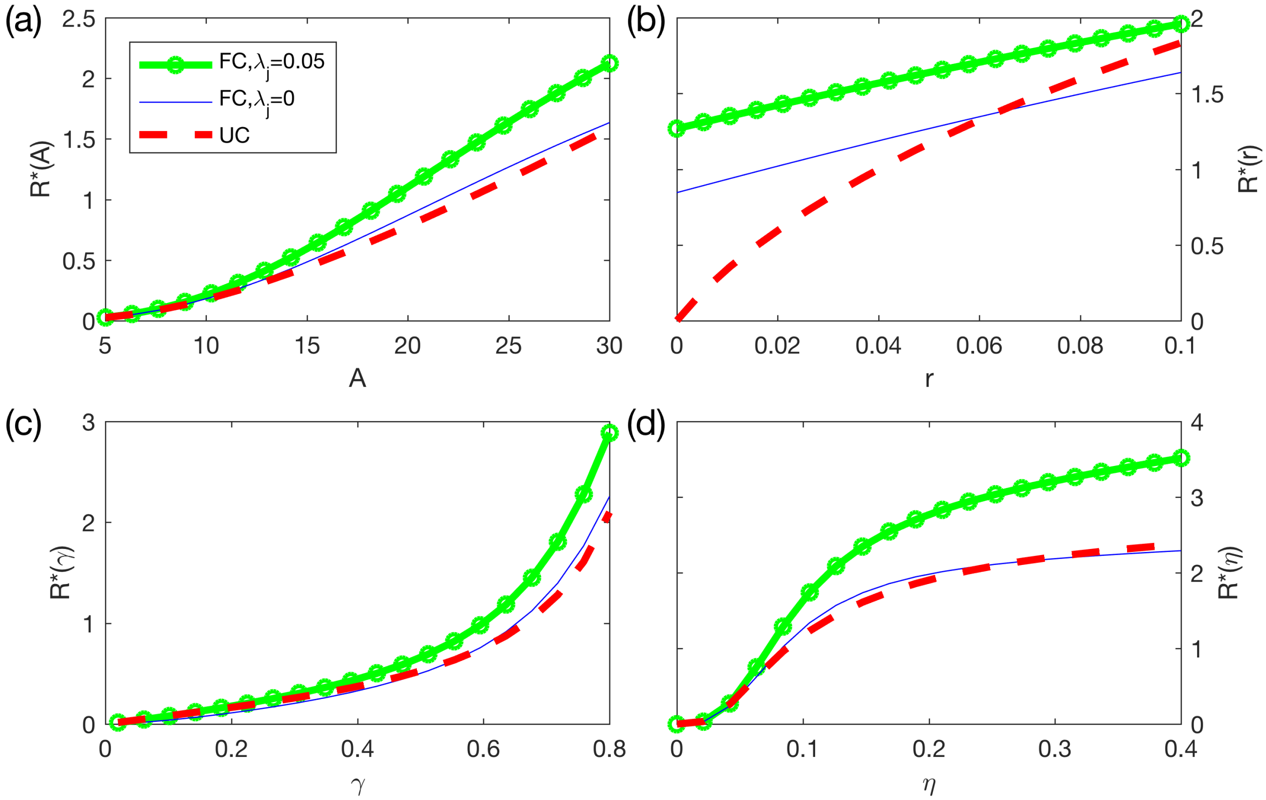


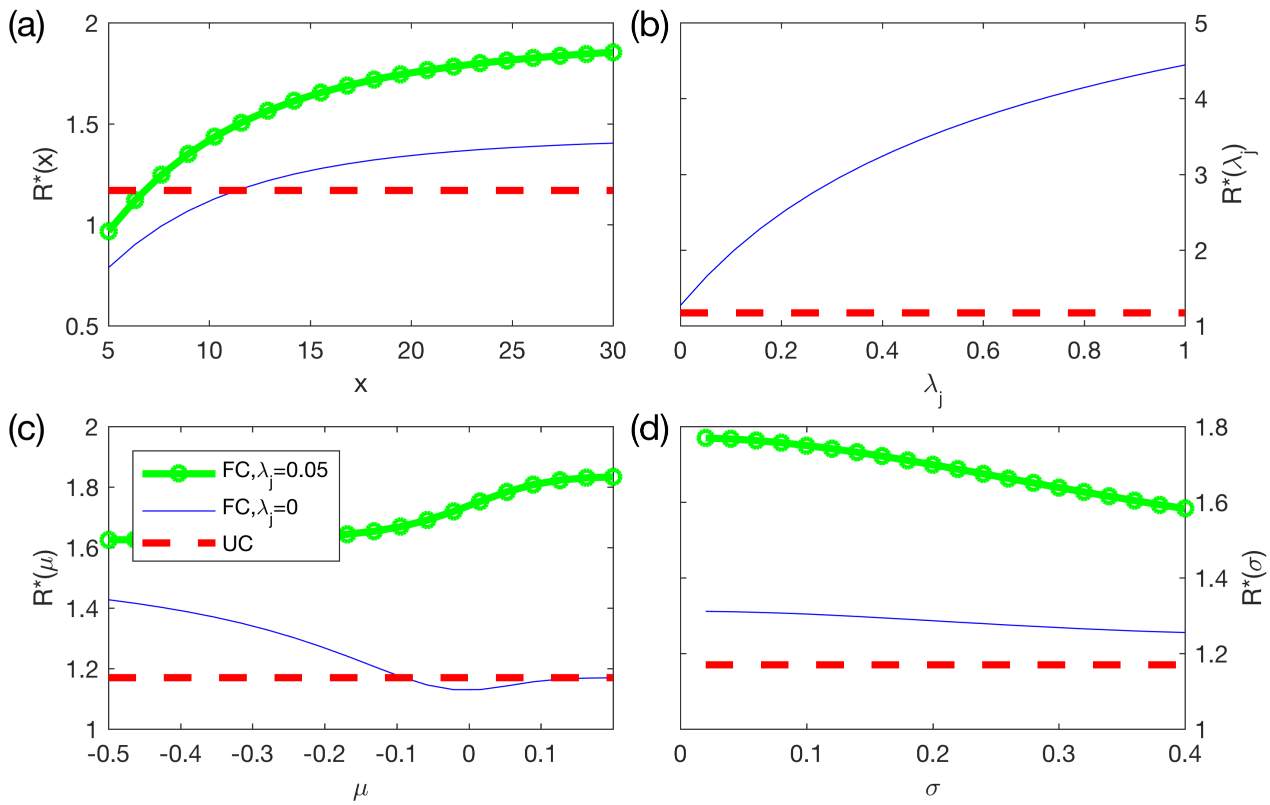
Figure 1.6 – Optimal investment on project characteristics and r- accelerable projects



Baseline: A=25; γ=0.7; x=15; μ=-0.2; σ=0.3; λ_j=0; r=0.05; η=0.1

E(payoff) = A;λ_d= η R^γ; at λ_j=0, R_{uc}=1.17 ; and R_{fc}=1.27

Figure 1.7 – Optimal investment on AIP cash flow parameters - accelerable projects



Baseline: A=25; γ=0.7; x=15; μ=-0.2; σ=0.3; λ_j=0; r=0.05; η=0.1

E(payoff) = A;λ_d= η R^{γ}; at λ_j=0, R_{uc}=1.17 ; and R_{fc}=1.27

Figure 1.8 – Project value on project characteristics and r- accelerable projects

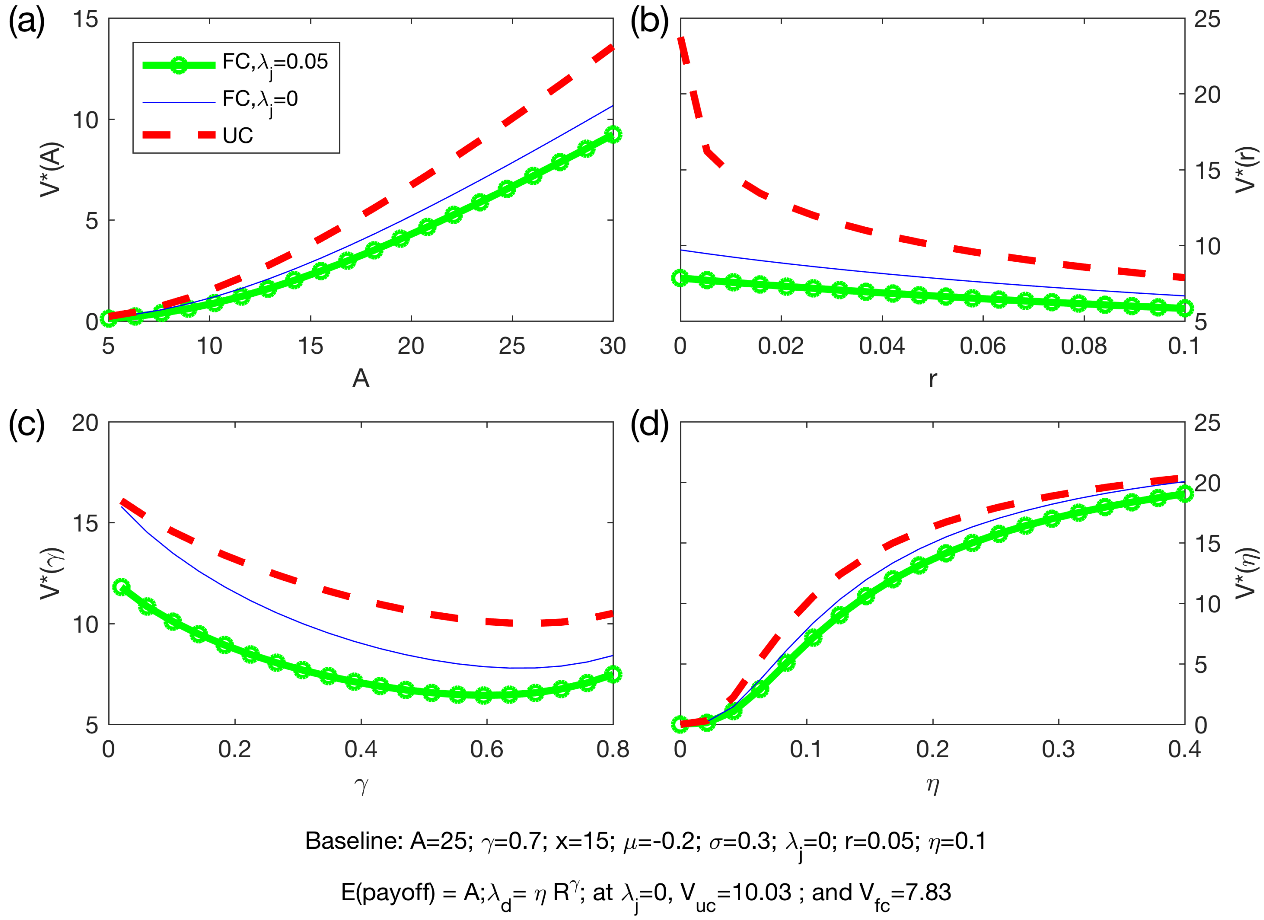
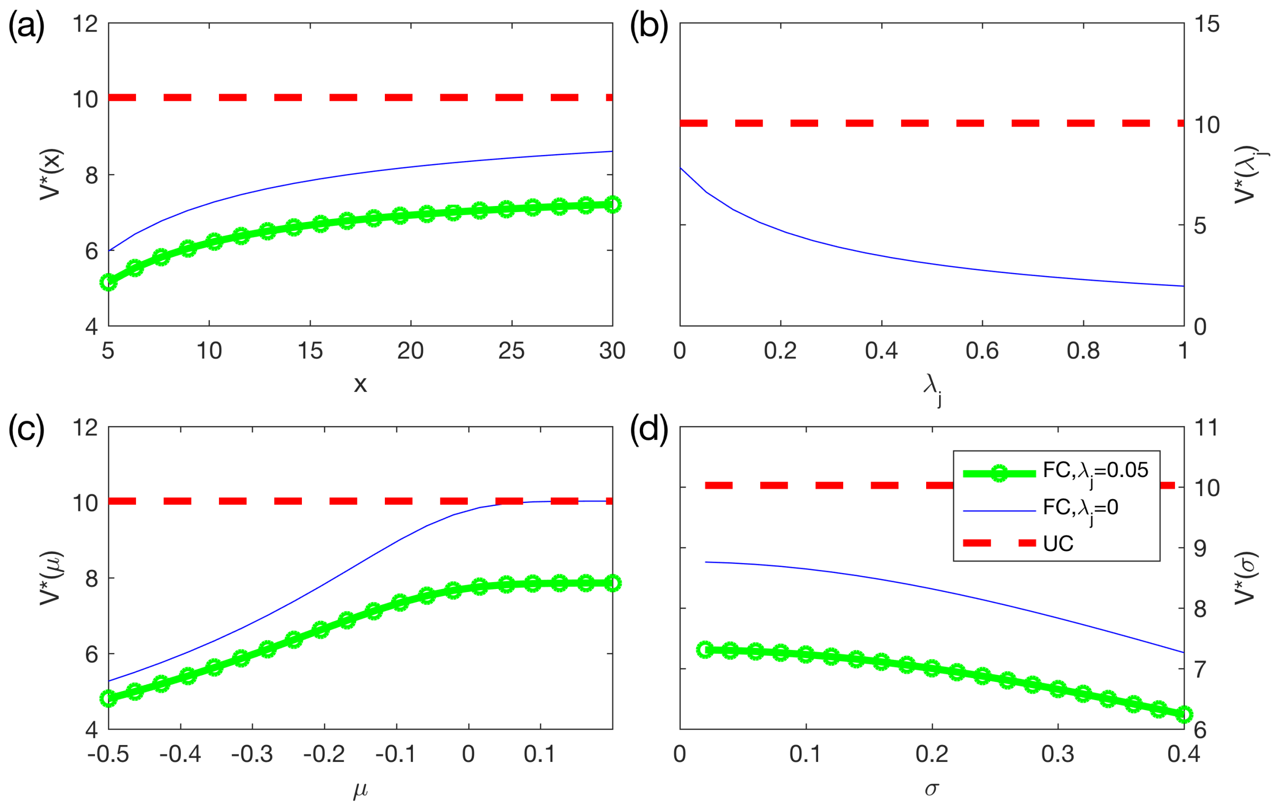


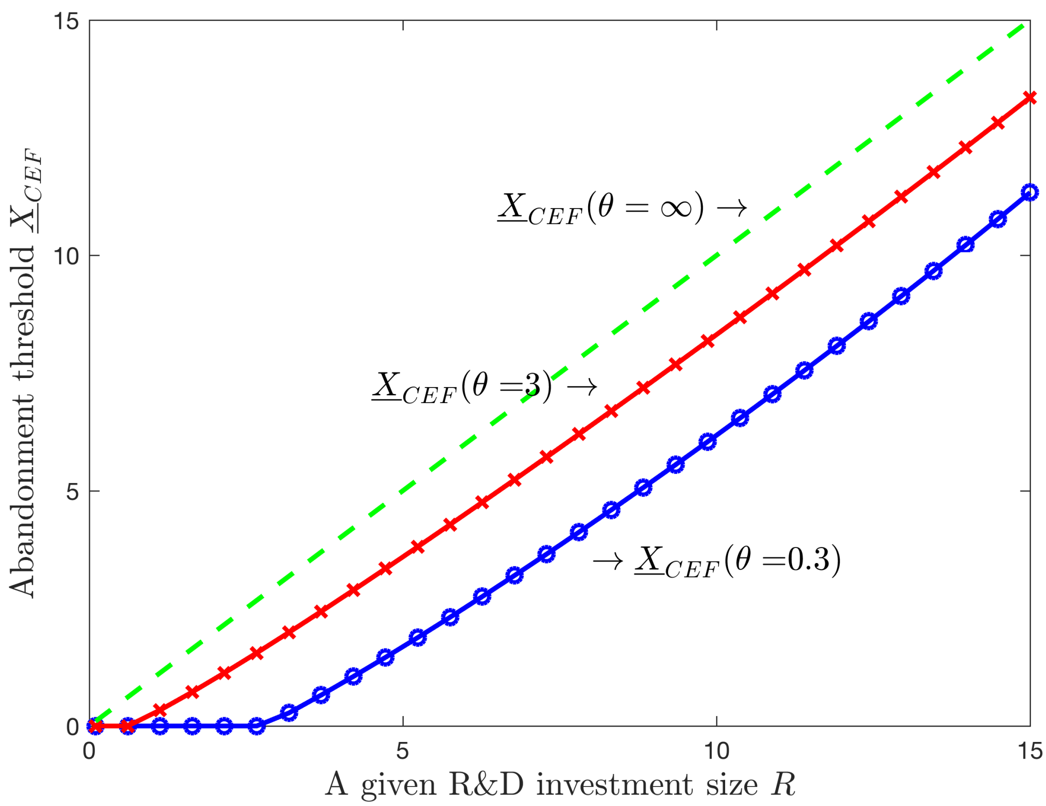
Figure 1.9 – Project value on AIP cash flow parameters - accelerable projects



Baseline: A=25; γ=0.7; x=15; μ=-0.2; σ=0.3; λ_j=0; r=0.05; η=0.1

E(payoff) = A;λ_d= η R^{γ}; at λ_j=0, V_{uc}=10.03 ; and V_{fc}=7.83

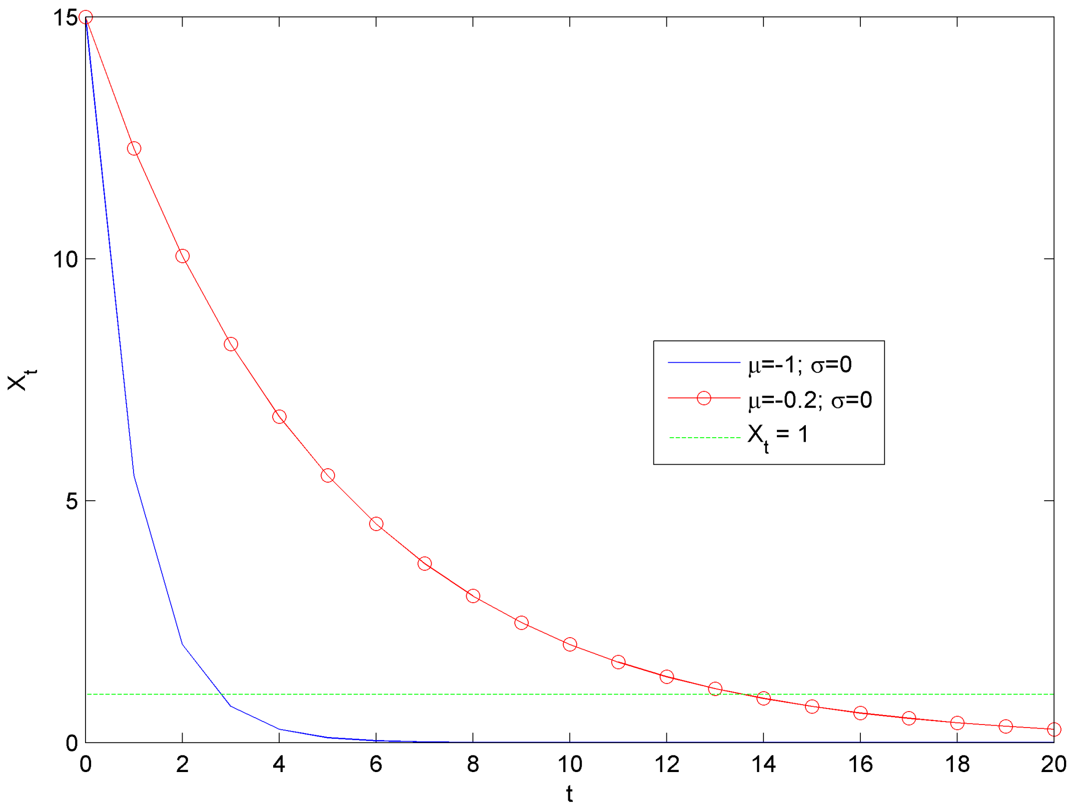
Figure 1.10 – Optimal abandonment threshold for a CEF monopoly



Baseline: A=25; η=0.1; β=0.7; γ=0; μ=-0.2; σ=0.3; X=15; r=0.05

External financing cost = θ * max((R-X),0)^2

Figure 1.11 – Deterministic cash flow over time



X_0 = 15; d $X1_t$ = -1 $X1_t$ dt; d $X2_t$ = -0.2 $X2_t$ dt

2 CHAPTER 2: DUOPOLY MODELS OF FINANCIAL CONSTRAINTS, R&D INVESTMENT, AND COMPETITION

2.1 Introduction

In Chapter One of the dissertation, I study a firm's R&D investment decision in isolation. However, often times, firms compete with each other on R&D projects. In this chapter, I use two duopoly models to study how strategic interactions in innovation affect firms' R&D strategies. Studies of competition and innovation (Aghion, Bloom, Blundell, Griffith, and Howitt (2005), Bena (2008)) suggest that competition motivates innovation especially when competition is not fierce, but the theory models is these papers do not study competition between firms with different financial constraints. My duopoly models are built on the monopoly model in Chapter One and take a step forward to analyze competition among unconstrained firms (homogeneous competitions), and between a constrained firm and an unconstrained firm (heterogeneous competitions). They provides answers to the following questions. Does innovation competition motivate R&D investment for firms that differ in their financial constraints when they compete head to head? Does competition reinforce or hinder the incentive from financial constraints to induce R&D investment? How does a firm's R&D investment depend on its opponent's cash flows and project characteristics?

Firms which compete in innovative projects target similar consumers. The first company that successfully innovates is expected to take a significant market share and earn a large profit. Essentially, innovation competition introduces an obsolescence risk to both firms. Recent papers that study implications for an obsolescence risk of a growth option on risk premium (Berk, Green, and Naik (2004)) or firm boundary (Hackbarth, Mathews, and Robinson (2014)) take this risk as exogenous. In my models, the emergence and magnitude of an obsolescence risk are the results of strategic interactions and determined by market equilibria. The rest of the setup is similar to the monopoly model. Furthermore, I assume the two competing firms know the assets in place cash flows and project characteristics of each firm and make simultaneous decisions on their R&D investment.

The winner-takes-all nature of R&D competition may lead one to believe that such a competition makes a company more aggressive in its R&D strategy. In addition, intuition might suggest that a competitor which does not have to rely on internal capital should always invest more aggressively than a financially more constrained competitor. My findings in this chapter contribute to the literature by uncovering the key elements that matter in understanding the effect of competition and financial constraints on R&D investment.

The widely accepted notion that competition enhances innovation holds only if the project is accelerable or at least one competitor is financially constrained. This benchmark result highlights the importance of project characteristics if we want to identify the effect of competition on innovation. When the project is accelerable, the marginal benefit of winning the competition at a monopoly's optimal level exceeds the marginal cost of investment, which induces investment. Meanwhile, when a duopoly firm is financially constrained, regardless of whether the project is accelerable or not, competition has a positive effect on innovation. This positive effect from competition intensifies if the firm can change the discovery timing in a more influential way by scaling up the investment.

Strategic interactions among firms are critical to their R&D choices if the projects under competition are accelerable. Most surprisingly, I find that a constrained firm may preempt an unconstrained competitor regarding R&D investment. When the probability is high that the constrained firm will be forced out of competition due to a shortage of funds, an unconstrained firm may stop escalating the speed contest, sit on the sideline with a small amount of investment, and count on the possibility to become a monopoly after the constrained firm terminates the project.

The framework in this chapter also provides novel implications regarding R&D project value. The extent to which the value is reduced due to frictions in the financial markets and innovation competition depends on different characteristics of a firm and its competitor. For example, the project value is lower for an unconstrained firm if its financially constrained competitor has higher growth or lower volatility. When two heterogeneous firms compete on the same R&D project, the difference between their project values is larger when the constrained competitor has less capital and higher cash flow risk and expects a bigger decline in its future cash flow. Furthermore, this chapter illustrates the market value of eliminating R&D competition. Such a value enhancement corresponds to the positive difference between the value of a project carried out by an unconstrained firm alone and the sum of project values of two competitors. This chapter contributes to the current debate about whether antitrust policy should be applicable to the domain of research and innovation.

2.2 Duopoly model with homogenous competition

2.2.1 Model setup

This benchmark duopoly model assumes away any financial frictions, so we can focus on studying the market equilibrium when two firms compete on a level playing field financially.

Upon the arrival of an R&D opportunity, two firms $i \in \{1, 2\}$ simultaneously decide whether to capture this opportunity or not by investing in a new project. The firm i that invests needs to choose its investment level R_i. With infinitely high adjustment cost, i has to commit to investing R_i in each period until any of the three things happens: (1) the firm i abandons the project at time $\tau_{c,i}$, (2) the project reaches its discovery at the random time $\tau_{d,i}$, or (3) the competitor (call it firm j) has reached its discovery at time $\tau_{d,j}$ before firm i does. I denote ψ_i as the indicator on investment abandonment, that is, $\psi_{t,i} = 1_{\{t>\tau_{c,i}\}}$. The abandonment decision is permanent, and the scrap value of a half-developed project is assumed to be zero.

Further assume that if both firms invest in this R&D opportunity, then the firm that makes the discovery first gets a high profit by claiming the full market share, whereas the remaining competitor receives nothing[15]. Same as in Chapter One, the random discovery time τ_d is modeled as the first arrival time of a Poisson process with intensity parameter λ_d, which may or may not depend on the firm's investment decision.

[15] This winner-takes-all sharing rule is widely accepted in the patent race literature and captures the nature of innovation competition.

The project characterization also follows Chapter One: an R&D project may be accelerable or/and scalable. Accelerable projects can be expedited by more investment, i.e. the expected discovery time can be shortened by higher R:

$$E(\tau_{d,i}) = \frac{1}{\lambda_{d,i}} = \frac{1}{\eta_i I_i(R_i)}, \text{ and } \frac{\partial I_i(R_i)}{\partial R_i} > 0, \ \forall i \in I; \tag{2.1}$$

while the profit for scalable projects can be magnified by a higher level of investment:

$$E(\tilde{u}_i) = A_i f_i(R_i), \text{ and } \frac{\partial f_i(R_i)}{\partial R_i} > 0, \ \forall i \in I. \tag{2.2}$$

To simplify the analysis and to solve for numerical solutions, I assume $I_i(R) = R_i^{\gamma_i}$ and $f_i(R_i) = R_i^{\beta_i}$ with $\gamma_i > 0, \beta_i > 0$, and $\gamma_i + \beta_i < 1$.

Both firms are financially unconstrained. If the cash flow from a firm's assets in place $X_{t,i}$ is not enough to cover its R&D investment cost R_i, then firm i can issue new equity to fill the instantaneous financing gap $(R_i - X_{t,i})^+$ without any additional cost. To simplify and be consistent with Chapter One, saving is not allowed. Any residual cash flow $(R_i - X_{t,i})^+$ is paid out as dividend in each period. The assets in place cash flow evolves according to a Geometric Brownian Motion with time invariant drift and volatility:

$$dX_{t,i} = \mu_i X_{t,i} dt + \sigma_i X_{t,i} dZ_{t,i} \tag{2.3}$$

The two firms may differ in their assets in place cash flows (X_i,μ_i, σ_i) or the ability to develop an R&D project $(A_i, \beta, \eta_i, \gamma_i)$. All parameter values are common knowledge to both firms, so this is a duopoly market with complete information.

2.2.2 Equilibrium

The strategy set for firm i includes whether to initiate the project, the level of investment, and the abandonment time:

$$S_i = \{1_{\text{initiate the project}}, R_i, \tau_{c,i}\} \in \{\{0,1\}, \mathbb{R}^+, \mathbb{R}^+\}. \tag{2.4}$$

Given the dynamic feature of the game, sub-game perfect Nash equilibrium is the appropriate equilibrium concept in this context, defined according to Fudenberg and Tirole (1991).

Equilibrium Definition A strategy profile $s = \{S_1, S_2\}$ is a sub-game perfect equilibrium (SPE) if at any point in time, given the investment levels of its own and its rival's, as well as whether the project has been abandoned by either competitor, neither firm has any incentive to deviate from the strategies specified in s in that subgame.

In order to figure out the SPE, we need to specify the utilities with each strategy profile. This game has perfect and complete information, so I solve firms' problems by backward induction which consists of three steps:

1. Given any pair of investment levels $\{R_i, R_j\}$ and the cash flows from assets in place $\{X_i, X_j\}$, firm i decides when to abandon the R&D project. I argue that financially unconstrained firms do not voluntarily abandon an on-going project. The reason is simple. If the competitor j has not abandoned its project yet, then firm i should not abandon its own project. It is because the distance to discovery[16] and the

[16] The distance to discovery only depends on the intensities of the Poisson process which are determined by (λ_i, λ_j) in this competitive setup.

discovery payoff has not changed since the project initiation, and the value of the project does not depend on cash flows from assets in place. If the competitor j has somehow abandoned its project, then firm i has a higher probability of success due to the elimination of competition which leads to a higher expected payoff by keeping investing in the R&D project. Therefore $\underline{X}_i = 0$ is a dominant strategy $\forall i \in I$, and $\tau_{c,i} = \infty$.

2. Provided any investment strategy R_j of the competitor, firm i chooses its R_i to maximize its firm value

$$\max_{\tau_{c,i}} E(\int_0^{\tau_{d,i}\wedge\tau_{d,j}} (X_{i,t} - R_i)e^{-rt}dt + \int_{\tau_{d,i}\wedge\tau_{d,j}}^{\infty} X_{i,t}e^{-rt}dt + 1_{\{\tau_{d,i}<\tau_{d,j}\}}e^{-r\tau_{d,i}}\tilde{A}f(R_i)) \tag{2.5}$$

Due to the independence of cash flows from the R&D project and from assets in place, this problem is equivalent to finding the optimal R_i which maximizes the project value:

$$V_i(R_i^*|R_j) = \max_{R_i} U(R_i|R_j) = \max_{R_i} E(\int_0^{\tau_{d,i}\wedge\tau_{d,j}} (-R_i)e^{-rt}dt + 1_{\{\tau_{d,i}<\tau_{d,j}\}}e^{-r\tau_{d,i}}\tilde{A}f(R_i)) \tag{2.6}$$

The solution is in the form of best response function $R_i(R_j)$. The fixed point of the best response correspondences (R_i^*, R_j^*) is part of the sub-game perfect Nash equilibrium profile. Denote V_i^* as the project value in the equilibrium.

3. Firm i will carry out an R&D project if and only if $V_i^* > 0$.

Step 1 and Step 3 are straightforward, I focus on Step 2 in the following analysis. We can figure out solutions to Step 2 by writing down the value functions for each firm, as shown in Proposition 7.

Proposition 7. *When two financially unconstrained firms compete against each other in an R&D project, their project values in the equilibrium are*

$$V_{UC1} = \frac{u_{UC1}\lambda_{d,UC1} - R_{UC1}}{r + \lambda_{d,UC1} + \lambda_{d,UC2}} \tag{2.7}$$

$$V_{UC2} = \frac{u_2\lambda_{d,2} - R_{UC2}}{r + \lambda_{d,UC1} + \lambda_{d,UC2}} \tag{2.8}$$

where (R_{UC1}, R_{UC2}) *are a pair of equilibrium investment levels,* u_i *is the expected one-time project payoff for Firm* i *and* $\lambda_{d,i}$ *is Firm* i*'s discovery rate.*

Proof. See the appendix.

As a basic check for Proposition 7, project values are independent of a firm's own and its rival's assets in place cash flows when both duopoly firms are unconstrained. The intuition is that when the capital market runs perfectly and the information is complete, the value of an investment project only depends on the project's characteristics (and potentially its competing projects' characteristics), but not on the characteristics of the firm which carries out such a project.

In addition, the project value for a UC duopoly is similar to a UC monopoly, with the modification that the competitor's success intensity is included in the discount rate. This is consistent with the interpretation that competition is another source of obsolescence risk (or termination risk). Such a risk reduces project values by heavier discounting.

To understand the existence and uniqueness of a Nash equilibrium in Step 2 of this duopoly model, I look more closely at firms' best response functions[17]. Based on Proposition 7, we can get some useful properties of a firm's investment behavior as the best response of its competitor:

Corollary 3. *In a homogeneous duopoly, for any pure investment strategy of the competitor j, firm i always has a unique pure strategy best response. Furthermore, this best response is weakly increasing and convex in the rival's investment scale.*

Proof. See the appendix.

The first half of Corollary 3 helps to establish our focus on pure strategy equilibrium. The second half ensures the existence and the uniqueness of a pure strategy Nash Equilibrium in Step 2. The positive effect of a rival's investment behavior is directly tied to the head-to-head kind of race we use to model innovation competition. When a firm competes against a more aggressive rival on a "Win or Perish" basis, we may expect the firm to act more aggressively. Here is the intuition: by reducing investment on the margin, a firm does not save much on the cost, but it risks losing the race against the rival and getting nothing instead of the entire market share in the competition. This is not an optimal decision.

Figure 2.1 illustrates the Nash equilibrium (R^*_{UC1}, R^*_{UC2}) of the homogeneous duopoly model, with each firm taking the baseline parameter values specified in Table 2.1. The

[17] Without the properties of best response functions, the challenge to prove the existence of a pure strategy equilibrium is two-folded: (1) this is an infinite strategic form game, and (2) we cannot exclude the possibility of a mixed strategy equilibrium. These two challenges prevent us from using Kakutani's Fixed Point Theorem. We cannot readily use Debreu-Glicksberg-Fan Theorem to prove the existence of pure strategy equilibrium in this infinite game, because we cannot establish the concavity of V_i in R_i on $R_i \in \mathbb{R}^+$.

two best response functions $R^*_{UC2}(R_{UC1})$ and $R^*_{UC1}(R_{UC2})$ are indeed increasing and concave, as suggested by Corollary 3. The unique equilibrium in this example is symmetric as we would expect from a duopoly with identical firms.

[insert Table 2.1 here]

[insert Figure 2.1 here]

One may ask if there is some equilibria in which only one firm invests and the other firm stays out of the competition. The answer is no in this model. Given that there is no fixed cost of investment upfront in the model and the investment size is infinitely divisible, a firm always chooses a positive level of R&D investment. This is because the marginal benefit of investment is infinitely large at $R = 0$, and it is much larger than the marginal cost of investment. A firm always benefits from at least investing a tiny amount to obtain this growth option and increases its chance of winning the race. Section 2.2.5 discusses what happens to the equilibrium when fixed investment cost is introduced in the model.

Next, I consider the effect of competition on firms' investment decisions in more details, and examine how an unconstrained firm's own and rival project characteristics matter for R&D investment decisions.

2.2.3 The effect of competition on investment

I row one step back from Corollary 3 and consider the effect of competition on investment. This effect depends on project characteristics. Competition has no effect on innovation if the project is only scalable but not accelerable. Evidently from Proposition 7, if discovery speed λ_d does not depend on a firm's R&D investment level R, then R_i^* is not part of the discount. Thus R_i^* shall simply maximize the average instantaneous payoff of the project. Obviously, such an investment level satisfies the same first order condition as a monopoly firm. In other words, if an innovation project cannot be accelerated with heavier investment, then the optimal investment level in this duopoly is the same as in the monopoly model. This is because when an innovative project is only scalable, competition does not effectively change a firm's tradeoff between a higher investment cost and a larger expected payoff. Instead, competition reduces the marginal benefit and cost equally by imposing the risk to terminate.

However, if the project is accelerable, intuition suggests that both firms shall be motivated to speed up its discovery process, comparing with the monopoly case. A firm in a competitive environment expedites the R&D process by heavier investment, in order to capture the entire market share and lower the probability of having to terminate the project because it becomes worthless. Figure 2.1 shows that the duopoly investment is more than the monopoly scale. The motivational effect in this baseline parameterization is quite significant: equilibrium investment level changes from 6.8 to 35.7. We can relate the effect of competition on innovation with project characteristics, as stated in the following result.

Corollary 4. *Competition among unconstrained companies motivates firms to increase R&D investment if the project is accelerable, but it does not change R&D investment incentive if the project is only scalable.*

Proof. See the appendix.

Previous studies regarding the effect of competition on innovation give mixed predictions. Early industrial organization theories agree with the Schumpeterian view, and suggest that innovation declines with competition (the "Schumpeterian effect") due to a reduction of the post-entry rents for new entrants[18], or by reducing the monopoly rents that reward new innovation[19]. However, the replacement effect[20], and the efficiency effects[21] predict more innovation with competition[22]. Most of the empirical work in this area supports a positive effect of competition on innovation[23], but some later ones find a negative relationship[24]. More recently, some models point out non-monotonic relationships between innovation and competition. For example, Aghion, Bloom, Blundell, Griffith, and Howitt (2005) finds an inverted-U relationship, but Tishler and Milstein (2009) find a U-shaped relationship.

Corollary 4 shows the importance of project characteristics in studying the effect of competition on innovation. For accelerable projects, when the market structure changes from a monopoly to a duopoly for some exogenous reason, the "Escape-Competition" effect in Aghion, Bloom, Blundell, Griffith, and Howitt (2005) prevails, but the Schumpeterian effect is not there. If we interpret one of the duopoly firms as an entrant to the monopoly, the new entrant is at a technological par with the incumbent in my model. By construction, a successful innovation by the entrant makes her the winner, and com-

[18]e.g., Salop (1977), and Dixit and Stiglitz (1977).

[19]e.g. models of endogenous growth, such as Romer (1990), Aghion and Howitt (1992), and Grossman and Helpman (1991).

[20]Such as Arrow (1962).

[21]as in in Gilbert and Newbery (1982) and Reinganum (1983).

[22]When there is agency cost, Hart 1983 argues increased competition may induce more efforts from the managers to innovate when managers draw private benefit from keeping their job.

[23]e.g. Geroski (1994), Nickell (1996), Blundell, Griffith, and Van Reenen (1999), and Bloom, Draca, and Van Reenen (2015).

[24]e.g. Hashmi (2013).

petition does not deter an entrant from innovating because the post-innovation rent is high. To map into Aghion, Bloom, Blundell, Griffith, and Howitt (2005), duopoly firms in my model are always in the neck-and-neck industry structure. In other words, firms compete head-to-head without a step-by-step innovation.

However, for projects which are only scalable, neither Escape-Competition effect nor Schumpeterian effect of competition are in place. What Corollary 4 helps to uncover is that, absent of financial market frictions, a head-to-head duopoly competition does not affect a firm's R&D investment incentive if the innovation project is not accelerable. This is not very surprising if we narrow our attention to speed competition, but it is an omitted result in the industrial organization literature as well as the patent race literature.

2.2.4 More about project characteristics

We've established the positive effect of competition on innovation for accelerable projects in Section 2.2.3. A natural question is: what project characteristics will lead to a stronger positive effect of competition on innovation? Although the following comparative statics cannot be analytically proved, numerical solutions from a wide range of parameters values suggest:

Conjecture 2. *The positive effect of competition on R&D investment is stronger if the innovation project has a higher expected payoff, or if it is less costly to accelerate.*

Figure 2.2 shows how the effect of competition on R&D investment changes with project characteristics around the baseline parameterization. Panel (a) and Panel (b) confirm the first part of Conjecture 2 that a better project payoff intensifies the R&D race: an unconstrained firm chooses a much higher investment in the duopoly equilibrium than in a monopoly market if A or β is high. Of course, if the project is not accelerable (which is not plotted), then the two lines which indicate the investment levels in duopoly and monopoly will always overlap. Panel (c) and Panel (d) show a very strong effect of competition on R&D investment in projects that have higher accelerability.

[insert Figure 2.2 here]

What happens to the positive effect of competition on R&D investment if firms in a duopoly have distinct abilities to innovate? Consider a duopoly in which one firm has a technological advantage over the other. For example, it has better outcomes in a successful discovery ($A \uparrow$ or $\beta \uparrow$), or it is able to get success sooner ($\eta \uparrow$ or $\gamma \uparrow$). We may reinterpret a duopoly with firms very close in their technologies as a "neck-and-neck" competition, and a duopoly with firms differ in their technologies as a "leader-follower" competition[25]. I propose the following conjecture, which is based on numerical solutions.

Conjecture 3. *The positive effect of competition on R&D investment is stronger in a neck-and-neck competition than in a leader-follower competition.*

This result is similar to Aghion, Bloom, Blundell, Griffith, and Howitt (2005) in that a leader-follower competition has a weaker impact on innovation than a neck-and-neck competition. However, it differs from that study and show that there is a positive effect of competition on innovation even in a leader-follower competition.

[25] Aghion, Bloom, Blundell, Griffith, and Howitt (2005) and Bena (2008) use the same terminologies differently.

2.2.5 Extension with a fixed cost of investment

If we modify the model and add a fixed cost of R&D investment upfront, then an analysis on project values can give us some prediction on entry decisions.

From Proposition 7 and Figure 2.3, it is clear that the project value is lower for a duopoly firm than for a monopoly firm. This is because competition makes a firm deviate from its first best strategy and also sometimes force a project termination. Now suppose a firm has to incur a fixed cost C to obtain this R&D option, and the fixed cost is at a level such that $C \in (V_{UC}^{duopoly}, V_{UC}^{monopoly})$. Then in a pure strategy equilibrium, one firm invests at a monopoly level while the other firm does not invest at all[26]. One interesting observation from Figure 2.3 is that the project characteristics related with accelerability (η in Panel (c) and γ in Panel (d)) may have a negative effect on duopoly project value. The intuition is that higher project accelerability exacerbates the R&D war which reduces project value.

[insert Figure 2.3 here]

In reality, firms often differ in their technologies when pursuing the same R&D opportunity. We confidently conjecture that there exists a range of R&D fixed cost that prevent a low technology firm from entering the competition against a high technology incumbent[27]. When the two duopoly firms differ in their R&D technologies (any one of A, β, η, γ, or a combination), a fixed cost to start the R&D project may make the project value for a low technology firm to be negative in a duopoly. Yet the project value re-

[26] We will also have at least one mixed strategy equilibrium.
[27] Numerical solutions verify this conjecture.

mains positive for the high technology firm in a duopoly. Thus, there will be a unique pure strategy equilibrium that only the high technology firm invests. In other words, an entry barrier benefits an incumbent who has a better technology by preventing an R&D war. Absent of the entry barrier, such a war unambiguously leads to a lower payoff for the incumbent. While this is obvious, the social welfare implication of a barrier to enter is not clear and it depends on one's view regarding whether there is too much or too little R&D effort in the economy[28]. If one believes that there is too little innovation in the economy, say due to the positive externality of innovation, then the barrier to enter a new market is harmful. If one believes that there is already too much effort in R&D wars than socially optimal because of the huge deadweight lost from unsuccessful innovation, then a fixed cost of investment on R&D projects can be helpful to reduce the waste of resources.

2.3 Duopoly model with heterogenous competition

2.3.1 Model setup

The model setup follows Section 2.2, except with one variation: one of the two firms is financially constrained. The definition of financial constraints follows Chapter One. A firm subject to financial constraints faces infinite cost of raising capital externally. The other firm in the duopoly is financially unconstrained and can fund its investment from the capital market with no extra cost. We call this duopoly a heterogenous duopoly.

Same as in Section 2.2, duopoly firms in this setting compete in an R&D race, and the winner takes the entire market share. Comparing with its unconstrained rival, the FC

[28]See Jones and Williams (2000) for a reference on such a debate in the literature.

firm may not have sufficient funds to keep investing in the R&D project even if doing so is optimal absent of the constraints. When that happens, the FC competitor will have to abandon the project involuntarily, and the duopoly becomes a UC monopoly. We are again interested in finding a sub-game perfect Nash equilibrium, and a equilibrium strategy profile is defined in the same way as in Section 2.2. We will mainly focus on Step 2 of the backward induction for this sequential game.

2.3.2 Value functions and best responses

Similar to the argument in *Lemma* 1 from Chapter One, the two firms in this heterogeneous competition do not abandon their projects voluntarily. Since the UC competitor's cash flow has no impact on its own decision and thus its rival's decision, the only state variable for both the FC and UC firms is the constrained firm's assets in place cash flow. The maximization problem in Step 2 for an unconstrained firm is:

$$\max_{R_{UC}} E\Big[\int_0^{\tau}(-R_{UC})e^{-rt}dt + (1 - 1_{\{\tau_{d,FC}<\tau_{d,UC}\wedge\tau_{c,FC}\wedge\tau_j)\}})e^{-r\tau_{UC}}\tilde{A}f(R_{UC})\Big],$$

where $\tau_{d,UC}$ and $\tau_{d,FC}$ is the time of discovery for the UC and FC firms, and $\tau_{c,FC}$ is the constraint-hitting time for the FC firm, τ is the time that the UC firm stops investing. If $\tau_{d,FC} < \tau_{d,UC} \wedge \tau_{c,FC} \wedge \tau_j$, then $\tau = \tau_{d,FC}$; otherwise, $\tau = \tau_{d,UC}$.

The problem for a constrained firm is

$$\max_{R_{FC}} E\Big[\int_0^{\tau_{d,UC}\wedge\tau_{d,FC}\wedge\tau_{c,FC}\wedge\tau_j}(-R_{FC})e^{-rt}dt + 1_{\{\tau_{d,FC}<\tau_{d,UC}\wedge\tau_{c,FC}\wedge\tau_j)\}}e^{-r\tau_{d,FC}}\tilde{A}f(R_{FC})\Big].$$

The project values for the duopoly firms can be simplified as follows.

Proposition 8. *In a heterogeneous duopoly, the R&D project values for the financially constrained firm and the unconstrained firm are*

$$V_{UC}(X_{FC}) = V_{UC}^{mon}(\frac{X_{FC}}{R_{FC}})^{\alpha} + \frac{u_{UC}\lambda_{d,UC} - R_{UC} + \lambda_j V_{UC}^{mon}}{r+\lambda_{d,UC}+\lambda_{d,FC}+\lambda_j}(1-(\frac{X_{FC}}{R_{FC}})^{\alpha}) \quad (2.9)$$

$$V_{FC}(X_{FC}) = \frac{\lambda_{d,FC}u_{FC} - R_{FC}}{r+\lambda_{d,UC}+\lambda_{d,FC}+\lambda_j}(1-(\frac{X_{FC}}{R_{FC}})^{\alpha}) \quad (2.10)$$

respectively, where (R_{FC}, R_{UC}) *are the equilibrium investment levels,* $\lambda_{d,UC}$ *and* u_{UC} *(*$\lambda_{d,FC}$ *and* u_{FC}*) are the equilibrium discovery intensity and the expected one-time project payoff of the UC firm (the FC firm), and* λ_j *is the rate of a jump in the FC firm' assets in place that wipes out all of its future cash flows.* $\alpha = \frac{1}{2} - \frac{\mu_{FC}}{\sigma_{FC}^2} - \sqrt{(\frac{1}{2}-\frac{\mu_{FC}}{\sigma_{FC}^2})^2 + \frac{2(r+\lambda_{d,UC}+\lambda_{d,FC}+\lambda_j)}{\sigma_{FC}^2}}$, V_{UC}^{mon} *is the monopoly project value for the UC firm, i.e.* $V_{UC}^{mon} = \frac{\mu_{UC}\lambda_{d,UC} - R_{UC}}{r+\lambda_{d,UC}}$.

Proof. See the appendix.

We know (and clearly from Proposition 8) there are two possibilities with regard to the market structure at the time of a discovery. It is a duopoly market if both firms were still developing the project right before the discovery τ_d. It is a monopoly if the constrained firm had already been forced to abandon the project due to its shortage of funds and left the competition. Eq (2.9) shows that for an unconstrained firm, the value of an R&D project is the weighted average of the two cases, with $1-(\frac{X_{FC}}{R_{FC}})^{\alpha}$ being the probability of the first case, and $(\frac{X_{FC}}{R_{FC}})^{\alpha}$ being the probability of the second case. In the first case, the project value coincides with Eq (2.7) in Proposition 7, as if the FC competitor is a UC firm in the homogeneous duopoly. In the second case, the project value is simply the monopoly value. Eq (2.10) shows the value of an R&D project for the constrained firm is also a weighted average of the two cases, with the value being zero in the second case and thus it is omitted from the expression.

Proposition 8 also suggests that Corollary 4 from the homogeneous duopoly model holds here for the UC firm. Eq (2.9) indicates the optimal R&D investment for the UC firm satisfies$\lambda_{UC} u'_{UC} = 1$ if the project is not accelerable ($\lambda'_d = 0$). This is the same condition for the monopoly case. It strengthens the conclusion that competition alone does not incentivize a UC firm to innovate more in an R&D race. We need project accelerability to generate the impact of competition on innovation.

[insert Figure 2.4 here]

The heterogeneous model offers new insights regarding the strategic interactions between the duopoly firms. I show in *Figure 2.4* an example of firms' best responses to their competitor's R&D investment. Model parameter values follow *Table* 2.1. The green circled line plots the FC firm's optimal R&D investment as a function of its UC rival's investment scale. It is monotonically increasing, concave, and converging to a fixed level in this example. The monotonicity and concavity is similar to a UC firm's best response in the homogeneous duopoly described in *Figure 2.1.* The intuition behind such similarity is that the FC firm has very little chance in winning if it backs down from competing head to head with the unconstrained competitor. As long as the project still has a positive NPV, the FC firm is willing to incur a higher cost of investment and risk hitting the constraints sooner. The convergence of FC firm's investment as UC firm invests above certain level is related with the limitation of the FC firm in the competition.

More interestingly, the UC firm's best response follows an inverted-U shape, plotted by the blue solid line in *Figure 2.4*. When the FC firm invests at a low level, the UC firm responds positively as FC invests more heavily in the race. After the FC firm reaches a certain level of investment, the UC firm reduces its R&D investment as the FC firm becomes more aggressive. The decreasing part of UC's best response is driven by its consideration that the constrained rival will run out of money soon. As the FC firm becomes more and more aggressive in the R&D strategy, the UC firms just wants to stay in the game and wait for the FC firm to burn out of the money and give up in the race. I conjecture these facts remain true in reasonable ranges of parameter values.

Conjecture 4. *In a heterogeneous duopoly, the best response of a constrained firm with regard to its unconstrained firm's investment is increasing and concave regardless of the project type. The best response of an unconstrained firm follows an inverted-U shape if the project is accelerable, and it remains at the monopoly level if the project is only scalable.*

Figure 2.1 and Figure 2.4 together suggest that a firm reacts to an unconstrained competitor's larger investment by making higher efforts in R&D, regardless of whether itself is financially constrained or not. It is because a UC rival would never abandon the project once it is started, thus a firm which competes with a financially capable rival faces a war of R&D that never ends with the other competitor quitting. In such cases, a firm acts more aggressively in the race hoping to expedite its discovery and offset the increased probability of losing the contest. Although it will be interesting to study whether a firm in competition always reacts in an inverted U shape fashion with regard to a constrained rival's investment, it is not in the scope of this chapter. We will need a homogeneous duopoly model with two constrained firms to have a full picture of the answer. I leave this under-explored area to future work[29].

2.3.3 Equilibrium

The equilibrium R&D investment levels in the heterogeneous duopoly depend on project characteristics as well as the constrained firm's assets in place cash flow. Due to the complexity of value functions in Proposition 8, we turn to numerical solutions to understand the properties of such equilibria.

In the baseline example plotted by Figure 2.4, the market equilibrium is marked as the cross[30] of two best response functions. In this duopoly equilibrium, $R^*_{UC} = 31.2$ and $R^*_{FC} = 31.3$. Both firms invest significantly higher than their monopoly levels, in the magnitude of multiple times (see the intercepts of the best responses or Chapter 1, which are 6.8 and 13.8 for UC and FC respectively). Notice that the constrained firm still invests more than the unconstrained firm even in a competitive setting. For a more

[29] The duopoly with two constrained competitors shall be interesting to investigate. However, this is a two-dimension free boundary problem which is very challenging to solve, and this chapter will not discuss equilibria in this case.

[30] Numerical solutions suggest the fixed point for firms' best response correspondences is unique when model parameters are within reasonable ranges.

detailed comparison, *Table* 2.2 lists the equilibrium investment scales and project values for firms in different scenarios.

[insert Table 2.2 here]

Not surprisingly, both the UC and the FC firms experience a large drop in project values comparing with the monopoly case. The unconstrained firm's project value drops from 52.6 in the monopoly to 23.1 in the heterogeneous duopoly, and the constrained firm's project value drops from 37.6 to 20.4. The obsolescence risk associated with the rival's success leads to involuntary termination of the R&D investment. Therefore, the R&D race distorts firms from their first best investment strategies by pushing firms to invest more aggressively and induces value reduction.

Table 2.2 also shows that competition between two unconstrained firms is more fierce than the heterogeneous duopoly. For an unconstrained firm, competition with another unconstrained firm leads to more aggressive investment (35.5 in the homogeneous duopoly vs. 31.2 in the heterogeneous duopoly) and lower project value (21.7 vs 23.1).

We conjecture $R_{FC}^{\text{mply}} < R_{FC}^{\text{hetero-dply}}$, $R_{UC}^{\text{mply}} < R_{UC}^{\text{hetero-dply}} < R_{UC}^{\text{homo-dply}}$, $V_{FC}^{\text{mply}} > V_{FC}^{\text{hetero-dply}}$, and $V_{UC}^{\text{mply}} > V_{UC}^{\text{hetero-dply}} > V_{UC}^{\text{homo-dply}}$, but the comparison between the UC and FC firms depend on project and FC firm's cash flow characteristics.

2.3.4 Comparative statics on equilibrium investment and project value

To study market equilibria more broadly, I apply comparative statics analysis to study the effects of changes in one competitor's characteristics on equilibrium investments and corresponding project values.

2.3.4.1 FC firm's assets in place cash flow

While the UC firm's cash flow has no effect on a heterogeneous duopoly, the FC firm's assets in place cash flow influences market equilibria through its effect on FC firm's investment motives. *Figure 2.5* plots some investment comparative statics regarding changes in the FC firm's cash flow characteristics in the baseline example. A market equilibrium is represented by a pair of investment scales that share the same horizontal value, with the UC firm's equilibrium investment on the blue solid line and the FC firm on the red dotted line.

[insert Figure 2.5 here]

Panel (a) of *Figure 2.5* shows the equilibrium investments in a heterogeneous duopoly when we vary the FC firm's cash flow at the arrival of the R&D opportunity, while keeping everything else at the baseline levels. Both firms' equilibrium investments increase with X_{FC}, but the constrained firm's investment increases faster. It is because relaxation of financial constraints reduces marginal cost of investment for the FC firm, and it intensifies the competition in equilibrium. Such pattern is shared by Panel (d), which plots the equilibrium investments with regard to the likelihood of a jump in FC firm's cash flow. The catastrophe risk motivates UC firm through its positive effect on the FC firm.

From Panel (b) and (c) of *Figure 2.5,* we can see that the race becomes more intense when the FC firm's cash flow has lower volatility and higher growth rate (or lower decline rate). As the FC's volatility increases, both firms reduce their R&D investments. However, the unconstrained firm reduces its investment even more than the constrained competitor. This provides more evidence to the result from the best response analysis that a financially more capable competitor may invest less aggressively than its constrained rival. As the head and head competition escalates, the UC duopoly scales down its investment to just keep the growth option alive and count on making the discovery after the FC firm is driven out of the market. In that scenario, the UC firm's investment comes down to its monopoly level and the effect of competition is softened. A similar argument applies with a decrease in the growth rate of the FC firm's cash flow. In addition, the UC firm's investment converges to its monopoly level as $\mu_{FC} \downarrow$ and $\sigma_{FC} \uparrow$.

[insert Figure 2.6 here]

Figure 2.6 plots comparative statics on project values in the baseline example. Although equilibrium investments for the two firms always move in the same direction, the project values usually move in the opposite ways. Panel (a), Panel (b), and Panel (c) indicate an increase in project values for the constrained firm as its cash flow improves ($X_{FC} \uparrow$, $\mu_{FC} \uparrow$, $\sigma_{FC} \downarrow$), and a decrease in the UC firm's value. Both firms' values shall converge to a homogeneous duopoly. On the other hand, Panel (d) demonstrates that the effect of a catastrophe risk on the FC firm's assets in place is negative for both firms' project values. It is because with the deterioration of a catastrophe risk, the FC firm becomes more aggressive in order to win the war before the jump happens, which leads to a more fierce race. However, no firm gains as $\lambda_j \uparrow$.

2.3.4.2 Project characteristics - scalability

My model allows us to study how various dimensions of project characteristics affect R&D investment when firms face different financial constraints. We start by looking at the effect of one firm's project scalability (A and β) on market equilibrium in a heterogeneous duopoly. *Figure 2.7* plots such comparative statics in the baseline example.

[insert Figure 2.7 here]

Panel (a) and Panel (b) graph market equilibria as functions of project scaling factor A of the UC firm and FC firm respectively. Both firms react to the UC firm's project quality measured by A positively in Panel (b). Nevertheless, the UC firm stops increasing its R&D investment when the project of the FC competitor improves beyond a certain level in Panel (a). This pattern highlights the different investment incentives for the UC and FC firms of their rival's project scalability characteristics in a duopoly competition.

The pattern is consistent with Panel (c) and Panel (d) in which β changes[31]. In Panel (c), R_{UC} even decreases in equilibrium as $\beta_{FC} \uparrow$, but R_{FC} keeps increasing mildly as $\beta_{UC} \uparrow$. When an FC rival is better capable in scaling up the R&D project, the UC firm acts less aggressively in the race. But when a UC rival has a more scalable technology, then the FC firm acts more aggressively. The "less-constrained less aggressive"

[31] Keep in mind that in this baseline parameterization, both firms' R&D projects are accelerable. As long as the project is accelerable, the scalability aspects of projects have effects on the equilibrium. Otherwise, scalability only has an effect on the FC firm's investment in equilibrium.

phenomenon extends our analysis regarding the two firms' best responses. I conjecture the monotonic best response of the FC firm and the inverse-U shaped best response of the UC firm stays true when we allow firms have different technologies in their R&D projects.

Implications on project value with regard to changes in one firm's project scalability is straight forward. *Figure 2.8* plots the typical comparative statics on project values with regard to changes in one firm's project scalability using baseline parameterization. This set of graphs show the intuitive result that the project value increases as the self firm's project scalability of improves, but decreases as the rival firm's project scalability becomes better. We shall expect the same result hold in both heterogeneous duopoly and homogeneous duopoly (where two unconstrained firm compete in a race), for any project characteristics.

[insert Figure 2.8 here]

2.3.4.3 Project characteristics - accelerability

I then investigate the effect of one firm's project accelerability (η and γ) on market equilibria and project values in a heterogeneous duopoly. I show in *Figure 2.9* and *Figure* 2.10 such comparative statics with baseline values at the benchmark.

[insert Figure 2.9 here]

Similar to the plots on project scalability, Figure 2.9 shows that the effect of an increase of accelerability on firm's own investment is larger than on the rival firm's investment in general. We can observe from Panel (a) and Panel (c) that the slope of

R_{FC} is steeper than the slope of R_{UC} as η_{FC} or γ_{FC} changes in a wide range. In Panel (b) and Panel (d), the slope of R_{UC} is steeper than R_{FC} as we change the UC firm's accelerability. In other words, a firm usually acts more responsively to changes in its own project characteristics than to its rival's project characteristics in equilibria, regardless of whether it is the FC or the UC firm. The intuition is that these variations change the equilibrium through a direct effect on the firm which experiences such variations, and through the strategic interaction on the rival firm. Furthermore, project accelerability also has limited effects on the FC firm's own investment. For example, notice from Panel (a) of *Figure 2.7,* as $\eta_{FC} \uparrow\uparrow$, its effect on R_{FC} gradually dies down.

Figure 2.10 shows the corresponding comparative statics on project values with baseline parameterization. Panel (a) and Panel (b) demonstrate that if a firm takes a very long time to develop the R&D project ($\eta_{FC} \to 0$ or $\eta_{UC} \to 0$), the project becomes worthless. In such cases, the other duopoly firm achieves its monopoly project value ($V_{UC} \to V_{UC}^{mply}$ or $V_{FC} \to V_{FC}^{mply}$). Panel (c) and Panel (d) illustrate the effects of γ, and there is convergence on project values as it gets more and more difficult for one firm to accelerate ($\gamma_{FC} \to 0$ or $\gamma_{UC} \to 0$) its R&D project. The two firms are essentially competing on different types of projects in the race now. For example, as $\gamma_{FC} \to 0$, the FC firm is developing a scalable only project, while the UC firm is developing a project that is both scalable and accelerable. Everything else equal, the firm which has better technology in developing a competing project, has higher project value even if it is constrained. This figure offers some evidence on the flexibilities of my model, which can be used to study duopoly equilibria when the two firms differ in both financial constraints and R&D technologies.

[insert Figure 2.10 here]

From both *Figure 2.8* and *Figure2.10,* when the constrained firm has a superior technology (larger A, β, η, or γ) comparing with its unconstrained rival, the constrained firm could achieve a higher project value $V_{FC}^{hetero-dply} > V_{UC}^{hetero-dply}$.

To conclude the discussion on comparative statics, I find that in a market with heterogeneous competitors, firms invest more heavily in R&D in equilibria if their own project quality or their rival's project quality gets better $(A \uparrow, \eta \uparrow, \beta \uparrow)$ and if the FC competitor's cash flow is improved with higher level $(X \uparrow)$, lower cash flow risk $(\sigma \downarrow)$, and deteriorates at a slower speed $(\mu \uparrow)$. A firm's own project value increases while its opponent's project value decreases as one firm's project quality improves.

2.3.5 Implications on preemption

Preemption occurs when a firm invests more aggressively than its competitor(s)[32], usually to deter entry or dampen investment incentive of its competitor(s). We see in the heterogeneous duopoly model that a financially constrained firm may surprisingly preempt a UC firm in a head to head R&D race. The best response analysis uncovers a motive for the UC firm to stop escalating competition when the probability of a failure from its constrained competitor rises beyond a certain level.

The implication on preemption is closely related with the over-investment result in the monopoly model in Chapter 1. Even though financial constraints could still drive a firm to invest heavily in a competitive environment, the degree of preemption from an

[32] Alternatively, a preemption in a heterogeneous duopoly can be understood as a firm investing more aggressively than it would in a homogeneous duopoly or in a monopoly.

FC firm is at most mild because a UC firm reacts more actively to the competition than an FC firm. Yet it is a novel result that an FC firm is not always intimidated by its more financially powerful competitor. The reason why competition doesn't completely eliminate the incentive from constraints is simple. When a unconstrained duopoly figures it is very likely for its constrained rival to blows up before success, it saves money from not accelerating the project and instead waits for the R&D competition to turn into a monopoly. The constrained firm does not deviate from its aggressive strategy either, because if it cuts back the R&D investment then the unconstrained competitor would intensify its R&D to win the contest.

As a summary of findings from more numerical solutions, a constrained firm is motivated to preempt a financially more capable rival when its cash flow risk is high ($\sigma \uparrow$, $\lambda_j \uparrow$), its cash flows deteriorate faster ($\mu \downarrow$ with $\mu < 0$), its cash flows are abundant at the project arrival ($X \uparrow$), and its project quality is better ($A \uparrow$, $\beta \uparrow$, $\eta \uparrow$, $\gamma \uparrow$). Recall that in the monopoly model, over-investment happens when AIP cash flow risk is low and decline more slowly which are opposite from the occurrence of preemption. Such a disparity arises as a result of strategic interactions in duopoly equilibria.

In addition, the analysis on project values in the duopoly model reveals a new incentive for an innovative firm to acquire a growth option similar to its own from its rival. Take the baseline case as an example, Figure 2.2 indicates a large increase in project value if it were carried out by either the UC firm alone ($V = 23.1 \to 52.6$) or the FC firm alone ($V = 20.4 \to 37.6$), as opposed to both firms. More importantly, the UC monopoly value of the project exceeds the sum of the project values for two duopoly which indicates a possibly profitable and feasible acquisition. The duopoly model can provide new insights on how the incentive of R&D driven acquisition depends on market participants' characteristics, and I leave it for future research.

The duopoly model does not have free entry, and the two firms working on innovation take it as given that no other potential competitor can capture such a growth option. This simplifying assumption can be explained by the barrier of entry set by certain prior knowledge or human capital to develop the project. We can take one step back, relax that assumption and think about the entry decision. When firms can enter the innovative market by paying some fixed cost and there is potentially a large pool of contestants, then the market profit shall be competed away. If firms are identical except for financial constraints, an unconstrained firm values the project more highly than a constrained firm, so no constrained firms will be in such an R&D race. However, when firms differ in their technologies in the R&D race (A, β, η, and γ) or in their assets in place cash flow, then a constrained firm may have a higher project value and be the only types of firms in an R&D race with free entry. From the fact that a more constrained makes R&D investment in a competitive environment with endogenous entry decision, it may suggest its project is superior.

2.4 Conclusion

In this chapter, I build two duopoly models on R&D investment. The model for homogeneous duopoly studies how two unconstrained firms invest in a race, and it helps to establish the effect of competition on innovation. I find that project values always decrease in the duopoly comparing with a monopoly, but firms' investment decisions differ from an unconstrained monopoly only if the project is accelerable. The "Escape-competition" motive for innovation is absent if competitors are financially unconstrained and the project they compete on cannot be expedited with heavier investment. On the other hand, I conjecture with confidence that the positive effect of competition on R&D investment when projects are accelerable is stronger if the project has a higher expected payoff, or if it is less costly to speed up.

The model for heterogeneous duopoly explores the strategic interactions between a financially constrained firm and an unconstrained firm as they compete on an R&D project. I find that the best response for the unconstrained firm follows an inverted U shape while the best response for the constrained firm is monotonically increasing. This is because an unconstrained firm optimally lower its investment and wait for its constrained competitor to quit the competition, as the constrained firm becomes more aggressive. As a result, a constrained company may preempt an unconstrained rival in an R&D race. The model provides several testable implications with regard to preemption, and it has potentials to generate new insights into firms' R&D investment strategies and project values in a competitive environment.

Two open questions that emerge from my work are particularly interesting. First, what would be the optimal cash holding policies for innovative firms and how does it depend on R&D payoff structures, competition, and firms' assets in place? Second, will firms' optimal R&D scale decisions depend monotonically on their costs of capital, especially when there is competition? I discuss potential ways to approach these relevant but challenging questions in the paper, but I leave the full exploration to my future work.

Table 2.1 – Baseline parameter values used in the numerical analyses

Parameter	Value
discount rate	$r = 0.05$
discovery rate ($\lambda_d = \eta R^\gamma$)	$\eta = 0.05$, $\gamma = 0.7$
expected project payoff ($u = AR^\beta$)	$A = 100$, $\beta = 0.01$
AIP cash flow level at the project's arrival	$X = 100$
decline rate of AIP cash flow	$\mu = -0.2$
volatility of AIP cash flow	$\sigma = 0.3$
catastrophe risk of AIP cash flow	$\lambda_j = 0.1$

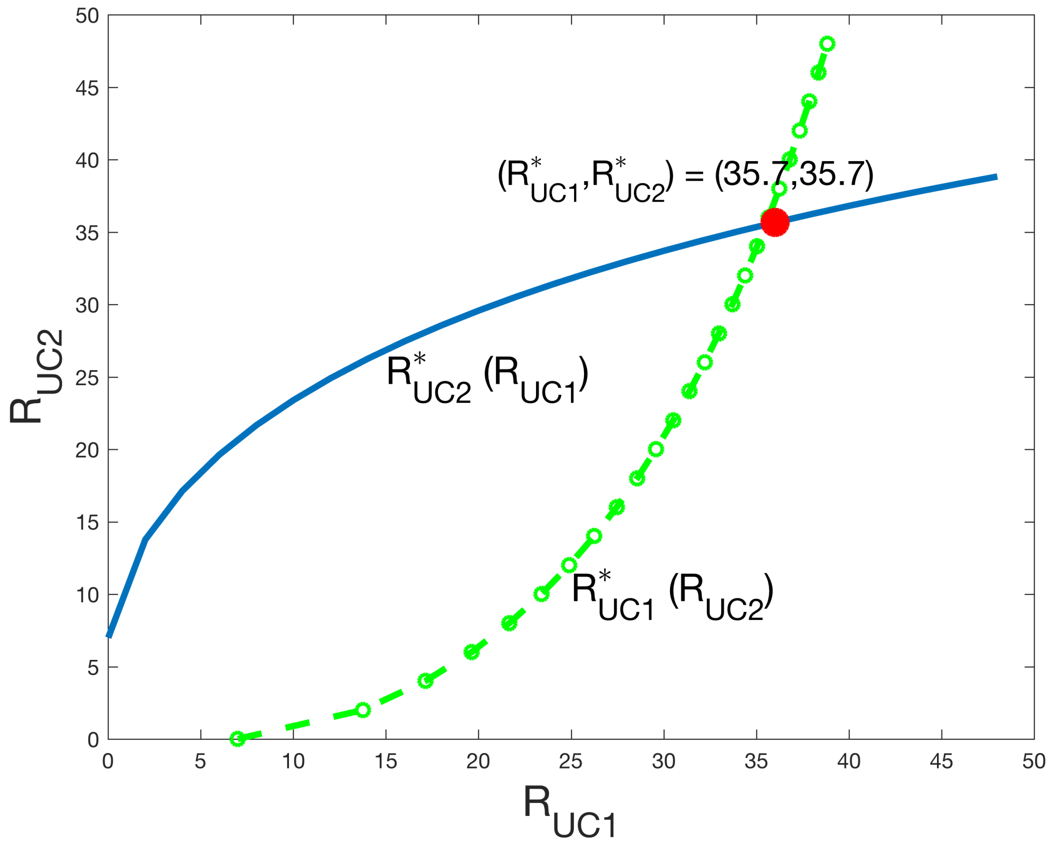
Note: Cash flows from assets in place and project characteristics follow the monopoly model setup in Chapter 1. In particular, assets in place cash flow of the constrained firm follows $dX = \mu X dt + \sigma X dZ_t - X_t dq_1$; the R&D project cash flow follows $dF = -Rdt + \tilde{u} dq_2$; the expected one-time project payoff is $E(\tilde{u}) = A \times R^\beta$; and the discovery rate is $\lambda_d = \eta \times R^\gamma$.

Table 2.2 – Equilibrium R&D investment and project values

	The constrained firm	The unconstrained firm
Monopoly	$R = 13.8$ ($V = 37.6$)	6.8 (52.6)
Heterogeneous duopoly	29.4 (21.8)	24.4 (25.3)
Homogeneous duopoly	-	35.5 (21.7)

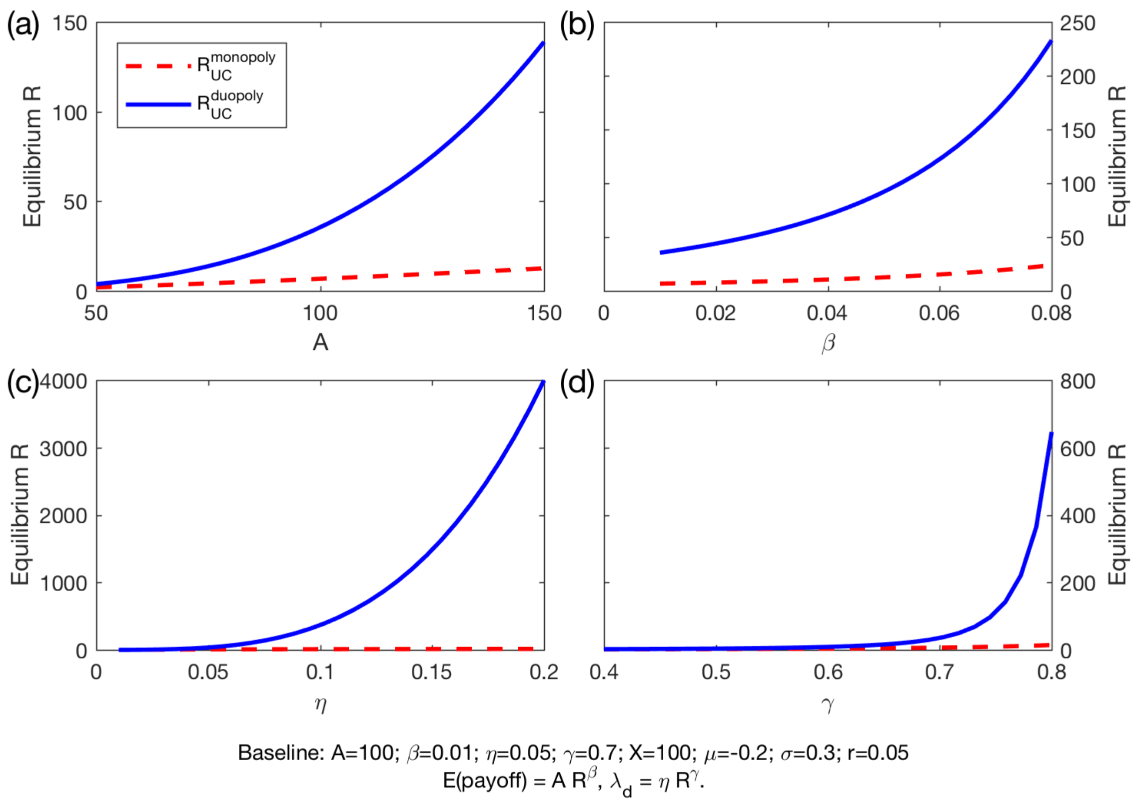
Note: The parameter values in all three setups follow Figure 2.1. In both the homogeneous duopoly and the heterogeneous duopoly, firms are assumed to have the same technologies in pursuing an R&D project. Numbers in the parentheses are equilibrium project values. Numbers outside of the parentheses are equilibrium R&D investment levels.

Figure 2.1 – Best responses in a homogeneous duopoly



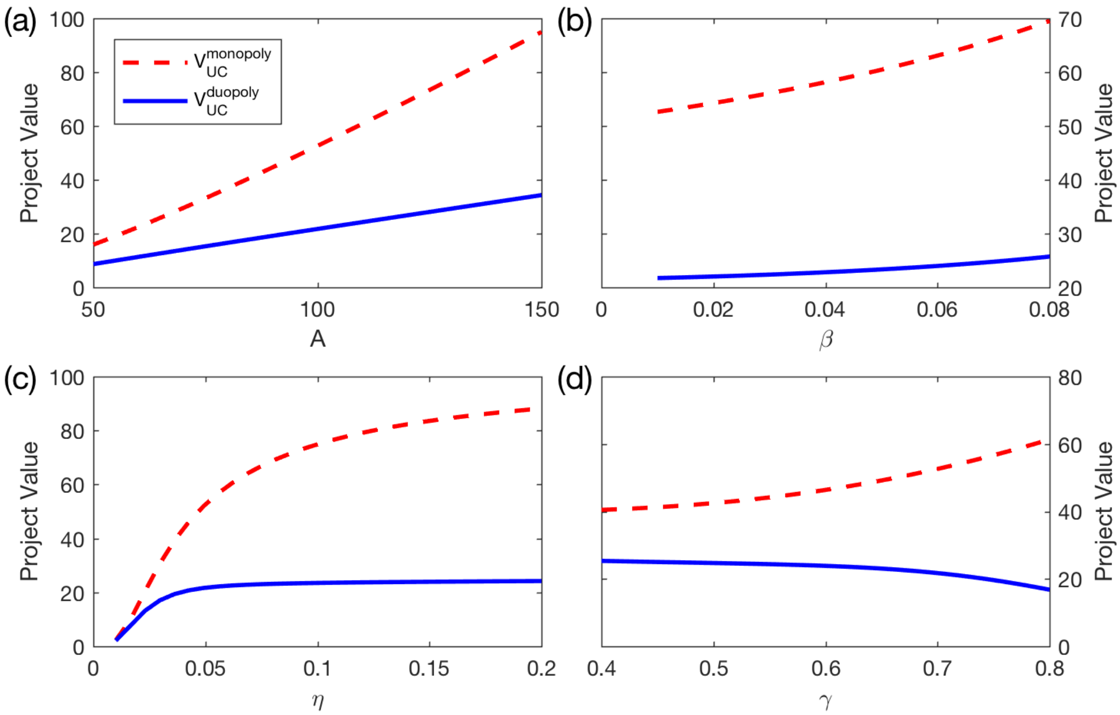
Note: This graph illustrates two representative best response functions in a homogeneous duopoly. Both firms are financially unconstrained, and have their project and assets in place cash flows parameter values set as in Table 2.1. Consistent with Corollary 3 on page 74, the best response functions are increasing and concave in the rivals' investment level. The unique pure strategy Nash equilibrium is denoted by the solid circle on the graph. Consistent with Corollary 4 on page 76, $R_{UC}^{\text{duopoly}} > R_{UC}^{\text{monopoly}}$ $(= 6.8)$, provided that the project is accelerable.

Figure 2.2 – An unconstrained firm's equilibrium investment in monopoly vs. homogeneous duopoly



Note: $R_{UC}^{duopoly}$ is the equilibrium investment for an unconstrained firm in a duopoly competition with an identical unconstrained firm. This group of comparative statics assumes the project characteristics (A, β, η, γ) are the same for the two duopoly firms. That is, there is no advantage of a duopoly firm over the other in terms of innovation efficiency.

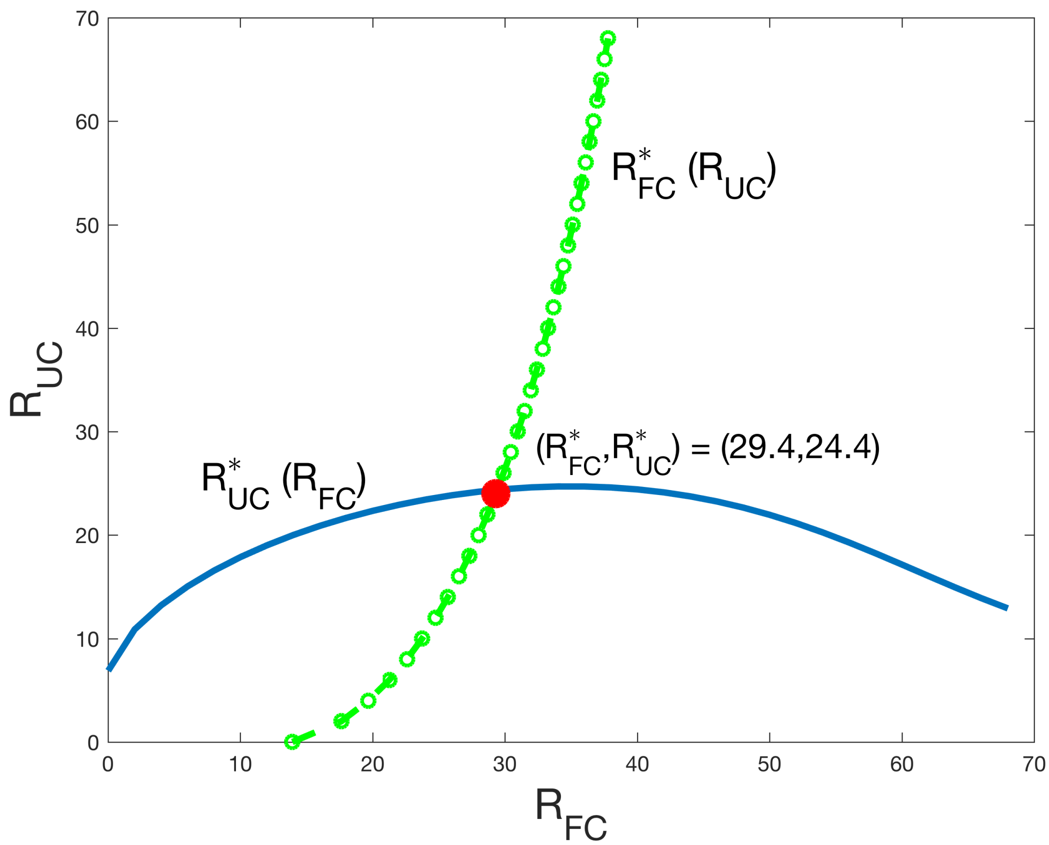
Figure 2.3 – An unconstrained firm's project value in monopoly vs. homogeneous duopoly



Baseline: A=100; β=0.01; η=0.05; γ=0.7; X=100; μ=-0.2; σ=0.3; r=0.05
E(payoff) = A R^{β}, $\lambda_d = \eta R^{\gamma}$.

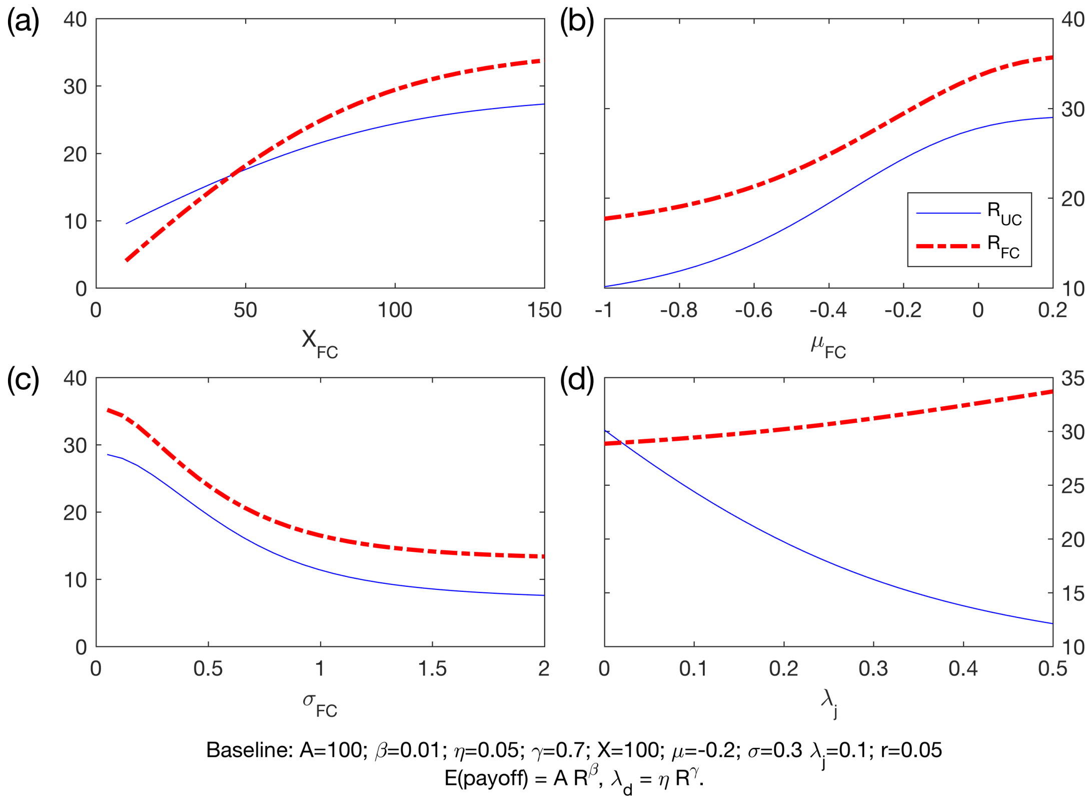
Note: $V_{UC}^{duopoly}$ is the equilibrium project value for an unconstrained firm in a duopoly competition with an identical unconstrained firm. Project values in these plots correspond to investment strategies in Figure 2.2.

Figure 2.4 – Best responses in a heterogeneous duopoly



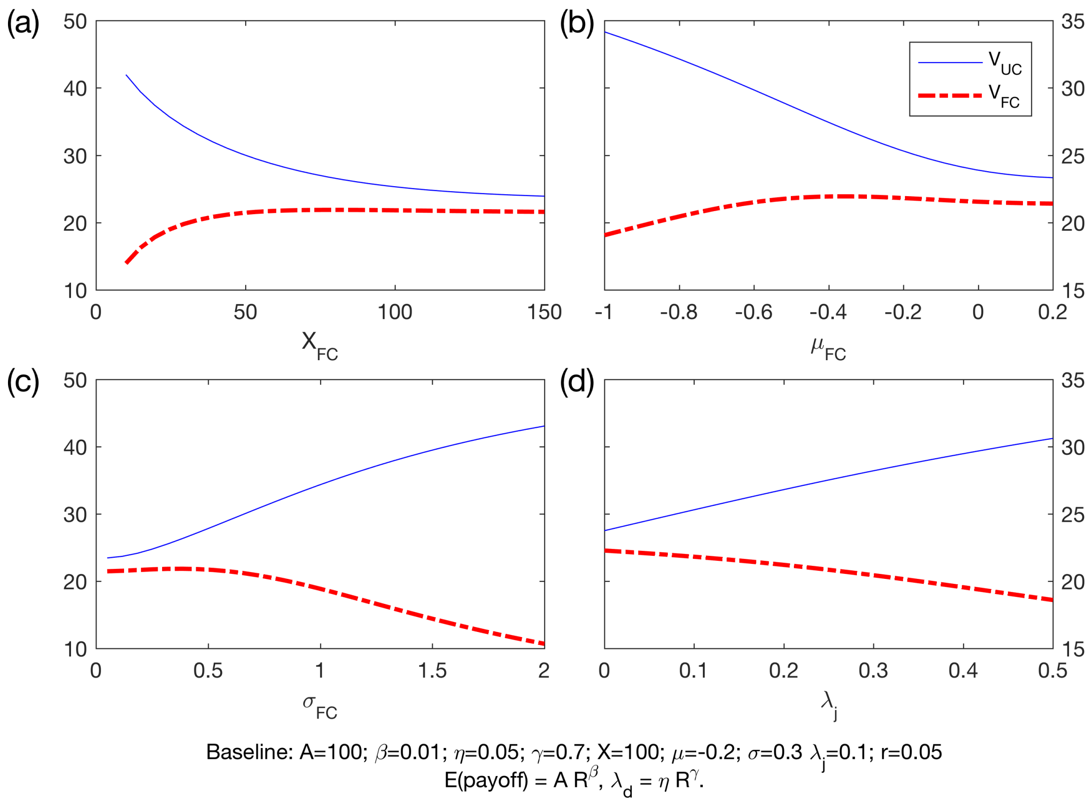
Note: This graph illustrates an example of best response functions in a heterogeneous duopoly. The green circled line depicts the best response of an FC firm with regard to its UC competitor's R&D investment. It is increasing and concave. The blue solid line depicts the other best response, and it is non-monotonic. Parameter values in this example follow Figure 2.1. The intercepts of the best response functions denote R&D investment in the monopoly.

Figure 2.5 – Equilibrium investments in heterogeneous duopoly on changes in FC's AIP



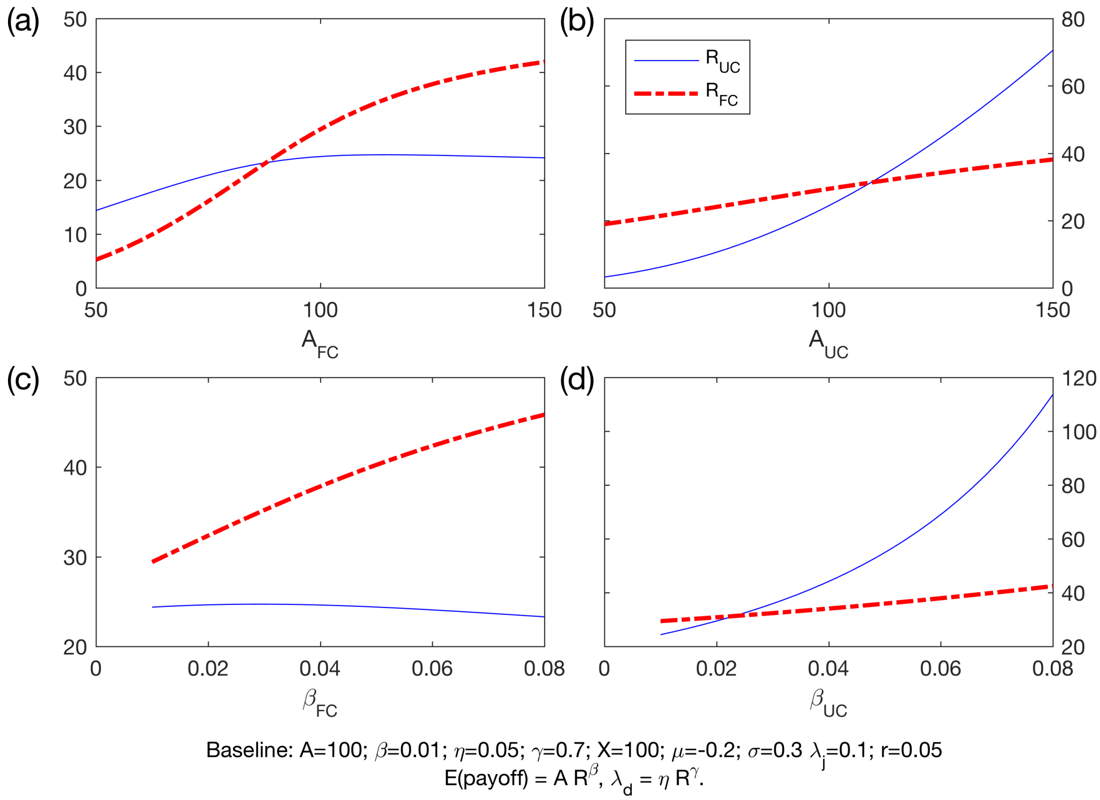
Note: All parameter values are set according to Figure 2.1, except the x-axis variables.

Figure 2.6 – Project values in heterogeneous duopoly on changes in FC's AIP



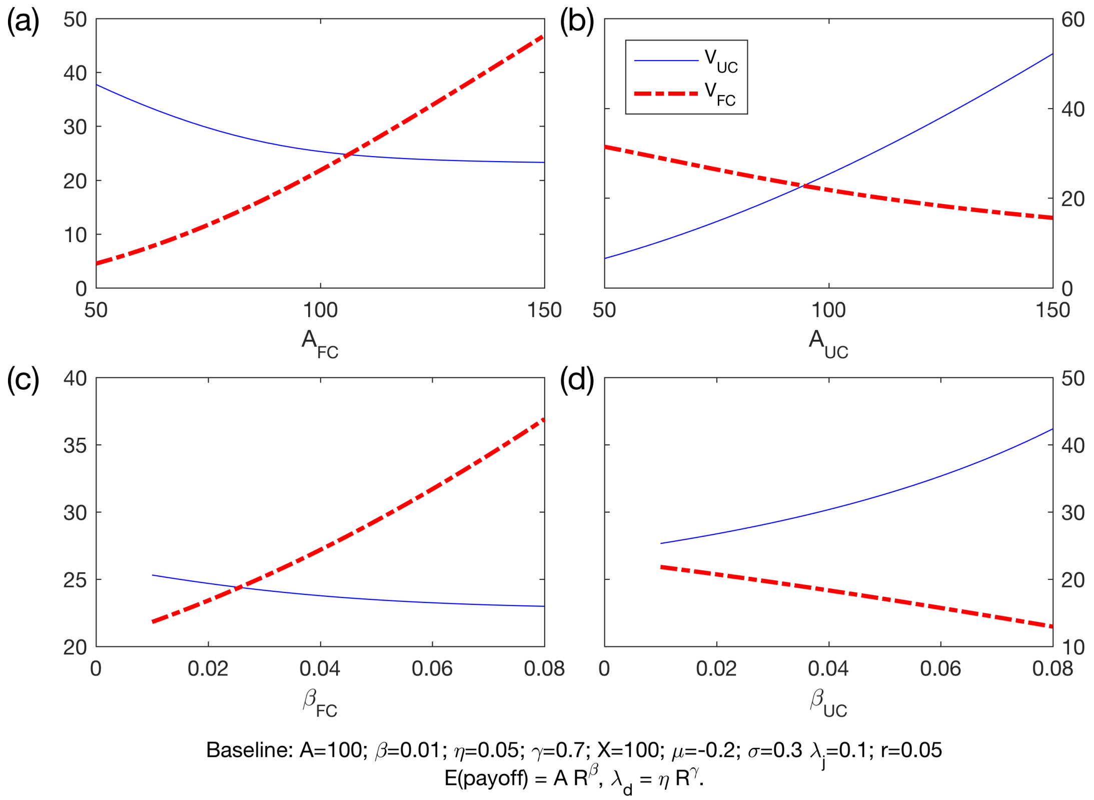
Note: All parameter values are set according to Figure 2.1, except the x-axis variables.

Figure 2.7 – Equilibrium investments in heterogeneous duopoly on changes in scalability



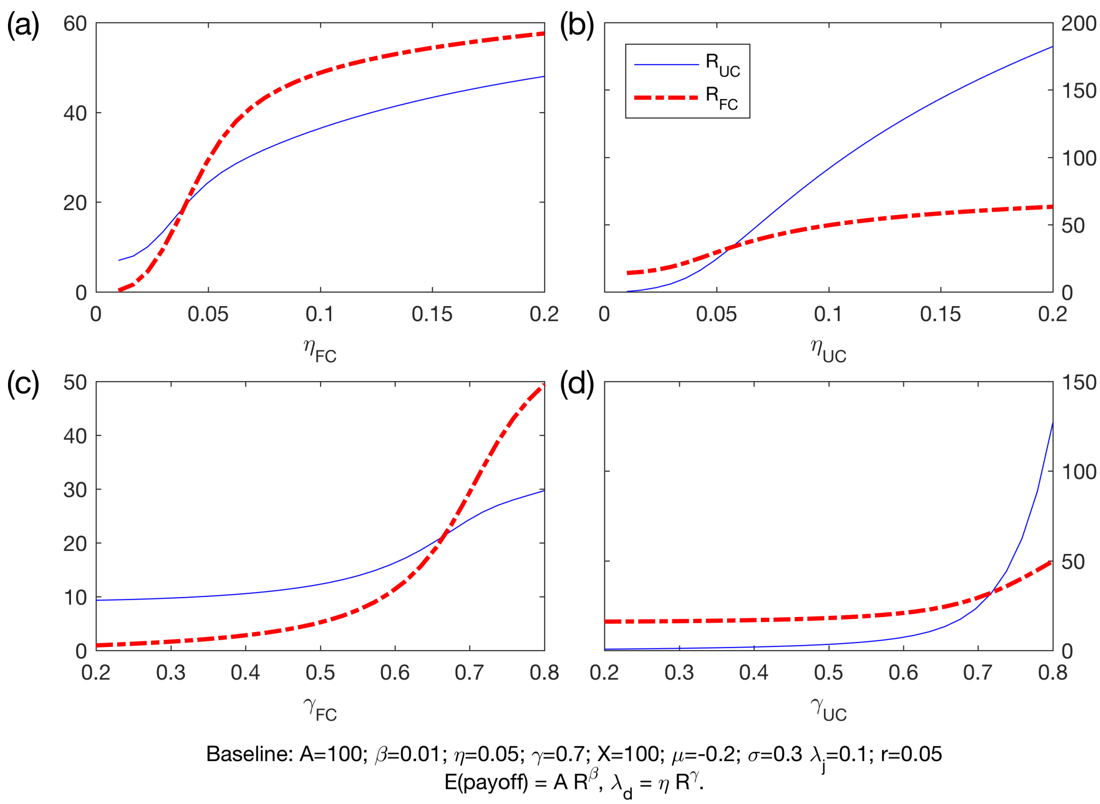
Note: All parameter values are set according to Figure 2.1, except the x-axis variables.

Figure 2.8 – Project values in heterogeneous duopoly on changes in scalability



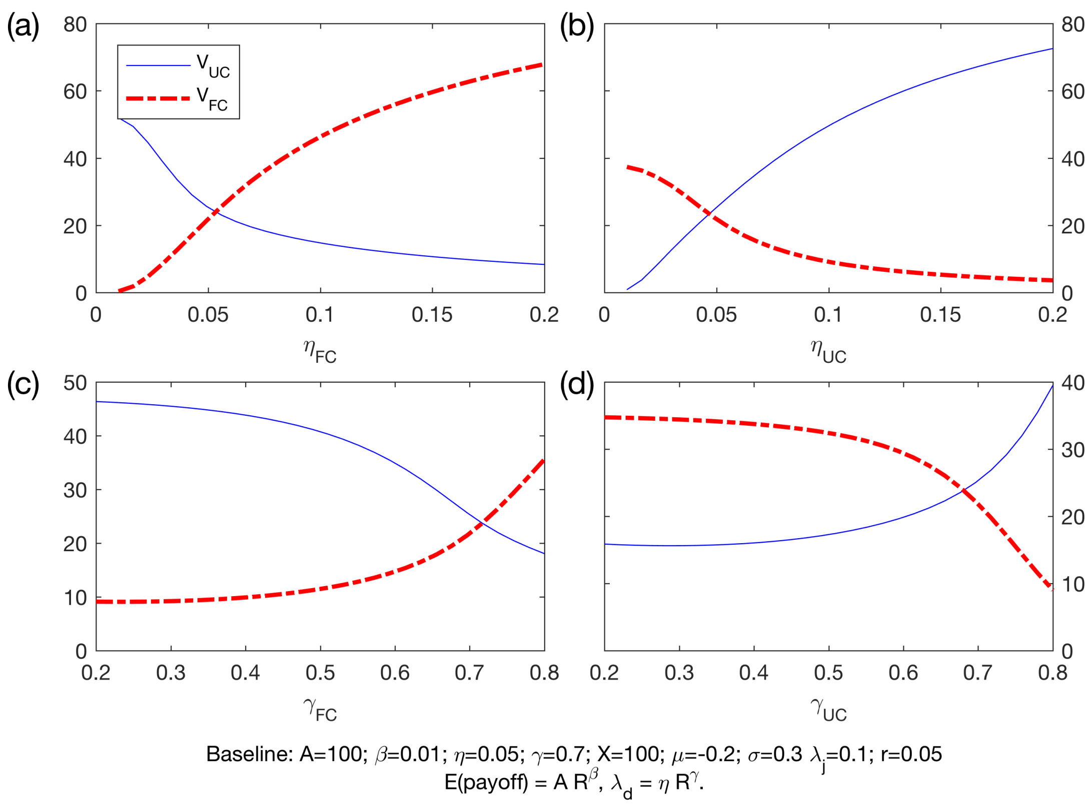
Note: All parameter values are set according to Figure 2.1, except the x-axis variables.

Figure 2.9 – Equilibrium investments in heterogeneous duopoly on changes in accelerability



Note: All parameter values are set according to Figure 2.1, except the x-axis variables.

Figure 2.10 – Project values in heterogeneous duopoly on changes in accelerability



Note: All parameter values are set according to Figure 2.1, except the x-axis variables.

Appendices to Chapter One

Appendix A: proofs

Proof of Lemma 1 on page 19: Since project value decreases with the abandonment threshold, thus no voluntary abandonment is optimal. This is obvious by taking the first order derivative of the project value in Eq (1.10) of Proposition 1 on page 17 with regard to the abandonment threshold, we get

$$\frac{\partial V}{\partial \underline{X}} = \frac{Af(R)\lambda_d - R}{\lambda_d + r}\alpha_1 X^{\alpha_1}\underline{X}^{-1-\alpha_1} < 0, \ \forall Af(R)\lambda_d - R > 0,$$

The lowest possible value of $\underline{X}$ is R for the FC monopoly firm and is 0 for the UC monopoly firm.

Q.E.D.

Proof of Corollary 1 on page 23: To find the optimal R&D size for a UC monopoly, I use the first order condition, i.e. $FOC \equiv -r - h(R) + Ah'(R)r + Rh'(R) = 0$. By applying implicit function theorem, we have

$$\frac{\partial FOC}{\partial R} = Ah"r + Rh" + h' - h' = h"(Ar + R) < 0$$

$$\frac{\partial FOC}{\partial r} = Ah' - 1 > 0 \Rightarrow \frac{\partial R^*_{UC}}{\partial r} > 0, \text{ and } \frac{\partial FOC}{\partial A} = h'r > 0 \Rightarrow \frac{\partial R^*_{UC}}{\partial A} > 0$$

The other results in this corollary can be obtained similarly.

Q.E.D.

Proof of Lemma 4 on page 51: Apply implicit function theorem on Eq (1.38),

$$\frac{\partial \underline{X}}{\partial \gamma} = -\frac{\overbrace{\int_{\underline{X}}^{R} \frac{(R-t)^2}{t^{\alpha_2+1}} dt}^{+}}{\underbrace{-\frac{R-\lambda_d A f(R)+\gamma(R-\underline{X})^2}{\underline{X}^{\alpha_2+1}}}_{+}} > 0$$

Similarly,

$$\frac{\partial \underline{X}}{\partial A} = -\frac{\overbrace{-\int_{\underline{X}}^{\infty} \frac{\lambda_d f(R)}{t^{\alpha_2+1}} dt}^{-}}{\underbrace{-\frac{R-\lambda_d A f(R)+\gamma(R-\underline{X})^2}{\underline{X}^{\alpha_2+1}}}_{+}} < 0, \quad \frac{\partial \underline{X}}{\partial \lambda_d} = -\frac{\overbrace{-\int_{\underline{X}}^{\infty} \frac{A f(R)}{t^{\alpha_2+1}} dt}^{-}}{\underbrace{-\frac{R-\lambda_d A f(R)+\gamma(R-\underline{X})^2}{\underline{X}^{\alpha_2+1}}}_{+}} < 0$$

I haven't been able to derive conditions under which we can pin down the signs of the comparative statics of the AIP cash flow in Conjecture 1. However, for the reference of future work,

$$\frac{\partial \underline{X}}{\partial \alpha_2} = -\frac{-\int_{\underline{X}}^{\infty}[R-\lambda_d A f(R)+g(t)]t^{-\alpha_2-1}\ln t dt}{\underbrace{-\frac{R-\lambda_d A f(R)+\gamma(R-\underline{X})^2}{\underline{X}^{\alpha_2+1}}}_{+}}$$

To determine comparative statics for AIP cash flow related parameters, denote $B = \frac{1}{2} - \frac{\mu}{\sigma^2}$ and $C = B^2 + \frac{2(r+\lambda_d)}{\sigma^2}$, we have

$$\frac{\partial \alpha_2}{\partial \mu} = -\frac{1}{\sigma^2}(1 + C^{-\frac{1}{2}}B) < 0$$

$$\frac{\partial \alpha_2}{\partial \sigma^2} = \frac{\mu}{(\sigma^2)^2}(1 + \frac{1}{2}C^{-\frac{1}{2}} \times 2B) > 0$$

$$\frac{\partial \alpha_2}{\partial r} = C^{\frac{1}{2}}\frac{1}{\sigma^2} > 0$$

Q.E.D.

Proof of Proposition 2 on page 22: Given the project is started, it must be the project has positive a NPV, and $Af(R)\lambda_d > R$ for the FC monopoly. Therefore the second term in Eq (1.14) is negative, and $\frac{Af'(R)-1}{\lambda_d+r} > 0$ at R^*_{FC} for the first order condition to hold. Since $Af'(R) - 1 = 0|_{R^*_{UC}}$, it must be$Af'(R)|_{R^*_{FC}} > Af'(R)|_{R^*_{UC}}$. The concavity of f suggests that $R^*_{FC} < R^*_{UC}$. To check with the second order condition,

$$\begin{aligned}
\frac{\partial^2 V_{FC}}{\partial R^2} &= \underbrace{\frac{Af''(R)\lambda_d}{\lambda_d + r}}_{<0}\underbrace{(1 - (\frac{X}{R})^{\alpha_1})}_{>0} + \underbrace{\frac{Af'(R) - 1}{\lambda_d + r}}_{>0} \times \underbrace{2\alpha_1 X^{\alpha_1} R^{-1-\alpha_1}}_{<0} \\
&\quad + \underbrace{\frac{Af(R)\lambda_d - R}{\lambda_d + r}}_{<0}\alpha_1(-1 - \alpha_1)\underbrace{X^{\alpha_1} R^{-2-\alpha_2}}_{>0} \\
&< 0.
\end{aligned}$$

Q.E.D.

Proof of Proposition 4 on page 33: The over-investment is shown by numerical solutions, such as those around the baseline parameters plotted in Figure 1.7 and Figure 1.6. Meanwhile, if we have

$$C^{-\frac{1}{2}}\frac{h'}{\sigma^2}\ln(\frac{X}{R}) + \frac{\alpha_1}{R}\Big|_{R^*_{FC}} > 0 \tag{2.11}$$

then $\frac{\partial V2}{\partial R}\Big|_{R^*_{FC}} > 0$. For R^*_{FC} to be optimal, it must be $\frac{\partial V1}{\partial R}\Big|_{R^*_{FC}} < 0$. Since $\frac{\partial V1}{\partial R}$ is decreasing in R and $\frac{\partial V1}{\partial R}\Big|_{R^*_{UC}} = 0$, thus $R^*_{FC} > R^*_{UC}$.

Q.E.D.

Proof of Corollary 1 on page 23: Recall

$$V_{FC}(X) = \frac{Af(R)\lambda_d - R}{\lambda_d + r}(1 - (\frac{X}{R})^{\alpha_1})$$

FOC leads to the R^*_{FC} which satisfies the following condition in Eq (2.12),

$$\begin{aligned} V' &= \frac{\partial V_{FC}(X)}{\partial R} = (1 - (\frac{X}{R})^{\alpha_1})(\frac{Af'(R)\lambda_d - 1}{\lambda_d + r}) + \alpha_1 X^{\alpha_1} R^{-\alpha_1 - 1}\frac{Af(R)\lambda_d - R}{\lambda_d + r} && (2.12) \\ &\Rightarrow (1 - (\frac{X}{R})^{\alpha_1})Af'(R)\lambda_d + \alpha_1(\frac{X}{R})^{\alpha_1}\frac{Af(R)\lambda_d}{R} = 1 + (\alpha_1 - 1)(\frac{X}{R})^{\alpha_1} \\ &\Rightarrow [(Af'(R)\lambda_d - 1) - \alpha_1\frac{Af(R)\lambda_d - R}{R}] = (Af'(R)\lambda_d - 1) \times (\frac{X}{R})^{-\alpha_1} && (2.13) \end{aligned}$$

By Eq(2.12), I can show that $\frac{\partial V'}{\partial R} < 0$ if $\alpha_1 < -1$. Thus the solution satisfies the second order condition. Furthermore,

$$\frac{\partial V'}{\partial r} = (\underbrace{-(\frac{X}{R})^{\alpha_1}\ln(\frac{X}{R})(Af'(R)\lambda_d - 1)}_{-} + \underbrace{\frac{1}{R}(\overbrace{Af(R)\lambda_d - R}^{+})\overbrace{[\alpha_1(\frac{X}{R})^{\alpha_1}\ln(\frac{X}{R}) + (\frac{X}{R})^{\alpha_1}]}^{-\text{ given }\alpha_1<-1})}_{-}\underbrace{\frac{\partial \alpha_1}{\partial r}}_{-} > 0$$

Thus,

$$\frac{\partial R^*_{FC}}{\partial r} = -\frac{\frac{\partial FOC}{\partial r}}{\frac{\partial FOC}{\partial R}} > 0$$

We also have

$$\frac{\partial V'}{\partial X} = \underbrace{\alpha_1 X^{\alpha_1 - 1} R^{-\alpha_1}}_{-}[\underbrace{-\frac{Af'(R)\lambda_d - 1}{\lambda_d + r}}_{-} + \underbrace{\alpha_1 \frac{Af(R)\lambda_d - R}{\lambda_d + r}}_{-}] > 0 \Rightarrow \frac{\partial R^*_{FC}}{\partial X} > 0$$

Similarly,

$$\begin{aligned}
\frac{\partial V'}{\partial A} &= \frac{\lambda_d}{\lambda_d + r}[(1 - (\frac{X}{R})^{\alpha_1})f'(R) + \alpha_1 \frac{f(R)}{R}(\frac{X}{R})^{\alpha_1}] \\
&= \frac{1}{(\lambda_d + r)A}[1 + (\alpha_1 - 1)(\frac{X}{R})^{\alpha_1}]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial V'}{\partial \alpha_1} &= \frac{1}{\lambda_d + r}(\frac{X}{R})^{\alpha_1}\{(\frac{Af(R)\lambda_d}{R} - 1)(1 + \alpha_1 \ln\frac{X}{R}) - (Af'(R)\lambda_d - 1)\ln\frac{X}{R}\} \\
&= \frac{1}{\lambda_d + r}[(\frac{X}{R})^{\alpha_1}\frac{Af(R)\lambda_d - R}{R} - \ln\frac{X}{R}(Af'(R)\lambda_d - 1)]
\end{aligned}$$

$$\frac{\partial \alpha_1}{\partial \mu} = -\frac{1}{\sigma^2}(1 - \frac{1}{2}C^{-\frac{1}{2}} \times 2B) = -\frac{1}{\sigma^2}(1 - C^{-\frac{1}{2}}B) < 0 \Rightarrow \frac{\partial V'}{\partial \mu} < 0 \Rightarrow \frac{\partial R^*_{FC}}{\partial \mu} < 0$$

$$\frac{\partial \alpha_1}{\partial \sigma^2} = \frac{\mu}{(\sigma^2)^2}(1 - \frac{1}{2}C^{-\frac{1}{2}} \times 2B) > 0 \Rightarrow \frac{\partial V'}{\partial \sigma^2} > 0 \Rightarrow \frac{\partial R^*_{FC}}{\partial \sigma^2} > 0$$

$$\frac{\partial \alpha_1}{\partial r} = -C^{\frac{1}{2}}\frac{1}{\sigma^2} < 0 \Rightarrow \frac{\partial V'}{\partial r} = \frac{dV'}{dr} + \frac{\partial V'}{\partial \alpha_1}\frac{\partial \alpha_1}{\partial r} < 0 \Rightarrow \frac{\partial R^*_{FC}}{\partial r} < 0$$

Q.E.D.

Proof of Proposition 5 on page 41:

$$\begin{aligned}
FV(X_0) &= \sup_{R,\tau_c} E[\int_0^{\tau_d \wedge \tau_c \wedge \tau_j} e^{-rt}(X_t - R)\mathrm{dt} + e^{-r\tau_d}\tilde{A}f(R)1_{\{\tau_d < \tau_c \wedge \tau_j\}} + \int_{\tau_d \wedge \tau_c \wedge \tau_j}^{\lambda_j} e^{-rt}X_t\mathrm{dt}] \\
&= \sup_{R,\tau_c} E\int_0^{\tau_c} e^{-rt}(-R)1_{\{t<\tau_d \wedge \tau_j\}}\mathrm{dt} + E[E(e^{-r\tau_d}1_{\{\tau_d<\tau_c\wedge\tau_j\}}Af(R)\Big|\tau_c]) \\
&\quad +E\int_0^{\lambda_j} e^{-rt}X_t\mathrm{dt}) \\
&= \sup_{R,\tau_c} E\int_0^{\tau_c} e^{-rt}(-R)P(t<\tau_d \wedge \tau_j)\mathrm{dt} + E[Af(R) \times \int_0^{\tau_c \wedge \tau_j} e^{-r\tau_s}\lambda_s e^{-\lambda_s\tau_s}\mathrm{d}\tau_\mathrm{s}] \\
&\quad +E\int_0^{\infty} e^{-rt}e^{-\lambda_j t}X_0 e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma B_t}\mathrm{dt} \\
&= \sup_{R,\tau_c} E\int_0^{\tau_c} e^{-(r+\tau_d+\tau_j)t}(-R)\mathrm{dt} + E[Af(R) \times \int_0^{\tau_c} \lambda_d e^{-(r+\lambda_d)t}P(\tau_j > t)\mathrm{dt}] \\
&\quad +\frac{X}{r+\lambda_j-\mu} \\
&= \sup_{R,\tau_c} E\int_0^{\tau_c} e^{-(r+\lambda_d+\lambda_j)t}(-R + Af(R)\lambda_d)\mathrm{dt} + \frac{X_0}{r+\lambda_j-\mu}
\end{aligned}$$

R&D project value is simply the first part, i.e.

$$V(X) = \sup_{R,\tau_c} E \int_0^{\tau_c} e^{-(r+\lambda_d+\lambda_j)t}(-R + Af(R)\lambda_d)\mathrm{dt} \tag{2.14}$$

Similar to Section 1.4.2 on page 15:

$$rVdt = EdV - Rdt = E\mathscr{D}V + \lambda_d(Af(R) - V)dt + \lambda_j(0 - V)dt - Rdt$$

Thus, HJB is

$$(r + \lambda_d + \lambda_j)V = \mu XV_X + \frac{1}{2}\sigma^2 X^2 V_{XX} + \lambda_d Af(R) - R$$

With boundary conditions of $\lim_{X\to\infty} V(X) = \dfrac{Af(R)\lambda_d - R}{\lambda_d + \lambda_j + r}$, and $V(\underline{X}) = 0$, we could solve for the value function

$$V(X;\ R(X_0), \underline{X}(R)) = \frac{Af(R)\lambda_d - R}{\lambda_d + \lambda_j + r}(1 - (\frac{X}{\underline{X}})^{\alpha_1'}) \tag{2.15}$$

where $\alpha_1' = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2(\lambda_d+\lambda_j+r)}{\sigma^2}} < 0$.

By taking $\lambda_j + r$ as r in Sec 1.4.6, we could use Lemma 1 to conclude that a firm does not actively withdraw from R&D projects. i.e. $\underline{X}_{UC} = 0$ and $\underline{X}_{FC} = R$. As argued before, the disappearance of AIP cash flow doesn't affect a UC firm's decision on R&D and has no consequence for a UC firm's project valuation. Together,

$$
\begin{aligned}
V_{FC}(X) &= \sup_R \quad \frac{Af(R)\lambda_d - R}{\lambda_d + \lambda_j + r}(1 - (\frac{X}{R})^{\alpha_1'}) \\
V_{UC}(X) &= \sup_R \quad \frac{Af(R)\lambda_d - R}{\lambda_d + r}
\end{aligned}
\tag{2.16}
$$

Q.E.D.

Appendix B: Expected hitting time of a geometric Brownian motion

The passage time T_b of a Brownian motion $W = \{W_t, \mathscr{F}_t; 0 \leq t < \infty\}$ with drift μ to the level $b \neq 0$ has the density function[33]

$$P[T_b \in dt] = \frac{|b|}{\sqrt{2\pi t^3}} \exp[-\frac{(b-\mu t)^2}{2t}]dt, \; t > 0.$$

We can derive the density function for a geometric Brownian motion X_t with drift μ and volatility σ hitting a fixed boundary. Define

$$\tau_c = \inf\{t : X_t \leq R\} = \inf\{t | \exp(W_t) \leq R\} = \inf\{t | W_t \leq \ln R\}.$$

The passage time τ_c follows an inverse Gaussian distribution with mean $\frac{\ln R - \ln X_0}{\mu - \frac{1}{2}\sigma^2}$ and a shape parameter $\frac{(\ln R - \ln X_0)^2}{\sigma^2}$. i.e. the density function of τ_c follows

$$f_{\tau_c}(t) = \frac{\ln X_0 - \ln R}{\sqrt{2\pi t^3}\sigma} \exp(-\frac{(t(\mu - \frac{1}{2}\sigma^2) - \ln R + \ln X_0))^2}{2t\sigma^2})$$

Figure 1.11 shows the evolution of X_t for a deterministic process with $\sigma = 0$, $X_0 = 15$, and $\underline{X} = 1$. It takes about three periods to hit the fixed boundary when $\mu = -1$. With a much lower decline rate $\mu = -0.2$, it takes about 13 periods.

[insert Figure 1.11 here]

For a stochastic process ($\sigma \neq 0$), we use multilevel Monte Carlo method introduced by Giles (2008) to simulate the expected hitting time. Set the baseline assets in place

[33] See Karatzas and Shreve (1991) Page 196-197 for a reference.

Table 2.3 – Expected hitting time with endogenously chosen R on a scalable project

	$\mu = -1$	$\mu = -0.8$	$\mu = -0.6$	$\mu = -0.4$	$\mu = -0.2$	$\mu = 0$
$\sigma = 0.3$	$E(\tau_c)$ =2.9	3.2	4.1	5.9	10.6	44.3
$\sigma = 0.5$	2.7	2.8	3.3	5.3	8.4	21.7
	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$	$\sigma = 0.5$	$\sigma = 0.6$
$\mu = -0.1$	67.3	46.3	36.8	23.5	19.1	13.1
$\mu = -0.2$	13	11.6	11.5	9.8	8.9	7.4

Baseline parameter values: $X_0 = 15$, $\sigma = 0.3$, $\mu = -0.2$, $A = 25$, $\beta = 0.7$ (the project's expected payoff:$Af(R) = AR^{\beta}$), $\lambda_d = 0.1$.

cash flow parameters as in Table 1.2 on page 54. Table 2.3 shows the expected time to hit $\underline{X} = R$ for an endogenously chosen investment level R. The numbers of periods correspond to a FC monopoly's abandonment time for the scalable project. The range of expected hitting time is reasonable, which helps to ensure the parameters in our numerical analysis are sensible.

[insert Table 2.3 here]

Proof of Proposition 7 on page 73: For firm i, from the contingent claim method or the martingale property of value functions at the optimum, the HJB equation before either firm discovers ($t < \tau_{d,i} \wedge \tau_{d,j}$) is

$$rV_i dt = E\mathscr{D}V_i - R_i dt + \lambda_{d,i}(u_i - V_i)dt + \lambda_{d,j}(0 - V_i)dt \tag{2.17}$$

From the backward induction analysis above, we know the value of the project does not depend on any state variable for unconstrained firms, so $E\mathscr{D}V_i = 0$[34]. Then Eq(2.7) and Eq(2.8) follow from Eq(2.17) for the two firms.

Q.E.D.

Proof of Corollary 3 on page 74: Firstly, I prove the existence and uniqueness of the best response function $R_1^*(R_2)$ for scalable only and accelerable only project respectively.

Suppose the project is only scalable, i.e. $\lambda_1' = 0,\ u_1' > 0$, the optimal R_1 is unique because of the monotonicity and concavity of u_1:

$$V_1 = \frac{u_1\lambda_1 - R_1}{r + \lambda_1 + \lambda_2} \Rightarrow (FOC)\ \ u_1'\lambda_1 - 1 = 0 \Rightarrow R_1 = f(\lambda_1) \tag{2.18}$$

[34] Or we can use $\underline{X}_i = 0,\ \forall i$, and solve the ODE with boundary conditions.

Check the second order derivative:

$$\frac{\partial^2 V_1}{\partial R_1^2} = \frac{1}{r+\lambda_1+\lambda_2}\frac{\partial(u_1'\lambda_1 - 1)}{\partial R_1} = \frac{\lambda_1 u_1"}{r+\lambda_1+\lambda_2} < 0 \tag{2.19}$$

Furthermore, the optimal R_1 does not depend on firm 2's parameters. With $\lim_{R_1\to 0} u_1' = \infty$, then $\lim_{R_1\to 0} V_1' > 0$. Since we also have $V_1(R_1 = 0) = 0$, then the optimal R_1 is always positive.

Now suppose the project is only accelerable, i.e. $\lambda_1' > 0,\ u_1' = 0$. I can also show that given R_2, the value of R_1 that maximizes V_1 is unique. It is because:

(1) The first order derivative $\dfrac{\partial V_1}{\partial R_1} = \dfrac{\lambda_1'(u_1(\lambda_2+r)+R_1)}{(r+\lambda_1+\lambda_2)^2} - \dfrac{1}{r+\lambda_1+\lambda_2}$ is continuous on $R_1 \in \mathbb{R}^+$;

(2) $\dfrac{\partial V_1}{\partial R_1}|_{R_1\to 0} > 0\ \forall R_2$, if $\lambda_i' = \infty$ when $R_i \to 0$ and if λ_i is concave;

(3) $\dfrac{\partial V_1}{\partial R_1}|_{R_1\to\infty} < 0\ \forall R_2$;

(4) $\dfrac{\partial V_1'}{\partial R_1^2} = \dfrac{\lambda_1"(u_1(r+\lambda_2)+R_1) - 2(r+\lambda_1+\lambda_2)\lambda_1' V_1'}{(r+\lambda_1+\lambda_2)^2} < 0$ when $V_1' \geq 0$.

(2) and (3) indicate that there is at least one R_1 where first order derivative equals zero. (4) ensures such R_1 maximizes V_1 and it is unique given the other firm's investment R_2. In addition, we know $V_1(R_1 = 0) = 0$, so (2) also indicates that the optimal $R_1 > 0$ for any R_2.

Secondly, without loss of generality, I will show that the best response $R_1(R_2)$ is increasing in its argument R_2 for accelerable projects.

For accelerable projects, I have shown that $R_1^* = \{R_1 : V_1' = 0\}$ and $V_1(R_1^*) > 0$. Denote $f(R_1^*, R_2) = \lambda_1'(u_1(\lambda_2 + r) + R_1^*) - (r + \lambda_1 + \lambda_2) = 0$ as the FOC for R_1. The implicit function theorem suggests

$$\frac{\partial R_1^*}{\partial R_2} = -\frac{\frac{\partial f}{\partial R_2}}{\frac{\partial f}{\partial R_1^*}} = -\frac{\lambda_2'(\lambda_1' u_1 - 1)}{\lambda_1"(u_1(\lambda_2 + r) + R_1^*)} \begin{cases} \geq 0 & \text{if } \lambda_1' u_1 \geq 1 \text{ at } R_1^* \\ < 0 & \text{if } \lambda_1' u_1 < 1 \text{ at } R_1^* \end{cases}$$

I argue that $\frac{\partial R_1^*}{\partial R_2} > 0$ and I prove it by contradiction. Define the numerator of V_1' as $S_1 = \lambda_1' u_1(r + \lambda_2) - r - \lambda_1 - \lambda_2 + \lambda_1' R_1$. If $\lambda_1' u_1 < 1$ at R_1^*, then $S_1 < \lambda_1' R_1 - \lambda_1 = R_1(\lambda_1' - \frac{\lambda_1}{R_1}) < 0$. The last inequality comes from the fact that $\lambda_1 = 0$ when $R_1 = 0$ and λ_1 is increasing and concave in R_1. Since $\text{sign}(S_1) = \text{sign}(V_1')$, so $V_1' < 0$ at R_1^*. The only situation where $V_1' < 0$ at the optimum is that R_1 takes a corner solution at zero. However, we learn from the first half of the proof that $R_1 = 0$ is never optimal. Therefore, it cannot be that $\lambda_1' u_1 < 1$. With $\lambda_1' u_1 \geq 1$, we have $\frac{\partial R_1^*}{\partial R_2} > 0$.

For scalable projects that λ_1 is fixed, it is obvious R_1 is not a function of R_2.

Finally, the concavity of the best response for accelerable projects follows from taking the derivative of $\frac{\partial R_1^*}{\partial R_2}$:

$$\begin{aligned} \frac{\partial^2 R_1^*}{\partial R_2^2} &= \frac{\partial(\frac{\partial R_1^*}{\partial R_2})}{\partial R_2} \\ &= -\frac{\overbrace{\lambda_2"(\lambda_1' u_1 - 1) \times \lambda_1"(u_1(\lambda_2 + r) + R_1^*)}^{\geq 0} - \overbrace{\lambda_2'(\lambda_1' u_1 - 1) \times \lambda_1" u_1 \lambda_2'}^{\leq 0}}{\left(\lambda_1"(u_1(\lambda_2 + r) + R_1^*)\right)^2} \\ &\leq 0 \end{aligned}$$

Q.E.D.

Proof of Corollary 4 on page 76: For scalable projects that λ_i is fixed, $R_1^* = \{R_1 : \lambda_1 u_1' = 1\}$. It is obvious that R_1 does not depend on R_2. Neither does R_1 depend on project characteristics of firm 2. For accelerable projects, we show in the proof of Corollary 3 that $\frac{\partial R_1^*}{\partial R_2} > 0$, which is a more general version of the effect of competition on investment.

Q.E.D.

Proof of Proposition 8 on page 83:We argue the only state variable for both firms' project value is the cash flow of FC firm X_{FC}. For any unconstrained firm, AIP cash flow level doesn't affect its choice of investment level, therefore it should not have an impact on its constrained rival's investment scale. However, constrained firm's decision is bounded by its AIP cash flow level, thus its unconstrained rival's decision in the equilibrium also depends on X_{FC}.

From the contingent claim method, the HJB equation for the FC firm while both firms operate on the R&D project s the following ODE:

$$rV_{FC}dt = E\mathscr{D}V_{FC} - R_{FC}dt + \lambda_{FC}(u_{FC} - V_{FC})dt + \lambda_{UC}(0 - V_{FC})dt + \lambda_j(0 - V_{FC})dt \quad (2.20)$$

where $\mathscr{D}V_{FC} = V_{X_{FC}}\mu_{FC}X_{FC}dt + \frac{1}{2}V_{X_{FC}X_{FC}}\sigma_{FC}^2 X_{FC}^2 dt$. The solution of Eq (2.20) has the solution form $V_{FC}(X_{FC}) = c_1 X_{FC}^{\alpha_1} + c_2 X_{FC}^{\alpha_2} + V_p$, where α_1, α_2 are the solutions for the quadratic function $\frac{1}{2}\sigma_{FC}^2\alpha(\alpha - 1) + \mu_{FC}\alpha - (r + \lambda_{FC} + \lambda_{UC} + \lambda_j) = 0$, and $V_p = \frac{\lambda_{FC}u_{FC} - R_{FC}}{r + \lambda_{FC} + \lambda_{UC} + \lambda_j}$ is one particular solution.

The boundary conditions are

$$\lim_{X_{FC}\to\infty} V_{FC}(X_{FC}) = \frac{\lambda_{FC}u_{FC} - R_{FC}}{r + \lambda_{UC} + \lambda_{FC} + \lambda_j} \tag{2.21}$$

$$V_{FC}(X_{FC} \to \underline{X}_{FC}) = 0 \tag{2.22}$$

where $\underline{X}_{FC}$ is the abandonment threshold and $\underline{X}_{FC} \geq R_{FC}$. The first boundary condition eliminates the term associated with a positive α in the expression of $V_{FC}(X_{FC})$, and thus $V_{FC}(X_{FC}) = c_2 X_{FC}^{\alpha_2} + V_p$ with $\alpha_2 < 0$. The second boundary condition gives $c_2 = -V_p \underline{X}_{FC}^{-\alpha_2}$[35] and Eq (2.10) follows.

Similarly, the HJB equation for the UC firm is

$$rV_{UC}dt = E\mathscr{D}V_{UC} - R_{UC}dt + \lambda_{UC}(u_{UC} - V_{UC})dt + \lambda_{FC}(0 - V_{UC})dt \tag{2.23}$$

with boundary conditions

$$\lim_{X_{FC}\to\infty} V_{UC}(X_{FC}) = \frac{u_{UC}\lambda_{UC} - R_{UC}}{r + \lambda_{UC} + \lambda_{FC}}$$

$$V_{UC}(X_{FC} = \underline{X}_{FC}) = \frac{u_{UC}\lambda_{UC} - R_{UC}}{r + \lambda_{UC}}$$

We follow the same steps as before, and then we can get Eq (2.9).

Q.E.D.

[35] We don't use the smooth pasting condition $\frac{\partial V_{FC}(X_{FC})}{\partial X_{FC}}|_{\underline{X}_{FC}} = 0$ because it is not clear whether there is an optimal voluntary abandonment. We do not use $\lim_{X_{FC}\to 0} V_{FC}(X_{FC}) \neq \infty$ to rule out the term in V_{FC} that is associated with the negative root because value function may change as $X_{FC} \to \infty$.

References

Aghion, Philippe, Nicholas Bloom, Richard Blundell, Rachel Griffith, and Peter Howitt, 2005, Competition and innovation: An inverted u relationship, *Quarterly Journal of Economics*.

Aghion, Philippe, and Peter Howitt, 1992, A model of growth through creative destruction, *Econometrica* 60, 323–351.

Almeida, Heitor, and Murillo Campello, 2007, Financial constraints, asset tangibility, and corporate investment, *Review of Financial Studies* 20, 1429–1460.

Almeida, Heitor, Po-Hsuan Hsu, and Dongmei Li, 2013, Less is more: Financial constraints and innovative efficiency, *working paper*.

Arrow, Kenneth, 1962, *Economic welfare and the allocation of resources for invention* (Princeton University Press).

Bena, Jan, 2008, The effect of credit rationing on the shape of the competition-innovation relationship, *CERGE-EI Working Paper*.

Berk, Jonathan B, Richard C Green, and Vasant Naik, 2004, Valuation and return dynamics of new ventures, *Review of Financial Studies* 17, 1–35.

Bloom, Nicholas, Mirko Draca, and John Van Reenen, 2015, Trade induced technical change? the impact of chinese imports on innovation, it and productivity, *The Review of Economic Studies* p. rdv039.

Blundell, Richard, Rachel Griffith, and John Van Reenen, 1999, Market share, market value and innovation in a panel of british manufacturing firms, *The Review of Economic Studies* 66, 529–554.

Boyce, William E, and Richard C DiPrima, 2000, *Elementary differential equations and boundary value problems* (Wiley New York) 7th edn.

Brown, James R., Gustav Martinsson, and Bruce C. Petersen, 2013, Law, Stock Markets, and Innovation, *Journal of Finance* 68, 1517–1549.

Childs, Paul D, Steven H Ott, and Alexander J Triantis, 1998, Capital budgeting for interrelated projects: A real options approach, *Journal of Financial and Quantitative Analysis* 33, 305–334.

Dixit, Avinash K, and Robert Pindyck, 1994, *Investment under uncertainty* (Princeton university press).

Dixit, Avinash K, and Joseph E Stiglitz, 1977, Monopolistic competition and optimum product diversity, *The American Economic Review* 67, 297–308.

Duffie, Darrell, 2010, *Dynamic asset pricing theory* (Princeton University Press).

Faulkender, Michael, and Mitchell Petersen, 2012, Investment and capital constraints: repatriations under the american jobs creation act, *Review of Financial Studies* 25, 3351–3388.

Fazzari, Steven, R Glenn Hubbard, and Bruce C Petersen, 1988, Financing constraints and corporate investment, *Brookings Papers on Economic Activity.*

Fudenberg, Drew, and Jean Tirole, 1991, *Game Theory* (MIT Press).

Geroski, Paul, 1994, *Market structure, corporate performance, and innovative activity* (Oxford University Press).

Gilbert, Richard J, and David MG Newbery, 1982, Preemptive patenting and the persistence of monopoly, *The American Economic Review* 72, 514–526.

Giles, Michael B, 2008, Multilevel monte carlo path simulation, *Operations Research* 56, 607–617.

Grossman, Gene M, and Elhanan Helpman, 1991, *Innovation and growth in the global economy* (Cambridge, MA: MIT press).

Hackbarth, Dirk, Richmond Mathews, and David Robinson, 2014, Capital structure, product market dynamics, and the boundaries of the firm, *Management Science* 60, 2971–2993.

Hashmi, Aamir Rafique, 2013, Competition and innovation: the inverted-u relationship revisited, *Review of Economics and Statistics* 95, 1653–1668.

Jones, Charles I, and John C Williams, 2000, Too much of a good thing? the economics of investment in r&d, *Journal of Economic Growth* 5, 65–85.

Kaplan, Steven N, and Luigi Zingales, 1997, Do investment-cash flow sensitivities provide useful measures of financing constraints?, *The Quarterly Journal of Economics* 112, 169–215.

Karatzas, Ioannis, and Steven Shreve, 1991, *Brownian motion and stochastic calculus* . vol. 113 (Springer-Verlag).

Lamont, Owen, 1997, Cash flow and investment: Evidence from internal capital markets, *The Journal of Finance* 52, 83–109.

Li, Dongmei, 2011, Financial constraints, r&d investment, and stock returns, *Review of Financial Studies* 24, 2974–3007.

Liu, Hong, and Mark Loewenstein, 2002, Optimal portfolio selection with transaction costs and finite horizons, *Review of Financial Studies* 15, 805–835.

Meng, Rujing, 2008, A patent race in a real options setting: Investment strategy, valuation, capm beta, and return volatility, *Journal of Economic Dynamics and Control* 32, 3192–3217.

Nickell, Stephen J, 1996, Competition and corporate performance, *Journal of political economy* CIV, 724–746.

Phillips, Gordon M., and Alexei Zhdanov, 2013, R&D and the Incentives from Merger and Acquisition Activity, *Review of Financial Studies* 26, 34–78.

Rauh, Joshua D, 2006, Investment and financing constraints: Evidence from the funding of corporate pension plans, *The Journal of Finance* 61, 33–71.

Reinganum, Jennifer F, 1983, Uncertain innovation and the persistence of monopoly, *The American Economic Review* 73, 741–748.

Romer, Paul, 1990, Endogenous technological change, *journal of Political Economy* 98, 71–102.

Salop, Steven, 1977, The noisy monopolist: imperfect information, price dispersion and price discrimination, *The Review of Economic Studies* 44, 393–406.

Seru, Amit, 2014, Firm boundaries matter: Evidence from conglomerates and r&d activity, *Journal of Financial Economics* 111, 381–405.

Tishler, Asher, and Irena Milstein, 2009, R&d wars and the effects of innovation on the success and survivability of firms in oligopoly markets, *International Journal of Industrial Organization* 27, 519–531.

Whited, Toni M, 1992, Debt, liquidity constraints, and corporate investment: Evidence from panel data, *The Journal of Finance* 47, 1425–1460.