ABSTRACT

Title of dissertation: LIFT OF THE TRIVIAL REPRESENTATION TO A NONLINEAR COVER

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Let G be the real points of a simply laced, simply connected complex Lie group, and let $\widetilde{G}$ be the nonlinear two-fold cover of G. We discuss a set of small genuine representations of $\widetilde{G}$, denoted by Lift($\mathbb{C}$), which can be obtained from the trivial representation of G by a lifting operator. The representations in Lift($\mathbb{C}$) can be characterized by the following properties: (a) the infinitesimal character is $\rho/2$; (b) they have maximal τ-invariant; (c) they have a particular associated variety $\mathcal{O}$. When G is split and of type A_{n-1} or D_n, we have a full description for Lift($\mathbb{C}$). In this case, these representations are parametrized by pairs (central character, real form of $\mathcal{O}$), and exhaust all small representations with infinitesimal character $\rho/2$ and maximal τ-invariant.

LIFT OF THE TRIVIAL REPRESENTATION TO A NONLINEAR COVER

by

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Table of Contents

Chapter 1: Introduction

Assume that $G_{\mathbb{C}}$ is a simply connected, semisimple, simply laced complex Lie group, and G is a real form of $G_{\mathbb{C}}$ with nontrivial fundamental group. Then G has a nontrivial two-fold cover $\widetilde{G}$, which is not a matrix group (see [6], Proposition 3.6). For example, this holds when G is a split group, or $G = SU(p,q)$, $Spin(p,q)$, and most real forms of the exceptional groups. In fact, most real forms of $G_{\mathbb{C}}$ have a nontrivial two-fold cover (see [2]). The purpose of this paper is to discuss some small genuine representations of $\widetilde{G}$ and their properties. By a genuine representation we mean that a representation of $\widetilde{G}$ which does not factor through G.

In Chapter 2, we first introduce some basic invariants, such as infinitesimal character, τ-invariant, and associated variety, which are used to classify representations. These notions are quite general, and are defined on more general real reductive groups G, which can be linear or nonlinear, and are not necessarily simply laced. For each type, we fix an infinitesimal character λ. If G is simply laced or of type G_2 and F_4, λ is chosen to be $\rho/2$, where ρ is half of the sum of the positive roots. For type B_n and C_n, λ is defined as in [3], and is listed in Table 2.1. Then we define a class of representations of $\widetilde{G}$ denoted

$$\textstyle\prod_{\lambda}^{s}(\widetilde{G}) = \{\widetilde{\pi} \,|\, \widetilde{\pi} \in \widehat{\widetilde{G}}_{adm,\lambda},\ \widetilde{\pi} \text{ is genuine and has maximal } \tau\text{-invariant}\},$$

where $\widehat{\widetilde{G}}_{adm,\lambda}$ is the set of irreducible admissible representations of $\widetilde{G}$ with infinitesimal character λ. Here the superscript s stands for *small* in the sense that the representations in this set have maximal τ-invariant. There is a unique complex nilpotent orbit $\mathcal{O}$ which is the complex associated variety of every $\widetilde{\pi}$ from $\prod_{\lambda}^{s}(\widetilde{G})$. We calculate this orbit $\mathcal{O}$ explicitly for all types and list them in Table 2.1.

Denote

$$\prod_{\rho/2}^{\mathcal{O}}(\widetilde{G}) = \{\widetilde{\pi} \,|\, \widetilde{\pi} \in \widehat{\widetilde{G}}_{adm,\lambda}, \widetilde{\pi} \text{ is genuine and the associated variety of } \widetilde{\pi} \text{ is } \mathcal{O}\}.$$

Then we have

Theorem 1.0.1. $\prod_{\rho/2}^{s}(\widetilde{G}) = \prod_{\rho/2}^{\mathcal{O}}(\widetilde{G})$.

The proof of the theorem is based on truncated induction of representations of Weyl groups and the Springer correspondence.

This set of representations $\prod_{\rho/2}^{s}(\widetilde{G}) = \prod_{\rho/2}^{\mathcal{O}}(\widetilde{G})$ plays a significant role throughout this paper. First of all, we can attach to each $\widetilde{\pi} \in \prod_{\rho/2}^{s}(\widetilde{G})$ a pair $(\chi_{\widetilde{\pi}}, \mathcal{O}_{\widetilde{\pi}})$, where $\chi_{\widetilde{\pi}}$ is the central character of $\widetilde{\pi}$ and $\mathcal{O}_{\widetilde{\pi}}$ is the real associated variety of $\widetilde{\pi}$. Here, $\mathcal{O}_{\widetilde{\pi}}$ is one of the real forms of $\mathcal{O}$, and in Chapter 3, we will see that there are not many real groups which have nonempty intersection with $\mathcal{O}$ and the number of real forms of $\mathcal{O}$ is tiny as well. The notions of real associated variety and genuine central character will be discussed in more detail in Chapters 3 and 4.

In Chapter 5, we restrict our attention to simply laced split groups. For split groups, there is a well-understood family of representations, called the Shimura representations (see [3]). Starting with these, we construct other genuine representations in $\prod_{\rho/2}^{s}(\widetilde{G})$. There are standard ways to get new representations from old ones: the theory of cross actions and Cayley transforms. In our setting these are non-standard, because they involve half-integral roots. It is possible to start with a Shimura representation, and apply some cross actions and Cayley transforms to it, to obtain other representations in $\prod^s_{\rho/2}(\widetilde{G})$. The conditions which need to be satisfied are very rigid, and we get a small number of representations in $\prod^s_{\rho/2}(\widetilde{G})$. Let $\prod_{R_D}(\widetilde{G})$ denote the set of representations obtained this way. The map $\widetilde{\pi} \in \prod_{R_D}(\widetilde{G}) \to (\chi_{\widetilde{\pi}}, \mathcal{O}_{\widetilde{\pi}})$ leads to a bijection with the pairs

$$\{(\chi, \mathcal{O}_{\mathbb{R}}) \,|\, \chi \text{ is a genuine central character of } \widetilde{G}, \mathcal{O}_{\mathbb{R}} \text{ is a real form of } \mathcal{O}\}.$$

In the last part of Chapter 5, furthermore, by counting the elements in $\prod^s_{\rho/2}(\widetilde{G})$ using a Weyl group calculation, we show that every representation in $\prod^s_{\rho/2}(\widetilde{G})$ is produced this way for type A_{n-1} and D_n and hence we have a bijection $\prod^s_{\rho/2}(\widetilde{G}) \leftrightarrow \{(\chi, \mathcal{O}_{\mathbb{R}})\}$ for type A_{n-1} and D_n. We conjecture this is true for type E.

In Chapter 7, the key tool, the lift operator, comes in. A basic tool in representation theory of linear groups is endoscopic transfer, or lifting. This idea was extended to nonlinear two-fold covers of real groups later on by many people. In [4], a lifting operator, denoted by $\mathrm{Lift}_G^{\widetilde{G}}$, is defined on the level of global characters of representations. It takes stable representations of G to 0 or virtual genuine representations of $\widetilde{G}$. (By a stable representation we mean its global character is invariant under conjugation of $G_{\mathbb{C}}$.) Hence for every stable representation π of G, let $\mathrm{Lift}(\pi)$ denote the finite set of all irreducible genuine representations of $\widetilde{G}$ occurring in $\mathrm{Lift}_G^{\widetilde{G}}(\pi)$. There is a complete discussion of $\mathrm{Lift}(\pi)$ for one-dimensional representation π of $GL(n, \mathbb{R})$, which can be found in [5]. For example, when $G = GL(n, \mathbb{R})$,

$\mathrm{Lift}(\mathbb{C}) = T_n$, where $\mathbb{C}$ is the trivial representation, and T_n is the genuine unipotent representation coming from minimal parabolic subgroup and containing the pin representation as its lowest K-type. What we attempt to do is a similar analysis for other simply laced groups. Because of the setting in the beginning, the only one-dimensional representation of G is $\mathbb{C}$, the trivial representation. What we expect is that $\mathrm{Lift}(\mathbb{C})$ should give an interesting class of unitary representations, and the goal is to study these representations and their characters. The following theorem describes the properties that a representation occurring in $\mathrm{Lift}(\mathbb{C})$ should possess. More precisely, the irreducible representations in $\mathrm{Lift}(\mathbb{C})$ are the small representations that we discuss in the previous chapters.

Theorem 1.0.2. *The setting is as above and assume G is simply laced. Then $Lift(\mathbb{C}) \subseteq \prod^{s}_{\rho/2}(\widetilde{G}) = \prod^{\mathcal{O}}_{\rho/2}(\widetilde{G})$.*

Then a natural question arises – Is $\mathrm{Lift}(\mathbb{C}) = \prod^{s}_{\rho/2}(\widetilde{G})$? This conjecture is true in some cases, for example, when G is split.

Theorem 1.0.3. *The setting is as above and assume G is simply laced and split. Then $Lift(\mathbb{C}) = \prod^{s}_{\rho/2}(\widetilde{G}) \leftrightarrow \{(\chi, \mathcal{O}_{\mathbb{R}})\}$*

The proof is based on case-by-case calculation. At the end, we obtain a small number of representations in $\mathrm{Lift}(\mathbb{C})$ and they are very concrete, in terms of their lowest K-types, Langlands parameters, associated varieties, and so on.

Chapter 2: Some Small Representations

In this chapter, we will introduce a category of representations which plays an important role when we are talking about the lifting of one-dimensional representations. Before doing that, some notions are needed.

2.1 Invariants of a representation

Let's get started with the setting. Let G be a connected real Lie group, and suppose that the complexified Lie algebra of G, denoted $\mathfrak{g}$, is reductive. Here G is allowed to be nonlinear, which means it cannot be embedded into any $GL(n, \mathbb{C})$ (see [4], [6] for example). We fix a Cartan involution θ of G and let $K = G^\theta$ be the corresponding maximal compact subgroup. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. Let $\triangle = \triangle(\mathfrak{g}, \mathfrak{h})$ be the root system and W be the Weyl group of $\mathfrak{g}$.

Let $\mathcal{HC}(\mathfrak{g}, K)$ be the set of Harish-Chandra modules and let $\widehat{G}_{adm}$ denote the set of equivalence classes of irreducible admissible representations of G. Then $\widehat{G}_{adm}$ can be viewed as a subset of $\mathcal{HC}(\mathfrak{g}, K)$ by sending an irreducible admissible representation $\pi \in \widehat{G}_{adm}$ to its space V_π of K-finite vectors and then the latter can be regarded as an irreducible $(\mathfrak{g}, K)$-module. What we are going to do is to attach certain invariants to the representations in $\widehat{G}_{adm}$.

The most basic invariant is the *infinitesimal character* of a representation. The center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ can be identified with the W-invariant polynomials on $\mathfrak{h}$ via the Harish-Chandra homomorphism $\zeta : Z(\mathfrak{g}) \to U(\mathfrak{h})^W$. In this way, we have a map $infchar : \widehat{G}_{adm} \to \mathfrak{h}^*/W$, and the infinitesimal character of $\pi \in \widehat{G}_{adm}$ is identified with a weight $\lambda \in \mathfrak{h}^*$. For $\lambda \in \mathfrak{h}^*/W$, we denote by

$$\widehat{G}_{adm,\lambda} = \{\pi \in \widehat{G}_{adm} | infchar(\pi) = \lambda\}$$

and refer to the representations in $\widehat{G}_{adm,\lambda}$ as the irreducible admissible representations with infinitesimal character λ. Similarly, let $\mathcal{HC}(\mathfrak{g}, K)_\lambda$ denote the set of Harish-Chandra modules with infinitesimal character λ.

2.1.1 Primitive Ideals

Many invariants to be considered are actually invariants attached to the primitive ideals in $U(\mathfrak{g})$, though there are some invariants attached directly to an irreducible Harish-Chandra module. Thus let's first define

Definition 2.1.1. Let V be an irreducible $U(\mathfrak{g})$-module. The annihilator of V in $U(\mathfrak{g})$ is

$$\mathrm{Ann}(V) := \{X \in U(\mathfrak{g}) | Xv = 0, \forall v \in V\},$$

which is a two-sided ideal in $U((g)$. It is called be the *primitive ideal* in $U(\mathfrak{g})$ attached to V.

If two $U(\mathfrak{g})$-modules have the same primitive ideals, then their infinitesimal characters are the same, and hence it makes sense to talk about the primitive ideals with infinitesimal character λ. We set $\mathrm{Prim}(\mathfrak{g})_\lambda$ to be the set of primitive ideals in $U(\mathfrak{g})$ with infinitesimal character λ. For any $\pi \in \widehat{G}_{adm}$, let V_π be the corresponding Hairish-Chandra module and let $I_\pi :=\mathrm{Ann}(V_\pi)$, and hence we have a map $\widehat{G}_{adm} \to \mathrm{Prim}(\mathfrak{g})_\lambda$ sending π to I_π. This map is several-to-one in general.

2.1.2 Associated Variety and Gelfand-Kirillov Dimension

Given a finitely generated $\mathfrak{g}$-module V. Let $U_n(\mathfrak{g}) \subseteq U(\mathfrak{g})$ be the subspace of $U(\mathfrak{g})$ generated by the monomial of the form $X_1 \cdots X_m$ with $m \leqslant n$ and $X_i \in \mathfrak{g}$. There is a good filtration (see Section 4 in [8]) of V compatible with the graded action of $U(\mathfrak{g})$, i.e. $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V$ and $U_n(\mathfrak{g})V_i \subseteq V_{i+n}$ for all i, n. Then $\mathrm{gr}(V) = \bigoplus_{n>0} V_n/V_{n-1}$ is a finitely generated module for the associated graded algebra of $U(\mathfrak{g})$, namely the symmetric algebra $S(\mathfrak{g})$ by Poincaré-Birkhoff-Witt theorem. So one can define the *associated variety* of V, denoted $\mathrm{AV}(V)$, to be the support of the $S(\mathfrak{g})$-module $\mathrm{gr}(V)$ in $\mathfrak{g}^*$.

Moreover, let $\varphi_V(n) = \sum_{q \leqslant n} \dim_{\mathbb{C}} V_q$, which is finite since V is finitely generated. By a theorem of Hilbert and Serre, there is a polynomial $\overline{\varphi}_V(n)$, of degree at most $\dim \mathfrak{g}$, such that $\varphi_V(n) = \overline{\varphi}_V(n)$. (The proof can be found in [23]). Therefore, the integer $\deg(\overline{\varphi}_V(n))$ is defined to be the *Gelfand-Kirillov dimension* of V, denoted by $\mathrm{GKdim}(V)$.

An important lemma is stated below.

Lemma 2.1.2. *$AV(V \otimes F)$ $=AV(V)$ and $GK\mathrm{dim}(V \otimes F)$ $=GK\mathrm{dim}(V)$ for any finite-dimensional $\mathfrak{g}$-module F.*

Proof. Choose a good filtration $\{V_i\}$ on V, then we obtain a good filtration $\{V_i \otimes F\}$ on $V \otimes F$. With these filtrations, $\mathrm{gr}(V \otimes F)$ as a $S(\mathfrak{g})$ is a sum of copies of $\mathrm{gr}(V)$. Hence the lemma follows. $\square$

Now suppose $\pi \in \widehat{G}_{adm}$ and I_π is the primitive ideal attached to π, which can be regarded as a left $U(\mathfrak{g})$-module, and hence we define $\mathrm{AV}(I_\pi)$ and $\mathrm{GKdim}(I_\pi)$, $\mathrm{GKdim}(\pi)$ in usual sense, whereas $\mathrm{AV}(\pi)$ will be defined upon a K-invariant filtration, and we won't talk about this until Chapter 3. By Kostant's theory of harmonics, $\mathrm{AV}(I_\pi)$ consists of nilpotent elements in $\mathfrak{g}^*$, and hence is a union of finite number of closures of nilpotent coadjoint orbits. In fact, it's a single orbit. Let's record some remarkable facts as follows.

Theorem 2.1.3. *(1) (Borho, Brylinski, see [8]) There exists a unique (complex) nilpotent coadjoint orbit $\mathcal{O}$ such that $AV(I_\pi) = \overline{\mathcal{O}}$.*
(2) (See [9]) $2GK\mathrm{dim}(\pi)$ $=GK\mathrm{dim}(I_\pi)$ $=$ $\mathrm{dim}_{\mathbb{C}} \overline{\mathcal{O}}$, where $\overline{\mathcal{O}} = AV(I_\pi)$ is obtained from (1).

2.1.3 τ-invariant

Given $I \in \mathrm{Prim}(\mathfrak{g})_\lambda$. Put $\triangle(\lambda) = \{\alpha \in \triangle | <\lambda, \alpha^\vee> \in \mathbb{Z}\}$, the integral root system for λ, and let W_λ denote the Weyl group for $\triangle(\lambda)$. Choose $\triangle^+(\lambda) \subseteq \triangle(\lambda)$ a positive system making λ dominant. Write $\prod(\lambda) \subseteq \triangle^+(\lambda)$ for the set of simple roots. There is the Borho-Jantzen-Duflo τ-invariant attached to I, which is a subset of $\prod(\lambda)$ (see [22], [24]), denoted $\tau(I)$.

Since $G_{\mathbb{C}}$ is simply connected, we have an alternative definition for τ-invariant. Let $\pi \in \mathcal{HC}(\mathfrak{g}, K)_\lambda$ and F_γ be the finite-dimensional representation of G with highest weight γ. Also let $\triangle(F_\gamma)$ denote the set of all weights of F_γ. Consider the Zuckerman translation functor $\psi_\lambda^{\lambda+\gamma}(\pi) = P_{\lambda+\gamma}(\pi \otimes F_\gamma)$,where $P_{\lambda+\gamma}$ by definition is the projection on the representations with infinitesimal character $\lambda + \gamma$, and hence $\psi_\lambda^{\lambda+\gamma}(\pi)$ is a functor that projects $\pi \otimes F_\gamma$ on representations with infinitesimal character $\lambda + \gamma$. Let $\alpha \in \prod(\lambda)$, and λ_α be singular with respect to α and $\lambda - \lambda_\alpha$ is a sum of roots. Define $\psi_\alpha(\pi) := \psi_\lambda^{\lambda_\alpha}(\pi)$ be the translation functor of π to the α-wall. Then we define

$$\tau(\pi) = \{\alpha \in \prod(\lambda) |\, \psi_\alpha(\pi) = 0\}.$$

It turns out that τ-invariant is a measure of size of π: the bigger the τ-invariant, the smaller the representation.

Definition 2.1.4. We say that π has maximal τ-invariant if $\tau(\pi) = \prod(\lambda)$, or equivalently, $\psi_\alpha(\pi) = 0$ for all $\alpha \in \prod(\lambda)$.

Lemma 2.1.5. *Let F be a finite dimensional representation. Then $\psi_\alpha(F) = 0$ for every root α and hence F has maximal τ-invariant.*

Proof. Note that the infinitesimal character of every finite dimensional representation is regular.

Assume the setting in the Lemma. We have $\psi_\alpha(F) = P_{\lambda'}(F \otimes F') = 0$, where λ' is singular for α and F' is a finite dimensional representation, since $F \otimes$

F' is a virtual finite dimensional representation and each constituent has regular infinitesimal character. $\square$

Definition 2.1.6. We call a representation *small* if it has maximal τ-invariant.

The Gelfand-Kirillov dimension of an irreducible representation is a measure of the growth of K-types. Here is the proposition connecting these two measures.

Proposition 2.1.7. *([22]) Let $\pi \in \widehat{G}_{adm,\lambda}$. If I_π has max τ-invariant, then*

$$GK\dim(\pi) = |\triangle^+| - |\triangle^+(\lambda)|$$

2.1.4 Weyl Group Representations

There are some details of Weyl group representations that can be found in various places, for instance, [11], [17], and [20]. We recall some of the useful facts as follws.

In [15], Joseph has attached to $I \in$Prim$(\mathfrak{g})_\lambda$ a representation $\sigma_I \in \widehat{W_\lambda}$. In fact, the map from $I \in \text{Prim}(\mathfrak{g})_\lambda$ to σ_I is surjective onto the set of special representations of W_λ (see [11] for definition of a special Weyl group representation).

On the other hand, Springer provides a method for producing a representation of W from a nilpotent orbit $\mathcal{O}$, which is the well-known Springer correspondence. We write sp$(\mathcal{O})$ for the irreducible representation of W attached to $\mathcal{O}$. There is an algorithm to calculate the $sp(\mathcal{O})$ if given $\mathcal{O}$ by use of symbols (see [17]). Note that the map $\mathcal{O} \to \text{sp}(\mathcal{O})$ is injective, but not surjective usually.

Let W' be any subgroup of W generated by reflections. There is an operation called t*runcated induction* $j_{W'}^W$, taking irreducible representations of W' to those of

W.

Fact. $j^W_{W'} : \widehat{W'} \to \widehat{W}$ is injective.

The following proposition summarizes and connects all concepts stated above.

Proposition 2.1.8. *Let $\pi \in \widehat{G}_{adm,\lambda}$, $I = I_\pi$, W_λ be the integral Weyl group for λ. Then there is a unique nilpotent orbit $\mathcal{O}$ such that $\sigma = sp(\mathcal{O})$. Furthermore, this $\mathcal{O}$ is dense in $AV(I)$, that is, $AV(I) = \overline{\mathcal{O}}$. Thus, we have a commutative diagram:*

$$\begin{tikzcd} \mathcal{O} \arrow[r, "sp"'] & sp(\mathcal{O}) \\ I \arrow[u, "AV"] \arrow[r, "\sigma"'] & \sigma_I \arrow[u, "j"'] \end{tikzcd}$$

(The left vertical arrow in the diagram means $AV(I) = \overline{\mathcal{O}}$.)

2.2 Certain Properties to Characterize Small Representations of $\widetilde{G}$

In this section we assume that G is a real form of a simply connected, semisimple complex Lie group, and $\widetilde{G}$ is the nonlinear two-fold cover of G. First, we identify the kernel of the covering map $p : \widetilde{G} \to G$ with ± 1 and write $\widetilde{H}$ for the inverse image in $\widetilde{G}$ of a subgroup H of G. We define

Definition 2.2.1. A representation $\widetilde{\pi}$ of $\widetilde{H}$ is called genuine if $\widetilde{\pi}(-1) = -I$. If $\widetilde{\pi}$ is irreducible, then $\widetilde{\pi}$ is genuine if and only if $\widetilde{\pi}$ does not factor through H.

We focus on the genuine representations with a particular infinitesimal character λ. If G is simply laced or of type G_2 and F_4, λ is chosen to be $\rho/2$, where ρ

is half sum of the positive roots. For type B_n and C_n, λ is defined as in [3], and is listed in Table 2.1. We are interested in a special category of representations with certain properties, defined as follows.

Denote

$$\prod_\lambda^s(\widetilde{G}) = \{\widetilde{\pi} \,|\, \widetilde{\pi} \in \widehat{\widetilde{G}}_{adm,\lambda}, \, \widetilde{\pi} \text{ is genuine and has maximal } \tau\text{-invariant}\}$$

The following is the key Lemma.

Lemma 2.2.2. *There is a unique complex nilpotent orbit $\mathcal{O}$ such that $AV(I_{\widetilde{\pi}}) = \overline{\mathcal{O}}$ for every $\widetilde{\pi} \in \prod_\lambda^s(\widetilde{G})$. This $\mathcal{O}$ can be computed explicitly (case by case) and $\dim(\mathcal{O}) = 2GK\dim(\widetilde{\pi}) = 2(|\triangle^+| - |\triangle^+(\lambda)|)$, where $\triangle$ and $\triangle(\lambda)$ are the root system and integral root system, respectively.*

Proof. Let $\pi \in \prod_\lambda^s(\widetilde{G})$. Since $\widetilde{\pi}$ has maximal τ-invariant, $\sigma_{I_{\widetilde{\pi}}} = sgn_{W_\lambda}$, the sign representation of the integral Weyl group for λ. Then the truncated induction takes $sgn_{W_{\rho/2}}$ to a special representation of W, denoted $j(sgn) = j_{W_\lambda}^W(sgn)$, since sgn_{W_λ} is a special representation of $W_{\rho/2}$. Hence $j(sgn)$ defines a nilpotent orbit $\mathcal{O}$ of $\mathfrak{g}$ through the Springer Correspondence, i.e. $sp(\mathcal{O}) = j(sgn)$, and this $\mathcal{O}$ is dense in the associated variety of $I_{\widetilde{\pi}}$, which means $\mathrm{AV}(I_{\widetilde{\pi}}) = \overline{\mathcal{O}}$. The uniqueness of this $\mathcal{O}$ follows from either Theorem 2.1.3 (1) or the injectivity of the Springer correspondence.

From [22], for a representation at infinitesimal character λ with maximal τ-invariant, $\mathrm{GKdim}(\widetilde{\pi}) = |\triangle^+| - |\triangle^+(\lambda)|$, and hence $\dim \mathcal{O} = 2(|\triangle^+| - |\triangle^+(\lambda)|)$ by Theorem 2.1.3 (2). For exceptional groups, there is a unique complex nilpotent orbit of this dimension (see [12]), so it is exactly the one that we are looking for.

For classical types, there is an algorithm to calculate $j(sgn)$ and the corresponding $\mathcal{O}$ explicitly via the Springer correspondence. The parametrization sets of nilpotent orbits are partitions of n for type A_{n-1}, and are partitions of $2n(2n+1$, resp.) which even part occur with even (odd, resp.) multiplicity for type B_n and D_n (C_n, resp.) (see [11] and [12]). All of the nilpotent orbits and the corresponding Weyl group representations are listed in Table 2.1. $\square$

Because of this lemma, let $\mathcal{O}$ denote the complex nilpotent orbit such that $\mathrm{AV}(I_{\widetilde{\pi}}) = \overline{\mathcal{O}}$ for $\widetilde{\pi} \in \prod_{\lambda}^{\mathcal{O}}(\widetilde{G})$, and define

$$\prod\nolimits_{\rho/2}^{\mathcal{O}}(\widetilde{G}) = \{\widetilde{\pi} \,|\, \widetilde{\pi} \in \widehat{\widetilde{G}}_{adm,\lambda}, \widetilde{\pi} \text{ is genuine and } \mathrm{AV}(I_{\widetilde{\pi}}) = \overline{\mathcal{O}}\}$$

Then here is the main theorem of this chapter.

Theorem 2.2.3. $\prod_{\lambda}^{s}(\widetilde{G}) = \prod_{\lambda}^{\mathcal{O}}(\widetilde{G})$

Proof. It is clear that $\prod_{\lambda}^{s}(\widetilde{G}) \subseteq \prod_{\lambda}^{\mathcal{O}}(\widetilde{G})$ due to Lemma 2.2.2. Conversely, given a representation $\widetilde{\pi} \in \prod_{\lambda}^{\mathcal{O}}(\widetilde{G})$, we need to show that $\widetilde{\pi}$ has maximal τ-invariant, that is, to show that $\sigma_{I_{\widetilde{\pi}}} = sgn_{W_\lambda}$. This simply follows from the injectivity of the truncated induction. $\square$

Table 2.1: $\mathcal{O}$ and its corresponding Weyl group representation (using parameterizations in [11] and [12])

Type	n	$\|\triangle^+\|$	λ	$\triangle(\lambda)$	$\|\triangle^+(\lambda)\|$	$\dim\mathcal{O}$	$\mathcal{O}$	$j(sgn_{W_\lambda})$
A_{n-1}	$2m$	$\frac{n(n-1)}{2}$	$\frac{1}{4}(n, n-3, \cdots, -n+3, -n+1)$	$A_{m-1}\times A_{m-1}$	$\frac{n}{2}(\frac{n}{2}-1)$	$\frac{n^2}{2}$	$[2^m]$	$[2^m]$
	$2m+1$	$\frac{n(n-1)}{2}$	$\frac{1}{4}(n, n-3, \cdots, -n+3, -n+1)$	$A_{m-1}\times A_{m-1}$	$(\frac{n-1}{2})^2$	$\frac{n^2-1}{2}$	$[2^m\,1]$	$[2^m\,1]$
B_n	$2m$	n^2	$\frac{1}{2}(n, n-1, \cdots, 1)$	$B_m\times B_m$	$\frac{n^2}{2}$	n^2	$[2^n\,1]$	$(\phi; [2^m])$
	$2m+1$	n^2	$\frac{1}{2}(n, n-1, \cdots, 1)$	$B_{m+1}\times B_m$	$\frac{n^2+1}{2}$	n^2-1	$[2^{n-1}\,1^3]$	$(\phi; [2^m\,1])$
C_n		n^2	$\frac{1}{2}(2n-1, 2n-3, \cdots, 1)$	D_n	n^2-n	$2n$	$[2\,1^{2n-2}]$	$([1^n]; \phi)$
D_n	$2m$	n^2-n	$\frac{1}{2}(n-1, \cdots, 1, 0)$	$D_m\times D_m$	$\frac{1}{2}n(n-2)$	n^2	$[3\,2^{n-2}\,1]$	$\{\phi; [2^m]\}$
	$2m+1$	n^2-n	$\frac{1}{2}(n-1, \cdots, 1, 0)$	$D_{m+1}\times D_m$	$\frac{(n-1)^2}{2}$	n^2-1	$[3\,2^{n-3}\,1^3]$	$\{\phi; [2^m\,1]\}$
E_6		36	$\rho/2$	$A_1\times A_5$	16	40	$3A_1$	$\phi_{15,16}$
E_7		63	$\rho/2$	A_7	28	70	$4A_1$	$\phi_{15,28}$
E_8		120	$\rho/2$	D_8	56	128	$4A_1$	$\phi_{50,56}$
F_4		24	$(4, 3/2, 1, 1/2)^*$	B_4	16	16	A_1	$\phi_{2,16}''$
G_2		6	$\rho/2$	$A_1\times A_1$	2	8	$\widetilde{A_1}$	$\phi_{2,2}$

Chapter 3: Real Associated Variety

In the previous chapter, given $\pi \in \widehat{G}_{adm}$, we defined its complex associated variety $\mathrm{AV}(I_\pi)$. Now we want to attach nilpotent orbits directly to π. Notice that these notions are quite general and they can be defined linear and nonlinear groups.

Suppose (π, V) is the given finitely-generated $(\mathfrak{g}, K)$-module. As in Section 2.1.2, suppose $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V$ is a good filtration, and furthermore suppose this is a $K_{\mathbb{C}}$-invariant filtration since V is also a $K_{\mathbb{C}}$-module. Hence we have that $\mathrm{AV}(\pi) = \mathrm{AV}(V)$ is a closed subvariety of $(\mathfrak{g}/\mathfrak{k})^*$. Since V is also a $K_{\mathbb{C}}$-module, $\mathrm{AV}(\pi)$ is actually a $K_{\mathbb{C}}$-invariant subset of $(\mathfrak{g}/\mathfrak{k})^*$. Similarly, $\mathrm{AV}(\pi)$ consists of nilpotent elements, say, $\mathrm{AV}(\pi) \subseteq \mathcal{N}(\mathfrak{g}/\mathfrak{k})^* := \mathcal{N}(\mathfrak{g}^*) \cap (\mathfrak{g}/\mathfrak{k})^*$, where $\mathcal{N}(\mathfrak{g}^*)$ denotes the nilpotent cone of $\mathfrak{g}^*$. By a theorem of Kostant-Rallis, there are finitely many K orbits on $\mathcal{N}(\mathfrak{g}/\mathfrak{k})^*$, and hence we may write

$$\mathrm{AV}(\pi) = \overline{\mathcal{O}_1^{K_{\mathbb{C}}}} \cup \cdots \cup \overline{\mathcal{O}_j^{K_{\mathbb{C}}}},$$

for orbits $\mathcal{O}_i^{K_{\mathbb{C}}}$ of $K_{\mathbb{C}}$ on $\mathcal{N}(\mathfrak{g}/\mathfrak{k})^*$.

The next result of Vogan relates the complex associated variety and real associated variety.

Theorem 3.0.4. *(see [21], for example) Suppose $\pi \in \widehat{G}_{adm}$. Write*

$$AV(\pi) = \overline{\mathcal{O}_1^{K_{\mathbb{C}}}} \cup \cdots \cup \overline{\mathcal{O}_j^{K_{\mathbb{C}}}}, \text{ and } AV(I_\pi) = \overline{\mathcal{O}}.$$

Then each $\mathcal{O}_i^{K_{\mathbb{C}}}$ *is a Lagrangian submanifold of the canonical symplectic structure of* $\mathcal{O}$*. In particular, for each* i*, we have*

$$G \cdot \mathcal{O}_i^{K_{\mathbb{C}}} = \mathcal{O} \text{ and } GK\dim(\pi) = \dim(\mathcal{O}_i^{K_{\mathbb{C}}})$$

Next we introduce the Sekiguchi correspondence (see [12], chapter 9, for example).

Theorem 3.0.5. *(Sekiguchi) There is a natural one-to-one correspondence between nilpotent* G*-orbits in* $\mathfrak{g}_{\mathbb{R}}$ *and nilpotent* $K_{\mathbb{C}}$*-orbits in* $(\mathfrak{g}/\mathfrak{k})$*.*

Thus, by the Sekiguchi correspondence, $\mathrm{AV}(\pi)$ can be viewed as $\overline{\mathcal{O}_1} \cup \cdots \overline{\mathcal{O}_j}$, where each $\mathcal{O}_i$ is a G-orbit in $\mathfrak{g}_{\mathbb{R}}$ corresponding to $\mathcal{O}_i^{K_{\mathbb{C}}}$ via the Sekiguchi correspondence. Moreover, if $\mathrm{AV}(I_\pi) = \overline{\mathcal{O}}$, then we have $G_{\mathbb{C}} \cdot \mathcal{O}_i = \mathcal{O}$, and hence we say that each $\mathcal{O}_i$ is a real form of $\mathcal{O}$. Equivalently, we say $\{\mathcal{O}_i\}_{i=1}^l$ is the set of real forms of $\mathcal{O}$ if $\mathcal{O} \cap \mathfrak{g}_{\mathbb{R}} = \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_l$.

Resuming the setting of G and $\widetilde{G}$ in Section 2.2, recall that we defined a set of representations $\prod_\lambda^s(\widetilde{G})$, and the complex associated variety of each representation in this set is the closure of a particular $\mathcal{O}$ (see Table 2.1). In Tables 3.1, 3.3, 3.5, 3.7, we list all real groups G such that $\mathcal{O} \cap \mathfrak{g}_{\mathbb{R}}$ is nonempty, as well as the real forms of $\mathcal{O}$ with respect to each G. For classical groups, we parametrize the real nilpotent orbits by Young diagrams or signed Young diagrams, with or without a Roman numeral (see [12]). For exceptional groups, we also refer to [12] for the parameterization of real nilpotent ortbits. The number of real orbits is also listed in the tables, denoted $\#\mathcal{O}_i$.

Remark 3.0.6. It can be observed from the tables that there are not many real groups which have nonempty intersection with $\mathcal{O}$. More precisely, if G is not listed in Table 3.1 to 3.7, then $\mathcal{O} \cap \mathfrak{g}_{\mathbb{R}} = \phi$.

We have the following proposition saying that we can attach to each small representation defined in Section 2.2 a single real nilpotent orbit.

Proposition 3.0.7. *We resume the setting and notations in Section 2.2. Suppose $G_{\mathbb{C}}$ is a simply connected, semisimple complex Lie group, G is a real form of $G_{\mathbb{C}}$ and $\widetilde{G}$ is the nontrivial two-fold cover of G. For each $\widetilde{\pi} \in \prod_{\lambda}^{s}(\widetilde{G}) = \prod_{\lambda}^{\mathcal{O}}(\widetilde{G})$, there is a unique real nilpotent orbit $\mathcal{O}_{\widetilde{\pi}}$ such that $AV(\widetilde{\pi}) = \overline{\mathcal{O}_{\widetilde{\pi}}}$. This $\mathcal{O}_{\widetilde{\pi}}$ is one of the real forms of $\mathcal{O}$.*

Proof. By a result of Vogan (see [25]), if $\mathcal{O}_l$ is a real orbit of maximal dimension in $\mathrm{AV}(\widetilde{\pi})$, and the complement of $\mathcal{O}_l$ has codimension at least two in $\overline{\mathcal{O}_i}$. Then $\mathrm{AV}(\widetilde{\pi}) = \overline{\mathcal{O}_l}$. Since $\dim_{\mathbb{R}} \mathcal{O}_i = \dim_{\mathbb{C}} \mathcal{O}$ for each real form $\mathcal{O}_i$ of $\mathcal{O}$, we just need to pick a complex nilpotent orbit $\mathcal{O}'$, which is one step down smaller than $\mathcal{O}$, and see if the difference of $\dim \mathcal{O}$ and $\dim \mathcal{O}'$ is at least 2. This case by case check is shown in the following table (see [12] for the parameterization of nilpotent orbits).

Type	n	$\mathcal{O}$	$\dim \mathcal{O}$	$\mathcal{O}'$	$\dim \mathcal{O}'$	$\operatorname{codim}_{\overline{\mathcal{O}}} \overline{\mathcal{O}'}$
A_{n-1}	$2m$	$[2^m]$	$\frac{n^2}{2}$	$[2^{m-1}\,1^2]$	$2m^2-2$	2
A_{n-1}	$2m+1$	$[2^n\,1]$	$\frac{n^2-1}{2}$	$[2^{m-1}\,1^3]$	$2m^2+2m-4$	4
B_n	$2m$	$[2^m\,1]$	n^2	$[2^{n-1}\,1^5]$	n^2-4	4
B_n	$2m+1$	$[2^{n-1}\,1^3]$	n^2-1	$[2^{n-3}\,1^7]$	n^2-9	8
C_n		$[2\,1^{2n-2}]$	$2n$	$[1^{2n}]$	0	$2n$
D_n $(n \geqslant 4)$	$2m$	$[3\,2^{n-2}\,1]$	n^2	$[3\,2^{n-4}\,1^5]$	n^2-4	4
D_4	$2m+1$	$[3\,2^{n-3}\,1^3]$	n^2-1	$[3\,2^{n-5}\,1^7]$	n^2-9	8
E_6		$3A_1$	40	$2A_1$	32	8
E_7		$4A_1$	70	A_2	66	4
E_8		$4A_1$	128	A_2	114	14
F_4		A_1	16	0	0	16
G_2		$\widetilde{A_1}$	8	A_1	6	2

$\square$

Table 3.1: Type A_{n-1}, $\mathfrak{g} = \mathfrak{sl}_n$

Type	$\mathfrak{g}$	n	$\mathcal{O}$	inner class	$G = G_{\mathbb{R}}$	$\mathcal{O} \cap \mathfrak{g}_{\mathbb{R}}$	$\#\mathcal{O}_i$
A_{n-1}	$\mathfrak{sl}_n$	$2m$	$[2^m]$	unequal	$SL(n,\mathbb{R})$ (split)	$\begin{array}{\|c\|c\|}\hline \; & \; \\ \hline \; & \; \\ \hline \; & \; \\ \hline \; & \; \\ \hline \end{array}$ (I,II)	2
		$2m+1$	$[2^m\,1]$	unequal	$SL(n,\mathbb{R})$ (split)	$\begin{array}{\|c\|c\|}\hline \; & \; \\ \hline \; & \; \\ \hline \; & \; \\ \hline \; & \\ \cline{1-1} \end{array}$	1
		$2m$	$[2^m]$	equal	$SU(m,m)$ (quasisplit)	$\begin{array}{\|c\|c\|}\hline + & - \\ \hline + & - \\ \hline + & - \\ \hline + & - \\ \hline \end{array}$	1
		$2m+1$	$[2^m\,1]$	equal	$SU(m+1,m)$ (quasisplit)	$\begin{array}{\|c\|c\|}\hline + & - \\ \hline + & - \\ \hline + & - \\ \hline \pm & \\ \cline{1-1} \end{array}$	2

Table 3.3: Type B_n and Type C_n

Type	$\mathfrak{g}$	n	$\mathcal{O}$	inner class	$G = G_{\mathbb{R}}$	$\mathcal{O} \cap \mathfrak{g}_{\mathbb{R}}$	$\#\mathcal{O}_i$
B_n	$\mathfrak{so}_{2n+1}$	$2m$	$[2^n\, 1]$	equal	$Spin(n+1, n)$ (split)	$\begin{smallmatrix} + & - \\ + & - \\ + & - \\ + & - \\ + & \end{smallmatrix}$ (I, II)	2
		$2m+1$	$[2^{n-1}\, 1^3]$	equal	$Spin(n+1, n)$ (split)	$\begin{smallmatrix} + & - \\ + & - \\ + & \\ + & \\ - & \end{smallmatrix}$	1
		$2m+1$	$[2^{n-1}\, 1^3]$	equal	$Spin(n+2, n-1)$	$\begin{smallmatrix} + & - \\ + & - \\ + & \\ + & \\ + & \end{smallmatrix}$	1
C_n	$\mathfrak{sp}_{2n}$		$[2\, 1^{2n-2}]$	equal	$Sp(2n, \mathbb{R})$ (split)	$\begin{smallmatrix} \pm & \mp \\ + & \\ + & \\ + & \\ + & \end{smallmatrix}$	2
				equal	$Sp(2p, 2q)$	$\begin{smallmatrix} + & - \\ + & \\ + & \\ - & \\ - & \end{smallmatrix}$	1

Table 3.5: Type D_n

Type	$\mathfrak{g}$	n	$\mathcal{O}$	inner class	$G = G_{\mathbb{R}}$	$\mathcal{O} \cap \mathfrak{g}_{\mathbb{R}}$	$\#\mathcal{O}_i$
D_n	$\mathfrak{so}_{2n}$	$2m$	$[3\,2^{n-1}\,1]$	equal	$Spin(n,n)$ (split)	$\begin{array}{ccc} \pm & \mp & \pm \\ + & - & \\ + & - & \\ + & - & \\ + & - & \\ \mp & & \end{array}$ (I, II)	4
		$2m+1$	$[3\,2^{n-3}\,1^3]$	unequal	$Spin(n,n)$ (split)	$\begin{array}{ccc} \pm & \mp & \pm \\ + & - & \\ + & - & \\ \pm & & \\ \mp & & \\ \mp & & \end{array}$	2
		$2m$	$[3\,2^{n-1}\,1]$	unequal	$Spin(n{+}1,n{-}1)$ (quasisplit)	$\begin{array}{ccc} + & - & + \\ + & - & \\ + & - & \\ + & & \end{array}$	1
		$2m+1$	$[3\,2^{n-3}\,1^3]$	equal	$Spin(n{+}1,n{-}1)$ (quasisplit)	$\begin{array}{ccc} + & - & + \\ + & - & \\ + & - & \\ + & & \\ + & & \\ - & & \end{array}$	1
		$2m+1$	$[3\,2^{n-3}\,1^3]$	unequal	$Spin(n{+}2,n{-}2)$	$\begin{array}{ccc} + & - & + \\ + & - & \\ + & - & \\ + & & \\ + & & \\ + & & \end{array}$	1

Table 3.7: Type E_6, E_7, E_8, F_4, G_2

Type	$\mathcal{O}$	inner class	$G = G_{\mathbb{R}}$	$\mathcal{O} \cap \mathfrak{g}_{\mathbb{R}}$	$\#\mathcal{O}_i$
E_6	$3A_1$	equal	$E_6(A_1 \times A_5)$ (quasisplit)	$\#4, \#5$	2
		unequal	$E_6(C_4)$ (split)	$\#3$	1
E_7	$4A_1$	equal	$E_7(A_7)$ (split)	$\#8, \#9$	2
E_8	$4A_1$	equal	$E_8(D_8)$(split)	$\#6$	1
F_4	A_1	equal	$F_4(B_4)$ (split)	$\#2$	1
G_2	$\widetilde{A_1}$	equal	$G_2(A_1 \times A_1)$(split)	$\#2$	1

Chapter 4: Genuine Central Character

4.1 Regular Character

The following material can be found in several places, for example, [6], [19], and [26]. Again, G is a real form of a simply connected, semisimple complex Lie group and $\widetilde{G}$ is the nonlinear two-fold cover of G. Let $\pi \in \widehat{G}_{adm,\lambda}$, where λ is a regular infinitesimal character. Then π can be specified by a parameter, which is called a λ-regular character, $\gamma = (H, \Gamma, \overline{\gamma})$, where H is a θ-stable Cartan subgroup of G, Γ is a character of H, and $\overline{\gamma}$ is an element in $\mathfrak{h}^*$ which defines the same infinitesimal character as λ, and there are certain compatibility conditions between $\overline{\gamma}$ and Γ. More precisely, $\pi = J(\gamma)$, the unique irreducible quotient of a standard representation $I(\gamma)$, which is parametrized by γ from a K-conjugacy class of regular characters for λ. Here the standard module $I(\gamma)$ is defined as follows. Write $H = TA$, where $T = H^\theta$ and A is the identity component of $\{h \in H | \theta(h) = h^{-1}\}$. Let $M = \mathrm{Cent}_G(A)$ and choose a parabolic subgroup $P = MN$, then we define $I(\gamma) = Ind_P^G(\sigma_M)$, where σ_M is some relative discrete series of M (see [7] or [1] for details.).

Recall that (see [1], for instance) when λ is a regular infinitesimal character, $\mathcal{HC}(\mathfrak{g}, K)_\lambda$ is parametrized by the set $\mathcal{P}_\lambda$ of K-conjugacy classes of λ-regular characters. Furthermore, the following two sets are bases of the Grothendieck group:

$$\{[J(\gamma)]\}_{\gamma\in\mathcal{P}_\lambda} \text{ and } \{[I(\gamma)]\}_{\gamma\in\mathcal{P}_\lambda}.$$

We have the following definition.

Definition 4.1.1. Define the change of basis matrix

$$[J(\delta)] = \sum_{\gamma\in\mathcal{P}_\lambda} M(\gamma,\delta)[I(\gamma)]$$

and the inverse matrix

$$[I(\delta)] = \sum_{\gamma\in\mathcal{P}_\lambda} m(\gamma,\delta)[J(\gamma)]$$

$M(\gamma,\delta)$ and $m(\gamma,\delta)$ are integers and $M(\gamma,\delta)$ are computed by the Kazhdan-Lusztig-Vogan algorithm when G is linear.

In particular, consider $\mathbb{C}$, the trivial representation, and write its standard module as $I(\gamma_0)$ with parameter γ_0. Then the coefficients $M(\gamma,\gamma_0)$ are ± 1.

Lemma 4.1.2. *([26]) There is an identity in the Grothendieck group*

$$\mathbb{C} = \sum_{\gamma} (-1)^{l(\gamma_0)-l(\gamma)} I(\gamma),$$

where $\gamma = (H,\Gamma,\overline{\gamma})$ runs over holomorphic characters Γ on H (see [1] for definition), and $l(\gamma)$ is the length function in [26].

The above notions can also be defined for nonlinear groups. More specifically, let λ be the regular infinitesimal character defined in Chapter 2. In this case, suppose that $\widetilde{\pi}$ is an irreducible genuine representation from $\widehat{\widetilde{G}}_{adm,\lambda}$. Then $\widetilde{\pi}$ is parametrized by a genuine λ- regular character $\gamma = (\widetilde{H},\Gamma,\overline{\gamma})$, where Γ is an irreducible genuine representation of $\widetilde{H} = p^{-1}(H)$. Note that in this case Γ can be replaced by a character of $Z(\widetilde{H})$, a central character of $\widetilde{H}$, because of the following proposition (see [3]).

Proposition 4.1.3. *Write $\prod_g(Z(\widetilde{H}))$ and $\prod_g(\widetilde{H})$ for equivalence classes of irreducible genuine representations of $Z(\widetilde{H})$ and $\widetilde{H}$, respectively, and let $n = |\widetilde{H}/Z(\widetilde{H})|^{\frac{1}{2}}$. For every $\chi \in \prod_g(Z(\widetilde{H}))$ there is a unique representation $\Gamma = \Gamma(\chi) \in \prod_g(\widetilde{H})$ for which $\Gamma|_{Z(\widetilde{H})}$ is a multiple of χ. The map $\chi \to \Gamma(\chi)$ is a bijection between $\prod_g(Z(\widetilde{H}))$ and $\prod_g(\widetilde{H})$. The dimension of $\Gamma(\chi)$ is n, and $Ind_{Z(\widetilde{H})}^{\widetilde{H}}(\chi) = n\Gamma$.*

When G is simply laced, a genuine representation of $\widetilde{H}$ is determined by the infinitesimal character and its restriction to $Z(\widetilde{G})$. We record the properties as follows (see [6]).

Proposition 4.1.4. *All the setting is as before, and also suppose G is simply laced, H is a Cartan subgroup of G, and H^0 is the identity component of H. Then*

(1) $Z(\widetilde{H}) = Z(\widetilde{G})\widetilde{H^0}$. In particular, a genuine character of $Z(\widetilde{H})$ is determined by its restriction to $Z(\widetilde{G})$ and its differential;

(2) A genuine regular character $\gamma = (\widetilde{H}, \Gamma, \overline{\gamma})$ of $\widetilde{G}$ is determined by $\overline{\gamma}$ and the restriction of Γ to $Z(\widetilde{G})$, and so is $\widetilde{\pi} = J(\gamma)$.

The second part of this proposition is basically a corollary of the first part. Consequently $\widetilde{G}$ typically has few genuine irreducible representations, denoted $\prod_g(\widetilde{G})$.

4.2 Action of Aut(G) on $\prod_g(\widetilde{G})$

In this section we want to see how an automorphism of G acts on $\prod_g(\widetilde{G})$. Let Aut(G) denote the automorphisim group of G, and

$$\mathrm{Int}(G) = \{\tau \in \mathrm{Aut}(G) \,|\, \tau = \tau_x \text{ for some } x \in G\}, \text{ where } \tau_x(g) = xgx^{-1} \text{ for } g \in G,$$

$$\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Int}(G).$$

Lemma 4.2.1. *There is a natural map from Out(G) to Aut$(Z(\widetilde{G}))$, which sends each $\tau \in$Aut(G) to $\widetilde{\tau} \in$ Aut$(Z(\widetilde{G}))$. When G is simply laced, This map is an embedding.*

Proof. The map $\tau \in \mathrm{Aut}(G) \to \widetilde{\tau} \in \mathrm{Aut}(Z(\widetilde{G}))$ is defined as follows. Every $\tau \in \mathrm{Aut}(G)$ can be lifted to an automorphism $\widetilde{\tau}$ of $\widetilde{G}$. Then by restricting $\widetilde{\tau}$ to $Z(\widetilde{G})$, we get an automorphism of $Z(\widetilde{G})$, which is also denoted by $\widetilde{\tau}$. This map is well-defined since if $\tau \in \mathrm{Int}(G)$, say, $\tau = \tau_x$ for some $x \in G$, $\widetilde{\tau}(\widetilde{z}) = \widetilde{x}\widetilde{z}\widetilde{x}^{-1} = \widetilde{z}\widetilde{x}\widetilde{x}^{-1} = \widetilde{z}$, for $\widetilde{z} \in Z(\widetilde{G})$.

The proof of the second assertion can be found in [4]. □

Let $\tau \in \mathrm{Aut}(G)$. Define an action of τ on $\prod_g(Z(\widetilde{G}))$ as follows. Let $\chi \in \prod_g(Z(\widetilde{G}))$, define $\chi^\tau(z) := \chi(\widetilde{\tau}(z))$, $z \in Z(\widetilde{G})$. When G is simply laced, we have an action of $\mathrm{Aut}(G)$ on $\prod_g(\widetilde{G})$. Due to Proposition 4.1.4 (2), every $\pi = J(\gamma)$, where $\gamma = (\widetilde{H}, \Gamma, \overline{\gamma})$, is determined by $\overline{\gamma}$ and $\chi := \Gamma|_{Z(\widetilde{G})}$. Then we can define $\pi^\tau := J(\gamma^\tau)$, where γ^τ is a regular character determined by $\overline{\gamma}$ and χ^τ.

The following is a corollary of Lemma 4.2.1.

Corollary 4.2.2. *Suppose G is simply laced. Let $\widetilde{\pi} \in \prod_g(\widetilde{G})$, and $\tau \in$Out(G), then $\widetilde{\pi}$ and $\widetilde{\pi}^\tau$ are inequivalent representations in $\prod_g(\widetilde{G})$.*

Chapter 5: $\prod_{\lambda}^{s}(\widetilde{G})$ – Split Case

In this chapter, the setting is as in Chapter 2, and furthermore we assume that G is split. Let H denote the split Cartan subgroup of G with Lie algebra $\mathfrak{h}$ (then $\widetilde{H} = p^{-1}(H)$ is the split Cartan subgroup of $\widetilde{G}$). It follows from Proposition 4.1.4 that there is a unique minimal principal representation of $\widetilde{G}$ coming from $\widetilde{H}$ if we fix a genuine central character and infinitesimal character. In [3], they are called genuine pseudospherical representations, or Shimura representations. We will show in this chapter that we can get more representations in $\prod_{\lambda}^{s}(\widetilde{G})$ by applying cross actions and Cayley transforms to the Shimura representations.

5.1 Shimura Representations

In [3], there is a set of minimal principal series denoted $\prod_{gs}(\widetilde{G})$, called Shimura representations. We list the lowest $\widetilde{K}$-types and the numbers of them in Table 5.1.

From this table, we can see there are few Shimura representations for each split group. We enumerate them as $\prod_{gs}(\widetilde{G}) = \{Sh_i\}_{i=1}^{k}$, where $k = 1, 2$, or 4. In fact, we have the following important properties for Shimura representations.

Proposition 5.1.1. *(1)* $\{Sh_i\} \subseteq \prod_{\lambda}^{s}(\widetilde{G})$.

(2) There is a bijection between $\{Sh_i\}$ *and* $\prod_{g}(Z(\widetilde{G}))$. *In particular, if* G *is simply*

Table 5.1: Shimura representations

Type	G	$\widetilde{K}$	$\lvert\Pi_{gs}(\widetilde{G})\rvert$	Lowest K-types of $\Pi_{gs}(\widetilde{G})$
A_{2m-1}	$SL(2m,\mathbb{R})$	$Spin(2m)$	2	$Spin_{\pm}$
A_{2m}	$SL(2m+1,\mathbb{R})$	$Spin(2m+1)$	1	$Spin$
B_{2m}	$Spin(2m+1,2m)$	$Spin(2m)\times Spin(2m+1)$	2	$1\otimes Spin_{\pm}$
B_{2m-1}	$Spin(2m,2m-1)$	$Spin(2m)\times Spin(2m-1)$	1	$1\otimes Spin$
C_n	$Sp(2n)$	$\widetilde{U(n)}$	2	$\det^{\frac{1}{2}}$
D_{2m}	$Spin(2m,2m)$	$Spin(2m)\times Spin(2m)$	4	$1\otimes Spin_{\pm}$, $Spin_{\pm}\otimes 1$
D_{2m+1}	$Spin(2m+1,2m+1)$	$Spin(2m+1)\times Spin(2m+1)$	2	$1\otimes Spin$, $Spin\otimes 1$
E_6	$E_6(C_4)$	$Sp(8)$	1	$\mathbb{C}^8$
E_7	$E_7(A_7)$	$SU(8)$	2	$\mathbb{C}^8, (\mathbb{C}^8)^*$
E_8	$E_8(D_8)$	$Spin(16)$	1	$\mathbb{C}^{16}$
F_4	$F_4(A_1\times C_3)$	$Sp(6)\times Spin(3)$	1	$1\otimes Spin$
G_2	$G_2(A_1\times A_1)$	$Spin(3)\times Spin(3)$	1	$Spin\otimes 1$

laced or of type G_2, there is a bijection between this and $\prod_g(Z(\widetilde{M})) = \prod_g(Z(\widetilde{H}))$, where M is as defined in Section 4.1, and $\widetilde{M} = p^{-1}(M)$.

(3) There is a bijection between $\prod_g(Z(\widetilde{G}))$ and $\{\mathcal{O}_i\}$, where $\{\mathcal{O}_i\}$ is the set of real forms of $\mathcal{O}$, the complex associated variety of every Shimura representation.

Proof. For the first part, it will be proved in Chapter 7 that every Shimura representation is in Lift($\mathbb{C}$) (Theorem 7.3.3) and hence every Shimura representation is in $\prod_\lambda^s(\widetilde{G})$ by Theorem 7.2.3.

The second part can be observed from Table 5.1, and the bijection sends every Shimura representation to its central character (see [3] for complete proof). The third part can be observed from the Table in Chapter 6. □

Remark 5.1.2. We can attach to each Sh_i a pair $(\chi_i, \mathcal{O}_i)$, where χ_i is the central character of Shi, and $\overline{\mathcal{O}_i}$ =AV(Sh_i). Then for each $\tau \in$Out(G), Sh_i^τ is associated to the pair $(\chi_i^\tau, \mathcal{O}_i^\tau)$, (i.e. as τ permutes the central characters, it also permutes the real associated varieties).

5.2 Constructing Representations in $\prod_\lambda^s(\widetilde{G})$

We need to recall some basic tools: cross-action, Cayley and inverse Cayley transforms before starting to construct new representations (see [6] and [19]).

5.2.1 Translation Functors across a Nonintegral Wall

Most of the material in this section can be found in [27] and [19]. Fix λ to be the infinitesimal character defined in Chapter 2. In order to compute characters for nonlinear groups, we need a family of infinitesimal characters containing λ, denoted $\mathcal{F}(\lambda)$. But we need to recall some notation first.

Let $\triangle^+(\lambda)$ be the positive root system making λ dominant, $R(\lambda)$ be the integral root system for λ, W_λ be the integral Weyl group for λ. Let P be the integral weight lattice, i.e. $P = \{\gamma \in \mathfrak{h}^* \mid\ <\gamma, \alpha^\vee> \in \mathbb{Z} \text{ for } \alpha \in \triangle\}$ and let $W_P(\lambda) = \{w \in W \mid w\lambda - \lambda \in P\}$. Then let $\mathcal{F}(\lambda)$ be a family of representatives of $(W \cdot \lambda + P)/P$ containing λ, and hence it's clear that $\mathcal{F}(\lambda)$ is indexed by $W/W_P(\lambda)$: if $\gamma \in \mathcal{F}(\lambda)$, then $\gamma = y\lambda$ modulo P for some $y \in W$ which is unique modulo $W_P(\lambda)$. So we can write $\mathcal{F}(\lambda) = \{\gamma_y = y\lambda \mid y \in W/W_P(\lambda)\}$. In particular, $\lambda = \gamma_1$. There is an obvious action of W on $\mathcal{F}(\lambda)$: $w * \gamma_y := w^{-1}(\gamma_y + \mu(y, w)) = \gamma_{yw}$, by picking some $\mu(y, w) \in P$. We fix once and for all integral wights $\mu(y, w) \in P$ satisfying the above conditions and we want to use them to define the following. First let α be a nonintegral simple root in $\triangle^+$, s_α be the corresponding simple reflection. Then we define:

(a) the nonintegral wall-crossing functors ψ_α and ϕ_α, where $\psi_\alpha(X) := \psi_{\gamma_y}^{\gamma_{ys_\alpha}}(X)$, a functor realizes an equivalence of categories between $\mathcal{HC}(\mathfrak{g}, K)_{\gamma_y}$ and $\mathcal{HC}(\mathfrak{g}, K)_{\gamma_{ys_\alpha}}$; its inverse is ϕ_α(see [26]) ;

(b) the cross action of W: let $\gamma = (H', \Gamma, \overline{\gamma})$ be a (γ_y)-regular character, $w \in W$, then the regular character $w \times \gamma = (H', w \times \Gamma, w \times \overline{\gamma})$ is defined by $w \times \overline{\gamma} = \overline{\gamma} + \mu(y, w)$ and $w \times \Gamma = \Gamma \otimes \mu(y, w) \otimes \partial\rho(w)$, where $\partial\rho(w) := w \cdot (\rho_i - 2\rho_{ic}) - (\rho_i - 2\rho_{ic})$, ρ_i (resp. ρ_{ic}) denotes the half-sum of positive imaginary (resp. compact imaginary) roots in $\triangle^+(\overline{\gamma})$. Note that $w \times \overline{\gamma}$ defines the same infinitesimal character as γ_{yw}.

Remark 5.2.1. Let $\alpha_1, \cdots, \alpha_p$ be simple roots and $s_1, \cdots, s_p$ be the corresponding reflections. If $w = s_p \cdots s_1 \in W_P(\lambda)$, we can define $\mu(1, w) = w\lambda - \lambda$, which is equal to $\mu(1, s_1) + \mu(s_1, s_2) + \mu(s_1 s_2, s_3) + \cdots + \mu(s_1 s_2 \cdots s_{p-1}, s_p)$. Thus, $w \times \overline{\gamma} = \overline{\gamma}$ if and only if $w \in W_P(\lambda)$, where $\overline{\gamma} \sim \lambda$.

We also need some basic facts about Cayley and inverse Cayley transforms. The related concepts can be found in various references (e.g. [19], [26]). Here we just introduce some notation and quote some important facts.

Let $\gamma = (H, \Gamma, \overline{\gamma})$ be a λ-regular character. Assume α is a nonintegral root, then we can define Cayley (or inverse Cayley) transform on γ (see Section 5 of [19]) through α if α is noncompact imaginary (real, resp.) and this action is denoted by $c^\alpha(\gamma) = \gamma^\alpha$ (or $c_\alpha(\gamma) = \gamma_\alpha$, resp.) Note that after Cayley (inverse Cayley, resp.) transform, we get a new λ-regular character, say, $\gamma^\alpha = (H^\alpha, \Gamma^\alpha, \overline{\gamma^\alpha})$ (or $\gamma_\alpha = (H_\alpha, \Gamma_\alpha, \overline{\gamma_\alpha})$, resp.), which has infinitesimal character λ and $I(\gamma^\alpha)$ (or $I(\gamma_\alpha)$, resp.) has the same central character as the original representation $I(\gamma)$. For convenience, we call both operators c^α and c_α Cayley transforms through the root α.

Now we are ready to state the result of Vogan describing translation functors across a nonintegral wall.

Theorem 5.2.2. *([19]) Let γ be a genuine λ-regular character of G. Suppose α is a nonintegral simple root in $\triangle^+(\overline{\gamma})$. Then, with the translation functor ψ_α defined by the weight μ_α fixed above, we have:*

$\psi_\alpha(J(\gamma)) = J((\gamma + \mu_\alpha)^\alpha) = J((s_\alpha \times \gamma)^\alpha)$ *if α is noncompact imaginary,*

$\psi_\alpha(J(\gamma)) = J((\gamma + \mu_\alpha)_\alpha) = J((s_\alpha \times \gamma)_\alpha)$ *if α is real satisfying the parity condition,*

$\psi_\alpha(J(\gamma)) = J(\gamma + \mu_\alpha) = J(s_\alpha \times \gamma)$ *otherwise.*

5.2.2 Construction related to Dynkin Diagrams

Now we would like to restrict the simply laced groups, so $\lambda = \rho/2$. Assume the setting as before, i.e. G is split, and $\widetilde{G}$ is the two-fold cover of G, and so on. As described in [18], to the Dynkin diagram D of $\widetilde{G}$, we attach a finite abelian group denoted by R_D as follows. Let $\prod$ be the set of simple roots.

$R_D = \{S \subseteq \prod \mid S$ is strongly orthogonal, so that any $\beta \notin S$ is adjacent to an even number of elements in $S\}$.

In Table 5.2, we list the elements in R_D for simply laced groups using Dynkin diagrams. Note that the root is in the element in R_D if and only if the corresponding node is filled.

Lemma 5.2.3. *There is a one-to-one correspondence between R_D and $(Z(G_{\mathbb{C}})_2)^\wedge$, the characters of elements in $Z(G_{\mathbb{C}})$ of order 2. The latter is isomorphic to $P/(2P+R)$ as group, and hence R_D can be parametrized by the elements in $P/(2P+R)$.*

Proof. Denote $Z = Z(G_{\mathbb{C}})$ and $Z_2 = Z(G_{\mathbb{C}})_2$.

From the exact sequence

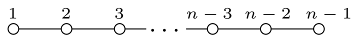
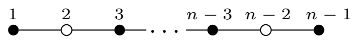
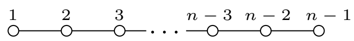
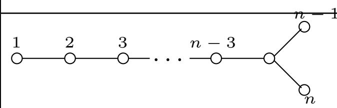
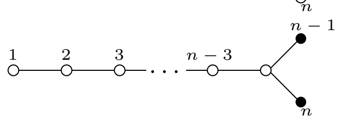
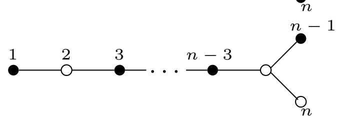
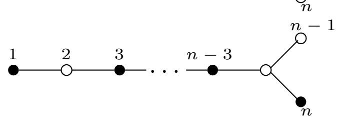
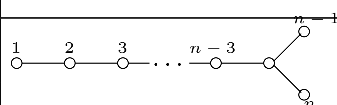
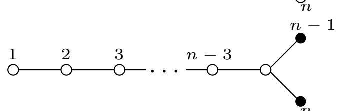
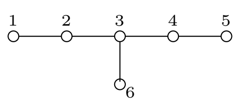
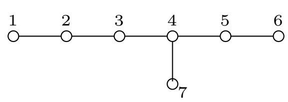
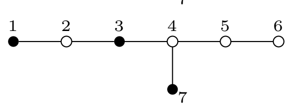
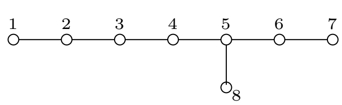
$$1 \to Z_2 \to Z \to Z/Z_2 \to 1,$$

we have another exact sequence

$$1 \to (Z/Z_2)^\wedge \to Z^\wedge \to Z_2^\wedge \to 1.$$

Notice that $Z^\wedge \simeq P/R$. Write $Z_2 = \{exp(2\pi i\tau^\vee) | \tau^\vee \in X_* \otimes \mathbb{C}, exp(2\pi i(2\tau^\vee)) = 1\}$.

Table 5.2:

Type	n	R_D	$\lvert R_D \rvert$
A_{n-1}	$2m$	 $\{\phi\}$	2
		 $\{\alpha_1, \alpha_3, \cdots, \alpha_{n-1}\}$	
A_{n-1}	$2m+1$	 $\{\phi\}$	1
D_n	$2m$	 $\{\phi\}$	4
		 $\{\alpha_{n-1}, \alpha_n\}$	
		 $\{\alpha_1, \alpha_3, \cdots, \alpha_{n-3}, \alpha_{n-1}\}$	
		 $\{\alpha_1, \alpha_3, \cdots, \alpha_{n-3}, \alpha_n\}$	
D_n	$2m+1$	 $\{\phi\}$	2
		 $\{\alpha_{n-1}, \alpha_n\}$	
E_6		 $\{\phi\}$	1
E_7		 $\{\phi\}$	2
		 $\{\alpha_1, \alpha_3, \alpha_7\}$	
E_8		 $\{\phi\}$	1

Then $(Z/Z_2)^\wedge \simeq (2P+R)/R$, since $\gamma \in P$ such that $\gamma|_{Z_2} = 1$ (i.e. $\gamma(exp(2\pi i\tau^\vee)) = exp(2\pi i < \gamma, \tau^\vee >) = 1$) if and only if $< \gamma, \tau^\vee >= 1$, if and only if $\gamma \in 2P+R$. Therefore, $Z_2^\wedge \simeq P/(2P+R)$ from the above exact sequence.

Associate to each $S = \{\alpha_1, \cdots, \alpha_p\} \in R_D$ an element $w_S = s_{\alpha_1} \cdots s_{\alpha_p} \in W$, then we have a map sending elements in R_D to $P/(2P+R)$ by $S \to w_S(\rho/2) - \rho/2$. This is a bijection by counting the elements in R_D and $P/(2P+R)$ case by case.

□

We will show that we can get a subset of representations in $\prod^s_{\rho/2}(\widetilde{G})$ from each Shimura representation Sh_i by a sequence of Cayley transforms or wall-crossings through the simple roots in $S \in R_D$.

Associate to each $S = \{\alpha_1, \cdots, \alpha_p\} \in R_D$ an element $w_S = s_{\alpha_1} \cdots s_{\alpha_p} \in W$, and let $c_S = c_{\alpha_1} \cdots c_{\alpha_p}$ and $\psi_S = \psi_{\alpha_1} \cdots \psi_{\alpha_p}$ be the corresponding Cayley transform and wall-crossing functor respectively.

Lemma 5.2.4. *For every w_S, $S \in R_D$, $w_S \in W_P(\rho/2)$, and hence $w_S \times \overline{\gamma} = \overline{\gamma}$, where $\gamma \sim \rho/2$, by Remark 5.2.1.*

Proof. Let $S = \{\alpha_1, \cdots, \alpha_p\} \in R_D$. Then $w_S(\rho/2) - \rho/2 = s_{\alpha_1} \cdots s_{\alpha_p}(\rho/2) - \rho/2 = - < \rho/2, \alpha_1^\vee > \alpha_1 - \cdots - < \rho/2, \alpha_p^\vee > \alpha_p$.

For each simple root $\beta \notin S$, β is adjacent to even numbers of α_i's, and hence $< w_S(\rho/2) - \rho/2, \beta^\vee > \in \mathbb{Z}$. For $\beta = \alpha_i$ some i, $< w_S(\rho/2) - \rho/2, \beta^\vee >= - < \rho/2, \alpha_i^\vee >< \alpha_i, \alpha_i^\vee > \in \mathbb{Z}$. Therefore, $w_S \in W_P(\rho/2)$. □

We would like to take a look at the effects of Cayley transforms on the τ-invariant. Note that for every root $\alpha \in \prod(\rho/2)$, there exists a positive root system

$\Psi_\alpha \subseteq \triangle$ such that $\Psi_\alpha \supseteq \triangle_\lambda^+$ and α is simple in Ψ_α (cf [27] Lemma 3.1), that is, we can apply a sequence of cross actions (across nonintegral walls) through a set of roots $Q = \{\beta_1, \cdots, \beta_q\}$, to move α to a chamber in which it is simple. More precisely, let γ be a λ-regular character. Let $w = s_{\beta_1} \cdots s_{\beta_q}$, and hence α is simple in $\Psi_\alpha = w(\triangle^+)$. Let $X = J(\gamma)$ and $X' = \psi_Q(J(\gamma))$, where $\psi_Q = \psi_{\beta_1} \cdots \psi_{\beta_q}$ is a sequence of nonintegral wall-crossings in Theorem 5.2.2. Since ψ_Q is an equivalence of categories, we in fact have $\tau(X) = \tau(X')$ and the following Theorem is extremely helpful.

Theorem 5.2.5. *(cf. [27] Theorem 4.12) Assume the settings as above for G and $\widetilde{G}$ as before, and X, X', w are as in the previous paragraph, so α is simple for $w \times \gamma$. Put $l = \dfrac{2 < \lambda, \alpha^\vee >}{< \alpha, \alpha^\vee >}$, then we have*

a) If α is real and $\gamma'(m_\alpha) \neq (-1)^l \epsilon_\alpha$ (cf. [27]) Proposition 4.5), then $\alpha \notin \tau(X)$.

b) If α is real and $\gamma'(m_\alpha) = (-1)^l \epsilon_\alpha$, then $\alpha \in \tau(X)$.

c) If α is complex and $\theta(\alpha) \in \triangle_{\gamma'}^+$, then $\alpha \notin \tau(X)$.

d) If α is complex and $\theta(\alpha) \notin \triangle_{\gamma'}^+$, then $\alpha \in \tau(X)$.

e) If α is noncompact imaginary, then $\alpha \notin \tau(X)$.

The following lemma is a corollary of this theorem.

Lemma 5.2.6. *Let $S \in R_D$ and $S' \subseteq S$. Let Sh be a Shimura representation. Then $c_{S'}(Sh)$ has maximal τ-invariant if and only if $S' = S$.*

Proof. This can be proved case by case using 5.2.5. □

Theorem 5.2.7. *Fix some Sh_i with central character χ_i as above. Suppose Sh_i is specified by the $\rho/2$-regular character $\gamma = (H, \Gamma, \overline{\gamma})$. Let $S \in R_D$. Then*

(1) $c_S(Sh_i)$ and $\psi_S(Sh_i)$ are in $\prod^s_{\rho/2}(\widetilde{G})$;

(2) $c_S(Sh_i)$ has central character χ_i, so let $\pi_i := c_S(Sh_i)$, where the subscript indicates π_i has the same central character as Sh_i. Then $\psi_S(Sh_i)$ can be denoted π_j for some j and $j \neq i$ if $S \neq \phi$. More precisely, if π_i is specified by the regular character $\gamma_i = \gamma_S = (H_S, \Gamma_i, \overline{\gamma_i})$ with central character χ_i and π_j is specified by the regular character $\gamma_j = \gamma'_S = (H'_S, \Gamma_j, \overline{\gamma_j})$ with central character χ_j, then $H_S = H'_S$, $\overline{\gamma_i} \sim \overline{\gamma_j} \sim \overline{\gamma} \sim \rho/2$ and $\chi_i \neq \chi_j$ if $S \neq \phi$.

Proof. For the first part, $c_S(Sh_i)$ is in $\prod^s_{\rho/2}(\widetilde{G})$ due to Lemma 5.2.6 and the fact that the infinitesimal character doesn't change under the action of Cayley transforms. On the other hand, ψ_S is a series of nonintegral wall-crossings ψ_α, and in each step, $\psi_\alpha(X) = P^{\gamma_{ys_\alpha}}_{\gamma_y}(X \otimes F_{\mu(y,s_\alpha)})$, the projection of $X \otimes F_{\mu(y,s_\alpha)}$ on to the Harish-Chandra modules at infinitesimal character γ_{ys_α}, where γ_y, γ_{ys_α}, and $\mu(y, s_\alpha)$ are described in the beginning of Section 5.2.1. Note that $Sh_i \in \prod^s_{\rho/2}(\widetilde{G}) = \prod^{\mathcal{O}}_{\rho/2}(\widetilde{G})$, we have $AV(Sh_i) = \overline{\mathcal{O}}$ and hence $AV(Sh_i \otimes F) = \overline{\mathcal{O}}$ for any finite dimensional F by Lemma 2.1.2. Therefore, $AV(\psi_\alpha(Sh_i)) = \overline{\mathcal{O}}$ and $AV(\psi_S(Sh_i)) = \overline{\mathcal{O}}$ by the same argument. By Lemma 5.2.4, $\psi_S(Sh_i))$ has infinitesimal character $\rho/2$ and hence $\psi_S(Sh_i)) \in \prod^s_{\rho/2}(\widetilde{G})$.

For the second part of the proof, first, observe that according to the Theorem 5.2.2, each step of the wall-crossings in ψ_S also goes through the same Cayley transform as in c_S, and hence $H_S = H'_S$.

Then we want to check that $\overline{\gamma_i}$ and $\overline{\gamma_j}$ define the same infinitesimal character, that is, $\rho/2$. Since Cayley transforms don't change infinitesimal characters, $\overline{\gamma_i}$ defines the same infinitesimal character as $\overline{\gamma} \sim \rho/2$. On the other hand, $w_S \in W_P(\rho/2)$ by Lemma 5.2.4, and hence $\overline{\gamma_j} = w_S \times \overline{\gamma} = \overline{\gamma} \sim \rho/2$.

Finally note that Cayley transforms don't change the central characters, and hence $c_S(Sh_i)$ and Sh_i have the same central character χ_i. However, since $w_S\overline{\gamma} - \overline{\gamma} \sim w_S(\rho/2) - \rho/2$ defines a nontrivial element in $P/(2P+R)$ if $S \neq \phi$ by Lemma 5.2.3. Therefore, $\chi_j/\chi_i \neq 1$ if $S \neq \phi$. $\square$

Now we denote

$$\textstyle\prod_{R_D}(\widetilde{G}) = \{\pi \mid \pi = c_S(Sh) \text{ where } Sh \text{ is a Shimura representation and } S \in R_D\}$$

Remark 5.2.8. (1) $\prod_{R_D}(\widetilde{G}) \subseteq \prod^s_{\rho/2}(\widetilde{G})$.

(2) If $|Z(\widetilde{G})^\wedge| = p$, then $|\prod_{R_D}(\widetilde{G})| = p^2$.

5.2.3 Example

In this section, we describe the representations in $\prod_{RD}(\widetilde{G})$ for type A_{n-1} and D_n, where n is even, by describing their lowest K-types and Langlands parameters. In the following content, we use the highest weight of the lowest K-type to stand for the lowest K-type.

Example 5.2.9. Let $G = SL(n, \mathbb{R})$, where $n = 2m$. Before describing representations in $\prod_{RD}(\widetilde{G})$, we consider a bigger group $G' = GL(n, \mathbb{R})$.

First we recall the definition of the Speh representations of $\widetilde{G'}$ (See [5] for details).

Let $L \cong GL(m, \mathbb{C})$ be an θ-stable Levi subgroup of G', $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a parabolic subalgebra of $\mathfrak{g}$, $S = \dfrac{n(n-1)}{2}$. Then for $k \in \{0, 1, 2, ...\} \cup \{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, ...\}$ and $\nu \in i\mathbb{R}$, Speh$(k,\nu) = R_{\mathfrak{q}}^S(\chi(k,\nu))$ is defined to be the irreducible unitary representation of $\widetilde{G'}$ obtained from the S-th cohomological induction from the one-dimensional character $\chi(k,\nu)(z) = \left(\frac{z}{|z|}\right)^k |z|^\nu$ of L. We write Speh$(k,0)$ =Speh(k). Note that when $n=2$, Speh$(k.\nu)$ is the relative discrete series representation of $\widetilde{GL}(2,\mathbb{R})$ with infinitesimal character $(\frac{k+\nu}{2}, \frac{-k+\nu}{2})$.

In [5] and [13], there are two genuine unitary irreducible representations of G' under consideration, denoted T_n and Speh(1/2). T_n is defined to be the unique irreducible subquotient of $\mathrm{Ind}_P^{\widetilde{G'}}(\delta_n \otimes \rho/2)$, where P is the minimal parabolic subgroup of $\widetilde{G'}$, which contain $\widetilde{M}$, the preimage of diag$\{\pm 1, \cdots, \pm 1\}$ in $\widetilde{G'}$, and δ_n is the representation of $\widetilde{M}$ restricted from the pin representation of $Pin(n)$ with highest weight $(\frac{1}{2}, \cdots, \frac{1}{2}; \frac{1}{2})$. It is shown in [13] that T_n has infinitesimal character $\rho/2$ and the lowest K-type is $(\frac{1}{2}, \cdots, \frac{1}{2}; \frac{1}{2})$. Furthermore, T_n has maximal τ-invariant, and hence $T_n \in \prod_{\rho/2}^s(\widetilde{G'})$.

On the other hand, Speh(1/2) is the irreducible quotient of $\mathrm{Ind}_{P_1}^{\widetilde{G'}}(\sigma)$, where $P_1 = \widetilde{M_1}N$, $M_1 \cong GL(2,\mathbb{R})^m$, and

$$\sigma = \mathrm{Speh}(\tfrac{1}{2}, \tfrac{n-2}{2}) \otimes \mathrm{Speh}(\tfrac{1}{2}, \tfrac{n-6}{2}) \otimes \cdots \otimes \mathrm{Speh}(\tfrac{1}{2}, 1) \otimes \mathrm{Speh}(\tfrac{1}{2}, -1) \otimes \cdots \otimes \mathrm{Speh}(\tfrac{1}{2}, \tfrac{-n+2}{2})$$

(see [5]), It is easy to show that this representation has infinitesimal $\rho/2$ and the lowest K-type is $(\frac{3}{2}, \cdots, \frac{3}{2}; \frac{1}{2})$. Furthermore, we claim that Speh(1/2) has maximal τ-invariant by using Theorem 5.2.5. Let $\prod(\rho/2) = \{e_i - e_{i+2} | i = 1, 3, \cdots, n-2\}$ be the set of integral simple roots, all of which are complex roots for Speh(1/2). Let $\alpha_i = e_i - e_{i+1}, i = 1, 3, \cdots, n-2$. Note that Speh(1/2) is specified by the Cartan $\widetilde{H}$ with $H = (\mathbb{C}^\times)^m$, obtained from the split Cartan through a sequence of

Cayley transforms $c_{\alpha_1}c_{\alpha_3}\cdots c_{\alpha_{n-2}}$, and hence the Cartan involution of Speh(1/2) is $\theta = -s_{\alpha_1}s_{\alpha_3}\cdots s_{\alpha_{n-2}}$. For the complex root $e_i - e_{i+2}$, calculate

$$\theta(e_i - e_{i+2}) = -s_{\alpha_1}s_{\alpha_3}\cdots s_{\alpha_{n-2}}(e_i - e_{i+2}) = -s_{\alpha_i}s_{\alpha_{i+2}}(e_i - e_{i+2}) =$$

$$-s_{\alpha_{i+2}}(e_{i+1} - e_{i+2}) = e_{i+3} - e_{i+1},$$

which is a negative root in the big root system, and hence for $\alpha \in \prod(\rho/2)$, we have $\theta(\alpha)$ is negative and hence α is in the τ-invariant by Theorem 5.2.5.

Note that the restriction of each of these two representations to $\widetilde{G} = \widetilde{SL}(n, \mathbb{R})$ is the sum of two inequivalent irreducible representations of $\widetilde{G}$. More precisely, pick $y \in G' \setminus G$, let $\tau = \tau_y$ be the conjugation action on G by y. Let χ denote the central character of T_n, and for $\widetilde{g} \in \widetilde{G}$, define $\chi^\tau(\widetilde{g}) = \chi(y\widetilde{g}y^{-1})$. Then χ and χ^τ define different genuine characters of $\widetilde{G}$ and hence $T_n|_{\widetilde{G}} = \widetilde{\pi} + \widetilde{\pi}^\tau$, where χ and χ^τ are the central characters of $\widetilde{\pi}$ and $\widetilde{\pi}^\tau$, respectively. In fact, $\widetilde{\pi}$ and $\widetilde{\pi}^\tau$ are the two Shimura representations of $\widetilde{G}$. We know that the lowest K-type of T_n is $(\frac{1}{2}, \cdots, \frac{1}{2}; \frac{1}{2})$ and its restriction to $Spin(n)$ is the sum of the representations of highest weights $(\frac{1}{2}, \cdots, \frac{1}{2})$ and $(\frac{1}{2}, \cdots \frac{1}{2}, -\frac{1}{2})$, which are the lowest L-types of Sh_1 and Sh_2, respectively.

Similarly, the restriction of Speh(1/2) to $\widetilde{G}$ is also a sum of two irreducible representations, parametrized by their lowest K-types $(\frac{3}{2}, \cdots, \frac{3}{2})$ and $(\frac{3}{2}, \cdots \frac{3}{2}, -\frac{3}{2})$. We have shown that they are in $\prod^s_{\rho/2}(\widetilde{G})$. From next section, we will know that $|\prod_{RD}(\widetilde{G})| = |\prod^s_{\rho/2}(\widetilde{G})| = 4$ and hence these two representations are the ones in $\prod_{RD}(\widetilde{G})$ besides the Shimura representations.

Example 5.2.10. Consider type D_n, where $n = 2m$. First consider a bigger group $\widetilde{G'} = \widetilde{\mathrm{Spin}}(2m+1, 2m)$, $K' = \mathrm{Spin}(2m+1) \times \mathrm{Spin}(2m)$. From [16], we have

4 representations of $\widetilde{G'}$, say $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$, with all K'-types specified (in the big table on the last page of this section). Let $\widetilde{G} = \widetilde{\mathrm{Spin}}(2n, 2n)$, $K = \mathrm{Spin}(2n) \times \mathrm{Spin}(2n)$, the maximal compact subgroup of $\widetilde{G}$. Since $\gamma(x_1, \cdots, x_m; y_1, \cdots, y_m) = (y_1, \cdots, y_m; x_1, \cdots, x_m)$ is an outer automorphism of $\widetilde{G}$, the K'-types parametrized by $(\lambda; \lambda')$ and $(\lambda'; \lambda)$ represent different representations when restricted to K, and hence restricting these Γ_i's to $\widetilde{G}$, we will get 16 representations, which are also listed in the big table. In fact, these 16 representations are contained in $\prod_{\rho/2}^{s}(\widetilde{G})$ (explain?) and hence are the 16 representations in $\prod_{R_D}(\widetilde{G})$ by the next section. Let $S_a = \{\phi\}$, $S_b = \{e_{n-1} \pm e_n\}$, $S_c = \{e_1 - e_2, e_3 - e_4, \cdots, e_{n-1} - e_n\}$, $S_d = \{e_1 - e_2, e_3 - e_4, \cdots, e_{n-1} + e_n\}$ be the elements in R_D. If Shimura representations are enumerated as $\{Sh_i\}_{i=1}^4$, then $c_{S_a}(Sh_i) = Sh_i$ and we denote

$$c_{S_b}(Sh_i) = \pi_i, c_{S_c}(Sh_i) = \delta_i, c_{S_d}(Sh_i) = \tau_i,$$

where the representations with the same subscript have the same central character. From the big table, Sh_1 is parametrized by its lowest K-type $(\frac{1}{2}, \cdots, \frac{1}{2}; 0 \cdots, 0)$, and similarly, Sh_2 is parametrized by $(\frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2}; 0 \cdots, 0)$, Sh_3 is parametrized by $(0 \cdots, 0; \frac{1}{2}, \cdots, \frac{1}{2})$, Sh_4 is parametrized by $(0 \cdots, 0; \frac{1}{2}, \cdots, -\frac{1}{2})$. Let $\sigma, \gamma \in \mathrm{Out}(G)$ such that the action of σ and γ on the K-types are as follows.

$$\sigma(\lambda_1, \cdots, \lambda_m; \lambda_{m+1}, \cdots, \lambda_n) = (\lambda_1, \cdots, -\lambda_n; \lambda_{m+1}, \cdots, -\lambda_n),$$

$$\gamma(\lambda_1, \cdots, \lambda_m; \lambda_{m+1}, \cdots, \lambda_n) = (\lambda_{m+1}, \cdots, \lambda_n; \lambda_1, \cdots, \lambda_m)$$

Then we have the action of σ and γ on representations and genuine central characters in terms of the K-types. For instance, $\sigma(Sh_1) = Sh_2$, $\sigma(Sh_3) = Sh_4$, $\gamma(Sh_1) = Sh_3$, and so on. The complete actions of the outer automorphisms on central characters are in the following table, assuming that χ_i is the corresponding central character of Sh_i.

	χ_1 $(0,\cdots,0)$	χ_2 $(1,0\cdots,0)$	χ_3 $(\frac{1}{2}\cdots,\frac{1}{2})$	χ_4 $(\frac{1}{2}\cdots,-\frac{1}{2})$
1	χ_1	χ_2	χ_3	χ_4
σ	χ_2	χ_1	χ_4	χ_3
γ	χ_3	χ_4	χ_1	χ_2
$\sigma\gamma$	χ_4	χ_3	χ_2	χ_1

The vectors under each χ_i in the first row are the representatives of

$$\prod_g(Z(\widetilde{G})) \cong P/(2P+R) \cong \mathbb{Z}^n \cup (\mathbb{Z}+\tfrac{1}{2})^n/\mathbb{Z}_e^n$$

By Lemma 5.2.3, each $S \in R_D$ corresponding to $w_S(\rho/2)-\rho/2 \in P/(2P+R)$. More precisely,

$$S_a \leftrightarrow (0,\cdots,0), S_b \leftrightarrow (1,0,\cdots,0), S_c \leftrightarrow (\tfrac{1}{2},\cdots,\tfrac{1}{2}), S_d \leftrightarrow (\tfrac{1}{2},\cdots,-\tfrac{1}{2})$$

Note that for each $S \in R_D$, $\psi_S(Sh_i) = (c_S(Sh_i))^\xi$ for some $\xi \in \mathrm{Out}(G)$ and then the central character of $\psi_S(Sh_i)$ is χ_i^ξ. In fact,

$$\psi_{S_a}(Sh_i) = c_{S_a}(Sh_i),\ \psi_{S_b}(Sh_i) = (c_{S_b}(Sh_i))^\sigma,\ \psi_{S_c}(Sh_i) = (c_{S_c}(Sh_i))^\gamma,$$
$$\psi_{S_d}(Sh_i) = (c_{S_d}(Sh_i))^{\sigma\gamma}.$$

Then, we have the following effects of ψ_S's on Shimura representations and 16 representations $\prod_{R_D}(\widetilde{G})$ are produced in this way. The representations produced are denoted by Sh_i, π_i, δ_i, τ_i as follows, where representations with the same subscript have the same central character.

	Sh_1	Sh_2	Sh_3	Sh_4
ψ_{S_a}	Sh_1	Sh_2	Sh_3	Sh_4
ψ_{S_b}	π_2	π_1	π_4	π_3
ψ_{S_c}	δ_3	δ_4	δ_1	δ_2
ψ_{S_d}	τ_4	τ_3	τ_2	τ_1

Tables 5.4-5.6 list all K-types and lowest K types of the small representations of $\widetilde{G} = Spin(n,n)$. In Tables 5.4 and 5.5, the case of even n is shown and the case of odd n is shown in Table 5.6. In Tables 5.4 and 5.5, $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n$ and $\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbb{Z}^n$ are decreasing sequences of nonnegative integers. By $\gamma \prec \lambda$ we mean that $\lambda_1 \geqslant \gamma_1 \geqslant \cdots \geqslant \lambda_n \geqslant \gamma_n \geqslant -\lambda_n$. In Table 5.6, $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n$, $\lambda' = (\lambda_1, \cdots, \lambda_n, 0) \in \mathbb{Z}^{n+1}$ and $\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbb{Z}^n$ are decreasing sequences of nonnegative integers. By $\gamma \prec \lambda'$ we mean that $\lambda_1 \geqslant \gamma_1 \geqslant \cdots \geqslant \lambda_n \geqslant \gamma_n \geqslant 0$. Again, the representations are described in [16].

Table 5.4: All K-types of representations in $\prod_{R_D}(\widetilde{G})$ when $G = Spin(n,n)$, $n = 2m$

(1)

$\widetilde{G'}$-rep	K'-type	L.K'.T	restriction to K	L.K.T	$\widetilde{G}$-rep
Γ_1	$V^+ = \bigoplus_\lambda (\lambda; \lambda + \frac{\mathbf{1}}{\mathbf{2}})$	$(\mathbf{0}; \frac{\mathbf{1}}{\mathbf{2}})$	$\bigoplus_\lambda \bigoplus_{\gamma \prec \lambda} (\gamma; \lambda + \frac{\mathbf{1}}{\mathbf{2}}), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}$	$(0, \cdots, 0; \frac{1}{2}, \cdots, \frac{1}{2})$	Sh_3
			$\bigoplus_\lambda \bigoplus_{\gamma \prec \lambda} (\gamma; \lambda + \frac{\mathbf{1}}{\mathbf{2}}), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z} + 1$	$(0, \cdots, 0; \frac{3}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$	π_4
Γ_1	$V^+ = \bigoplus_\lambda (\lambda + \frac{\mathbf{1}}{\mathbf{2}}; \lambda)$	$(\frac{\mathbf{1}}{\mathbf{2}}; \mathbf{0})$	$\bigoplus_\lambda \bigoplus_{\gamma \prec \lambda} (\lambda + \frac{\mathbf{1}}{\mathbf{2}}; \overrightarrow{\gamma}), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}$	$(\frac{1}{2}, \cdots, \frac{1}{2}; 0, \cdots, 0)$	Sh_1
			$\bigoplus_\lambda \bigoplus_{\gamma \prec \lambda} (\lambda + \frac{\mathbf{1}}{\mathbf{2}}; \gamma), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z} + 1$	$(\frac{3}{2}, \frac{1}{2}, \cdots, \frac{1}{2}; 0, \cdots, 0)$	π_2
Γ_2	$V^- = \bigoplus_\lambda (\lambda; \sigma(\lambda + \frac{\mathbf{1}}{\mathbf{2}}))$	$(\mathbf{0}; \sigma(\frac{\mathbf{1}}{\mathbf{2}}))$	$\bigoplus_\lambda \bigoplus_{\gamma \prec \lambda} (\gamma; \sigma(\lambda + \frac{\mathbf{1}}{\mathbf{2}})), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}$	$(0, \cdots, 0; \frac{1}{2}, \cdots, -\frac{1}{2})$	Sh_4
			$\bigoplus_\lambda \bigoplus_{\gamma \prec \lambda} (\gamma; \sigma(\lambda + \frac{\mathbf{1}}{\mathbf{2}})), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z} + 1$	$(0, \cdots, 0; \frac{3}{2}, \frac{1}{2}, \cdots, -\frac{1}{2})$	π_3
Γ_2	$V^- = \bigoplus_\lambda (\sigma(\lambda + \frac{\mathbf{1}}{\mathbf{2}}); \lambda)$	$(\sigma(\frac{\mathbf{1}}{\mathbf{2}}); \mathbf{0})$	$\bigoplus_\lambda \bigoplus_{\gamma \prec \lambda} (\sigma(\lambda + \frac{\mathbf{1}}{\mathbf{2}}); \gamma), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}$	$(\frac{1}{2}, \cdots, -\frac{1}{2}; 0, \cdots, 0)$	Sh_2
			$\bigoplus_\lambda \bigoplus_{\gamma \prec \lambda} (\sigma(\lambda + \frac{\mathbf{1}}{\mathbf{2}}); \gamma), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z} + 1$	$(\frac{3}{2}, \frac{1}{2}, \cdots, -\frac{1}{2}; 0, \cdots, 0)$	π_1

Table 5.5: All K-types of representations in $\prod_{R_D}(\widetilde{G})$ when $G = Spin(n,n)$, $n = 2m$

(2)

$\widetilde{G'}$-rep	K'-type	L.K'.T	restriction to K	L.K.T	$\widetilde{G}$-rep
Γ_3	$V_0^+ = \bigoplus_\lambda(\lambda + \mathbf{\frac{1}{2}}; \lambda + \mathbf{1})$	$(\mathbf{\frac{1}{2}}; \mathbf{1})$	$\bigoplus_\lambda \bigoplus_{\gamma \prec \lambda} (\gamma + \mathbf{\frac{1}{2}}; \lambda + \mathbf{1}), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}$ $\bigoplus_\lambda \bigoplus_{\gamma \prec \lambda} (\sigma(\gamma + \mathbf{\frac{1}{2}}); \lambda + \mathbf{1}), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}+1$	$(\frac{1}{2}, \cdots, \frac{1}{2}; 1, \cdots, 1)$	$\delta_1(m \text{ even})/\delta_2(m \text{ odd})$
			$\bigoplus_\lambda \bigoplus_{\gamma \prec \lambda} (\gamma + \mathbf{\frac{1}{2}}; \lambda + \mathbf{1}), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}+1$ $\bigoplus_\lambda \bigoplus_{\gamma \prec \lambda} (\sigma(\gamma + \mathbf{\frac{1}{2}}); \lambda + \mathbf{1}, \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}$	$(\frac{1}{2}, \cdots, - - \frac{1}{2}; 1, \cdots, 1)$	$\tau_2(m \text{ even})/\tau_1(m \text{ odd})$
Γ_3	$V_0^- = \bigoplus_\lambda(\lambda + \mathbf{1}; \lambda + \mathbf{\frac{1}{2}})$	$(\mathbf{1}; \mathbf{\frac{1}{2}})$	$\bigoplus_\lambda \bigoplus_{\gamma \prec \lambda} (\lambda + \mathbf{1}; \gamma + \mathbf{\frac{1}{2}}), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}$ $\bigoplus_\lambda \bigoplus_{\gamma \prec \lambda} (\lambda + \mathbf{1}; \sigma(\gamma + \mathbf{\frac{1}{2}})), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}+1$	$(1, \cdots, 1; \frac{1}{2}, \cdots, \frac{1}{2})$	$\delta_3(m \text{ even})/\delta_4(m \text{ odd})$
			$\bigoplus_\lambda \bigoplus_{\gamma \prec \lambda} (\lambda + \mathbf{1}; \gamma + \mathbf{\frac{1}{2}}), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}+1$ $\bigoplus_\lambda \bigoplus_{\gamma \prec \lambda} (\lambda + \mathbf{1}; \sigma(\gamma + \mathbf{\frac{1}{2}})), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}$	$(1, \cdots, 1; \frac{1}{2}, \cdots, -\frac{1}{2})$	$\tau_4(m \text{ even})/\tau_3(m \text{ odd})$
Γ_4	$V_0^- = \bigoplus_\lambda(\lambda + \mathbf{\frac{1}{2}}; \sigma(\lambda + \mathbf{1}))$	$(\mathbf{\frac{1}{2}}; \sigma(\mathbf{1}))$	$\bigoplus_\lambda \bigoplus_{\gamma \prec \lambda} (\gamma + \mathbf{\frac{1}{2}}; \sigma(\lambda + \mathbf{1})), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}$ $\bigoplus_\lambda \bigoplus_{\gamma \prec \lambda} (\sigma(\gamma + \mathbf{\frac{1}{2}}); \sigma(\lambda + \mathbf{1})), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}+1$	$(\frac{1}{2}, \cdots, \frac{1}{2}; 1, \cdots, -1)$	$\tau_1(m \text{ even})/\tau_2(m \text{ odd})$
			$\bigoplus_\lambda \bigoplus_{\gamma \prec \lambda} (\gamma + \mathbf{\frac{1}{2}}; \sigma(\lambda + \mathbf{1})), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}+1$ $\bigoplus_\lambda \bigoplus_{\gamma \prec \lambda} (\sigma(\gamma + \mathbf{\frac{1}{2}}; \sigma(\lambda + \mathbf{1})), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}$	$(\frac{1}{2}, \cdots, -\frac{1}{2}; 1, \cdots, -1)$	$\delta_2(m \text{ even})/\delta_1(m \text{ odd})$
Γ_4	$V_0^- = \bigoplus_\lambda(\sigma(\lambda + \mathbf{1}); \lambda + \mathbf{\frac{1}{2}})$	$(\sigma(\mathbf{1}); \mathbf{\frac{1}{2}})$	$\bigoplus_\lambda \bigoplus_{\gamma \prec \lambda} (\sigma(\lambda + \mathbf{1}); \gamma + \mathbf{\frac{1}{2}}), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}$ $\bigoplus_\lambda \bigoplus_{\gamma \prec \lambda} (\sigma(\lambda + \mathbf{1}); \sigma(\gamma + \mathbf{\frac{1}{2}})), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}+1$	$(1, \cdots, -1; \frac{1}{2}, \cdots, \frac{1}{2})$	$\tau_3(m \text{ even})/\tau_4(m \text{ odd})$
			$\bigoplus_\lambda \bigoplus_{\gamma \prec \lambda} (\sigma(\lambda + \mathbf{1}); \gamma + \mathbf{\frac{1}{2}}), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}+1$ $\bigoplus_\lambda \bigoplus_{\gamma \prec \lambda} (\sigma(\lambda + \mathbf{1}); \sigma(\gamma + \mathbf{\frac{1}{2}})), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}$	$(1, \cdots, -1; \frac{1}{2}, \cdots, -\frac{1}{2})$	$\delta_4(m \text{ even})/\delta_3(m \text{ odd})$

Table 5.6: All K-types of representations in $\prod_{R_D}(\widetilde{G})$, when $G = Spin(n,n)$, $n = 2m+1$

$\widetilde{G'}$-rep	K'-type	L.K'.T	restriction to K	L.K.T	$\widetilde{G}$-rep
Γ	$V = \bigoplus_{\lambda}(\lambda'; \lambda + \frac{\mathbf{1}}{\mathbf{2}})$	$(\mathbf{0}; \frac{\mathbf{1}}{\mathbf{2}})$	$\bigoplus_{\lambda} \bigoplus_{\gamma \prec \lambda'} (\gamma; \lambda + \frac{\mathbf{1}}{\mathbf{2}}), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}$	$(0, \cdots, 0; \frac{1}{2}, \cdots, \frac{1}{2})$	Sh_2
			$\bigoplus_{\lambda} \bigoplus_{\gamma \prec \lambda'} (\gamma; \lambda + \frac{\mathbf{1}}{\mathbf{2}}), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}+1$	$(0, \cdots, 0; \frac{3}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$	π_1
Γ	$V = \bigoplus_{\lambda}(\lambda + \frac{\mathbf{1}}{\mathbf{2}}; \lambda')$	$(\frac{\mathbf{1}}{\mathbf{2}}; \mathbf{0})$	$\bigoplus_{\lambda} \bigoplus_{\gamma \prec \lambda'} (\lambda + \frac{\mathbf{1}}{\mathbf{2}}; \gamma), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}$	$(\frac{1}{2}, \cdots, \frac{1}{2}; 0, \cdots, 0)$	Sh_1
			$\bigoplus_{\lambda} \bigoplus_{\gamma \prec \lambda'} (\lambda + \frac{\mathbf{1}}{\mathbf{2}}; \gamma), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}+1$	$(\frac{3}{2}, \frac{1}{2}, \cdots, \frac{1}{2}; 0, \cdots, 0)$	π_2

5.3 Exhaustion – Characterization of $\prod_{\lambda}^{s}(\widetilde{G})$

In the last section, we have shown that $\prod_{R_D}(\widetilde{G}) \subseteq \prod_{\rho/2}^{s}(\widetilde{G})$, and in this section we will show by counting the elements in $\prod_{\rho/2}^{s}(\widetilde{G})$ that this is in fact an equality when G is split.

Fix a central character $\widetilde{\chi}$ of $\widetilde{G}$. Let $\prod_{\rho/2}^{s}(\widetilde{G})_{\widetilde{\chi}}$ be the subset of representations in $\prod_{\rho/2}^{s}(\widetilde{G})$ with central character $\widetilde{\chi}$. The goal is to count $|\prod_{\rho/2}^{s}(\widetilde{G})_{\widetilde{\chi}}|$. Take a block $\mathcal{B}$ of representations with central character $\widetilde{\chi}$ and infinitesimal character $\rho/2$, and consider $\prod(\rho/2), \triangle(\rho/2), W(\rho/2)$, the simple roots of the integral root system, the integral root system for $\rho/2$, and the integral Weyl group, respectively. Let $\mathbb{Z}[\mathcal{B}]$ be the $\mathbb{Z}$-span of the set of standard modules $I(\gamma_j)$, where each γ_j is a $\rho/2$-regular character in $\mathcal{B}$. Then $W(\rho/2)$ acts on $\mathbb{Z}[\mathcal{B}]$ by the coherent continuation action ([10]) and this action is denoted by $w \cdot I(\gamma)$, or simply $w \cdot \gamma$ for $w \in W(\rho/2)$ and $\gamma \in \mathcal{B}$.

Consider $\{J(\gamma)|\gamma \in \mathcal{B}\}$, the set of irreducible quotients of $\{I(\gamma)|\gamma \in \mathcal{B}\}$, as another basis of $\mathbb{Z}[\mathcal{B}]$, we have

Lemma 5.3.1. *Let $\alpha \in \prod(\rho/2)$, $\gamma \in \mathcal{B}$, then $s_\alpha \cdot J(\gamma) = -J(\gamma)$ if and only if $\alpha \in \tau(J(\gamma))$.*

Proof. Let $\alpha \in \prod(\rho/2)$ and let λ be an infinitesimal character which is singular for α. Define a coherent family with $\pi(\rho/2) = J(\gamma)$. Then we have the identity

$$\pi(\rho/2) + \pi(s_\alpha(\rho/2)) = \psi_{\lambda}^{\rho/2} \circ \psi_{\rho/2}^{\lambda}(\pi(\rho/2))$$

Notice that $\alpha \in \tau(J(\gamma))$ if and only if $\psi_{\rho/2}^{\lambda}(\pi(\rho/2)) = 0$, which is equivalent to

$\psi_{\lambda}^{\rho/2} \circ \psi_{\rho/2}^{\lambda}(\pi(\rho/2)) = 0$, since the functor of push-to or push-off walls is injective. We conclude that $\alpha \in \tau(J(\gamma))$ if and only if $J(\gamma) = \pi(\rho/2) = -\pi(s_\alpha \rho/2) = -s_\alpha \cdot (J(\gamma))$ (by definition of coherent continuation). $\square$

Proposition 5.3.2. $|\prod_{\rho/2}^{s}(\widetilde{G})_{\widetilde{\chi}}| = \dim Hom_{W(\rho/2)}(sgn, \mathbb{Z}[\mathcal{B}])$

Proof. Let $\pi = J(\gamma)$. Then by the previous Lemma, $\pi \in \prod_{\rho/2}^{s}(\widetilde{G})_{\widetilde{\chi}}$ if and only if $s_\alpha \cdot \pi = -\pi$ for all $\alpha \in \prod(\rho/2)$, which is equivalent to saying that $W(\rho/2)$ acts on π as the sign representation. Thus the proposition follows. $\square$

Therefore, to count the left hand side, we just need to count the right hand side in this Lemma. More precisely, we need to analyze the $W(\rho/2)$-representation $\mathbb{Z}[\mathcal{B}]$ in order to count the right hand side.

The first observation is that it makes counting more convenient if we consider a special block $\mathcal{D}$, which is equivalent to $\mathcal{B}$, instead of $\mathcal{B}$, but at an infinitesimal character other than $\rho/2$, and in a different chamber.

The following Lemma tells what this block is.

Lemma 5.3.3. *Let $\rho = \rho(\triangle^+)$, and $\lambda_0 = \rho/2$. Then we can find $w \in W$ and let $(\triangle')^+ = w\triangle^+$, $\lambda = w\rho - \frac{1}{2}\lambda_i^\vee((\triangle')^+)$, where $\lambda_i^\vee$ is the fundamental weight for α_i, such that (1) $< \lambda, \alpha_j^\vee >= 1$ when $i = j$ and $< \lambda, \alpha_j^\vee >= 1/2$ elsewhere; (2) $\triangle(\lambda) = \triangle(\lambda_0)$, (and $\prod(\lambda) = \prod(\lambda_0)$ as well). Therefore, for type A_{n-1}, we can always move to a block $\mathcal{D}$ (through nonintegral wall-crossing equivalence) with infinitesimal character $\lambda_{\mathcal{D}} = \lambda$, such that every root in $\prod(\lambda)$ is simple for all root system; for type $D_n, n \geqslant 4, E_6, E_7, E_8$, we can move to a block $\mathcal{D}$ with infinitesimal character λ, such that every root in $\prod(\lambda)$ is simple for all root system but one.*

Proof. This lemma is proved by a case by case calculation. □

The following table gives a summary of this special block $\mathcal{D}$. In the table $\prod_{\mathcal{D}}$ denotes the simple roots in the chamber of $\mathcal{D}$. Notice that the integral root system is fixed and so is $\prod(\lambda) = \prod(\rho/2)$, the simple integral roots. For type D_n, E_6, E_7 and E_8, let α denote the only integral root in $\prod(\rho/2)$ but not simple for the whole system.

Type	$\prod(\rho/2)$	$\prod_{\mathcal{D}}$	α
A_{n-1} (n even)	$e_i - e_{i+2}$, $1 \leqslant i \leqslant n-2$	$\prod \cup \{e_{n-1} - e_2\}$	N/A
A_{n-1} (n odd)	$e_i - e_{i+2}$, $1 \leqslant i \leqslant n-2$	$\prod \cup \{e_n - e_2\}$	N/A
D_n, (n even)	$e_i - e_{i+2}$, $1 \leqslant i \leqslant n-2$, $e_{n-3} + e_{n-1}, e_{n-2} + e_n$	$e_i - e_{i+2}$, $1 \leqslant i \leqslant n-2$, $e_{n-1} - e_2, e_{n-2} + e_n$	$e_{n-3} + e_{n-1}$
D_n, (n odd)	$e_i - e_{i+2}$, $1 \leqslant i \leqslant n-2$, $e_{n-3} + e_{n-1}, e_{n-2} + e_n$	$e_i - e_{i+2}$, $1 \leqslant i \leqslant n-2$, $e_n - e_2, e_{n-3} + e_{n-1}$	$e_{n-2} + e_n$
E_6		$-e_3 + e_5, -e_1 + e_3$, $-e_3 - e_5, -e_2 + e_4$, $\frac{1}{2}(1,-1,1,-1,1,-1,-1,1)$, $\frac{1}{2}(1,1,1,-1,1,1,1,-1)$	$e_2 + e_4$
E_7			
E_8		$\frac{1}{2}(1,1,1,1,1,1,-1,-1)$,	

In particular, we decompose s_α into a product of simple reflections (with respect to the chamber of $\mathcal{D}$) for type D_n for later use.

When n is even,

$$s_\alpha = s_{e_{n-1}-e_{n-2}} s_{e_{n-2}-e_n} s_{e_{n-2}+e_n} s_{e_{n-1}-e_{n-3}} s_{e_{n-1}-e_{n-2}} s_{e_{n-1}-e_{n-3}} s_{e_{n-2}+e_n} s_{e_{n-2}-e_n} s_{e_{n-1}-e_{n-2}}$$

.

When n is odd,

$$s_\alpha = s_{e_{n-2}-e_n} s_{e_n-e_{n-3}} s_{e_{n-2}+e_n} s_{e_{n-1}-e_{n-3}} s_{e_{n-1}+e_{n-3}} s_{e_n-e_{n-3}} s_{e_{n-3}+e_{n-1}} s_{e_{n-3}-e_{n-1}} s_{e_{n-2}-e_n} s_{e_n-e_{n-3}} s_{e_{n-2}-e_n}.$$

Due to the equivalence of block $\mathcal{B}$ and block $\mathcal{D}$, we'll focus on analyzing $\mathbb{Z}[\mathcal{D}]$ from now on and then count $\dim_{W(\lambda)}(sgn, Z[\mathcal{D}])$.

Now take a closer look at the coherent continuation action of $W(\lambda)$ on $\mathbb{Z}[\mathcal{D}]$. The following proposition gives explicit formulas for the action of $W(\rho/2)$ on $\{I(\gamma)|\gamma \in \mathcal{B}\}$, which can be found in [29].

Proposition 5.3.4. *Fix $\gamma \in \mathcal{B}$ and $\alpha \in \prod(\rho/2)$. Furthermore, suppose α is simple in the whole root system (See Theorem 4.12 in [27]). Let $s := s_\alpha \in W(\rho/2)$.*

(a) If α is complex or real for γ, then $s \cdot \gamma = s \times \gamma$.

(b) If α is compact imaginary for γ, then $s \cdot \gamma = -\gamma$.

(c) If α is noncompact imaginary for γ, then $s \cdot \gamma = -s \times \gamma + c_\alpha(\gamma)$.

From Proposition 5.3.4, we can see the coherent continuation action is closely related to the cross action, so we also consider the cross action of $W(\lambda)$ on $\mathbb{Z}[\mathcal{D}]$. Notice that two λ-regular characters $\gamma_i = (\widetilde{H_i}, \Gamma_i, \overline{\gamma_i})$ and $\gamma_j = (\widetilde{H_j}, \Gamma_j, \overline{\gamma_j})$ from $\mathcal{D}$ are in the same cross action orbit if and only if $\widetilde{H_i} = \widetilde{H_j}$. Indeed, if $\widetilde{H_i} = \widetilde{H_j} = \widetilde{H}$, then Γ_i and Γ_j agree on $Z(\widetilde{G})$, since $Z(\widetilde{H}) = Z(\widetilde{G})\widetilde{H_0}$ (by Proposition 4.1.4 (1)). Since γ_i and γ_j are in the same block, $\overline{\gamma_i}$ and $\overline{\gamma_j}$ define the same infinitesimal character, say, $\overline{\gamma_i} \sim \overline{\gamma_j} \sim \lambda$ and hence $\gamma_j = w \times \gamma_i$ for some $w \in W(\lambda)$. Enumerate the Cartan subgroups of $\widetilde{G}$ as $\{\widetilde{H_1}, \cdots, \widetilde{H_l}\}$, and pick a λ-regular character γ_j specified by $\widetilde{H_j}$, then $\{\gamma_1, \cdots, \gamma_l\}$ is a set of representatives of the cross action orbits of $W(\lambda)$ on

$\mathbb{Z}[\mathcal{D}]$.

Let $W_{\gamma_j} = \{w \in W(\lambda) \,|\, w \times \gamma_j = \gamma_j\}$ be the cross stabilizer of γ_j in $W(\lambda)$. Then we have the following proposition.

Proposition 5.3.5. $\mathbb{Z}[\mathcal{D}] \simeq \oplus_j Ind_{W_{\gamma_j}}^{W(\lambda)}(\epsilon_j)$, *where* ϵ_j *is a one-dimensional representation of* W_{γ_j} *such that for* $w \in W_{\gamma_j}$, $w \cdot \gamma_j = \epsilon_j(w)\gamma_j +$ *other terms from more split Cartan subgroups.*

Proof. Since W_{γ_j} is generated by $\{s_\beta | \beta \in \prod(\rho/2)\}$, it suffices to show that $s_\beta \cdot I(\gamma_j) = \pm I(s_\beta \times \gamma_j) +$ other terms from more split Cartan subgroups. This is clear when β is simple for the whole root system by Proposition 5.3.4.

Consider $\alpha \in \prod(\rho/2)$, which is not simple (as listed in the table). Let T_α be the corresponding Hecke operator, the $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-linear map from $\mathbb{Z}[\mathcal{D}][q^{1/2}, q^{-1/2}]$ to itself (defined in [29]). We can decompose $T_\alpha = T_{\alpha_1} T_{\alpha_2} \cdots T_{\alpha_m}$, where α_j's are simple. Here the α's are allowed to be non-integral. Notice that

$$T_\alpha = -\phi_\alpha \psi_\alpha + q,$$

up to a sign, where ψ_α and ϕ_α are the functors of push-to and push-off walls, respectively, and also,

$$\phi_\alpha \psi_\alpha(I(\gamma)) = I(\gamma) + s_\alpha \cdot I(\gamma).$$

We conclude that $s_\alpha \cdot I(\gamma) = -T_\alpha(1)(\gamma)$, up to a sign. Using definition 9.4 in [19], each $T_{\alpha_j}(I(\gamma))$ can be calculated explicitly, and so can $T_\alpha(I(\gamma))$. Therefore, it is not hard to see that $s_\alpha \cdot I(\gamma_j) = \pm I(s_\alpha \times \gamma_j) +$ other terms from more split Cartan subgroups. $\square$

By this proposition and Frobenius reciprocity, the multiplicity of $sgn_{W(\lambda)}$ in $\mathbb{Z}(\mathcal{D})$ is $[sgn_{W(\lambda)} : \mathbb{Z}[\mathcal{D}]] = [sgn_{W(\lambda)}|_{W_{\gamma_j}} : \epsilon_j]$, which is equal to 0 or 1, since $sgn_{W(\lambda)}|_{W_{\gamma_j}}$ is one-dimensional. This means that we have reduced our goal to count the number of γ_j's which make $[sgn_{W(\rho/2)}|_{W_{\gamma_j}} : \epsilon_j] = 1$, which is called condition $(*)$. Equivalently, condition $(*)$ is

$$sgn_{W(\lambda)}|_{W_{\gamma_j}} = \epsilon_j \qquad (*)$$

To reach this goal, we need to analyze ϵ_j for each j. By [6],

$$W_{\gamma_j} = W^C(\overline{\gamma_j})^\theta \ltimes (W^i((\overline{\gamma_j}) \times W^r(\overline{\gamma_j})).$$

So we can decompose $\epsilon_j = \epsilon_j^C \otimes \epsilon_j^i \otimes \epsilon_j^r$, where $\epsilon_j^C, \epsilon_j^i, \epsilon_j^r$ are characters of $W^C(\overline{\gamma_j})^\theta, W^i(\overline{\gamma_j}), W^r(\overline{\gamma_j})$, respectively. Notice that in the linear case (or say, when $\mathcal{B}$ is a block with integral infinitesimal character λ), we have $\epsilon_j = sgn_i$ for all j (see [10]).

Proposition 5.3.6. *If $\widetilde{G}$ has type A_n, $\epsilon_j = sgn_i$ for all γ_j.*

Proof. We are in a block $\mathcal{D}$ where every $\beta \in \prod(\rho/2)$ is simple for the whole root system. every $w \in W_{\gamma_j} = W^C(\overline{\gamma_j})^\theta \ltimes (W^i(\overline{\gamma_j}) \times W^r(\overline{\gamma_j}))$ can be written as $w = w_C w_i w_r$. Note that $W^i(\overline{\gamma_j})$ is generated by $\{s_\beta | \beta \text{ is compact imaginary for } \gamma_j\}$. By Proposition 5.3.4, $s_\beta \cdot \gamma_j = -\gamma_j$ for compact imaginary β, so $\epsilon_j(w_i) = sgn_i(w_i)$ and hence $\epsilon_j^i = sgn_i$.

Secondly, $W^r(\overline{\gamma_j})$ is generated by $\{s_\beta | \beta \text{ is a nonparity real root for } \gamma_j\}$. By Proposition 5.3.4 again, $s_\beta \cdot \gamma_j = s_\beta \times \gamma_j = \gamma_j$ for nonparity real β, so $\epsilon_j(w_r) = w_r$ and hence $\epsilon_j^r = 1$. Similar argument shows that $\epsilon_j^C = 1$. Thus we conclude that $\epsilon_j = sgn_i$. $\square$

Lemma 5.3.7. *For type A_{n-1}, if there is a real integral root for γ_j, then γ_j doesn't satisfy condition* $(*)$.

Analyzing ϵ_j for type D_n, $n \geqslant 4$, requires more work. We need to consider $\epsilon_j(s_\alpha)$ first, where α is the only root in $\prod(\rho/2)$ which is not simple. Recall that in the above table, we decompose $s_\alpha = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_m}$, a product of simple reflections. Since the coherent continuation is not defined for s_{α_j} on $\mathbb{Z}[\mathcal{D}]$ when α_j is non-integral, we pass to the level of Hecke operators (as the discussion in the proof of Proposition 5.3.5), that is, consider the decomposition $T_\alpha = T_{\alpha_1}T_{\alpha_2}\cdots T_{\alpha_m}$. Here, we need to be more careful.

Let $\gamma \in \mathcal{D}$, $T_\alpha(\gamma) = T_{\alpha_1}T_{\alpha_2}\cdots T_{\alpha_m}(\gamma)$. Note that on the right hand side, the Hecke operation is calculated step by step. In each step, we have to deal with some $T_{\alpha_k}(\delta)$, where δ is the parameter of a standard module not necessarily belonging to block $\mathcal{D}$. In fact, this is an "abstract" Hecke operation, and it should be denoted by $T_{\alpha_k}\cdot_a(\delta)$. Taking an inner automorphism ϕ_k of $\mathfrak{g}$ sending $(\lambda, \mathfrak{h}^*)$ to $(\overline{\delta}, \mathfrak{h}^*_\delta)$, we define $T_{\alpha_k}\cdot_a(\delta) := T_{\phi_k(\alpha_k)}(\delta)$. Here $\phi_k(\alpha_k)$ is a simple root in the chamber of δ, and hence we can use the formulas in Definition 9.4 of [19] to calculate $T_{\phi_k(\alpha_k)}(\delta)$ in each step.

$$\begin{aligned} T_\alpha(\gamma) &= T_{\alpha_1}\cdot_a T_{\alpha_2}\cdot_a\cdots\cdot_a T_{\alpha_m}(\gamma) \\ &= (T_{\phi_1(\alpha_1)}(T_{\phi_2(\alpha_2)}(\cdots(T_{\phi_m(\alpha_m)}(\gamma)))\cdots) \\ &= p_1(q)\cdots p_m(q)\phi_1(\alpha_1)\times(\phi_2(\alpha_2\times\cdots(\phi_m(\alpha_m)\times\gamma)) + \text{higher terms}, \end{aligned}$$

where $p_j \in \mathbb{Z}[q, q^{-1}]$

Let c_γ be the number of occurrences of complex roots in $\{\phi_j(\alpha_j), 1 \leqslant j \leqslant m\}$, and t_γ be the number of occurrences imaginary roots in $\{\phi_j(\alpha_j), 1 \leqslant j \leqslant m\}$. It turns out that

$$s_\alpha(\gamma) = -T_\alpha(1)(\gamma), \text{ when } \alpha \text{ is real or imaginay}$$

$$s_\alpha(\gamma) = (-1)^{c_\gamma} T_\alpha(1)(\gamma), \text{ when } \alpha \text{ is complex}$$

An easy calculation shows that

$$s_\alpha \cdot \gamma = (-1)^{t_\gamma} s_\alpha \times \gamma + \text{ terms from more split Cartans.} \tag{5.3.1}$$

When n is even, then $m = 9$ and

$$\{\alpha_j\} = \{e_{n-1} - e_{n-2}, e_{n-2} - e_n, e_{n-2} + e_n, e_{n-3} - e_{n-1}, e_{n-1} - e_{n-2}, e_{n-3} - e_{n-1}, e_{n-2} + e_n, e_{n-2} - e_n, e_{n-1} - e_{n-2}\}$$

$$\{\phi_j(\alpha_j)\} = \{e_{n-3} + e_{n-2}, e_{n-3} + e_n, e_{n-3} - e_n, e_{n-2} + e_{n-1}, e_{n-3} - e_{n-1}, e_{n-3} - e_{n-2}, e_{n-1} + e_n, e_{n-1} - e_n, e_{n-1} - e_{n-2}\}$$

When n is odd, then $m = 11$ and

$$\{\alpha_j\} = \{e_{n-2} - e_n, e_n - e_{n-3}, e_{n-2} - e_n, e_{n-3} - e_{n-1}, e_{n-3} + e_{n-1}, e_n - e_{n-3}, e_{n-3} + e_{n-1}, e_{n-3} - e_{n-1}, e_{n-2} - e_n, e_n - e_{n-3}, e_{n-2} - e_n\}$$

$$\{\phi_j(\alpha_j)\} = \{e_n - e_{n-2}, e_{n-3} + e_n, e_{n-3} + e_{n-2}, e_{n-1} + e_n, e_n - e_{n-1}, e_{n-2} + e_n, e_{n-2} + e_{n-1}, e_{n-2} - e_{n-1}, e_n - e_{n-3}, e_{n-2} - e_{n-3}, e_{n-2} - e_n\}$$

Due to the remark above Proposition 5.3.5, we can choose each γ_j properly and calculate the ϵ_j's according to the chosen γ_j's. In fact, our goal is to rule out γ_j's satisfying either of the following conditions.

- (R) If there is a real integral root, then choose γ_j such that α is real for γ_j.

- (C) If there are no real integral roots, and there is an orthogonal set of 4 nonintegral roots of the form $\{e_p \pm e_q . e_r \pm e_s\}$, where $e_p \pm e_q$ are both imaginary, or both real, and one of $\{e_r \pm e_s\}$ is real, whereas the other is imaginary, then choose γ_j such that this quadruple is $\{e_{n-3} \pm e_{n-2}, e_{n-1} \pm e_n\}$. In this case α is a complex root.

With the setting, we have the following key Lemma.

Lemma 5.3.8. *Suppose that $\widetilde{G}$ has type D_n, $n \geqslant 4$. If γ_j satisfies condition (R) and (C) then γ_j doesn't satisfy condition* $(*)$.

Proof. To show that the chosen γ_j fails to satisfy condition $(*)$, we will pick a $w \in W_{\gamma_j}$ and show that $\epsilon_j(w)$ and $sgn(w)$ do not coincide. In either case, we have to calculate $\epsilon_j(s_\alpha)$ for γ_j. By Equation 5.3.1, we just need to count the number t_{γ_j} for the chosen γ_j.

Suppose that n is even. If γ_j satisfies condition (R), $e_{n-3} - e_{n-1}$ is also a real integral root, and the roots $e_{n-2} \pm e_{n-1}, e_{n-2} \pm e_{n-3}, e_n \pm e_{n-1}, e_n \pm e_{n-3}$ can be arranged so that each of them is either real or complex. Therefore, $t_{\gamma_j} = 0$, and hence $\epsilon_j(s_\alpha) = 1$. This result follows for the odd case by applying the same argument. Since $s_\alpha \in W_{\gamma_j}$ and $\mathrm{sgn}(s_\alpha) = -1$, γ_j fails to satisfy condition $(*)$.

Suppose that γ_j satisfies condition (C). It can be easily counted that $t_{\gamma_j} = 3$, which implies $\epsilon_j(s_\alpha) = -1$. Let $w = s_{e_{n-3}-e_{n-1}} s_{e_{n-3}+e_{n-1}} s_{e_{n-2}-e_n} s_{e_{n-2}+e_n}$. We claim that $w \in W_{\gamma_j}$ (later). When n is even (odd, respectively), we have $e_{n-3} - e_{n-1}, e_{n-2} \pm e_n$ ($e_{n-3} \pm e_{n-1}, e_{n-2} - e_n$, respectively) are simple and complex, so $\epsilon_j(s_\beta) = 1$ for every β from these three root, and hence $\epsilon_j(w) = \epsilon_j(s_\alpha) = -1$. But it's easily seen that $sgn(w) = 1$. Therefore γ_j fails to satisfy condition $(*)$.

$\square$

Theorem 5.3.9. *For type A_{n-1} and D_n, $n \geqslant 4$, we have $\prod_\lambda^s(\widetilde{G}) = \prod_{R_D}(\widetilde{G})$.*

Proof. As stated in the beginning of the section, we have shown in the preceding sections that $\prod_{R_D}(\widetilde{G}) \subseteq \prod_\lambda^s(\widetilde{G})$. To show this is an equality, we just need to show $|\prod_\lambda^s(\widetilde{G})| = |\prod_{R_D}(\widetilde{G})|$.

By Proposition 5.3.2, fixing a genuine central character, we calculate $\dim_{W(\lambda)}(sgn, \mathbb{Z}[\mathcal{D}])$, with λ, $\mathcal{D}$ defined earlier in this section. It comes down to counting the number of γ_j's in Proposition 5.3.5 satisfying condition $(*)$.

For type A_{n-1}, we claim that if the real rank of the Cartan subgroup H_j is at least $n/2$ (when n is even) or $(n-1)/2$ (when n is odd) , then there exists a real integral root for γ_j, and hence such γ_j can be ruled out by Lemma 5.3.7.

When n is even, we enumerate all Cartan subgroups as $\{H_{n/2-1}, H_{n/2}, \cdots, H_{n-2}, H_{n-1}\}$, where the real rank of H_j is j. Let γ_{n-1} be the parameter of the principal series, and $\alpha_k = e_{2k-1} - e_{2k}$, $1 \leqslant k \leqslant n/2$, then we pick $\gamma_{n-1-k} = c_{\alpha_k} \cdots c_{\alpha_2} c_{\alpha_1}(\gamma_{n-1})$ to be the representative of the cross action orbit specified by H_{n-1-k}, $1 \leqslant k \leqslant n/2$. Notice that when $k \leqslant n/2 - 2$, $e_{n-2} - e_n$ is a real integral root for γ_{n-1-k}, which means that we can rule out γ_j, for $n/2+1 \leqslant j \leqslant n-1$. Only $\gamma_{n/2-1}$ and $\gamma_{n/2}$ are not ruled out, and they are exactly the γ_j's satisfying condition $(*)$ since the number of $\prod_{R_D}(\widetilde{G})$ with a fixed central character is also 2. Hence the theorem follows for type A_{n-1}, when n is even.

When n is odd, we enumerate all Cartan subgroups as $\{H_{(n-1)/2}, H_{(n+1)/2}, \cdots, H_{n-2}, H_{n-1}\}$, where the real rank of H_j is j. Let γ_{n-1} be the parameter of the principal series, and $\alpha_k = e_{2k-1} - e_{2k}$, $1 \leqslant k \leqslant (n-1)/2$, then we pick $\gamma_{n-1-k} = c_{\alpha_k} \cdots c_{\alpha_2} c_{\alpha_1}(\gamma_{n-1})$ to be the representative of the cross action orbit specified by H_{n-1-k}, $1 \leqslant k \leqslant (n-1)/2$. Notice that when $k \leqslant (n-3)/2$, $e_{n-2} - e_n$ is a real integral root for γ_{n-1-k}, which means that we can rule out γ_j, for $(n+1)/2 \leqslant j \leqslant n-1$. Only $\gamma_{(n-1)/2}$ is not ruled out, and hence it is exactly the only one satisfying condition $(*)$ since the number of $\prod_{R_D}(\widetilde{G})$ with a fixed central character is also 1. Hence the theorem follows for type A_{n-1}, when n is odd.

For type D_n, when n is even, we enumerate all Cartan subgroups as $\{H_j^d, 0 \leqslant j \leqslant n\}$, where the real rank of H_j^d is j, and we use the superscript d to distinguish Cartan subgroups of the same real rank but not conjugate to each other. For example, when $n = 4$, there are three Cartan subgroups of real rank 2, and they are labeled by H_2^1, H_2^2, H_2^3, all of which are isomorphic to $\mathbb{R} \times S^1 \times \mathbb{C}^\times$.

Let γ_n be the parameter of the principle series. We start with a set of orthogonal nonintegral real roots $R(\gamma_n) = \{\alpha_k, \beta_k, 1 \leqslant k \leqslant n/2\}$ of γ_n, where $\alpha_k = e_{2k-1} - e_{2k}$, $\beta_k = e_{2k-1} + e_{2k}$, and obtain γ_j^d by taking Cayley transforms through the roots in $R(\gamma_n)$. We attach to each γ_j^d a set of real roots $R(\gamma_j^d) = \{\beta \in R(\gamma_n) \,|\, \beta \text{ is real for } \gamma_j^d\}$. Now let γ_0 be the parameter of the discrete series with $R(\gamma_0) = \phi$, γ_2^1 be the parameter two steps up from γ_0, with $R(\gamma_2^1) = \{\alpha_{n/2}, \beta_{n/2}\}$, $\gamma_{n/2}^2$ be the parameter with $R(\gamma_{n/2}^2) = \{\beta_1, \cdots, \beta_{n/2}\}$, and $\gamma_{n/2}^3$ be the parameter with $R(\gamma_{n/2}^3) = \{\beta_1, \cdots, \beta_{n/2-1}, \alpha_{n/2}\}$. Observe that when $n = 4$, γ_2^1 is the representative from the middle Cartan subgroup H_2^1; when $n > 4$, choose $\gamma_{n/2}^1$ to be the representative from $H_{n/2}^1$ with $R(\gamma_{n/2}^1) = \{\beta_2, \cdots, \beta_{n/2}, \alpha_{n/2}\}$. Note that it is possible that there exists $\gamma^2_{n/2}$, $d > 3$. In this case, choose $\gamma^d_{n/2}$ such that $\{\alpha_{n/2-1}, \alpha_{n/2}, \beta_{n/2-1}, \beta_{n/2}\} \subseteq R(\gamma^d_{n/2})$.

Now we claim that if γ^d_j is not one of these four, then it satisfies either condition (R) or (C), and hence can be ruled out by Lemma 5.3.8.

When $j \geqslant n/2+2$, γ^d_j can be chosen so that $\{\alpha_{n/2-1}, \alpha_{n/2}, \beta_{n/2-1}, \beta_{n/2}\} \subseteq R(\gamma^d_j)$ and hence $e_{n-2} \pm e_n$ are real integral roots of γ^d_j.

Now suppose $n > 4$. We observe that $\gamma^1_{n/2}$ satisfies condition (C) since there is a a quadruple $\{\alpha_{n/2-1}, \alpha_{n/2}, \beta_{n/2-1}, \beta_{n/2}\}$, where $\alpha_{n/2}, \beta_{n/2}, \beta_{n/2-1}$ are real, and $\alpha_{n/2-1}$ is imaginary for $\gamma^1_{n/2}$. For $d > 3$, since $\{\alpha_{n/2-1}, \alpha_{n/2}, \beta_{n/2-1}, \beta_{n/2}\} \subseteq R(\gamma^d_{n/2})$, $e_{n-2} \pm e_n$ are real imaginary roots of $\gamma^d_{n/2}$, and hence $\gamma^d_{n/2}$ satisfies condition (R).

Any $\gamma^d_{n/2+1}$ is obtained from some $\gamma^{d'}_{n/2}$ by an inverse Cayley transform, that is, we can choose $\gamma^d_{n/2+1}$ such that $R(\gamma^d_{n/2+1})$ is obtained from $R(\gamma^{d'}_{n/2})$ by adding a root. But adding a root to $R(\gamma^{d'}_{n/2})$ would result in either a real integral roots or a quadruple as described in condition (C) for $\gamma^d_{n/2+1}$.

Finally we observe γ^d_j, $j > n/2$. Every $\gamma^d_{n/2}$ can be obtained from some $\gamma^{d'}_{n/2}$ by a sequence of Cayley transforms through roots in $R(\gamma^{d'}_{n/2})$, that is, γ^d_j is chosen such that $R^d_{\gamma_j}$ is obtained by removing roots from $R(\gamma^{d'}_{n/2})$. It turns out that when $j > n/2$, there would be a quadruple as described in condition (C) for all γ^d_j, except γ_0 and γ^1_2. We conclude that γ_0, γ^1_2, $\gamma^2_{n/2}$, $\gamma^3_{n/2}$ are the γ_j's satisfying condition $(*)$ since the number of $\prod_{R_D}(\widetilde{G})$ with a fixed central character is exactly 4. Hence the theorem follows for type D_n, $n \geqslant 4$, when n is even.

For type D_n, when n is odd, we again enumerate all Cartan subgroups as $\{H^d_j, 1 \leqslant j \leqslant n\}$, where the real rank of H^d_j is j.

Let γ_n be the parameter of the principle series. We start with a set of orthogonal nonintegral real roots $R(\gamma_n) = \{\alpha_k, \beta_k, 1 \leqslant k \leqslant (n-1)/2\}$ of γ_n, where $\alpha_k = e_{2k-1} - e_{2k}$, $\beta_k = e_{2k-1} + e_{2k}$, and obtain γ_j^d by taking Cayley transforms through the roots in $R(\gamma_n)$. We attach to each γ_j^d a set of real roots $R(\gamma_j^d) = \{\beta \in R(\gamma_n) \,|\, \beta \text{ is real for } \gamma_j^d\ \}$. Now let γ_1 be the representative of the fundamental series with $R(\gamma_1) = \phi$, $\gamma^1_{(n-1)/2}$ be the parameter with $R(\gamma^1_{(n-1)/2}) = \{\beta_1, \cdots, \beta_{(n-1)/2}\}$. Note that there exists $\gamma^d_{(n-1)/2}$, $d > 1$. In this case, we choose $\gamma^d_{n/2}$ such that $\{\alpha_{(n-1)/2}, \beta_{(n-1)/2-1}\} \subseteq R(\gamma^d_{n/2})$.

Now we claim that if γ_j^d is a parameter other than γ_1 and $\gamma^1_{(n-1)/2}$, then it satisfies either condition (R) or (C), and hence can be ruled out by Lemma 5.3.8.

When $j \geqslant (n-1)/2+1$, γ_j^d can be chosen so that $\{\alpha_{(n-1)/2}, \beta_{(n-1)/2-1}\} \subseteq R(\gamma_j^d)$ and hence $e_{n-2} \pm e_n$ are real integral roots of γ_j^d. For the same reason, $e_{n-2} \pm e_n$ are also real integral roots of $\gamma^d_{(n-1)/2}$, for $d > 1$.

The remaining parts to deal with are γ_j^d's, $j < (n-1)/2$. Every $\gamma^d_{(n-1)/2}$ can be obtained from some $\gamma^{d'}_{(n-1)/2}$ by a sequence of Cayley transforms through roots in $R(\gamma^{d'}_{(n-1)/2})$, that is, γ_j^d is chosen such that $R(\gamma_j^d)$ is obtained by removing roots from $R(\gamma^{d'}_{(n-1)/2})$. It turns out that when $j < (n-1)/2$, there would be a quadruple as described in condition (C) for all γ_j^d, except γ_1. We conclude that γ_0 and $\gamma^1_{(n-1)/2}$ are the γ_j's satisfying condition $(*)$ since the number of $\prod_{R_D}(\widetilde{G})$ with a fixed central character is exactly 2. Hence the theorem follows for type D_n, $n \geqslant 4$, when n is odd.

$\square$

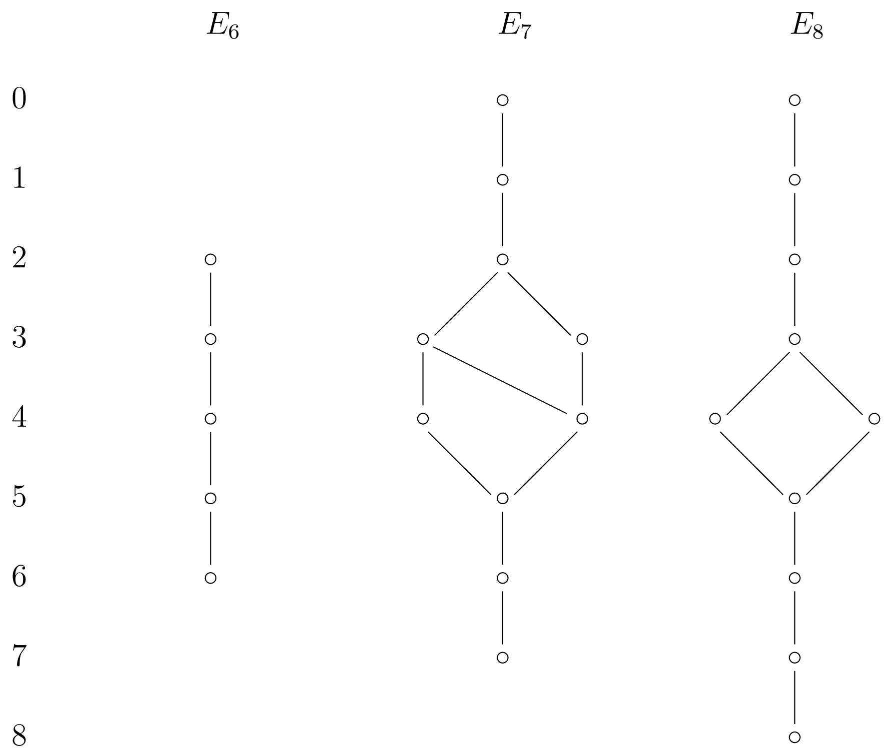
Table 5.8:

	Counting for γ_j's satisfying $(*)$		representations in $\prod_{R_D}(\widetilde{G})$	
Type	Cartan subgroups	real rank	Cartan subgroup	real rank
A_{n-1}	$(\mathbb{C}^\times)^{\frac{n}{2}-1} \times S^1$	$\frac{n}{2}-1$	$(\mathbb{R}^\times)^{n-1}$	$n-1$
(n even)	$\mathbb{R}^\times \times (\mathbb{C}^\times)^{\frac{n}{2}-1}$	$\frac{n}{2}$	$(\mathbb{C}^\times)^{\frac{n}{2}-1} \times S^1$	$\frac{n}{2}-1$
A_{n-1} (n odd)	$(\mathbb{C}^\times)^{\frac{n-1}{2}}$	$\frac{n-1}{2}$	$(\mathbb{R}^\times)^{n-1}$	$n-1$
	$(S^1)^n$	0	$(\mathbb{R}^\times)^n$	n
D_n	$\mathbb{R}^\times \times \mathbb{C}^\times \times (S^1)^{n-3}$	2	$(\mathbb{R}^\times)^{n-3} \times \mathbb{C}^\times \times S^1$	$n-2$
(n even)	$\mathbb{R}^\times \times (\mathbb{C}^\times)^{\frac{n}{2}-1} \times S^1$	$\frac{n}{2}$	$\mathbb{R}^\times \times (\mathbb{C}^\times)^{\frac{n}{2}-1} \times S^1$	$\frac{n}{2}$
	$\mathbb{R}^\times \times (\mathbb{C}^\times)^{\frac{n}{2}-1} \times S^1$	$\frac{n}{2}$	$\mathbb{R}^\times \times (\mathbb{C}^\times)^{\frac{n}{2}-1} \times S^1$	$\frac{n}{2}$
D_n	$\mathbb{C}^\times \times (S^1)^{n-2}$	1	$(\mathbb{R}^\times)^n$	n
(n odd)	$(\mathbb{C}^\times)^{\frac{n-1}{2}} \times S^1$	$\frac{n-1}{2}$	$(\mathbb{C}^\times)^{\frac{n-1}{2}} \times S^1$	$\frac{n-1}{2}$

We compare the Cartan subgroups where the γ_j's satisfying condition $(*)$ when counting $|\prod^s_{\rho/2}(\widetilde{G})|$ come from and the ones where the representations in $\prod^s_{\rho/2}(\widetilde{G}) = \prod_{R_D}(\widetilde{G})$ actually come from. Table 5.8 is a summary.

We would like to do the same thing in type E, parallel to the case of type D_n. Like in type D_n, we can also move to a block $\mathcal{D}$ where all integral simple roots are simple but one, say α. Then there come some difficulties. First, to decompose s_α into a product of simple reflections is never easy, and after having done with that, we have to keep track of a sequence of inner automorphisms when trying to calculate the coherent continuation action $s_\alpha \cdot \gamma$, where γ is a standard module parameter.

Figure 5.1:



Even though this complication has not been solved yet, we strongly believe that the counting $|\prod_{\rho/2}^{s}(\widetilde{G})| = |\prod_{R_D}(\widetilde{G})|$ in Theorem 5.3.9 holds for type E_6, E_7 and E_8.

We conjecture that when we count the number of γ_j satisfying condition $(*)$, those satisfying condition (R) or (C) should be ruled out.

Conjecture 5.3.10. *For type E_6, E_7 and E_8, γ_j does not satisfy condition $(*)$ if it satisfy condition (R) or (C).*

The Cartan diagrams of type E_6 to E_8 are listed in Figure 5.1.

For type E_6, it can be shown that for every Cartan subgroup H_j of real rank greater than 2, γ_j can be chosen to satisfy condition (R). If Conjecture 5.3.10 is true, then the only γ_j satisfying condition $(*)$ comes from the fundamental Cartan $(\mathbb{C}^\times)^2 \times (S^1)^\times$.

For type E_8, it can be shown that if the real rank of the Cartan H_j is at least 4, then γ_j can be chosen to satisfy condition (R); if the real rank of H_j is greater than 0, then γ_j can be chosen to satisfy condition (C). It turns out that the only γ_j satisfying condition $(*)$ comes from the compact Cartan $(S^1)^8$ if Conjecture 5.3.10 is true.

For type E_7, we enumerate all Cartans as $\{H_0, H_1, H_2, H_3^1, H_3^2, H_4^1, H_4^2, H_5, H_6, H_7\}$. Here we denote $H_4^1 = (\mathbb{R}^\times)^2 \times (\mathbb{C}^\times)^2 \times S^1$, $H_4^2 = \mathbb{R}^\times \times (\mathbb{C}^\times)^3$, $H_3^1 = \mathbb{R}^\times \times (\mathbb{C}^\times)^2 \times (S^1)^2$, $H_3^2 = (\mathbb{C})^3 \times S^1$. Here every the real rank of each H_j (or H_j^d) is j. It can be shown that for H_7, H_6, H_5, H_4^1, γ_j can be chosen to satisfy condition (R), and for H_1, H_2, H_3^1, γ_j can be chosen to satisfy condition (C). This means that these Cartans can be ruled out if Conjecture 5.3.10 is true. We know that the number of representations in $\prod_{R_D}(\widetilde{G})$ with a fixed central character is two, so we expect we will get two γ_j's satisfying condition $(*)$. We expect one is from H_0, the compact Cartan, as usual, but we have not had any clues that the other comes from H_3^2 or H_4^2.

Chapter 6: Relation to the pairs $(\chi, \mathcal{O}_{\mathbb{R}})$

6.1 Number of Genuine Central Characters and Real Associated Varieties

In Chapter 3 we discuss the real group G such that $\mathcal{O} \cap \mathfrak{g}_{\mathbb{R}} \neq \phi$ (see Remark 3.0.6). In Table 6.1, we list the center of these groups, and then the number of genuine central characters of $\widetilde{G}$, compared to the number of real forms of $\mathcal{O}$, which is denoted $\#\{\mathcal{O}_i\}$.

Therefore we have the following observation, which follows from Tables 5.1 and 6.1.

Lemma 6.1.1. *Suppose G is simply laced and split, then $|\prod_g \widetilde{(G)}| = \#\{\mathcal{O}_i\}$, which also matches the number of Shimura representations.*

Denote

$$\mathcal{CO}(\widetilde{G}) = \{(\chi, \mathcal{O}_{\mathbb{R}}) \mid \chi \text{ is a genuine central character of } \widetilde{G}, \mathcal{O}_{\mathbb{R}} \text{ is a real form of } \mathcal{O}\}$$

Then we have the following theorem.

Theorem 6.1.2. *Suppose G is a real form of a simply connected, semisimple complex Lie group, and $\widetilde{G}$ is the nontrivial two-fold cover of G. Moreover, suppose G is*

Table 6.1:

G	$Z(G)$	$Z(\widetilde{G})$	$\lvert \prod_g (Z(\widetilde{G}))\rvert$	$\#\{\mathcal{O}_i\}$
$SL(2m,\mathbb{R})$ (split)	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$ (m even) $\mathbb{Z}_4$ (m odd)	2	2
$SL(2m+1,\mathbb{R})$ (split)	1	$\mathbb{Z}_2$	1	1
$SU(m,m)$ (quasisplit)	$\mathbb{Z}_{2m}$		$2m$	$m+1$
$SU(m+1,m)$ (quasisplit)	$\mathbb{Z}_{2m+1}$		$2m+1$	$m+1$
$Spin(2m+1,2m)$ (split)	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2	2
$Spin(2m+2,2m+1)$ (split)	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2	1
$Spin(2m+3,2m)$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2	1
$Sp(2n,\mathbb{R})$ (split)	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$ (n even) $\mathbb{Z}_4$ (n odd)	2	2
$Spin(2m,2m)$ (split)	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (m even) $\mathbb{Z}_2 \times \mathbb{Z}_4$ (m odd)	4	4
$Spin(2m+1,2m+1)$ (split)	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2	2
$Spin(2m+1,2m-1)$ (quasisplit)	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2	2
$Spin(2m+2,2m)$ (quasisplit)			4	2
$Spin(2m+3,2m-1)$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2	2
$E_6(A_1 \times A_5)$ (quasisplit)	$\mathbb{Z}_3$		3	2
$E_6(C_4)$ (split)	1	$\mathbb{Z}_2$	1	1
$E_7(A_7$ (split)	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2	2
$E_8(D_8)$ (split)	1	$\mathbb{Z}_2$	1	1
$F_4(B_4)$ (split)	1	$\mathbb{Z}_2$	1	1
$G_2(A_1 \times A_1)$	1	$\mathbb{Z}_2$	1	1

simply laced and split. Then there is a one-to-one correspondence between $\prod_{\rho/2}^{s}(\widetilde{G})$ *and* $\mathcal{CO}(\widetilde{G})$ *.*

Proof. We will show this case by case.

First, the theorem is obvious for type A_{n-1}, n is odd, E_6, and E_8, since $|\{Sh_i\}| = \#\{\mathcal{O}_i\} = |\mathcal{CO}(\widetilde{G})| = 1$; and just map the unique Shimura representation to the unique pair $(\chi, \mathcal{O}_{\mathbb{R}} \in \mathcal{CO}(\widetilde{G})$.

For type D_n, $n = 2m$, i.e. $G = Spin(2m, 2m)$. Using the same notations as in Example 5.2.10. Suppose $\mathcal{O}_i$ is the real associated variety of Sh_i, then each Sh_i is corresponding to the pair $(\chi_i, \mathcal{O}_i)$. From the first table in Example 5.2.10, we have the action of Out(G) on genuine central characters of $\widetilde{G}$, and hence Out(G) also permutes the representations and their real associated variety. More precisely, for $\xi \in$Out(G), $\chi^{\xi} = \chi_j$ if and only if $\mathcal{O}_i^{\xi} = \mathcal{O}_j$. For example, all of the representations π, $i = 1, \cdots, 4$, have different real associated varieties, and similar for δ_i's and τ_i's. Furthermore, from Table 5.4 and 5.5 of Section 5.2, we observe that Sh_1, π_2, δ_3, τ_4 have the same asymptotic K-types, and hence they share the same real associated variety, say, $\mathcal{O}_1$, Similarly, $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_4$ are shared by four different representations from $\prod_{\rho/2}^{s}(\widetilde{G})$, respectively. Therefore, we have the precise correspondence between $\prod_{\rho/2}^{s}(\widetilde{G})$ and $\mathcal{CO}(\widetilde{G})$ illustrated in Table 6.3. (Note that every representation is parametrized by the highest weight of its lowest K-type.)

Similary, for type D_n, $n = 2m + 1$, we have the correspondence illustrated in Table 6.5, which follows from Table 5.6.

$\square$

Table 6.3:

	$\mathcal{O}_1$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_4$
χ_1	Sh_1 $(\frac{1}{2},\cdots,\frac{1}{2};0,\cdots,0)$	π_1 $(\frac{3}{2},\frac{1}{2},\cdots,-\frac{1}{2};0,\cdots,0)$	δ_1 $(\frac{1}{2},\cdots,\frac{1}{2};1,\cdots,1)$	τ_1 $(\frac{1}{2},\cdots,\frac{1}{2};1,\cdots,-1)$
χ_2	π_2 $(\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2};0,\cdots,0)$	Sh_2 $(\frac{1}{2},\cdots,-\frac{1}{2};0,\cdots,0)$	τ_2 $(\frac{1}{2},\cdots,-\frac{1}{2};1,\cdots,1)$	δ_2 $(\frac{1}{2},\cdots,-\frac{1}{2};1,\cdots,-1)$
χ_3	δ_3 $(1,\cdots,1;\frac{1}{2},\cdots,\frac{1}{2})$	τ_3 $(1,\cdots,-1;\frac{1}{2},\cdots,\frac{1}{2})$	Sh_3 $(0,\cdots,0;\frac{1}{2},\cdots,\frac{1}{2})$	π_3 $(0,\cdots,0;\frac{3}{2},\frac{1}{2},\cdots,-\frac{1}{2})$
χ_4	τ_4 $(1,\cdots,1;\frac{1}{2},\cdots,-\frac{1}{2})$	δ_4 $(1,\cdots,-1;\frac{1}{2},\cdots,-\frac{1}{2})$	π_4 $(0,\cdots,0;\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2})$	Sh_4 $(0,\cdots,0;\frac{1}{2},\cdots,-\frac{1}{2})$

Table 6.5:

	$\mathcal{O}_1$	$\mathcal{O}_2$
χ_1	Sh_1 $(\frac{1}{2},\cdots,\frac{1}{2};0,\cdots,0)$	π_1 $(\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2};0,\cdots,0)$
χ_2	π_2 $(0,\cdots,0;\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2})$	Sh_2 $(0,\cdots,0;\frac{1}{2},\cdots,\frac{1}{2})$

Chapter 7: Lifting of the Trivial Representation

In this chapter, we will restrict the attention to the simply laced real groups. More precisely, the setting is stated in the beginning of the introduction. Let $G_{\mathbb{C}}$ be a simply connected, semisimple, simply laced complex Lie group, and G be a connected real form of $G_{\mathbb{C}}$ with nontrivial fundamental group, and consider the nontrivial two-fold cover $\widetilde{G}$ of G. Now we're going to introduce the key tool, the lifting operator, which relates genuine characters of $\widetilde{G}$ to characters of G. By character we mean the character of a representation, viewed as a function on the regular semisimple elements.

7.1 Lifting Operator

Now suppose $\pi \in \widehat{G}_{adm}$, with character Θ_π viewed as a function on G', the set of regular semisimple elements of G.

Definition 7.1.1. Let $\pi \in \widehat{G}_{adm}$, with character Θ_π. We say π and Θ_π are *stable* if Θ_π is invariant under conjugation of $G_{\mathbb{C}}$, that is, $\Theta_\pi(g) = \Theta_\pi(g')$ if $g, g' \in G'$ and $g' = xgx^{-1}$ for some $x \in G(\mathbb{C})$.

Suppose H is a Cartan subgroup of G and Φ^+ is a set of positive roots of H in G. For $h \in H$ we have the Weyl denominator

$$|D(h)|^{\frac{1}{2}} = |\prod_{\alpha\in\Phi^+}(1-\alpha^{-1}(h))||e^{\rho}(h)| \text{ (see [4]).}$$

Definition 7.1.2. (see [4]) Suppose $\pi \in \widehat{G}_{adm}$ and π is stable. For $\widetilde{g} \in \widetilde{G}'$, define

$$\mathrm{Lift}_G^{\widetilde{G}}(\Theta_\pi)(\widetilde{g}) = \sum_{\{h\in G | h^2 = p(\widetilde{g})\}} \Delta(h,\widetilde{g})\Theta_\pi(h).$$

Here $\Delta(h,\widetilde{g})$ is a certain function on $G' \times \widetilde{G'}$ satisfying the following conditions:

$$\Delta(h,\widetilde{g}) = 0 \text{ unless } h^2 = p(\widetilde{g})$$

$$|\Delta(h,\widetilde{g})| = |D(h)|^{\frac{1}{2}}/|D(\widetilde{g})|^{\frac{1}{2}}$$

$$\Delta(xhx^{-1}, \widetilde{x}\widetilde{g}\widetilde{x}^{-1}) = \Delta(h,\widetilde{g}) \quad (\widetilde{x}\in\widetilde{G}, x = p(\widetilde{x}))$$

$$\Delta(h,-\widetilde{g}) = -\Delta(h,\widetilde{g})$$

By section 5 in [4], since $G_{\mathbb{C}}$ is simply connected and semisimple, the function Δ is canonical.

The following theorem is a special case of the main theorem of [4]. Since $G_{\mathbb{C}}$ is simply connected and semisimple, a simplified version of Section 5 in [4] applies.

Theorem 7.1.3. *Assume the setting in the beginning of this chapter. Then there is a canonical function (see Section 5 in [4]) $\Delta(h,\widetilde{g})$ satisfying the conditions in Definition 7.1.2, such that for all stable admissible representation π of G,*

$$\mathit{Lift}(\Theta_\pi)(\widetilde{g}) = \sum_{\{h\in G | h^2 = p(\widetilde{g})\}} \Delta(h,\widetilde{g})\Theta_\pi(h)$$

is the character of a genuine virtual representation $\widetilde{\pi}$ of $\widetilde{G}$, or 0. We say $\widetilde{\pi}$ is the lift of π and write $\widetilde{\pi} = \mathit{Lift}_G^{\widetilde{G}}(\pi)$, where $\Theta_{\widetilde{\pi}} = \mathit{Lift}_G^{\widetilde{G}}(\Theta_\pi)$.

Because of this theorem, for a stable admissible representation π of G, we will denote $\mathrm{Lift}_G^{\widetilde{G}}(\pi)$ as a set as follows. If $\mathrm{Lift}_G^{\widetilde{G}}(\pi) = \sum_{\widetilde{\pi}} a_{\widetilde{\pi}}\widetilde{\pi}$, for $a_{\widetilde{\pi}} \in \mathbb{Z}$ and $\widetilde{\pi} \in \prod_g(\widetilde{G})$, the set of genuine irreducible representations of $\widetilde{G}$, then the set

$$\text{Lift}_G^{\widetilde{G}}(\pi) = \{\widetilde{\pi} \in \textstyle\prod_g(\widetilde{G}) \,|\, a_{\widetilde{\pi}} \neq 0\},$$

and this is a finite set of irreducible genuine representations due to Theorem 7.1.3.

Lifting of regular characters is also defined (see Theorem 19.1 in [4]), and is written as $\widetilde{\gamma}$ =$\text{Lift}_G^{\widetilde{G}}(\gamma)$. Lifting of a stable sum of standard modules is also well-understood in [4] (see Corollary 19.8). We quote the important result as follows. Let $I_G^{st}(\gamma)$ be a stable sum of standard modules of G with parameter γ, W_i be the imaginary Weyl group, then

Theorem 7.1.4. *([4]) Let $\{\widetilde{\gamma}_1, \cdots, \widetilde{\gamma}_n\}$ be the set of constituents of $Lift_G^{\widetilde{G}}(w\gamma)$ as w runs over W_i, considered without multiplicity. Then*

$$Lift_G^{\widetilde{G}}(I_G^{st}(\gamma)) = C(H) \sum_{i=1}^{n} I_{\widetilde{G}}(\widetilde{\gamma}_i),$$

where $C(H) = c(H)/c(H_s)$, $c(H) = |H_2^0||H/Z_0(H)|^{\frac{1}{2}}$, H_s is the maximally split Cartan subgroup of G, H_2^0 is the subgroup of elements of order 2 in the identity component of H, $Z_0(H) = p(Z(\widetilde{H}))$. Note that all the constituents have distinct central characters, so are a fortiori distinct, and that $C(H)$ is normalized so that $C(H_s) = 1$.

Now restrict the attention to one-dimensional representations. Since G is connected, the only one-dimensional representation is $\mathbb{C}$, the trivial representation. We are interested in the representations in the set $\text{Lift}_G^{\widetilde{G}}(\mathbb{C})$, which will be written as $\text{Lift}(\mathbb{C})$ for simplicity. It's not surprising that the representations in $\text{Lift}(\mathbb{C})$ are small in the sense that we discussed in Definition 2.1.6 in Chapter 2.

7.2 Properties of representations in Lift($\mathbb{C}$)

In [5], when $G = GL(n, \mathbb{R})$, $\widetilde{\pi} =$Lift$_G^{\widetilde{G}}(\pi)$ has infinitesimal character $\rho/2$ and maximal τ-invariant for one-dimensional representation π of G, assuming $\widetilde{\pi} \neq 0$. The same is true for various simply laced connected group G. Before stating the result, we need some Lemmas.

Lemma 7.2.1. *Let π be a stable admissible virtual representation of G with infinitesimal character λ. Assume that Lift$(\pi) \neq \phi$, then every $\widetilde{\pi} \in$Lift(π) has infinitesimal character $\lambda/2$.*

Proof. In fact, we just need to show the case for standard modules since standard modules span virtual modules.

Let $I_G^{st}(\gamma)$ be a stable sum of standard modules, and $\gamma = (H, \Gamma, \lambda)$ be the regular character parametrizing it. By Definition 17.5 in [4], we can define formal lifting data of $\widetilde{G}$ if Lift$_H^{\widetilde{H}}(\Gamma) \neq 0$, say, Lift$_G^{\widetilde{G}}(\gamma) = \{\widetilde{\gamma}_1, \cdots, \widetilde{\gamma}_n\}$, where each $\widetilde{\gamma}_i = (\widetilde{H}, \widetilde{\Gamma}_i, \frac{1}{2}(\lambda - \mu))$ is a genuine regular character of $\widetilde{G}$. It turns out that each $\widetilde{\gamma}_i$ has infinitesimal character $\lambda/2$ since $G_{\mathbb{C}}$ is simple and simply connected, lifting is canonical and $\mu = 0$ (see chapter 5 in [4]). Now by Theorem 7.1.4, Lift$_G^{\widetilde{G}}(I_G^{st}(\gamma))$ is a sum of $I_{\widetilde{G}}(\widetilde{\gamma}_i)$ and hence the infinitesimal character of Lift$_G^{\widetilde{G}}(I_G^{st}(\gamma))$ is $\lambda/2$. $\square$

Lemma 7.2.2. *Let π be a representation of G described as in Lemma 7.2.1 and F be a finite dimensional representation of $\widetilde{G}$ (as well as of G) with the set of weights $\triangle(F)$, then*

$$\textit{Lift}(\pi) \otimes F = \textit{Lift}(\pi \otimes F')$$

for some virtual finite dimensional representation F' with weights $\triangle(F') = 2\triangle(F)$.

Proof. The proof is analogous to the case of $GL(n, \mathbb{R})$, see [5]. $\square$

Now we are ready to prove the following result.

Theorem 7.2.3. *Let $0 \neq \widetilde{\pi} \in$Lift$(\mathbb{C})$. Then $\widetilde{\pi}$ has infinitesimal character $\rho/2$ and maximal τ-invariant.*

Proof. The first assertion of the theorem is a corollary of Lemma 7.2.1 since the infinitesimal character of $\mathbb{C}$ is ρ.

To show that $\widetilde{\pi}$ has maximal τ-invariant, we will show that $\psi_\alpha(\widetilde{\pi}) = 0$ for all $\alpha \in \prod(\rho/2)$, where ψ_α is the Zuckerman translation functor to the α-wall. Fix $\alpha \in \prod(\rho/2)$ and let λ be singular with respect to α and suppose $\gamma = \rho/2 - \lambda$ is a weight. (Indeed, Let $c =< \rho/2, \alpha^\vee >\in \mathbb{Z}$ and λ can be chosen to be $\rho/2 - c\lambda_\alpha$, where λ_α is the fundamental weight for α, and hence it's easy to check that λ is singular for α. Thus, $\gamma = \rho/2 - \lambda = c\lambda_\alpha$ is a weight.) Since $G_\mathbb{C}$ is simply connected, γ is the highest weight of a finite dimensional representation, say, F. Let $\triangle(F)$ be the set of its weights. The goal is to show that none of the constituents in $\widetilde{\pi} \otimes F$ have infinitesimal character λ.

Let F' be the virtual finite dimensional representation as in Lemma 7.2.2, and hence by the same Lemma we have

$$\widetilde{\pi} \otimes F \in \text{Lift}(\mathbb{C} \otimes F') = \text{Lift}(\sum_{\mu \in \triangle(F)} \pi(\rho + 2\mu)),$$

assuming that $\mathbb{C} = \pi(\rho)$ and each $\pi(\rho + 2\mu)$ has infinitesimal character $\rho + 2\mu$. By Lemma 2.1.5, none of these $\pi(\rho + 2\mu)$'s have infinitesimal character 2λ, and hence none of the representations of $\mathrm{Lift}(\mathbb{C} \otimes F')$ have infinitesimal character λ, and we conclude that none of the constituents of $\widetilde{\pi} \otimes F$ has infinitesimal character λ. Then the theorem follows. $\square$

We will apply the same notation in Chapter 2 to $\widetilde{G}$, except that $\prod_\lambda^s(\widetilde{G})$ and $\prod_\lambda^{\mathcal{O}}(\widetilde{G})$ denote the sets of genuine representations of $\widetilde{G}$ possessing the respective properties. Since we just care about simply laced groups, $\lambda = \rho/2$. Therefore we have the theorem.

Corollary 7.2.4. *Assume the setting in the beginning of the chapter for G and $\widetilde{G}$. Then*

$$\mathit{Lift}(\mathbb{C}) \subseteq \textstyle\prod_{\rho/2}^s(\widetilde{G}).$$

Then from Theorem 2.2.3, we have the picture:

$$\mathrm{Lift}(\mathbb{C}) \subseteq \textstyle\prod_{\rho/2}^s(\widetilde{G}) = \prod_{\rho/2}^{\mathcal{O}}(\widetilde{G})$$

Corollary 7.2.5. *$\mathit{Lift}_G^{\widetilde{G}}(\mathbb{C}) = 0$ if G is not in Tables 3.1, 3.5, and 3.7 in Chapter 3. Therefore, if $\mathit{Lift}_G^{\widetilde{G}}(\mathbb{C}) \neq 0$ then G is quasisplit with one exception.*

7.3 Characterization of Lift($\mathbb{C}$) for split groups

In this section we suppose that G is split. We want to describe $\mathrm{Lift}(\mathbb{C})$ more explicitly. In fact we will show that every representation in $\prod_{R_D}(\widetilde{G})$ (defined in Section 5.2.2) is in $\mathrm{Lift}(\mathbb{C})$.

Let $I_G(\gamma_0)$ be the standard module of $\mathbb{C}$ and write $\mathbb{C}$ as a sum of standard modules, say, $\mathbb{C} = \sum\limits_{\gamma} M(\gamma, \gamma_0) I_G(\gamma)$. Then after lifting, we get

$$\mathrm{Lift}_G^{\widetilde{G}}(\mathbb{C}) = \sum_{\text{some } \widetilde{\gamma}} c_{\widetilde{\gamma}} I_{\widetilde{G}}(\widetilde{\gamma}) \text{ some coefficient } c_{\widetilde{\gamma}} \tag{7.3.1}$$

Next let $\widetilde{I}_{Sh_i}$ denote the standard module of the Shimura representation Sh_i with central character χ_i. Then we can write $\widetilde{I}_{Sh_i} = I_{\widetilde{G}}(\widetilde{\gamma}_i)$, where $\widetilde{\gamma}_i = (\widetilde{H}, \widetilde{\Gamma}_i, \overline{\widetilde{\gamma}_i})$ with $\widetilde{\Gamma}_i|_{Z(\widetilde{G})} = \chi_i$ and $\overline{\widetilde{\gamma}_i} \sim \rho/2$. Then we have the Lemma.

Lemma 7.3.1. *$Lift_G^{\widetilde{G}}(I_G(\gamma_0)) \neq 0$ and $Lift_G^{\widetilde{G}}(I_G(\gamma_0)) = \sum_i I_{\widetilde{G}}(\widetilde{\gamma}_i)$.*

Proof. This is a corollary of Theorem 7.1.4. □

Let $S \in R_D$ and $\pi_i = c_S(Sh_i) \in \prod_{R_D}(\widetilde{G})$. Now define $\mathcal{I}_{S,i}$ to be the family of standard modules $\widetilde{I}$ of $\widetilde{G}$ obtained from $\widetilde{I}_{Sh_i}$ by a sequence of inverse Cayley transforms $c_{S'}$, where S' is a subset of S. Then we have the following Lemma.

Lemma 7.3.2. *Fix $S \in R_D$ and fix a Shimura representation Sh_i with central character χ_i. Also assume the above notations and let $\widetilde{\nu}_i$ be the regular character specifying π_i, say, the standard module of π_i, denoted $\widetilde{I}_{\pi_i}$, is equal to $I_{\widetilde{G}}(\widetilde{\nu}_i)$. Then*

(1) for every $S' \subseteq S$, $M(c_{S'}(\gamma_0), \gamma_0) \neq 0$, i.e. every $I_G(c_{S'}(\gamma_0))$ occurs in the character formula of $\mathbb{C}$;

(2) for every $S' \subseteq S$, $I_G(c_{S'}(\gamma_0))$ is a stable sum of standard modules and $Lift(I_G(c_{S'}(\gamma_0))) \neq 0$. Moreover, $Lift(I_G(c_{S'}(\gamma_0))) = c_{S'}(Lift_G^{\widetilde{G}}((I_G(\gamma_0)))) = \sum_i I_{\widetilde{G}}(c_{S'}(\widetilde{\gamma}_i))$.

(3) For a genuine regular character $\widetilde{\gamma}$ of $\widetilde{G}$, $m(\widetilde{\nu}_i, \widetilde{\gamma}) \neq 0$ if and only if $\widetilde{\gamma} = c_{S'}(\widetilde{\gamma}_i)$ for some $S' \subseteq S$. In this case, we have $m(\widetilde{\nu}_i, \widetilde{\gamma}) = 1$. This means that all of the standard modules in $\mathcal{I}_{S,i}$ appear in Equation 7.3.1, and they are the only standard modules of $\widetilde{G}$ containing $\pi_i = c_S(Sh_i)$.

Proof. The third part of this Lemma is obvious when $S = \{\phi\}$. In this case, $\mathcal{I}_{S,i} = \{\widetilde{I}_{Sh_i}\}$, and $\widetilde{I}_{Sh_i}$ appears in Equation 7.3.1 because of Lemma 7.3.1 and no other standard modules rather that $\widetilde{I}_{Sh_i}$ since $\widetilde{I}_{Sh_i}$ is the only standard module coming from the split Cartan $\widetilde{H}$ with central character χ_i.

□

Theorem 7.3.3. $\prod_{R_D}(\widetilde{G}) \subseteq$ *Lift*$(\mathbb{C})$.

Proof. We just need to show that the coefficient of each $\pi \in \prod_{R_D}(\widetilde{G})$ in Equation 7.3.1 is nonzero. By Lemma 7.3.2 (3), one needs to compute the coefficients of $\widetilde{I} \in \mathcal{I}_{S,i}$ for every $S \in R_D$ and every χ_i.

It is obvious that every Shimura representation is in Lift($\mathbb{C}$) since the only standard modules containing Shimura representations in Equation 7.3.1 are $\widetilde{I}_{Sh_i}$'s. Therefore, we just need to show the theorem when $|\prod_{R_D}(\widetilde{G})| > 1$. Also, it suffices to compute the coefficients of every $\widetilde{I} \in \mathcal{I}_{S,i}$ for nonempty S. Let $\pi_i = c_S(Sh_i)$. By Lemma 4.1.2, 7.1.4, 7.3.2, the coefficient of π_i in Lift($\mathbb{C}$) is

$$c_{\pi_i} = \sum_{S' \subseteq S} \sum_{\gamma_{S'}} (-1)^{l(\gamma_0) - l(\gamma_{S'})} C(H_{S'}), \tag{7.3.2}$$

where $\gamma_{S'} = c_{S'}(\gamma)$ is defined on $H_{S'}$. So we compute the coefficient c_{π_i} case by case as follows (see Table 7.1 to 7.4).

For type A_{n-1}, $n = 2m$ is even:

When $S = \{\alpha_1, \alpha_3, \cdots, \alpha_{n-1}\}$,

$$c_{\pi_i} = \sum_{k=0}^{m-1} (-1)^k C_k^m + (-1)^m \cdot 2,$$

which is 1 when m is even, and is -1 when m is odd.

For type D_n, $n = 2m$ is even:

When $S = \{\alpha_{n-1}, \alpha_n\}$, $c_{\pi_i} = 1 - 2 \cdot 1 + 2 = 1$.

When $S = \{\alpha_1, \alpha_3, \cdots, \alpha_{n-3}, \alpha_{n-1}\}$ or $\{\alpha_1, \alpha_3, \cdots, \alpha_{n-3}, \alpha_n\}$,

$$c_{\pi_i} = \sum_{k=0}^{m-1} (-1)^k C_k^m + (-1)^m \cdot 2,$$

which is 1 when m is even, and is -1 when m is odd.

For type D_n, $n = 2m + 1$ is odd:

This case is the same as the case when n is even and $S = \{\alpha_{n-1}, \alpha_n\}$.

For type E_7:

When $S = \{\alpha_1, \alpha_3, \cdots, \alpha_7\}$, $c_{\pi_i} = 1 - 3 + 3 - 2 = -1$.

Since $c_{\pi_i} \neq 0$ for every $\pi = c_S(Sh_i)$, $S \in R_D$, we conclude that $\prod_{R_D}(\widetilde{G}) \subseteq$Lift($\mathbb{C}$).

$\square$

Table 7.1: Type A_{n-1}, $n = 2m$

S	S'	$\#\{S'\}$	$H_{S'}$	$C(H_{S'})$
$\{\alpha_1, \alpha_3, \cdots, \alpha_{n-1}\}$	$\{\phi\}$	1	$(\mathbb{R}^\times)^{n-1}$	1
	$\{\alpha_i\}$, $i = 1, 3, \cdots n-1$	C_1^m	$(\mathbb{R}^\times)^{n-3} \times \mathbb{C}^\times$	1
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\{\alpha_{i_1}, \cdots, \alpha_{i_k}\}$	C_k^m	$(\mathbb{R}^\times)^{n-1-2k} \times (\mathbb{C}^\times)^k$	1
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$S - \{\alpha_i\}$	C_{m-1}^m	$\mathbb{R}^\times \times (\mathbb{C}^\times)^{m-1}$	1
	S	1	$(\mathbb{C}^\times)^{m-1} \times S^1$	2

Table 7.2: Type D_n, $n = 2m$

S	S'	$\#\{S'\}$	$H_{S'}$	$C(H_{S'})$
	$\{\phi\}$	1	$(\mathbb{R}^\times)^n$	1
$\{\alpha_{n-1}, \alpha_n\}$	$\{\alpha_i\},\ {\scriptstyle i=n-1,n}$	2	$(\mathbb{R}^\times)^{n-2} \times \mathbb{C}^\times$	1
	S	1	$(\mathbb{R}^\times)^{n-3} \times \mathbb{C}^\times \times S^1$	2
	$\{\phi\}$	1	$(\mathbb{R}^\times)^n$	1
	$\{\alpha_i\},\ {\scriptstyle i=1,3,\cdots n-1}$	C_1^m	$(\mathbb{R}^\times)^{n-2} \times \mathbb{C}^\times$	1
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\{\alpha_1, \alpha_3, \cdots, \alpha_{n-3}, \alpha_{n-1}\}$	$\{\alpha_{i_1}, \cdots, \alpha_{i_k}\}$	C_k^m	$(\mathbb{R}^\times)^{n-2k} \times (\mathbb{C}^\times)^k$	1
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$S - \{\alpha_i\}$	C_{m-1}^m	$(\mathbb{R}^\times)^2 \times (\mathbb{C}^\times)^{m-1}$	1
	S	1	$\mathbb{R}^\times \times (\mathbb{C}^\times)^{m-1} \times S^1$	2
	$\{\phi\}$	1	$(\mathbb{R}^\times)^n$	1
	$\{\alpha_i\},\ {\scriptstyle i=1,3,\cdots n}$	C_1^m	$(\mathbb{R}^\times)^{n-2} \times \mathbb{C}^\times$	1
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\{\alpha_1, \alpha_3, \cdots, \alpha_{n-3}, \alpha_n\}$	$\{\alpha_{i_1}, \cdots, \alpha_{i_k}\}$	C_k^m	$(\mathbb{R}^\times)^{n-2k} \times (\mathbb{C}^\times)^k$	1
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$S - \{\alpha_i\}$	C_{m-1}^m	$(\mathbb{R}^\times)^2 \times (\mathbb{C}^\times)^{m-1}$	1
	S	1	$\mathbb{R}^\times \times (\mathbb{C}^\times)^{m-1} \times S^1$	2

Table 7.3: Type D_n, $n = 2m + 1$

S	S'	$\#\{S'\}$	$H_{S'}$	$C(H_{S'})$
$\{\alpha_{n-1}, \alpha_n\}$	$\{\phi\}$	1	$(\mathbb{R}^\times)^n$	1
	$\{\alpha_i\}$, $i = n-1, n$	2	$(\mathbb{R}^\times)^{n-2} \times \mathbb{C}^\times$	1
	S	1	$(\mathbb{R}^\times)^{n-3} \times \mathbb{C}^\times \times S^1$	2

Table 7.4: Type E_7

S	S'	$\#\{S'\}$	$H_{S'}$	$C(H_{S'})$
$\{\alpha_1, \alpha_3, \cdots, \alpha_7\}$	$\{\phi\}$	1	$(\mathbb{R}^\times)^7$	1
	$\{\alpha_i\}$, $i = 1, 3, 7$	3	$(\mathbb{R}^\times)^5 \times \mathbb{C}^\times$	1
	$\{\alpha_i, \alpha_j\}$, $\{i, j\} \in \{1, 3, 7\}$	3	$(\mathbb{R}^\times)^3 \times (\mathbb{C}^\times)^2$	1
	S	1	$(\mathbb{R}^\times)^2 \times (\mathbb{C}^\times)^2 \times S^1$	2

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