ABSTRACT

Title of dissertation: Partonic Contributions to the Proton Spin in Lattice QCD

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Dissertation directed by: Dr. Xiangdong Ji
Department of Physics
University of Maryland
College Park, MD 20742

In Feynman's parton picture, the proton spin can be understood as sum of the contributions from the spin and orbital angular momentum of the quark and gluon partons. However, in gauge theories, there is no local gauge-invariant notion of the spin or orbital angular momentum of the gauge particles. It is shown that in the infinite momentum frame of the proton, the gluons can be equivalent to free radiation, which is analogous to the Weizsäcker-Williams approximation in electrodynamics, and therefore one can talk about gluon helicity and longitudinal orbital angular momentum. We will justify the physical meaning of the Jaffe-Manohar sum rule for the longitudinal proton spin which uses the free-field expression of the QCD angular momentum operator in the light-cone gauge. Furthermore, it is discovered that each term in the Jaffe-Manohar sum rule can be related to the matrix element of a gauge-invariant, but frame-dependent operator through a factorization formula in large-momentum effective field theory. This provides a new approach for the nonperturbative calculation of the proton spin content in lattice QCD, and can be applied to the other parton observables as well. We present all the matching coefficients for the proton spin sum rule and non-singlet quark distributions at one-loop order in perturbation theory. These results will be useful for a first direct lattice calculation of the corresponding parton properties, especially the gluon helicity and parton orbital angular momentum.

Partonic Contributions to the Proton Spin in Lattice QCD

by

Yong Zhao

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Advisory Committee:
Dr. Xiangdong Ji, Chair/Advisor
Dr. Paulo Bedaque
Dr. Rabindra Mohapatra
Dr. Carter Hall
Dr. Da-Lin Zhang

Dedication

To my parents and younger sister.

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Chapter 1

Introduction

Fifty years ago, quarks and color symmetry were introduced to study the hadron structure and strong interaction. Later on, quantum chromodynamics (QCD), an $SU(3)$ gauge theory, was established as the fundamental theory that describes the interactions between quarks and the strong force mediator—gluons. The quarks bind together to form baryons and mesons, and asymptotic freedom of QCD makes it impossible to separate them as free particles. At the energy scale of $\Lambda_{\text{QCD}} \sim 200$ MeV for baryons and mesons, the coupling α_s is strong and thus the physical properties of the proton are governed by nonpeturbative effects. For example, the proton is understood to be made of two up and one down quarks, but its mass (~ 1 GeV) is way larger than the sum of the (current) masses of the three (~ 10 MeV), which indicates that the interaction accounts for the most contribution. Due to this reason, it remains a challenging task in theoretical physics to quantitate the hadron structure in terms of the quark and gluon degrees of freedom.

1.1 The Proton Spin Problem

The proton has a spin of $1/2$ when at rest. For a moving proton, the projection of spin along the direction of its motion—the helicity—is conserved and has quantized values of $\pm 1/2$. In 1974, Ellis and Jaffe [1] suggested that the polarized proton does not contain any polarized strange quarks, and based only on $SU(3)$ flavor symmetry they could predict the total quark spin contribution to be about 60% from hyperon β-decay [2]. In a quark model it is reasonable to attribute the proton spin to the spin and orbital motions of the quarks, and the Ellis-Jaffe prediction was once a "folklore" in particle physics [2].

High-energy scattering provides another window to look at the proton. In this case, the proton is moving at almost the speed of light, and the quarks can be approximated as a beam of free partons that were introduced by Feynman [3]. In 1987, the European Muon Collaboration (EMC) at CERN measured the quark spin from polarized deep-inelastic (DIS) muon scattering with a fixed proton target, and discovered that it was consistent with zero [4, 5], which sharply contradicted the Ellis-Jaffe prediction and thus generated the famous "proton spin crisis", or, proton spin problem. Ever since that, an enormous amount of experimental efforts have been dedicated to measure the separate contributions of different quark flavors as well as the gluon helicity. These include the spin programs at SLAC, CERN (SMC and COMPASS), DESY (HERMES), JLab and RHIC (STAR, PHENIX, and BRAHMS) [6]. The electron-ion collider (EIC), which will become the next QCD frontier, will give more precise answer to the spin structure of the proton [7].

1.2 Spin Structure Functions and Parton Distributions

The quark and gluon spin contributions can be understood from polarized inclusive and semi-inclusive DIS and proton-proton scattering experiments. The former measures a spin-dependent structure function $g_1(x, Q^2)$, while the latter is directly related to the polarized parton distribution functions for different quark flavors and the gluon.

1.2.1 Inclusive and Semi-inclusive Deep Inelastic Scattering

In a fixed-target lepton-proton scattering experiment, the high-energy lepton exchanges a hard virtual photon with the static proton, and strikes out a quark that hadronizes into observable particles. For totally inclusive processes, the longitudinal spin asymmetry of the cross section is related to $g_1(x, Q^2)$ in the scaling limit [8]:

$$A_1 = \frac{\sigma_{\frac{1}{2}} - \sigma_{\frac{3}{2}}}{\sigma_{\frac{1}{2}} + \sigma_{\frac{3}{2}}} \approx \frac{g_1(x, Q^2)}{F_1(x, Q^2)} \; , \tag{1.1}$$

where x is the Bjorken variable, and $-Q^2$ is the invariant mass of the exchanged photon. $\sigma_{\frac{3}{2}}$ and $\sigma_{\frac{1}{2}}$ are the cross sections for the absorption of a transversely polarized photon with spin parallel and antiparallel to the spin of the longitudinally polarized proton, and $F_1(x, Q^2)$ is a well-known DIS structure function. In the parton model,

$$g_1^p = \frac{1}{2} \sum_q e_q^2 \left\{ \Delta q + \frac{\alpha_s}{2\pi} \left[\Delta C_q \otimes \Delta q + \Delta C_g \otimes \Delta g \right] \right\} \; , \tag{1.2}$$

where p stands for proton, and e_q is the unit of charge carried by the quark of flavor q. $\Delta q(x)$ and $\Delta g(x)$ are the polarized quark and distribution functions,

$$\begin{aligned} \Delta q(x) &= q_+(x) + \bar{q}_+(x) - q_-(x) - \bar{q}_-(x) \; , \\ \Delta g(x) &= g_+(x) - g_-(x) \; , \end{aligned}$$

with $+$ and $-$ meaning that the spin of the parton is parallel or antiparallel to that of the proton. There are only three light flavors considered here because it is assumed that Q^2 is below the threshold of the production of heavy quarks, otherwise contributions from the latter should be taken as negligible. ΔC_q and ΔC_g are spin-dependent Wilson coefficients calculable in perturbative QCD, and the convolution "$\otimes$" is defined to be

$$(\Delta C \otimes q)(x, Q^2) = \int_x^1 \frac{dy}{y} \; \Delta C(\frac{x}{y}) q(y, Q^2) \; . \tag{1.3}$$

By parametrizing the polarized quark and gluon distributions according to the *ansatz* in Ref. [9], one can fit these distributions and obtain the flavor structure of proton spin. This method requires large statistics and a good understanding of the unpolarized parton distributions, and the earliest analysis came in 1995 [9]. On the other hand, semi-inclusive experiments that measure the cross section of specific hadron prodcutions can be used to tag the flavor of the struck quark, and the longitudinal spin asymmetry is rewritten as [6]

$$A_1^h(x, Q^2) \approx \frac{\sum_{q,h} e_q^2 \Delta q(x, Q^2) \int_{z_{\min}}^1 dz D_f^h(z, Q^2)}{\sum_{q,h} e_q^2 q(x, Q^2) \int_{z_{\min}}^1 dz D_f^h(z, Q^2)} \; , \tag{1.4}$$

where h stands for the hadron, and D_f^h is a fragmentation function for the struck quark to produce a hadron h with momentum fraction z. Here $q(x, Q^2)$ is the unpolarized quark distribution, and $z_{\min}$ (~ 0.2) is determined by kinematical cuts applied when measuring the asymmetries. In this way, one can also reconstruct the flavor content of the quark spin [10].

Actually, the quantity being analyzed immediately in the EMC results was the first moment of g_1. According to operator product expansion (OPE) [11],

$$\begin{aligned}\int_0^1 dx g_1^p(x,Q^2) &= \frac{1}{2}\sum_q e_q^2\left\{\Delta\Sigma_q + \frac{\alpha_s}{2\pi}\left[\Delta\Sigma_q\int_0^1 dx\Delta C_q(x) + \Delta G\int_0^1 dx\Delta C_g(x)\right]\right\}\\ &= \left(\frac{1}{12}g_A^{(3)} + \frac{1}{36}g_A^{(8)}\right)\left[1+\sum_{l\geq 1} c_l^{\rm NS}\alpha_s^l(Q^2)\right]\\ &\quad + \frac{1}{9}g_A^{(0)}\left[1+\sum_{l\geq 1} c_l^{\rm S}\alpha_s^l(Q^2)\right] + \mathcal{O}(\frac{1}{Q^2})\ ,\end{aligned} \tag{1.5}$$

where the flavor- non-singlet and singlet Wilson coefficients $c_l^{\rm NS}$, $c_l^{\rm S}$ are calculable in l-loop perturbative QCD, and $\mathcal{O}(1/Q^2)$ are higher-twist contributions. $g_A^{(3)}$, $g_A^{(8)}$, and $g_A^{(0)}$ are the isovector, $SU(3)$ octet, and flavor-singlet charges, respectively:

$$\begin{aligned} g_A^{(3)}S^\mu &= \frac{2\langle P,S|\bar\psi\gamma^\mu\gamma^5 t^3\psi|P,S\rangle}{2P^0} = (\Delta\Sigma_u - \Delta\Sigma_d)\frac{S^\mu}{P^0}\ ,\\ g_A^{(8)}S^\mu &= \frac{2\sqrt{3}\langle P,S|\bar\psi\gamma^\mu\gamma^5 t^8\psi|P,S\rangle}{2P^0} = (\Delta\Sigma_u + \Delta\Sigma_d - 2\Delta\Sigma_s)\frac{S^\mu}{P^0}\ ,\\ g_A^{(0)}S^\mu &= \frac{\langle P,S|\bar\psi\gamma^\mu\gamma^5\psi|P,S\rangle}{2P^0} = (\Delta\Sigma_u + \Delta\Sigma_d + \Delta\Sigma_s)S^\mu = \Delta\Sigma\frac{S^\mu}{P^0}\ ,\end{aligned} \tag{1.6}$$

where t^a is an $SU(3)$-flavor generator in the fundamental representation, $|P,S\rangle$ is a proton state of momentum P, spin vector S^μ with $S^2 = -M^2$, $S\cdot P = 0$, normalized to $\langle P,S|P',S'\rangle = 2P^0(2\pi)^3\delta^{(3)}(\vec{P}-\vec{P}')\delta_{SS'}$, and

$$\begin{aligned}\Delta\Sigma_q\frac{S^\mu}{P^0} &= S^\mu\int_0^1 dx\Delta q(x) = \frac{\langle P,S|\bar q\gamma^\mu\gamma^5 q|P,S\rangle}{2P^0}\ ,\\ \Delta G &= \int_0^1 dx\Delta g(x)\ .\end{aligned} \tag{1.7}$$

The flavor-singlet axial vector current

$$j_\mu^5 = \bar\psi\gamma_\mu\gamma_5\psi = \bar u\gamma_\mu\gamma_5 u + \bar d\gamma_\mu\gamma_5 d + \bar s\gamma_\mu\gamma_5 s \tag{1.8}$$

has an anomalous dimension starting at two loops, and $g_A^{(0)}$ is multiplicatively renormalized:

$$g_A^{(0)}(Q^2) = g_A^{(0)}|_{\rm inv}/E(\alpha_s) \ , \qquad E(\alpha_s) = \exp\left[\int_0^{\alpha_s(Q^2)} d\alpha_s' \gamma(\alpha_s')/\beta(\alpha_s')\right] \ , \qquad (1.9)$$

where $g_A^{(0)}|_{\rm inv}$ is $g_A^{(0)}(Q^2=\infty)$ and thus renormalization group invariant. Note that there is no gluonic contribution in Eq. (1.5) because the first moment of the Wilson coefficient $\Delta C_g(x)$ is zero in the $\overline{\rm MS}$ factorization scheme. This became a matter of dispute over the EMC results and will be explained in the following discussions.

For massless u, d and s quarks, $SU(3)$ flavor symmetry ensures that the non-singlet axial charges $g_A^{(3)}$ and $g_A^{(8)}$ are strictly conserved. As $g_A^{(3)}$ and $g_A^{(8)}$ are the nucleon matrix elements in baryon β-decays, the latter can measure these two constants even though they are at low energy. By performing a weighted least squares two-parameter (F and D) fit to the modified Particle Data Group data of baryon β-decays [12], Jaffe and Manohar obtained [2]

$$g_A^{(3)} = F + D = 1.28 \pm 0.07, \qquad g_A^{(8)} = 3F - D = 0.60 \pm 0.12 \ . \qquad (1.10)$$

Before the EMC experiment, the idea that the strange quark content of the proton is very small was a corollary of the famous Okubo-Zweig-Iizuka (OZI) rule [2], and according to the Ellis-Jaffe ansatz [1] one would have predicted that

$$\Delta\Sigma(Q^2_{\rm EMC})_{\rm EJ} = g_A^{(0)} \approx g_A^{(8)} = \ 0.60 \pm 0.12 \ . \qquad (1.11)$$

However, the EMC experiment measured the first moment of g_1 [4, 5], and found the total quark spin of the proton to be

$$\Delta\Sigma(Q^2_{\rm EMC}) = g_A^{(0)} = 0.13 \pm 0.19 \ ,$$

which was consistent with zero and significantly smaller than the Ellis-Jaffe prediction. This is how the "proton spin crisis" came into being.

The smallness of $\Delta\Sigma(Q^2_{\rm EMC})$ could be explained by a violation of the OZI rule as Δs may contribute considerably to the proton spin,

$$\Delta s(Q^2_{\rm EMC}) = -0.16 \pm 0.08 \ ,$$

or the breaking of the $SU(3)$ flavor symmetry that makes the two parameter fit of the axial charges inaccurate [2]. It was also pointed out in Ref. [13] that an instanton-induced axial $U(1)$ symmetry breaking will lead to a polarized condensate that contributes to $g_1(x)$ with support only at $x = 0$. However, the kinematical region of all the inelastic scattering experiments can only reach a minimal value of x which can be extremely small but nonzero, so it is actually "$g_A^{(0)} - \lambda$"—where λ is the contribution from the polarized condensate—that they extract from the first moment of g_1. To provide an independent measurement of Δs and evaluate the effects of dynamical axial $U(1)$ symmetry breaking, one can turn to elastic Z^0 exchange processes such as νp scattering as it can probe the complete $g_A^{(0)}$ [13, 14]. An analysis in Ref. [14] showed that the value for $\Delta\Sigma$ is consistent with the EMC result if one assumes $SU(3)$ flavor symmetry. This indicates a nonzero negative contribution from Δs and small value of λ, but more extensive studies are still needed to make such a statement.

Meanwhile, another interesting idea prevailing around 1988 was that the smallness of the "quark spin" measured by the EMC experiment is due to a large cancellation from the gluon helicity ΔG through the $U(1)$ axial triangle anomaly [16–18], which corresponds to the Feynman diagram shown in Fig. 1.1. Thus the flavor-singlet charge calculated at one-loop order of QCD is

$$\frac{\langle P,S|\bar{\psi}\gamma^{\mu}\gamma^{5}\psi|P,S\rangle}{2P^{0}} = \left(\Delta\Sigma' - n_f\frac{\alpha_s}{2\pi}\Delta G\right)\frac{S^{\mu}}{P^{0}}\ , \tag{1.12}$$

where n_f is the number of active quark flavors and $n_f = 3$ for the EMC experiment. $\Delta\Sigma'$ is regarded as the renormalization-group-invariant intrinsic quark spin, and the scaling violation comes from the gluonic term. The inclusion of the gluonic contribution is due to the redefinition of the Wilson coefficient $\Delta C_g(x)$, which refers to the Adler-Bardeen factorization scheme [16–18].

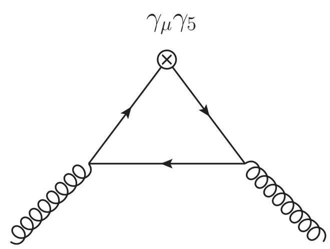

Figure 1.1: Triangle anomaly contribution to the flavor-singlet axial vector current.

It was argued that $\Delta G(Q^2)$ scales like $1/\alpha_s(Q^2)$ at leading order in the limit of $Q^2 \to \infty$, and therefore its contribution to $g_A^{(0)}$ can be important and the total quark spin $\Delta\Sigma'$ can be consistent with the Ellis-Jaffe prediction [17, 18]. However, to have such a cancellation in $g_A^{(0)}$ it requires a large ΔG ($\sim$ 10), while Jaffe and Manohar showed that this will lead to a large intrinsic heavy quark spin [2] which contradicts the quark model even worse. Nevertheless, in recent years, the actual contribution from ΔG is measured to be much smaller ($\sim$ 0.2) than anticipated, which we will elaborate in the following subsection. Therefore, in the rest of this paper we will ignore the dispute over the triangle anomaly, and only consider the total quark spin in the $\overline{\mathrm{MS}}$ factorization scheme, i.e., the flavor-singlet charge $g_A^{(0)}$.

1.2.2 Proton-proton Scattering

Polarized deep inelastic proton-proton scattering provides another window to measure the spin of quarks with different flavors as well as the gluon helicity. RHIC is the first and only polarized proton-proton collider in the world, and a typical observable of interest at RHIC is the spin-dependent cross section for $pp \to \text{jet} + X$ with transverse momentum p_T [19],

$$\frac{d\Delta\sigma}{dp_T} \equiv \frac{1}{2}\left(\frac{d\sigma_{++}}{dp_T} - \frac{d\sigma_{+-}}{dp_T}\right) , \tag{1.13}$$

where the superscripts "++" and "+−" denote the same and opposite helicity combinations of the proton beams. The above cross section can be factorized into a convolution of polarized parton densities and hard scattering cross sections:

$$\frac{d\Delta\sigma}{dp_T} = \sum_{ab}\int dx_a dx_b \Delta q_a(x,\mu)\Delta q_b(x,\mu)\frac{d\Delta\hat{\sigma}^{ab\to\text{jet}+X}}{dp_T}(x_a P_a, x_b P_b, \mu) , \tag{1.14}$$

where μ is the factorization scale, a, b run over all quark flavors and the gluon, and P_a, P_b are the momenta of the scattering protons.

When jet $= W^{\pm}$, the cross section is dominated by the channels $u\bar{d} \to W^+$ and $d\bar{u} \to W^-$ with no fragmentation, and therefore the RHIC data can provide complementary and precise information on the polarized distributions of the up and down quarks and their antiquarks [19]. When jet $= \pi^0$, the longitudinal double spin asymmetry is sensitive to the gluon polarization distribution, which is key to the determination of its x-dependence.

In 2009, the DSSV (D. de Florian, Sassot, Stratmann and Vogelsang) group [20] made a global analysis of the data from the inclusive and semi-inclusive experiments of SMC, HERMES and COMPASS, as well as the proton-proton scattering at RHIC. Their results showed that the (truncated) total quark spin and gluon polarization of the proton are

$$\begin{aligned} \Delta\Sigma(Q^2 = 10\ \mathrm{GeV}^2) &= \int_{0.001}^{1} dx\ \Delta\Sigma(x, Q^2 = 10\ \mathrm{GeV}^2) = 0.366^{+0.042}_{-0.062}\ , \\ \Delta G(Q^2 = 10\ \mathrm{GeV}^2) &= \int_{0.001}^{1} dx\ \Delta g(x, Q^2 = 10\ \mathrm{GeV}^2) = 0.013^{+0.702}_{-0.314}\ . \end{aligned} \tag{1.15}$$

Especially, within the kinematical range of the RHIC experiments, $0.05 \leq x \leq 0.2$,

$$\int_{0.05}^{0.2} dx\ \Delta g^{\mathrm{RHIC}}(x, Q^2 = 10\ \mathrm{GeV}^2) = 0.005^{+0.129}_{-0.164}\ , \tag{1.16}$$

which shows that the gluon polarization is consistent with zero.

Later on the DSSV group included the new data from the 2009 run of RHIC and re-analyzed the gluon polarization [21]. In contrast to their earlier results [20], the new analysis supports a positive definite distribution $\Delta g(x, Q^2)$ at $Q^2 = 10\ \mathrm{GeV}^2$, and the truncated first moment of Δg is

$$\int_{0.05}^{0.2} dx\ \Delta g^{\mathrm{RHIC}}(x, Q^2 = 10\ \mathrm{GeV}^2) = 0.195 \pm 0.070 \tag{1.17}$$

within 90% confidential level. Note that the exact value and error of the truncated first moment of Δg was not given in Ref. [21], while the result provided above is obtained by reading the pixels in the plot of the change of its $\Delta\chi^2$ profile.

Since the small x region is still the most important source of uncertainty for $\Delta g(x, Q^2)$, EIC will provide the missing information needed to fully determine the gluon polarization [7].

1.3 Sum rules for the proton spin

With the total quark spin measured to be about one third, and the gluon helicity not likely to be significantly larger than 0.2, it is natural to attribute the rest of the proton spin to the orbital motion of the quarks and gluons.

In the past 25 years, two well-known sum rules have been proposed to analyze the proton spin structure. The first, proposed by Jaffe and Manohar [2], was motivated from a free-field expression of QCD angular momentum boosted to the infinite momentum frame (IMF) of the proton. The second, usually called Ji's sum rule, is the frame-independent and manifestly gauge-invariant decomposition of the proton spin [22].

1.3.1 The Jaffe-Manohar sum rule

The Jaffe-Manohar sum rule is defined in the light-cone gauge $A^+ = 0$, and states that the proton spin can be decomposed into four parts,

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma(\mu) + \Delta G(\mu) + L_q^z(\mu) + L_g^z(\mu) \ , \tag{1.18}$$

where the individual terms are the spin and OAM of the quarks and gluons, respectively, and μ is a renormalization scale. All the four terms are defined to be the proton matrix elements of free-field angular momentum operators in the IMF or on the light-cone plane [2]:

$$\begin{aligned}\vec{J} = &\int d^3\xi\ \psi^\dagger \frac{\vec{\Sigma}}{2}\psi + \int d^3\xi\ \psi^\dagger \vec{\xi} \times (-i\vec{\nabla})\psi \\ &+ \int d^3\xi\ \vec{E}_a \times \vec{A}^a + \int d^3\xi\ E_a^i\ \vec{\xi} \times \vec{\nabla} A^{i,a} \ ,\end{aligned} \tag{1.19}$$

where $E^i = F^{i+}$, a and i are the color and spatial indices. Here the the light-cone coordinates $\xi^\pm = (x^0 \pm x^3)/\sqrt{2}$ are used.

In light-cone quantization, each term in Eq. (1.18) can be expressed as sum of the spin and OAM over all Fock states, so the Jaffe-Manohar sum rule has a clear partonic interpretation. However, the free-field form of the angular momentum in gauge theories faces two conceptual problems: all terms except the first one are gauge dependent, and it is unclear why the light-cone gauge operator is measurable in experiments.

$\Delta\Sigma$ and ΔG in the Jaffe-Manohar sum rule are known to be the quark spin and gluon polarization measured in polarized DIS experiments. It is not obvious that ΔG is just the gluon spin, as in OPE there is no local gauge-invariant operator for the first moment of the polarized gluon distribution $\Delta g(x)$. To understand this, let us look at the definition of $\Delta g(x)$ from QCD factorization theorems [23]:

$$\begin{aligned}\Delta g(x) &= \frac{i}{2xP^+}\int \frac{d\xi^-}{2\pi} e^{-ixP^+\xi^-} \\ &\quad \times \langle P,S|F_a^{+\alpha}(0,\xi^-,0_\perp)\mathcal{L}^{ab}(\xi^-,0)\tilde{F}^+_{\alpha,b}(0,0,0_\perp)|P,S\rangle \ , \qquad (1.20)\end{aligned}$$

where $\tilde{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}F_{\mu\nu}$, and $\mathcal{L}(\xi^-,0) = P\exp[-ig\int_0^{\xi^-} d\eta^- \mathcal{A}^+(0,\eta^-,0_\perp)]$ with $\mathcal{A}^+ \equiv T^a A_a^+$ is a light-cone gauge link defined in the adjoint representation of $SU(3)$. The n-th ($n \geq 2$) moments of $\Delta g(x)$ give rise to the matrix elements of all the leading-twist gluonic operators in the spin-dependent part of the OPE [11]. Since the first moment of $\Delta g(x)$ is understood to be the total gluon polarization, we can define the gauge-invariant gluon spin operator as [24]

$$S_g^{\rm inv} = \int dx \frac{i}{x}\int \frac{d\xi^-}{2\pi}e^{-ixP^+\xi^-}\ F_a^{+\alpha}(0,\xi^-,0_\perp)\mathcal{L}^{ab}(\xi^-,0)\tilde{F}^+_{\alpha,b}(0,0,0_\perp)\ . \qquad (1.21)$$

In the light-cone gauge $A^+ = 0$, the gauge link becomes unit one. After integration by parts,

$$S_g^{\rm inv} = \left[\vec{E}_a(0) \times \vec{A}^a(0)\right]^3\Big|_{A^+=0} \ , \tag{1.22}$$

which is exactly the free-field gluon spin operator.

As for L_q^z and L_g^z, they originate from the transverse motion of the quarks and gluons, so they should be related to higher-twist effects. The free-field form of OAM is also called “canonical OAM”, and recent theoretical developments found that they are related to twist-three generalized parton distributions (GPD’s) [25–27], which have been studied and can be extracted from two-photon processes such as deep virtual Compton scattering (DVCS) [28, 29].

1.3.2 Ji’s sum rule

Ji’s sum rule takes a different form from Eq. (1.19), as the total QCD angular momentum is decomposed into three gauge-invariant parts [22]:

$$\begin{aligned} \vec{J} \ &= \ \int d^3x \ \psi^\dagger \frac{\vec{\Sigma}}{2}\psi + \int d^3x \ \psi^\dagger \vec{x} \times (-i\vec{\nabla} - g\vec{A})\psi \\ &\quad + \int d^3x \ \vec{x} \times (\vec{E} \times \vec{B}) \ , \end{aligned} \tag{1.23}$$

where the total gluon angular momentum in the second line cannot be gauge-invariantly decomposed into local spin and OAM operators [30]. In this way, Ji’s sum rule reads:

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma(\mu) + \mathcal{L}_q^z(\mu) + J_g^z(\mu) \ . \tag{1.24}$$

Unlike the Jaffe-Manohar sum rule, each term in Eq. (1.23) is gauge invariant and frame independent, which is what one would expect from physical observables.

When Ji's sum rule was first proposed, it immediately received a lot of attention because each term can be measured through twist-two GPD's from DVCS experiments [22,31]. The quark and gluon angular momenta satisfy

$$
\begin{aligned}
J_{q,g} &= \frac{1}{2}\left[A_{q,g}(0) + B_{q,g}(0)\right] \ , \\
J_q + J_g &= \frac{1}{2} \ , \qquad (1.25)
\end{aligned}
$$

where $A_{q,g}(0)$ and $B_{q,g}(0)$ are form factors of the symmetrized quark and gluon energy-momentum tensors.

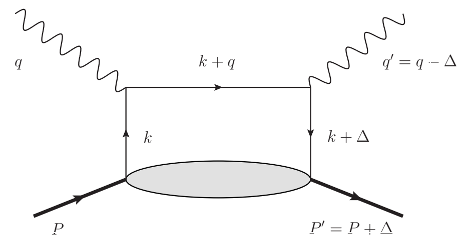

Figure 1.2: Dominant scattering process in DVCS.

In a DVCS process as shown in Fig. 1.2, the Compton amplitude depends on four twist-two GPD's, H, $\tilde{H}$, E and $\tilde{E}$. In the light-cone gauge, they are defined to be the off-forward matrix elements of the light-cone correlations:

$$
\begin{aligned}
\int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle P'|\bar{\psi}(-\frac{\lambda n}{2})\gamma^\mu \psi(\frac{\lambda n}{2})|P\rangle &= H(x,\Delta^2,\xi)\bar{U}(P')\gamma^\mu U(P) \\
&\quad + E(x,\Delta^2,\xi)\bar{U}(P') i\frac{\sigma^{\mu\nu}\Delta_\nu}{2M} U(P) + \cdots \ , \\
\int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle P'|\bar{\psi}(-\frac{\lambda n}{2})\gamma^\mu \gamma_5 \psi(\frac{\lambda n}{2})|P\rangle &= \tilde{H}(x,\Delta^2,\xi)\bar{U}(P')\gamma^\mu U(P) \\
&\quad + \tilde{E}(x,\Delta^2,\xi)\bar{U}(P') i\frac{\sigma^{\mu\nu}\Delta_\nu}{2M} U(P) + \cdots \ ,
\end{aligned}
$$

(1.26)

where n^μ is a vector along the light-cone direction, and the skewness parameter

$\xi = -n \cdot \Delta / n \cdot (P + P')$. Here the "$\cdots$" represents higher-twist contributions. The quark angular momentum is related to these GPD's through the sum rule [22, 31]

$$\int_{-1}^{1} dx\ x \left[H(x, \xi, \Delta^2) + E(x, \xi, \Delta^2)\right] = A_q(\Delta^2) + B_q(\Delta^2)\ . \tag{1.27}$$

To obtain J_q, one can extrapolate the sum rule to $\Delta^2 = 0$, and then $J_g = 1/2 - J_q$. Since the quark spin $\Delta\Sigma$ has been precisely measured in inclusive and semi-inclusive scattering experiments, one can subtract it from J_q to determine $\mathcal{L}_q$ in Ji's sum rule. In the IMF, J_g can be further decomposed into three parts,

$$J_g^z = \Delta G + L_g^z + J_{\text{pot}}^z\ , \tag{1.28}$$

where the so called "potential" angular momentum J_{pot}^z is the matrix element of the operator

$$\vec{J}_{\text{pot}} = g \int d^3x\ \psi^\dagger \vec{x} \times \vec{A} \psi\ . \tag{1.29}$$

J_{pot}^z is also related to twist-three GPD's that can be measured in hard exclusive processes [27].

1.4 Theoretical Understanding of the Proton Spin Content

Since the 1970's, there has been a lot of proposals to calculate the proton spin content. The early attempts were model calculations which give predictions for parton spin and OAM in terms of free parameters that can be fitted from known experimental results (see Appendix A). Instead of modeling the baryons, one would expect to do a first-principle calculation of the proton matrix elements. Till now, the only practical nonperturbative approach to solve QCD is the lattice theory developed by K. Wilson. In this section we discuss the development of lattice QCD calculation of the proton spin content.

Since the Jaffe-Manohar sum rule is defined in the light-cone coordinates (or IMF) and the $A^+ = 0$ gauge, the real time dependence makes it not feasible for lattice QCD calculations because the latter is formulated in the Euclidean space with imaginary time. Nevertheless, unlike the other three operators, the quark spin is gauge invariant and frame independent, so one can calculate its matrix in a finite momentum frame with any gauge conditions that can be fixed on the lattice.

In 1995, the first lattice calculation of the flavor-singlet axial charge $g_A^{(0)}$ was carried out using the improved Wilson action with quenched approximation [32,33]. In this calculation, the spin of a specific quark flavor is divided into the connected and disconnected insertions, which correspond to the valence and sea contributions respectively. The connected insertions obey the OZI rule, so the strange quark spin originates solely from the disconnected insertion. In Ref. [32], the result for $\beta = 6$ is

$$\Delta\Sigma = \Delta\Sigma_u + \Delta\Sigma_d + \Delta\Sigma_s = +0.79(11) - 0.42(11) - 0.12(1) = +0.25(12) \ , \quad (1.30)$$

while in Ref. [33], at $\beta = 5.7$,

$$\Delta\Sigma = \Delta\Sigma_u + \Delta\Sigma_d + \Delta\Sigma_s = +0.638(54) - 0.347(46) - 0.109(30) = +0.18(10) \ . \quad (1.31)$$

With improved computational power, simulation with dynamical fermions became available. In 1999, a re-analysis of $g_A^{(0)}$ was done with $n_f = 2$ heavy dynamical quarks [34], and the result for $\beta = 5.6$ is

$$\Delta\Sigma = \Delta u + \Delta d + \Delta s = +0.62(7) - 0.29(6) - 0.12(7) = +0.20(12) \ . \quad (1.32)$$

Since the disconnected sea contribution has a larger uncertainty compared to the connected insertions, simulation of the strange quark spin with light dynamical quarks has been studied with improved statistics in recent years [35–38]. The values for $\Delta\Sigma_s$ are $\{-0.020(10)(4), -0.031(17), -0.0227(34), -0.019(11)\}$ for $\mu^2 = \{7.4\ \text{GeV}^2, 4\ \text{GeV}^2, 0, 0\}$ in the $\overline{\text{MS}}$ renormalization scheme.

The earliest attempt to calculate the gluon polarization in lattice QCD was carried out by evaluating the matrix element of the topological current

$$K_\mu = \epsilon_{\mu\alpha\lambda\sigma}\text{Tr}A^\alpha(F^{\lambda\sigma} - \frac{2}{3}A^\lambda A^\sigma) \tag{1.33}$$

or $\text{Tr}F_{\mu\nu}\tilde{F}^{\mu\nu}$ in the $A^0 = 0$ gauge [39]. However, it was soon pointed out that what Ref. [39] measured is actually not the gluon polarization ΔG [40,41]. The topological current is not gauge invariant, although its forward matrix element coincides with the ΔG at one-loop order in perturbation theory [2]. ΔG in parton physics is defined to be the matrix element of K_μ in the light-cone gauge [2], while it was proved in Ref. [42] nonperturbatively that the result fixed in the $A^0 = 0$ gauge differs from ΔG by $1/P^0$ corrections, where P^0 is the energy of the nucleon.

Since it is not possible to directly calculate ΔG or the polarized gluon distribution $\Delta g(x)$ in lattice QCD, there has been little progress in this direction in the past two decades, let alone the calculation of the quark and gluon canonical OAM.

The quark OAM and gluon angular momentum in Ji's sum rule, however, are accessible on the Euclidean lattice. Since they are gauge invariant and frame independent, one can calculate them on the equal-time plane, and then analytically continue to the Euclidean space with imaginary time. Therefore, there has been consistent effort in calculating the quark OAM in Ji's sum rule [43–49]. The most recent calculation that includes both the connected and disconnected insertions was accomplished on a quenched lattice [50]. In the $\overline{\text{MS}}$ scheme at $\mu = 2$ GeV,

$$\begin{aligned} \Delta\Sigma &= \text{CI}(u+d) + 2\text{DI}(u/d) + \text{DI}(s) = +0.62(9) - 0.24(2) - 0.12(1) = 0.25(12) \ , \\ 2L_q^{\text{Bel}} &= \text{CI}(u+d) + 2\text{DI}(u/d) + \text{DI}(s) = +0.01(10) + 0.16(1) + 0.14(1) = 0.28(10) \ , \end{aligned} \tag{1.34}$$

where CI and DI stand for connected and disconnected insertions repsectively.

To summarize, there has been significant progress in determining the proton spin content in Ji's sum rule in lattice QCD, but the calculation for the Jaffe-Manohar sum rule still remains as a challenging task nowadays.

Chapter 2

A Physical Sum Rule for the Proton Spin

In this chapter, we justify the physical meaning of the Jaffe-Manohar sum rule in quantum field theory.

2.1 Poincaré symmetry and the QCD angular momentum

In quantum field theory, Noether's theorem states that there is a conserved current associated with each continuous symmetry, and the charge of the conserved current is a generator of the symmetry group. Poincaré group is the basic symmetry group for relativistic quantum fields, as it includes translation and Lorentz symmetries. For a generic field ϕ_r with Lagrangian density

$$\mathcal{L} = \mathcal{L}[\phi_r, \partial_\mu \phi_r] \ , \tag{2.1}$$

translational invariance leads to the conserved energy-momentum tensor

$$T^{\mu\nu}(x) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \partial^\nu \phi_r(x) - g^{\mu\nu} \mathcal{L} \ , \tag{2.2}$$

and Lorentz invariance gives rise to the conserved angular momentum density

$$M^{\mu\nu\lambda} = x^\nu T^{\mu\lambda} - x^\lambda T^{\mu\nu} - i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} (\Sigma^{\nu\lambda})_r{}^s \phi_s(x) \ , \tag{2.3}$$

where $\Sigma^{\mu\nu}$ is a generator in the spinor space. Under an infinitesimal Lorentz transformation, $x^\mu \to x'^\mu = (g^\mu{}_\nu + \omega^\mu{}_\nu) x^\nu$,

$$\phi_r(x) \to \phi'_r(x') = \phi_r(x) - \frac{i}{2} \omega_{\mu\nu} (\Sigma^{\mu\nu})_r{}^s \phi_s(x) \ . \tag{2.4}$$

For scalars, spin-1/2 fermions and spin-1 bosons,

$$
\begin{aligned}
\phi(x) \qquad & \Sigma^{\mu\nu} = 0 \ , \\
\psi_r(x) \qquad & (\Sigma^{\mu\nu})_r{}^s = \frac{1}{2}(\sigma^{\mu\nu})_r{}^s \ , \\
A_\alpha(x) \qquad & (\Sigma^{\mu\nu})_\alpha{}^\beta = i(g^\mu{}_\alpha g^{\nu\beta} - g^\nu{}_\alpha g^{\mu\beta}) \ .
\end{aligned} \tag{2.5}
$$

In canonical quantization, the commutation relations of the fields and their conjugate momenta are defined at equal time. Accordingly, the charge for a conserved current j^μ is defined to be

$$
Q \equiv \int d^3x \ j^0(x) \ .
$$

Meanwhile, in light-cone quantization, the commutation relations are defined at equal light-cone time, and thus the conserved charges are

$$
Q' \equiv \int d\xi^- d^2\xi_\perp j^+(\xi) \ . \tag{2.6}
$$

Therefore, in canonical quantization the four-momentum and Lorentz generators are

$$
\begin{aligned}
P^\mu &= \int d^3x \ T^{0\mu}(x) \ , \\
\mathcal{J}^{\mu\nu} &= \int d^3x \ M^{0\mu\nu}(x) \ ,
\end{aligned} \tag{2.7}
$$

and their commutation relations with the field operator generate the infinitesimal Poincaré transformation of the latter,

$$
\begin{aligned}
i[P^\mu, \ \phi_r] &= \partial^\mu \phi_r \ , \\
i[\mathcal{J}^{\mu\nu}, \ \phi_r] &= (x^\mu \partial^\nu - x^\nu \partial^\mu)\phi_r - i(\Sigma^{\mu\nu})_r{}^s \phi_s \ .
\end{aligned} \tag{2.8}
$$

$\mathcal{J}^{\mu\nu}$ is anti-symmetric with $\mathcal{J}^{0i}$ $(i = 1, 2, 3)$ being the boost generator and $\mathcal{J}^{ij} = \epsilon^{ijk} J^k$ the angular momentum operator.

For the QCD Lagrangian

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/)\psi \ , \tag{2.9}$$

Noether's theorem leads to the canonical energy-momentum and angular momentum density tensors,

$$\begin{aligned} T^{\mu\nu}_{\rm can} &= \bar{\psi} i\gamma^\mu \partial^\nu \psi - F^{\mu\alpha}_a \partial^\nu A^a_\alpha - g^{\mu\nu}\mathcal{L}_{\rm QCD} \ , \\ M^{\mu\nu\lambda}_{\rm can} &= i\bar{\psi}\gamma^\mu \left(x^\nu \partial^\lambda - x^\lambda \partial^\nu\right)\psi + \bar{\psi}\gamma^\mu \Sigma^{\nu\lambda}\psi \\ & -F^{\mu\alpha}_a \left(x^\nu \partial^\lambda A^a_\alpha - x^\lambda \partial^\nu A^a_\alpha\right) - (F^{\mu\nu}_a A^\lambda_a - F^{\mu\lambda}_a A^\nu_a) \\ & +(x^\nu g^{\mu\lambda} - x^\lambda g^{\mu\nu})\mathcal{L}_{\rm QCD} \ . \end{aligned} \tag{2.10}$$

As a result, the canonical momentum and angular momentum operators are

$$\begin{aligned} \vec{P}_{\rm can} &= \int d^3x \ \psi^\dagger(-i\vec{\nabla})\psi + \int d^3x \ E^i \vec{\nabla} A^i \ , \\ \vec{J}_{\rm can} &= \int d^3x \ \psi^\dagger \frac{\vec{\Sigma}}{2}\psi + \int d^3x \ \psi^\dagger \left(\vec{x} \times \frac{\vec{\nabla}}{i}\right)\psi \\ & + \int d^3x \ \vec{E} \times \vec{A} + \int d^3x \ E^i \left(\vec{x} \times \vec{\nabla}\right) A^i \ . \end{aligned} \tag{2.11}$$

According to the previous discussion, the Jaffe-Manohar sum rule is based on the canonical form of the QCD angular momentum.

The canonical energy-momentum tensor is generally not symmetric, and each term except the quark spin in Eq. (2.11) is gauge dependent. This can be improved by adding a divergence term to $T^{\mu\nu}_{\rm can}$:

$$T^{\mu\nu}_{\rm Bel} = T^{\mu\nu}_{\rm can} + \partial_\lambda H^{\lambda\mu\nu} \ , \tag{2.12}$$

where the totally anti-symmetric super potential $H^{\lambda\mu\nu}$ is

$$H^{\lambda\mu\nu} = \frac{1}{2}\left[\frac{\partial\mathcal{L}}{(\partial_\mu\phi_r)}(\Sigma^{\nu\lambda})_r{}^s\phi_s + \frac{\partial\mathcal{L}}{(\partial_\lambda\phi_r)}(\Sigma^{\mu\nu})_r{}^s\phi_s - \frac{\partial\mathcal{L}}{(\partial_\nu\phi_r)}(\Sigma^{\lambda\mu})_r{}^s\phi_s\right] . \tag{2.13}$$

$T^{\mu\nu}_{\rm Bel}$ is called the *Belinfante-Rosenfeld* tensor, which is symmetric and manifestly gauge invariant. For QCD,

$$T^{\mu\nu}_{\rm Bel} = \frac{1}{2}[\bar\psi i D^{(\mu}\gamma^{\nu)}\psi + \bar\psi i \overleftarrow{D}^{(\mu}\gamma^{\nu)}\psi] - F_a^{\mu\alpha}F_\alpha^{\nu,a} - g^{\mu\nu}\mathcal{L}_{\rm QCD} \ , \tag{2.14}$$

where $D_\mu = \partial_\mu + igA_\mu$, $\overleftarrow{D}_\mu = -\overleftarrow{\partial}_\mu + igA_\mu$, and $A^{(\mu}B^{\nu)}$ means that the Lorentz indices μ,ν are symmetrized. Accordingly, the *Belinfante-Rosenfeld* angular momentum density tensor is

$$\begin{aligned}
M^{\mu\nu\lambda}_{\rm Bel} &= x^\nu T^{\mu\lambda}_{\rm Bel} - x^\lambda T^{\mu\nu}_{\rm Bel} \\
&= \frac{1}{2}\epsilon^{\mu\nu\lambda\beta}\bar\psi\gamma_\beta\gamma_5\psi + \bar\psi\gamma^\mu(x^\nu i D^\lambda - x^\lambda i D^\nu)\psi \\
&\quad + F_a^{\mu\alpha}(x^\nu F_\alpha^{\lambda,a} - x^\lambda F_\alpha^{\nu,a}) + \partial_\beta S^{[\mu,\beta][\nu,\lambda]} \\
&\quad + (x^\nu g^{\mu\lambda} - x^\lambda g^{\mu\nu})\mathcal{L}_{\rm QCD} \ ,
\end{aligned} \tag{2.15}$$

where $S^{[\mu,\beta][\nu,\lambda]}$ is a super potential with anti-symmetrized indices μ,β and ν,λ.

After the *Belinfante-Rosenfeld* procedure, one obtains

$$\begin{aligned}
\vec{P}_{\rm Bel} &= \int d^3x\ \psi^\dagger(-i\vec\nabla - e\vec{A})\psi + \int d^3x\ \vec{E}\times\vec{B} \ , \\
\vec{J}_{\rm Bel} &= \int d^3x\ \psi^\dagger\frac{\vec\Sigma}{2}\psi + \int d^3x\ \psi^\dagger\vec{x}\times(-i\vec\nabla - e\vec{A})\psi \\
&\quad + \int d^3x\ \vec{x}\times(\vec{E}\times\vec{B}) \ ,
\end{aligned} \tag{2.16}$$

where each term is gauge invariant. $\vec{P}_{\rm Bel}$ and $\vec{J}_{\rm Bel}$ are actually equivalent to $\vec{P}_{\rm can}$ and $\vec{J}_{\rm can}$ as their differences are merely two surface terms that vanish after the integration. The quark momentum and OAM operators in Eq. (2.16) are also called "mechanical" in literature, and $\vec{J}_{\rm Bel}$ is the starting point of Ji's sum rule for the proton spin.

While each term in $\vec{J}_{\rm Bel}$ is gauge invariant, the total gluon angular momentum cannot be further decomposed into local gauge-invariant spin and OAM parts. This has been a standard textbook point of view [30], and it leaves us with a great puzzle: how can the gluon spin be measured from high-energy scattering experiments.

2.2 Gauge-invariant decomposition of the proton spin

The gluon spin puzzle has motivated resurrected attempts to define the gauge-invariant parton spin and OAM in recent years [51–57]. In particular, in 2008, Chen *et al.* reinvented the concept of gauge symmetry by proposing to decompose the gauge potential $\vec{A}$ into the so-called "physical" and "pure" gauge parts [52], which we denote by $\vec{A}_\perp$ and $\vec{A}_\parallel$ respectively,

$$\vec{A} = \vec{A}_\perp + \vec{A}_\parallel \ , \tag{2.17}$$

where $\vec{A}_\perp$ satisfies a generalized Coulomb condition,

$$\partial^i A^i_\perp - ig[A^i, A^i_\perp] = 0 \ , \tag{2.18}$$

and and $\vec{A}_\parallel$ generates null chromo- electric and magnetic field strength,

$$\partial^\mu A^{\nu,a}_\parallel - \partial^\nu A^{\mu,a}_\parallel - gf^{abc}A^{\mu,b}_\parallel A^{\nu,c}_\parallel = 0 \ , \tag{2.19}$$

These conditions were found by Treat in 1973 [58] in an attempt at a gauge-invariant formulation of the quantized Yang-Mills theory [59].

Under a gauge transformation $U(x)$,

$$\begin{aligned}
\vec{A}_\perp &\rightarrow U(x)\vec{A}_\perp U^\dagger(x) \ , \\
\vec{A}_\parallel &\rightarrow U(x)\vec{A}_\parallel U^\dagger(x) + \frac{i}{g}U(x)\vec{\nabla}U^\dagger(x) \ .
\end{aligned} \tag{2.20}$$

In this way, Chen *et al.* use $\vec{A}_\perp$ and $\vec{A}_\parallel$ to construct gauge-invariant spin and OAM operators of quarks and gluons [52],

$$\begin{aligned}
\vec{J} &= \int d^3x \ \psi^\dagger \frac{\vec{\Sigma}}{2}\psi + \int d^3x \ \psi^\dagger \vec{x} \times (-i\vec{\nabla} - e\vec{A}_\parallel)\psi \\
&\quad + \int d^3x \ \vec{E} \times \vec{A}_\perp + \int d^3x \ E^i \vec{x} \times \vec{\nabla} A^i_\perp \ ,
\end{aligned} \tag{2.21}$$

and thus redefine the proton spin sum rule. The difference between the quark OAM in Eq. (2.21) and $\vec{J}_{\text{Bel}}$ is that $\vec{A}$ in the covariant derivative is replaced by $\vec{A}_\parallel$.

Later on Chen *et al.* proposed that the QCD momentum be decomposed in a similar way [53]:

$$\vec{P} = \int d^3x \ \psi^\dagger(-i\vec{\nabla} - e\vec{A}_\parallel)\psi + \int d^3x \ E^i \vec{\mathcal{D}}_\parallel A^i_\perp \ , \tag{2.22}$$

where $\mathcal{D}^\mu_\parallel \equiv \partial^\mu - ig[A^\mu_\parallel,]$ and acts on the adjoint representation. $\vec{\mathcal{D}}_\parallel$ was also used to replace the partial derivative in the gluon OAM in Eq. (2.21) to "improve" the latter [53]. With the redefinition of quark and gluon momenta, Chen *et al.* concluded that the gluons carry about one fifth of the nucleon momentum in the asymptotic limit, which is contradictory to conventional QCD prediction of one half [53].

Following the work by Chen *et al.*, Wakamatsu [8] proposed to generalize the procedure of separating the "pure" and "physical" parts of the potential so that one can impose alternative conditions on the latter and still maintain the gauge symmetry of Chen *et al.*'s decomposition. By requiring the transformation properties of $A^\mu_\perp$ and $A^\mu_\parallel$ in Eq. (2.20), as well as the covariant version of Eq. (2.19), Wakamatsu decomposed the angular momentum tensor into gauge invariant parts and claimed that this procedure is Lorentz covariant or frame independent. Since $A^\mu_\perp$ and $A^\mu_\parallel$ are not completely fixed in his approach, one can recover Chen *et al.*'s result by imposing the generalized Coulomb condition in Eq. (2.18), or the Bashinsky-Jaffe decomposition with the light-cone condition [51]. Besides, Wakamatsu pointed out that there are two distinct decompositions depending on whether one attributes the potential angular momentum—which is $\psi^\dagger \vec{x} \times \vec{A}_\perp \psi$ in this case— to the quark or gluon OAM.

A deeper discussion of the gauge-invariance of the above proposals is available in Appendix B.

2.3 Canonical or mechanical orbital angular momentum?

As has been thoroughly discussed in a recent review [60], all these different proposals, including the Jaffe-Manohar and Ji's sum rules, can be classified into two categories. If one fixes the generalized Coulomb gauge condition, then Chen *et al.*'s decomposition will reduce to the canonical form; so does the Bashinsky-Jaffe decomposition in the light-cone gauge. On the other hand, if the potential angular momentum is attributed to the quark OAM, then the decomposition will be similar to Ji's form except that the gluon angular momentum is further separated into its spin and OAM.

Since the potential angular momentum itself is gauge invariant, one needs further reason to decide whether the canonical or mechanical form is a more physical operator description of the proton spin. As the OAM is naturally linked to momentum through its classical definition $\vec{x} \times \vec{P}$, such discrepancy also leads to the debate on whether the canonical momentum should be chosen as a physical observable over the mechanical one [53].

Supporters for the canonical form of momentum and angular momentum argue that their matrix elements are gauge invariant despite the fact that these operators are not [61,62]. However, a general proof in the path integral formalism showed that the gluon spin in the free-field form has different matrix elements in the light-cone and covariant gauges [63]. A recent one-loop calculation in the Coulomb gauge also invalidates this argument [64]. Therefore, it is not likely that the canonical operators are the real physical observables in a general sense. To understand this, let us give a simple proof in non-relativistic quantum mechanics [24].

In a non-relativistic quantum theory with external electromagnetic fields, the Hamiltonian is

$$H = \frac{(\vec{P} - e\vec{A})^2}{2m} + e\phi \ , \tag{2.23}$$

where $\vec{P}$ is the canonical momentum and we have customarily called $\phi \equiv A^0$. It has the eigenvalue system

$$H\psi_n(\vec{r}) = E_n\psi_n(\vec{r}) \ , \tag{2.24}$$

with energy eigenvalues E_n and eigen wave functions $\psi_n(\vec{r})$.

Under a time-independent gauge transformation,

$$A^\mu(\vec{r}) \to A^{\mu\prime}(\vec{r}) = A^\mu(\vec{r}) + \partial^\mu\chi(\vec{r}) \ , \tag{2.25}$$

we obtain a new Hamiltonian,

$$H' = \frac{(\vec{P} - e\vec{A}')^2}{2m} + e\phi \ , \tag{2.26}$$

which is manifestly different. However, since energy is a physical observable, it should remain the same under a gauge transformation, thus

$$H'\psi'_n(\vec{r}) = E_n\psi'_n(\vec{r}) \ , \tag{2.27}$$

where $\psi'_n = e^{ie\chi(\vec{r})}\psi_n(\vec{r})$ is the new eigen wave function. As a result, while the charged particle probability density

$$\rho_n(\vec{r}) = \psi^*_n(\vec{r})\psi_n(\vec{r}) \tag{2.28}$$

is gauge invariant, the expectation value of the canonical momentum $\vec{P}$ is not:

$$\left\langle \psi'_m \left| \vec{P} \right| \psi'_n \right\rangle = \left\langle \psi_m \left| \vec{P} \right| \psi_n \right\rangle + e \left\langle \psi_m \left| \nabla\chi(\vec{r}) \right| \psi_n \right\rangle \ . \tag{2.29}$$

Actually, in Ref. [62] it was claimed that the gauge transformation leaves the physical states invariant, but this is not true according to Eq. (2.27): the wave function is a not a gauge-independent quantity. It is the matrix element of the mechanical momentum $\vec{p} = \vec{P} - e\vec{A}$ that is gauge invariant:

$$\left\langle \psi'_m \left| \vec{P} - e\vec{A}' \right| \psi'_n \right\rangle = \left\langle \psi_m \left| \vec{P} - e\vec{A} \right| \psi_n \right\rangle \ , \tag{2.30}$$

which corresponds to the covariant derivative in quantization, $\vec{p} = \vec{P} - e\vec{A} \equiv -i\vec{D}$.

There is a simple example from Feynman's lectures on physics that demonstrates why the covariant derivative corresponds to the observed momentum for a charged particle [65]: Consider a charged particle near a solenoid with wave function $\psi(\vec{r}, t)$. The solenoid does not have any current in the beginning. At some point in time, a current passes through the solenoid and a stable magnetic field is established. During the process, the particle gets a momentum kick because the changing magnetic field induces an electric field which exerts a force on the charged particle. From the Schrödinger equation, the wave function must be continuous in time. Therefore, the momentum kick on the particle cannot be obtained from the partial derivative acting on the wave function, which must be also continuous in time. Instead, it comes from the establishment of the vector potential $\vec{A}$ in the system.

Nevertheless, the distinction between canonical and mechanical momentum is camouflaged in parton physics. As has been explained in the previous chapter, the simple parton picture emerges in the IMF of the proton, so parton physics is formulated in the IMF, or equivalently on the light-cone plane.

The gauge-invariant longitudinal quark distribution from QCD factorization

$$q(x) = \frac{1}{2P^+} \int \frac{d\xi^-}{2\pi} e^{ixP^+\xi^-} \langle PS|\overline{\Psi}(\xi^-, 0_\perp)\gamma^+\Psi(0, 0_\perp)|PS\rangle \ , \tag{2.31}$$

measures the probability to find a quark parton with momentum $k^+ = xP^+$. Here $\Psi(\xi)$ is a gauge-invariant quark field defined through multiplication of a light-cone gauge link,

$$\Psi(\xi) = \exp\left(-ig \int_0^\infty A^+(\xi + \eta^-) d\eta^-\right) \psi(\xi) \ . \tag{2.32}$$

The gauge link ensures that whenever a partial derivative (canonical momentum) of colored quarks appear, the gauge potential A^μ must be present simultaneously to make it a covariant derivative (mechanical momentum), $D^\mu = \partial^\mu + igA^\mu$. Indeed, taking the moments of $q(x)$, one gets

$$\int x^{n-1}q(x)dx \quad \sim \quad \langle P|\psi^\dagger(0)\overbrace{iD^+...iD^+}^{n-1}\psi(0)|P\rangle \ . \tag{2.33}$$

Especially, the average quark momentum

$$\langle x\rangle \sim \langle P|\psi^\dagger\ i(\partial^+ + igA^+)\psi|P\rangle \ . \tag{2.34}$$

We can see that the parton momentum distribution refers to the gauge-invariant mechanical momentum! The mechanical momentum structure is clearly seen through Feynman diagrams in Fig. 2.1: Gauge symmetry requires that a parton with mechanical momentum $k^+ = xP^+$ includes the sum of all diagrams with towers of longitudinal gluon A^+ insertions.

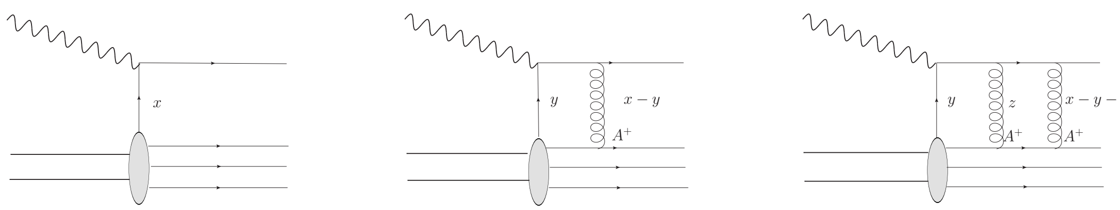

Figure 2.1: DIS process in which the gauge invariance involving the longitudinal quark mechanical momentum xP^+ is achieved through insertions of gluons with longitudinal polarization A^+.

The inclusion of longitudinal gluons guarantees the gauge-invariance of QCD factorization, but the simple parton picture is still not clear as the quark is always accompanied by an infinite number of unphysical gluons. Only in the light-cone gauge $A^+ = 0$, the gauge link disappears and all the covariant derivatives in Eq. (2.33) become partial ones, i.e., the canonical momentum, and the simple quark field becomes physical. Therefore, the parton picture is both frame (the IMF) and gauge (the light-cone gauge) dependent, and in this picture the distinction between canonical and mechanical momentum is overlooked.

However, such statement cannot be applied to the longitudinal OAM of the quarks and gluons. The reason is simple: the longitudinal component of the mechanical angular momentum involves the transverse components of the gluon field that does not vanish in the light-cone gauge,

$$L_q^z = \int d^3x\ \psi^\dagger \left[x^1(i\partial^2 - gA^2) - x^2(i\partial^1 - gA^1)\right] \psi\ . \tag{2.35}$$

Therefore, the mechanical OAM does not have a simple partonic interpretation even in the light-cone gauge. If one prefers to use the parton picture as a standard for defining the physical spin sum rule, then the canonical OAM should be prior to the mechanical one.

2.4 Frame-dependence and the infinite momentum frame limit

It is well known that the Coulomb gauge eliminates all the unphysical degrees of freedom in QED, so Chen *et al.*'s decomposition appears to bear much physical significance when it was first proposed. However, it was soon criticized that angular momentum operators in Eq. (2.21)—except for the quark spin—are nonlocal and frame dependent, which does not satisfy the requirement for physical observables [66, 67]. Nevertheless, this problem turns out to be the crucial point for us to disentangle the intricacy and unravel the physical meaning of parton observables. In particular, we will focus on the gluon spin operator and show how it acquires physical significance when boosted to the IMF.

Before we discuss the frame-dependence of angular momentum operators in Eq. (2.21), let us review the representations of the Lorentz group. For a massless particle, the representation of the homogeneous Lorentz group can be induced from the representation of its little group $ISO(2)$, which is different from that for a massive particle ($SO(3)$) [68]. The $ISO(2)$ group consists of translations and rotations in two dimensions, and for photons they correspond to the gauge symmetry and helicity respectively. Therefore, with the elimination of the redundant degrees of freedom, the free photon state is distinguished by the eigen value of the helicity operator, which is invariant under any Lorentz transformation. In other words, the free photon state does not form an irreducible representation of the $SO(3)$ group, so spin (s^2 or s^z) is not a good quantum number for the photon. Only when the z axis is chosen along the direction of propagation of the photon, s^z coincides with the helicity and thus it can be regarded as a physical observable. Therefore, our discussion in this paper is limited to the longitudinal gluon spin.

In the bound state proton, the gluons are off-shell and have unphysical longitudinal degrees of freedom. Although one can define gauge-invariant gluon spin operators, it is not clear whether they carry the physical meaning of spin or helicity. In addition, for nonlocal operators such $\vec{E} \times \vec{A}_\perp$ in Chen *et al.*'s proposal, their transformation under a Lorentz boost is nontrivial and strongly related to the dynamics. To understand this, we can first look at the example of QED.

In QED, $\vec{E} \times \vec{A}_\perp$ is equivalent to $\vec{E} \times \vec{A}$ in the Coulomb gauge

$$\vec{\nabla} \cdot \vec{A} = 0 \ . \tag{2.36}$$

Before we impose the above condition, we find that under a Lorentz boost Λ along the z direction,

$$\vec{\nabla} \cdot \vec{A}(x) = \partial_i A^i(x) = \partial_i [\Lambda^i{}_\mu A'^\mu(\Lambda^{-1}x)] = (\Lambda^{-1})^\nu{}_i \Lambda^i{}_\mu \partial'_\nu A'^\mu(x') \ , \tag{2.37}$$

where $x' = \Lambda^{-1}x$. Eq. (2.37) shows that if $\vec{A}$ satisfies the Coulomb condition in the original frame, then $\vec{A}'$ will not satisfy the same condition in the new frame. In other words, the Coulomb condition is not a frame-independent condition.

In Chen *et al.*'s proposal, $\vec{A}_\perp$ and $\vec{A}_\parallel$ are subject to the conditions,

$$\vec{\nabla} \cdot \vec{A}_\perp = 0 \ , \qquad \vec{\nabla} \times \vec{A}_\parallel = 0 \ . \tag{2.38}$$

With the boundary conditions that $\vec{A}_\perp$ and $\vec{A}_\parallel$ decrease faster than $1/|\vec{x}|$ when $|\vec{x}| \to \infty$, the solution is unique, i.e., the Helmholtz decomposition,

$$\begin{aligned} A^i_\perp(x) &= A^i(x) - \nabla^i \frac{1}{\nabla^2} \vec{\nabla} \cdot \vec{A} = A^i(x) + \nabla^i_x \int d^3y \frac{\vec{\nabla}_y \cdot \vec{A}(y)}{4\pi|\vec{x} - \vec{y}|} \ , \\ A^i_\parallel(x) &= \nabla^i \frac{1}{\nabla^2} \vec{\nabla} \cdot \vec{A} = -\nabla^i_x \int d^3y \frac{\vec{\nabla}_y \cdot \vec{A}(y)}{4\pi|\vec{x} - \vec{y}|} \ . \end{aligned} \tag{2.39}$$

According to Eq. (2.37), under the Lorentz boost Λ, $\vec{A}_\perp$ will not remain as the solution to Eq. (2.38) in the new reference frame. Instead, for an arbitrary four vector V^μ,

$$\begin{aligned} V^0 &= \frac{1}{\sqrt{2}}(V^+ + V^-) = \frac{1}{\sqrt{2}}\left(\lambda V'^+ + \lambda^{-1} V'^-\right) \ , \\ V^3 &= \frac{1}{\sqrt{2}}(V^+ - V^-) = \frac{1}{\sqrt{2}}\left(\lambda V'^+ - \lambda^{-1} V'^-\right) \ , \end{aligned} \tag{2.40}$$

where $\lambda > 1$ is a boost factor. In this way, for $i = 1, 2$,

$$
\begin{aligned}
A^i_\perp(x) &= A'^i(\Lambda^{-1}x) - \partial'^i \frac{1}{\partial'^2_\perp + \frac{1}{2}\left[\lambda^2(\partial'^+)^2 - 2\partial'^-\partial'^+ + \lambda^{-2}(\partial'^-)^2\right]} \\
&\quad \times \left(\vec{\nabla}_\perp \cdot \vec{A}'_\perp + \frac{1}{2}[\lambda^2\partial'^+ A'^+ - \partial'^- A'^+ - \partial'^+ A'^- + \lambda^{-2}\partial'^- A'^-]\right) \\
&\overset{\lambda\to\infty}{=} A'^i(x') - \partial'^i \frac{1}{(\partial'^+)^2}\partial'^+ A'^+ \ . \qquad (2.41)
\end{aligned}
$$

If x were originally fixed on the equal-time plane, then under the infinite boost x' will be on the light-cone plane.

Recall the gauge-invariant gluon spin defined as a GIE of the light-cone gauge,

$$
S_g^{\rm inv} = \int dx \frac{i}{x} \int \frac{d\xi^-}{2\pi} e^{-ixP^+\xi^-} \ F_a^{+\alpha}(\xi^-, 0_\perp)\mathcal{L}^{ab}(\xi^-, 0)\tilde{F}^+_{\alpha,b}(0, 0_\perp) \ .
$$

In QED, there is no need of the gauge link, so the gauge-invariant photon spin [64]

$$
\begin{aligned}
\hat{S}_\gamma^{\rm inv}(0) &= i \int\frac{dx}{x}\int\frac{d^2k_\perp}{(2\pi)^3}\int d\xi^- d^2\xi_\perp e^{-i(xP^+\xi^- - \vec{k}_\perp\cdot\vec{\xi}_\perp)} \left[ixP^+ A^i(\xi) - ik^i_\perp A^+(\xi)\right] \tilde{F}_i{}^+(0) \\
&= -\int \frac{dk^+ d^2k_\perp}{(2\pi)^3}\left[\tilde{A}^i(k) - \frac{k^i_\perp}{k^+}\tilde{A}^+(k)\right]\tilde{F}_i{}^+(0) \\
&= \left[\vec{E}(0) \times \left(\vec{A}(0) - \vec{\partial}\frac{1}{\partial^+}A^+(\xi^-, 0_\perp)\right)\right]^3 , \qquad (2.42)
\end{aligned}
$$

where $\partial^+ = \partial/\partial\xi^-$, $E^i = F^{i+}$ with $i = 1, 2$, and $k^+ = xP^+$. The first equality is obtained with integration by parts, and the ξ^- coordinate on A^+ in the last line is taken to 0 after operation of the inverse derivative, which is understood with a Fourier transform

$$
\begin{aligned}
\frac{1}{\partial^+} f(\xi'^-) &= \int dk^+ \frac{1}{ik^+} e^{ik^+\xi^-} \int \frac{d\xi'^-}{2\pi} e^{-ik^+\xi'^-} f(\xi'^-) \\
&= \frac{1}{2}\int d\xi'^- {\rm sgn}(\xi^- - \xi'^-) f(\xi'^-) \ . \qquad (2.43)
\end{aligned}
$$

However, for a general first-order differential equation

$$
\partial^+ \mathcal{F}(\xi^-) = f(\xi^-) \ ,
$$

there is no unique solution if one does not impose a boundary condition. In other words, to have well-defined Green's function $1/\partial^+$, one must require that $\mathcal{F}(\pm\infty) = 0$. Unfortunately, this condition cannot always be satisfied in the light-cone coordinates, so we make a little change by defining

$$\frac{1}{\partial^+} f(\xi'^-) = \frac{1}{2}\left[\int_{-\infty}^{\xi^-} d\xi'^- - \int_{\xi^-}^{\infty} d\xi'^-\right] f(\xi'^-) + \frac{1}{2}\left[\mathcal{F}(\infty) + \mathcal{F}(-\infty)\right] . \qquad (2.44)$$

In this way, the operation of $1/\partial^+$ is independent of the boundary conditions.

Since $E^i = F^{i0}$ will be reduced to F^{i+} under the infinite boost, by comparison we can easily find that $\hat{S}_\gamma^{\rm inv}$ is just the IMF limit of $\left(\vec{E} \times \vec{A}_\perp\right)^3$!

The case for QCD is a bit more complicated. So far no exact solutions to Eqs. (2.18), (2.19) have been obtained, so it is not possible for us to directly study the IMF limit of $A_\perp^\mu$. Instead, we can first take the IMF limit of these equations, and then seek the solutions. Since $A_\perp^\mu = A^\mu - A_\parallel^\mu$, we just need to solve for $A_\parallel^\mu$ first.

Recall Eq. (2.19) of $A_\parallel^\mu$, and now let us choose $\mu = +$ and $\nu = i$ $(i = 1, 2)$,

$$\partial^+ A_\parallel^{i,a} - \partial^i A_\parallel^{+,a} - g f^{abc} A_\parallel^{+,b} A_\parallel^{i,c} = 0 . \qquad (2.45)$$

We first want to show that $A_\parallel^+ = A^+$ so that this nonlinear equation can be reduced to a linear one. Our strategy is to rewrite Eq. (2.19) in terms of $A_\perp^\mu$,

$$\partial^\mu A_\perp^\nu - \partial^\nu A_\perp^\mu = \partial^\mu A^\nu - \partial^\nu A^\mu + ig[A^\mu - A_\perp^\mu, A^\nu - A_\perp^\nu] ,$$

First we choose $\mu = i$ and hit both sides by ∂^i, then the sum over i gives

$$\nabla^2 A_\perp^\nu - \partial^\nu \partial^i A_\perp^i = \nabla^2 A^\nu - \partial^\nu \partial^i A^i + ig\partial^i[A^i - A_\perp^i, A^\nu - A_\perp^\nu] . \qquad (2.46)$$

Substituting Eq. (2.18) into the above equation, we have

$$\nabla^2 A_\perp^\nu = \nabla^2 A^\nu - \partial^\nu \partial^i A^i + ig\partial^i [A^i - A_\perp^i, A^\nu - A_\perp^\nu] - ig\partial^\nu [A^i, A_\perp^i] \ . \tag{2.47}$$

With a proper boundary condition, the solution at leading order of g is

$$A_{\perp,\nu}^{(0)} = A_\nu - \frac{1}{\nabla^2}\partial_\nu(\nabla\cdot\vec{A}) \ , \tag{2.48}$$

or alternatively in momentum space

$$\tilde{A}_{\perp,\nu}^{(0)} = \tilde{A}_\nu - \frac{k_\nu(\vec{k}\cdot\tilde{\vec{A}})}{\vec{k}^2}. \tag{2.49}$$

At the next-to-leading order, $A_{\perp,\nu}^{(1)}$ satisfies

$$\begin{aligned}\nabla^2 A_{\perp,\nu}^{(1)} &= ig\partial_i[A_i - A_{\perp,i}^{(0)}, A_\nu - A_{\perp,\nu}^{(0)}] - ig\partial_\nu[A_i, A_{\perp,i}^{(0)}] \\ &= ig\partial_i[A_i, A_\nu] - ig\partial_i[A_i, A_{\perp,\nu}^{(0)}] - ig\partial_i[A_{\perp,i}^{(0)}, A_\nu] \\ &\quad -ig\partial_\nu[A_i, A_{\perp,i}^{(0)}] + ig\partial_i[A_{\perp,i}^{(0)}, A_{\perp,\nu}^{(0)}]\end{aligned} \tag{2.50}$$

The solution is given by

$$A_{\perp,\nu}^{(1)} = \frac{ig}{\nabla^2}\left(\partial_i[A_i - A_{\perp,i}^{(0)}, A_\nu - A_{\perp,\nu}^{(0)}] - \partial_\nu[A_i, A_{\perp,i}^{(0)}]\right) \ . \tag{2.51}$$

At higher orders, $A_{\perp,\nu}^{(n)}(n \geq 2)$ satisfies

$$\begin{aligned}\nabla^2 A_{\perp,\nu}^{(n)} &= -ig\partial_i[A_i, A_{\perp,\nu}^{(n-1)}] - ig\partial_i[A_{\perp,i}^{(n-1)}, A_\nu] - ig\partial_\nu[A_i, A_{\perp,i}^{(n-1)}] \\ &\quad +ig\sum_{m=0}^{n-1}\partial_i[A_{\perp,i}^{(m)}, A_{\perp,\nu}^{(n-1-m)}] \ ,\end{aligned} \tag{2.52}$$

so we obtain a recursive solution for higher-order terms (with $n \geq 2$):

$$\begin{aligned}A_{\perp,\nu}^{(n)} &= \frac{ig}{\nabla^2}\Big(-\partial_i[A_i, A_{\perp,\nu}^{(n-1)}] - \partial_i[A_{\perp,i}^{(n-1)}, A_\nu] - \partial_\nu[A_i, A_{\perp,i}^{(n-1)}] \\ &\quad + \sum_{m=0}^{n-1}\partial_i[A_{\perp,i}^{(m)}, A_{\perp,\nu}^{(n-1-m)}]\Big) \ ,\end{aligned} \tag{2.53}$$

Now let us take $\nu = +$ and boost the system to the IMF,

$$A_\perp^{(0),+} \to A^+ - \partial^+ \frac{1}{(\partial^+)^2}(\partial^+ A^+) = 0 \ . \tag{2.54}$$

Plugging this into Eq. (2.51), we find that in the IMF limit,

$$\begin{aligned} A_\perp^{(1),+} &\to \frac{ig}{\nabla^2}\left(\partial^i[A^i - A_\perp^{i,(0)}, A^+] - \partial^+[A^i, A_\perp^{i,(0)}]\right) \\ &\to \frac{ig}{(\partial^+)^2}\left(\partial^+[A^+ - A_\perp^{+,(0)}, A^+] - \partial^+[A^+, A_\perp^{+,(0)}]\right) \\ &= 0 \ . \end{aligned} \tag{2.55}$$

Following the same procedure, we have for $n \geq 2$,

$$A_\perp^{(n),+} \to 0 \tag{2.56}$$

under an infinite Lorentz boost.

Therefore, we prove that $A_\perp^+ = 0$ in the IMF limit perturbatively. Substituting this into Eq. (2.19), we obtain a first-order inhomogeneous linear equation for $A_\parallel^i$:

$$\partial^+ A_\parallel^{i,a} - g f^{abc} A^{+,b} A_\parallel^{i,c} \ = \ \partial^i A^{+,a} \ . \tag{2.57}$$

Its solution is easy to construct as a geometric series expansion [64]:

$$A_\parallel^{i,a} = \frac{1}{\partial^+}\left[1 + \left(-ig\mathcal{A}^+ \frac{1}{\partial^+}\right) + ... + \left(-ig\mathcal{A}^+ \frac{1}{\partial^+}\right)^n + ...\right]^{ab} (\partial^i A^{+,b}) \ . \tag{2.58}$$

The last two factors of the n-th order term can be explicitly expressed as

$$
\begin{aligned}
I_n &= \frac{1}{\partial^+_{n-1}}\left(-ig\mathcal{A}^+\frac{1}{\partial^+_n}\right)^{b_n b}\partial^i A^{+,b}(\xi^-_{n+1}) \\
&= -ig\frac{1}{2}\left[\int_{-\infty}^{\xi^-_{n-1}} d\xi^-_n - \int_{\xi^-_{n-1}}^{\infty} d\xi^-_n\right]\mathcal{A}^+_{b_n b}(\xi^-_n) \\
&\quad \times\frac{1}{2}\left[\int_{-\infty}^{\xi^-_n} d\xi^-_{n+1} - \int_{\xi^-_n}^{\infty} d\xi^-_{n+1}\right]\partial^i A^{+,b}(\xi^-_{n+1}) + \text{boundary term} \\
&= -ig\frac{1}{2}\left[\int_{-\infty}^{\xi^-_{n-1}} d\xi^-_{n+1} - \int_{\xi^-_{n-1}}^{\infty} d\xi^-_{n+1}\right]\partial^i A^{+,b}(\xi^-_{n+1})\int_{\xi^-_{n-1}}^{\xi^-_{n+1}} d\xi^-_n\,\mathcal{A}^+_{b_n b}(\xi^-_n) \\
&\quad -\int_{-\infty}^{\infty} d\xi^-_n\int_{-\infty}^{\infty} d\xi^-_{n+1}\left(-ig\mathcal{A}^+(\xi^-_n)\right)^{b_n b}A^{+,b}(\xi^-_{n+1}) + \text{boundary term} \\
&= \frac{1}{\partial^+_{n-1}}\partial^i A^{+,b}(\xi^-_{n+1})\int_{\xi^-_{n+1}}^{\xi^-_{n-1}} d\xi^-_n\left(-ig\mathcal{A}^+(\xi^-_n)\right)^{b_n b}\ ,
\end{aligned}
\tag{2.59}
$$

where the second term in the last but second line is a constant, so it can be absorbed into the boundary term as a redefinition of $1/\partial^+_{n-1}$. In this way, the n-th term in Eq. (2.58) is

$$
\begin{aligned}
A_\parallel^{(n),i,a}(\xi^-) &= \frac{1}{\partial^+}\left(-ig\mathcal{A}^+\frac{1}{\partial^+}\right)^n_{ab}(\partial^i A^{+,b}) \\
&= \frac{1}{\partial^+}\partial^i A^{+,b}(\xi^-_{n+1})\int_{\xi^-_{n+1}}^{\xi^-} d\xi^-_1\int_{\xi^-_{n+1}}^{\xi^-_1} d\xi^-_2\cdots\int_{\xi^-_{n+1}}^{\xi^-_{n-1}} d\xi^-_n \\
&\quad \times\left(-ig\mathcal{A}^+(\xi^-_1)\right)^{ab_1}\left(-ig\mathcal{A}^+(\xi^-_2)\right)^{b_1 b_2}\cdots\left(-ig\mathcal{A}^+(\xi^-_n)\right)^{b_n b}\ ,
\end{aligned}
\tag{2.60}
$$

where $1/\partial^+$ brings the coordinate in $\partial^i A^{+,b}(\xi^-_{n+1})$ to ξ^-. Let us denote $\xi'^- = \xi^-_{n+1}$, then we have

$$
\begin{aligned}
A_{\parallel}^{i,a}(\xi^-) &= \frac{1}{\partial^+}\partial^i A^{+,b}(\xi'^-)\Bigg[\delta^{ab}+\sum_{n=1}^{\infty}\int_{\xi'^-}^{\xi^-} d\xi_1^- \int_{\xi'^-}^{\xi_1^-} d\xi_2^- \cdots \int_{\xi'^-}^{\xi_{n-1}^-} d\xi_n^- \\
&\quad \times \left(-ig\mathcal{A}^+(\xi_1^-)\right)^{ab_1}\left(-ig\mathcal{A}^+(\xi_2^-)\right)^{b_1b_2}\cdots\left(-ig\mathcal{A}^+(\xi_n^-)\right)^{b_nb}\Bigg] \\
&= \frac{1}{\partial^+}\partial^i A^{+,b}(\xi'^-)\ \mathcal{P}\mathrm{exp}\left[-ig\int_{\xi'^-}^{\xi^-} d\eta^-\mathcal{A}^+(\eta^-)\right]^{ab} \\
&= \frac{1}{\partial^+}\partial^i A^{+,b}(\xi'^-)\ \mathcal{P}\mathrm{exp}\left[-ig\int_{\xi^-}^{\xi'^-} d\eta^-\mathcal{A}^+(\eta^-)\right]^{ba}, \\
&= \frac{1}{\partial^+}\Big(\partial^i A^{+,b}(\xi'^-)\mathcal{L}^{ba}(\xi'^-,\xi^-)\Big)\ ,
\end{aligned}
\tag{2.61}
$$

where we have used the unitarity of Wilson lines,

$$
\mathcal{L}^{\dagger}(x,y)=\mathcal{L}^{-1}(x,y)=\mathcal{L}(y,x)\ . \tag{2.62}
$$

An alternative way to solve Eq. (2.57) is to multiply both sides by a gauge link $\mathcal{L}(\xi^-,-\infty)$. After some manipulations [64], one finds that

$$
\partial^+\left(A_{\parallel}^{i,a}\mathcal{L}^{ad}(\xi^-,-\infty)\right)=(\partial^i A^{+,a})\mathcal{L}^{ad}(\xi^-,\infty)\ . \tag{2.63}
$$

Then the solution to $A_{\parallel}^{i,a}$ is formally given by

$$
\begin{aligned}
A_{\parallel}^{i,a}(\xi^-) &= \left[\frac{1}{\partial^+}\Big(\partial^i A^{+,b}(\xi'^-)\mathcal{L}^{bd}(\xi'^-,-\infty)\Big)\right](\mathcal{L}^{-1})^{da}(\xi^-,-\infty) \\
&= \frac{1}{\partial^+}\Big(\partial^i A^{+,b}(\xi'^-)\mathcal{L}^{ba}(\xi'^-,\xi^-)\Big)\ ,
\end{aligned}
\tag{2.64}
$$

where $\mathcal{L}^{-1}$ is absorbed into the square bracket because it does not depend on the integration variable ξ'^-. Therefore,

$$
A_{\perp}^{i,a}(\xi^-)=A^{i,a}(\xi^-)-\frac{1}{\partial^+}\Big(\partial^i A^{+,b}(\xi'^-)\mathcal{L}^{ba}(\xi'^-,\xi^-)\Big)\ , \tag{2.65}
$$

which is equivalent to the result derived in Ref. [56].

Now recall the gauge-invariant gluon spin operator in Eq. (1.21). Since

$$F_a^{+i} = \partial^+ A^{i,a} - \partial^i A^{+,a} - g f^{abc} A^{+,b} A^{i,c} \ , \tag{2.66}$$

after integration by parts, we have

$$\begin{aligned}
S_g^{\rm inv} &= i \int \frac{dx}{x} \int \frac{d^2 k_\perp}{(2\pi)^3} \int d\xi^- d^2\xi_\perp e^{-i(xP^+\xi^- - \vec{k}_\perp \cdot \vec{\xi}_\perp)} \\
& \times \left[-(-ixP^+) A^{i,a}(\xi) - i k_\perp^i A^{+,a}(\xi) - g f^{ade} A^{+,d}(\xi) A^{\alpha,e}(\xi) \right] \\
& \times \mathcal{L}(\xi^-, 0_\perp; 0, 0_\perp)^{ab} \tilde{F}_i^{+,b}(0, 0_\perp) \\
& + i \int \frac{dx}{x} \int \frac{d^2 k_\perp}{(2\pi)^3} \int d\xi^- d^2\xi_\perp e^{-i(xP^+\xi^- - \vec{k}_\perp \cdot \vec{\xi}_\perp)} \\
& \times \left(ig A^{i,a}(\xi) A^{+,c}(\xi^-, 0_\perp) [T^c]^{ad} \right) \mathcal{L}(\xi^-, 0_\perp; 0, 0_\perp)^{db} \tilde{F}_i^{+b}(0, 0_\perp) \ ,
\end{aligned} \tag{2.67}$$

where one sums over $i = 1, 2$, and the second term comes from the derivative of the gauge link $\mathcal{L}$. Since the integration over $k_\perp$ will give us $\delta(\xi_\perp)$, it doesn't make a difference to change the coordinate of $A^{+,c}(\xi^-, 0_\perp)$ to $A^{+,c}(\xi)$. Moreover, after exchanging the dummy color indices,

$$\left(ig A^{i,a}(\xi) A^{+,c}(\xi^-, 0_\perp) [T^c]^{ad} \right) \mathcal{L}(\xi^-, 0_\perp; 0, 0_\perp)^{db} = g f^{ade} A^{+,d}(\xi) A^{\alpha,e}(\xi) \mathcal{L}(\xi^-, 0_\perp; 0, 0_\perp)^{ab} \ . \tag{2.68}$$

Therefore, Eq. (2.67) can be simplified as

$$\begin{aligned}
S_g^{\rm inv} &= i \int \frac{dx}{x} \int \frac{d^2 k_\perp}{(2\pi)^3} \int d\xi^- d^2\xi_\perp e^{-i(xP^+\xi^- - \vec{k}_\perp \cdot \vec{\xi}_\perp)} \\
& \times \left[-(-ixP^+) A^{i,a}(\xi) - i k_\perp^i A^{+,a}(\xi) \right] \ \mathcal{L}(\xi^-, 0_\perp; 0, 0_\perp)^{ab} \tilde{F}_i^{+,b}(0, 0_\perp) \\
&= - \int \frac{dk^+}{2\pi} \int d\xi^- e^{-ixP^+\xi^-} \left[A^{i,a}(\xi^-, 0_\perp) - \frac{1}{ik^+} \partial^i A^{+,a}(\xi^-, 0_\perp) \right] \\
& \times \mathcal{L}(\xi^-, 0_\perp; 0, 0_\perp)^{ab} \tilde{F}_i^{+,b}(0, 0_\perp) \\
&= \left[\vec{E}^a(0) \times \left(\vec{A}^a(0) - \frac{1}{\partial^+} (\vec{\partial} A^{+,b}) \mathcal{L}^{ba}(\xi^-, 0) \right) \right]^3 \ ,
\end{aligned} \tag{2.69}$$

which is exactly the IMF limit of $(\vec{E}\times\vec{A}_\perp)^3$ in Chen *et al.*'s proposal [64].

If one starts from Wakamatsu's proposal and fixes $A_\perp^\mu$ with the axial gauge condition $A^z = 0$, then $A_\parallel^\mu$ will be a nonlocal operator that involves a gauge-link along the z axis. We can easily prove that the IMF limit of $(\vec{E}\times\vec{A}_\perp)^3$ in this proposal is still the gluon spin defined in Eq. (1.21). Apart from the gluon spin operator, we can also prove that the OAM operators in Chen *et al.*'s proposal have the IMF limit as light-cone correlation operators—although we do not know the counterparts to them in parton physics—and reduce to the free-field form in the light-cone gauge. This reflects why the IMF and light-cone gauge are natural for parton physics.

2.5 Weizsäcker-Williams approximation

From the previous section, one learns that although the physical meaning of gluon spin—or more rigorously, helicity—is ambiguous when the proton is at rest, it becomes clear in the IMF and light-cone gauge. It is on the basis that the gluons can be approximated as free massless particles in the IMF limit, i.e., when the parton picture emerges. Such approximation is actually known as the Weizsäcker-Williams' equivalent photon method in electrodynamics [69].

It is known that in electromagnetic theory the vector potential can be uniquely separated into the longitudinal and transverse parts, $\vec{A}=\vec{A}_\parallel+\vec{A}_\perp$, and the transverse part is gauge invariant [30, 70–72]: Given $\vec{A}$, $\vec{A}_\perp$ can be uniquely constructed as a functional of $\vec{A}$ with an appropriate boundary condition. Thus $\vec{E}\times\vec{A}_\perp$ is can be regarded as the gauge-invariant part of the gauge particle spin [52].

However, it is also important to realize that separating $\vec{A}$ and $\vec{E}$ into longitudinal and transverse parts is in general not physically meaningful. In the first place, the physics of $\vec{E}$ is to apply force to electric charge and there is no charge that responds separately to $\vec{E}_\parallel$ and $\vec{E}_\perp$ [64]. Second, in a different frame, one sees different transverse and longitudinal separations, and therefore the notion has no Lorentz covariance [66, 67]. As we shall see later, the frame-dependence of both parts is dynamical, and cannot be calculated without solving the theory. To explain this point, let us consider the example of a point charge.

For a static charge, the electric field is purely longitudinal,

$$\vec{E} = \vec{E}_\parallel = \vec{\nabla} A^0 \ , \qquad \vec{\nabla} \times \vec{E}_\parallel = 0 \ .$$

As the charge moves with velocity β, the field lines start to contract in the transverse direction due to special relativity, which is shown in Fig. 2.2. The moving charge

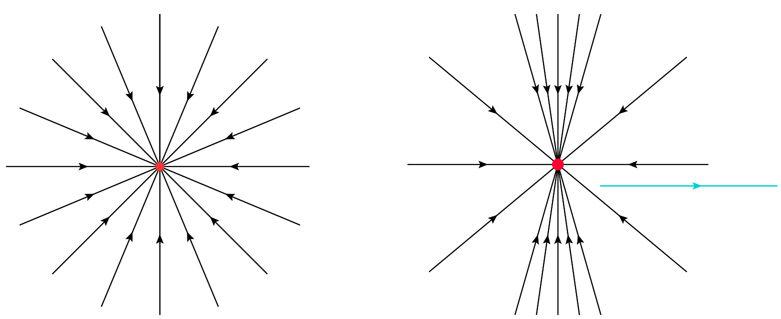

Figure 2.2: Contraction of the electric field lines of a moving point charge. The light-blue arrow indicates the direction of motion of the charge.

forms an electric current that generates transverse magnetic fields,

$$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{A}_\perp \ ,$$

which requires $\vec{A}$ to have a nontrivial transverse part $\vec{A}_\perp$. Since the magnetic field cannot be constant in time, this means that $\vec{A}_\perp(t)$ will in turn generate a transverse electric field

$$\vec{E}_\perp = -\frac{\partial \vec{A}_\perp}{\partial t} .$$

If we define

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} ,$$

the electromagnetic field in the transverse direction gets enhanced by a factor of $\beta\gamma$, while the electric field in the longitudinal direction is suppressed by a factor of γ^{-2} [69]. In the limit of $\beta \to 1$ (or $\gamma \to \infty$), $\vec{E}_\perp \sim \vec{B}$, and $|\vec{E}_\perp| \gg |\vec{E}_\parallel|$, so the electromagnetic field can be approximated as free radiation. If one considers a charged target being scattered by a high-energy charged particle, the cross section can be equivalent to that of the scattering of free photons. As shown in Fig. 3.20, when the charged particle moves very fast, the virtuality k^2 of the photon can be ignored, and the relationship between the two differential cross section is [30]

$$d\sigma_b = d\sigma_a \cdot n(\vec{k}) d^3p' , \tag{2.70}$$

where $n(\vec{k})$ is the number density of the photons. This is called the equivalent-photon method or Weizsäcker-Williams approximation in electrodynamics [69].

In analogy to QED, the Weizsäcker-Williams approximation is also a valid picture for the gluons in a proton moving at extremely large momentum [73]. For free radiation, there are only two transverse degrees of freedom of the gauge field. In the case of a beam of free gluons propagating along the z direction, the gauge-dependent field components $A^{1,2}$ thus acquire physical meaning. Therefore, $(\vec{E} \times$

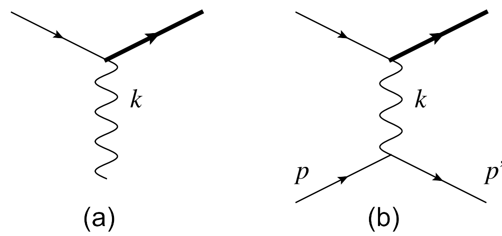

Figure 2.3: Weizsäcker-Williams approximation in electrodynamics.

$\vec{A})^3 = E^1A^2 - E^2A^1$ can be clearly interpreted as the longitudinal gluon spin (or gluon helicity) operator as long as one fixes it in a physical gauge condition that leaves $A^{1,2}$ intact, such as $A^+ = 0$ in the light-cone coordinates. Any other version of the gluon spin operator that has the IMF limit as $(\vec{E} \times \vec{A})^3$ in the light-cone gauge can be considered as an element of a universality class [74], which we will elaborate in the next chapter.

To gain more physical insights from QED, we will review how the photon spin and OAM are measured in atomic physics in Appendix C.

To conclude this chapter, we establish the free-field form of QCD angular momentum as the basis of a physical sum rule, i.e., the Jaffe-Manohar sum rule, for the proton spin in high-energy scattering experiments. Our future discussions will be focusing on how to obtain this sum rule from first principle calculations.

Chapter 3

Universality Class of Operators for the Spin Sum Rule

3.1 New window for lattice QCD calculation

In the last chapter we have justified the physical meaning of the Jaffe-Manohar sum rule, and our ultimate goal is to calculate it on the theoretical side. Since the proton spin structure is intrinsically nonperturbative, one has to rely on lattice QCD to do the calculations.

However, from a practical perspective, the Jaffe-Manohar sum rule places a great hurdle for lattice calculation, because the explicit usage of light-cone coordinates and gauge brings real-time dependence, as has been discussed in the first chapter. One may avoid this difficulty by using normal space-time coordinates with a "physical" gauge that does not involve time, and calculating with a proton at infinite momentum [2]. However, the largest momentum attainable on the lattice with spacing a is constrained by the lattice cutoff π/a, and usually the nucleon momentum is taken to be much smaller than this value to reduce the noise in numerical computations.

The angular momentum operators defined in Eq. (2.21), instead, provide a solution to this problem. Since they are constructed to be time independent, one can calculate their matrix elements directly in lattice QCD [64]. By exploiting their dependence on the nucleon momentum, one may eventually extract out the result in the IMF limit. However, taking the IMF limit of the matrix elements is not a trivial task in quantum field theory, and it will be the main subject for the rest of this paper.

Before we go on to discuss the procedure of taking the IMF limit, it is necessary to note that there can be more than one proposals whose Weizsäcker–Williams approximation is the Jaffe-Manohar sum rule. This allows us to talk about a universality class of operators that can be used to define the gluon spin [74], as well as the quark and gluon OAM. Since any operator in the universality class is essentially the free-field operators defined in a specific gauge, this universality class is actually a group of "physical" gauge conditions that eliminate the unphysical degrees of freedom. For instance, in the Coulomb gauge, the condition $\vec{k} \cdot \vec{A} = 0$ yields $\epsilon^3 = 0$ for a beam of gluons propagating along the z direction with momentum $\vec{k}$. Under an infinite Lorentz boost, the transverse components of A^μ are not affected, whereas the t- and z- components are transformed into the $+$-component so that the Coulomb condition reduces to the light-cone condition $A^+ = 0$. The universality class of operators will offer more options for lattice calculations, and therefore it is worthwhile for a thorough investigation.

3.2 A universality class of operators for the gluon spin

In this section, we discuss the matrix elements of the gluon spin operator with different gauge choices, which asymptotically approach the physical gluon helicity ΔG in the IMF limit. We start with the Coulomb gauge that has been considered in Ref. [64].

Let us begin with the standard definition of ΔG as the matrix element of the gauge-invariant gluon spin operator defined in Eq. (1.21) [23],

$$\begin{aligned}\Delta G \frac{S^+}{P^+} &= \frac{1}{2P^+}\int dx \frac{i}{x}\int \frac{d\xi^-}{2\pi} e^{-ixP^+\xi^-}\langle PS|F_a^{+\alpha}(\xi^-)\mathcal{L}^{ab}(\xi^-,0)\tilde{F}_{\alpha,b}^{\ +}(0)|PS\rangle \\ &= \frac{1}{2P^+}\langle PS|\epsilon^{ij}F^{i+}(0)A_\perp^j(0)|PS\rangle \ . \qquad (3.1)\end{aligned}$$

In the second line we defined [56, 60, 64]

$$A_\perp^\mu \equiv \frac{1}{\mathcal{D}^+}F^{+\mu} \, , \qquad (3.2)$$

and introduced the antisymmetric tensor in the transverse plane ϵ^{ij} ($\epsilon^{12} = -\epsilon^{21} = 1$). The boundary condition for the integral operator $1/\mathcal{D}^+$ is related to the $i\epsilon$-prescription for the $1/x$ pole, and we have derived the explicit result in the previous chapter. In the light-cone gauge $A^+ = 0$, $A_\perp^\mu$ reduces to A^μ.

The matrix element in Eq. (3.1), being nonlocal in the light–cone direction, cannot be readily evaluated in lattice QCD. However, one can relate ΔG to the following matrix element in the Coulomb gauge [64],

$$\Delta\tilde{G}(P^z,\mu)\frac{S^z}{P^0} = \frac{1}{2P^0}\langle PS|\epsilon^{ij}F^{i0}(0)A^j(0)|PS\rangle \ , \qquad (3.3)$$

which is local and time independent, hence measurable on the lattice. In Eq. (3.3), the momentum P^z is assumed to be large but finite. $\epsilon^{ij}F^{i0}A^j = (\vec{E}\times\vec{A})^3$ is the gluon helicity operator identified by Jaffe and Manohar [2]. As is well–known, this operator is not gauge invariant, so the matrix element in Eq. (3.3) depends on the gauge choice.

The scales that matter in the nucleon state are the nucleon mass M, $\Lambda_{\rm QCD}$, and the nucleon momentum P^z. M and $\Lambda_{\rm QCD}$ are associated with the infrared (IR) physics of the system, while P^z is the only large scale among the three. The matrix elements of the gluon spin in the Coulomb gauge is expected to have a nontrivial dependence on P^z, as has been shown at the operator level in the previous chapter. In perturbation theory, P^z only affects the ultraviolet (UV) behavior of the loop integrals, resulting in a logarithmic dependence in the matrix elements. Therefore, the IMF limit of the matrix element is not well defined, and the procedure needs special treatment.

For the external onshell quark state $|PS\rangle_q$, a one–loop calculation using dimensional regularization (in $d = 4 - 2\epsilon$ dimensions) yields [64, 75]

$$\Delta\tilde{G}(P^z,\mu)\frac{S^z}{P^0} = \frac{\langle PS|\epsilon^{ij}F^{i0}A^j|PS\rangle_q}{2P^0}\bigg|_{\vec{\nabla}\cdot\vec{A}=0} = \frac{\alpha_s C_F}{4\pi}\left(\frac{5}{3\epsilon_{UV}} - \frac{1}{9} + \frac{4}{3}\ln\frac{4P_z^2}{m^2}\right)\frac{S^z}{P^0}\,, \tag{3.4}$$

where we defined $1/\epsilon_{UV} \equiv 1/\epsilon - \gamma_E + \ln 4\pi + \ln(\mu^2/m^2)$. μ is the renormalization scale, and m is the quark mass to regularize the collinear divergence. $C_F = (N_c^2-1)/2N_c$ as usual. However, if we follow the procedure in Ref. [76] to take the $P^z \to \infty$ limit before the loop integration [64],

$$\Delta\tilde{G}(\infty,\mu) = \frac{\langle PS|\epsilon^{ij}F^{i0}A^j|PS\rangle_q}{2P^0}\bigg|_{\vec{\nabla}\cdot\vec{A}=0} = \frac{\alpha_s C_F}{4\pi}\left(\frac{3}{\epsilon_{UV}} + 7\right)\frac{S^z}{P^0}\,. \tag{3.5}$$

On the other hand, in the same regularization scheme Eq. (3.1) is evaluated in the light-cone gauge as [77]

$$\Delta G(\mu) = \frac{\langle PS|\epsilon^{ij}F^{i+}A^j|PS\rangle_q}{2P^+}\bigg|_{A^+=0} = \frac{\alpha_s C_F}{4\pi}\left(\frac{3}{\epsilon_{UV}} + 7\right)\frac{S^+}{P^+}\,. \tag{3.6}$$

We see that the coefficients of $1/\epsilon_{UV}$ (anomalous dimension) are different. The reason for this discrepancy is that the IMF limit $P^z \to \infty$ and the large loop momentum limit $l^\mu \to \infty$ in the one-loop integral do not commute: One can actually recover the exact light-cone gauge result in Eq. (3.6) from the Coulomb gauge calculation by sending $P^z \to \infty$ *before* doing the l-integral [64]. On a lattice, P^z is restricted to be less than the cutoff, which is tantamount to taking the $l^\mu \to \infty$ limit first. Thus, the matrix element in Eq. (3.3), evaluated in the Coulomb gauge, fails to capture the UV properties of ΔG. Nevertheless, since the IR physics characterized by $\ln m^2$ is not affected by the order of limits, one can correct the discrepancy via the

$$\frac{1}{\epsilon_{UV}} + \frac{16}{3} = \ln \frac{4P_z^2}{m^2} \,. \tag{3.7}$$

This observation paves the way for evaluating ΔG on a Euclidean lattice.

The Coulomb gauge is not the unique possibility in order to match with ΔG. For instance, consider the temporal axial gauge $A^0 = 0$. In this gauge one can identically write

$$A_\perp^\mu = \frac{1}{\mathcal{D}^0} F^{0\mu} \,. \tag{3.8}$$

Taking the IMF limit, one trivially recovers Eq. (3.2),

$$\frac{1}{\mathcal{D}^0} F^{0\mu} \to \frac{1}{\mathcal{D}^+} F^{+\mu} \,. \tag{3.9}$$

Alternatively, one may choose the $A^z = 0$ gauge so that

$$A_\perp^\mu = \frac{1}{\mathcal{D}^z} F^{z\mu} \,. \,, \tag{3.10}$$

which has the same IMF limit in Eq. (3.9).

However, the matrix elements of $\vec{E} \times \vec{A}$ are generally different in different gauges. To one-loop order, we find

$$\Delta\tilde{G}(P^z,\mu) = \frac{\langle PS|\epsilon^{ij}F^{i0}A^j|PS\rangle_q}{2P^0}\bigg|_{A^0=0} = \frac{\alpha_s C_F}{4\pi}\left(\frac{3}{\epsilon_{UV}} + 7\right)\frac{S^z}{P^0}\,, \tag{3.11}$$

$$\Delta\tilde{G}(P^z,\mu) = \frac{\langle PS|\epsilon^{ij}F^{i0}A^j|PS\rangle_q}{2P^0}\bigg|_{A^z=0} = \frac{\alpha_s C_F}{4\pi}\left[\frac{2}{\epsilon_{UV}} + 4 + \frac{P^z}{P^0}\ln\frac{(P^0+P^z)^2}{m^2}\right]\frac{S^z}{P^0}\,. \tag{3.12}$$

Eq. (3.11) agrees with the previous result in Eq. (3.6) in the light–cone gauge (see, also, Ref. [78]). On the other hand, Eq. (3.12) features yet another anomalous dimension together with logarithmic frame dependence. Here again, the order of limits matters: If one takes the $P^z \to \infty$ limit before the loop integration, one recovers Eq. (3.6) from the $A^z = 0$ gauge calculation. At large but finite momentum, part of the divergence $1/\epsilon_{UV}$ is transferred to the logarithm $\ln P_z^2$, keeping the sum of their coefficients unchanged. The following matching condition then establishes the connection between Eq. (3.12) and Eq. (3.6),

$$\frac{1}{\varepsilon_m} + 3 = \frac{P^z}{P^0}\ln\frac{(P^0+P^z)^2}{m^2} \approx \ln\frac{4P_z^2}{m^2}\,. \tag{3.13}$$

The constant term is different from the Coulomb gauge case in Eq. (3.7).

Thus, for the purpose of obtaining ΔG, one can broadly generalize the approach of Ref. [64]: Evaluate the “naive” gluon helicity operator Eq. (3.3) either in the Coulomb gauge, or $A^0 = 0$, or $A^z = 0$ gauge and perform an appropriate matching. However, this does not mean that any gauge choice is allowed. For instance, in the $A^x = 0$ gauge where

$$A_\perp^\mu = \frac{1}{\mathcal{D}^x}F^{x\mu}\,, \tag{3.14}$$

or in the Landau (or covariant) gauge $\partial \cdot A = 0$ where

$$A_\perp^\mu = A^\mu - \frac{1}{\partial^2}\partial^\mu \partial \cdot A\,, \tag{3.15}$$

Neither of the above has the same IMF limit as Eq. (3.9). This is also reflected in their one-loop matrix elements

$$\frac{\langle PS|\epsilon^{ij}F^{i0}A^j|PS\rangle_q}{2P^0}\bigg|_{A^x=0} = \frac{C_F\alpha_s}{4\pi}\left(\frac{3}{2\epsilon_{UV}}+\frac{7}{2}\right)\frac{S^z}{P^0}\,, \tag{3.16}$$

$$\frac{\langle PS|\epsilon^{ij}F^{i0}A^j|PS\rangle_q}{2P^0}\bigg|_{\partial\cdot A=0} = \frac{C_F\alpha_s}{4\pi}\left(\frac{2}{\epsilon_{UV}}+4\right)\frac{S^z}{P^0}\,, \tag{3.17}$$

which do not agree with the light–cone gauge result.[1] Moreover, the logarithm of P^z is absent, so the matrix element is the same even one take the $P^z \to \infty$ before the l-integral and there is no possibility of matching.

The above analysis suggests that there is a class of gauges (similar to the universality class of second order phase transitions) which flows to the "fixed point" of light-cone gauge in the IMF limit, and thus can be used to compute ΔG. This class of gauges clearly do not include all possible gauge conditions. To see what gauges are permitted, we consider the Weizsäcker–Williams approximation [69] in the IMF. The gluon field is dominated by quasi-free radiation in the sense that $\vec{B}_\perp \sim \vec{E}_\perp \gg \vec{E}_{||}$. Thus we have in effect a beam of gluons with momentum xP^z. For these on-shell gluons, gauge transformation only affects the time component and the third spatial component (we consider only the example of Abelian gauge theory),

$$A^\mu \to A^\mu + \lambda k^\mu\,, \tag{3.18}$$

[1] Interestingly, Eq. (3.16) is exactly one half of Eq. (3.11).

where $k^\mu = (k^0, 0, 0, k^z)$. Thus the transverse part of the polarization vector is physical,

$$A^\mu \sim \varepsilon^\mu(xP^z) = \frac{1}{\sqrt{2}}(0, 1, \mp i, 0)\ . \tag{3.19}$$

The longitudinal gluon spin operator $(\vec{E} \times \vec{A})^3$ is independent of gauge transformations which leave $A^{1,2}$ invariant. Although Eq. (3.18) seems to guarantee this for Weizsäcker–Williams gluon field, it allows for only a subclass of gauges. For the gauge choices that are incompatible with the notion that Weizsäcker–Williams gluon fields $A^{1,2}$ shall not be affected, they will not "flow" into the fixed point—the light-cone gauge—in the IMF limit.

The axial gauge $A^z = 0$ and the temporal gauge $A^0 = 0$ have no effect on the gluon polarization vector. Therefore, they can be used to calculate the gluon helicity. In the Coulomb gauge, one has $\vec{k} \cdot \vec{A} = k^z A^z = 0$. This is similar to the axial gauge $A^z = 0$.

On the other hand, the obvious counterexample is $A^x = 0$ or $A^y = 0$. A less trivial one is the covariant gauge, in which the condition $k \cdot A = k^+ A^- = 0$ itself is consistent with having nonzero transverse components. However, actually the Weizsäcker–Williams field in the covariant gauge has only the A^+ component. This can be seen from an example of the Weizsäcker–Williams field associated with a fast-moving pointlike charge [74]. In the covariant gauge we have

$$A^\mu(\xi) = -e \ln \xi_\perp^2 \delta(\xi^-) \delta^\mu_+\ . \tag{3.20}$$

Eq. (3.20) indeed satisfies $\partial \cdot A = \partial_+ A^+ = 0$, but has vanishing transverse components. Therefore, the covariant gauge does not belong to the universality class.

3.3 Axial gauges and topological current

The temporal axial gauge $A^0 = 0$ seems to have a special status since the matrix element in Eq. (3.11) coincides with that in the $A^+ = 0$ gauge. Therefore, in this section we explore strategies to measure ΔG in the $A^0 = 0$ gauge where there is no logarithmic matching, or more generally, in non-lightlike axial gauges $n \cdot A = 0$ with $n^2 \neq 0$ (see, also, Ref. [78]). As we shall see, the matrix element of the topological current allows us to find more operators in the universality class [74], and some of them do not even have the form of spin operator in a particular gauge.

First, note that in the $A^0 = 0$ gauge, the operator $\epsilon^{ij} F^{i0} A^j$ is the same as

$$\epsilon^{ij} \left(F^{i0} A^j - \frac{1}{2} A^0 F^{ij} \right) . \tag{3.21}$$

Likewise, in the $A^+ = 0$ gauge the operator $\epsilon^{ij} F^{i+} A^j$ is the same as

$$\epsilon^{ij} \left(F^{i+} A^j - \frac{1}{2} A^+ F^{ij} \right) . \tag{3.22}$$

Actually, the forward matrix elements of these operators are gauge invariant to one-loop,

$$\frac{\langle PS | \epsilon^{ij} \left(F^{i0} A^j - \frac{1}{2} A^0 F^{ij} \right) | PS \rangle_q}{2P^0} = \frac{\alpha_s C_F}{4\pi} \left(\frac{3}{\epsilon_{UV}} + 7 \right) \frac{S^z}{P^0} , \tag{3.23}$$

as can be explicitly checked in all the gauges mentioned in the previous section (See, also, Ref. [79]). This in particular means that the logarithm of P^z which appears in some gauges is canceled by the contribution from the extra term $\epsilon^{ij} A^0 F^{ij}$. The reason is that Eqs. (3.21) and (3.22) are a part of the topological current in QCD,

$$\begin{aligned} K^\mu &= \epsilon^{\mu\nu\rho\lambda}\left(A_\nu^a F_{\rho\lambda}^a + \frac{g}{3} f_{abc} A_\nu^a A_\rho^b A_\lambda^c\right) , \qquad (3.24) \\ K^+ &= 2\epsilon^{ij}\left(F_a^{i+} A_a^j - \frac{1}{2} F_a^{ij} A_a^+ - \frac{g}{2} f_{abc} A_a^+ A_i^b A_j^c\right) , \\ K^z &= 2\epsilon^{ij}\left(F_a^{i0} A_a^j - \frac{1}{2} F_a^{ij} A_a^0 - \frac{g}{2} f_{abc} A_a^0 A_i^b A_j^c\right) , \end{aligned}$$

which satisfies $\partial_\mu K^\mu = F_a^{\mu\nu}\tilde{F}_{\mu\nu}^a$. The forward matrix element of Eq. (3.24) is perturbatively gauge invariant [2,17] and the $\mathcal{O}(gAAA)$ term starts to contribute only at two loops for quark external states.

Nonperturbatively, however, there is gauge dependence due to anomaly [2,42, 80]. In axial gauges $n \cdot A = 0$, this dependence has been precisely calculated in Ref. [42]. The *non*-forward matrix element of K^μ in a polarized nucleon state is given by

$$\langle PS|K^\mu|P{+}q,S\rangle\Big|_{n\cdot A=0} \xrightarrow{q^\mu\to 0} 4\left(S^\mu - \frac{q\cdot S}{q\cdot n}n^\mu\right)\Delta G\,(n,P) + \frac{in^\mu}{q\cdot n}\langle PS|F_a^{\mu\nu}\tilde{F}_{\mu\nu}^a|PS\rangle\,, \qquad (3.25)$$

where

$$\int_0^\infty d\lambda \langle PS|n^\tau F_{\tau\nu}(\lambda n)\mathcal{L}\tilde{F}^{\nu\mu}(0)|PS\rangle \equiv 2S^\mu \Delta G(n,P)\,. \qquad (3.26)$$

The matrix element in Eq. (3.26) is the same as in Eq. (3.1) except for the direction of the Wilson line. Expanding around the deviation from the light–cone n^2, one finds the relation [42]

$$\Delta G(n,P) = \Delta G + \mathcal{O}\left(\frac{n^2}{(P\cdot n)^2}\right) , \qquad (3.27)$$

which is valid at large momentum (assuming $P\cdot n \neq 0$).

From Eq. (3.25) one can read off various representations of ΔG. For the $\mu = z$ component in the $A^0 = 0$ gauge, the ambiguity (gauge dependence) in the $q^\mu \to 0$ limit drops out. One can safely take the forward limit and find

$$\langle PS|\epsilon^{ij} A^i \partial^0 A^j |PS\rangle \,\Big|_{A^0=0} = 2S^z \Delta G + \mathcal{O}(1/P_z^2)\,. \tag{3.28}$$

This result extends Eq. (3.11) to all orders in perturbation theory. Similarly, taking $\mu = 0$ in the $A^z = 0$ gauge, one gets

$$\langle PS|\epsilon^{ij} A^i \partial^z A^j |PS\rangle \,\Big|_{A^z=0} = 2S^0 \Delta G + \mathcal{O}(1/P_z^2)\,, \tag{3.29}$$

which is related to Eq. (3.12) by replacing F^{i0} with F^{iz}. In the IMF limit, the t–component and z–component of a quantity have similar scaling properties as they both approach the "+" direction. Note that the operator on the left hand side of Eq. (3.29) does not have a straightforward gluon spin interpretation.

Moreover, Eqs. (3.26) and (3.27) directly give

$$\begin{aligned} \int_0^\infty d\xi^0 \langle PS|F^0{}_\nu(\xi^0)\mathcal{L}\tilde{F}^{\nu 0}(0)|PS\rangle &= \langle PS|\vec{A}^a \cdot \vec{B}^a|PS\rangle \Big|_{A^0=0} \\ &= 2S^0 \Delta G + \mathcal{O}(1/P_z^2)\,. \end{aligned} \tag{3.30}$$

$$\begin{aligned} \int_0^\infty d\xi^z \langle PS|F^z{}_\nu(\xi^z)\mathcal{L}\tilde{F}^{\nu z}(0)|PS\rangle &= \langle PS|\epsilon^{ij}\left(F^{i0}A^j - \frac{1}{2}A^0 F^{ij}\right)|PS\rangle \Big|_{A^z=0} \\ &= 2S^z \Delta G + \mathcal{O}(1/P_z^2)\,. \end{aligned} \tag{3.31}$$

The operator in Eq. (3.30) is similar to an operator written down by Jaffe [110], except that it includes the z–component as well. Eq. (3.31) coincides with the operator introduced in Ref. [81]. All the matrix elements in Eqs. (3.28)–(3.31) are measurable on the lattice. In particular, the operators in Eqs. (3.29) and (3.30) can be readily transcribed into Euclidean space as they do not contain temporal indices ∂^0, A^0. Note that all these operators yield the gluon helicity ΔG without logarithmic corrections at large P^z.

3.4 Matching the gluon spin to the lattice

Before we introduce our systematic approach to calculate parton observables from the universality class of operators in the next chapter, let us first consider the gluon spin. In order to relate $\Delta\tilde{G}_{\rm lat}$ measured on the lattice to $\Delta G_{\overline{\rm MS}}$ defined in the continuum theory in the $\overline{\rm MS}$ scheme, one has to perform a perturbative matching. The matching coefficients depend on the operators chosen and the UV regularization scheme, which is independent of the IR regulator. In the case of the operators in Eqs. (3.28)-(3.31), the perturbative matching is particularly simple because there are no logarithms $\ln P^z/\mu$ involved.

To figure out the matching coefficients, we should first consider the mixing of ΔG with the quark spin $\Delta\Sigma$. This can be read off from Eq. (3.6), but here we use a different regularization of the IR and collinear divergences in order to keep in line with the gluon matrix element calculated below, and also with typical lattice computations [82]. Namely, we now assume that the quark is massless and slightly off-shell $P^2 < 0$. This affects the finite term of the matrix element

$$\langle PS|\epsilon^{ij}F^{i+}A^j|PS\rangle_q\Big|_{A^+=0} = \frac{\alpha_s C_F}{4\pi}\left(\frac{3}{\epsilon_v}+4\right)\langle PS|\bar{q}\gamma_5\gamma^+q|PS\rangle_q^{\rm tree}\,, \quad (3.32)$$

where $1/\epsilon_v \equiv 1/\epsilon - \gamma_E + \ln 4\pi + \ln\mu^2/(-P^2)$. Due to the fact that K^μ transforms as a Lorentz vector and its forward matrix element is one-loop gauge invariant, Eq. (3.32) immediately implies that the same coefficient should appear in the (quark) matrix element of all the operators in Eqs. (3.28)–(3.31), e.g.,

$$\left.\frac{\langle PS|\epsilon^{ij}F^{i0}A^j|PS\rangle_q}{2S^z}\right|_{A^0=0} = \left.\frac{\langle PS|\epsilon^{ij}A^i\partial^z A^j|PS\rangle_q}{2S^0}\right|_{A^z=0} = \frac{\alpha_s C_F}{4\pi}\left(\frac{3}{\epsilon_v}+4\right) . \tag{3.33}$$

Next we compute the one-loop matrix element in the gluon external state $|Ph\rangle_g$ ($h=\pm 1$ is the helicity). In the light–cone gauge with the Mandelstam-Leibbrandt prescription for the propagator pole $1/k^+ \to 1/(k^+ + i\epsilon k^-)$ [83,84], the contribution from the irreducible diagrams is calculated to be (see Appendix D)

$$\left.\frac{\langle Ph|\epsilon^{ij}F^{i+}A^j|Ph\rangle_g}{2P^+}\right|_{A^+=0}^{\mathrm{irr}} = h\frac{\alpha_s N_c}{2\pi}\left(2+\frac{\pi^2}{3}\right) . \tag{3.34}$$

Note that there is no divergence here. The self-energy insertion in the external gluon legs is divergent and reads (cf. Ref. [85])

$$\left.\frac{\langle Ph|\epsilon^{ij}F^{i+}A^j|Ph\rangle_g}{2P^+}\right|_{A^+=0}^{\mathrm{self}} = h\frac{\alpha_s N_c}{2\pi}\left(\frac{11}{6\epsilon_v}-\frac{\pi^2}{3}+\frac{67}{18}\right) + h\frac{\alpha_s N_f}{2\pi}\left(-\frac{1}{3\epsilon_v}-\frac{5}{9}\right) , \tag{3.35}$$

where the two terms correspond to the gluon and quark loop contributions, respectively. Combining these results, we find

$$\langle Ph|\epsilon^{ij}F^{i+}A^j|Ph\rangle_g\big|_{A^+=0} = \left[1+\frac{\alpha_s}{4\pi}\left(\frac{\beta_0}{\epsilon_v}+\frac{103N_c-10N_f}{9}\right)\right]\langle Ph|\epsilon^{ij}F^{i+}A^j|Ph\rangle_g^{\mathrm{tree}} , \tag{3.36}$$

where $\beta_0 = \frac{11N_c}{3} - \frac{2N_f}{3}$ is the coefficient of the one-loop QCD beta function. By the same reasoning as in Eq. (3.33), we immediately obtain[2]

$$\langle Ph|\epsilon^{ij}F^{i0}A^j|Ph\rangle_g\big|_{A^0=0} = \left[1+\frac{\alpha_s}{4\pi}\left(\frac{\beta_0}{\epsilon_v}+\frac{103N_c-10N_f}{9}\right)\right]\langle Ph|\epsilon^{ij}F^{i0}A^j|Ph\rangle_g^{\mathrm{tree}} , \tag{3.37}$$

[2] The agreement of the divergent part in Eqs. (3.36) and (3.37) was explicitly checked in Ref. [78].

and similarly for the other matrix elements in Eqs. (3.29)–(3.31). Note that, *a priori*, the one-loop calculation of the latter two matrix elements Eqs. (3.30) and (3.31) could be complicated, not least because the non-Abelian part of the operator $\mathcal{O}(gAAA)$ would contribute already at one-loop for gluon external states. Yet, the above discussion guarantees that the final result is identical to the one computed in the light-cone gauge Eq. (3.36). In the $\overline{\rm MS}$ scheme, $1/\epsilon_v$ is replaced by $\ln \mu^2/(-P^2)$. In lattice perturbation theory the logarithms become $\ln 1/(a^2P_E^2)$ with P_E being the Euclidean momentum of the proton. Since the anomalous dimension is renormalization-scheme independent, it should be the same in these two cases. The matching of the constant terms can be done in a standard manner [82].[3]

To conclude , in this chapter we first extended the matching method of Ref. [64] to a broad class of gauges. Not only the Coulomb gauge, but also other gauge choices that maintain the transverse components of the on-shell gluon fields do qualify, and for some of them the gluon spin matrix element does not have logarithmic corrections at large momentum. We then focused our attention on non-light-like axial gauges. All the matrix elements in Eqs. (3.28)–(3.31) can be used to compute ΔG in lattice QCD, and we have computed the one-loop matching coefficients on the continuum theory side.

The implementation of the Coulomb gauge and axial gauges on a lattice may pose technical problems. The usual periodic boundary condition on gauge field con-

[3]We note that there exists an exact matching scheme [86] which goes beyond the one-loop matching considered here.

figurations is incompatible with the condition $n \cdot A = 0$ because of nonvanishing Polyakov loops. In order to circumvent this and fix the residual gauge symmetry, ideally one should impose antisymmetric boundary condition in the direction specified by the vector n^μ. Or else, one has to confront the problem of lattice Gribov copies [87, 88].

It is worthwhile to mention that our approach has been taken in a recent attempt to calculate the gluon polarization in lattice QCD [89]. In this calculation $\vec{A}_\perp$ is fixed with the Coulomb condition, and the unrenormalized lattice matrix element indicates a nonzero gluon spin contribution at small proton momenta. By going to larger momentum and performing a renormalization of the lattice matrix elements, one can expect to obtain ΔG in the near future.

Chapter 4

A Large Momentum Effective Field Theory Approach

Acknowledging that lattice QCD can only calculate the matrix elements of time-independent operators with finite nucleon momentum, we need to figure out a way to relate the results to the parton observables defined in the IMF. As has been mentioned in the last chapter, the solution is a perturbative matching which is yet to be systemized as a general approach. Therefore, in this chapter we formulate this approach in the frame work of *large-momentum effective field theory* (LaMET) [90], and will use it to obtain the factorization formula for the proton spin content.

4.1 Large momentum effective field theory

Suppose one is to calcualte some light-cone quantity or parton observable $\mathcal{O}$. Instead of computing it directly, one defines, in the LaMET framework, a quasi-observable $\tilde{\mathcal{O}}$ that depends on a large hadron momentum P^z. In general, both the parton and quasi- observables suffer from UV and IR divergences. If $P^z \to \infty$ is taken prior to UV regularization, the quasi-observable $\tilde{\mathcal{O}}$ becomes the parton observable $\mathcal{O}$ by construction. On the other hand, what one can calculate in lattice QCD is the quasi-observable at finite P^z, with UV regularization imposed first. As shown in the previous chapter, the result may have logarithmic dependence on P^z so that its IMF limit is not well defined.

The difference between $\mathcal{O}$ and $\tilde{\mathcal{O}}$ is just a matter of order of limits. Since P^z remains as a large scale of the system, taking the $P^z \to \infty$ limit shall not change the IR property of the quasi-observable $\tilde{\mathcal{O}}$, whereas it only affects the UV physics. Therefore, $\mathcal{O}$ captures all the nonperturbative physics in $\tilde{\mathcal{O}}$, and the difference between them should be IR free. This is exactly the situation in an effective field theory set-up. The difference is that the role of heavy degree of freedom is played by the large momentum of the external state, so it cannot be arranged into a Lagrangian formalism. Nevertheless, one can bridge the quasi- and parton observables by a factorization formula,

$$\tilde{\mathcal{O}}(P^z/\Lambda) \;=\; Z\left(P^z/\Lambda, \mu/\Lambda\right) \mathcal{O}(\mu) + \frac{c_2}{P_z^2} + \frac{c_4}{(P^z)^4} + \cdots \;, \tag{4.1}$$

where Λ is a UV cutoff imposed on the quasi-observable, and c_i's are higher-twist contributions suppressed by powers of $1/P_z^2$. This formula means that the quasi-observable $\tilde{\mathcal{O}}(P^z/\Lambda)$ can be factorized into the parton observable $\mathcal{O}(\mu)$ and a matching coefficient Z, which is completely perturbative, up to power-suppressed corrections. Within this context, Feynman's parton model can be regarded as an effective theory for the nucleon moving at large momentum [90].

According to Eq. (4.1), the momentum dependence of the quasi-observables can be studied through a "renormalization group" equation. One can define the anomalous dimension [90]

$$\gamma(\alpha_s) = \frac{1}{Z}\frac{dZ}{d\ln P^z} \;, \tag{4.2}$$

and obtain the renormalization group equation,

$$\frac{\partial \tilde{\mathcal{O}}(P^z/\Lambda)}{\partial \ln P^z} = \gamma(\alpha_s)\tilde{\mathcal{O}}(P^z/\Lambda) + O(\frac{1}{P_z^2}) \;. \tag{4.3}$$

Finally, the above equation can be used to sum the large logarithms involving P^z to solve for $\tilde{\mathcal{O}}(P^z/\Lambda)$ in terms of $\mathcal{O}(\mu)$.

A lattice calculation of the quasi-observables will give the IR (nonperturbative) as well as UV (perturbative) contributions, and the factorization formula will help us to correct the UV part to obtain the parton observables. This approach can in principle be applied to all parton physics with variations of the factorization formula. When there is operator mixing in the quasi-observables, the matching coefficient Z will become a matrix; when the parton observable is a distribution, the factorization will take a convolutional form.

4.2 Factorization formula for the Jaffe-Manohar spin sum rule

With LaMET, we can start with any quasi-observable fulfilling the above criteria to calculate the proton spin content. These quasi-observables are just the universality class of operators that have the correct Weizsäcker-Williams approximation as free-field operators in the light-cone gauge [74]. A possible choice of the "physical" gauge condition is the nonlocal operators introduced by Chen *et al.* in Eq. (2.21) [52,53]. From Eq. (2.18), one can show order by order that $\vec{A}_\perp = \vec{A}$ if one fixes $\vec{A}$ in the Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$. Therefore, Chen *et al.*'s decomposition of angular momentum corresponds to choosing the Coulomb gauge as the "physical" gauge.

The advantage of the expression in Eq. (2.21) is that it is time independent and thus allows for direct calculations in lattice QCD. Suppose we evaluate the matrix elements of these quasi-observables with finite momentum P^z, we should have

$$\frac{1}{2} = \frac{1}{2}\Delta\widetilde{\Sigma}(\mu, P^z) + \Delta\widetilde{G}(\mu, P^z) + \widetilde{L}_q(\mu, P^z) + \widetilde{L}_g(\mu, P^z) \ , \tag{4.4}$$

where the dependence on P^z is expected since Eq. (2.21) is a frame-dependent expression.

Based on the effective theory argument, for all the quasi-observables defined in Eq. (2.21) we can relate them to the corresponding parton observables through the following factorization formula [91]:

$$\begin{aligned}
\Delta\widetilde{\Sigma}(\mu, P^z) &= \Delta\Sigma(\mu) \ , \\
\Delta\widetilde{G}(\mu, P^z) &= z_{qg}\Delta\Sigma(\mu) + z_{gg}\Delta G(\mu) + O\left(\frac{M^2}{P_z^2}\right) \ , \\
\widetilde{L}_q(\mu, P^z) &= P_{qq}L_q(\mu) + P_{gq}L_g(\mu) + p_{qq}\Delta\Sigma(\mu) + p_{gq}\Delta G(\mu) + O\left(\frac{M^2}{P_z^2}\right) \ , \\
\widetilde{L}_g(\mu, P^z) &= P_{qg}L_q(\mu) + P_{gg}L_g(\mu) + p_{qg}\Delta\Sigma(\mu) + p_{gg}\Delta G(\mu) + O\left(\frac{M^2}{P_z^2}\right) \ ,
\end{aligned} \tag{4.5}$$

where M is the proton mass. All the matrix elements on both sides of Eq. (4.5) are renormalized in the $\overline{\text{MS}}$ scheme and thus there is no Λ dependence. $\Delta\widetilde{\Sigma}(\mu, P^z)$ is the same as $\Delta\Sigma(\mu)$ because the quark spin operator is gauge invariant and should have the same matrix elements in the Coulomb and light-cone gauges. z_{ij}, P_{ij} and p_{ij}'s are the matching coefficients to be calculated in perturbative QCD.

In the remainder of this section, we show how to obtain all the matching coefficients in Eq. (4.5) at one-loop order. First, let us take z_{qg} and z_{gg} as an example. To obtain z_{qg} and z_{gg}, we need to calculate the matrix elements of $\vec{E}_a \times \vec{A}_\perp^a$ at finite P^z and in the IMF limit (before UV regularization). To ensure gauge invariance and angular momentum conservation in our calculation, we use on-shell and massless external quarks and gluons, and regularize the UV and IR/collinear divergences with dimensional regularization ($d = 4 - 2\epsilon$). One may think of using the off-shellness of external quarks and gluons as IR/collinear regulator, and then take the on-shell limit. However, in this case one needs to take into account the contribution from the ghost and gauge-fixing terms. This is because the total angular momentum operator in QCD from Noether's theorem contains not only the terms presented in our paper, but also the ghost and gauge-fixing terms—which are called BRS-exact "alien" operators in Ref. [92]—from the QCD Lagrangian. The matrix elements of BRS-exact operators vanish in a physical on-shell state, but not in an off-shell state. Therefore, one has to be careful with these contributions when starting from off-shell external states and then going to the on-shell limit, in order to have angular momentum conservation. Considering matrix elements of on-shell states simply avoids such complications.

Since the angular momentum operators we consider are all gauge invariant, we can work in an arbitrary gauge, and for simplicity we choose the Coulomb gauge $\vec{\nabla} \cdot \vec{A}^a = 0$. As mentioned before, the Coulomb gauge condition is equivalent to the condition for $\vec{A}_\perp$ in QCD (see Eq. (2.18)) . So in our calculation, we treat $\vec{A}^a_\perp$ as the transverse component of $\vec{A}^a$. In Appendix E we provide the explicit calculation of the one-loop matrix elements of all the angular momentum operators in Eq. (2.21).

At tree level, $\Delta\widetilde{G}^{\text{tree}} = \Delta G^{\text{tree}}$. At one-loop level, the Feynman diagram for the quark matrix element is shown in Fig. E.2, while the Feynman diagrams for the gluon matrix element are listed in Fig. E.3. The one-loop matrix elements of

$\vec{E}_a \times \vec{A}_\perp^a$ are

$$\Delta\widetilde{G}^{\text{1-loop}} = \frac{\alpha_s C_F}{4\pi}\left(\frac{5}{3}\frac{1}{\epsilon'_{UV}} + \frac{4}{3}\ln\frac{P_z^2}{\mu^2} - \frac{3}{\epsilon'_{IR}} + R_1\right)\Delta\Sigma^{\text{tree}}$$
$$+\frac{\alpha_s}{4\pi}\left[\frac{4C_A - 2n_f}{3}\frac{1}{\epsilon'_{UV}} - \frac{11C_A - 2n_f}{3}\frac{1}{\epsilon'_{IR}} + C_A\left(\frac{7}{3}\ln\frac{P_z^2}{\mu^2} + R_2\right)\right]\Delta G^{\text{tree}} , \tag{4.6}$$

where

$$\frac{1}{\epsilon'} = \frac{1}{\epsilon} - \gamma_E + \ln 4\pi , \qquad C_A = N_c . \tag{4.7}$$

R_1 and R_2 are finite constants that depend on the regularization scheme,

$$R_1 = \frac{8}{3}\ln 2 - \frac{64}{9}, \qquad R_2 = \frac{14}{3}\ln 2 - \frac{121}{9} , \tag{4.8}$$

The corresponding IMF (or light-cone) matrix elements are [93]

$$\Delta G^{\text{1-loop}} = \frac{\alpha_s C_F}{4\pi}\left(\frac{3}{\epsilon'_{UV}} - \frac{3}{\epsilon'_{IR}}\right)\Delta\Sigma^{\text{tree}}$$
$$+\frac{\alpha_s}{4\pi}\left[\frac{11C_A - 2n_f}{3}\frac{1}{\epsilon'_{UV}} - \frac{11C_A - 2n_f}{3}\frac{1}{\epsilon'_{IR}}\right]\Delta G^{\text{tree}} . \tag{4.9}$$

Apparently the anomalous dimensions (coefficients of $1/\epsilon'_{UV}$'s) are different between $\Delta\widetilde{G}^{\text{1-loop}}$ and $\Delta G^{\text{1-loop}}$, but the IR or collinear divergence (coefficients of $1/\epsilon'_{IR}$'s) is the same for both. In the $\overline{\text{MS}}$ scheme, we subtract the $1/\epsilon'_{UV}$ terms, and then substitute the $1/\epsilon'_{IR}$ terms in $\Delta\widetilde{G}$ with ΔG, and obtain the relation:

$$\Delta\widetilde{G}^{\text{1-loop}} = \frac{\alpha_s C_F}{4\pi}\left(\frac{4}{3}\ln\frac{P_z^2}{\mu^2} + R_1\right)\Delta\Sigma^{\text{tree}}$$
$$+\frac{\alpha_s C_A}{4\pi}\left(\frac{7}{3}\ln\frac{P_z^2}{\mu^2} + R_2\right)\Delta G^{\text{tree}} + \Delta G^{\text{1-loop}} . \tag{4.10}$$

Since

$$\Delta\widetilde{G} \approx \Delta\widetilde{G}^{\text{tree}} + \Delta\widetilde{G}^{\text{1-loop}} , \qquad \Delta G \approx \Delta G^{\text{tree}} + \Delta G^{\text{1-loop}} , \tag{4.11}$$

we have at $O(\alpha_s)$

$$\Delta\widetilde{G} \;=\; \frac{\alpha_s C_F}{4\pi}\left(\frac{4}{3}\ln\frac{P_z^2}{\mu^2}+R_1\right)\Delta\Sigma+\left[1+\frac{\alpha_s C_A}{4\pi}\left(\frac{7}{3}\ln\frac{P_z^2}{\mu^2}+R_2\right)\right]\Delta G\;. \tag{4.12}$$

Therefore, the matching coefficients for $\Delta\widetilde{G}$ are:

$$\begin{aligned} z_{qg}(\mu/P^z) &= \frac{\alpha_s C_F}{4\pi}\left(\frac{4}{3}\ln\frac{P_z^2}{\mu^2}+R_1\right)\;, \\ z_{gg}(\mu/P^z) &= 1+\frac{\alpha_s C_A}{4\pi}\left(\frac{7}{3}\ln\frac{P_z^2}{\mu^2}+R_2\right)\;. \end{aligned} \tag{4.13}$$

As one can see, z_{qg} and z_{gg} are both dependent on P^z/μ only. Following the same procedure, we calculate all the matching coefficients in Eq. (4.5) at one-loop order. The results are as follows,

$$\begin{aligned} P_{qq} &= 1+\frac{\alpha_s C_F}{4\pi}\left(-2\ln\frac{P_z^2}{\mu^2}+R_3\right) \;, & P_{gq} &= 0 \;, \\ P_{qg} &= \frac{\alpha_s C_F}{4\pi}\left(2\ln\frac{P_z^2}{\mu^2}-R_3\right) \;, & P_{gg} &= 1 \;, \\ p_{qq} &= \frac{\alpha_s C_F}{4\pi}\left(-\frac{1}{3}\ln\frac{P_z^2}{\mu^2}+R_4\right) \;, & p_{gq} &= 0 \;, \\ p_{qg} &= \frac{\alpha_s C_F}{4\pi}\left(-\ln\frac{P_z^2}{\mu^2}-R_1-R_4\right) \;, & p_{gg} &= \frac{\alpha_s C_A}{4\pi}\left(-\frac{7}{3}\ln\frac{P_z^2}{\mu^2}-R_2\right) \;, \end{aligned} \tag{4.14}$$

where

$$R_3 \;=\; -4\ln 2+\frac{28}{3}\;,\qquad R_4 \;=\; -\frac{2}{3}\ln 2+\frac{13}{9}\;. \tag{4.15}$$

At this stage, we are able to match the quasi-observables in Eq. (2.21) evaluated at a large finite momentum to the parton spin and OAM. The next step is to perform a similar matching procedure to extract the continuum theory matrix elements from the lattice QCD simulations. To be more specific, we need to calculate the one-loop matrix elements of the quasi-observables in lattice perturbation theory [82], and compare them to the continuum theory matrix elements renormalized in the $\overline{\mathrm{MS}}$ scheme. The soft and collinear divergences in the quasi-observables are unchanged in the lattice theory because the discretization of space-time does not affect the long range physics. Therefore, the matching is still completely perturbative, but to keep track of the soft and collinear divergences we should use dimensional regularization to handle them. Since the renormalization in lattice and continuum theories only differs by scheme, the anomalous dimension should be the same as in Eq. (4.14) because it is scheme independent. However, the finite constants in the matching coefficients are scheme dependent and should be precisely calculated in lattice perturbation theory.

4.3 Factorization formula for parton distributions

Apart from the proton spin content, parton distributions can also be calculated directly on the Euclidean lattice with the LaMET approach [81]. In this approach one computes, instead of the light-cone distribution, a related quantity which may be called quasi-distribution. In the case of unpolarized quark density, the quasi-distribution is [81]

$$\tilde{q}(x,\Lambda,P^z) = \int \frac{dz}{4\pi} e^{izk^z} \langle P|\overline{\psi}(z)\gamma^z \exp\left(-ig\int_0^z dz' A^z(z')\right)\psi(0)|P\rangle \ , \qquad (4.16)$$

where $x = k^z/P^z$. The above quantity is time-independent, and thus can be simulated on the lattice. However, the result is not the light-cone distribution extracted from the experimental data, $q(x,\mu^2)$. Instead, it approaches the light-cone distribution in the IMF limit, so LaMET allows us to relate the two (for the non-singlet case) through the factorization formula [81, 94]

$$\tilde{q}_{\rm NS}(x,\Lambda,P^z)=\int dy\, Z\left(\frac{x}{y},\frac{\Lambda}{P^z},\frac{\mu}{P^z}\right)q_{\rm NS}(y,\mu)+\mathcal{O}((M/P^z)^2)\ . \tag{4.17}$$

According to LaMET, Z is entirely perturbative. One has yet to prove that the above relation holds to all orders in perturbation theory. However, as a first step, we test this factorization at next-to-leading order at this stage. Of course the choice of quasi-distributions is not unique, one can define more than one possible quasi-distributions which have similar properties as $\tilde{q}(x,\Lambda,P^z)$. Here we focus on the simplest type suitable for lattice QCD calculations.

4.3.1 One-loop result for unpolarized quasi-quark distribution

In this subsection, we consider the one-loop correction to the unpolarized quasi-quark distribution $\tilde{q}(x,\Lambda,P^z)$. The one-loop calculation for non-singlet quark distribution is similar to QED because the non-Abelian property has no effect in the non-singlet case.

Since the one-loop result is gauge-invariant, we can perform the calculation in any gauge. The simplest choice is the axial gauge $A^z=0$ [95–97] where the gauge link in Eq. (4.16) becomes unity. In the axial gauge, the relevant Feynman diagrams are shown in Fig. 4.1, where the non-local operator is depicted as a dashed line. The diagrams contain UV as well as soft and collinear divergences. We use the quark mass m to regulate the collinear divergence. The soft divergence is expected to cancel between the diagrams. The UV divergence is regulated by a transverse-momentum cut-off Λ. Of course this cut-off violates rotational symmetry. However, it is expected to yield the correct leading-logarithmic behavior.

Figure 4.1: One loop corrections to quasi quark distribution.

The one-loop diagrams generate the following result

$$\tilde{q}(x, \Lambda, P^z) = (1 + \tilde{Z}_F^{(1)} + \ldots)\delta(x-1) + \tilde{q}^{(1)}(x) + \ldots \tag{4.18}$$

with

$$\tilde{q}^{(1)}(x) = \frac{\alpha_s C_F}{2\pi} \begin{cases} \frac{1+x^2}{1-x} \ln \frac{x(\Lambda(x)-xP^z)}{(x-1)(\Lambda(1-x)+P^z(1-x))} + 1 - \frac{xP^z}{\Lambda(x)} \\ +\frac{x\Lambda(1-x)+(1-x)\Lambda(x)}{(1-x)^2 P^z} \ , & x > 1 \ , \\ \\ \frac{1+x^2}{1-x} \ln \frac{P_z^2}{m^2} + \frac{1+x^2}{1-x} \ln \frac{4x(\Lambda(x)-xP^z)}{(1-x)(\Lambda(1-x)+(1-x)P^z)} - \frac{4x}{1-x} \\ +1 - \frac{xP^z}{\Lambda(x)} + \frac{x\Lambda(1-x)+(1-x)\Lambda(x)}{(1-x)^2 P^z} \ , & 0 < x < 1 \ , \\ \\ \frac{1+x^2}{1-x} \ln \frac{(x-1)(\Lambda(x)-xP^z)}{x(\Lambda(1-x)+(1-x)P^z)} - 1 - \frac{xP^z}{\Lambda(x)} \\ +\frac{x\Lambda(1-x)+(1-x)\Lambda(x)}{(1-x)^2 P^z} \ , & x < 0 \ . \end{cases} \tag{4.19}$$

for finite P^z, where $\Lambda(x) = \sqrt{\Lambda^2 + x^2 P_z^2}$ and the logarithm with collinear divergences is related to the standard Altarelli-Parisi kernel [73]. The wave function renormalization correction depends on P^z as well

$$\tilde{Z}_F^{(1)} = \frac{\alpha_s C_F}{2\pi} \int dy \begin{cases} -\frac{1+y^2}{1-y} \ln \frac{y(\Lambda(y)-yP^z)}{(y-1)(\Lambda(1-y)+P^z(1-y))} - 1 - \frac{y\Lambda(1-y)+(1-y)\Lambda(y)}{(1-y)^2 P^z} \\ +\frac{y^2 P^z}{\Lambda(y)} + \frac{y(1-y)P^z}{\Lambda(1-y)} + \frac{\Lambda(y)-\Lambda(1-y)}{P^z} \ , & y > 1 \ , \\ \\ -\frac{1+y^2}{1-y} \ln \frac{P_z^2}{m^2} - \frac{1+y^2}{1-y} \ln \frac{4y(\Lambda(y)-yP^z)}{(1-y)(\Lambda(1-y)+(1-y)P^z)} + \frac{4y^2}{1-y} + 1 \\ -\frac{y\Lambda(1-y)+(1-y)\Lambda(y)}{(1-y)^2 P^z} + \frac{y^2 P^z}{\Lambda(y)} + \frac{y(1-y)P^z}{\Lambda(1-y)} + \frac{\Lambda(y)-\Lambda(1-y)}{P^z} \ , & 0 < y < 1 \ , \\ \\ -\frac{1+y^2}{1-y} \ln \frac{(y-1)(\Lambda(y)-yP^z)}{y(\Lambda(1-y)+(1-y)P^z)} + 1 - \frac{y\Lambda(1-y)+(1-y)\Lambda(y)}{(1-y)^2 P^z} \\ +\frac{y(1-y)P^z}{\Lambda(1-y)} + \frac{y^2 P^z}{\Lambda(y)} + \frac{\Lambda(y)-\Lambda(1-y)}{P^z} \ , & y < 0 \ . \end{cases} \tag{4.20}$$

It is easy to check that the result satisfies the vector current conservation

$$\int_{-\infty}^{+\infty} dx\, \tilde{q}(x, \Lambda, P^z) = 1 \tag{4.21}$$

to one-loop order. Since the constituent of quark in a quasi-distribution does not have a parton interpretation, the parton momentum fraction x extends from $-\infty$ to $+\infty$. However, it is interesting to see that the collinear divergence exists only for $0 < x < 1$.

In field theory calculations, one often takes the ultraviolet cut-off to be larger than any other scale in the problem. In other words, one shall take the limit $\Lambda \to \infty$ and keep only the leading contribution and ignore the power-suppressed ones. This in principle shall also be the case in lattice QCD calculations. Thus the actual field theoretical result for the quasi-distribution shall be

$$\tilde{q}^{(1)}(x,\Lambda,P^z) = \frac{\alpha_s C_F}{2\pi}\begin{cases} \frac{1+x^2}{1-x}\ln\frac{x}{x-1}+1+\frac{\Lambda}{(1-x)^2P^z} \ , & x>1 \ , \\ \frac{1+x^2}{1-x}\ln\frac{P_z^2}{m^2}+\frac{1+x^2}{1-x}\ln\frac{4x}{1-x}-\frac{4x}{1-x}+1+\frac{\Lambda}{(1-x)^2P^z} \ , & 0<x<1 \ , \\ \frac{1+x^2}{1-x}\ln\frac{x-1}{x}-1+\frac{\Lambda}{(1-x)^2P^z} \ , & x<0 \ , \end{cases} \tag{4.22}$$

and

$$\tilde{Z}_F^{(1)} = \frac{\alpha_s C_F}{2\pi}\int dy \begin{cases} -\frac{1+y^2}{1-y}\ln\frac{y}{y-1}-1-\frac{\Lambda}{(1-y)^2P^z} \ , & y>1 \ , \\ -\frac{1+y^2}{1-y}\ln\frac{P_z^2}{m^2}-\frac{1+y^2}{1-y}\ln\frac{4y}{1-y}+\frac{4y^2}{1-y}+1-\frac{\Lambda}{(1-y)^2P^z} \ , & 0<y<1 \ , \\ -\frac{1+y^2}{1-y}\ln\frac{y-1}{y}+1-\frac{\Lambda}{(1-y)^2P^z} \ , & y<0 \ . \end{cases} \tag{4.23}$$

One shall note several interesting features of the above result: First, there are contributions in the regions $x > 1$ and $x < 0$. The physics behind this is clear: when the parent particle has a finite momentum P^z, it can have backward emissions, so the constituent parton can have momentum larger than P^z, and even negative. This is very different from the IMF case, where the momentum fraction is restricted to $0 < x < 1$. Second, there is a linear divergence arising from the self-energy of the gauge link, which can be easily seen if one goes to a non-axial gauge, e.g., the Feynman gauge. This linear divergence is usually ignored in dimensional regularization. However, it is present in lattice regularization. Third, there is no logarithmic UV divergence. Instead, there is a logarithmic dependence on P^z in the region $0 < x < 1$. We will see later on that this logarithmic dependence can be translated into the renormalization scale dependence of the light-cone parton distribution throught the factorization formula. Finally, all soft divergences are cancelled between the quasi-and light-cone distributions. However, there are remaining collinear divergences reflected by the logarithm of quark mass.

On the other hand, in the same regularization scheme one can calculate the light-cone parton distribution by taking the limit $P^z \to \infty$. This is done following the spirit of Ref. [76], and the result is

$$q(x,\mu) = (1 + Z_F^{(1)} + \dots)\delta(x-1) + q^{(1)}(x) + \dots \tag{4.24}$$

with

$$q^{(1)}(x) = \frac{\alpha_S C_F}{2\pi} \begin{cases} 0 \ , & x > 1 \text{ or } x < 0 \ , \\ \frac{1+x^2}{1-x} \ln \frac{\mu^2}{m^2} - \frac{1+x^2}{1-x} \ln{(1-x)^2} - \frac{2x}{1-x} \ , & 0 < x < 1 \ , \end{cases} \tag{4.25}$$

and

$$Z_F^{(1)} = \frac{\alpha_S C_F}{2\pi} \int dy \begin{cases} 0 \ , & y > 1 \text{ or } y < 0 \ , \\ -\frac{1+y^2}{1-y} \ln \frac{\mu^2}{m^2} + \frac{1+y^2}{1-y} \ln{(1-y)^2} + \frac{2y}{1-y} \ , & 0 < y < 1 \ , \end{cases} \tag{4.26}$$

where the integrand of $\delta Z^{(1)}$ is exactly opposite to that of $q^{(1)}(x)$. This result agrees exactly with that derived from the light-cone definition of parton distribution in the transverse momentum cut-off scheme. Also the collinear divergence is clearly the same as in the quasi-parton distribution. This shows that at one-loop level, the quasi-parton distribution captures all the IR physics of the parton distributions in the IMF. Moreover, the collinear divergence comes only from the diagram where the intermediate gluon has a cut that leads to a partonic interpretation.

4.3.2 Factorization at next-to-leading order

Now we are ready to derive the factorization formula at next-to-leading order for the non-singlet parton distribution. In the IMF or on the light-cone plane, the momentum fraction in parton distributions and splitting functions is limited to $0 < x < 1$. However, in the present case, the splitting in the quasi-distribution is not constrained to this region, it can be in $-\infty < x < \infty$. Therefore, the connection between the two distributions is reflected through the following factorization formula up to power corrections for large P^z,

$$\tilde{q}(x, \Lambda, P^z) = \int_0^1 \frac{dy}{y} Z\left(\frac{x}{y}, \frac{\Lambda}{P^z}, \frac{\mu}{P^z}\right) q(y, \mu) + \mathcal{O}\left(\Lambda^2/P_z^2, M^2/P_z^2\right) \ , \tag{4.27}$$

where the integration range is determined by the support of the quark distribution $q(y)$ on the light cone.

The Z factor has a perturbative expansion in α_s,

$$Z\left(\xi, \frac{\Lambda}{P^z}, \frac{\mu}{P^z}\right) = \delta(\xi - 1) + \frac{\alpha_s}{2\pi} Z^{(1)}\left(\xi, \frac{\Lambda}{P^z}, \frac{\mu}{P^z}\right) + \dots \ . \tag{4.28}$$

Before we proceed, it is important to note the existence of a linear divergence coupled to the axial-gauge singularity, which is a double pole at $\xi = 1$ in the one-loop corrections to the quasi-quark distribution, as one can see from Eqs. (4.22) and (4.23). As mentioned earlier, if one chooses a covariant gauge like the Feynman gauge, the diagrams in Fig. 4.1 do not give axial singularities, but now one has extra diagrams with gauge link where the axial-gauge singularity originates from. Obviously the linear divergence is absent in dimensional regularization, but it is present in the cutoff regularization. Unlike the case of light-cone parton distribution, where one encounters at most a single pole at $\xi = 1$ and can appropriately regularize it by a plus-prescription, singularity at the double pole cannot be completely regularized. In general, it only reduces the double pole to a single pole, which still yields singular contribution after integration over ξ. However, it turns out that in our case this singularity can be removed with a principal value prescription, which corresponds to a regularization of the Wilson line self-energy. Although the linear divergence can also be singled out (this can be done on the lattice by varying the lattice spacing with fixed P^z and x) and subtracted [98], we propose to include this linearly divergent term within our factorization formula, in order to simplify the extraction of light-cone distribution from lattice data.

Now we are ready to write down the Z factor matching the quasi-quark distribution to the light-cone quark distribution. For $\xi > 1$, one has

$$Z^{(1)}(\xi)/C_F = \left(\frac{1+\xi^2}{1-\xi}\right)\ln\frac{\xi}{\xi-1} + 1 + \frac{1}{(1-\xi)^2}\frac{\Lambda}{P^z}\ , \tag{4.29}$$

whereas for $0 < \xi < 1$,

$$Z^{(1)}(\xi)/C_F = \left(\frac{1+\xi^2}{1-\xi}\right)\ln\frac{P_z^2}{\mu^2} + \left(\frac{1+\xi^2}{1-\xi}\right)\ln\left[4\xi(1-\xi)\right] - \frac{2\xi}{1-\xi} + 1 + \frac{\Lambda}{(1-\xi)^2P^z}\ , \tag{4.30}$$

and for $\xi < 0$,

$$Z^{(1)}(\xi)/C_F = \left(\frac{1+\xi^2}{1-\xi}\right)\ln\frac{\xi-1}{\xi} - 1 + \frac{\Lambda}{(1-\xi)^2P^z}\ . \tag{4.31}$$

Near $\xi = 1$, one has an additional term coming from the self energy correction

$$Z^{(1)}(\xi) = \delta Z^{(1)}(2\pi/\alpha_s)\delta(\xi-1)\ , \tag{4.32}$$

which can be extracted from Eqs. (4.23) and (4.26), and provides a plus-regularization for the singularity at $\xi = 1$. Thus the large logarithmic dependence on P^z in $\tilde{q}(x, \Lambda, P^z)$ can be translated into the renormalization scale dependence through the above factorization formula. On the lattice, the matching coefficient Z must be recalculated up to a constant accuracy using the standard approach, where the longitudinal and transverse momentum cutoffs are done in a way consistent with lattice symmetry [82, 100, 101].

So far, we have considered only the quark contribution. One can start with an antiquark to do the one-loop calculation. In this case, there is also a contribution to $\tilde{q}(x, \Lambda, P^z)$ from $\bar{q}(x)$. However, the antiquark distribution has the property

$$\bar{q}(x) = -q(-x) \ , \tag{4.33}$$

which is related to quark distribution at negative x. By including both quark and antiquark contributions, one obtains the following factorization formula with the integration region extended to $-1 < y < 1$,

$$\tilde{q}(x, \Lambda, P^z) = \int_{-1}^{1} \frac{dy}{|y|} Z\left(\frac{x}{y}, \frac{\Lambda}{P^z}, \frac{\mu}{P^z}\right) q(y, \mu) + \mathcal{O}\left(\Lambda^2/P_z^2, M^2/P_z^2\right) \ , \tag{4.34}$$

where negative y indicates the antiquark contribution. The above is the complete one-loop factorization formula for the non-singlet case, which replaces Eq. (4.27) and the Z-factor in Ref. [81]. In Appendix F we also provide the factorization formulas for the polarized and transversity distributions for the non-singlet case [94]. Note that the polarized gluon distribution function can also be matched to the quasi distribution defined in Ref. [81] similarly with the LaMET approach.

We have constructed at one-loop level a factorization formula matching the quasi-parton distribution to the light-cone parton distribution. The factorization formula then allows one to extract the parton distribution $q(x, \mu^2)$ from the lattice calculation of the time-independent $\tilde{q}(x, \Lambda, P^z)$ in a state with increasingly large P^z. P^z cannot be larger than the lattice cutoff π/a, but should be much larger than the mass of the nucleon.

Of course it remains to be shown that there exists such a formula to all orders in perturbation theory, which is attempted recently in Ref. [102]. Besides, Ref. [102] offers a different perspective by proposing to extract the quasi-parton distributions from the QCD factorization of lattice "cross sections". Meanwhile, the first direct lattice calculation of the isovector sea-quark parton distributions [103] is done using the formalism developed in Ref. [81] and our factorization formula in Eq. (4.34).

In conclusion, we have shown in detail how each term in the Jaffe-Manohar sum rule can be extracted from the lattice matrix elements of corresponding quasi-observables through the LaMET approach. The factorization formulas we have obtained will be of great importance to the first lattice calculation of the gluon polarization and parton OAM. Since the data on quark and gluon spin is being collected by the state-of-the-art hadron physics programs, while the OAM are also related to observables that can be measured in high-energy scattering, we will eventually be able to compare the proton spin structure in both theory and experiment.

Appendices

Appendix A

Model Calculations of the Proton Spin Content

In this appendix we discuss several models that have been used to calculate the quark and gluon contributions to the proton spin.

A.1 Quark models

The simple $SU(6)$ symmetric non-relativistic quark model (NRQM) was the first to predict the quark spin contribution. In this model, the proton is made of three constituent quarks which are current quarks dressed with gluons and the quark sea. If a constituent quark is not distinguished from a current quark, then the NRQM predicts $\Delta\Sigma = 1$ [2]. Otherwise, the quark OAM contributes to the proton spin, and with the OZI rule the NRQM leads to $\Delta\Sigma = g_A^{(0)} = g_A^{(8)}$, that is, the Ellis-Jaffe prediction [1,2] that can be extracted from hyperon decays. However, this was cast into doubt after the EMC spin crisis.

Relativistic quark models, for example, the MIT bag model [104], are capable of generating the spin contribution from the quarks. In the MIT bag model, the proton is an ensemble of three current quarks confined in a finite square well potential. The quarks satisfy the Dirac equation and certain boundary conditions on the surface of the potential, and they can only form a color-singlet. According to the

MIT bag model, the wave function of the quark is given by

$$\psi(\vec{r}) = \frac{N}{\sqrt{4\pi}} \begin{pmatrix} f(\vec{r}) \\ \\ i\sigma \cdot \hat{r} g(\vec{r}) \end{pmatrix} , \tag{A.1}$$

where N is a renormalization factor, and f and g are normalized as

$$\int d^3r(f^2 + g^2) = 1 . \tag{A.2}$$

In the limit of $SU(3)$ flavor symmetry, for all the quarks in the $1s$ ground state of the proton [2, 105],

$$\Delta\Sigma = 3F - D = N^2 \int d^2r \; r^2 \left(f^2 - \frac{1}{3} g^2 \right) = 0.65 , \tag{A.3}$$

which is not much different from the Ellis-Jaffe prediction. In the ground state of the bag-model proton, the other 35% contribution comes from the quark OAM.

Another type of relativistic quark models is the covariant spectator quark (CSQ) model, where the proton is a bound state of a dressed quark and a pair of spectator quarks that obey spin-flavor symmetry [106]. The dressed quark is off-shell and is involved in the interaction of the proton with external sources, while the spectator quarks are on-shell. The pair of spectator quarks are also referred to as a "diquark", but actually they do not interact with each other. In the CSQ model, the proton wave function is constructed to be Lorentz covariant in the Dirac spinor representation, and can include S-, P-, D-, and higher OAM components with free parameters to be fitted by known nucleon structure functions [107]. With the parameters fixed, one can calculate the polarized quark distributions of different flavors [107].

In the light-cone representation of the quark-spectator model, the relation between the polarized and unpolarized valence quark distributions was obtained by taking into account of flavor asymmetry and the Wigner rotation effect [108]. As for the origin of the quark OAM, Ref. [109] proposed that it is transferred from the quark spin through the Melosh-Wigner rotation. In the light-cone representation, the quark OAM distribution is equal to the polarized quark distribution times a Melosh-Wigner rotation factor. In Ref. [109], sum rules for the quark OAM were obtained for the quark-spectator model, and the estimated values are

$$L_u + L_d = 0.04 \sim 0.42 \ . \tag{A.4}$$

The total quark OAM calculated from the sum rule in Ref. [109] is

$$L_q = \delta\Sigma - \Delta\Sigma = 0.76 \pm 0.26 \ , \tag{A.5}$$

where $\delta\Sigma$ is the proton tensor charge.

In all the quark models, the gluon spin and OAM do not show up in the lowest order wave function of the proton. Since the quarks interact strongly via gluons, one should be able to calculate their contribution in QCD with the quark models. In fact, the baryon mass difference $M_\Delta - M_N$ was well explained by the lowest order exchange of transverse magnetic gluons, and it has motivated Jaffe to calculate the spin carried by them [110]. The results of ΔG in NRQM and the MIT bag model are [110]

$$\Delta G_{\rm NRQM}(Q^2 = 0.25 \text{ GeV}^2) \approx -0.7, \qquad \Delta G_{\rm bag}(Q^2 = 0.25 \text{ GeV}^2) \approx -0.4 \ , \tag{A.6}$$

which are negative and do not satisfy the requirement by the axial anomaly analysis discussed previously. Later on, in Ref. [111], it was argued that one should include the "self angular momentum" effects. Based on the Isgur–Karl (IK) quark model [112–114], the gluon spin contribution was calculated to be very different from Eq. (A.6) [111],

$$\Delta G_{\rm IK}(Q^2 = 0.25\ {\rm GeV}^2) \approx 0.24\ . \tag{A.7}$$

In addition to calculating the total gluon spin, it was proposed in Ref. [115] that one can also calculate the polarized gluon distribution $\Delta g(x)$ from the MIT bag model. It was argued that at the leading non-vanishing order only the one-body exchange Feynman diagram gives rise to the the matrix elements of the nonlocal operator for $\Delta g(x)$. The result in Ref. [115] suggests that ΔG is of the order of 0.2 or 0.3 at low-energy scales.

A.2 Chiral models

Another general approach to the proton spin problem is based on the chiral-soliton models, where the chiral symmetry is spontaneously broken and the proton is treated as a collective excitation, i.e., soliton or Skyrmion.

In 1988, based on the simplest version of the $SU(3)$ Skyrme model, Brodsky *et al.* predicted that the flavor-singlet axial charge vanishes at the leading-order in the $1/N_c$ expansion [116]. Therefore, the quark spin contribution to the proton should be of order $1/N_c$, which is consistent with the EMC result. Later on, Ryzak introduced an effective operator that corresponds to the flavor-singlet axial current and calculated its baryonic matrix element [117]. His result on the total quark spin

is

$$\Delta u + \Delta d + \Delta s = 0.2 \pm 0.1 \ . \tag{A.8}$$

The chiral-soliton model has had much improvement since it was first applied to the calculation of the quark spin in the proton. One of the variants is the chiral model that includes vector mesons as auxiliary "gauge fields" [118, 119] whose presence in the effective Lagrangian respects the chiral $U(3) \times U(3)$ symmetry. The vector mesons lead to an additional term to the $U(1)$ axial vector current, and its contribution to the flavor-singlet axial charge is

$$g_A^{(0)} = 0.30 \ , \tag{A.9}$$

which changes only by about 10% on the variation of the parameters in this model [119]. Another variant is the chiral quark model [118, 119], which includes quarks explicitly in the Lagrangian. This model gives the prediction of the flavor-singlet axial charge in terms of adjustable parameters, and is capable of generating the value

$$g_A^{(0)} \approx 0.33 \ . \tag{A.10}$$

Actually, the traditional chiral quark model, or chiral bag model [120], treats the proton as soliton outside the bag and allows it to interact with the bag quarks via the pion clouds at the boundary. With the bag radius fixed to be $R = 0.6$ fm, the chiral bag model predicts [120]

$$g_A^{(0)} = 0.30 \ . \tag{A.11}$$

Since the up-to-date result of the quark spin contribution is 36.6% [20], the predictions of the chiral models are consistent with experiments. However, it must be pointed out that the consistency is based on the proper fixing of the parameters in all of these models.

Appendix B

Gauge-invariant extension

According to Wakamatsu [55], Eqs. (2.19), (2.20) are adequate to ensure the gauge-invariance and frame-independence of the decomposition, although they do not completely fix the gauge field $A^\mu_\perp$ or $A^\mu_\parallel$. For practical calculations, one can fix $A^\mu_\perp$ by a further constraint, such as the generalized Coulomb condition in Eq. (2.18) or the light-cone condition which lead to Chen *et al.* and Jaffe-Bashinsky's spin decompositions respectively. Since gauge-invariance is ensured before one exactly fixes $A^\mu_\perp$ and $A^\mu_\parallel$, these proposals should be gauge equivalent [55]. In other words, the physical matrix elements of the angular momentum operators should be the same in different proposals. To show the gauge-invariance of the gluon spin operator $\vec{E} \times \vec{A}_\perp$ with $\vec{A}_\perp$ given by the light-cone condition, Wakamatsu calculated its one-loop anomalous dimension in the Feynman gauge, and found that the result is the same as that in the light-cone gauge [121].

We agree that operators defined with $A^\mu_\perp$ and $A^\nu_\parallel$ have gauge-invariant physical matrix elements, but disagree that different proposals are gauge equivalent. Let us first take the photon propagator as an example [24]. In QED, the propagator of the "physical" gauge field

$$D^{\mu\nu}_\perp(k) = -i \int d^4x \; e^{-ik\cdot(x-y)} \langle T\left[A^\mu_\perp(x) A^\nu_\perp(y)\right]\rangle \tag{B.1}$$

should be gauge invariant at the operator level.

Now let us fix $A_\perp^\mu(x)$ with a certain condition, such as the Coulomb and light-cone conditions. In both cases, $A_\perp^\mu(x)$ can be defined in the momentum space with a projection operator $\mathcal{P}^{\mu\nu}$,

$$A_\perp^\mu(x) = \int \frac{d^4k}{(2\pi)^4} \mathcal{P}^\mu{}_\nu(k) \tilde{A}_\perp^\mu(k), \tag{B.2}$$

where for the Coulomb gauge,

$$\mathcal{P}_{\rm C}^{\mu\nu}(k) = g^{\mu\nu} - n\cdot k \frac{n^\mu k^\nu + n^\nu k^\mu}{(n\cdot k)^2 - k^2} + n^2 \frac{k^\mu k^\nu}{(n\cdot k)^2 - k^2} + k^2 \frac{n^\mu n^\nu}{(n\cdot k)^2 - k^2}\ , \tag{B.3}$$

and for the light-cone gauge,

$$\mathcal{P}_{\rm LC}^{\mu\nu}(k) = g^{\mu\nu} - \frac{\eta^\mu k^\nu + \eta^\nu k^\mu}{\eta\cdot k} + k^2 \frac{\eta^\mu \eta^\nu}{(\eta\cdot k)^2}\ , \tag{B.4}$$

with $n^\mu = (1,0,0,0)$ and $\eta^\mu = (1,0,0,-1)/\sqrt{2}$.

The photon propagator can be rewritten as

$$D_\perp^{\mu\nu}(k) = P^\mu{}_\alpha(k) \left(-i \int d^4x\ e^{-ik\cdot(x-y)} \langle T\left[A^\alpha(x) A^\beta(y)\right]\rangle \right) P_\beta{}^\nu(k)\ . \tag{B.5}$$

With $A_\perp^\mu$ defined by projecting A^μ onto $\mathcal{P}_{\rm C}^{\mu\nu}$ and $\mathcal{P}_{\rm LC}^{\mu\nu}$, we calculate the photon propagator perturbatively in the general covariant, axial, and Coulomb gauges, and find that it is just the same Coulomb or light-cone propagator respectively. It is easy to verify this at tree level; at higher orders, since the vacuum polarization satisfies the Ward identity, we can write down the general form of the radiative corrections to the gluon propagator $(k^2 g^{\mu\nu} - k^\mu k^\nu)$ and find that after contraction with $\mathcal{P}_{\rm C}^{\mu\nu}$ and $\mathcal{P}_{\rm LC}^{\mu\nu}$ it just returns the Coulomb or light-cone propagator.

However, gauge-invariant as they are, the photon propagators obtained from $\mathcal{P}_{\rm C}^{\mu\nu}$ and $\mathcal{P}_{\rm LC}^{\mu\nu}$ are obviously different. Moreover, the anomalous dimension of the gluon spin operator with $A_\perp^\mu$ fixed by the Coulomb condition is also calculated in Refs. [64, 75], and the result is different from that in the light-cone gauge [77]. This indicates that the two different choices for $A_\perp^\mu$ are not gauge equivalent, so the claim by Wakamatsu is not correct [55]. The reason behind the discrepancy is simple: $A_\perp^\mu$ fixed by different conditions are not related by a simple homogeneous gauge transformation as shown in Eq. (2.20).

Since the different proposals are not gauge equivalent, one may think that there are an infinite number of ways to decompose the proton spin, and there is no first principle telling us which one has the most physical meaning. The gauge-invariant gluon spin operator defined in Eq. (1.21) [56] also can be regarded as one of them if

$$A_\perp^{\mu,a}(x) = -\frac{1}{2}\int_{-\infty}^{\infty} \epsilon(y^- - x^-)\mathcal{P}\mathrm{exp}\left(-ig\int_{y^-}^{x^-} A^+(y'^-, \vec{x}_\perp)dy'^-\right)_{ab} F_b^{+\mu}(y^-, \vec{x}_\perp)\ . \tag{B.6}$$

The common feature of these proposals is that one achieves manifest gauge-invariance at the cost of locality. Furthermore, as has been demonstrated by our example of the photon propagator, they lead to gauge-invariant results which are exactly the same as what one obtains in the gauge that fixes $A_\perp^\mu$. In general, the gluon spin is not a gauge-invariant quantity, and all the above discussions are merely manipulating the gauge field so that one extends the result in a specific gauge to all. This idea is called *gauge-invariant extension* (GIE) [24] as it transcends the traditional sense of gauge symmetry. Therefore, Chen *et al.*'s decomposition can be considered as the GIE of the generalized Coulomb gauge, while the Jaffe-Bashinsky decomposition is the GIE of the light-cone gauge.

It should be pointed out that the GIE of a gauge-dependent quantity is in fact not gauge invariant. Besides, there are also problems with this idea [24]:

First, operators constructed from GIE are in general nonlocal. In quantum field theory, locality is a fundamental property of the fields, while $A_\perp^\mu$ and $A_\parallel^\nu$ are constructed to be nonlocal quantities. The nonlocality of the GIE operators makes it hard to interpret their physical meaning, although the latter becomes clear in a fixed gauge condition. While local gauge-invariant operators often have simple classifications in terms of representations of the Lorentz group, the nonlocal operators usually involves geometric lines or space integrals that do not transform in a proper way as tensors.

Second, the gauge conditions where GIE starts from can be frame dependent, which does not satisfy the requirement for physical observables in special relativity. For example, the Coulomb gauge condition is not Lorentz invariant, which means that under a boost transformation $\vec{A}_\perp$ will transform into a quantity that is different from the $\vec{A}_\perp$ defined in the new frame.

Third, the scale evolution of the nonlocal operators is complicated. The renormalization of Wilson lines is a highly nontrivial task in quantum field theory [122], and the difficulty increases exponentially as the geometric lines in the GIE operators can be arbitrary. Besides, since there are an infinite number of nonlocal operators that have the same quantum number, they can mix under scale evolution, which becomes intractable for perturbative calculations.

Finally, the GIE operators are in general not measurable. So far, the only example is offered in high-energy scattering where certain partonic GIE operators—such as the gluon polarization—may be measured. For GIE operators in the Coulomb gauge, there is no known physical measurement for any of them.

Appendix C

Photon spin and orbital angular momentum in atomic physics

It has often been claimed that the photon spin and OAM in electromagnetism can be separately defined and measured, and therefore must be individually gauge invariant. This has served as another important motivation to look for a gauge-invariant definition of the gluon spin. In this section, we discuss the examples in optics, pointing out that this is possible only for optical modes with a fixed frequency.

We first recall a bit of history about photon's angular momentum. For a circularly polarized plane wave, R. Beth [123] was the first to measure its spin angular momentum by measuring the torque exerted on the quartz wave plate it passed through. As we explained above, this is simply the gauge-invariant helicity. In 1992, L. Allen et al. pointed out that Laguerre-Gaussian laser modes also have a well-defined orbital angular momentum [124]. Based on this, several experiments [125–127] have been set up to observe and measure the orbital angular momentum of a Laguerre-Gaussian photon.

For radiation field with $e^{-i\omega t}$ time-dependence, using Maxwell's equation,

$$\vec{B} \;=\; -\frac{i}{\omega}\nabla\times\vec{E}\ , \tag{C.1}$$

one always has the gauge-invariant decomposition

$$\begin{aligned}\vec{J} \;&=\; \frac{1}{2}\int d^3x\ [\vec{x}\times(\vec{E}^*\times\vec{B}+\vec{E}\times\vec{B}^*)] \\ &=\; -\frac{i\epsilon_0}{\omega}\int d^3x[\vec{E}_i^*(\vec{x}\times\nabla)\vec{E}_i+\vec{E}^*\times\vec{E}]\ .\end{aligned} \tag{C.2}$$

The two terms may be identified as the orbital and spin angular momentum, respectively, and are manifestly gauge invariant. In particular, this is true for photon electric and magnetic multipoles which are often used in transitions between atomic or nuclear states. The photon OAM can be defined without referring to the gauge potential at all [69].

It can be easily checked from the above equation that the spin equals ∓ 1 for left-handed or right-handed circularly polarized light, and 0 for linearly polarized light. In paraxial approximation, a Laguerre-Gaussian mode with azimuthal angular dependence of $\exp(il\phi)$ is an eigen mode of the operator $L_z = -i\partial/\partial\phi$, and carries OAM of $l\hbar$. It is remarkable that the experimentalists are able to find ways to detect the effects of the OAM alone in recent years [125–127].

Clearly, the above procedure only applies to a specific type of radiation field. In the case of QCD, the gluons in the nucleon cannot be of this type. In particular, they are off-shell and do not satisfy the on-shell equations of motion. The best one can do is to go to the IMF where the gluons appear as on-shell radiation; in this way, the gluon helicity and OAM could be defined and measured "naturally" in the light-cone gauge. Thus we are back to the previous discussions.

Appendix D

Gluonic matrix element of the gluon spin in the light-cone gauge

In this appendix we display some intermediate steps leading to the result in Eq. (3.34). We treat the external gluon to be off–shell $P^2 < 0$. After some algebra, the one–loop matrix element in the light–cone gauge reduces to (see, also, Ref. [78])

$$\frac{\langle Ph|\epsilon^{ij}F^{i+}A^j|Ph\rangle_g}{2P^+} \sim h\frac{ig^2N_c}{P^+}\int\frac{d^dk}{(2\pi)^d}\frac{\frac{16}{d-2}k_\perp^2 P^+ - k^+(P+k)^2 - 2\frac{P^++k^+}{P^+-k^+}k^+(k^2-P^2)}{k^2k^2(P-k)^2}\,. \tag{D.1}$$

We use the Mandelstam–Leibbrandt prescription for the pole in the last term of the numerator $1/k^+ \to 1/(k^+ + i\epsilon k^-)$. The following formulas are useful:

$$\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2(P-k)^2(P^+-k^+)} = \frac{i}{(4\pi)^2P^+}\frac{\pi^2}{6}\,, \tag{D.2}$$

$$\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2k^2(P-k)^2(P^+-k^+)} = \frac{-i}{(4\pi)^2P^+P^2}\frac{1}{\epsilon_{IR}}\,, \tag{D.3}$$

where ϵ_{IR} is an IR regulator.

Appendix E

Matrix elements of parton spin and OAM

E.1 The quark spin

The quark spin operator is

$$S_q^z = \int d^3x\ \psi^\dagger \frac{\Sigma^3}{2}\psi = \frac{1}{2}\int d^3x\ \bar{\psi}\gamma^3\gamma^5\psi\ , \tag{E.1}$$

which is gauge invariant and the same as that in the Jaffe-Manohar form of spin sum rule. Therefore, there is no need of matching for the quark spin.

E.2 The gluon spin

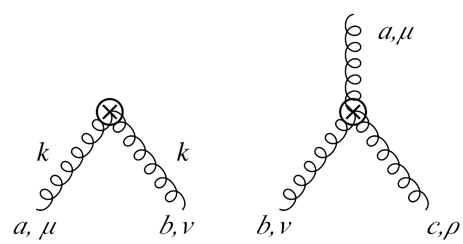

Figure E.1: Vertices from the gluon spin operator.

The quasi gluon spin operator

$$S_g = \int d^3x \vec{E}_a \times \vec{A}_\perp^a \tag{E.2}$$

includes two- and three-gluon vertices as shown in Fig. E.1, and their Feynman rules are:

$$\delta^{ab}\epsilon_{ijm}(ik^0 g^{j\mu} g_\perp^{m\nu} - ik^0 g^{j\nu} g_\perp^{m\mu} - ik^j g^{0\mu} g_\perp^{m\nu} + ik^j g^{0\nu} g_\perp^{m\mu})\ , \tag{E.3}$$

$$g\epsilon_{ijm}f^{abc}(g^{0\mu}g^{j\nu}g_\perp^{m\rho} - g^{0\mu}g^{j\rho}g_\perp^{m\nu} + g^{0\nu}g^{j\rho}g_\perp^{m\mu} - g^{0\nu}g^{j\mu}g_\perp^{m\rho} + g^{0\rho}g^{j\mu}g_\perp^{m\nu} - g^{0\rho}g^{j\nu}g_\perp^{m\mu}) \ , \tag{E.4}$$

where, $i, j, m = 1, 2, 3$, and $g_\perp^{\mu\nu}(k)$ is a projection operator that projects any four-vector to its transverse components with respect to k^μ,

$$g_\perp^{\mu\nu}(k) \ = \ g^{\mu\nu} - n\cdot k\frac{n^\mu k^\nu + n^\nu k^\mu}{\vec{k}^2} + \frac{k^\mu k^\nu}{\vec{k}^2} + \frac{n^\mu n^\nu k^2}{\vec{k}^2} \ , \tag{E.5}$$

with $n^\mu = (1, 0, 0, 0)$.

E.2.1 Matrix element in the quark state

To extract out the z_{qg} factor, we need to calculate the matrix element of S_g in a free quark state, as shown in Fig. E.2.

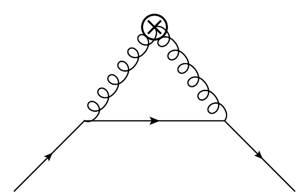

Figure E.2: Matrix element of S_g in a free quark state.

This diagram gives

$$
\begin{aligned}
\langle p,s\left|S_g^z\right|p,s\rangle^{(1)} &= \bar{u}(p)\int\frac{d^4k}{(2\pi)^4}(-ig\gamma^\beta\tau^a)\frac{i}{\not{p}-\not{k}}iD_{\beta\nu}(k) \\
&\quad\times[ik^0(g^{\mu 1}g_\perp^{\nu 2}-g^{\mu 2}g_\perp^{\nu 1})-ig^{\mu 0}(k^1g_\perp^{\nu 2}-k^2g_\perp^{\nu 1})] \\
&\quad\times iD_{\mu\alpha}(k)(-ig\gamma^\alpha\tau^a)u(p) \\
&\quad+\bar{u}(p)\int\frac{d^4k}{(2\pi)^4}(-ig\gamma^\beta\tau^a)\frac{i}{\not{p}-\not{k}}iD_{\beta\nu}(k) \\
&\quad\times[ik^0(g_\perp^{\mu 1}g^{\nu 2}-g_\perp^{\mu 2}g^{\nu 1})+ig^{\nu 0}(k^1g_\perp^{\mu 2}-k^2g_\perp^{\mu 1})] \\
&\quad\times iD_{\mu\alpha}(k)(-ig\gamma^\alpha\tau^a)u(p)\ ,
\end{aligned}
\tag{E.6}
$$

where

$$
D^{\mu\nu}(k) = \frac{1}{k^2+i\epsilon}\left[-g_\perp^{\mu\nu}(k)+\frac{n^\mu n^\nu k^2}{\vec{k}^2}\right]\ . \tag{E.7}
$$

The second term in the square brackets of Eq. (E.7) is the instantaneous Coulomb interaction.

With a massless external quark state, we expect to encounter collinear divergence in the one-loop integral. As we work in the dimensional regularization scheme, for a scaleless integral, the UV and collinear poles cancel each other and the result is zero. However, for our purpose we need to separate the UV divergence from the collinear divergence, and therefore we introduce an arbitrary "mass" parameter m to regularize scaleless integrals. For example,

$$
\begin{aligned}
\int\frac{d^dk}{k^2(k+p)^2} &= \int\frac{d^dk}{(k^2-m^2)(k+p)^2}-m^2\int\frac{d^dk}{k^2(k^2-m^2)(k+p)^2} \\
&\sim \left(\frac{1}{\epsilon_{UV}}-\frac{1}{\epsilon_{IR}}\right)\ ,
\end{aligned}
\tag{E.8}
$$

where $\epsilon_{UV}>0$, and $\epsilon_{IR}<0$.

Since the operator $\vec{E}_a \times \vec{A}_\perp^a$ is frame dependent, its matrix element should also be frame dependent. Actually, when evaluating the loop integral in Eq. (E.6), we encounter noncovariant integrals such as

$$I = \int \frac{d^d k}{k^2 (k+p)^2 \vec{k}^2} \,, \tag{E.9}$$

To regularize this type of integrals, we adopt the split dimensional regularization used in Ref. [128] to achieve the one-loop renormalization of QED in the Coulomb gauge. In practice, we choose the time and space dimensions to be 1 and $d-1$, and integrate over the time and space loop momentum separately. A more systematic treatment of split dimensional regularization is provided by Ref. [129]. In our calculation, the integral in Eq. (E.9) is evaluated as

$$\begin{aligned}
I &= \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k+p)^2 \vec{k}^2} \\
&= i \int_0^1 dx \int \frac{dk^4 d^{d-1}\vec{k}}{(2\pi)^d} \frac{1}{(k_4^2 + \vec{k}^2)^2 (\vec{k} - x\vec{p})^2} \\
&= i \int_0^1 dx \int_0^1 dy \; 2(1-y) \int \frac{dk^4 d^{d-1}\vec{k}}{(2\pi)^d} \frac{1}{\left[(1-y)k_4^2 + \vec{k}^2 + x^2 y (1-y) \vec{p}^2\right]^3} \\
&= i \int_0^1 dx \int_0^1 dy \; 2\sqrt{1-y} \int \frac{d^d k_E}{(2\pi)^d} \frac{1}{[k_E^2 + x^2 y(1-y)\vec{p}^2]^3} \\
&= \frac{i}{(4\pi)^{d/2}} \int_0^1 dx \int_0^1 dy \; \sqrt{1-y} \frac{\Gamma(1+\epsilon)}{(x^2 y (1-y) \vec{p}^2)^{1+\epsilon}} \\
&= \frac{i}{16\pi^2 \vec{p}^2} \left[\frac{1}{\epsilon_{IR}} - \gamma_E - \ln \vec{p}^2 - 2 - 2\ln 2 \right] \,.
\end{aligned} \tag{E.10}$$

The result of Eq. (E.6) is

$$\begin{aligned}
\langle p, s \,| S_g^z |\, p, s \rangle^{(1)} &= \frac{\alpha_s C_F}{4\pi} \left(\frac{5}{3} \frac{1}{\epsilon'_{UV}} - \frac{3}{\epsilon'_{IR}} + \frac{4}{3} \ln \frac{\vec{p}^2}{\mu^2} + \frac{8}{3} \ln 2 - \frac{64}{9} \right) \\
&\quad \times \langle p, s | \Sigma^3 | p, s \rangle^{\text{tree}} \,,
\end{aligned} \tag{E.11}$$

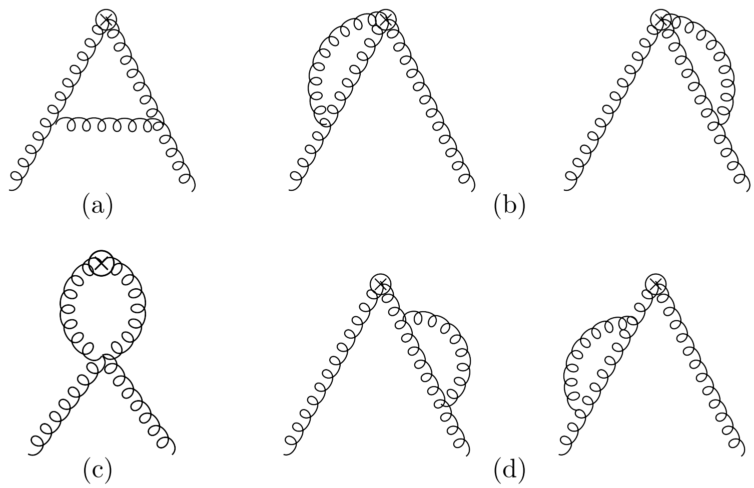

Figure E.3: Diagrams contributing to one-loop gluon matrix element of gluon spin.

where μ is the renormalization scale.

According to Ref. [93], the corresponding IMF (or light-cone) matrix element is

$$\langle p,s \left| \int d^3x (\vec{E}\times\vec{A})^3 \right|_{A^+=0} \left| p,s\right\rangle^{(1)} = \frac{\alpha_s C_F}{4\pi}\left[\frac{3}{\epsilon'_{UV}} - \frac{3}{\epsilon'_{IR}}\right]\langle p,s|\Sigma^3|p,s\rangle^{\rm tree} \ . \quad \text{(E.12)}$$

E.2.2 Matrix element in the gluon state

To extract out the matching factor z_{gg}, we need to calculate the matrix elements of S_g in the free gluon state. The relevant Feynman diagrams are shown in Fig. E.3.

Fig. E.3(a) gives

$$\begin{aligned}
\left\langle k,\lambda \left| S_g^z \right| k,\lambda \right\rangle_a^{(1)} &= \epsilon_\nu^{*a}(k,\lambda) \int \frac{d^4q}{(2\pi)^4} (-g f^{acd}) \\
&\times \left[g^{\nu\lambda'}(2k-q)^{\rho'} - g^{\nu\rho'}(k+q)^{\lambda'} + g^{\rho'\lambda'}(2q-k)^\nu \right] iD_{\rho'\beta}(q) \\
&\times [iq^0(g^{1\alpha}g_\perp^{2\beta} - g^{2\alpha}g_\perp^{1\beta}) - ig^{0\alpha}(q^1 g_\perp^{2\beta} - q^2 g_\perp^{1\beta}) - \alpha \leftrightarrow \beta] \\
&\times iD_{\rho\alpha}(q)(-g f^{bdc}) \\
&\times \left[g^{\mu\lambda}(2k-q)^\rho + g^{\mu\rho}(-k-q)^\lambda + g^{\rho\lambda}(2q-k)^\mu \right] \\
&\times iD_{\lambda\lambda'}(k-q)\ \epsilon_\mu^b\ , \qquad \text{(E.13)}
\end{aligned}$$

where $\epsilon_\mu^a(k,\lambda)$ is the polarization vector of a gluon with color a and polarization λ. The momentum of the gluon is along the z direction, i.e., $k^\mu = (k^0, 0, 0, k^0)$. For physical polarizations, the Lorentz indices μ and ν are restricted to run over 1, 2.

We will encounter the same types of integrals in the calculation of the Feynman diagram in Fig. E.2, but the structure of the integrand is much more complicated in this case. Here we show how to caclulate the most difficult one:

$$I_0 = \int \frac{d^4q}{(2\pi)^4} \frac{\vec{k}^4}{q^2(q-k)^2\vec{q}^2(\vec{q}-\vec{k})^2}\ . \qquad \text{(E.14)}$$

We call this a four-point integral where four means the total power of quadratic terms in the denominator. One can first integrate over q^0 and get

$$\begin{aligned}
I_0 &= \int \frac{d^dq}{(2\pi)^d} \frac{\vec{k}^4}{q^2(q-k)^2\vec{q}^2(\vec{q}-\vec{k})^2} \\
&= \frac{i}{2} \int \frac{d^{d-1}q}{(2\pi)^{d-1}} \frac{\vec{q}\cdot\vec{k}\vec{k}^2}{(\vec{q}^2)^{3/2}(\vec{q}-\vec{k})^2 \left[\vec{q}^2 - \frac{(\vec{q}\cdot\vec{k})^2}{\vec{k}^2} \right]}\ . \qquad \text{(E.15)}
\end{aligned}$$

Then without loss of generality one can choose $\vec{k} = (0, 0, k^3)$ and integrate over q^3 (of dimension 1) and then integrate over $\vec{q}_\perp$ (of dimension $d-2$) to obtain the final result:

$$I_0 = \frac{1}{(4\pi)^{2-\epsilon}} \frac{\sqrt{\pi} 2^{2\epsilon+3}(1+\epsilon)}{(1+2\epsilon)} \frac{\Gamma[\epsilon]\Gamma[-2\epsilon]}{\Gamma[-1/2-2\epsilon]} (\vec{k}^2)^{-\epsilon} . \tag{E.16}$$

For integrals of more than four points, we can use integration by parts and tensor reduction to reduce them into simpler forms including I_0. In this way, we calculate the Feynman diagram in Fig. E.3(a):

$$\begin{aligned}\langle k,\lambda | S_g^z | k,\lambda\rangle_a^{(1)} &= \epsilon_\nu^{*a}(k,\lambda) \left[2ik^0(g^{\mu 1}g^{\nu 2} - g^{\nu 1}g^{\mu 2})\right] \epsilon_\mu^a(k,\lambda) \\ &\quad \times \frac{\alpha_s C_A}{4\pi} \left[\frac{3}{\epsilon_{UV}} + \frac{2}{\epsilon_{IR}^2} + \frac{1}{\epsilon_{IR}} \left(-2\ln\frac{\vec{k}^2}{\pi\mu^2} - 2\gamma_E + 2 \right) \right. \\ &\quad \left. + \left(\ln\frac{\vec{k}^2}{\pi\mu^2} + \gamma_E \right)^2 - 5\left(\ln\frac{\vec{k}^2}{\pi\mu^2} + \gamma_E \right) - \frac{7}{6}\pi^2 + \frac{47}{3} \right] .\end{aligned} \tag{E.17}$$

The Feynman diagrams in Fig. E.3(b) gives

$$\begin{aligned}\langle k,\lambda | S_g^z | k,\lambda\rangle_b^{(1)} &= 2\cdot\frac{1}{2}\epsilon_\nu^{*a}(k,\lambda) \int \frac{d^4k}{(2\pi)^4} (gf^{bcd}) \\ &\quad \times \left(g^{0\nu}g^{1\rho}g_\perp^{2\sigma} - g^{0\nu}g^{1\sigma}g_\perp^{2\rho} + g^{0\rho}g^{1\sigma}g_\perp^{2\nu} \right. \\ &\quad \left. - g^{0\rho}g^{1\nu}g_\perp^{2\sigma} + g^{0\sigma}g^{1\nu}g_\perp^{2\rho} - g^{0\sigma}g^{1\rho}g_\perp^{2\nu} - 1 \leftrightarrow 2 \right) \\ &\quad \times iD_{\rho\alpha}(k-q) iD_{\sigma\beta}(q)(-gf^{adc}) \\ &\quad \times \left[g^{\mu\beta}(k+p)^\alpha + g^{\beta\alpha}(k-2p)^\mu + g^{\alpha\mu}(p-2k)^\beta \right] \epsilon_\mu^a(k,\lambda) ,\end{aligned} \tag{E.18}$$

where the 1/2 is a symmetry factor. The result is

$$\begin{aligned}\left\langle k,\lambda\left|S_g^z\right|k,\lambda\right\rangle_b^{(1)} &= \epsilon_\nu^{*a}(k,\lambda)\left[2ik^0(g^{\mu 1}g^{\nu 2}-g^{\nu 1}g^{\mu 2})\right]\epsilon_\mu^a(k,\lambda)\\ &\quad\times\frac{\alpha_s C_A}{4\pi}\left[-\frac{8}{3}\frac{1}{\epsilon_{UV}}+\frac{8}{3}\left(\ln\frac{\vec{k}^2}{\pi\mu^2}+\gamma_E\right)-\frac{104}{9}\right]\ .\end{aligned} \tag{E.19}$$

Fig. E.3(c) vanishes at the integrand the level because the inserted operator contracts with anti-symmetrized Lorentz indices while the four-gluon vertex contracts with symmetrized ones.

To renormalize the gluonic matrix elements of $\vec{E}_a\times\vec{A}_\perp^a$ we need to know the wavefunction renormalization of the gluon field in the Coulomb gauge. Since the matrix elements are evaluated onshell, we calculated the wavefunction renormalization factor as the residue of the gluon propagator at $k^2 = 0$, where k^μ is the momentum of the gluon.

Because of the noncovariant nature of the Coulomb gauge condition, the gluon self-energy $\Pi^{\mu\nu}(k)$ is highly nontrivial,

$$\Pi^{\mu\nu}(k)\ \neq\ \Pi(k^2)\left(k^2g^{\mu\nu}-k^\mu k^\nu\right)\ . \tag{E.20}$$

Instead, it satisfies the Ward identity that includes the ghost contribution [129]:

$$k^\mu\Pi_{\mu\nu}^{ab}(k)+(k^2g_{\mu\nu}-k_\mu k_\nu)H^{\mu,ab}(k)\ =\ 0\ , \tag{E.21}$$

where $H^{\mu,ab}(k)$ corresponds to the diagram in Fig. E.4.

In our calculation, we evaluated the gluon self-energy as

$$-i\Pi^{ij}(k)\ =\ -i\left[A(k^2,\vec{k}^2)\delta^{ij}+B(k^2,\vec{k}^2)k^ik^j\right]\ , \tag{E.22}$$

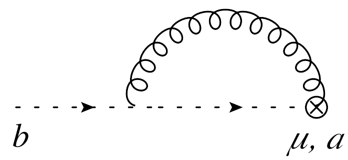

Figure E.4: Ghost-loop needed for the Ward identity.

where A and B depend on k^2 and $\vec{k}^2$. The gluon propagator with its one-loop correction is

$$\begin{aligned} iD^{ij}(k) &= -\frac{i}{k^2}g_\perp^{ij}(k) + \left(-\frac{i}{k^2}g_\perp^{il}(k)\right)(-i)\left[A(k^2,\vec{k}^2)\delta^{lm} + B(k^2,\vec{k}^2)k^l k^m\right] \\ &\quad \times \left(-\frac{i}{k^2}g_\perp^{mj}(k)\right) \\ &= -\frac{i}{k^2}g_\perp^{ij}(k)\left[1 - \frac{A(k^2,\vec{k}^2)}{k^2}\right] \\ &\approx -\frac{ig_\perp^{ij}(k)}{k^2 + A(k^2,\vec{k}^2)} \ . \end{aligned} \tag{E.23}$$

Gauge invariance requires that

$$A(k^2 = 0, \vec{k}^2) \ = \ 0 \ . \tag{E.24}$$

Therefore, the onshell gluon wavefunction renormalization factor

$$Z_A \ = \ \left(1 - \frac{A(k^2,\vec{k}^2)}{k^2}\right)^{-1}\Bigg|_{k^2=0} \ \approx \ 1 + \frac{A(k^2,\vec{k}^2)}{k^2}\Bigg|_{k^2=0} \ = \ 1 + \frac{dA(k^2,\vec{k}^2)}{dk^2}\Bigg|_{k^2=0} \ . \tag{E.25}$$

In the Coulomb gauge, we calculate the gluon self-energy that comes from Fig. E.3(d) and the tad-pole and ghost-loop diagrams. To be noted, the ghost-loop diagrams contributes an energy-divergent integral whose integrand has no dependence on the time component of the loop momentum, e.g.,

$$\int \frac{d^4q}{(2\pi)^4}\frac{1}{\vec{q}^2(\vec{q}+\vec{k})^2} \ . \tag{E.26}$$

Such type of integrals cannot be regularized by the split dimensional regularization we have used. The energy-divergent integrals also exist in Fig. E.3(d) and the tadpole diagram. Nevertheless, in a more generalized version of the split dimensional regularization, Leibbrandt showed that they can be consistently regularized [129]. Meanwhile, it is shown in Refs. [130, 131], and also confirmed by our calculation, that such energy divergences get cancelled among contributions from gluon and ghost loops at the integrand level.

The gluon wavefunction renormalization also receives gauge-invariant contributions from the quark loops, which can be found in standard textbooks. The result for δZ_A is

$$\begin{aligned} \delta Z_A &= \frac{\alpha_s C_A}{4\pi}\left[\frac{1}{\epsilon_{UV}} - \frac{2}{\epsilon_{IR}^2} + \frac{1}{\epsilon_{IR}}\left(2\ln\frac{\vec{k}^2}{\pi\mu^2} + 2\gamma_E - \frac{17}{3}\right) - \left(\ln\frac{\vec{k}^2}{\pi\mu^2} + \gamma_E\right)^2 \right. \\ &\quad \left. + \frac{14}{3}\left(\ln\frac{\vec{k}^2}{\pi\mu^2} + \gamma_E\right) + \frac{7}{6}\pi^2 - \frac{158}{9}\right] + \frac{\alpha_s}{4\pi}\frac{2n_f}{3}\left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}}\right) . \end{aligned} \tag{E.27}$$

Combining the results from Eqs. (E.17), (E.19), and (E.27), we obtain the one-loop onshell gluonic matrix element of S_g as

$$\begin{aligned} \left\langle k,\lambda\left|S_g^z\right|k,\lambda\right\rangle^{(1)} &= \frac{\alpha_s}{4\pi}\left[\frac{4C_A - 2n_f}{3}\frac{1}{\epsilon'_{UV}} - \frac{11C_A - 2n_f}{3}\frac{1}{\epsilon'_{IR}} \right. \\ &\quad \left. + C_A\left(\frac{7}{3}\ln\frac{\vec{k}^2}{\mu^2} + \frac{14}{3}\ln 2 - \frac{121}{9}\right)\right]\left\langle k,\lambda\left|S_g^z\right|k,\lambda\right\rangle^{\text{tree}} . \end{aligned} \tag{E.28}$$

According to Ref. [93], the corresponding IMF (or light-cone) matrix element

is

$$\langle k,\lambda \left| \int d^3x (\vec{E}\times\vec{A})^3 \right|_{A^+=0} \left| k,\lambda\right\rangle^{(1)} = \frac{\alpha_s}{4\pi}\left[\frac{11C_A-2n_f}{3}\frac{1}{\epsilon'_{UV}} - \frac{11C_A-2n_f}{3}\frac{1}{\epsilon'_{IR}}\right] \times \left\langle k,\lambda \left| S^z_g \right| k,\lambda\right\rangle^{\text{tree}} . \tag{E.29}$$

E.3 The quark orbital angular momentum

The Quark OAM operator is

$$L_q = \int d^3x\ \psi^\dagger \vec{x}\times(-i\vec{\nabla}-e\vec{A}_\parallel)\psi\ . \tag{E.30}$$

At tree level, the matrix element of L_q can be evaluated umambiguously in a wave packet

$$|\Psi\rangle = \int \frac{d^3p}{(2\pi)^3}\Phi(p)|p\rangle\ , \tag{E.31}$$

and its matrix element is

$$\langle\Psi|L_q|\Psi\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\Phi^*(p')\Phi(p)\langle p'|\int d^3x\ \psi^\dagger\vec{x}\times(-i\vec{\nabla}-e\vec{A}_\parallel)\psi|p\rangle\ . \tag{E.32}$$

In the Coulomb gauge, $\vec{A}_\parallel = 0$, and the above matrix element reduces to

$$\begin{aligned}\langle\Psi|L_q|\Psi\rangle &= \int \frac{d^3pd^3p'}{(2\pi)^3}\Phi^*(p')\Phi(p)\left(-i\vec{\nabla}_{\vec{p}}\delta^{(3)}(\vec{p}'-\vec{p})\right)\times u^\dagger(p')\ \vec{p}\ u(p) \\ &= \int \frac{d^3p}{(2\pi)^3}\left(i\vec{\nabla}_{\vec{p}}\Phi^*(p')\Phi(p)\right)\Big|_{\vec{p}'=\vec{p}}\times u^\dagger(p)\ \vec{p}\ u(p) \\ &\quad + \int \frac{d^3p}{(2\pi)^3}\Phi^*(p)\Phi(p)u^\dagger(p)\ \vec{p}\times(-i\vec{\nabla}_{\vec{p}})u(p)\ ,\end{aligned} \tag{E.33}$$

where the second equality is obtained through integration by parts. The first term is called a nonlocal contribution, while the second term a local one [132]. The nonlocal term contributes to the OAM only, while the local term contributes to both the quark spin and OAM at higher orders. To ensure that the quark OAM operator is multiplicatively renormalizable, both the local and nonlocal terms should contribute the same to the quark OAM.

E.3.1 Matrix element in the quark state

Let us first calculate the one-loop quark matrix element of the local term,

$$
\begin{aligned}
\langle p,s|L_q|p,s\rangle^{(1)}_{\rm local} &= \int \frac{d^4k}{(2\pi)^4}\bar{u}(p)(-ig\tau^a\gamma^\mu)\frac{i}{\not{p}-\not{k}}\gamma^0 \\
&\quad (\vec{p}-\vec{k})\times(-i\vec{\nabla}_{\vec{p}-\vec{k}})\frac{i}{\not{p}-\not{k}}(-ig\tau^a\gamma^\nu)iD_{\mu\nu}(k)u(p) \\
&= g^2C_F\int \frac{d^4k}{(2\pi)^4}\bar{u}(p)\frac{1}{\not{p}-\not{k}}\gamma^0\frac{1}{\not{p}-\not{k}} \\
&\quad \times(\vec{p}-\vec{k})\times\vec{\gamma}\frac{1}{\not{p}-\not{k}}\gamma^\nu D_{\mu\nu}(k)u(p) \\
&\quad +g^2C_F\int \frac{d^4k}{(2\pi)^4}\bar{u}(p)\frac{1}{\not{p}-\not{k}}\gamma^0\frac{1}{\not{p}-\not{k}}\gamma^\nu \\
&\quad \times D_{\mu\nu}(k)(\vec{p}-\vec{k})\times\vec{\nabla}_{\vec{p}}u(p)\ .
\end{aligned}
\tag{E.34}
$$

One can prove that

$$
\begin{aligned}
\bar{u}(p)\left(\gamma^0\vec{p}-p^0\vec{\gamma}\right)\times\vec{\nabla}_{\vec{p}}u(p) &= \frac{1}{2}\bar{u}(p)\gamma^0\vec{\gamma}\times\not{p}\vec{\nabla}_{\vec{p}}u(p)\ , \\
\not{p}\vec{\nabla}_{\vec{p}}u(p) &= -\left(\frac{\vec{p}}{p^0}\gamma^0-\vec{\gamma}\right)u(p)\ .
\end{aligned}
\tag{E.35}
$$

Using this trick, we get

$$
\begin{aligned}
\langle p,s|L_q|p,s\rangle^{(1)}_{\rm local} \;=\; & \frac{\alpha_s C_F}{4\pi}\left[-\frac{2}{3}\frac{1}{\epsilon_{UV}}+\frac{2}{\epsilon_{IR}}-\frac{4}{3}\left(\ln\frac{\vec{p}^2}{\pi\mu^2}+\gamma_E\right)+\frac{64}{9}\right] \\
& \times u^\dagger(p)\Sigma^3 u(p) \\
& +\frac{\alpha_s C_F}{4\pi}\left[-\frac{1}{3}\frac{1}{\epsilon_{UV}}-\frac{2}{3}\frac{1}{\epsilon_{IR}}+\left(\ln\frac{\vec{p}^2}{\pi\mu^2}+\gamma_E\right)-\frac{17}{3}\right] \\
& \times u^\dagger(p)\Sigma^3 u(p) \\
& +\frac{\alpha_s C_F}{4\pi}\left[\frac{1}{3}\frac{1}{\epsilon_{UV}}+\frac{2}{\epsilon_{IR}^2}+\frac{1}{\epsilon_{IR}}\left(-2\ln\frac{\vec{p}^2}{\pi\mu^2}-\gamma_E+\frac{23}{3}\right)\right. \\
& \left.+\left(\ln\frac{\vec{p}^2}{\pi\mu^2}+\gamma_E\right)^2-8\left(\ln\frac{\vec{p}^2}{\pi\mu^2}+\gamma_E\right)-\frac{7}{6}\pi^2+\frac{100}{3}\right] \\
& \times u^\dagger(p)\vec{p}\times(-i\vec{\nabla}_{\vec{p}})u(p)\ ,
\end{aligned}
\tag{E.36}
$$

where the first line comes from the first term of Eq. (E.34), and the second and third lines come from the second term of Eq. (E.34).

The one-loop quark matrix element of the nonlocal term is

$$
\begin{aligned}
\langle p,s|L_q|p,s\rangle^{(1)}_{\rm nonlocal} \;=\; & \int\frac{d^4k}{(2\pi)^4}\bar{u}(p)(-ig\tau^a\gamma^\mu)\frac{i}{\not{p}-\not{k}}\gamma^0(\vec{p}-\vec{k})\frac{i}{\not{p}-\not{k}} \\
& (-ig\tau^a\gamma^\nu)iD_{\mu\nu}(k)u(p) \\
=\; & \frac{\alpha_s C_F}{4\pi}\left[\frac{1}{3}\frac{1}{\epsilon_{UV}}+\frac{2}{\epsilon_{IR}^2}+\frac{1}{\epsilon_{IR}}\left(-2\ln\frac{\vec{p}^2}{\pi\mu^2}-2\gamma_E+\frac{23}{3}\right)\right. \\
& \left.+\left(\ln\frac{\vec{p}^2}{\pi\mu^2}+\gamma_E\right)^2-8\left(\ln\frac{\vec{p}^2}{\pi\mu^2}+\gamma_E\right)-\frac{7}{6}\pi^2+\frac{100}{3}\right] \\
& \times u^\dagger(p)\ \vec{p}\ u(p)\ ,
\end{aligned}
\tag{E.37}
$$

which is exactly the same as the contribution to the quark OAM from the local term.

The wavefunction renormalization of quarks is highly nontrivial in the Coulomb gauge. To obtain the onshell wavefunction renormalization factor, we employed the conservation of the vector current

$$j^\mu \;=\; \bar\psi\gamma^\mu\psi\ , \tag{E.38}$$

so that the first-order renormalization constant is

$$\begin{aligned}\delta Z_q \;=\; \frac{\alpha_s C_F}{4\pi}\Bigg[&-\frac{1}{\epsilon_{UV}}-\frac{2}{\epsilon_{IR}^2}+\frac{1}{\epsilon_{IR}}\left(2\ln\frac{\vec p^2}{\pi\mu^2}+2\gamma_E-5\right)\\ &-\left(\ln\frac{\vec p^2}{\pi\mu^2}+\gamma_E\right)^2+6\left(\ln\frac{\vec p^2}{\pi\mu^2}+\gamma_E\right)+\frac{7}{6}\pi^2-24\Bigg]\ .\end{aligned} \tag{E.39}$$

Combining the above results in Eqs. (E.36), (E.37), and (E.39), we have

$$\begin{aligned}\langle p,s|L_q|p,s\rangle^{(1)} \;=\;& \frac{\alpha_s C_F}{4\pi}\left[-\frac{1}{\epsilon'_{UV}}+\frac{4}{3}\frac{1}{\epsilon'_{IR}}-\frac{1}{3}\ln\frac{\vec p^2}{\mu^2}-\frac{2\ln 2}{3}+\frac{13}{9}\right]\\ &\times\langle p,s|\Sigma^3|p,s\rangle^{\rm tree}\\ &+\frac{\alpha_s C_F}{4\pi}\left[-\frac{2}{3}\frac{1}{\epsilon'_{UV}}+\frac{8}{3}\frac{1}{\epsilon'_{IR}}-2\ln\frac{\vec p^2}{\mu^2}-4\ln 2+\frac{28}{3}\right]\\ &\times\langle p,s|L_q|p,s\rangle^{\rm tree}\ .\end{aligned} \tag{E.40}$$

According to Ref. [93], the corresponding IMF (or light-cone) matrix element is

$$\begin{aligned}\langle p,s|\int d^3x\,\psi^\dagger\left(\vec x\times\frac{\vec\nabla}{i}\right)^3\psi\Bigg|_{A^+=0}|p,s\rangle^{(1)} \;=\;& \frac{\alpha_s C_F}{4\pi}\left[-\frac{4}{3}\frac{1}{\epsilon'_{UV}}+\frac{4}{3}\frac{1}{\epsilon'_{IR}}\right]\langle p,s|\Sigma^3|p,s\rangle^{\rm tree}\\ &+\frac{\alpha_s C_F}{4\pi}\left[-\frac{8}{3}\frac{1}{\epsilon'_{UV}}+\frac{8}{3}\frac{1}{\epsilon'_{IR}}\right]\langle p,s|L_q^z|p,s\rangle^{\rm tree}\ .\end{aligned} \tag{E.41}$$

E.3.2 Matrix element in the gluon state

Now let us look at the one-loop matrix elements of the quark operator in free gluon states. The Feynman diagrams are shown in Fig. E.5.

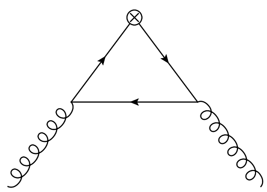

Figure E.5: Diagrams contributing to one-loop gluon matrix element of quark OAM.

Since only quark propagators are involved in the loop, the integrals arising from these diagrams are scaleless and no logarithmic dependence on $\vec{k}^2$ will show up. As a consequence, the result is frame independent and must be the same as that in the IMF limit [93]:

$$\begin{aligned} \langle k,\lambda|L_q|k,\lambda\rangle^{(1)} &= \frac{\alpha_s}{4\pi}\left(\frac{2n_f}{3}\frac{1}{\epsilon'_{UV}} - \frac{2n_f}{3}\frac{1}{\epsilon'_{IR}}\right)\langle k,\lambda|S_g|k,\lambda\rangle^{\text{tree}} \\ &\quad+\frac{\alpha_s}{4\pi}\left(\frac{2n_f}{3}\frac{1}{\epsilon'_{UV}} - \frac{2n_f}{3}\frac{1}{\epsilon'_{IR}}\right)\langle k,\lambda|L_g|k,\lambda\rangle^{\text{tree}} . \end{aligned} \tag{E.42}$$

E.4 The gluon OAM

The gluon OAM operator is

$$L_g = \int d^3x E^{i,a}\vec{x}\times\vec{\nabla}A_\perp^{i,a} . \tag{E.43}$$

The Feynman rule of the gluon orbital angular momentum can also be obtained from the wave packet interpretation.

For the two-gluon vertex,

$$\begin{aligned}
\langle\Psi|L_g|\Psi\rangle &= \sum_\lambda \int \frac{d^3k d^3k'}{(2\pi)^6}\Phi^*(\vec{k}')\Phi(\vec{k}) \\
&\quad \times\langle k',\lambda|(\partial^i A_c^0 - \partial^0 A_c^i - g f^{cde} A_d^0 A_e^i)\vec{x}\times\vec{\nabla} A_\perp^{i,c}|k,\lambda\rangle \\
&= \sum_\lambda \int \frac{d^3k d^3k'}{(2\pi)^6}\Phi^*(\vec{k}')\Phi(\vec{k}) \\
&\quad \times\Big[i\left(-k'^0 g_\perp^{i\mu}(k) g^{i\nu} + k'^i g_\perp^{i\mu}(k) g^{0\nu}\right)\left(\vec{\nabla}_{\vec{k}}\delta^{(3)}(\vec{k}-\vec{k}')\right)\times\vec{k}) \\
&\quad -i\left(-k^0 g_\perp^{i\nu}(k') g^{i\mu} + k^i g_\perp^{i\nu}(k') g^{0\mu}\right)\left(\vec{\nabla}_{\vec{k}'}\delta^{(3)}(\vec{k}-\vec{k}')\right)\times\vec{k}'\Big] \\
&\quad \times\epsilon_\nu^*(k',\lambda)\epsilon_\mu(k,\lambda)\ . \qquad \text{(E.44)}
\end{aligned}$$

For the three-gluon vertex, the Feynman rule is

$$\begin{aligned}
g f^{abc}\Big[&\left(g_\perp^{i\mu}(k_1) g^{0\nu} g^{i\rho} - g_\perp^{i\mu}(k_1) g^{0\rho} g^{i\nu}\right)\left(\vec{\nabla}_{\vec{k}_1}\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\right)\times\vec{k}_1 \\
&+ \left(g_\perp^{i\nu}(k_2) g^{0\rho} g^{i\mu} - g_\perp^{i\nu}(k_2) g^{0\mu} g^{i\rho}\right)\left(\vec{\nabla}_{\vec{k}_2}\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\right)\times\vec{k}_2 \\
&+ \left(g_\perp^{i\rho}(k_3) g^{0\mu} g^{i\nu} - g_\perp^{i\rho}(k_3) g^{0\nu} g^{i\mu}\right)\left(\vec{\nabla}_{\vec{k}_3}\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\right)\times\vec{k}_3\Big]\ .
\end{aligned} \qquad \text{(E.45)}$$

E.4.1 Matrix element in the quark state

The matrix element of the gluon OAM operator in a free quark state has been calculated following similar procedure to that for the quark OAM. The Feynman diagram we consider is the same as Fig. E.2, and is divided into local and nonlocal parts:

$$
\begin{aligned}
&\langle p,s|L_g|p,s\rangle^{(1)}_{\rm local}\\
= \; & -g^2 C_F \bar{u}(p)\int \frac{d^4k}{(2\pi)^4}\frac{\gamma^\beta}{k^4(p-k)^2}\\
&\times\Bigg[\left(g^\perp_{\beta\nu}(k)-\frac{k^2}{\vec{k}^2}n_\beta n_\nu\right)\vec{\nabla}_{\vec{k}}g^{i\mu}_\perp(k)\left(k^0 g^{i\nu}-k^i g^{0\nu}\right)\times\vec{k}g^\perp_{\mu\alpha}(k)\\
&\quad -g^\perp_{\beta\nu}(k)\vec{\nabla}_{\vec{k}}g^{i\nu}_\perp(k)\left(k^0 g^{i\mu}-k^i g^{0\mu}\right)\times\vec{k}\left(g^\perp_{\mu\alpha}(k)-\frac{k^2}{\vec{k}^2}n_\mu n_\alpha\right)\Bigg](\not p-\not k)\gamma^\alpha u(p)\\
&-g^2 C_F \bar{u}(p)\int \frac{d^4k}{(2\pi)^4}\frac{\gamma^\beta}{k^4(p-k)^2}\\
&\times\Bigg[\left(g^\perp_{\beta\nu}(k)-\frac{k^2}{\vec{k}^2}n_\beta n_\nu\right)\left(k^0 g^{i\mu}_\perp(k)g^{i\nu}\right)\vec{\nabla}_{\vec{k}}g^\perp_{\mu\alpha}(k)\times\vec{k}\\
&\quad -\vec{\nabla}_{\vec{k}'}g^\perp_{\beta\nu}(k')\times\vec{k}'\Big|_{k'=k}\left(k^0 g^{i\nu}_\perp(k)g^{i\mu}\right)\left(g^\perp_{\mu\alpha}(k)-\frac{k^2}{\vec{k}^2}n_\mu n_\alpha\right)\Bigg](\not p-\not k)\gamma^\alpha u(p)\\
&-g^2 C_F \bar{u}(p)\int \frac{d^4k}{(2\pi)^4}\frac{\gamma^\beta}{k^4(p-k)^2}\\
&\times\Bigg[\left(g^\perp_{\beta\nu}(k)-\frac{k^2}{\vec{k}^2}n_\beta n_\nu\right)\left(k^0 g^{i\mu}_\perp(k)g^{i\nu}\right)g^\perp_{\mu\alpha}(k)\\
&\quad -g^\perp_{\beta\nu}(k)\left(k^0 g^{i\nu}_\perp(k)g^{i\mu}\right)\left(g^\perp_{\mu\alpha}(k)-\frac{k^2}{\vec{k}^2}n_\mu n_\alpha\right)\Bigg](\not p-\not k)\gamma^\alpha\vec{k}\times\nabla_{\vec{p}}u(p)\;.
\end{aligned}
\tag{E.46}
$$

Using the same trick in Eq. (E.35), we get

$$
\begin{aligned}
\langle p,s|L_g|p,s\rangle^{(1)}_{\rm local} \;=\; & \frac{\alpha_s C_F}{4\pi}\left[-\frac{1}{\epsilon'_{UV}}+\frac{1}{\epsilon'_{IR}}\right]u^\dagger(p,s)\Sigma^3 u(p,s)\\
&+\frac{\alpha_s C_F}{4\pi}\left[\frac{1}{3}\frac{1}{\epsilon'_{UV}}+\frac{2}{3}\frac{1}{\epsilon'_{IR}}-\ln\frac{\vec{p}^2}{\mu^2}-2\ln 2+\frac{17}{3}\right]\\
&\times u^\dagger(p,s)\Sigma^3 u(p,s)\\
&+\frac{\alpha_s C_F}{4\pi}\left[\frac{2}{3}\frac{1}{\epsilon'_{UV}}-\frac{8}{3}\frac{1}{\epsilon'_{IR}}+2\ln\frac{\vec{p}^2}{\mu^2}+4\ln 2-\frac{28}{3}\right]\\
&\times u^\dagger(p)\vec{p}\times(-i\vec{\nabla}_{\vec{p}})u(p)\;,
\end{aligned}
\tag{E.47}
$$

where the first line comes from the first two integrals in Eq. (E.46), and the second and third lines come from the last.

The nonlocal part of the matrix element is

$$\begin{aligned}
\langle p,s|L_g|p,s\rangle^{(1)}_{\rm nonlocal} &= -g^2 C_F \bar{u}(p)\int \frac{d^4k}{(2\pi)^4}\frac{\gamma^\beta}{k^4(p-k)^2} \\
&\quad \times\Bigg[\left(g^\perp_{\beta\nu}(k)-\frac{k^2}{\vec{k}^2}n_\beta n_\nu\right)\left(k^0 g_\perp^{i\mu}(k)g^{i\nu}\right)g^\perp_{\mu\alpha}(k) \\
&\quad -g^\perp_{\beta\nu}(k)\left(k^0 g_\perp^{i\nu}(k)g^{i\mu}\right)\left(g^\perp_{\mu\alpha}(k)-\frac{k^2}{\vec{k}^2}n_\mu n_\alpha\right)\Bigg] \\
&\quad \times(\not{p}-\not{k})\gamma^\alpha\vec{k}\,u(p) \\
&= \frac{\alpha_s C_F}{4\pi}\left[\frac{2}{3}\frac{1}{\epsilon'_{UV}}-\frac{8}{3}\frac{1}{\epsilon'_{IR}}+2\ln\frac{\vec{p}^2}{\mu^2}+4\ln 2-\frac{28}{3}\right] \\
&\quad \times u^\dagger(p)\,\vec{p}\,u(p)\ ,
\end{aligned} \tag{E.48}$$

which is exactly the same as the local contribution to the quark OAM. Therefore, combining the above results, we have

$$\begin{aligned}
\langle p,s|L_g^z|p,s\rangle^{(1)} &= \frac{\alpha_s C_F}{4\pi}\left[-\frac{2}{3}\frac{1}{\epsilon'_{UV}}+\frac{5}{3}\frac{1}{\epsilon'_{IR}}-\ln\frac{\vec{p}^2}{\mu^2}-2\ln 2+\frac{17}{3}\right] \\
&\quad \times\langle p,s|\Sigma^3|p,s\rangle^{\rm tree} \\
&\quad +\frac{\alpha_s C_F}{4\pi}\left[\frac{2}{3}\frac{1}{\epsilon'_{UV}}-\frac{8}{3}\frac{1}{\epsilon'_{IR}}+2\ln\frac{\vec{p}^2}{\mu^2}+4\ln 2-\frac{28}{3}\right] \\
&\quad \times\langle p,s|L_q^z|p,s\rangle^{\rm tree}\ .
\end{aligned} \tag{E.49}$$

According to Ref. [93], the corresponding IMF (or light-cone) matrix element

is

$$\langle p,s|\int d^3x\vec{E}^{i,a}\vec{x}\times\vec{\nabla}\vec{A}^{i,a}\bigg|_{A^+=0}|p,s\rangle^{(1)} = \frac{\alpha_s C_F}{4\pi}\left[-\frac{5}{3}\frac{1}{\epsilon'_{UV}}+\frac{5}{3}\frac{1}{\epsilon'_{IR}}\right]\langle p,s|\Sigma^3|p,s\rangle^{\rm tree} + \frac{\alpha_s C_F}{4\pi}\left[\frac{8}{3}\frac{1}{\epsilon'_{UV}}-\frac{8}{3}\frac{1}{\epsilon'_{IR}}\right]\langle p,s|L_q|p,s\rangle^{\rm tree}\,. \tag{E.50}$$

E.4.2 Matrix element in the gluon state

The matrix element of the gluon OAM operator in a free gluon state is also divided into local and nonlocal parts. The same Feynman diagrams as shown in Fig. E.3 are calculated. For the diagram in Fig. (E.3(a)),

$$\begin{aligned}
\langle k,\lambda|L_g|k,\lambda\rangle^{(1),a}_{\rm local} &= 2g^2C_A\epsilon^{*b}_\nu(k,\lambda)\int\frac{d^4q}{(2\pi)^4}\frac{\epsilon_{lm}q^m}{q^4(k-q)^2} \\
&\times\left[g^{\nu\lambda'}(2k-q)^{\rho'}+g^{\nu\rho'}(-k-q)^{\lambda'}+g^{\lambda'\rho'}(2q-k)^\nu\right] \\
&\times\left(g_{\perp,\rho'\beta}(q)-\frac{n_{\rho'}n_\beta q^2}{\vec{q}^2}\right)\frac{\partial}{\partial q^l}\left\{(q'^0g^{i\beta}-q'^ig^{0\beta})g^{i\alpha}_\perp(q)g_{\perp,\rho\alpha}(q)\right. \\
&\times\left.\left[g^{\mu\lambda}(2k-q')^\rho+g^{\mu\rho}(-k-q)^\lambda+g^{\rho\lambda}(q+q'-k)^\mu\right]\right\}\Big|_{q'=q} \\
&\times\left(g_{\perp,\lambda\lambda'}(q)-\frac{n_\lambda n_{\lambda'}q^2}{\vec{q}^2}\right)\epsilon^a_\mu(k,\lambda) \\
&+2g^2C_A\epsilon^{*b}_\nu(k,\lambda)\int\frac{d^4q}{(2\pi)^4}\frac{\epsilon_{lm}q^m}{q^4(k-q)^2} \\
&\times\left[g^{\nu\lambda'}(2k-q)^{\rho'}+g^{\nu\rho'}(-k-q)^{\lambda'}+g^{\lambda'\rho'}(2q-k)^\nu\right] \\
&\times\left(g_{\perp,\rho'\beta}(q)-\frac{n_{\rho'}n_\beta q^2}{\vec{q}^2}\right)(q^0g^{i\beta}-q^ig^{0\beta})g^{i\alpha}_\perp(q)g_{\perp,\rho\alpha}(q) \\
&\times\frac{\partial}{\partial k^l}\left\{\left[g^{\mu\lambda}(k+k'-q)^\rho+g^{\mu\rho}(-k-q)^\lambda+g^{\rho\lambda}(2q-k')^\mu\right]\right. \\
&\times\left.\left(g_{\perp,\lambda\lambda'}(q)-\frac{n_\lambda n_{\lambda'}q^2}{\vec{q}^2}\right)\epsilon^a_\mu(k,\lambda)\right\}\;,
\end{aligned} \tag{E.51}$$

where $l, m = 1, 2$, and $\epsilon_{12} = -\epsilon_{21} = 1$.

$$\begin{aligned}
\langle k,\lambda|L_g|k,\lambda\rangle^{(1),a}_{\text{nonlocal}} &= 2g^2 C_A \epsilon^{*b}_{\nu}(k,\lambda) \int \frac{d^4q}{(2\pi)^4} \frac{\epsilon_{lm} q^m}{q^4 (k-q)^2} \\
&\times \left[g^{\nu\lambda'}(2k-q)^{\rho'} + g^{\nu\rho'}(-k-q)^{\lambda'} + g^{\lambda'\rho'}(2q-k)^{\nu} \right] \\
&\times \left(g_{\perp,\rho'\beta}(q) - \frac{n_{\rho'} n_{\beta} q^2}{\vec{q}^2} \right) (q^0 g^{i\beta} - q^i g^{0\beta}) g^{i\alpha}_{\perp}(q) g_{\perp,\rho\alpha}(q) \\
&\times \left[g^{\mu\lambda}(2k-q)^{\rho} + g^{\mu\rho}(-k-q)^{\lambda} + g^{\rho\lambda}(2q-k)^{\mu} \right] \\
&\times \left(g_{\perp,\lambda\lambda'}(q) - \frac{n_{\lambda} n_{\lambda'} q^2}{\vec{q}^2} \right) \epsilon^a_{\mu}(k,\lambda) \ .
\end{aligned} \tag{E.52}$$

By applying the identities

$$k^{\mu}\epsilon_{\mu}(k,\lambda) = 0, \qquad \delta^{\mu l}\epsilon_{\mu}(k,\lambda) + p^{\mu}\partial_l \epsilon_{\mu}(k,\lambda) = 0 \ , \tag{E.53}$$

we obtain

$$\begin{aligned}
\langle k,\lambda|L_g|k,\lambda\rangle^{(1),a}_{\text{local}} &= \epsilon^{*a}_{\nu}(k,\lambda) \left[2ik^0 (g^{\mu 1} g^{\nu 2} - g^{\nu 1} g^{\mu 2}) \right] \epsilon^a_{\mu}(k,\lambda) \\
&\times \frac{\alpha_s C_A}{4\pi} \left[-\frac{16}{15}\frac{1}{\epsilon_{UV}} + \frac{11}{3}\frac{1}{\epsilon_{IR}} - \frac{13}{5}\left(\ln \frac{\vec{k}^2}{\pi\mu^2} + \gamma_E \right) + \frac{357}{25} \right] \\
&+ \epsilon^{*a}_{\nu}(k,\lambda) \left(-2ik^0 g^{\mu i} g^{\nu i} \right) \vec{k} \times \vec{\nabla}_{\vec{k}} \epsilon^a_{\mu}(k,\lambda) \\
&\times \frac{\alpha_s C_A}{4\pi} \left[\frac{7}{5}\frac{1}{\epsilon_{UV}} + \frac{2}{\epsilon^2_{IR}} - \frac{1}{\epsilon_{IR}} \left(2\ln \frac{\vec{k}^2}{\pi\mu^2} + 2\gamma_E - \frac{17}{3} \right) \right. \\
&\left. + \left(\ln \frac{\vec{k}^2}{\pi\mu^2} + \gamma_E \right)^2 - \frac{106}{15} \left(\ln \frac{\vec{k}^2}{\pi\mu^2} + \gamma_E \right) - \frac{7}{6}\pi^2 + \frac{6362}{225} \right] \ .
\end{aligned} \tag{E.54}$$

$$
\begin{aligned}
\langle k,\lambda|L_g|k,\lambda\rangle^{(1),a}_{\text{nonlocal}} &= \epsilon^{*a}_{\nu}(k,\lambda)\left(-2ik^0 g^{\mu i} g^{\nu i}\right)\vec{k}\ \epsilon^{a}_{\mu}(k,\lambda) \\
&\times\frac{\alpha_s C_A}{4\pi}\left[\frac{7}{5}\frac{1}{\epsilon_{UV}}+\frac{2}{\epsilon^2_{IR}}-\frac{1}{\epsilon_{IR}}\left(2\ln\frac{\vec{k}^2}{\pi\mu^2}+2\gamma_E-\frac{17}{3}\right)\right. \\
&\quad\left.+\left(\ln\frac{\vec{k}^2}{\pi\mu^2}+\gamma_E\right)^2-\frac{106}{15}\left(\ln\frac{\vec{k}^2}{\pi\mu^2}+\gamma_E\right)-\frac{7}{6}\pi^2+\frac{6362}{225}\right]\ .
\end{aligned} \tag{E.55}
$$

For the diagram in Fig. (E.3(b)),

$$
\begin{aligned}
\langle k,\lambda|L_g|k,\lambda\rangle^{(1),b}_{\text{local}} &= 2\cdot\frac{1}{2}g^2 C_A\left(\frac{\partial}{\partial k^l}\epsilon^{*b}_{\nu}(k,\lambda)\right)\int\frac{d^4q}{(2\pi)^4}\frac{\epsilon_{lm}}{q^2(k-q)^2} \\
&\times\left[g^{i\nu}_{\perp}(k)(g^{0\alpha'}g^{i\beta'}-g^{0\beta'}g^{i\alpha'})k^m-2g^{i\alpha'}_{\perp}(k)(g^{0\beta'}g^{i\nu}-g^{0\nu}g^{i\beta'})q^m\right] \\
&\times\left(g_{\perp,\alpha\alpha'}(q)-\frac{n_\alpha n_{\alpha'}q^2}{\vec{q}^2}\right)\left(g_{\perp,\beta\beta'}(k-q)-\frac{n_\beta n_{\beta'}(k-q)^2}{(\vec{k}-\vec{q})^2}\right) \\
&\times\left[g^{\mu\beta}(2p-k)^\alpha+g^{\beta\alpha}(2k-p)^\mu-g^{\alpha\mu}(p+k)^\beta\right]\epsilon^{a}_{\mu}(k,\lambda)\ .
\end{aligned} \tag{E.56}
$$

$$
\begin{aligned}
\langle k,\lambda|L_g|k,\lambda\rangle^{(1),b}_{\text{nonlocal}} &= 2\cdot\frac{1}{2}g^2 C_A\epsilon^{*b}_{\nu}(k,\lambda)\int\frac{d^4q}{(2\pi)^4}\frac{\epsilon_{lm}}{q^2(k-q)^2} \\
&\times\left[g^{i\nu}_{\perp}(k)(g^{0\alpha'}g^{i\beta'}-g^{0\beta'}g^{i\alpha'})k^m-2g^{i\alpha'}_{\perp}(k)(g^{0\beta'}g^{i\nu}-g^{0\nu}g^{i\beta'})q^m\right] \\
&\times\left(g_{\perp,\alpha\alpha'}(q)-\frac{n_\alpha n_{\alpha'}q^2}{\vec{q}^2}\right)\left(g_{\perp,\beta\beta'}(k-q)-\frac{n_\beta n_{\beta'}(k-q)^2}{(\vec{k}-\vec{q})^2}\right) \\
&\times\left[g^{\mu\beta}(2p-k)^\alpha+g^{\beta\alpha}(2k-p)^\mu-g^{\alpha\mu}(p+k)^\beta\right]\epsilon^{a}_{\mu}(k,\lambda)\ .
\end{aligned} \tag{E.57}
$$

By applying the same identities in Eq. (E.53), we obtain

$$\begin{aligned}
\langle k,\lambda|L_g|k,\lambda\rangle^{(1),b}_{\text{local}} &= \epsilon^{*a}_\nu(k,\lambda)\left[2ik^0(g^{\mu1}g^{\nu2}-g^{\nu1}g^{\mu2})\right]\epsilon^a_\mu(k,\lambda) \\
&\quad\times\frac{\alpha_s C_A}{4\pi}\left[-\frac{4}{15}\frac{1}{\epsilon_{UV}}+\frac{4}{15}\left(\ln\frac{\vec{k}^2}{\pi\mu^2}+\gamma_E\right)-\frac{188}{225}\right] \\
&\quad+\epsilon^{*a}_\nu(k,\lambda)\left(-2ik^0g^{\mu i}g^{\nu i}\right)\vec{k}\times\vec{\nabla}_{\vec{k}}\epsilon^a_\mu(k,\lambda) \\
&\quad\times\frac{\alpha_s C_A}{4\pi}\left[-\frac{12}{5}\frac{1}{\epsilon_{UV}}-\frac{12}{5}\left(\ln\frac{\vec{k}^2}{\pi\mu^2}+\gamma_E\right)-\frac{268}{25}\right]\ .
\end{aligned} \tag{E.58}$$

$$\begin{aligned}
\langle k,\lambda|L_g|k,\lambda\rangle^{(1),b}_{\text{nonlocal}} &= \epsilon^{*a}_\nu(k,\lambda)\left(-2ik^0g^{\mu i}g^{\nu i}\right)\vec{k}\ \epsilon^a_\mu(k,\lambda) \\
&\quad\times\frac{\alpha_s C_A}{4\pi}\left[-\frac{12}{5}\frac{1}{\epsilon_{UV}}-\frac{12}{5}\left(\ln\frac{\vec{k}^2}{\pi\mu^2}+\gamma_E\right)-\frac{268}{25}\right]\ .
\end{aligned} \tag{E.59}$$

After including the self-energy corrections, we obtain

$$\begin{aligned}
\langle k,\lambda|L_g|k,\lambda\rangle^{(1)} &= \frac{\alpha_s C_A}{4\pi}\left[-\frac{4}{3}\frac{1}{\epsilon'_{UV}}+\frac{11}{3}\frac{1}{\epsilon'_{IR}}-\frac{7}{3}\ln\frac{\vec{k}^2}{\mu^2}-\frac{14\ln 2}{3}+\frac{121}{9}\right] \\
&\quad\times\langle k,\lambda|S_g|k,\lambda\rangle^{\text{tree}} \\
&\quad+\frac{\alpha_s}{4\pi}\left[-\frac{2n_f}{3}\frac{1}{\epsilon'_{UV}}+\frac{2n_f}{3}\frac{1}{\epsilon'_{IR}}\right]\langle k,\lambda|L_g|k,\lambda\rangle^{\text{tree}}\ .
\end{aligned} \tag{E.60}$$

According to Ref. [93], the corresponding IMF (or light-cone) matrix element is

$$\begin{aligned}
&\langle p,s|\int d^3x\vec{E}^{i,a}\vec{x}\times\vec{\nabla}\vec{A}^{i,a}\Big|_{A^+=0}|p,s\rangle^{(1)} \\
&= \frac{\alpha_s C_A}{4\pi}\left[-\frac{11}{3}\frac{1}{\epsilon'_{UV}}+\frac{11}{3}\frac{1}{\epsilon'_{IR}}\right]\langle k,\lambda|S_g|k,\lambda\rangle^{\text{tree}} \\
&\quad+\frac{\alpha_s}{4\pi}\left[-\frac{2n_f}{3}\frac{1}{\epsilon'_{UV}}+\frac{2n_f}{3}\frac{1}{\epsilon'_{IR}}\right]\langle k,\lambda|L_g|k,\lambda\rangle^{\text{tree}}\ .
\end{aligned} \tag{E.61}$$

One can check that all the one-loop matrix elements add up to zero, and thus verify that the total angular momentum of QCD is conserved and needs no renormalization.

Appendix F

Factorization formulas for the polarized and transversity distributions

Here we present the results for the polarized and transversity distributions. For the polarized quark distribution, the quasi distribution $\Delta\tilde{q}^{(1)}(x)$ can be obtained by replacing γ^z with $\gamma^z\gamma^5$ in Eq. (1). The one-loop result then reads

$$\Delta\tilde{q}^{(1)}(x) = \frac{\alpha_s C_F}{2\pi}\begin{cases} \frac{1+x^2}{1-x}\ln\frac{x(\Lambda(x)-xP^z)}{(x-1)(\Lambda(1-x)+P^z(1-x))} + 1 - \frac{xP^z}{\Lambda(x)} \\ +\frac{x\Lambda(1-x)+(1-x)\Lambda(x)}{(1-x)^2P^z}\ , & x > 1\ , \\ \\ \frac{1+x^2}{1-x}\ln\frac{P_z^2}{m^2} + \frac{1+x^2}{1-x}\ln\frac{4x(\Lambda(x)-xP^z)}{(1-x)(\Lambda(1-x)+(1-x)P^z)} \\ -\frac{4}{1-x} + 2x + 3 - \frac{xP^z}{\Lambda(x)} + \frac{x\Lambda(1-x)+(1-x)\Lambda(x)}{(1-x)^2P^z}\ , & 0 < x < 1\ . \\ \\ \frac{1+x^2}{1-x}\ln\frac{(x-1)(\Lambda(x)-xP^z)}{x(\Lambda(1-x)+(1-x)P^z)} - 1 - \frac{xP^z}{\Lambda(x)} \\ +\frac{x\Lambda(1-x)+(1-x)\Lambda(x)}{(1-x)^2P^z}\ , & x < 0\ . \end{cases} \tag{F.1}$$

Taking the limit $\Lambda \to \infty$ yields

$$\Delta\tilde{q}^{(1)}(x) = \frac{\alpha_s C_F}{2\pi} \begin{cases} \frac{1+x^2}{1-x}\ln\frac{x}{x-1} + 1 + \frac{\Lambda}{(1-x)^2 P^z}\ , & x > 1\ , \\ \\ \frac{1+x^2}{1-x}\ln\frac{P_z^2}{m^2} + \frac{1+x^2}{1-x}\ln\frac{4x}{1-x} \\ -\frac{4}{1-x} + 2x + 3 + \frac{\Lambda}{(1-x)^2 P^z}\ , & 0 < x < 1\ , \\ \\ \frac{1+x^2}{1-x}\ln\frac{x-1}{x} - 1 + \frac{\Lambda}{(1-x)^2 P^z}\ , & x < 0\ . \end{cases} \tag{F.2}$$

The result for the light-cone distribution is again given by taking $P^z \to \infty$,

$$\Delta\tilde{q}^{(1)}(x) = \frac{\alpha_S C_F}{2\pi} \begin{cases} 0\ , & x > 1 \text{ or } x < 0\ , \\ \frac{1+x^2}{1-x}\ln\frac{IMF^2}{m^2} - \frac{1+x^2}{1-x}\ln(1-x)^2 - \frac{2}{1-x} + 2x\ , & 0 < x < 1\ . \end{cases} \tag{F.3}$$

Note that as in the unpolarized case, the collinear singularity in the quasi polarized quark distribution is exactly the same as in the light-cone distribution.

Similarly, for the transversity distribution, the quasi distribution $\delta\tilde{q}^{(1)}(x)$ is obtained by replacing γ^z with $\gamma^z\gamma^\perp\gamma^5$ in Eq. (1). The one-loop result is

$$\delta\tilde{q}^{(1)}(x) = \frac{\alpha_s C_F}{2\pi} \begin{cases} \frac{2x}{1-x}\ln\frac{x(\Lambda(x)-xP^z)}{(x-1)(\Lambda(1-x)+P^z(1-x))} + \frac{x\Lambda(1-x)+(1-x)\Lambda(x)}{(1-x)^2 P^z}\ , & x > 1\ , \\ \\ \frac{2x}{1-x}\ln\frac{P_z^2}{m^2} + \frac{2x}{1-x}\ln\frac{4x(\Lambda(x)-xP^z)}{(1-x)(\Lambda(1-x)+(1-x)P^z)} \\ -\frac{4x}{1-x} + \frac{x\Lambda(1-x)+(1-x)\Lambda(x)}{(1-x)^2 P^z}\ , & 0 < x < 1\ , \\ \\ \frac{2x}{1-x}\ln\frac{(x-1)(\Lambda(x)-xP^z)}{x(\Lambda(1-x)+(1-x)P^z)} + \frac{x\Lambda(1-x)+(1-x)\Lambda(x)}{(1-x)^2 P^z}\ , & x < 0\ . \end{cases} \tag{F.4}$$

The limit $\Lambda \to \infty$ gives

$$\delta\tilde{q}^{(1)}(x) = \frac{\alpha_s C_F}{2\pi} \begin{cases} \frac{2x}{1-x} \ln \frac{x}{x-1} + \frac{\Lambda}{(1-x)^2 P^z} \ , & x > 1 \ , \\ \frac{2x}{1-x} \ln \frac{P_z^2}{m^2} + \frac{2x}{1-x} \ln \frac{4x}{1-x} - \frac{4x}{1-x} + \frac{\Lambda}{(1-x)^2 P^z} \ , & 0 < x < 1 \ , \\ \frac{2x}{1-x} \ln \frac{x-1}{x} + \frac{\Lambda}{(1-x)^2 P^z} \ , & x < 0 \ , \end{cases} \tag{F.5}$$

and the result in the IMF is

$$\delta q^{(1)}(x) = \frac{\alpha_s C_F}{2\pi} \begin{cases} 0 \ , & x > 1 \text{ or } x < 0 \ , \\ \frac{2x}{1-x} \ln \frac{\mu^2}{m^2} - \frac{2x}{1-x} \ln (1-x)^2 - \frac{2x}{1-x} \ , & 0 < x < 1 \ . \end{cases} \tag{F.6}$$

One can construct similar matching conditions as in Eq. (4.34) for the polarized and transversity distributions. We just list the results for the matching factors here, noting that the quark self-energy is the same. For the polarized quark distribution, one has for $\xi > 1$,

$$\Delta Z^{(1)}(\xi)/C_F = \left(\frac{1+\xi^2}{1-\xi}\right) \ln \frac{\xi}{\xi - 1} + 1 + \frac{1}{(1-\xi)^2} \frac{\Lambda}{P^z} \ , \tag{F.7}$$

while for $0 < \xi < 1$,

$$\Delta Z^{(1)}(\xi)/C_F = \left(\frac{1+\xi^2}{1-\xi}\right) \ln \frac{P_z^2}{\mu^2} + \left(\frac{1+\xi^2}{1-\xi}\right) \ln \left[4\xi(1-\xi)\right] - \frac{2}{1-\xi} + 3 + \frac{\Lambda}{(1-\xi)^2 P^z} \ , \tag{F.8}$$

and for $\xi < 0$,

$$\Delta Z^{(1)}(\xi)/C_F = \left(\frac{1+\xi^2}{1-\xi}\right) \ln \frac{\xi - 1}{\xi} - 1 + \frac{\Lambda}{(1-\xi)^2 P^z} \ . \tag{F.9}$$

The linearly divergent term is the same as in the unpolarized case.

Finally, in the factorization formula for transversity distribution, one has the matching factor for $\xi > 1$,

$$\delta Z^{(1)}(\xi)/C_F = \left(\frac{2\xi}{1-\xi}\right) \ln \frac{\xi}{\xi - 1} + \frac{1}{(1-\xi)^2} \frac{\Lambda}{P^z} \ , \tag{F.10}$$

whereas for $0 < \xi < 1$,

$$\delta Z^{(1)}(\xi)/C_F = \left(\frac{2\xi}{1-\xi}\right)\ln\frac{P_z^2}{\mu^2} + \left(\frac{2\xi}{1-\xi}\right)\ln\left[4\xi(1-\xi)\right] - \frac{2\xi}{1-\xi} + \frac{\Lambda}{(1-\xi)^2 P^z}\ , \tag{F.11}$$

and for $\xi < 0$,

$$\delta Z^{(1)}(\xi)/C_F = \left(\frac{2\xi}{1-\xi}\right)\ln\frac{\xi-1}{\xi} + \frac{\Lambda}{(1-\xi)^2 P^z}\ . \tag{F.12}$$

One again has an linearly divergent contribution. Near $\xi = 1$, one needs to include an extra contribution from self energy just like in the unpolarized case.

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