ABSTRACT

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Despite the abundance of available literature that starts with the seminal paper of Wang and Davison almost forty years ago, when dealing with the problem of decentralized control for linear dynamical systems, one faces a surprising lack of general design methods, implementable via computationally tractable algorithms. This is mainly due to the fact that for decentralized control configurations, the classical control theoretical framework falls short in providing a systematic analysis of the stabilization problem, let alone cope with additional optimality criteria.

Recently, a significant leap occurred through the theoretical machinery developed in "Rotkowitz and Lall, IEEE-TAC, vol. 51, 2006, pp. 274-286" which unifies and consolidates many previous results, pinpoints certain tractable decentralized control structures, and outlines the most general known class of convex problems in decentralized control. The decentralized setting is modeled via the structured sparsity constraints paradigm, which proves to be a simple and effective way to formalize many decentralized configurations where the controller feature a given sparsity pattern. Rotkowitz and Lall propose a computationally tractable algorithm for the

design of $\mathcal{H}^2$ optimal, decentralized controllers for linear and time–invariant systems, provided that the plant is strongly stabilizable. The method is built on the assumption that the sparsity constraints imposed on the controller satisfy a certain condition (named *quadratic invariance*) with respect to the plant and that some decentralized, strongly stablizable, stabilizing controller is available beforehand.

For this class of decentralized feedback configurations modeled via sparsity–constraints, so called *quadratically invariant*, we provided complete solutions to several open problems. Firstly, the strong stabilizability assumption was removed via the so–called *coordinate–free parametrization* of all, sparsity constrained controllers. Next we have addressed the unsolved problem of stabilizability/stabilization via sparse controllers, using a particular form of the celebrated Youla parametrization. Finally, a new result related to the optimal disturbance attenuation problem in the presence of stable plant perturbations is presented. This result is also valid for quadratically invariant, decentralized feedback configurations. Each result provides a computational, numerically tractable algorithms which is meaningful in the synthesis of sparsity–constrained optimal controllers.

Optimal Control with Information Pattern Constraints

by

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Table of Contents

List of Figures iv

List of Abbreviations v

1 Introduction and a Brief Outline 1
 1.1 Brief Historical Perspective 2
  1.1.1 The Decentralized Stabilization Problem after Davison 3
  1.1.2 Optimal Control with Sparsity Constrained Controllers 5
 1.2 Outline of this Thesis's Contributions 6

2 A Coordinate-Free Parametrization of All Sparse, Stabilizing Controllers 9
 2.1 Introduction 9
 2.2 Preliminaries 11
  2.2.1 Basic Concepts 11
  2.2.2 Feedback Control Systems 13
 2.3 The Coordinate–Free Approach 15
  2.3.1 The Parametrization of All Stabilizing Controllers
   via the Coordinate–Free Approach 16
  2.3.2 The Disturbance Attenuation Problem
   via the Coordinate–Free Approach 18
 2.4 Sparsed Controllers via Information Pattern Constraints 21
  2.4.1 ***Notation*** 21
  2.4.2 The Decentralized Disturbance Attenuation Problem 24
  2.4.3 Sparsity Constraints on the Q-Parameter 26
  2.4.4 Quadratic Invariance 27
 2.5 Main Result 30
  2.5.1 The Coordinate–free Parametrization of All Stabilizing,
   Decentralized Controllers 32
  2.5.2 The Decentralized, Optimal Disturbance Attenuation Problem 33
  2.5.3 The Decentralized, Mixed $\mathcal{H}^2$ Sensitivity Problem 34

3 Optimal Disturbance Attenuation
 in the Presence of Stable, Additive Plant Perturbations 45
  3.0.4 Preliminaries and Notation 46
 3.1 Main Result 48
  3.1.1 Numerical Example. 51
  3.1.2 Numerical Considerations 53
  3.1.3 The Stable Plant Case 54
 3.2 Quadratically Invariant, Sparsity Constrained Controllers 55

4 Necessary and Sufficient Conditions for Stabilizability subject to Quadratic Invariance 66
4.1 Preliminaries 68
4.1.1 The Control Problem 68
4.1.2 Coprime and Doubly Coprime Factorization for LTI Systems . 69
4.1.3 The Youla Parametrization of All Stabilizing Controllers . . . 70
4.2 Feedback Control Configurations with Sparsity Constraints 71
4.2.1 Conformal Block Partitioning 71
4.2.2 Sparsity Constraints 72
4.2.3 Quadratic Invariance 74
4.3 Main Result 76
4.3.1 Outline of the Sparse Controller Synthesis Algorithm 79
4.3.2 A Numerical Example 82
4.3.3 Sparse Controller Synthesis as An Exact Model–Matching Problem 84
4.3.4 Parametrization of All Sparse, Stabilizing Controllers 86
4.3.5 Numerical Example – Continued 87
4.4 A Meaningful, Particular Case 88
4.4.1 The Output Decoupled, Left Coprime Factorization 89
4.4.2 The Input Decoupled, Right Coprime Factorization 91
4.4.3 Input/Output Decoupled DCFs 92
4.4.4 Stabilizability with Sparse Controllers 93
4.4.5 The Youla Parametrization 94
4.4.6 The Model–Matching Problem for Sparse Controller Synthesis 96
4.4.7 Sparse, Optimal Controller Synthesis 98
4.5 Conclusions 99
4.5.1 State–space Representations of LTI Systems 100

5 Future Research Ideas 113
5.1 2–Inverses for Binary Matrices 113
5.1.1 Reliable Decentralized Stabilization 114
5.2 Strong Stabilization and Controller Order Reduction via Fundamental Spaces Analysis 116

Bibliography 120

List of Figures

2.1 Feedback interconnection between the generalized plant and the controller . . . 12

2.2 Input/Output representation for $\Theta(P, K)$ . . . 14

2.3 Standard unity feedback interconnection . . . 15

3.1 Standard unity feedback interconnection . . . 47

3.2 Standard unity feedback interconnection with stable additive perturbation . . . 49

4.1 Standard unity feedback configuration . . . 69

List of Abbreviations

TFM	Transfer Function Matrix
DCF	Doubly Coprime Factorization
LCF	Left Coprime Factorization
RCF	Right Coprime Factorization

Chapter 1

Introduction and a Brief Outline

Despite the abundance of available literature starting with the seminal paper [4] of Wang and Davison almost forty years ago, when dealing with the problem of decentralized control for linear dynamical systems, one faces a surprising lack of general, design methods, computationally tractable algorithms and plug–and–play software to address the decentralized stabilization problem, yet alone supplemental optimality criteria.

Recently, a significant leap occurred through the theoretical machinery developed in [79] which unifies and consolidates many previous results, pinpoints certain tractable decentralized control structures, and outlines the most general known class of convex problems in decentralized control. The decentralized setting is modeled via the structured sparsity constraints paradigm which proves to be a simple and effective way to formalize many decentralized configurations. The authors of [79] propose a computationally tractable algorithm for the design of $\mathcal{H}^2$ optimal, decentralized controllers for linear and time–invariant systems, provided that the plant is strongly stabilizable. The method also premises that the sparsity constraints imposed on the controller satisfy a certain condition (named *quadratic invariance*) with respect to the plant and that some decentralized, stable, stabilizing controller is available beforehand.

The power of these new results has only begun to be exploited and we believe that it brings along an excellent opportunity for further research. For the framework of decentralized information patterns modeled via sparsity constraints, we propose a set of open problems directed toward numerical algorithms for solving key issues in the control of decentralized, linear systems.

1.1 Brief Historical Perspective

Throughout this document we make the leading assumption that all systems are linear, finite–dimensional, time–invariant and with continous time. The most handy means of describing the dynamical behavior of a system satisfying all these assumptions is the state–space representation. A continuous–time state–space system is given by the equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1.1}$$

where $t \in \mathbb{R}, x(t) \in \mathbb{R}^{n_s}$ is the state, $u(t) \in \mathbb{R}^{n_u}$ is the input, $y(t) \in \mathbb{R}^{n_y}$ is the output of the system, and A, B, C, D are constant matrices with real entries. The input–output behavior of the system (1.1) is conveniently described by the transfer function matrix which is the n_u rows by n_y columns, rational matrix function

$$P(s) \stackrel{def}{=} C(sI - A)^{-1}B + D. \tag{1.2}$$

Notice that $P(s)$ is obtained by taking the Laplace transform in (1.1) and making explicit $y(s)$ as a function of $u(s)$ in the form $y(s) = G(s)u(s)$ (where now $y(s)$ and $u(s)$ are viewed as the Laplace transforms of $y(t)$ and $u(t)$, respectively).

A classical result in linear control theory is that the poles of a controllable and observable linear system can be arbitrarily placed (assuming complex conjugated pole-pairing) by state variable feedback. This result has been extended to get the class of linear, stabilizing controllers such that the poles of the closed-loop system consisting of a controllable and observable linear system with the controller can be freely assigned [1]. These results make up for the backbone of most practical synthesis procedures.

1.1.1 The Decentralized Stabilization Problem after Davison

A natural extension of the pole-placement problem arises when the set of admissible controllers is restricted to *decentralized feedback* control. The conclusive results were given in the early 70's by Wang and Davison [4] and Corfmat and Morse [5, 6]. We give a brief synopsis next.

For a linear system the problem of decentralized pole placement can be formulated as follows: given the linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \textstyle\sum_{i=1}^{N} B_i u_i(t), \\ y_i(t) &= C_i x(t), \end{aligned} \tag{1.3}$$

where $i = 1, .., N$ indexes the input and output variables of the various controllers, the i-th controller employs dynamic compensation of the form

$$\begin{aligned} u_i(t) &= M_i z_i(t) + F_i y_i(t) + G_i v_i(t). \\ \dot{z}_i(t) &= H_i z_i(t) + L_i y_i(t) + R_i v_i(t). \end{aligned} \tag{1.4}$$

The decentralized pole-placement problem is to find matrices $M_i, F_i, G_i, H_i, L_i, R_i$ such that the closed-loop system described by (1.4) has prespecified poles. Of course, if (C_i, A, B_i) is controllable and observable for some i, the problem is trivial. The interesting case is to assume that (1.3) is controllable from all controls $u_1, \dots, u_N$, but not from any single control u_i, with a similar observability assumption. For illustrative simplicity, consider first the special case $M_i = 0$ in (1.4). This corresponds to nondynamic decentralized output feedback. If F denotes the collection of feedback matrices $(F_1, \dots, F_N)$, then the pole-placement problem is to determine F such that the matrix

$$A_F \overset{def}{=} A + \sum_{i=1}^{N} B_i F_i C_i$$

has a specified arbitrary set of eigenvalues. A necessary condition for pole placement in this case is that the polynomials $\alpha(\lambda) \overset{def}{=} \det(\lambda I - A_F)$ have no common factor.

What is much more interesting is that this condition is both necessary and sufficient ([4]) for pole placement with dynamic decentralized compensation. More generally, since the zeros of $\alpha(\lambda)$ (termed *the fixed modes* of the system) are invariant under decentralized dynamic compensation, it follows that a necessary and sufficient condition for stabilizability is that the roots of $\alpha(\lambda)$ have strictly negative real parts. A transfer function caracterisation of fixed modes along with the concept of degree of a fixed mode was later given by Anderson in [24].

Further research effort was made by Davison et all. [7]–[16] to outline the canonical invariants of decentralized linear systems (1.3), show the connection between the decentralized fixed modes and the classical concept of transmission zeros and develop robust decentralized feedback allocation procedures.

Anderson showed in [23] that oposed to the classical (centralized) case, certain time-invariant systems which cannot be stabilized by decentralized time-invariant controllers, (namely those with unstable decentralized fixed modes), can thus be stabilized by decentralized time-varying (in fact periodic), linear controllers. Various applications of this fact such as simultaneous stabilization, and reliable decentralized control (maintaining stability when *any* of the controllers fail), strong decentralized stabilization were later treated by Ozguler et all et all in [30] – [35].

1.1.2 Optimal Control with Sparsity Constrained Controllers

The study of the computational complexity of decentralized control problems has proved certain problems to be intractable. Tsitsiklis et all [26, 27] showed that the problem of computing a stabilizing decentralized static output feedback is NP-complete.

For certain information structures, the optimal control problem may have a tractable solution, and in particular, it was shown by Voulgaris [42] that the so-called *one-step delay information sharing pattern* problem has this property. Also in [42] a solution is given for the first time to the $\mathcal{H}^2$, $\mathcal{H}^1$ and $\mathcal{H}^\infty$ control synthesis problems for this particular decentralized configuration. A class of structured spatio–temporal systems has also been analyzed in [43], and shown to be reducible to a convex program. Several information structures are identified in [52] for which the problem of minimizing multiple objectives is reduced to a finite–dimensional convex optimization problem.

The new approach in [79] shows that the key, necessary and sufficient condition that allows optimal stabilizing decentralized controllers to be synthesized via convex programming is for the constraints set of the controller to be preserved under feedback. This is a significant extension to previous theoretical results and at the same time a unifying framework since all previously studied tractable structures of [41, 2, 52, 42, 43] can be shown to be particular cases that satisfy this property.

1.2 Outline of this Thesis's Contributions

Chapter 2. The solution proposed by Rotkowitz and Lall to the optimal $\mathcal{H}^2$ disturbance attenuation via sparse controllers, is based on the so-called *Q–parametrization* of all stabilizing controllers of a given linear, strongly–stabilizable plant introduced by Zames in [61] and later extended to the case of nonlinear plants by Anantharam and Desoer in [62]. The merit of Rotkowitz's method from [79] is that it manages to cast the sparsity constraints imposed on the decentralized controller as convex constraints on the Q–parameter.

The first contribution of this thesis is to provide a new parametrization of all decentralized controllers that satisfy pre-selected, quadratically invariant sparsity constraints and stabilize a given linear time-invariant plant. Unlike the prior work of Rotkowitz and Lall, that hinges on Zames's Q–parameterization, this chapter adopts a recently developed *coordinate-free* approach that does not require the plant to be strongly stabilizable. Hence, the approach proposed here also extends the applicability of the work in [79] to the case where the plant is not strongly stabilizable. In this chapter, we show how the new parameterization can be used in the design of sparsity constrained controllers that attain the optimal disturbance attenuation, and how the computational scheme from [79, Theorem 29] can still be employed here. Moreover, with our new parametrization, we are able to deal with cost-function of a more general type and in doing so we provide along the way a tractable solution to the mixed $\mathcal{H}^2$ sensitivity problem from [57, pp. 139], with sparsity constrained controllers.

Chapter 3. The main result of this chapter proves that the minimal gain attainable by causal feedback in the optimal disturbance attenuation problem, is not influenced by linear, stable, additive plant perturbations. Furthermore, this is shown to hold true, irrespective of the used norm (*e.g.* for 1-D, LTI systems it could be any of the $\mathcal{L}^p$ or ℓ^p induced norms, respectively). It follows as a direct consequence that for the optimal synthesis procedure, it is sufficient to solve the disturbance attenuation problem only for the anti–stable component of the plant. The solution obtained for the anti–stable component of the plant can than be used to retrieve the optimal solution for the entire plant, via a simple algebraic, feedback transformation. More importantly, we also prove the validity of our result for an important class of decentralized control systems, namely decentralized configurations that are *invariant under feedback* ([79]). Finally, since the proof of the main result is completed without any assumption on the coprime factorizability of the plant, it also encompasses the case of linear, n-D systems ([69]).

Chapter 4. All the available algorithms for optimal synthesis via sparse controllers ([79] and the one from Chapter 2 of this thesis) rely indispensably on the fact that some stabilizing controller that verifies the imposed sparsity constraints is *a priori* known, while synthesis methods for such a controller, (needed to initialize the aforementioned optimization schemes) are not yet available. This provided the motivation to the work presented here, as in this chapter we develop necessary and sufficient conditions for such a plant to be *stabilizable* with a controller having the pre–selected sparsity pattern. These conditions are formulated in terms of the existence of a special type of doubly coprime factorization of the plant, which we have named the *input/output decoupled, doubly coprime factorization.* More importantly, the set of all decentralized stabilizing controllers is characterized via the Youla parametrization. The sparsity constraints on the controller are also recast as convex constraints on the Youla parameter. Furthermore, using the powerful tools from [79] and the Youla parametrization, we present improved, tractable formulations of the optimal disturbance attenuation problem and optimal mixed sensitivity problem with sparsity constrained controllers.

Chapter 2

A Coordinate-Free Parametrization of All Sparse, Stabilizing Controllers

2.1 Introduction

The authors of [79] propose a computationally tractable procedure for the design of $\mathcal{H}^2$ optimal sparsity constrained controllers for a given class of linear time-invariant plants. The method is anchored on a convex parametrization, whose existence can be determined by an algebraic test (quadratic invariance) involving only the sparsity pattern of the plant and the sparsity constraints to be imposed on the controller. The solution involves the so-called *Q–parametrization* of all stabilizing controllers of a given linear, strong–stabilizable plant introduced by Zames in [61] and later extended to the case of nonlinear plants by Anantharam and Desoer in [62]. The merit of the method from [79] is that it manages to cast the sparsity constraints imposed on the decentralized controller as convex constraints on the Q–parameter.

Contribution. The main contribution presented in this chapter is to provide a new parametrization of all decentralized controllers that satisfy pre-selected, quadratically invariant sparsity constraints and stabilize a given linear time-invariant plant. Unlike prior work that hinges on Zames's Q–parameterization, in this chapter we adopt a recently developed *coordinate-free* approach that does not require the plant to be strongly stabilizable. Hence, the approach proposed here also extends the applicability of the work in [79] to the case where the plant is non-strongly stabilizable. In this chapter, we show how the new parameterization can be used in the design of norm-optimal sparsity constrained controllers, and how the computational scheme from [79, Theorem 29] can still be employed here.

As oposed to the Youla parametrization which is built on the doubly coprime factorization of the plant, Zames's Q–parameterization essentially relies on the *apriori* knowledge of some stabilizable and stable controller. While necessary and sufficient conditions for strong stabilizability have been generalized to the case of MIMO plants by Vidyasagar in [63], computing such a stable controller is still a daunting task even in the case of centralized controllers. This happens because the available techniques *e.g.* [64, 107, 114], depend on the solutions of some non–standard Riccati matrix equations which apart from being difficult to compute are in general not even guaranteed to exist. Furthermore, general conditions for asserting strong stabilizability via sparsity constrained controllers are currently unavailable (see also [73]), as is the case for methods for computing such stable controllers. This situation gives our results the added practical significance of relying on the apriori knowledge of any decentralized, stabilizing controller instead of a decentralized, *stable* controller (as in [79]), whose synthesis may be impractical. The coordinate free method was pioneered by the authors of [71, 72, 74] and later extended in [67, 68, 70, 69].

2.2 Preliminaries

Throughout this chapter we make the leading assumption that all systems are discrete–time, finite–dimensional, linear time–invariant (LTI).

2.2.1 Basic Concepts

In many instances (*e.g.* convolution operators) the set of all proper, stable, linear systems forms a commutative ring. That is, the fact that parallel and serial connections of systems that are proper, stable and linear yield proper, stable and linear systems. The seminal work of Desoer et al. [75] and Vidyasagar et al. [81] show that this abstract ring setup encompasses within a single framework a broad class of linear systems (lumped or distributed linear systems, 1-D as well as n-D systems). For consistency and ease of reference, we will adopt the notation from [81] also used in [67, 68, 70, 69] whose results we use extensively in this chapter.

As in [67, Section 2.2], [68, Section II.A] we denote with $\mathcal{A}$ the set of transfer functions of all proper, stable, linear systems. Note that $\mathcal{A}$ has a commutative ring structure. The set of all possible transfer functions, which we denote with $\mathcal{F}$, is the ring of fractions of $\mathcal{A}$:

$$\mathcal{F} \stackrel{def}{=} \left\{ n/d \middle| n, d \in \mathcal{A}, \text{ where } d \text{ is not a divisor of zero} \right\} \tag{2.1}$$

By a natural extension of notation, we use $\mathcal{A}^{n_y \times n_u}$ to denote the set of matrices with n_y rows by n_u columns and whose entries are in $\mathcal{A}$, that is the set of proper, stable transfer function matrices of dimension $n_y \times n_u$. Similarly, $\mathcal{F}^{n_y \times n_u}$ is the set

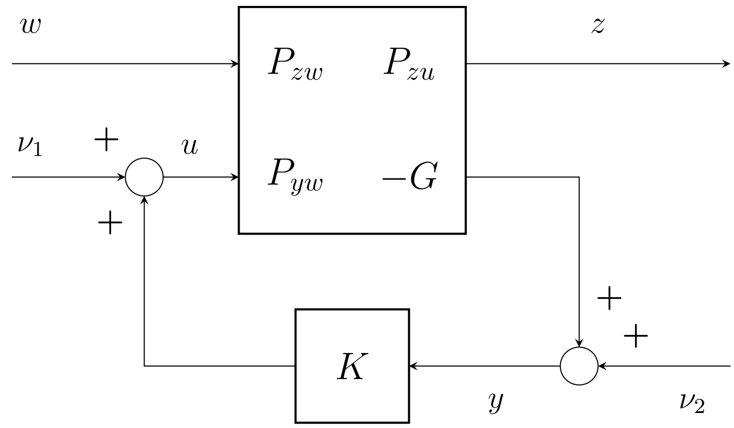

Figure 2.1: Feedback interconnection between the generalized plant and the controller

of all possible transfer function matrices (TFMs) of size $n_y \times n_u$. Henceforth we may omit the indices n_y and n_u whenever their values are clear from the context.

Throughout this chapter, we require that both the plant and the controller are proper according to the following definition, which is adapted from [67, Definition 2.3], [68, Definition 2].

Definition 2.2.1. *[67, Definition 2.3], [68, Definition 2] Let $\mathcal{Z}$ be a prime ideal of $\mathcal{A}$, with $\mathcal{A} \neq \mathcal{Z}$ and $\mathcal{Z}$ including all the divisors of zero of $\mathcal{A}$. Define, the subsets $\mathcal{P}$ and $\mathcal{P}_s$ of $\mathcal{F}$ as follows:*

$$\mathcal{P} \stackrel{def}{=} \left\{ \frac{n}{d} \in \mathcal{F} \middle| n \in \mathcal{A}, d \in \mathcal{A} - \mathcal{Z} \right\}, \tag{2.2}$$

$$\mathcal{P}_s \stackrel{def}{=} \left\{ \frac{n}{d} \in \mathcal{F} \middle| n \in \mathcal{Z}, d \in \mathcal{A} - \mathcal{Z} \right\} \tag{2.3}$$

A transfer function in the set $\mathcal{P}$ ($\mathcal{P}_s$) is called proper *(*strictly proper*). Similarly,* *if all entries of a transfer function matrix over $\mathcal{F}$ are in $\mathcal{P}$ ($\mathcal{P}_s$), then the respective transfer function matrix is called proper (strictly proper).*

The set $\mathcal{A}$ and the ideal $\mathcal{Z}$ (of all divisors of zero of $\mathcal{A}$) for discrete-time LTI systems is characterized in [68, (5) pp.745] and [68, (6) pp.745], respectively.

2.2.2 Feedback Control Systems

In Figure 1, we depict the standard feedback interconnection between a generalized plant P and a controller K. Here, w is the vector of reference signals, while ν_1 and ν_2 are the disturbance signals and sensor noise, respectively. In addition, u are the controls, y are the measurements and z the regulated outputs (in general some error signals). The integers n_w, n_u, n_y and n_z denote the dimensions of the vectors w, u, y and z, respectively. The generalized plant P is proper and it belongs to the set $\mathcal{P}^{(n_y+n_z)\times(n_u+n_w)}$ while the proper controller K belongs to the set $\mathcal{P}^{n_u\times n_y}$. The transfer function matrix of the generalized plant P is conformably partitioned such that $P_{zw} \in \mathcal{P}^{n_z\times n_w}$, $P_{zu} \in \mathcal{P}^{n_z\times n_u}$, $P_{yw} \in \mathcal{P}^{n_y\times n_w}$ and $P_{yu} \in \mathcal{P}^{n_y\times n_u}$. For convenience of notation, henceforward we adopt the following convention:

$$G \stackrel{def}{=} -P_{yu} \tag{2.4}$$

Assuming that the loop is *well posed* – that is $(I + KG)$ is invertible over $\mathcal{F}^{n_u\times n_u}$ – then we denote the transfer matrix function from $[w^T \nu_1^T \nu_2^T]^T$ to $[z^T u^T y^T]^T$ from Figure 2.1 with $\Theta(P, K)$, where we adopt the superscript T to denote matrix transposition. For the input–output equations of the standard feedback interconnection from Figure 1 we refer to [69, pp.230], while the explicit algebraic expression of

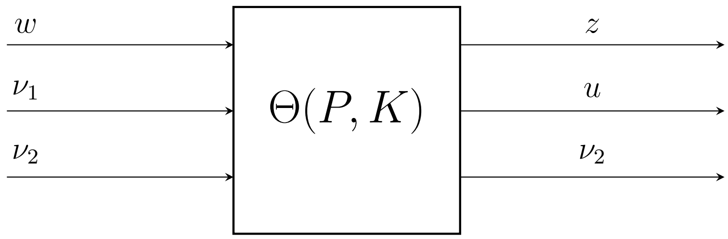

Figure 2.2: Input/Output representation for $\Theta(P, K)$

$\Theta(P, K)$ can be found in [69, pp.231]. If the transfer function matrix $\Theta(P, K)$ belongs to the set $\mathcal{A}$, then we say that K is a *a stabilizing controller* of P, or equivalently that K *stabilizes* P. If a stabilizing controller of P exists, we say that P is *stabilizable.*

Of particular interest is the feedback system displayed in Figure 2, where the proper transfer function matrices $K \in \mathcal{P}^{n_u \times n_y}$ and $G \in \mathcal{P}^{n_y \times n_u}$ represent the controller and the plant respectively. Denote by $H(G, K)$ the transfer function matrix from $[\nu_1^T \quad \nu_2^T \;]^T$ to $[y^T \quad u^T \;]^T$ (provided that $(I + KG)$ is invertible over $\mathcal{P}^{n_u \times n_u}$):

$$H(G, K) \stackrel{def}{=} \begin{bmatrix} (I + GK)^{-1} & -G(I + KG)^{-1} \\ K(I + GK)^{-1} & (I + KG)^{-1} \end{bmatrix} \tag{2.5}$$

If the transfer matrix $H(G, K)$ belongs to $\mathcal{A}$, we say that K is a *stabilizing controller* of G or equivalently that K *stabilizes* G. If a stabilizing controller of G exists, we call G *stabilizable.*

The following Lemma states that a proper controller K stabilizes the generalized plant P via the feedback configuration of Figure 1 if and only if K stabilizes the G block of P given in (2.4) via the feedback configuration of Figure 2.

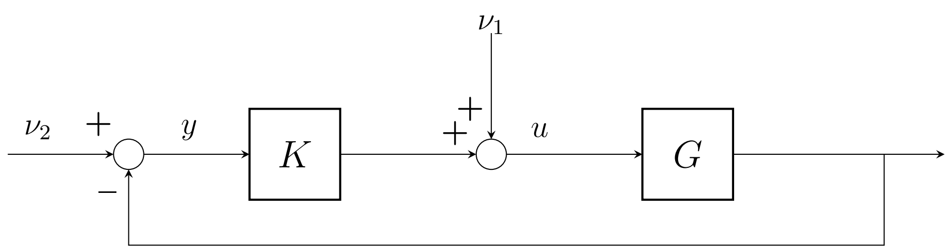

Figure 2.3: Standard unity feedback interconnection

Lemma 2.2.2. *[69, Lemma 1] Let P in the set $\mathcal{P}^{(n_u+n_y)\times(n_u+n_y)}$ be a proper, stabilizable, generalized plant (with the notation from (2.4) in effect). Let K in the set $\mathcal{P}^{n_u\times n_y}$ be any proper, stabilizing controller of P. Then the transfer function $\Theta(P,K)$ is in $\mathcal{A}$ if and only if $H(G,K)$ is in $\mathcal{A}$.*

Since Lemma 2.2.2 does not require coprime factorizability of the plant G, it generalizes the central result in [56, Theorem 4.3.2].

2.3 The Coordinate–Free Approach

This section gives a brief summary of the combined results in [67, 68, 70, 69], on the so–called *coordinate–free approach* to linear control design, where we emphasize results that are used throughout this chapter. The coordinate free approach is pursued in [71, 72, 74, 67, 68, 70, 69] (within the framework developed in Desoer et al. [75] and Vidyasagar et al. [81]), pertaining feedback stabilization when the coprime factorizability of the plant is not viable (*e.g.* for linear systems with multiple scales of time, also called n-D systems).

2.3.1 The Parametrization of All Stabilizing Controllers via the Coordinate–Free Approach

Unlike Youla's parametrization, the coordinate–free approach ([67, 68, 70, 69]) parametrizes directly all achievable stable closed–loop transfer function matrices $H(G, K)$ without prior knowledge of the doubly coprime factorization of the plant's transfer function matrix, but it has the drawback of requiring prior knowledge of some stabilizing controller. Once the parametrization of all closed–loop TFMs $H(G, K)$ is available, a parametrization of all stabilizing controllers is readily retrieved via an algebraic transformation.

From this point onward we assume that the plant G is strictly proper, that is

$$G \in \mathcal{P}_s^{n_y \times n_u} \tag{2.6}$$

It is known that the coordinate free parametrization ([68, Theorem 4]) may yield nonproper stabilizing controllers when the plant is not strictly proper. The following Remark details the important consequences of (2.6).

Remark 2.3.1. *[69] The assumption in (2.6) implies that the closed–loop system is well–posed [57, pp.119] for every stabilizing controller [68, Proposition 5]. Equally important, it ensures that every stabilizing controller of G is proper ([67, Proposition 6.2], [70, Proposition 1]).*

The next statement follows as a summary of [68, Proposition 4 and Proposition 5], which will be instrumental in the sequel.

Theorem 2.3.2. *Given positive integers* n_u *and* n_y*, and a strictly proper plant* G *in the set* $\mathcal{P}_s^{n_y \times n_u}$*, let* $\mathcal{H}_G$ *be the set of all stable closed–loop transfer functions* i.e.*:*

$$\mathcal{H}_G \stackrel{def}{=} \left\{ H(G,K) \in \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)} \middle| \, K \text{ stabilizes } G \right\}$$

A) *[68, Proposition 4] If* K_0 *is a stabilizing controller of* G*, then the following equality holds:*

$$\mathcal{H}_G = \left\{ \Omega(Q) \mid Q \in \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)} \right\},$$

where for any Q *in the set* $\mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$*,* $\Omega(Q)$ *is defined as*

$$\Omega(Q) \stackrel{def}{=} \left(H(G,K_0) - \begin{bmatrix} I_{n_y} & O \\ O & O \end{bmatrix} \right) Q \left(H(G,K_0) - \begin{bmatrix} O & O \\ O & I_{n_u} \end{bmatrix} \right) + H(G,K_0) \tag{2.7}$$

Here I_{n_y} *and* I_{n_u} *denote the identity matrices of dimension* n_y *and* n_u*, respectively.*

B)*[68, Proposition 5] For* $\Omega(Q)$ *defined in (2.7) consider the following conformable partition*

$$\Omega(Q) = \begin{matrix} & \overbrace{\phantom{\Omega_{11}(Q)}}^{n_y} \; \overbrace{\phantom{\Omega_{12}(Q)}}^{n_u} & \\ & \begin{bmatrix} \Omega_{11}(Q) & \Omega_{12}(Q) \\ \Omega_{21}(Q) & \Omega_{22}(Q) \end{bmatrix} & \begin{matrix} \}n_y \\ \}n_u \end{matrix} \end{matrix} \tag{2.8}$$

The set $\mathcal{K}_G$ *of all stabilizing controllers of* G *can be parametrized as:*

$$\mathcal{K}_G = \left\{ \Omega_{21}(Q)\Omega_{11}^{-1}(Q) = \Omega_{22}^{-1}(Q)\Omega_{21}(Q) \middle| Q \in \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)} \right\} \tag{2.9}$$

Furthermore, every controller in $\mathcal{K}_G$ *is proper.*

Remark 2.3.3. *Assumption (2.6) also guarantees that for any* Q *in the set* $\mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$*,* $\Omega(Q)$ *given in (2.7) is invertible and belongs to the set* $\mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$*. This, in turn, implies the invertibility of the* Ω_{11} *and* Ω_{22} *blocks in (2.8), which guarantees that the expression (2.9) for the controller is well defined.*

2.3.2 The Disturbance Attenuation Problem via the Coordinate–Free Approach

We denote the complex unit circle with $\mathbb{D}_0$ and by $*$ the matrix complex conjugate transposition.

We briefly remind some standard notation for the transfer functions of linear and time–invariant (LTI) systems (their input–output operators) in the discrete–time case. A rational function $G(e^{j\omega}) : \mathbb{D}_0 \longmapsto \mathbb{C}$ is called *real–rational* if the polynomials of the numerator and denominator have real coefficients. Correspondingly, a matrix–valued function $G(e^{j\omega}) : \mathbb{D}_0 \longmapsto \mathbb{C}^{n_y \times n_u}$ is qualified as real–rational if all its entries are real–rational.

We use $\mathcal{RH}_\infty$ to denote the set of real–rational transfer functions matrices that are analytic outside the open, complex unit disk. It can be shown that functions in $\mathcal{RH}_\infty$ are completely defined by their values on $\mathbb{D}_0$.

The so called $L_2(\mathbb{D}_0)$–norm or $\mathcal{H}^2$–norm is defined for any transfer function matrix of an LTI system with discrete–time belonging to $\mathcal{RH}_\infty$ as:

$$\|G\|_2 \stackrel{def}{=} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \text{Trace} \left[G^*(e^{j\theta}) G(e^{j\theta}) \right] d\theta \right\}^{\frac{1}{2}} \tag{2.10}$$

For the continuous–time counterpart definition, we refer to [77, pp. 35]. Since from this point on we refer exclusively to LTI systems and their $\mathcal{H}^2$ norm, the norm index is dropped.

A standard problem in control is the following: given the proper, stabilizable generalized plant P in the feedback configuration from Figure 1, design a proper, stabilizing controller K that minimizes the $\mathcal{H}^2$ norm of the transfer function from w to z, namely:

$$\min_{K \text{stabilizes } P} \quad \|f(P, K)\| \tag{2.11}$$

where (the transfer function from w to z) $f(P, K)$ is the lower–linear fractional transformation of the generalized plant P with controller K defined as follows:

$$f(P, K) \stackrel{def}{=} P_{zw} + P_{zu}\, K\big(I + GK\big)^{-1} P_{yw}. \tag{2.12}$$

The following result, [69, Theorem 1] is important in our approach, as it makes clear the equivalence between the disturbance attenuation problem (3.1) and the model–matching problem of minimizing the norm of some affine (and therefore convex) functional.

Theorem 2.3.4. *[69, Theorem 1] Let P be a proper, stabilizable, generalized plant with the block $G \in \mathcal{P}_s^{n_y \times n_u}$ strictly proper (2.4). Given* any *proper, stabilizing con-*

troller $K_0 \in \mathcal{P}^{n_u \times n_y}$, *consider the following optimization problem:*

$$\min_{Q \in \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}} \left\| T_1 - T_2 Q T_3 \right\| \tag{2.13}$$

If there exists an optimal solution Q^* *to (2.13) then* $K^* = \Omega_{21}(Q^*)\Omega_{11}^{-1}(Q^*)$ *is an optimal solution of problem (3.1). Conversely, if there exists an optimal solution* K^* *to (3.1) then there exists an optimal solution* Q^* *to (2.13) such that* $K^* = \Omega_{21}(Q^*)\Omega_{11}^{-1}(Q^*)$. *Here* $\Omega(Q)$ *is as defined in (2.7),* $\Omega_{11}(Q)$ *and* $\Omega_{21}(Q)$ *are the blocks in the first column of* $\Omega(Q)$ *with the conformable partition in (2.8), while* T_1, T_2 *and* T_3 *are the transfer function matrices defined below:*

$$\begin{aligned} T_1 &\overset{def}{=} P_{zw} + P_{zu}\, K_0(I+GK_0)^{-1}P_{yw}, \\ T_2 &\overset{def}{=} \begin{bmatrix} P_{zu}K_0(I+GK_0)^{-1} & P_{zu}(I+K_0G)^{-1} \end{bmatrix}, \\ T_3 &\overset{def}{=} \begin{bmatrix} (I+GK_0)^{-1}P_{yw} \\ K_0(I+GK_0)^{-1}P_{yw} \end{bmatrix}. \end{aligned} \tag{2.14}$$

Remark 2.3.5. *Since we use [69] throughout this chapter, we need to clarify that it contains a typo in Section III. Namely, the expression of the controller* K *in [69] is given as* $\Omega_{21}(Q)\Omega_{22}^{-1}(Q)$, *for some* Q *in* $\mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$. *According to [68, Proposition 5] the correct expression for the stabilizing controllers (as stated in (2.9) above) is* $K(Q) = \Omega_{21}(Q)\Omega_{11}^{-1}(Q)$ *for some proper, stable* Q. *References to the results in [69] are made in the sequel, but using the correct expression.*

2.4 Sparsed Controllers via Information Pattern Constraints

The core of this section consists of the decentralized counterpart of Theorem 4.51, where we use *sparsity constraints* to impose a pre-selected information structure on the controller. We will also use this formulation to develop decentralized versions of the feedback stabilization and disturbance attenuation problems.

The notation we introduce next is entirely concordant with the one used in [78, 79].

2.4.1 ***Notation***

For $p \geq 1$, we denote the set of integers ranging from 1 to p with $\overline{1,p}$. Throughout the sequel, we consider that the block $G \in \mathcal{P}_s^{n_y \times n_u}$ of the generalized plant P (2.4) is partitioned in p block–rows and m block–columns. The i-th block–row has n_y^i rows, while the j-th block–column has n_u^j columns. Hence, it holds that $\sum_{i=1}^{p} n_y^i = n_y$ and $\sum_{j=1}^{m} n_u^j = n_u$. For every pair (i, j) in the set $\overline{1,p} \times \overline{1,m}$, we denote by $[G]_{ij} \in \mathcal{P}_s^{n_y^i \times n_u^j}$ the transfer matrix formed by the i-th block–row and j-th block–column of G, leading to the following representation:

$$G = \begin{bmatrix} [G]_{11} & \dots & [G]_{1m} \\ \vdots & & \vdots \\ [G]_{p1} & \dots & [G]_{pm} \end{bmatrix}, \quad \text{with} \quad [G]_{ij} \in \mathcal{P}_s^{n_y^i \times n_u^j}. \qquad (2.15)$$

Here, we shall use this square bracketed notation for indexing the block transfer function matrices.

Analogously, the controller's transfer function matrix $K \in \mathcal{P}^{n_u \times n_y}$ is partitioned in m block–rows and p block–columns, where the j-th block–row has n_u^j rows and the i-th block–column has n_y^i columns. Correspondingly, $[K]_{ji}$ is the notation for the element of $\mathcal{P}^{n_u^j \times n_y^i}$ located at the intersection of the j-th block–row and i-th block–column of K.

For the boolean algebra, the operations $(+, \cdot)$ are defined as usual: $0 + 0 = 0 \cdot 1 = 1 \cdot 0 = 0 \cdot 0 = 0$ and $1 + 0 = 0 + 1 = 1 + 1 = 1 \cdot 1 = 1$. By a binary matrix we mean a matrix whose entries belong to the set $\{0, 1\}$. With the usual extension of notation, $\{0, 1\}^{m \times p}$ stands for the set of all binary matrices with m rows and p columns. The addition and multiplication of binary matrices are carried out in the usual way, keeping in mind that the binary operations $(+, \cdot)$ follow the boolean algebra.

Binary matrices are denoted by upper-case letters with the "bin" superscript to distinguish them from transfer function matrices over $\mathcal{F}$, which are represented in the sequel using plain upper-case font. Henceforth, we adopt the convention that transfer function matrices are indexed by blocks while binary matrices are indexed by each individual entry.

Furthermore, for binary matrices only, having the same dimensions, the notation $A^{\text{bin}} \leq B^{\text{bin}}$ means that $a_{ij} \leq b_{ij}$ holds elementwise for all i and j.

With the conformable block partitioning for K introduced earlier, for any $K \in \mathcal{P}^{n_u \times n_y}$, define $\text{Pattern}(K) \in \{0, 1\}^{m \times p}$ to be the binary matrix

$$\text{Pattern}(K)_{ij} \stackrel{def}{=} \begin{cases} 0 & \text{if the block } [K]_{ij} = 0 \\ 1 & \text{otherwise} \end{cases} \tag{2.16}$$

Conversely, for any binary matrix with m rows and p columns, $K^{\text{bin}} \in \{0, 1\}^{m \times p}$, we can define the following linear subspace of $\mathcal{F}^{n_u \times n_y}$:

$$\text{Sparse}(K^{\text{bin}}) \stackrel{def}{=} \left\{ K \in \mathcal{F}^{n_u \times n_y} \middle| \ \text{Pattern}(K) = K^{\text{bin}} \right\} \tag{2.17}$$

Hence $\text{Sparse}(K^{\text{bin}})$ is the subspace of all controllers K whose sparsity pattern is $K^{\text{bin}} = 0$. More specifically, $[K]_{ij} = 0$ holds if and only if $K^{\text{bin}}_{ij} = 0$ also holds. From a functional point of view, the binary value of $\text{Pattern}(K)_{ij}$ determines whether controller i may read the jth block-row of the output of P.

Let $K^{\text{bin}} \in \{0, 1\}^{m \times p}$ be the pre-specified sparsity pattern to be imposed on the controller. Define the subset $\mathcal{S}$ of $\mathcal{F}^{n_u \times n_y}$ as:

$$\mathcal{S} \stackrel{def}{=} \left\{ K \in \mathcal{P}^{n_u \times n_y} \middle| \text{Pattern}(K) \leq K^{\text{bin}} \right\}, \tag{2.18}$$

that is, the set of controllers whose transfer function matrices satisfy the imposed sparsity structure. With the terminology from [79], the subspace $\mathcal{S}$ (of admissible, decentralized proper controllers) will be called the *information constraint*.

The following matrix G^{bin}, in the set $\{0, 1\}^{p \times m}$, is the sparsity pattern of the plant, which is defined as:

$$G^{\text{bin}} \stackrel{def}{=} \text{Pattern}(G) \tag{2.19}$$

Finally, from the matrix multiplication of matrices over $\mathcal{F}$ we note that for any $K \in \mathcal{F}^{n_u \times n_y}$ and any $G \in \mathcal{F}^{n_y \times n_u}$ with arbitrary sparsity patterns the following holds:

$$\text{Pattern}(KG) \leq \text{Pattern}(K)\text{Pattern}(G). \tag{2.20}$$

2.4.2 The Decentralized Disturbance Attenuation Problem

Throughout this subsection, we consider that $\mathcal{S}$ is the set of controllers that satisfy a pre-selected sparsity pattern as specified by a binary matrix $K^{\text{bin}} \in \{0,1\}^{m \times p}$, as defined in (4.6). Assume that P is stabilizable by some proper controller K_0 that is in $\mathcal{S}$. The decentralized disturbance attenuation problem, as introduced in [79, (1)/pp. 276], is formulated by adding the sparsity constraint $K \in \mathcal{S}$ to problem (3.1), as follows:

$$\min_{\substack{K \text{stabilizes } P \\ K \in \mathcal{S}}} \left\| f(P, K) \right\|. \tag{2.21}$$

The following result is a corollary of Theorem 4.51.

Corollary 2.4.1. *Let $P \in \mathcal{P}^{(n_u \times n_y) \times (n_u \times n_y)}$ be a proper, generalized plant, and $\mathcal{S}$ a subspace of proper controllers. If there is a controller K_0 in $\mathcal{S}$ that stabilizes P, then there exists a controller K^* in S that is optimal for (2.21) if and only if there exists a Q^* in $\mathcal{A}^{(n_u+n_y) \times (n_u+n_y)}$ that is optimal for the following minimization:*

$$\min_{\substack{Q \in \mathcal{A}^{(n_u+n_y) \times (n_u+n_y)} \\ \Omega_{21}(Q)\Omega_{11}^{-1}(Q) \in \mathcal{S}}} \left\| T_1 - T_2 Q T_3 \right\|. \tag{2.22}$$

where $K^* = \Omega_{21}(Q^*)\Omega_{11}^{-1}(Q^*)$. *Here,* T_1, T_2 *and* T_3 *are as in (2.14) while* $\Omega(Q)$ *is as in (2.7).*

Proof. **Necessity:** Suppose that K^* is an optimal solution of (2.21). As a consequence of Remark 2.3.1 we know that K^* is proper. Due to Theorem 2.3.2 **B)**, it follows that there exists $Q^* \in \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$ such that $K^* = \Omega_{21}(Q^*)\Omega_{11}^{-1}(Q^*)$ and therefore $\Omega_{21}(Q^*)\Omega_{11}^{-1}(Q^*) \in \mathcal{S}$. From the argument in the proof of [69, Theorem 1] we get that $\|f(P, K^*)\| = \|T_1 - T_2Q^*T_3\|$. We claim now that Q^* is optimal for (2.22). Suppose it is not. Then there must exist $Q' \in \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$ such that $\|T_1 - T_2Q'T_3\| < \|T_1 - T_2Q^*T_3\|$. Furthermore, $K' = \Omega_{21}(Q')\ \Omega_{11}^{-1}(Q')$ is a stabilizing controller (Theorem 2.3.2 **B)**) and $\|f(P, K')\| = \|T_1 - T_2Q'T_3\|$ (proof of [69, Theorem 1]). (From Remark 2.3.1 it follows that K' is also proper.) But then $\|f(P, K')\| < \|f(P, K^*)\|$ which contradicts the initial hypothesis on the optimality of K^*. We conclude that Q^* is an optimal solution to (2.22).

Sufficiency: Suppose that $Q^* \in \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$ is an optimal solution of (2.22). It follows by Theorem 2.3.2 **B)** that $K^* = \Omega_{21}(Q^*)\Omega_{11}^{-1}(Q^*) \in \mathcal{S}$ is a proper (Remark 2.3.1) stabilizing controller of P. Furthermore, via the argument in the proof of [69, Theorem 1] we get that $\|T_1 - T_2Q^*T_3\| = \|f(P, K^*)\|$. We claim now that K^* is optimal for (2.21). Suppose it is not. Then, there must exist a proper, stabilizing controller $K' \in \mathcal{S}$ such that $\|f(P, K')\| < \|f(P, K^*)\|$. It follows by Theorem 2.3.2 **B)** that there exists $Q' \in \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$ such that $K' = \Omega_{21}(Q')\Omega_{11}^{-1}(Q')$. From [69, Theorem 1] we get that $\|f(P, K')\| = \|T_1 - T_2Q'T_3\|$ and this implies that $\|T_1 - T_2Q'T_3\| < \|T_1 - T_2Q^*T_3\|$, which contradicts the optimality of Q^* assumed at the beginning of the proof. Hence K^* is an optimal solution for (2.21).

$\square$

2.4.3 Sparsity Constraints on the Q-Parameter

Consider the following conformable partition of the parameter Q (from Theorem 2.3.2 and Theorem 4.51), where Q belongs to the set $\mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$:

$$Q = \begin{array}{c} \begin{array}{cc} \overbrace{\phantom{Q_{11}}}^{n_y} & \overbrace{\phantom{Q_{12}}}^{n_u} \end{array} \\ \left[\begin{array}{cc} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{array}\right] \end{array} \begin{array}{l} \\ \}n_y \\ \}n_u \end{array} \tag{2.23}$$

For $Q_{12} \in \mathcal{A}^{n_y\times n_u}$ we assume the same partition by blocks as the partition for G in (2.15). That is, Q_{12} is partitioned in p block–rows and m block–columns and the i-th block–row has n_y^i rows, while the j-th block–column has n_u^j columns. Hence for any $(i,j) \in \overline{1,p} \times \overline{1,m}$ we get that $[Q_{12}]_{ij} \in \mathcal{A}^{n_y^i \times n_u^j}$. Similarly, assume for $Q_{21} \in \mathcal{A}^{n_u\times n_y}$ the same partition by blocks as the controller K, namely: m block–rows and p block–columns and for any $(j,i) \in \overline{1,m} \times \overline{1,p}$, $[Q_{21}]_{ji} \in \mathcal{A}^{n_u^j \times n_y^i}$. It follows that Q_{11} is naturally partitioned in p block–rows by p block–columns and the i-th block–row has n_y^i rows, while the j-th block–column has n_y^j columns. Consequently, for any $(i,j) \in \overline{1,p} \times \overline{1,p}$ we get that $[Q_{11}]_{ij} \in \mathcal{A}^{n_y^i \times n_y^j}$. Similarly, Q_{22} has m block–rows and m block–columns and the i-th block–row has n_u^i rows, while the j-th block–column has n_u^j columns.

In the sequel, we will make use of the set $\mathcal{T} \subset \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$ defined as follows:

$$\mathcal{T} \stackrel{def}{=} \left\{ \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \in \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)} \middle| \text{Pattern}(Q_{11}) \leq \left(G^{\text{bin}}K^{\text{bin}}+I_m\right); \text{Pattern}(Q_{12}) \leq \right.$$

$$\left. \leq \left(G^{\text{bin}}K^{\text{bin}}G^{\text{bin}}+G^{\text{bin}}\right);\ \text{Pattern}(Q_{21}) \leq K^{\text{bin}};\ \text{Pattern}(Q_{22}) \leq K^{\text{bin}}G^{\text{bin}} \right\}. \tag{2.24}$$

The set $\mathcal{T}$ in (2.24) can be written in a more compact form as follows:

$$\mathcal{T} = \left\{ Q \in \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)} \middle| \text{Pattern}(Q) \leq Q^{\text{bin}} \right\} \tag{2.25}$$

where Q^{bin} is the following matrix:

$$Q^{\text{bin}} \stackrel{def}{=} \begin{bmatrix} \left(G^{\text{bin}}K^{\text{bin}} + I_m\right) & \left(G^{\text{bin}}K^{\text{bin}}G^{\text{bin}} + G^{\text{bin}}\right) \\ K^{\text{bin}} & K^{\text{bin}}G^{\text{bin}} \end{bmatrix}. \tag{2.26}$$

Remark 2.4.2. *Note that $\mathcal{T}$ is a linear space. The alternative characterization of $\mathcal{T}$ in (2.25) reveals that $\mathcal{T}$ is solely specified by the sparsity matrix Q^{bin}.*

2.4.4 Quadratic Invariance

This subsection comprises a few results from [79], slightly adapted for the scope of this chapter. Since we deal exclusively with discrete–time, LTI systems with finite dimensional state we will use the terms *proper* and *causal* interchangeably.

Definition 2.4.3. *[79, Definition 2] Suppose that a strictly causal plant $G \in \mathcal{P}_s^{n_y \times n_u}$ and $\mathcal{S}$, a subset of $\mathcal{P}^{n_u \times n_y}$, are given. The set $\mathcal{S}$ is called quadratically invariant* *under the plant G if*

$$KGK \in \mathcal{S}, \qquad \text{for all} \;\; K \in \mathcal{S}$$

For ease of reference, we restate the following result, in fact another necessary and sufficient condition for quadratic invariance:

Proposition 2.4.4. *[79, Theorem 26] A linear subspace of controllers $\mathcal{S}$ is quadratically invariant under G if and only if*

$$KGJ \in \mathcal{S}, \;\; \text{for all} \; K, J \; \in \mathcal{S}$$

The next Proposition is a slight adaptation of Lemma 5 in [79].

Proposition 2.4.5. *Suppose that a strictly causal plant $G \in \mathcal{P}_s^{n_y \times n_u}$ and a linear subspace $\mathcal{S} \subset \mathcal{P}^{n_u \times n_y}$ are given. If $\mathcal{S}$ is quadratically invariant under G then*

$$K(GJ)^n \in \mathcal{S}, \;\; \text{for all} \; K, J \; \in \mathcal{S}, \; n \in \mathbb{Z}_+$$

where $\mathbb{Z}_+$ denotes the set of positive integers.

Proof. The proof follows by induction. For $n = 1$ the statement is true due to the quadratic invariance assumption and Proposition 2.4.4. The induction hypothesis at step $n \in \mathbb{Z}_+$ is that $K(GJ)^n$ is in $\mathcal{S}$. Now, consider the following identity:

$$K(GJ)^{n+1} = (K + F)G(J + F) - KGJ - KGF - FGF \tag{2.27}$$

where $F \overset{def}{=} K(GJ)^n$. Because F is in $\mathcal{S}$ by assumption and $\mathcal{S}$ is a linear space, we conclude that both $K + F$ and $J + F$ belong to $\mathcal{S}$ as well. Since the factors on the right hand side of (2.27) belong to $\mathcal{S}$, from Proposition 2.4.4, it follows that their sum stays in (the linear space) $\mathcal{S}$, hence the conclusion that $K(GJ)^{n+1}$ is in $\mathcal{S}$. □

For any positive integer time-horizon T and an arbitrarily selected function $f : \mathbb{Z}_+ \to \mathbb{R}$, we define f_T as follows:

$$f_T(t) \stackrel{def}{=} \begin{cases} f(t), & \text{if } t \leq T \\ 0, & \text{if } t > T \end{cases} \tag{2.28}$$

In addition to the standard ℓ_p Banach spaces, we define the following extended space denoted as ℓ_e:

$$\ell_e \stackrel{def}{=} \left\{ f : \mathbb{Z}_+ \to \mathbb{R} \middle| f_T \in \ell_\infty, \text{ for all } T \in \mathbb{Z}_+ \right\}.$$

Given general topological spaces $\mathcal{X}$ and $\mathcal{Y}$, we denote with $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the set of all linear, continuous maps from $\mathcal{X}$ to $\mathcal{Y}$. We consider that the topology on ℓ_e is generated by the sufficient family of seminorms $\{\| \cdot \|_T \mid T \in \mathbb{Z}_+\}$ where $\|f\|_T \stackrel{def}{=} \|f_T\|_{\ell_2}$. We adopt the topology on $\mathcal{L}(\ell_e^{n_u}, \ell_e^{n_y})$ that is generated by the sufficient family of seminorms $\{\| \cdot \|_T \mid T \in \mathbb{Z}_+\}$, where for any element A of $\mathcal{L}(\ell_e^{n_u}, \ell_e^{n_y})$ we define $\|A\|_T \stackrel{def}{=} \|A_T\|_{\ell_2^{n_u} \to \ell_2^{n_y}}$. Here, $\| \cdot \|_{\ell_2^{n_u} \to \ell_2^{n_y}}$ denotes the induced norm on maps from $\ell_2^{n_u}$ to $\ell_2^{n_y}$ and A_T is the map defined as $A_T : f \mapsto g_T$, where $A : f \mapsto g$.

Definition 2.4.6. *[79, Definition 13] A subset $\mathcal{S}$ of $\mathcal{L}(\ell_e^{n_u}, \ell_e^{n_y})$ is* inert *with respect to G if the inequality $r((gk(0))) < 1$ is satisfied and the inclusion $(gk)_{ij} \in \ell_e$ holds for all $K \in \mathcal{S}$ and $(i,j) \in \overline{1,m} \times \overline{1,m}$, where (gk) is the impulse response matrix of (GK) and $r(\cdot)$ denotes the spectral radius.*

Remark 2.4.7. *[79] In the sequel, we will use the fact that the assumption (2.6) on strict causality of the plant $G \in \mathcal{P}_s^{n_y \times n_u}$ implies that any subset $\mathcal{S}$ of $\mathcal{P}^{n_y \times n_u}$ is* *inert with respect to G.*

The following Lemma is a modified version of the first implication of [79, Theorem 14] and will be used extensively in the proof of our main result.

Lemma 2.4.8. *Suppose that $G \in \mathcal{L}(\ell_e^{n_u}, \ell_e^{n_y})$ and that $\mathcal{S}$ is a quadratically invariant and inert (with respect to G) closed subspace. The following inclusions hold:*

$$K(I - GJ)^{-1} \in \mathcal{S} \quad \text{for all } K, J \in \mathcal{S}, \tag{2.29}$$

$$(I - KG)^{-1} J \in \mathcal{S} \quad \text{for all } K, J \in \mathcal{S} \tag{2.30}$$

Proof. We will only prove (2.29), since (2.30) follows analogously by adequately adapting the results in Proposition 2.4.5, [79, Theorem 7] and [79, Theorem 8]. For any arbitrary choice of K and J in $\mathcal{S}$ the following holds:

$$K(I - GJ)^{-1} = K \sum_{n=0}^{\infty} (GJ)^n = \sum_{n=0}^{\infty} K(GJ)^n$$

where the first equality follows from [79, Theorem 7] and [79, Theorem 8], while the second equality follows from the continuity of K. Finally, by Proposition 2.4.5, we get that $K(GJ)^n \in \mathcal{S}$ for all $n \in \mathbb{Z}_+$ and since the subspace $\mathcal{S}$ is closed, it follows that (2.29) holds. □

2.5 Main Result

The central results of this chapter are stated in Theorems 2.5.4 and 2.5.5. We start this section with Proposition 2.5.1 and Lemma 2.5.2 that will be used as preliminary results in the rest of the chapter.

Proposition 2.5.1. *Under the hypothesis of Theorem 2.3.2* **A)** *and* **B)***, for any* $Q \in \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$ *the first block column of* $\Omega(Q)$ *from (2.7), with the conformable partition defined in (2.8), can be written as follows:*

$$\begin{bmatrix} \Omega_{11}(Q) \\ \Omega_{21}(Q) \end{bmatrix} = \begin{bmatrix} I_{n_y} - G\Omega_{21}(Q) \\ \Omega_{21}(Q) \end{bmatrix} \tag{2.31}$$

where $\Omega_{21}(Q)$ *is given by the following expression:*

$$\Omega_{21}(Q) = \Big(I+K_0G\Big)^{-1}\Big(K_0+K_0GK_0+K_0Q_{11}+K_0Q_{12}K_0+Q_{21}+Q_{22}K_0\Big)\Big(I+GK_0\Big)^{-1} \tag{2.32}$$

Proof. The proof is algebraic and is presented in the Appendix section of this chapter. □

Lemma 2.5.2. *Let* $P \in \mathcal{P}^{(n_u+n_y)\times(n_u+n_y)}$ *be a proper, generalized plant. Assume that the* G *block of* P *is strictly proper* ($G \in \mathcal{P}_s^{n_y\times n_u}$) *and that* P *is stabilizable by a controller* K_0 *that is in* $\mathcal{S}$*. If* $\mathcal{S}$ *is quadratically invariant under* G *and* K_0 *is a stabilizing controller in* $\mathcal{S}$*, then the map* $K : \mathcal{T} \longmapsto \Big(\mathcal{K}_G \cap \mathcal{S}\Big)$ *defined below is* onto*:*

$$K(Q) \overset{def}{=} \Omega_{21}(Q)\Omega_{11}^{-1}(Q), \qquad Q \in \mathcal{T} \tag{2.33}$$

Here the set $\mathcal{K}_G$ *is as defined in (2.9), the set* $\mathcal{T}$ *is defined in (2.24), while* $\Omega_{11}(Q)$ *and* $\Omega_{21}(Q)$ *are the blocks in the first column of* $\Omega(Q)$ *from (2.7) with the conformable partition defined in (2.8).*

Proof. See the Appendix section of this chapter. □

Remark 2.5.3. *Lemma 2.5.2 is the centerpiece of our main result, as it bridges the gap between the sparsity constraint $K(Q) \in \mathcal{S}$ imposed on the controller and the convex sparsity constraint $Q \in \mathcal{T}$ on the Q–parameter. (Note that according to Remark 2.4.2, the set $\mathcal{T}$ is a linear subspace.)*

2.5.1 The Coordinate–free Parametrization of All Stabilizing, Decentralized Controllers

The following Theorem, which is the decentralized counterpart of Theorem 2.3.2, provides the parametrization of all decentralized, stabilizing controllers subject to pre-selected, quadratically invariant sparsity constraints.

Theorem 2.5.4. *Let $P \in \mathcal{P}^{(n_u \times n_y)\times(n_u \times n_y)}$ be a generalized plant with a strictly proper block G of P ($G \in \mathcal{P}_s^{n_y \times n_u}$). Given a set $\mathcal{S}$ of sparsity-constrained controllers that is quadratically invariant under G and a controller*[1] *K_0 in $\mathcal{S}$ that stabilizes P, the set of all stabilizing controllers in $\mathcal{S}$ is given by:*

$$\left(\mathcal{K}_G \cap \mathcal{S}\right) = \left\{\Omega_{21}(Q)\Omega_{11}^{-1}(Q)\middle|Q \in \mathcal{T}\right\}. \tag{2.34}$$

Notice that since this Theorem does not involve Zames's Q–parametrization, it is not conditional on the strongly stabilizability of the plant and it constitutes an extension of [79] as it only requires that the plant is stabilizable by a (not necessarily stable) controller that is in the sparsity constrained set $\mathcal{S}$.

Proof. (Of Theorem 2.5.4) The "$\subset$" inclusion in (2.34) follows from Theorem 2.3.2,

[1]Here K_0 is used in the expression for $\Omega(Q)$ as given in (2.7)

Lemma 2.2.2 and the fact that the function $K(\cdot)$ from (2.33) is a well defined function from $\mathcal{T}$ to $\left(\mathcal{K}_G \cap \mathcal{S}\right)$ (part **I** in the proof of Lemma 2.5.2). Finally, the "$\supset$" inclusion in (2.34) follows from Theorem 2.3.2, Lemma 2.2.2 and the fact that the function $K(\cdot)$ is onto (Lemma 2.5.2). $\square$

2.5.2 The Decentralized, Optimal Disturbance Attenuation Problem

The following Theorem uses the parametrization of Theorem 2.5.4 to cast problem (2.21) using a convex (model–matching) program. It is important to note this formulation improves on the approach outlined in [78, (4.8)/ pp.37] as it does not require stability constraints.

Theorem 2.5.5. *Let $P \in \mathcal{P}^{(n_u+n_y)\times(n_u+n_y)}$ be a proper, generalized plant whose G block is strictly proper ($G \in \mathcal{P}_s^{n_y \times n_u}$) and $\mathcal{S}$ be a preselected set of sparsity constrained controllers. In addition, suppose that P can be stabilized by a proper controller K_0 that is in $\mathcal{S}$. If $\mathcal{S}$ is quadratically invariant under G then the decentralized disturbance attenuation problem (2.21) is equivalent to the following convex (model–matching) program:*

$$\min_{Q \in \mathcal{T}} \quad \left\| T_1 - T_2 Q T_3 \right\| \tag{2.35}$$

where T_1, T_2 and T_3 are given in (2.14). An optimal solution K^* to (2.21) can always be obtained from the optimal Q in (2.35), denoted with Q^*, via $K^* =$

$\Omega_{21}(Q^*)\Omega_{11}^{-1}(Q^*)$.

Proof. It is a consequence of Corollary 2.4.1 and Lemma 2.5.2. □

A convenient feature of the equivalent convex formulation (2.35) from Theorem 2.5.5 is that the numerical technique from [79, Theorem 29] is readily available to numerically solve (2.35) (by employing existing tools from standard $\mathcal{H}_2$ synthesis). The only draw–back being that the optimization problem from (2.35) although similar, is larger in size than its (strongly stabilizable case) counterpart from [78, Section 4.5].

More specifically, the dimension of the Q parameter used in (2.35) is the order of the closed loop transfer function matrix $H(G, K)$ given in (2.5), while the the dimension of the analogous parameter in [78, Section 4.5] is the order of the controller K.

2.5.3 The Decentralized, Mixed $\mathcal{H}^2$ Sensitivity Problem

Let G be a strictly proper plant ($G \in \mathcal{P}_s^{n_y \times n_u}$), stabilizable with a proper, decentralized controller $K_0 \in \mathcal{S}$. As another classical control application, consider the mixed $\mathcal{H}^2$ sensitivity problem from [57, pp. 139], which consists in minimizing the weighted first block–column of $H(G, K)$ over all stabilizing controllers in $\mathcal{S}$, namely:

$$\min_{\substack{K \text{stabilizes } G \\ K \in \mathcal{S}}} \left\| \begin{bmatrix} W_e & O \\ O & \rho W_u \end{bmatrix} H(G,K) \begin{bmatrix} W_d \\ O \end{bmatrix} \right\|, \tag{2.36}$$

where W_e, W_d are preselected weighting transfer function matrices and ρ is an appropriately chosen positive real constant.

The problem stated in (2.36) is an extension of the original approach in [78, 79], where only the $K(I-GK)^{-1}$ entry of $H(G,K)$ (involved in the cost function) was employed via a change of variables.

The following Theorem shows how problem (2.36) can be solved via a convex (model–matching) program.

Theorem 2.5.6. *Let G be a strictly proper plant ($G \in \mathcal{P}_s^{n_y \times n_u}$) and $\mathcal{S}$ a pre-selected set of sparsity–constrained controllers that is quadratically invariant with respect to G. If K_0 is a controller in $\mathcal{S}$ that stabilizes G then the minimum norm control problem (2.36) can be solved via the following convex (model-matching) program:*

$$\min_{Q \in \mathcal{T}} \left\| V_1 + V_2 Q V_3 \right\| \tag{2.37}$$

Here $H(G,K)$ is the closed–loop TFM defined in (2.5), while V_1, V_2 and V_3 are the following TFMs:

$$
\begin{aligned}
V_1 &\stackrel{def}{=} \begin{bmatrix} W_e & O \\ O & \rho W_u \end{bmatrix} H(G, K_0) \begin{bmatrix} W_d \\ O \end{bmatrix}, \\
V_2 &\stackrel{def}{=} \begin{bmatrix} W_e & O \\ O & \rho W_u \end{bmatrix} \left(H(G, K_0) - \begin{bmatrix} I_{n_y} & O \\ O & O \end{bmatrix} \right) \begin{bmatrix} W_d \\ O \end{bmatrix}, \\
V_3 &\stackrel{def}{=} \begin{bmatrix} W_e & O \\ O & \rho W_u \end{bmatrix} \left(H(G, K_0) - \begin{bmatrix} O & O \\ O & I_{n_u} \end{bmatrix} \right) \begin{bmatrix} W_d \\ O \end{bmatrix}.
\end{aligned}
\tag{2.38}
$$

An optimal solution K^* to (2.36) can always be obtained from the optimal Q in (4.52), denoted with Q^*, via $K^* = \Omega_{21}(Q^*)\Omega_{11}^{-1}(Q^*)$. The sets $\mathcal{S}$ and $\mathcal{T}$, as well as $\Omega_{11}(Q)$ and $\Omega_{21}(Q)$ are as specified in the statement of Theorem 2.5.4.

Proof. It is a direct consequence of Theorem 2.3.2 and Lemma 2.5.2. □

We point out that the numerical technique from [79, Theorem 29] is again readily employable to compute the optimal solution of (4.52).

Appendix

Proof of Proposition 2.5.1 The following algebraic identities will prove to be useful. They hold true in any ring provided the inverses involved exist:

$$(I + AB)^{-1}A = A(I + BA)^{-1}, \tag{2.39}$$

$$(I + AB)^{-1} = I - A(I + BA)^{-1}B. \tag{2.40}$$

We start with the expression of the first block–column of (2.7) from Theorem 2.3.2.

$$
\begin{aligned}
&\Omega(Q)\begin{bmatrix} I_{n_y} \\ O_{n_u\times n_y} \end{bmatrix} = \\
&\overset{(2.7)}{=} \begin{bmatrix} \left(I+GK_0\right)^{-1}-I & -G\left(I+K_0G\right)^{-1} \\ K_0\left(I+GK_0\right)^{-1} & \left(I+K_0G\right)^{-1} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \times \\
&\qquad\qquad \times \begin{bmatrix} \left(I+GK_0\right)^{-1} \\ K_0\left(I+GK_0\right)^{-1} \end{bmatrix} + H(G,K_0)\begin{bmatrix} I_{n_y} \\ O_{n_u\times n_y} \end{bmatrix} \\
&\overset{(2.40,2.39)}{=} \begin{bmatrix} -GK_0\left(I+GK_0\right)^{-1} & -G\left(I+K_0G\right)^{-1} \\ K_0\left(I+GK_0\right)^{-1} & \left(I+K_0G\right)^{-1} \end{bmatrix} \times \\
&\times \begin{bmatrix} Q_{11}\left(I+GK_0\right)^{-1}+Q_{12}K_0\left(I+GK_0\right)^{-1} \\ Q_{21}\left(I+GK_0\right)^{-1}+Q_{22}K_0\left(I+GK_0\right)^{-1} \end{bmatrix} + H(G,K_0)\begin{bmatrix} I_{n_y} \\ O_{n_u\times n_y} \end{bmatrix}
\end{aligned}
\tag{2.41}
$$

$$\stackrel{(2.39,2.40,2.5)}{=} \begin{bmatrix} -G\Big(I+K_0G\Big)^{-1}\Big(K_0Q_{11}+K_0Q_{12}K_0+Q_{21}+Q_{22}K_0\Big)\Big(I+GK_0\Big)^{-1} \\ \Big(I+K_0G\Big)^{-1}\Big(K_0Q_{11}+K_0Q_{12}K_0+Q_{21}+Q_{22}K_0\Big)\Big(I+GK_0\Big)^{-1} \end{bmatrix} + \\ + \begin{bmatrix} I-G\big(I+K_0G\big)^{-1}K_0 \\ K_0\big(I+GK_0\big)^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} I-G\Big(I+K_0G\Big)^{-1}\Big(K_0+K_0GK_0+K_0Q_{11}+K_0Q_{12}K_0+Q_{21}+Q_{22}K_0\Big)\Big(I+GK_0\Big)^{-1} \\ \Big(I+K_0G\Big)^{-1}\Big(K_0+K_0GK_0+K_0Q_{11}+K_0Q_{12}K_0+Q_{21}+Q_{22}K_0\Big)\Big(I+GK_0\Big)^{-1} \end{bmatrix} \tag{2.42}$$

which is the desired expression.

Proof of Lemma 2.5.2 We divide the proof in two parts: in part **I)** we prove that the function $K(\cdot)$ is a well–defined function indeed, from $\mathcal{T}$ to $\big(\mathcal{C} \cap \mathcal{S}\big)$. In part **II)** we show that the function $K(\cdot)$ is *onto*.

I) The invertibility (for every $Q \in \mathcal{T}$) of the block $\Omega_{11}(Q)$ in the expression (2.9) of $K(\cdot)$ is guaranteed by the arguments stated in Remark 2.3.3.

Let $Q \in \mathcal{T}$ be arbitrary but fixed. Since $Q \in \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$, it follows by Theorem 2.3.2 ***ii)*** that $K(Q) \in \mathcal{C}$, so it only remains to show that $K(Q) \in \mathcal{S}$.

We expand the product in (2.32) to get that $\Omega_{21}(Q)$ (in the form provided by Proposition 2.5.1) is the sum of the following six terms:

$$\Omega_{21}(Q) = \underbrace{\big(I+K_0G\big)^{-1}K_0\big(I+GK_0\big)^{-1}}_{t_1} + \underbrace{\big(I+K_0G\big)^{-1}K_0GK_0\big(I+GK_0\big)^{-1}}_{t_2} +$$

$$\underbrace{\big(I+K_0G\big)^{-1}K_0Q_{11}\big(I+GK_0\big)^{-1}}_{t_3} + \underbrace{\big(I+K_0G\big)^{-1}K_0Q_{12}K_0\big(I+GK_0\big)^{-1}}_{t_4} +$$

$$\underbrace{\big(I+K_0G\big)^{-1}Q_{21}\big(I+GK_0\big)^{-1}}_{t_5} + \underbrace{\big(I+K_0G\big)^{-1}Q_{22}K_0\big(I+GK_0\big)^{-1}}_{t_6} \tag{2.43}$$

We prove next that $\Omega_{21}(Q)$ is in $\mathcal{S}$. We prove this by showing that each of the six terms in the sum of the right hand side of (2.43) are in $\mathcal{S}$. Since $\mathcal{S}$ is a (closed) linear subspace, it will follow that $\Omega_{21}(Q)$ stays in $\mathcal{S}$ as well. Remember that from the hypothesis $K_0 \in \mathcal{S}$ and define

$$\Delta_0 \stackrel{def}{=} K_0(I+GK_0)^{-1}. \tag{2.44}$$

It follows that Δ_0 belongs to $\mathcal{S}$, by the assumed quadratically invariance of $\mathcal{S}$ under G and [79, Theorem 14].

The first term in (2.43) is

$$t_1 = \Big((I+K_0G)^{-1}K_0\Big)(I+GK_0)^{-1} \stackrel{(2.44)}{=} \Delta_0(I+GK_0)^{-1}$$

which is in $\mathcal{S}$ by (2.29) from Lemma 2.4.8. The second term

$$t_2 = \Big((I+K_0G)^{-1}K_0\Big)G\Big(K_0(I+GK_0)^{-1}\Big) \stackrel{(2.44)}{=} \Delta_0G\Delta_0$$

which is in $\mathcal{S}$ because $\Delta_0 \in \mathcal{S}$, $\mathcal{S}$ is quadratically invariant under G and Definition 4.2.3.

We know that

$$K^{\rm bin}G^{\rm bin}K^{\rm bin} = K^{\rm bin} \tag{2.45}$$

holds true, as an immediate consequence of Definition 4.2.3.

From (2.24) we know that $\text{Pattern}(Q_{11}) = G^{\rm bin}K^{\rm bin} + I_m$ and so

$$\text{Pattern}\big(\Delta_0 Q_{11}\big) \overset{(4.8)}{\leq} \text{Pattern}(\Delta_0)\text{Pattern}(Q_{11}) = K^{\rm bin}\big(G^{\rm bin}K^{\rm bin} + I_m\big)$$
$$= K^{\rm bin}G^{\rm bin}K^{\rm bin} + K^{\rm bin} = K^{\rm bin} + K^{\rm bin} = K^{\rm bin} \tag{2.46}$$

because of (2.45) and the fact that $K^{\rm bin} + K^{\rm bin} = K^{\rm bin}$ (due to the way addition is defined for binary matrices). Define

$$W_{11} \overset{def}{=} \big(\Delta_0 Q_{11}\big). \tag{2.47}$$

Then, since $\text{Pattern}(W_{11}) \leq K^{\rm bin}$ we conclude $W_{11} \in \mathcal{S}$. The third term is

$$t_3 = \Delta_0 Q_{11}(I + GK_0)^{-1} \overset{(2.47)}{=} W_{11}(I + GK_0)^{-1}$$

and it belongs to $\mathcal{S}$ by (2.29) in Lemma 2.4.8.

From (2.24) we know that

$$\text{Pattern}(Q_{12}) = G^{\rm bin}K^{\rm bin}G^{\rm bin} + G^{\rm bin}.$$

The fourth term is $t_4 = \Delta_0 Q_{12}\Delta_0$. It follows that

$$\begin{aligned}
&\text{Pattern}\big(\Delta_0 Q_{12}\Delta_0\big) \overset{(4.8)}{\leq} \text{Pattern}\big(\Delta_0\big)\text{Pattern}\big(Q_{12}\big)\text{Pattern}\big(\Delta_0\big)\\
&= K^{\text{bin}}\big(G^{\text{bin}}K^{\text{bin}}G^{\text{bin}} + G^{\text{bin}}\big)K^{\text{bin}} = K^{\text{bin}}\big(G^{\text{bin}}K^{\text{bin}}G^{\text{bin}}\big)K^{\text{bin}} + K^{\text{bin}}G^{\text{bin}}K^{\text{bin}}\\
&= \big(K^{\text{bin}}G^{\text{bin}}K^{\text{bin}}\big)G^{\text{bin}}K^{\text{bin}}+K^{\text{bin}} \overset{(2.45)}{=} K^{\text{bin}}G^{\text{bin}}K^{\text{bin}}+K^{\text{bin}} \overset{(2.45)}{=} K^{\text{bin}}+K^{\text{bin}} = K^{\text{bin}}
\end{aligned} \tag{2.48}$$

From $\text{Pattern}(t_4) \leq K^{\text{bin}}$ we get that $t_4 \in \mathcal{S}$ as well.

From (2.24) we know that $\text{Pattern}(Q_{21}) = K^{\text{bin}}$ and so $Q_{21} \in \mathcal{S}$. Denote

$$W_{21} \overset{def}{=} Q_{21}(I + GK_0)^{-1}. \tag{2.49}$$

But then $W_{21} \in \mathcal{S}$ by (2.29) in Lemma 2.4.8. The fifth term is then

$$t_5 = (I + K_0G)^{-1}\Big(Q_{21}(I + GK_0)^{-1}\Big) \overset{(2.49)}{=} (I + K_0G)^{-1}W_{21}$$

which is in $\mathcal{S}$ by (2.30) from Lemma 2.4.8.

Finally, from (2.24) we write that $\text{Pattern}(Q_{22}) = K^{\text{bin}}G^{\text{bin}}$ and so

$$\begin{aligned}
&\text{Pattern}\big(Q_{22}\Delta_0\big) \overset{(4.8)}{\leq} \text{Pattern}\big(Q_{22}\big)\text{Pattern}\big(\Delta_0\big) = \big(K^{\text{bin}}G^{\text{bin}}\big)K^{\text{bin}}\\
&= K^{\text{bin}}G^{\text{bin}}K^{\text{bin}} \overset{(2.45)}{=} K^{\text{bin}}
\end{aligned} \tag{2.50}$$

Denote

$$W_{22} \stackrel{def}{=} Q_{22}\Delta_0. \tag{2.51}$$

Since Pattern$(W_{22}) \leq K^{\text{bin}}$ we get that $W_{22} \in \mathcal{S}$. Therefore the sixth and last term

$$t_6 = (I + K_0 G)^{-1}\big(Q_{22}\Delta_0\big) \stackrel{(2.51)}{=} (I + K_0 G)^{-1} W_{22}$$

and it belongs to $\mathcal{S}$ by (2.30) from Lemma 2.4.8.

We have just proved that $\Omega_{21}(Q) \in \mathcal{S}$ for any $Q \in \mathcal{T}$. It follows then by [79, Theorem 14] that

$$K(Q) \stackrel{(2.9)}{=} \Omega_{21}(Q)\Omega_{11}^{-1}(Q) \stackrel{(4.70)}{=} \Omega_{21}(Q)\Big(I - G\Omega_{21}(Q)\Big)^{-1} \in \mathcal{S}.$$

and the first part of the proof ends.

II) Let be $\overline{K} \in \Big(\mathcal{C} \,\cap\, \mathcal{S}\Big)$, arbitrarily chosen. We will prove that there exists a $\overline{Q} \in \mathcal{T}$ such that $K(\overline{Q}) = \overline{K}$. We show that such a $\overline{Q}$ is given by

$$\overline{Q} = \begin{bmatrix} -\big(I + G\overline{K}\big)^{-1} & -G\big(I + \overline{K}G\big)^{-1} \\ \overline{K}\big(I + G\overline{K}\big)^{-1} & I - \big(I + \overline{K}G\big)^{-1} \end{bmatrix}. \tag{2.52}$$

Note that

$$\overline{Q} \stackrel{(2.5)}{=} \begin{bmatrix} -I & O \\ O & I \end{bmatrix} H(G, \overline{K}) \begin{bmatrix} I & O \\ O & -I \end{bmatrix} + \begin{bmatrix} O & O \\ O & I \end{bmatrix}$$

and because $\overline{K} \in \mathcal{C}$ implies that $H(G, \overline{K}) \in \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$, we get that $\overline{Q}$ is in the set $\mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$ as well. Next, denote

$$\overline{\Delta} \stackrel{def}{=} \overline{K}\big(I + G\overline{K}\big)^{-1}. \tag{2.53}$$

By the hypothesis of quadratic invariance of $\mathcal{S}$ under G it follows via [79, Theorem 14] that $\overline{\Delta} \in \mathcal{S}$. Furthermore, because of the invertibility of the transformation in (2.53) as a function of $\overline{K}$ (see [79, pp. 15]), we get that

$$\overline{K} \stackrel{def}{=} \overline{\Delta}\big(I - G\overline{\Delta}\big)^{-1}. \tag{2.54}$$

Next, because of (2.52) and

$$\begin{aligned}
&G\overline{\Delta}-I \stackrel{(2.53)}{=} G\overline{K}\big(I+G\overline{K}\big)^{-1}-I \stackrel{(2.39)}{=} G\big(I+\overline{K}G\big)^{-1}\overline{K}-I \stackrel{(2.40)}{=} I-\big(I+G\overline{K}\big)^{-1}-I = -\big(I+G\overline{K}\big)^{-1},\\
&G\overline{\Delta}G - G = -\big(I + G\overline{K}\big)^{-1}G \stackrel{(2.39)}{=} -G\big(I + G\overline{K}\big)^{-1},\\
&\overline{\Delta}G \stackrel{(2.53)}{=} \overline{K}\big(I + G\overline{K}\big)^{-1}G \stackrel{(2.40)}{=} I - \big(I + \overline{K}G\big)^{-1}.
\end{aligned} \tag{2.55}$$

we get that

$$\overline{Q} = \begin{bmatrix} \big(G\overline{\Delta} - I\big) & \big(G\overline{\Delta}G - G\big) \\ \overline{\Delta} & \overline{\Delta}G \end{bmatrix} \tag{2.56}$$

Assuming for $\overline{Q}$ the same conformable partition from (2.23), denote $\overline{Q}_{11} \stackrel{def}{=} (G\overline{\Delta} - I)$, $\overline{Q}_{12} \stackrel{def}{=} (G\overline{\Delta}G - G)$, $\overline{Q}_{21} \stackrel{def}{=} \overline{\Delta}$ and $\overline{Q}_{22} \stackrel{def}{=} \overline{\Delta}G$. It follows that $\text{Pattern}(\overline{Q}_{11}) \leq \Big(G^{\text{bin}}K^{\text{bin}}+I_m\Big)$, $\text{Pattern}(\overline{Q}_{12}) \leq \Big(G^{\text{bin}}K^{\text{bin}}G^{\text{bin}}+G^{\text{bin}}\Big)$, $\text{Pattern}(\overline{Q}_{21}) \leq K^{\text{bin}}$ and $\text{Pattern}(\overline{Q}_{22}) \leq K^{\text{bin}}G^{\text{bin}}$. This proves via (2.24) that $\overline{Q} \in \mathcal{T}$.

It only remains to show that $K(\overline{Q}) = \overline{K}$. By plugging (2.56) in (2.32) we get

$$
\begin{aligned}
\Omega_{21}(\overline{Q}) &= \Big(I+K_0G\Big)^{-1}\Big(K_0+K_0GK_0+K_0(G\overline{\Delta}-I)+K_0(G\overline{\Delta}G-G)K_0+\overline{\Delta}+\overline{\Delta}GK_0\Big)\Big(I+GK_0\Big)^{-1} \\
&= \Big(I+K_0G\Big)^{-1}\Big(K_0G\overline{\Delta}+K_0G\overline{\Delta}GK_0+\overline{\Delta}+\overline{\Delta}GK_0\Big)\Big(I+GK_0\Big)^{-1} \\
&= \Big(I+K_0G\Big)^{-1}\Big(K_0G\overline{\Delta}(I+GK_0)+\overline{\Delta}(I+GK_0)\Big)\Big(I+GK_0\Big)^{-1} \\
&= \Big(I+K_0G\Big)^{-1}\Big(K_0G\overline{\Delta}+\overline{\Delta}\Big) \\
&= \Big(I+K_0G\Big)^{-1}\Big(I+K_0G\Big)\overline{\Delta} \\
&= \ \overline{\Delta}
\end{aligned}
\tag{2.57}
$$

Finally

$$
K(\overline{Q}) \overset{(4.70)}{=} \Omega_{21}(\overline{Q})\Big(I-G\Omega_{21}(\overline{Q})\Big)^{-1} \overset{(2.57)}{=} \overline{\Delta}\Big(I-G\overline{\Delta}\Big)^{-1} \overset{(2.54)}{=} \overline{K}
$$

hence the proof.

Chapter 3

Optimal Disturbance Attenuation in the Presence of Stable, Additive Plant Perturbations

Contribution. In this chapter we deal with the optimal disturbance attenuation problem ([58]) for linear and time invariant (LTI) systems. We look at performance criteria quantified by a given operatorial induced norm, or gain, of the lower linear fractional transformation of a generalized plant in feedback interconnection with the controller. Our main result proves that the minimal gain attainable by causal feedback is not influenced by linear, stable, additive plant perturbations. Furthermore, this is shown to hold true, irrespective of the used norm (*e.g.* for 1-D, LTI systems it could be any of the $\mathcal{L}^p$ or ℓ^p induced norms, respectively). It follows as a direct consequence that for the optimal synthesis procedure, it is sufficient to solve the disturbance attenuation problem only for the anti–stable component of the plant. The solution obtained for the anti–stable component of the plant can than be used to retrieve the optimal solution for the entire plant, via a simple algebraic, feedback transformation.

Furthermore, we also prove the validity of our result for an important class of decentralized control systems, namely decentralized configurations that are *quadratically invariant* ([79]) or *invariant under feedback*. Moreover, since the proof of the main result is completed without any assumption on the coprime factorizability of the plant, it also encompasses the case of linear, n-D systems ([69]).

3.0.4 Preliminaries and Notation

We have preferred to present our result in an abstract theoretic setting, which encompasses a large class of linear systems, including n-D systems ([69]). To this end we borrow entirely the notation from the previous chapter, specifically the notation from Subsection 2.2.1 and Subsection 2.2.2. For the proof of our main result, we employ solely the commutative ring algebra on the set of all stable, linear systems. That is, the fact that parallel and cascade connections of stable, linear systems are again stable, linear systems.

The so–called disturbance attenuation problem, which we state next, stands out as a central topic in systems control theory ([58]).

Problem 1. Consider a proper, stabilizable, generalized plant P in the feedback configuration of Figure 2.1. We wish to design a stabilizing controller K that minimizes a certain norm of the transfer function from w to z, namely

$$\min_{K \text{stabilizes } P} \left\| P_{zw} + P_{zu}\, K\big(I + GK\big)^{-1} P_{yw} \right\|. \tag{3.1}$$

The functional in (3.1) is called the lower–linear fractional transformation of the generalized plant P with controller K and it will be denoted it in the sequel with

$$f(P_{zw}, P_{zu}, G, P_{yw}, K) \stackrel{def}{=} P_{zw} + P_{zu}\, K\big(I + GK\big)^{-1} P_{yw}. \tag{3.2}$$

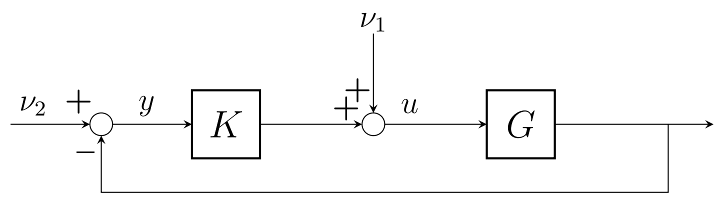

Figure 3.1: Standard unity feedback interconnection

In Figure 1, the transfer function matrices $K \in \mathcal{P}^{n_u \times n_y}$ and $G \in \mathcal{P}_s^{n_y \times n_u}$ represent the controller and the plant respectively, interconnected in the standard *unity feedback* configuration. We denote with $H(G, K)$ the transfer function matrix from $[\nu_2^T \quad \nu_1^T \quad]^T$ to $[y^T \quad u^T \quad]^T$ in Figure 2, (provided that the feedback loop is well–posed):

$$H(G, K) \stackrel{def}{=} \begin{bmatrix} (I + GK)^{-1} & -G(I + KG)^{-1} \\ K(I + GK)^{-1} & (I + KG)^{-1} \end{bmatrix} \tag{3.3}$$

If the transfer matrix $H(G, K)$ in (3.3) belongs to $\mathcal{A}$ we say that K is a *stabilizing controller* of G or equivalently that K *stabilizes* G. If a stabilizing controller of G exists, we say that G is *stabilizable.* With the exact same the notation introduced in the previous chapter, we remark that $H(G, K)$ is the transfer function matrix from $[\nu_2^T \ \nu_1^T \ ]^T$ to $[y^T \ u^T \ ]^T$ for the system in Figure 1, as well.

The following Lemma is a generalization of the result for LTI systems, from [56, Theorem 4.3.2]. The generalization comes from the fact that the next Lemma is proved in [69] using only the algebra of the abstract, commutative ring $\mathcal{A}$ and in doing so, unlike [56, Theorem 4.3.2], it does not assume the coprime factorizability of the plant.

Lemma 3.0.7. *[69, Lema 1] Given a proper, stabilizable, generalized plant P, the* *controller K is a stabilizing controller of P (in the feedback system from Figure 2.1) if and only if K is a stabilizing controller of G (in the feedback system from Figure 2.2).*

We introduce the following notation for the set of all stabilizing controllers of a given plant G (note that via Remark 2.3.1 any stabilizing controller of the strictly proper G is proper):

$$\mathcal{C}_G \stackrel{def}{=} \Big\{K \Big| K \in \mathcal{P}^{n_y \times n_u} \text{ and } K \text{ stablilizes } G\Big\}. \tag{3.4}$$

Using the notation in (3.2), (3.4) and Lemma 3.0.7, a more compact formulation of **Problem 1** is:

$$\min_{K \in \mathcal{C}_G} \quad \| f(P_{zw}, P_{zu}, G, P_{yw}, K) \| \,. \tag{3.5}$$

Whenever the feedback loop $H(G, K)$ is well–posed, we call the transfer function matrix from ν_2 to u, the *feedback transformation* of G with K and we denote it in the sequel with

$$h_G(K) \stackrel{def}{=} K\big(I + GK\big)^{-1}. \tag{3.6}$$

Remark 3.0.8. *Also note that $h_G(\cdot)$ from (3.6), seen as a function from $\mathcal{P}^{n_u \times n_y}$ to $\mathcal{P}^{n_u \times n_y}$ is bijective and its inverse has the expression $h_G^{-1}(K) = K(I - GK)^{-1}$.*

3.1 Main Result

Figure 2 represents the standard unity feedback configuration of the additively perturbed *nominal* plant G. The *stable* transfer function matrix G_s represents an

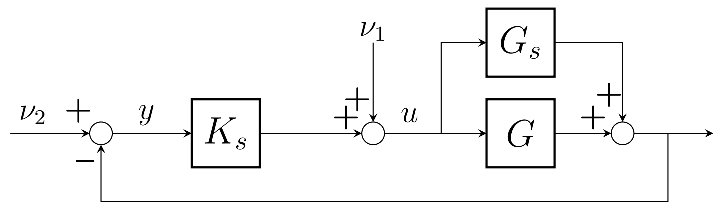

Figure 3.2: Standard unity feedback interconnection with stable additive perturbation

additive plant perturbation of the *nominal* plant G, while the K_s block represents the controller. The nominal plant G is not assumed to be stable. We are interested in how the solution to Problem 1 relates to the solution of the disturbance attenuation problem with the additively perturbed nominal plant, which we state next.

Problem 2. Let be the proper, stabilizable, generalized plant P, and the strictly proper, *stable* plant perturbation G_s of the nominal plant G be given. We wish to design a stabilizing controller K_s of $(G + G_s)$ that minimizes the following functional

$$\min_{K_s \in \mathcal{C}_{(G+G_s)}} \quad \|f(P_{zw}, P_{zu}, (G + G_s), P_{yw}, K_s)\| . \tag{3.7}$$

With the notation from (3.4), here $\mathcal{C}_{(G+G_s)}$ stands for the set of all stabilizing controllers of $(G + G_s)$ in the feedback interconnection from Figure 3 while $f(\cdot)$ is as defined in (3.2).

Lemma 3.1.1. A) *Consider the strictly proper, stabilizable plant G in the set $\mathcal{P}_s^{n_y \times n_u}$, any strictly proper,* stable *perturbation G_s also in the set $\mathcal{P}_s^{n_y \times n_u}$ and any* *stabilizing controller K of G. Then K_s, defined by the expression*

$$K_s \overset{def}{=} K(I - G_s K)^{-1}. \tag{3.8}$$

is a stabilizing controller of $(G + G_s)$. Furthermore $h_G(K) = h_{(G+G_s)}(K_s)$. Here, $h_G(K)$ is as defined in (3.6).

B) *Conversely, given the proper, stabilizable plant G in the set $\mathcal{P}_s^{n_y \times n_u}$, any strictly proper,* stable *perturbation G_s in the set $\mathcal{P}_s^{n_y \times n_u}$ and any stabilizing controller K_s of $(G + G_s)$, then K defined as*

$$K \overset{def}{=} K_s(I + G_s K_s)^{-1}. \tag{3.9}$$

is a stabilizing controller of G. Furthermore $h_{(G+G_s)}(K_s) = h_G(K)$.

Proof. The proof is given in the Appendix of this chapter. □

An immediate consequence of Lemma 3.1.1 is the following main result of this paper, connecting the optimal solution to Problem 1 to the optimal solution of Problem 2. Here follows the precise statement.

Corollary 3.1.2. *For* any *consistent norm over $\mathcal{A}$, consider the given proper, stabilizable, generalized plant P, and any strictly proper,* stable *plant perturbation G_s (of the nominal plant G). If K^* is the optimal solution to Problem 1 then*

$$K_s^* = K^*(I - G_s K^*)^{-1} \tag{3.10}$$

is the optimal solution to Problem 2.

Proof. The proof follows by contradiction. Suppose that K^* is the optimal solution to Problem 1. According to Lemma 3.1.1 **A)**, $h_G(K^*) = h_{(G+G_s)}(K_s^*)$ and so $\|f(P_{zw}, P_{zu}, G, P_{yw}, K^*)\| \overset{(3.2)}{=} \|P_{zw} + P_{zu}\, h_G(K^*)P_{yw}\| = \left\|P_{zw} + P_{zu}\, h_{(G+G_s)}(K_s^*)P_{yw}\right\| \overset{(3.2)}{=}$ $\|f(P_{zw}, P_{zu}, G + G_s, P_{yw},\ K_s^*)\|$. Suppose now that K_s^* is not the optimal solution for Problem 2, therefore there exists $\widetilde{K}_s$ a causal, stabilizing controller of $(G + G_s)$ such that $\left\|f(P_{zw}, P_{zu}, (G + G_s), P_{yw}, \widetilde{K}_s\right\| < \|f(P_{zw},\ P_{zu}, (G + G_s), P_{yw}, K_s^*)\|$. Then, according to Lemma 3.1.1 **B)**, $\widetilde{K} = \widetilde{K}_s(I + G_S\widetilde{K}_sG_s)^{-1}$ is a stabilizing controller of G and $h_G(\widetilde{K}) = h_{(G+G_s)}(\widetilde{K}_s)$, which in turn implies that $\|f(P_{zw}, P_{zu}, G,$ $P_{yw}, \widetilde{K})\Big\| = \Big\|f(P_{zw}, P_{zu}, (G + G_s), P_{yw}, \widetilde{K}_s)\Big\|$. But this would imply that $\|f(P_{zw}, P_{zu}, G, P_{yw},$ $\widetilde{K})\Big\| < \|f(P_{zw}, P_{zu}, G, P_{yw}, K^*)\|$ which is a contradiction with the assumed optimality of K^*. □

Remark 3.1.3. *As a consequence of Corollary 4.3.4, we remark that the optimal gain in the disturbance attenuation problem (3.1) is not affected by linear, stable, additive perturbations G_s of the nominal plant G, irrespective of the operatorial norm involved.*

3.1.1 Numerical Example.

We work out an illustrative numerical example, and show that it is sufficient to solve the disturbance attenuation problem only for the antistable part of the given plant. Consider the case of 1-D LTI, continuous–time systems along with the $\mathcal{H}_2$ norm. Consider the generalized plant $P(s)$ given below, where $n_w = 1$, $n_u = 1$,

$n_y = 1$ and $n_z = 1$:

$$P(s) = \begin{bmatrix} \frac{1}{s+10} & \frac{0.1(s+100)}{100s+1} \\ 1 & \frac{s-8}{(s-1)(s+2)(s+3)(s+4)(s+5)(s+6)} \end{bmatrix} \quad (3.11)$$

Compute the additive factorization of the P_{yu} block of P, into the sum $(G + G_s)$ of an antistable plus a stable factor (in fact a partial fraction expansion):

$$\frac{s-8}{(s-1)(s+2)(s+3)(s+4)(s+5)(s+6)} =$$
$$-\frac{0.002778}{s-1} + \frac{0.002778s^4 + 0.05833s^3 + 0.4889s^2 + 2.1s + 6}{(s+2)(s+3)(s+4)(s+5)(s+6)}$$

with $G(s) = -0.002778/(s-1)$.

We want to solve the optimal disturbance attenuation Problem 2 for the generalized plant $P(s)$ in (3.11). To this end, we compute the solution $K^*(s)$ to Problem 1, using the Matlab library function `h2syn` (see reference [83]). We obtain the following expression for $K^*(s)$:

$$K^*(s) = \frac{-144669.4215s^2 - 1440777.27s - 14393.3058}{s^3 + 113s^2 + 932.1405s - 960.0186}$$

We retrieve K_s^*, the solution to Problem 2, via (3.10). The numerator of $K_s^*(s)$ is given by: $(-144669.42149s^7 - 4334165.70248s^6 - 51253699.09091s^5 - 307516607.85125s^4 - 988916656.6116s^3 - 1616681573.55368s^2 - 1052386247.60308s - 10363180.16506)$ while the denominator of $K_s^*(s)$ is given by $(s^8 + 133s^7 + 3749s^6 + 48219s^5 + 346678s^4 + 1519556s^3 + 4398389.42148762s^2 + 8343771.32231411s - 604853.55371882)$. One would obtain the exact same expression of the optimal $\mathcal{H}_2$ controller $K_s^*(s)$, if one would run the `h2syn` Matlab routine ([83]) on the "complete", generalized plant $P(s)$ from (3.11).

3.1.2 Numerical Considerations

Remark 3.1.4. *For 1-D LTI systems, one can always obtain an additive factorization $G + G_s$ of the P_{yu} block of the generalized plant, such that the factor G is antistable while the factor G_s is stable (contains all the stable poles and only those). This factorization is readily implemented in the Matlab library function* `stabsep` *([83]). The factorization can be computed in terms of state–space realizations (see [60,] for complete details), as it only comes down to performing an orthogonal similarity transformation that brings the state matrix to an ordered Schur form ([98, 99]) and then solving a Sylvester matrix equation ([100]).*

After the additive factorization is performed, computational effort is spent to solve Problem 1 (from (3.5)) for the antistable part G, which has a smaller McMillan degree, in order to obtain the optimal K^*. Once that K^* is available, a (nonminimal) state–space realization of the optimal K_s^* is readily available in terms of the realizations K^* and G_s respectively, via the feedback transformation (3.10) (see [**?**, pp. 39] for the state–space formulas). This approach seems promising for the case in which the plant has a relatively much larger number of stable poles than unstable poles. Indeed, for the numerical example above we do obtain a slightly superior average running time than the time for computing the optimal controller for the entire plant P from (3.11).

Unfortunately, the operations involved are very badly conditioned from the numerical point of view. Firstly, the additive decomposition is badly conditioned, especially for the case when G_s has a large number of poles. Secondly, the feedback transformation necessary to retrieve K_s^*, the solution to Problem 2, via (3.10) is very badly conditioned due to the large number of poles/zeros cancelation that occur when computing (3.10).

3.1.3 The Stable Plant Case

We look at the particular case when the plant is stable to begin with (i.e. $G = 0$ and G_s is the given, stable plant). Then *any* stable Q (in the set $\mathcal{A}^{n_u \times n_y}$) will be a stabilizing controller for the feedback configuration in Figure 1 (with $G = 0$). It follows via Lemma 3.1.1 **A)** that for any stable Q in the set $\mathcal{A}^{n_u \times n_y}$, $K_s \overset{def}{=} Q(I - G_s Q)^{-1}$ is a stabilizing controller of G_s. This way we retrieve the classical result due to Zames and Desoer et al ([61, 90]) of parametrizing all stabilizing controllers of the stable plant G_s.

As expected, when $G = 0$, Problem 2 becomes an *open loop* problem, being equivalent with a *model matching* problem. Specifically, solve for the optimal Q^* the model matching problem

$$\min_{Q \text{ stable}} \quad \| P_{zw} + P_{zu} \, Q \, P_{yw} \| \tag{3.12}$$

and retrieve the solution to Problem 2 as $K_s^* \overset{def}{=} Q^*(I - G_s Q^*)^{-1}$.

3.2 Quadratically Invariant, Sparsity Constrained Controllers

From this point on, by simply instancing $\mathcal{A}$ as the $\mathcal{RH}_\infty$ set, we restrict our discussion to 1-D, LTI systems. We prove here the validity of our main result to an important class of decentralized configurations, namely decentralized configurations that are *quadratically invariant* or *invariant under feedback* ([78, 79]). This class of decentralized configurations is particularly important since it is the most general one for which there is available a computational method for solving the *decentralized* disturbance attenuation problem (the decentralized version of Problem 1).

The decentralized setting is formalized via sparsity constraints ([79, pp. 283]). We denote with $\mathcal{S}$ the set of admissible, decentralized controllers, that satisfy a pre–specified sparsity constraint. The set $\mathcal{S}$ can also be seen as a given linear subspace of $\mathcal{P}^{n_y \times n_u}$.

Given the proper, generalized plant P and the set $\mathcal{S}$, the decentralized disturbance attenuation problem (as introduced in [79, pp. 276]) is formulated by simply adding to Problem 1 from (3.1) the extra constraint $K \in \mathcal{S}$ on the stabilizing controllers. We will refer to it as **Problem 1'** :

$$\min_{\substack{K \in \mathcal{C}_G \\ K \in \mathcal{S}}} \left\| f(P_{zw}, P_{zu}, G, P_{yw}, K) \right\|. \tag{3.13}$$

Definition 3.2.1. *[79, Definition 13] Given the plant $G \in \mathcal{P}_s^{n_y \times n_u}$ and the set $\mathcal{S}$,* *we call $\mathcal{S}$* inert *with respect to G if it satisfies the definition in [79, Definition 13].*

Remark 3.2.2. *[79] Throughout this section, both for continuous–time and discrete–time, finite–dimensional, 1-D LTI systems, the constraint set $\mathcal{S}$ is always inert, since G is assumed strictly proper (Remark 2.3.1) and $\mathcal{S}$ is a subset of the set of finite–dimensional, proper 1-D, LTI systems. Note also, that for the case of sparsity constraints, $\mathcal{S}$ is a linear space.*

Definition 3.2.3. *[79, Definition 2] Given the plant $G \in \mathcal{P}_s^{n_y \times n_u}$ and the set $\mathcal{S} \subset \mathcal{P}^{n_u \times n_y}$, the set $\mathcal{S}$ is called* quadratically invariant *under the plant G if*

$$KGK \in \mathcal{S} \qquad \text{for all } \; K \in \mathcal{S}. \tag{3.14}$$

Remark 3.2.4. *For sparsity constraints, condition (4.9) can be elegantly formalized ([79, Theorem 26]) and it completely characterizes the class of invariant under feedback, decentralized configurations treated in this section. The standard hypothesis for the main result in [79, Theorem 14] (which we also assume here) is for the pre–specified, (inert) linear space $\mathcal{S}$ to be quadratically invariant under the P_{yu} block of the generalized plant. Note that quadratic invariance under $\mathcal{S}$ does not depend on the dynamics of P_{yu}, in the sense made precise by the following proposition.*

Proposition 3.2.5. *Given the set $\mathcal{S}$ and the additive factorization $P_{yu} = G + G_s$, with G antistable and G_s stable, P_{yu} is quadratically invariant under $\mathcal{S}$ if and only if both G and G_s respectively, are quadratically invariant under $\mathcal{S}$.*

Proof. The "If" part, follows immediately via the linearity of the K operator and the fact that $\mathcal{S}$ is a linear space. The "Only If" part follows directly from the fact that unless both G and G_s have the same sparsity pattern as P_{yu} they have an even sparser pattern. $\square$

As for the main result, we are interested in how the decentralized, optimal solution to **Problem 1'** from (3.13) relates to the decentralized, optimal solution in the presence of additive, stable plant perturbation, which we will refer to as **Problem 2'**:

$$\min_{\substack{K_s \in \mathcal{C}_{G+G_s} \\ K_s \in \mathcal{S}}} \left\| f(P_{zw}, P_{zu}, (G+G_s), P_{yw}, K_s) \right\|. \tag{3.15}$$

Lemma 3.2.6. A) *Consider $\mathcal{S}$ a given linear subspace of $\mathcal{P}^{n_u \times n_y}$, the plant G in the set $\mathcal{P}_s^{n_y \times n_u}$, the decentralized, stabilizing controller $K \in \mathcal{S}$ of G and the* stable *perturbation G_s belonging to $\mathcal{P}_s^{n_y \times n_u}$ such that $\mathcal{S}$ is quadratically invariant under $P_{yu} = G + G_s$. Then K_s given by $K_s = K(I - G_s K)^{-1}$, belongs to the set $\mathcal{S}$ and is an admissible decentralized, stabilizing controller of $(G + G_s)$.*

B) *Conversely, consider $\mathcal{S}$ a given linear subspace of $\mathcal{P}^{n_u \times n_y}$, the plant G in the set $\mathcal{P}_s^{n_y \times n_u}$, the* stable *perturbation G_s in the set $\mathcal{P}_s^{n_y \times n_u}$ and the decentralized, stabilizing controller $K_s \in \mathcal{S}$ of $G + G_s$. Assume that $\mathcal{S}$ is quadratically invariant under $P_{yu} = G + G_s$. Then the controller K given by $K = K_s(I + G_s K_s)^{-1}$, belongs to the set $\mathcal{S}$ and is an admissible decentralized, stabilizing controller of G.*

Proof. **A)** We get via Lemma 3.1.1 A) that K_s is a stabilizing controller for $(G+G_s)$.

It only remains to prove that it is an admissible, decentralized controller (i.e. K_s belongs to the set $\mathcal{S}$). For this we employ the main result from [79]. Note that via Remark 4.2.2 the set $\mathcal{S}$ is inert with respect to G_s. Also, via Proposition 3.2.5, it follows that $\mathcal{S}$ is quadratically invariant under G_s. Then, [79, Theorem 14] implies that $h^{-1}_{G_s}(K) \stackrel{def}{=} K(I - G_s K)^{-1}$ (defined in Remark 3.0.8) is a bijection from $\mathcal{S}$ to $\mathcal{S}$ and so $K \in \mathcal{S}$ implies $h^{-1}_{G_s}(K)$ belongs to $\mathcal{S}$, *i.e.* $K_s \in \mathcal{S}$.

B) Lemma 3.1.1 B) shows that K is a stabilizing controller for G, so it only remains to be shown that $K \in \mathcal{S}$. Noting that $\mathcal{S}$ is inert with respect to G_s and quadratic invariant under G_s (with the same arguments from point A) of this proof), we employ [79, Theorem 14] to get that $h_{G_s}(K_s) \stackrel{def}{=} K_s(I + G_s K_s)^{-1}$ (defined in (3.6)) is a bijection from $\mathcal{S}$ to $\mathcal{S}$. Finally, since $K_s \in \mathcal{S}$, we get that $h_{G_s}(K_s) \in \mathcal{S}$, *i.e.* $K \in \mathcal{S}$ and the proof ends. □

Corollary 3.2.7. *Consider the given proper, stabilizable, generalized plant P, the strictly proper,* stable *plant perturbation G_s and $\mathcal{S}$ a given linear subspace of $\mathcal{P}^{n_u \times n_y}$ such that $\mathcal{S}$ is quadratically invariant under $P_{yu} = G + G_s$. If $K^* \in \mathcal{S}$ is the optimal solution to the decentralized Problem 1' then $K^*_s = K^*(I - G_s K^*)^{-1}$ belonging to the set $\mathcal{S}$, is the optimal solution to the decentralized Problem 2'.*

Proof. The proof follows on the exact lines of the proof of Corollary 4.3.4, taking into account the conclusions of Lemma 3.2.6 and is omitted for brevity. □

Appendix

Proof of Theorem 3.1.1 Throughout this proof, we employ solely the commutative ring algebra on the set $\mathcal{A}$ of all stable, linear systems. That is, the fact that parallel and cascade connections of any two elements of $\mathcal{A}$ are again elements of $\mathcal{A}$.

We will make extensive use of the following identities, which hold true in any commutative ring, provided that the inverses involved exist.

$$(A+B)^{-1} = A^{-1} - A^{-1}(I+BA^{-1})^{-1}BA^{-1} \tag{3.16}$$

$$(I+AB)^{-1}A = A(I+BA)^{-1} \tag{3.17}$$

$$(I+AB)^{-1} = I - A(I+BA)^{-1}B \tag{3.18}$$

We start by pointing out that K_s is given by (3.8) if and only if K is given by (3.9). We prove next that if K_s is given by (3.8) (and equivalently K is given by (3.9)) then the identity $h_{(G+G_s)}(K_s) = h_G(K)$ holds.

$$\begin{aligned}
&h_{(G+G_s)}\big(K_s\big) \overset{(3.6)}{=} K_s\Big((I+G_sK_s)+GK_s\Big)^{-1} \overset{(3.16)}{=} \\
&h_{G_s}(K_s) - h_{G_s}(K_s)\Big(I+G\,h_{G_s}(K_s)\Big)^{-1}G\,h_{G_s}(K_s) = \\
&h_{G_s}(K_s)\Big(I-\big(I+G\,h_{G_s}(K_s)\big)^{-1}G\,h_{G_s}(K_s)\Big) \overset{(3.18)}{=} \\
&h_{G_s}(K_s)\Big(I-G\big(I+h_{G_s}(K_s)\,G\big)^{-1}h_{G_s}(K_s)\Big) \overset{(3.18)}{=} \\
&h_{G_s}(K_s)\Big(I+G\,h_{G_s}(K_s)\Big)^{-1} \overset{(3.6)}{=} h_G\Big(\,h_{G_s}(K_s)\Big) \overset{(3.9)}{=} \\
&h_G\big(K\big).
\end{aligned} \tag{3.19}$$

A) We prove here that if K is a stabilizable controller for G (and consequently $H(G,K)$ from (3.3) belongs to $\mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$), then K_s given in (3.8) is a stabilizing controller for $(G+G_s)$. Note that due to Remark 2.3.1, if K stabilizes the strictly proper G, then K is proper.

We consider $H(G,K)$ in (3.3) conformably partitioned into four blocks and we introduce the following index notation such that $H(G,K)_{(1,1)}$, $H(G,K)_{(1,2)}$, $H(G,K)_{(2,1)}$, and $H(G,K)_{(2,2)}$ denotes the transfer function matrices from ν_2 to y, from ν_1 to y, from ν_2 to u and from ν_1 to u respectively.

As pointed out before, if K_s is given by (3.8) then K has the expression in (3.9).

Note that $H(G+G_s,K_s)_{(2,1)} \overset{(4.4.6)}{=} H(G,K)_{(2,1)}$ which is stable, from the assumption on K to be stabilizable for G, hence

$$H(G+G_s,K_s)_{(2,1)} \in \mathcal{A}^{n_u\times n_y}. \tag{3.20}$$

In prealable we denote $\Delta \overset{def}{=} \big(I+K_sG_s\big)^{-1}$ in order to get the following identity

$$
\begin{aligned}
&H(G+G_s,K_s)_{(2,2)} \overset{(3.3)}{=} \Big(I+K_s\big(G+G_s\big)\Big)^{-1} = \\
&\Big(\big(I+K_sG_s\big)+K_sG\Big)^{-1} \overset{(3.16)}{=} \\
&\Delta - \Delta\big(I+K_sG\,\Delta\big)^{-1}K_sG\,\Delta = \\
&\Big(I-\Delta\big(I+K_sG\,\Delta\big)^{-1}K_sG\Big)\Delta \overset{(3.18)}{=} \big(I+\Delta K_sG\big)^{-1}\Delta \overset{(3.17)}{=} \\
&\Big(I+K_s\big(I+G_sK_s\big)^{-1}G\Big)^{-1}\big(I+K_sG_s\big)^{-1} \overset{(3.9)}{=} \\
&\big(I+KG\big)^{-1}\big(I+K_sG_s\big)^{-1} \overset{(3.18)}{=} \\
&\big(I+KG\big)^{-1}\Big(I-K_s\big(I+G_sK_s\big)^{-1}G_s\Big)^{-1} \overset{(3.9)}{=} \\
&\big(I+KG\big)^{-1}\big(I-KG_s\big)^{-1} \overset{(3.3,3.17)}{=} \\
&H(G,K)_{(2,2)} - H(G,K)_{(2,1)}G_s.
\end{aligned}
\tag{3.21}
$$

Note that both terms on the last line of (4.4.7) are stable from the assumption on K to be a stabilizable controller for G and the hypothesis on G_s to be stable. It follows that via (4.4.7) above, that

$$
H(G+G_s,K_s)_{(2,2)} \in \mathcal{A}^{n_u\times n_u}. \tag{3.22}
$$

The following identity holds

$$
\begin{aligned}
& -G\,H(G+G_s,K_s)_{(2,2)} \overset{(4.4.7)}{=} \\
& -G\,H(G,K)_{(2,2)} + GH(G,K)_{(2,1)}G_s \overset{(3.17)}{=} \\
& H(G,K)_{(1,2)} + \big(G(I+KG)^{-1}K\big)G_s \overset{(3.18)}{=} \\
& H(G,K)_{(1,2)} + \big(I-(I+GK)^{-1}\big)G_s \overset{(3.3)}{=} \\
& H(G,K)_{(1,2)} + \big(I-H(G,K)_{(1,1)}\big)G_s = \\
& H(G,K)_{(1,2)} + G_s - H(G,K)_{(1,1)}G_s.
\end{aligned}
\tag{3.23}
$$

It follows that

$$
\begin{aligned}
& H(G+G_s,K_s)_{(1,2)} = -(G+G_s)H(G+G_s,K_s)_{(2,2)} \\
& \overset{(4.4.9)}{=} H(G,K)_{(1,2)} + G_s - H(G,K)_{(1,1)}G_s - G_sH(G+G_s,K_s)_{(2,2)}
\end{aligned}
\tag{3.24}
$$

From the assumption on K to be a stabilizable controller for G, the hypothesis on G_s to be stable and (3.22) it follows that all terms on the last line of (3.24) are stable and consequently

$$H(G+G_s,K_s)_{(1,2)} \in \mathcal{A}^{n_y \times n_u}. \tag{3.25}$$

The following identity holds

$$
\begin{aligned}
&H(G+G_s,K_s)_{(1,1)} \overset{(3.3)}{=} \\
&\Big(I+(G+G_s)K_s\Big)^{-1} \overset{(3.18)}{=} \\
&I-(G+G_s)\big(I+K_s(G+G_s)\big)^{-1}K_s = \\
&I-GH(G+G_s,K_s)_{(2,1)}-G_sH(G+G_s,K_s)_{(2,1)} \overset{(4.4.6)}{=} \\
&\Big(I-GH(G,K)_{(2,1)}\Big)-G_sH(G,K)_{(2,1)} \overset{(3.18)}{=} \\
&H(G,K)_{(1,1)}-G_sH(G,K)_{(2,1)}
\end{aligned}
\tag{3.26}
$$

From the assumption on K to be a stabilizable controller for G and the hypothesis on G_s to be stable, it follows that all terms on the last line of (3.26) are stable and consequently

$$H(G+G_s,K_s)_{(1,1)} \in \mathcal{A}^{n_y\times n_y}. \tag{3.27}$$

Finally, from (3.20, 3.22, 3.25, 3.27,) we conclude that $H(G+G_s,K_s)$ belongs to the set $\mathcal{A}$ and so K_s is a stabilizing controller of $(G+G_s)$, which ends the proof of part **A)** of the Theorem. Finally note that due to Remark 2.3.1, if K_s stabilizes the strictly proper $(G+G_s)$, then K_s is proper.

B) We prove now that if K_s is a stabilizable controller for $(G+G_s)$ (and consequently $H(G+G_s,K_s)$ belongs to $\mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$) then K given in (3.9) is a stabilizing controller for G. Note that due to Remark 2.3.1, if K_s stabilizes the strictly proper $G+G_s$, then K_s is proper.

As pointed out before if K is given by (3.9) then K_s has the expression in

(3.8).

Note that $H(G,K)_{(2,1)} \overset{(4.4.6)}{=} H(G+G_s,K_s)_{(2,1)}$ which is stable, from the assumption on K_s to be stabilizable for $(G+G_s)$, hence

$$H(G,K)_{(2,1)} \in \mathcal{A}^{n_u \times n_y}. \tag{3.28}$$

The following identity holds

$$\begin{aligned}
&H(G,K)_{(2,2)} \overset{(4.4.7)}{=} \\
&H(G+G_s,K_s)_{(2,2)} + H(G,K)_{(2,1)}G_s \overset{(4.4.6)}{=} \\
&H(G+G_s,K_s)_{(2,2)} + H(G+G_s,K_s)_{(2,1)}G_s
\end{aligned} \tag{3.29}$$

Note that both terms on the last line of (3.29) are stable from the assumption on K_s to be a stabilizable controller for $(G+G_s)$ and the hypothesis on G_s to be stable. It follows via (3.29) above, that

$$H(G,K)_{(2,2)} \in \mathcal{A}^{n_u \times n_u}. \tag{3.30}$$

It follows from identity (3.26) that

$$H(G,K)_{(1,1)} = G_s H(G,K)_{(2,1)} + H(G+G_s,K_s)_{(1,1)} \tag{3.31}$$

All the terms on the right hand side of (3.31) are stable due to (3.28), the assumption on K_s to be a stabilizable controller for $(G+G_s)$ and the hypothesis on G_s to be stable, hence

$$H(G,K)_{(1,1)} \in \mathcal{A}^{n_y \times n_y}. \tag{3.32}$$

It follows from identity (3.24) that

$$H(G,K)_{(1,2)} = H(G+G_s,K_s)_{(1,2)} - G_s + H(G,K)_{(1,1)}G_s + G_sH(G+G_s,K_s)_{(2,2)}. \tag{3.33}$$

All the terms on the right hand side of (3.33) are stable due to the assumption on K_s to be a stabilizable controller for $(G+G_s)$, the hypothesis on G_s to be stable and (3.32), hence

$$H(G,K)_{(1,2)} \in \mathcal{A}^{n_y \times n_u}. \tag{3.34}$$

From (3.28, 4.1.2, 3.32, 3.34) it follows that K is a stabilizing controller for G and the proof ends.

Chapter 4

Necessary and Sufficient Conditions for Stabilizability subject to Quadratic Invariance

Contribution. Throughout this chapter, we deal exclusively with LTI systems and quadratically invariant, feedback configurations. Both available algorithms for the sparse, optimal controller synthesis ([79] and the ones presented in Chapter 2 of this thesis), rely crucially on the fact that some stabilizing controller that verifies the imposed sparsity constraints is *a priori* known, while synthesis methods for such a controller, (needed to initialize the aforementioned optimization schemes) are not yet available. This provided the motivation to the work presented here as in this chapter we provide necessary and sufficient conditions for such a plant to be *stabilizable* with a controller having the given sparsity pattern. These conditions are formulated in terms of the existence of a doubly coprime factorization of the plant with additional sparsity constraints on certain factors. We show that the computation of such a factorization is equivalent to solving an exact model–matching problem. We also give the parametrization of the set of all decentralized stabilizing controllers by imposing additional constraints on the Youla parameter. These constraints are for the Youla parameter to lie in the set of all stable transfer function matrices belonging to a certain linear subspace.

Outline of the Chapter. This chapter is organized as follows: after the introductive section we follow with a preliminaries section, introducing the feedback control stabilization problem and a short primer on coprime factorizations of LTI systems. The third section contains mostly notation and introduces the notion of sparsity constraints for linear systems along with a summary of the main results on quadratic invariance from [79]. The fourth section contains the main results of this paper. We provide a necessary and sufficient condition for a plant to be *stabilizable* with a controller satisfying a pre–selected sparsity pattern that is quadratically invariant with respect to the plant. These conditions are formulated in terms of the existence of a doubly coprime factorization of the plant with additional sparsity constraints on certain factors. We prove that the computation of this particular doubly coprime factorization (when it does exist) is equivalent to solving an exact model–mathing problem. Along the way we obtain the set of all decentralized stabilizing controllers, characterized via the Youla parametrization. The sparsity constraints on the controller are recast as a linear subspace type of constraint on the Youla parameter. Applications to optimal controller synthesis are presented as conclusions, following the main results presented here and the optimal synthesis tools introduced in [79]. The fifth section, revisits the results of the previous section, under the hypothesis that the given plant admits a special type of doubly coprime factorization which we have dubbed *Input/Output Decoupled*. It turns out that this hypothesis is a *generic* property, meaning that it is valid for *almost all* plants. We show how it spectacularly simplifies all the result from the fourth section while providing additional insight into the sparse stabilization problem and the Youla parametrization of all sparse, stabilizing controllers.

4.1 Preliminaries

Throughout this paper we make the leading assumptions that all systems are linear and time invariant (LTI), finite dimensional, proper, with either continuous or discrete–time. We deal with the frequency domain input/output operators of LTI systems. These operators are transfer function matrices (TFM), meaning matrices with all entries real–rational functions. By $\mathbb{R}(\lambda)$ we denote the set of all real–rational functions and by $\mathbb{R}(\lambda)^{n_y \times n_u}$ the set of $n_y \times n_u$ matrices having all entries in $\mathbb{R}(\lambda)$. The undeterminate λ is either s for continuous–time systems or z for discrete–time systems, respectively. Almost everywhere in the sequel, the λ argument following a TFM is omitted if it is clear from the context.

This paper gives a unified treatment for both the continuous and discrete-time cases. Henceforth, we will denote by Ω the open left half complex plane or the open unit disk, according to the type of system: continuous or discrete–time, respectively. The standard interpretation of Ω in systems theory is related to the *stability domain* of linear systems. We qualify a TFM $G(\lambda)$ as *stable* if all its poles are in Ω.

4.1.1 The Control Problem

In Fig.1 we depict the standard feedback interconnection between a plant and a controller, with the plant G belonging to $\mathbb{R}(\lambda)^{n_y \times n_u}$ and the controller K in the set $\mathbb{R}(\lambda)^{n_u \times n_y}$. Here, ν_1 and ν_2 are the disturbances and sensor noise, respectively. In addition, u is the control and y are the measurements. The integers n_u and n_y denote the dimensions of u and y respectively. Denote by $H(G, K) \in \mathbb{R}(\lambda)^{(n_u+n_y)\times(n_u+n_y)}$

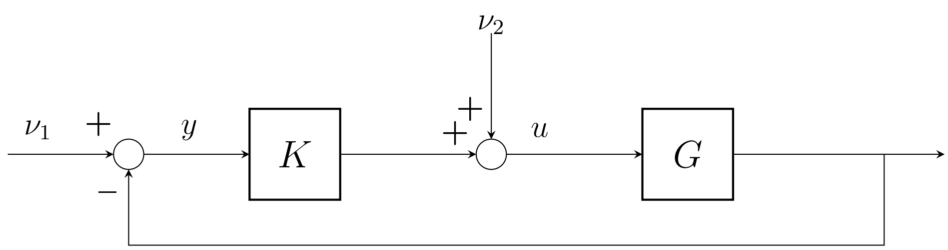

Figure 4.1: Standard unity feedback configuration

the TFM from $[\nu_1^T \;\; \nu_2^T\,]^T$ to $[y^T \;\; u^T\,]^T$ (provided that the feedback loop is *well–posed*, *i.e.* $(I + KG)$ is invertible as a TFM). For the complete expressions of $H(G, K)$ in terms of G and K, we refer the reader to [82, Ch. 5.1, (7)]. If the transfer matrix $H(G, K)$ is *stable* we say that K is a *stabilizing controller* of G or equivalently that K *stabilizes* G. If a stabilizing controller of G exists, we say that G is *stabilizable*.

4.1.2 Coprime and Doubly Coprime Factorization for LTI Systems

Let $G(\lambda)$ be an arbitrary $n_y \times n_u$ TFM and Ω the stability domain in the complex plane. A *right coprime factorization* (RCF) of G over Ω is a fractional representation of the form $G = NM^{-1}$, with N and M having poles only in Ω, and for which $YM + XN = I$ holds for certain TFMs X and Y with poles in Ω ([82, Ch. 4, Corollary 17]). Analogously, a *left coprime factorization* (LCF) of G (over Ω) is defined by $G = \widetilde{M}^{-1}\widetilde{N}$, where $\widetilde{N}$ and $\widetilde{M}$ are TFMs having poles only in Ω and satisfying $\widetilde{M}\widetilde{Y} + \widetilde{N}\widetilde{X} = I$ for certain TFMs $\widetilde{X}$ and $\widetilde{Y}$ with all poles in Ω. Due to the natural interpretation of the coprime factorizations as fractional representations, the invertible $\widetilde{M}$ and M factors are sometimes called the "denominator" TFMs of the coprime factorization.

Definition 4.1.1. *[82, Ch.4, Remark pp. 79] A collection of eight TFMs* $\big(M(\lambda), N(\lambda), \widetilde{M}(\lambda), \widetilde{N}(\lambda),\ X(\lambda), Y(\lambda),\ \widetilde{X}(\lambda), \widetilde{Y}(\lambda)\big)$ *having all poles in* Ω *is called a* doubly coprime factorization *(DCF) of* $G(\lambda)$ *over* Ω *if the "denominator" TFMs* $\widetilde{M}(\lambda)$ *and* $M(\lambda)$ *are invertible and satisfy* $G(\lambda) = \widetilde{M}(\lambda)^{-1}\widetilde{N}(\lambda) = N(\lambda)M(\lambda)^{-1}$ *and*

$$\begin{bmatrix} Y(\lambda) & X(\lambda) \\ -\widetilde{N}(\lambda) & \widetilde{M}(\lambda) \end{bmatrix} \begin{bmatrix} M(\lambda) & -\widetilde{X}(\lambda) \\ N(\lambda) & \widetilde{Y}(\lambda) \end{bmatrix} = I_{n_y+n_u}. \tag{4.1}$$

To avoid excessive terminology throughout this paper, we will simply refer to doubly coprime factorizations over Ω simply as doubly coprime factorizations (DCFs).

4.1.3 The Youla Parametrization of All Stabilizing Controllers

The following theorem is a central result in linear systems theory. We state it next, as it stands at the core of our main result.

Theorem 4.1.2. (Youla) *[82, Ch.5, Theorem 1] Given a plant with the TFM* $G \in \mathbb{R}(\lambda)^{n_y \times n_u}$*, and* any *of its DCF (4.1), the set of* all *controllers* K *stabilizing* G *(in the standard feedback configuration from Figure 4.1) is given by*

$$\begin{aligned} K &= \big(\widetilde{X} + MQ\big)\big(\widetilde{Y} - NQ\big)^{-1} \\ &= \big(Y - Q\widetilde{N}\big)^{-1}\big(X + Q\widetilde{M}\big) \end{aligned} \tag{4.2}$$

with Q *any stable TFM in the set* $\mathbb{R}^{n_u \times n_y}(\lambda)$*.*

Definition 4.1.3. *Given the plant* G *and a certain DCF (4.1) of* G*, when taking the Youla–parameter* Q *equal to zero in (4.2) we get* $K = \widetilde{X}\,\widetilde{Y}^{-1} = Y^{-1}X$*, which is called the* central controller *(associated with the corresponding DCF (4.1)).*

4.2 Feedback Control Configurations with Sparsity Constraints

Throughout this paper, the information constraints that are to be imposed on the controller are modeled via *sparsity constraints* ([79, pp. 283]). The precise formulation of the sparsity constrained stabilization problem is achieved by imposing a certain pre–selected sparsity pattern on the set of admissible stabilizing controllers. The notation we introduce next is entirely concordant with the one used in [78, 79].

4.2.1 Conformal Block Partitioning

For $p \geq 1$, we denote the set of integers from 1 to p as $\overline{1,p}$. Throughout the sequel we consider that the transfer function matrix $G(\lambda) \in \mathbb{R}(\lambda)^{n_y \times n_u}$ is partitioned in p block–rows and m block–columns. The i-th block–row has n_y^i rows, while the j-th block–column has n_u^j columns. Obviously, $\sum_{i=1}^{p} n_y^i = n_y$ and $\sum_{j=1}^{m} n_u^j = n_u$. For every pair (i,j) in the set $\overline{1,p} \times \overline{1,m}$, we denote by $[G]_{ij} \in \mathbb{R}^{n_y^i \times n_u^j}(\lambda)$ the $n_y^i \times n_u^j$ TFM at the intersection of the i-th block–row and j-th block–column of $G(\lambda)$. Accordingly,

$$G(\lambda) = \begin{bmatrix} [G]_{11} & \dots & [G]_{1m} \\ \vdots & & \vdots \\ [G]_{p1} & \dots & [G]_{pm} \end{bmatrix}, \quad \text{with} \quad [G]_{ij} \in \mathbb{R}^{n_y^i \times n_u^j}(\lambda). \tag{4.3}$$

Henceforth, we shall use this square bracketed notation for block indexing of transfer function matrices.

Analogously, the controller's transfer function matrix $K(\lambda) \in \mathbb{R}^{n_u \times n_y}(\lambda)$ is partitioned in m block–rows and p block–columns, where the j-th block–row has

n_u^j rows and the i-th block–column has n_y^i columns. Correspondingly, $[K]_{ji}$ is the notation for the $n_u^j \times n_y^i$ TFM at the intersection of the j-th block–row and i-th block–column of $K(\lambda)$.

4.2.2 Sparsity Constraints

For the boolean algebra, the operations $(+, \cdot)$ are defined as usual: $0 + 0 = 0 \cdot 1 = 1 \cdot 0 = 0 \cdot 0 = 0$ and $1 + 0 = 0 + 1 = 1 + 1 = 1 \cdot 1 = 1$. By a binary matrix we mean a matrix whose entries belong to the set $\{0, 1\}$. With the usual extension of notation, $\{0, 1\}^{m \times p}$ stands for the set of all binary matrices with m rows and p columns. The addition and multiplication of binary matrices is carried out in the usual way, keeping in mind that the binary operations $(+, \cdot)$ follow the boolean algebra.

Binary matrices are denoted by capital letters with the "bin" superscript, in order to be distinguished from transfer function matrices over $\mathbb{R}(\lambda)$, which are represented in the sequel by plain capital letters. Henceforth, we adopt the convention that the transfer function matrices are indexed by blocks while binary matrices are indexed by each individual entry.

Furthermore, for binary matrices only, having the same dimensions, the notation $A^{\text{bin}} \leq B^{\text{bin}}$ means that $a_{ij} \leq b_{ij}$ for all i and j.

With the conformable block partitioning for K introduced in Subsection 4.2.1, for any $K \in \mathbb{R}(\lambda)^{n_u \times n_y}$, define $\text{Pattern}(K) \in \{0,1\}^{m \times p}$ to be the binary matrix

$$\text{Pattern}(K)_{ij} \stackrel{def}{=} \begin{cases} 0 & \text{if the block } [K]_{ij} = 0; \\ 1 & \text{otherwise .} \end{cases} \tag{4.4}$$

Conversely, for any binary matrix with m rows and p columns, $K^{\text{bin}} \in \{0,1\}^{m \times p}$, we can define the following linear subspace of $\mathbb{R}(\lambda)^{n_u \times n_y}$:

$$\text{Sparse}(K^{\text{bin}}) \stackrel{def}{=} \left\{ K \in \mathbb{R}(\lambda)^{n_u \times n_y} \middle| \text{ Pattern}(K) = K^{\text{bin}} \right\} \tag{4.5}$$

Hence $\text{Sparse}(K^{\text{bin}})$ is the set of all controllers K in the set $\mathbb{R}(\lambda)^{n_u \times n_y}$ for which $[K]_{ij} = 0$ whenever $K^{\text{bin}}_{ij} = 0$. Accordingly, the binary value of $\text{Pattern}(K)_{kl}$ determines whether controller k may read the block-row l of the output of the plant G.

Let $K^{\text{bin}} \in \{0,1\}^{m \times p}$ be the pre-specified sparsity pattern to be imposed on the controller. Define the linear subspace $\mathcal{S}$ of $\mathbb{R}(\lambda)^{n_u \times n_y}$ as:

$$\mathcal{S} \stackrel{def}{=} \left\{ K \in \mathbb{R}(\lambda)^{n_u \times n_y} \middle| \text{ Pattern}(K) \leq K^{\text{ bin}} \right\}, \tag{4.6}$$

that is, the set of controllers whose transfer function matrices satisfy the imposed sparsity structure. With the terminology from [79], the linear space $\mathcal{S}$ (of admissible, sparsity constrained, causal controllers) will be called the *information constraint.*

The following matrix $G^{\text{ bin}}$ in the set $\{0,1\}^{p \times m}$ is the sparsity pattern of the plant which is defined as:

$$G^{\text{ bin}} \stackrel{def}{=} \text{Pattern}(G) \tag{4.7}$$

Finally, from the matrix multiplication of matrices over $\mathbb{R}(\lambda)$ we note that for any $K \in \mathbb{R}(\lambda)^{n_u \times n_y}$ and any $G \in \mathbb{R}(\lambda)^{n_y \times n_u}$ with arbitrary sparsity patterns

$$\text{Pattern}(K\,G) \leq \text{Pattern}(K)\,\text{Pattern}(G). \tag{4.8}$$

4.2.3 Quadratic Invariance

Assumption 1. From this point on we make the assumption on the plant G to be strictly proper, *i.e.* for any of the entries of the transfer function matrix G (which is a rational function) the degree of the denominator is strictly greater than the degree of the numerator.

Definition 4.2.1. *[79, Definition 13] Given the plant $G \in \mathbb{R}(\lambda)^{n_y \times n_u}$ and the subset $\mathcal{S}$ of $\mathbb{R}(\lambda)^{n_u \times n_y}$, we call $\mathcal{S}$* inert *with respect to G if it satisfies the definition in [79, Definition 13].*

Remark 4.2.2. *[79] Throughout this section, both for continuous–time and discrete–time systems, the constraint set $\mathcal{S}$ is always inert with respect to the plant G, since G is assumed strictly proper and $\mathcal{S}$ is a subset of the set of proper LTI systems. Note also, that $\mathcal{S}$ is a closed set since it is a linear subspace (4.6).*

Definition 4.2.3. *[79, Definition 2] Given the plant $G \in \mathbb{R}(\lambda)^{n_y \times n_u}$ and the set $\mathcal{S} \subset \mathbb{R}(\lambda)^{n_u \times n_y}$, the set $\mathcal{S}$ is called* quadratically invariant *under the plant G if*

$$KGK \in \mathcal{S} \qquad \text{for all} \;\; K \in \mathcal{S}. \tag{4.9}$$

Definition 4.2.4. *Define the feedback transformation of G with K, as the following* *function from* $\mathbb{R}(\lambda)^{n_u \times n_y}$ *to* $\mathbb{R}(\lambda)^{n_u \times n_y}$

$$h_G(K) \stackrel{def}{=} K\big(I + GK\big)^{-1}. \tag{4.10}$$

Proposition 4.2.5. *The feedback transformation* $h_G(\cdot)$ *from (4.10) is an invertible function from* $\mathbb{R}(\lambda)^{n_u \times n_y}$ *to* $\mathbb{R}(\lambda)^{n_u \times n_y}$ *and its inverse is given by*

$$h_G^{-1}(K) \stackrel{def}{=} K\big(I - GK\big)^{-1}. \tag{4.11}$$

Proof. First note that $h_G(\cdot)$ from (4.10) is indeed a well–posed function from $\mathbb{R}(\lambda)^{n_u \times n_y}$ to $\mathbb{R}(\lambda)^{n_u \times n_y}$, due to fact that the inverse of $\big(I + GK\big)$ exists for any K in $\mathbb{R}(\lambda)^{n_u \times n_y}$. This is guaranteed by the fact that K is proper and G is strictly proper (Assumption 1). The rest of the proof follows by direct algebraic computations and is omitted for brevity. □

We restate next, for ease of reference, the main result from [78, 79], frequently invoked throughout the next section.

Theorem 4.2.6. *[79, Theorem 14] Given the plant* $G \in \mathbb{R}(\lambda)^{n_y \times n_u}$, *the set* $\mathcal{S} \subseteq \mathbb{R}(\lambda)^{n_u \times n_y}$ *closed, inert with respect to* G *and quadratically invariant under* G, *then*

$$\mathcal{S} \text{ is quadratically invariant under } G \Longleftrightarrow h_K(\mathcal{S}) = \mathcal{S}. \tag{4.12}$$

Assumption 2. Throughout this entire paper, we assume that the set $\mathcal{S}$ that defines the sparsity constraints to be imposed on the controller is quadratically invariant under the plant G.

4.3 Main Result

In this section we develop a necessary and sufficient condition for a plant to be *stabilizable* with a controller satisfying a pre–selected sparsity pattern that is quadratically invariant with respect to the plant. These conditions are formulated in terms of the existence of a doubly coprime factorization of the plant featuring additional sparsity constraints on certain factors. This result has an especially important computational value, as it turns out that such a factorization (when it exists) is equivalent to solving for the Youla parameter a TFM linear equation (an exact model matching problem) .

The following preparatory result will be needed.

Proposition 4.3.1. *Given any DCF (4.1) of the plant G denote by $K = \widetilde{X}\,\widetilde{Y}^{-1} = Y^{-1}X$ the "central" controller (from Definition 4.1.3). Then the following identities hold*

$$\begin{aligned} MY &= \big(I + KG\big)^{-1}, \\ MX &= \big(I + KG\big)^{-1}K, \\ \widetilde{Y}\widetilde{M} &= \big(I + GK\big)^{-1}, \\ \widetilde{X}\widetilde{M} &= K\big(I + GK\big)^{-1}. \end{aligned} \tag{4.13}$$

Proof. See the Appendix. □

The next Theorem makes out for the main result of this paper.

Theorem 4.3.2. *Given a plant G in the set $\mathbb{R}(\lambda)^{n_y \times n_u}$ then G is stabilizable with a sparsity constrained controller K belonging to the set $\mathcal{S}$ if and only if there exists a DCF (4.1) of G such that*

$$\text{Pattern}(\widetilde{X}\widetilde{M}) \leq K^{\text{bin}} \quad \text{or} \quad \text{Pattern}(MX) \leq K^{\text{bin}}. \tag{4.14}$$

Proof. Throughout the proofs, we shall make use of the following identities (that hold true in any ring, provided the inverses involved exist).

$$(I + AB)^{-1}A = A(I + BA)^{-1}, \tag{4.15}$$

$$(I + AB)^{-1} = I - A(I + BA)^{-1}B. \tag{4.16}$$

"Necessity". Suppose that there exists a stabilizing controller K in the set $\mathcal{S}$. Then as a consequence of Youla's Theorem 4.1.2, there exists a DCF (4.1) of the plant G for which K is the central controller. According to Proposition 4.3.1 we get from (4.13) that

$$\widetilde{X}\widetilde{M} = K\big(I + GK\big)^{-1}. \tag{4.17}$$

We apply the Pattern operator (4.4) on both sides of equation (4.17) and using Definition 4.2.4 get that $\text{Pattern}(\widetilde{X}\widetilde{M}) = \text{Pattern}(h_G(K))$. But $h_G(K)$ belongs to $\mathcal{S}$ because of Assumption 2 and Theorem 4.2.6 and so $\text{Pattern}(h_G(K)) \leq K^{\text{bin}}$.

For $\text{Pattern}(MX)$ we employ (4.13) and identity (4.15) to get that $\text{Pattern}(MX) = \text{Pattern}(h_G(K))$. Then by the same arguments as before we also get that $\text{Pattern}(MX) \leq K^{\text{bin}}$.

"Sufficiency". Suppose that $\text{Pattern}(\widetilde{X}\widetilde{M}) \leq K^{\text{bin}}$ holds, hence $\widetilde{X}\widetilde{M}$ belongs to the set $\mathcal{S}$. Take each side of (4.17) as an argument for $h_G^{-1}(\cdot)$ in order to get via

Definition 4.2.4 that $h_G^{-1}(\widetilde{X}\widetilde{M}) = h_G^{-1}(h_G(K))$ and equivalently that $K = h_G^{-1}(\widetilde{X}\widetilde{M})$. Furthermore, via Proposition 4.2.5, Assumption 2 and Theorem 4.2.6 we get that $h_G^{-1}(\mathcal{S}) = \mathcal{S}$ which in turn implies that $h_G^{-1}(\widetilde{X}\widetilde{M})$ belongs to the set $\mathcal{S}$. This means that $K = h_G^{-1}(\widetilde{X}\widetilde{M})$ is also in $\mathcal{S}$.

The sufficiency of the second condition (Pattern$(MX) \leq K^{\text{bin}}$) follows by a similar line of reasoning and so is omitted for brevity.

□

Kronecker Products and Linear Matrix Equations([88, Chapter 13]) Given two matrices $P \in \mathbb{R}(\lambda)^{a\times b}$ and $S \in \mathbb{R}(\lambda)^{c\times d}$ let the Kronecker product of P and S be denoted by $P \otimes S$ and belonging to the set $\mathbb{R}(\lambda)^{ac\times bd}$. Given the matrix P, we write P in terms of its columns as

$$P = \begin{bmatrix} p_1 & p_2 & \dots & p_a \end{bmatrix}$$

and then associate a column vector $\text{vec}(P) \in \mathbb{R}(\lambda)^{ab}$ defined as

$$\text{vec}(P) \stackrel{def}{=} \begin{bmatrix} p_1 \\ \vdots \\ p_a \end{bmatrix}. \tag{4.18}$$

All the presented results related to matrix vectorization and Kronecker products do not depend in any way on the ring of matrices involved, therefore they are valid for the ring of TFMs (matrices over the field of real–rational functions).

Proposition 4.3.3. *[88, Theorem 13.26] Let* $P \in \mathbb{R}(\lambda)^{a\times b}$*,* $R \in \mathbb{R}(\lambda)^{b\times c}$ *and* $S \in$

$\mathbb{R}(\lambda)^{c\times d}$. *Then*

$$\mathrm{vec}(PRS) = (S^T \otimes P)\mathrm{vec}(R) \tag{4.19}$$

4.3.1 Outline of the Sparse Controller Synthesis Algorithm

In this subsection, given the plant G we provide a numerically tractable algorithm (based on Theorem 4.3.2 above) for the computation of a sparse, stabilizing controller, belonging to the set $\mathcal{S}$ (when such a controller exists). We start with *any* DCF (4.1) of the plant, which can be computed using the standard state–space techniques from [85]. If this DCF satisfies relations (4.14) then according to Theorem 4.3.2 its associated central controller will be in the set $\mathcal{S}$.

Suppose now that this DCF we start with does not satisfy (4.14), which is generically speaking the case. An immediate consequence of Youla's Theorem 4.1.2 states that for any Youla parameter Q (stable TFM, belonging to the set $\mathbb{R}(\lambda)^{n_u\times n_y}$) the following identity represents another DCF of the plant G

$$\begin{bmatrix} (Y - Q\widetilde{N}) & (X + Q\widetilde{M}) \\ -\widetilde{N} & \widetilde{M} \end{bmatrix} \begin{bmatrix} M & -(\widetilde{X} + MQ) \\ N & (\widetilde{Y} - NQ) \end{bmatrix} = I_{n_y+n_u}. \tag{4.20}$$

We want to find that particular Youla parameter Q, for which the factors of the newly obtained DCF (4.20) statisfy the relations (4.14), namely that

$$\mathrm{Pattern}\Big((\widetilde{X} + MQ)\widetilde{M}\Big) \leq K^{\mathrm{bin}} \quad \text{or} \quad \mathrm{Pattern}\Big(M(X + Q\widetilde{M})\Big) \leq K^{\mathrm{bin}}$$

or equivalently

$$\mathrm{Pattern}\big(MQ\widetilde{M} + \widetilde{X}\widetilde{M}\big) \le K^{\mathrm{bin}} \quad \text{or} \quad \mathrm{Pattern}\big(MQ\widetilde{M} + MX\big) \le K^{\mathrm{bin}}. \tag{4.21}$$

Corollary 4.3.4. *Given a plant G in the set $\mathbb{R}(\lambda)^{n_y \times n_u}$ then G is stabilizable with a sparsity constrained controller K belonging to the set $\mathcal{S}$ if and only if, starting from* any *DCF (4.1) of G, there exists a Youla parameter Q (stable TFM, belonging to the set $\mathbb{R}(\lambda)^{n_u \times n_y}$) such that (4.21) holds.*

Proof. "Sufficiency" If there exists a Youla parameter Q, such that (4.21) holds, then exactly as in the "Sufficiency" part of the proof of Theorem 4.3.2, the controller (depending on Q) $K = h_G^{-1}\Big((\widetilde{X} + MQ)\widetilde{M}\Big)$ will belong to the set $\mathcal{S}$.

"Necessity" Suppose that a stabilizing controller K of G, belonging to the set $\mathcal{S}$ does exist and we consider K fixed. Then, a direct consequence of Youla's Theorem 4.1.2 states that for any DCF (4.1), there exist a (unique) Youla parameter Q (depending on the DCF), such that $K = \big(\widetilde{X} + MQ\big)\big(\widetilde{Y} - NQ\big)^{-1}$ (is the central controller associated with the DCF (4.20) of G). Then exactly as in the "Necessity" part of the proof of Theorem 4.3.2, it follows that (4.20) must satisfy (4.21). □

Remark 4.3.5. *We will provide our further argumentation only for the first relation from (4.21), since all the needed results for the second relation from (4.21) follow by a similar line of reasoning.*

The intuition behind the equation (4.21) is the following: we want to find the Youla parameter Q for which certain block–entries in the factor $MQ\widetilde{M}$ are identical with the corresponding block–entries in $-\widetilde{X}\widetilde{M}$, such that they cancel out in the sum $\widetilde{X}\widetilde{M} + MQ\widetilde{M}$. (This is a so called exact model–matching problem.) The block–entries of $-\widetilde{X}\widetilde{M}$ are precisely those identified by the zero entries of the K^{bin} matrix and only those, because that is necessary and sufficient for making $\text{Pattern}\big(MQ\widetilde{M} + \widetilde{X}\widetilde{M}\big) \leq K^{\text{bin}}$. Using the same argument, we observe that the entries equal to one of K^{bin} do not make out for additional constraints, since their corresponding block–entries of $(\widetilde{X}\widetilde{M} + MQ\widetilde{M})$ can be any stable TFM.

We take a look now at the vectorization (4.18) of these relations, meaning the exact model–matching of certain block–entries of $\text{vec}(MQ\widetilde{M})$ with the corresponding block–entries of $\text{vec}(-\widetilde{X}\widetilde{M})$. These block–entries will now be precisely those identified by the zero entries of the $\text{vec}(K^{\text{bin}})$ matrix and only those. We know via Proposition 4.3.3 that $\text{vec}(MQ\widetilde{M}) = (\widetilde{M}^T \otimes M)\text{vec}(Q)$ and so the problem will become an exact model-matching of certain entries of the vector $(\widetilde{M}^T \otimes M)\text{vec}(Q)$ with the corresponding entries in the vector $\text{vec}(-\widetilde{X}\widetilde{M})$. This is a linear system of equations in the unknown $\text{vec}(Q)$. We emphasize that the equivalent problem in <u>not</u> the entire system of equations $(\widetilde{M}^T \otimes M)\text{vec}(Q) = \text{vec}(-\widetilde{X}\widetilde{M})$, but only a subset of its linear equations, consisting only in the block–rows identified by the zero entries of $\text{vec}(K^{\text{bin}})$.

For illustrative simplicity, we outline all these ideas in a numerical example before proceeding to the formal statement of the results.

4.3.2 A Numerical Example

Suppose we are given as input data the plant G and K^{bin}, as

$$G(\lambda) = \begin{bmatrix} \frac{1}{\lambda-1} & \frac{1}{\lambda+2} & 0 \end{bmatrix} \quad \text{and} \quad K^{\mathrm{bin}} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T \tag{4.22}$$

where all blocks in the partition (4.3) of G are 1×1 and both Assumptions 1 and 2 are met. We can start up our synthesis algorithm with *any* DCF (4.1) of the plant which can be computed for instance via the classical state–space formulas from [85]:

$$M(\lambda) = \begin{bmatrix} \frac{\lambda-1}{\lambda+5} & \frac{\lambda-1}{\lambda+6} & 0 \\ 0 & \frac{\lambda+2}{\lambda+6} & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad -\widetilde{X}(\lambda) = \begin{bmatrix} \frac{40}{\lambda+5} \\ -\frac{8}{3}\frac{1}{\lambda+6} \\ 0 \end{bmatrix},$$

$$N(\lambda) = \begin{bmatrix} \frac{1}{\lambda+5} & \frac{2}{\lambda+6} & \frac{1}{\lambda+2} \end{bmatrix}, \quad \widetilde{Y}(\lambda) = \frac{\lambda^2 + 17\lambda + 66 + 2/3}{(\lambda+5)(\lambda+6)} \tag{4.23}$$

and also

$$-\widetilde{N}(\lambda) = \begin{bmatrix} \frac{\lambda+2}{(\lambda+3)(\lambda+4)} & \frac{\lambda-1}{(\lambda+3)(\lambda+4)} & 0 \end{bmatrix}, \quad \widetilde{M}(\lambda) = \frac{(\lambda-1)(\lambda+2)}{(\lambda+3)(\lambda+4)}. \tag{4.24}$$

The remaining factors X and Y that complete the DCF (4.1) of G are not needed in view of Remark 4.3.5. By looking at (4.23) we can see that $\mathrm{Pattern}(\widetilde{X}\widetilde{M}) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$. We need to find a Youla parameter Q, such that $\big(MQ\widetilde{M} + \widetilde{X}\widetilde{M}\big)$ has a zero in the second row entry. This is necessary and sufficient such as to make $\mathrm{Pattern}\big(MQ\widetilde{M} + \widetilde{X}\widetilde{M}\big) \leq K^{\mathrm{bin}}$.

We discuss next the exact model–matching problem $MQ\widetilde{M} = -\widetilde{X}\widetilde{M}$. Linear matrix equations of this type (also named Sylvester matrix equations) can be solved for Q via Proposition (4.3.3), by solving for $\mathrm{vec}(Q)$ the following equivalent linear system of TFM equations.:

$$(\widetilde{M}^T \otimes M)\mathrm{vec}(Q) = \mathrm{vec}(-\widetilde{X}\widetilde{M}) \tag{4.25}$$

(For this particular example, it happens that $\mathrm{vec}(Q) = Q$ and also $\mathrm{vec}(K^{\mathrm{bin}}) = K^{\mathrm{bin}}$, but this does not change the mechanic of the algorithm for the general case.) We reiterate the important fact that we do <u>not</u> need to solve the linear problem from (4.25). We must solve only a subset of linear equations from (4.25), composed precisely from the rows identified by the zero entries in the $\mathrm{vec}(K^{\mathrm{bin}})$ binary matrix. The only zero in $\mathrm{vec}(K^{\mathrm{bin}})$ is in the second row, hence we must solve only the equation in the second row of (4.25):

$$\widetilde{M}^T(\lambda)\begin{bmatrix} 0 & \frac{\lambda+2}{\lambda+6} & 1 \end{bmatrix} Q(\lambda) = -\frac{8}{3}\frac{1}{\lambda+6}\widetilde{M}(\lambda) \tag{4.26}$$

We choose a solution Q for (4.26)

$$Q = \frac{(\lambda+6+8/3)}{(\lambda+2)(\lambda+6)}\begin{bmatrix} (\lambda+5) & (\lambda+6) & \frac{(\lambda+2)(\lambda+6)}{(\lambda+6+8/3)} \end{bmatrix}^T$$

yielding the following controller $K = \big(\widetilde{X} + MQ\big)\big(\widetilde{Y} - NQ\big)^{-1}$

$$K = \frac{1}{\lambda^3 + 19\lambda^2 + (103+1/3)\lambda + (146+2/3)}\begin{bmatrix} -40(\lambda+2)(\lambda+6) \\ 0 \\ (\lambda+2)(\lambda+5)(\lambda+6) \end{bmatrix} \tag{4.27}$$

which has the desired sparsity pattern.

4.3.3 Sparse Controller Synthesis as An Exact Model–Matching Problem

For the remaining part of this section only, we briefly revisit the assumptions made in Subsection 4.2.1. Specifically, we make the assumption that all the blocks in the conformal partition (4.3) of the plant G have the size 1×1, meaning that $\forall i \in \overline{1,p}$ and $\forall j \in \overline{1,m}$ it holds that $n_y^i = n_u^j = 1$. This hypothesis does not imply any loss of generality whatsoever, since all the vectorization and matrix Kronecker product results can be naturally adapted when the factors involved are conformally block–partitioned. However, this hypothesis does considerably simplify the notation while outlining all the essential ideas needed for the proof of the general case (for any conformal block–partition (4.3) of G).

As a consequence of the assumption made at the beginning of the current subsection we get (see Subsection 4.2.1) that $G \in \mathbb{R}(\lambda)^{p\times m}$, $K \in \mathbb{R}(\lambda)^{m\times p}$ and consequently $K^{\text{bin}} \in \{0,1\}^{m\times p}$. Define n_G as the number of the zero entries in the K^{bin} binary matrix (and also in the $\text{vec}(K^{\text{bin}}) \in \{0,1\}^{mp\times 1}$ binary vector). (It follows that the number of one entries in K^{bin} is equal to $(mp - n_G)$.)

Let $(i_1, i_2, \ldots, i_{n_G})$ be the row indices of the zero entries in $\text{vec}(K^{\text{bin}})$. Let I_{mp} denote the $(mp) \times (mp)$ identity matrix and $0^{n_G\times 1}$ be the zero column vector with n_G rows. For any index $i \in \overline{1,(mp)}$ we denote with e_i^T the i-th row of I_{mp}. We define next the $n_G \times (mp)$ matrix Φ made out by selecting n_G rows of I_{mp}

$$\Phi \stackrel{def}{=} \begin{bmatrix} e_{i_1} & e_{i_2} & \ldots & e_{i_{n_G}} \end{bmatrix}^T \tag{4.28}$$

such that

$$\Phi \operatorname{vec}(K^{\text{bin}}) = 0^{n_G \times 1}. \tag{4.29}$$

Theorem 4.3.6. *Given a plant G in the set $\mathbb{R}(\lambda)^{p\times m}$ then G is stabilizable with a sparsity constrained controller K belonging to the set $\mathcal{S}$ if and only if, starting from* any *DCF (4.1) of G, there exists a Youla parameter Q (stable TFM, belonging to the set $\mathbb{R}(\lambda)^{m\times p}$) such that* $\operatorname{vec}(Q)$ *is a stable solution to the linear system of TFM equations*

$$\Phi\big(M^T \otimes \widetilde{M}\big)\operatorname{vec}(Q) = -\Phi \operatorname{vec}\big(\widetilde{X}\widetilde{M}\big), \tag{4.30}$$

where Φ is the matrix defined in (4.28).

Proof. We remind here that the $\operatorname{vec}(\cdot)$ operator (4.18) is linear. Also note that the $\operatorname{vec}(\cdot)$ operator and the $\operatorname{Pattern}(\cdot)$ operator (4.4) are commutative.

We prove next that the existence of a Youla parameter (*stable* TFM, belonging to the set $\mathbb{R}(\lambda)^{m\times p}$) to satisfy the first relation in (4.21) (see also Remark 4.3.5) is equivalent with $\operatorname{vec}(Q)$ being a *stable* solution to (4.30). The rest of the proof will follow via Corollary 4.3.4.

$$
\begin{aligned}
&\mathrm{Pattern}\big(\widetilde{X}\widetilde{M} + MQ\widetilde{M}\big) \leq K^{\mathrm{bin}} \iff \\
&\mathrm{vec}\Big(\mathrm{Pattern}\big(\widetilde{X}\widetilde{M} + MQ\widetilde{M}\big)\Big) \leq \mathrm{vec}(K^{\mathrm{bin}}) \iff \\
&\mathrm{Pattern}\Big(\mathrm{vec}\big(\widetilde{X}\widetilde{M} + MQ\widetilde{M}\big)\Big) \leq \mathrm{vec}(K^{\mathrm{bin}}) \quad \overset{(4.18)}{\iff} \\
&\mathrm{Pattern}\Big(\mathrm{vec}\big(\widetilde{X}\widetilde{M}\big) + \mathrm{vec}\big(MQ\widetilde{M}\big)\Big) \leq \mathrm{vec}(K^{\mathrm{bin}}) \quad \overset{\textit{Prop. } 4.3.3}{\iff} \\
&\mathrm{Pattern}\Big(\mathrm{vec}\big(\widetilde{X}\widetilde{M}\big) + \big(M^T \otimes \widetilde{M}\big)\mathrm{vec}(Q)\Big) \leq \mathrm{vec}(K^{\mathrm{bin}}) \iff \\
&\mathrm{Pattern}\Big(\Phi\Big(\mathrm{vec}\big(\widetilde{X}\widetilde{M}\big) + \big(M^T \otimes \widetilde{M}\big)\mathrm{vec}(Q)\Big)\Big) \leq \Phi\,\mathrm{vec}(K^{\mathrm{bin}}) \overset{(4.29)}{\iff} \\
&\mathrm{Pattern}\Big(\Phi\,\mathrm{vec}\big(\widetilde{X}\widetilde{M}\big) + \Phi\big(M^T \otimes \widetilde{M}\big)\mathrm{vec}(Q)\Big)\Big) \leq 0^{n_G \times 1} \iff \\
&\Phi\big(M^T \otimes \widetilde{M}\big)\mathrm{vec}(Q) = -\Phi\,\mathrm{vec}\big(\widetilde{X}\widetilde{M}\big)
\end{aligned}
\tag{4.31}
$$

□

4.3.4 Parametrization of All Sparse, Stabilizing Controllers

In this subsection we present a particularly important corollary of Theorems 4.3.2 and 4.3.6. Given the plant G in the set $\mathbb{R}(\lambda)^{p\times m}$, suppose G stabilizable with a sparsity constrained controller K belonging to the set $\mathcal{S}$. We provide next the parametrization of *all* stabilizing controllers of G, belonging to the set $\mathcal{S}$. We achieve this parametrization, starting from a DCF (4.1) of G satisfying (4.21) and imposing additional constraints on the Youla parameter, constraints that guarantee that the resulted controller will belong to $\mathcal{S}$. The constraints are for the Youla parameter to lie in the set of all stable TFMs belonging to a certain linear subspace.

Here comes the precise statement.

Corollary 4.3.7. *Given a plant G in the set $\mathbb{R}(\lambda)^{p\times m}$ stabilizable with a sparsity constrained controller K belonging to the set $\mathcal{S}$, and consequently a DCF (4.1) of G satisfying the first relation in (4.21), the set of all stabilizing controllers of G belonging to the set $\mathcal{S}$ is given by $K = \big(\widetilde{X} + MQ\big)\big(\widetilde{Y} - NQ\big)^{-1}$ where the Youla parameter Q (*stable *TFM, belonging to the set $\mathbb{R}(\lambda)^{m\times p}$) is such that*

$$\mathrm{vec}(Q) \in \mathrm{Null}\Big(\Phi\big(M^T \otimes \widetilde{M}\big)\Big), \tag{4.32}$$

where Φ is the matrix defined in (4.28). We make here the elementary observation that Q is stable if and only if vec(Q) *is stable.*

Proof. The DCF we start with satisfies the first relation (4.21), meaning $\mathrm{Pattern}(\widetilde{X}\widetilde{M}) \leq K^{\mathrm{bin}}$ and equivalently $\mathrm{vec}(\mathrm{Pattern}(\widetilde{X}\widetilde{M})) \leq \mathrm{vec}(K^{\mathrm{bin}})$. Then for any Youla parameter Q, we get via Theorem 4.3.6 that $K = \big(\widetilde{X} + MQ\big)\big(\widetilde{Y} - NQ\big)^{-1}$ belongs to the set $\mathcal{S}$ if and only if $\Phi(M^T \otimes \widetilde{M})\mathrm{vec}(Q) = -\Phi\ \mathrm{vec}(\widetilde{X}\widetilde{M})$. Now, because $\mathrm{vec}(\mathrm{Pattern}(\widetilde{X}\widetilde{M})) \leq \mathrm{vec}(K^{\mathrm{bin}})$, due to the way the Φ matrix is defined in (4.28) and due to (4.29), we get that $\Phi\ \mathrm{vec}(\widetilde{X}\widetilde{M}) = 0^{n_G\times(mp)}$, hence the proof. □

4.3.5 Numerical Example – Continued

In this subsection we will illustrate numerically the result of Corollary 4.3.7. We start with the same data from Subsection 4.3.2 but with a different DCF of the plant. The factors $\widetilde{M}$, $\widetilde{N}$ will still be as in (4.24) and M, N will be as in (4.23) but

$\widetilde{X}$ and $\widetilde{Y}$ will be given by

$$\widetilde{X} = \begin{bmatrix} -\dfrac{40}{(\lambda+5)} & 0 & 1 \end{bmatrix}^T,$$

$$\widetilde{Y} = \frac{\lambda^3 + 19\lambda^2 + (103 + 1/3)\lambda + (146 + 2/3)}{(\lambda+2)(\lambda+5)(\lambda+6)}.$$

which is the DCF satisfying the first relation in (4.21) since it is the DCF for which the sparse controller given in (4.27) is the central controller. For the argument stated in Remark 4.3.5, the remaining factors X and Y of the DCF are not needed.

For this example (as well as for what is presented in Subsection 4.3.2), the Φ matrix defined in (4.28) is given by $\Phi = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$. Furthermore, the set of all stable TFMs in the null space of $\big(\Phi(M^T \otimes \widetilde{M})\big)$ is given by

$$\mathcal{Q} \overset{def}{=} \Big\{ Q \in \mathbb{R}(\lambda)^{3\times 1} \Big| \; Q(\lambda) = \begin{bmatrix} \alpha(\lambda) & \beta(\lambda) & -\dfrac{\lambda+6}{\lambda+2}\beta(\lambda) \end{bmatrix}^T$$
$$\text{with } \alpha(\lambda), \beta(\lambda) \text{ stable, real} - \text{rational functions} \Big\} \tag{4.33}$$

The set of *all* stabilizing controllers of G, belonging to the set $\mathcal{S}$ is given by $K = \big(\widetilde{X} + MQ\big)\big(\widetilde{Y} - NQ\big)^{-1}$, with $Q \in \mathcal{Q}$.

4.4 A Meaningful, Particular Case

In this section we look at the same stabilization problem (see Subsection 4.1.1) via sparse controllers, but with the additional hypothesis that the given plant satisfies a particular criteria. Specifically, we look at the case when the plant G admits both a left coprime factorization $G = \widetilde{M}^{-1}\widetilde{N}$ over Ω and a right coprime factorization $G = NM^{-1}$ (see Subsection 4.1.2) such that both "denominators" $\widetilde{M}$ and M are block–diagonal. As it turns out such a factorization is guaranteed to exist for *almost all* plants, meaning that it is a *generic* property. Furthermore, for any given plant it is quite easy to check if such a factorization exists and if this is the case, it is also easy to compute. The advantages it brings are important. Firstly it makes all the equivalent results presented in the previous section far less complicated, since now vectorization is not needed. Secondly, it makes possible to characterize the set of all decentralized stabilizing controllers via the Youla parametrization, while the sparsity constraints on the controller are recast as sparsity constraints on the Youla parameter.

Notation: For p transfer function matrices $\widetilde{M}_i$ of sizes $n_y^i \times n_y^i$ respectively, where $i \in \overline{1,p}$ and $\sum_{i=1}^{p} n_y^i = n_y$, we shall use the common notation $\mathrm{diag}_{i\in\overline{1,p}}\left\{\widetilde{M}_i\right\}$ for the $n_y \times n_y$ block matrix that has the $\widetilde{M}_i$ TFMs on its block–diagonal.

4.4.1 The Output Decoupled, Left Coprime Factorization

We start with the given plant $G \in \mathbb{R}(\lambda)^{n_y \times n_u}$, having the block partition from (4.3) in Subsection 4.2.1. For every index $i \in \overline{1,p}$, we can always perform the left coprime factorization of the i-th block–row of G in (4.3), and we get

$$\begin{bmatrix} [G]_{i1} & \dots & [G]_{im} \end{bmatrix} = \widetilde{M}_i^{*\,-1}\widetilde{N}_i^{*}, \tag{4.34}$$

with $\widetilde{M}_i^{*} \in \mathbb{R}(\lambda)^{n_y^i \times n_y^i}$ and $\widetilde{N}_i^{*} \in \mathbb{R}(\lambda)^{n_y^i \times n_u}$. Writing in matrix form, relations (4.34) for all the p block–rows of G, we get

$$G = \begin{bmatrix} \widetilde{M}_1^{*\,-1} & & O \\ & \ddots & \\ O & & \widetilde{M}_p^{*\,-1} \end{bmatrix} \begin{bmatrix} \widetilde{N}_1^* \\ \vdots \\ \widetilde{N}_p^* \end{bmatrix}. \tag{4.35}$$

Denote $\widetilde{M}^* \overset{def}{=} \operatorname{diag}_{i \in \overline{1,p}} \left\{ \widetilde{M}_i^* \right\}$ and the remaining factor on the right hand side of (4.35) with $\widetilde{N}^* \overset{def}{=} \begin{bmatrix} \widetilde{N}_1^{*T} & \dots & \widetilde{N}_p^{*T} \end{bmatrix}^T$ respectively, such that (4.35) becomes $G = \widetilde{M}^{*\,-1} \widetilde{N}^*$.

Assumption 3. From here on, we assume that given the plant G and the factorization in (4.35), the TFM

$$\begin{bmatrix} \widetilde{M}^*(\lambda) & \widetilde{N}^*(\lambda) \end{bmatrix} \tag{4.36}$$

has full row rank for all λ outside the stability domain Ω.

Remark 4.4.1. *[87] The condition in (4.36) guarantees that the factorization (4.35) of G is indeed* coprime. *In this case, we will call (4.35) an* output decoupled right coprime *factorization of G.*

Remark 4.4.2. *Note that (4.36) is* generically *true, meaning that it holds for* almost all *pairs of TFMs $\widetilde{M}^*$ and $\widetilde{N}^*$, with $\widetilde{M}^*$ invertible (as a TFM).*

Remark 4.4.3. *Condition (4.36) needs to be checked only at those (finite number of) points λ that are* unstable *poles of G. This is because these values and only these are the unstable zeroes of $\widetilde{M}^*$ (note that $\widetilde{M}^*$ has only stable poles). Hence for any unstable λ_0 that is not a pole of G, it follows that $\widetilde{M}^*(\lambda_0)$ is invertible and so the rank (4.36) condition is satisfied.*

A sufficient condition for Assumption 3 to hold, would be for any two block-rows of G not to have a common unstable pole, in which case the rank condition is satisfied due to the coprimeness of each of the p factorizations in (4.34). This condition is by no means necessary, as even in the case of common unstable poles the row rank might be held by the $\widetilde{N}^*$ factor.

4.4.2 The Input Decoupled, Right Coprime Factorization

By interchanging the roles of the block–rows of G with its block–columns and applying the exact same procedure as at the beginning of Subsection 4.4.1, one can compute the following factorization of G (where the pair N_j^*, M_j^* is a right coprime factorization of the j-th block–column of G):

$$G(\lambda) = \begin{bmatrix} N_1^* & \dots & N_m^* \end{bmatrix} \begin{bmatrix} M_1^{*-1} & & \\ & \ddots & \\ & & M_m^{*\,-1} \end{bmatrix} \tag{4.37}$$

The N_j^* and M_j^* in (4.37) are right coprime for any j in $\overline{1,m}$. Denote $M^* \stackrel{def}{=} \mathrm{diag}_{j\in\overline{1,m}} \left\{ M_j^* \right\}$ and the remaining factor on the right hand side of (4.37) with N^*, such that (4.37) becomes $G = N^* M^{*-1}$.

The following assumption is the "right" correspondent of Assumption 3 and from this point onward it will be considered to hold true.

Assumption 4. Assume that given the plant G and the factorization in (4.37), the TFM

$$\begin{bmatrix} N^{*T}(\lambda) & M^{*T}(\lambda) \end{bmatrix}^T \tag{4.38}$$

has full column rank for all λ outside the stability domain Ω.

Remark 4.4.4. *The condition in (4.38) ensures that the factorization (4.37) of G is coprime ([87]). In this case, we call (4.37) an* input–decoupled right coprime factorization *of the plant G. Note that (4.38) need only be checked (for the finite number of points) λ that are unstable poles of the plant G. All other comments made in Remarks 4.4.2 and 4.4.3 can be adapted by simply interchanging the role of the block–rows of the plant G with its block–columns.*

4.4.3 Input/Output Decoupled DCFs

In this subsection, given plant $G(\lambda)$ satisfying Assumptions 3 and 4, we are interested in computing (when it exists) a doubly coprime factorization (4.1), where both denominators $M(\lambda)$ and $\widetilde{M}(\lambda)$ are simultaneously in block–diagonal form. We have named this type of DCF *Input/Output Decoupled* DCFs.

It turns out, that provided that there exists an Output–Decoupled Left Coprime Factorization (or equivalently Assumption 3 is met) *and* there exists an Input–Decoupled Right Coprime factorization (whose existence in equivalent with Assumption 4), then there exists a DCF (4.1) of G where both denominators $M(\lambda)$ and $\widetilde{M}(\lambda)$ are simultaneously in block–diagonal form.

Lemma 4.4.5. *[82, Theorem 60, Ch. 4] Given any $G \in \mathbb{R}(\lambda)^{n_y \times n_u}$ partitioned as in (4.3), satisfying Assumptions 3 and 4, there always exists a DCF of G such that* *the "denominators"* $\widetilde{M}^*$ *and* M^* *from the left and right–coprime factorizations of* G *respectively, (*$G = \widetilde{M}^{*^{-1}}\widetilde{N} = N^*M^{*-1}$*) are in block diagonal form. We call such a DCF, an* Input/Output Decoupled *DCF.*

The only downside of this result from [82, Theorem 60, Ch. 4] is that it only deals with the existence of the respective DCF, while providing no clue on how to actually compute one. All the results presented in this section are founded on the Input/Output Decoupled DCF. Therefore, we provide an entire section, later in the paper, devoted to computational, state–space methods for the Input/Output Decoupled DCF.

4.4.4 Stabilizability with Sparse Controllers

The following preliminary result will be needed later.

Proposition 4.4.6. *For any Input/Output Decoupled DCF of* $G(\lambda)$

$$\text{Pattern}(\widetilde{N}) = \text{Pattern}(N) = G^{\text{bin}}. \tag{4.39}$$

Proof. Since the DCF is Input/Output Decoupled, we get that $\text{Pattern}(\widetilde{M}) = I_p$ and $\text{Pattern}(M) = I_m$. Furthermore, $\text{Pattern}(\widetilde{N}) = \text{Pattern}(\widetilde{M}G) \leq I_p\,\text{Pattern}(G) = G^{\text{bin}}$. Similarly, $\text{Pattern}(N) = \text{Pattern}(GM) \leq G^{\text{bin}}$. □

The following theorem is an immediate consequence of the main result from Theorem 4.3.2.

Theorem 4.4.7. *Given an arbitrary plant* $G(\lambda)$ *in the set* $\mathbb{R}(\lambda)^{n_y \times n_u}$, $G(\lambda)$ *is stabilizable with a sparsity constrained controller* $K(\lambda)$ *belonging to the set* $\mathcal{S}$*, if and* *only if there exists an Input/Output Decoupled DCF of* $G(\lambda)$ *(as in Lemma 4.4.5) such that*

$$\text{Pattern}(\widetilde{X}) = K^{\text{bin}} \quad \text{or} \quad \text{Pattern}(X) = K^{\text{bin}}. \tag{4.40}$$

Proof. It follows directly from Theorem 4.3.2 and the fact that $\text{Pattern}(\widetilde{M}) = I_p$ and $\text{Pattern}(M) = I_m$. □

4.4.5 The Youla Parametrization

The following Theorem is the main result of this section.

Theorem 4.4.8. *Given a plant* G*, stabilizable with a sparsity constrained controller* K *in the set* $\mathcal{S}$*, and an Input/Output Decoupled DCF* $(M^*(\lambda), N^*(\lambda), \widetilde{M}^*(\lambda), \widetilde{N}^*(\lambda), X(\lambda), Y(\lambda), \widetilde{X}(\lambda), \widetilde{Y}(\lambda))$ *satisfying (4.40) from Theorem 4.4.7, the set of* all *sparsity constrained, stabilizing controllers belonging to the set* $\mathcal{S}$ *is given by*

$$\begin{aligned} K &= \big(\widetilde{X} + M^*Q\big)\big(\widetilde{Y} - N^*Q\big)^{-1} \\ &= \big(Y - Q\widetilde{N}^*\big)^{-1}\big(X + Q\widetilde{M}^*\big) \end{aligned} \tag{4.41}$$

where the Youla–parameter Q *is any stable TFM in the set* $\mathcal{S}$*.*

Proof. "$\supset$" We show, that for any Q stable, in the set $\mathcal{S}$ the controller produced by (4.41) is a sparsity constrained, stabilizing controller of G, belonging to the set $\mathcal{S}$. That any such K is a stabilizing controller, is an immediate consequence of the Youla Theorem 4.1.2, so it only remains to show that K belongs to $\mathcal{S}$. Since $\widetilde{X} \in \mathcal{S}$ (from (4.40)), $Q \in \mathcal{S}$ (from the hypothesis) and M^* is block–diagonal (because the DCF is Input/Output Decoupled), it follows that $\text{Pattern}(\widetilde{X} + M^*Q) \leq K^{\text{bin}}$.

Since $\mathrm{Pattern}(\widetilde{M}) = I_p$ we get that $\mathrm{Pattern}\big((\widetilde{X} + M^*Q)\widetilde{M}\big) \leq K^{\mathrm{bin}}$ or equivalently $(\widetilde{X} + M^*Q)\widetilde{M} \in \mathcal{S}$. Then exactly as in the "Sufficiency" proof of Corrolary 4.3.4 we get that

$$\big(\widetilde{X} + M^*Q\big)\big(\widetilde{Y} - N^*Q\big)^{-1} = h_G^{-1}\big((\widetilde{X} + M^*Q)\widetilde{M}\big)$$

and since $h_G^{-1}(\cdot)$ is a bijection from $\mathcal{S}$ to $\mathcal{S}$, we get that $\big(\widetilde{X}+M^*Q\big)\big(\widetilde{Y}-N^*Q\big)^{-1} \in \mathcal{S}$.

"$\subset$" To complete the proof, we show next that any sparsity constrained, stabilizing controller K in the set $\mathcal{S}$ is of the form (4.41), with Q stable, in the set $\mathcal{S}$. Let K belonging to the set $\mathcal{S}$ be an arbitrarily chosen but fixed, sparsity constrained, stabilizing controller of G. It follows from Youla's Theorem 4.1.2 applied for our Input/Output Decoupled DCF, that there exists a Youla parameter Q, stable TFM in $\mathbb{R}(\lambda)^{n_u \times n_y}$ such that

$$\begin{aligned} K &= \big(\widetilde{X} + M^*Q\big)\big(\widetilde{Y} - N^*Q\big)^{-1} \\ &= \big(Y - Q\widetilde{N^*}\big)^{-1}\big(X + Q\widetilde{M^*}\big). \end{aligned} \tag{4.42}$$

It only remains to prove that Q belongs to the set $\mathcal{S}$. Employing Proposition 4.3.1 for the particular Input/Output Decoupled DCF for which K is a central controller we get that

$$\big(\widetilde{X} + M^*Q\big)\widetilde{M}^* = h_G(K)$$

or equivalently

$$\widetilde{X} + M^*Q = K\big(I + GK\big)^{-1}\widetilde{M}^{*-1} \tag{4.43}$$

Note that $\mathrm{Pattern}\Big(K\big(I+GK\big)^{-1}\Big) = K^{\mathrm{bin}}$ due to the fact that K belongs to the set $\mathcal{S}$, Assumption 2 and Theorem 4.2.6. Also remember that M^* and so $\mathrm{Pattern}(M^*Q) = \mathrm{Pattern}(Q)$. Also remember that $\widetilde{M}^{*-1}$ is block–diagonal and that $\mathrm{Pattern}(\widetilde{X}) =$

K^{bin} (from the hypothesis that the Input/Output Decoupled DCF we started with, satisfies (4.40)). With all these in mind, we apply the Pattern($\cdot$) operator from (4.4) to (4.43) in order to get that Pattern(Q) must satisfy the following (binary) matrix equation:

$$K^{\mathrm{bin}} + \mathrm{Pattern}(Q) = K^{\mathrm{bin}}. \tag{4.44}$$

Furthermore, note that Pattern(Q) is a solution to (4.44) if and only if Pattern(Q) $\leq$ K^{bin}, or equivalently if and only if $Q \in \mathcal{S}$, and the proof ends.

□

4.4.6 The Model–Matching Problem for Sparse Controller Synthesis

In this subsection we deal with the problem of actually computing (when it does exist) an Input/Output Decoupled DCF of G that also satisfies the conditions (4.40), yielding a sparse controller. Just like in Subsection 4.3.3 we dealt with the general case, it turns out that the problem is equivalent with solving an exact model–matching problem. Only that now, due to the particularities of the Input/Output Decoupled DCF, the exact model matching problem can be formulated in a more compact way and is easier to solve.

We define the binary matrix $K_{\perp}^{\mathrm{bin}}$ belonging to the set $\{0,1\}^{m\times p}$ as

$$\left(K_{\perp}^{\mathrm{bin}}\right)_{ij} \overset{def}{=} \begin{cases} 1 & \text{if } K_{ij}^{\mathrm{bin}} = 0, \\ 0 & \text{otherwise}. \end{cases} \tag{4.45}$$

With $K_{\perp}^{\mathrm{bin}} \in \{0,1\}^{m\times p}$ defined in (4.45) we introduce the linear subspace $\mathcal{S}_{\perp}$

of $\mathbb{R}(\lambda)^{n_u \times n_y}$ as

$$\mathcal{S}_\perp \stackrel{def}{=} \left\{ A \in \mathbb{R}(\lambda)^{n_u \times n_y} \middle| \text{Pattern}(A) \le K_\perp^{\text{bin}} \right\}. \tag{4.46}$$

We start with any Input/Output Decoupled DCF of G, and we want to compute (when it exists) an Input/Output Decoupled DCF that additionally satisfies the conditions (4.40) from Theorem 4.4.7.

Theorem 4.4.9. *Given an arbitrary plant $G(\lambda)$ in the set $\mathbb{R}(\lambda)^{n_y \times n_u}$, partitioned as in (4.3) take any input/output decoupled DCF of $G(\lambda)$ which we consider fixed. There always exists an additive factorization of the $\widetilde{X}$ factor as $\widetilde{X} = \widetilde{X}_K + \widetilde{X}_{K_\perp}$, such that* $\text{Pattern}(\widetilde{X}_K) = K^{\text{bin}}$ *and* $\text{Pattern}(\widetilde{X}_{K_\perp}) = K_\perp^{\text{bin}}$*. Then $G(\lambda)$ is stabilizable with a sparsity constrained controller $K(\lambda)$ belonging to the set $\mathcal{S}$, if and only if there exist a solution $Q \in \mathcal{S}_\perp \cap \mathcal{RH}^\infty$ to the TFM equation*

$$\widetilde{X}_{K_\perp} = -MQ. \tag{4.47}$$

Proof. If a controller K belonging to $\mathcal{S}$ does exist, it can be obtained from any DCF for an adequate Youla parameter Q (depending on K). With this argument in mind, we start out with a (fixed) input/output decoupled DCF of G. According to Theorem 4.4.7, a controller $K \in \mathcal{S}$ exists if and only if there exists a Youla parameter Q such that $\text{Pattern}(\widetilde{X} + MQ) = K^{\text{bin}}$ or equivalently

$$\text{Pattern}(\widetilde{X}_K + \widetilde{X}_{K_\perp} + MQ) = K^{\text{bin}}. \tag{4.48}$$

For any Youla parameter Q, there always exists an additive factorization of the Q factor as $Q = Q_K + Q_{K_\perp}$, such that $\text{Pattern}(Q_K) = K^{\text{bin}}$ and $\text{Pattern}(Q_{K_\perp}) =$

$K_{\perp}^{\text{bin}}$ and consequently (because M is block–diagonal) $\text{Pattern}(MQ_K) = K^{\text{bin}}$ and $\text{Pattern}(MQ_{K_\perp}) = K_{\perp}^{\text{bin}}$. We rewrite (4.48) accordingly in order to obtain that

$$\text{Pattern}(\widetilde{X}_K + \widetilde{X}_{K_\perp} + MQ_K + MQ_{K_\perp}) = K^{\text{bin}}. \tag{4.49}$$

Since $\text{Pattern}(\widetilde{X}_K + MQ_K) = K^{\text{bin}}$ and $\text{Pattern}(\widetilde{X}_{K_\perp} + MQ_{K_\perp}) = K_{\perp}^{\text{bin}}$, it follows that (4.49) is equivalent with

$$\widetilde{X}_{K_\perp} + MQ_{K_\perp} = 0 \tag{4.50}$$

hence the proof.

□

4.4.7 Sparse, Optimal Controller Synthesis

In this section we point out how the Youla parametrization from Theorem 4.4.8 can be directly employed within the powerful tools developed in [79] for the synthesis of the $\mathcal{H}^2$ optimal controller satisfying sparsity constraints that are quadratically invariant with respect to the plant. If G is stabilizable with a K in the set $\mathcal{S}$, then we can compute an Input/Output Decoupled DCF (as in Lemma 4.4.5), satisfying (4.40) from Theorem 4.4.7. Following Corrolary 7 [82, pp.110] and Theorem 4.4.8, the set of all $H(G,K)$ with K stabilizing, $K \in \mathcal{S}$ admits the affine parametrization

$$H(G,K) = \begin{bmatrix} I - N^*X & -\widetilde{N}^*Y \\ D^*X & D^*Y \end{bmatrix} + \begin{bmatrix} -N^* & \widetilde{N}^* \\ D^* & -D^* \end{bmatrix} Q \begin{bmatrix} \widetilde{D}^* & \widetilde{N}^* \\ \widetilde{D}^* & \widetilde{N}^* \end{bmatrix} \tag{4.51}$$

where the Youla parameter Q is any stable TFM in the set $\mathcal{S}$. This immediately implies that the sparsity constrained disturbance attenuation problem (as introduced in [79, (1)/pp. 276]), or the sparsity constrained mixed $\mathcal{H}^2$ sensitivity problem (from [57, pp. 139]) can be ultimately written in the form of the following model–matching problem

$$\min_{Q \in \mathcal{S} \cap \mathcal{RH}^\infty} \left\| T_1 \ + \ T_2 Q T_3 \right\| \tag{4.52}$$

where T_1, T_2 and T_3 are certain TFMs (resulting from (4.51) and the problem's data). At this point the numerical technique from [79, Theorem 29] is readily available to numerically solve (4.52) (by employing existing tools from standard $\mathcal{H}_2$ synthesis).

4.5 Conclusions

In this paper we have provided necessary and sufficient conditions for the stabilizability of a given plant, with a controller satisfying sparsity constraints that are quadratically invariant with respect to the plant. These conditions are formulated in terms of the existence of a specific input/output decoupled doubly coprime factorization of the plant with additional sparsity constraints on certain factors . Along the way have obtained the set of all decentralized stabilizing controllers, characterized via the Youla parametrization. The sparsity constraints on the controller are also recast as convex constraints on the Youla parameter. In order to achieve this, it is noteworthy that the constraints on the Youla parameter become linear subspace constraints on the Youla parameter, only from this particular input/output decoupled doubly coprime factorization with supplemental sparsity constraints on certain factors. Solving the stabilization problem provides the missing link for fully exploiting the powerful optimal synthesis methods for sparse controllers from [79].

State–space Computation of the Input/Output Decoupled Douby Coprime Factorization

4.5.1 State–space Representations of LTI Systems

Given any n–dimensional state–space representation (A, B, C, D) of an LTI system, its input–output description is given by the *transfer function matrix* (TFM) which is the $n_y \times n_u$ matrix with real, rational functions entries

$$G(\lambda) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \overset{def}{=} D + C(\lambda I_n - A)^{-1}B, \tag{4.53}$$

where A, B, C, D are $n \times n$, $n \times n_u$, $n_y \times n$, $n_y \times n_u$ real matrices, respectively while n is also called *the order* of the realization (4.53). For any real, invertible, $n \times n$ matrix Z, we call a *similarity transformation* of the realization (4.53) the following state–space realization of G

$$\left[\begin{array}{c|c} Z^{-1}AZ & Z^{-1}B \\ \hline CZ & D \end{array}\right] = G(\lambda). \tag{4.54}$$

The undeterminate λ is either s for continuous–time systems or z for discrete–time systems, respectively. Frequently in the sequel, the λ argument following a TFM is omitted if it is clear from the context. The D matrix in *any* realization (4.53) of $G(\lambda)$ is called the gain at infinity of $G(\lambda)$, and it will be denoted in the sequel with (the somehow abusive but straightforward notation) $G(\infty)$.

By $\mathbb{C}$ we denote the complex plane. The identity matrix of size $n \times n$ is denoted by I_n, while the subscript is dropped if the size is clear from the context. By $\Lambda(A)$ we mean the union of eigenvalues of the square matrix A (multiplicities counting). By $\mathbb{R}(\lambda)$ we denote the set of all real rational functions and by $\mathbb{R}(\lambda)^{n_y \times n_u}$ the set of $n_y \times n_u$ matrices having all entries in $\mathbb{R}(\lambda)$.

It is well known that for any proper, $n_y \times n_u$ TFM $G(\lambda)$ there exist (A, B, C, D), a state–space representation such that (4.53) holds and furthermore such a quartet of matrices is not unique. A realization (4.53) of order n, (or the pair (A, B)) is called controllable if rank $\begin{bmatrix} A - \lambda I & B \end{bmatrix} = n$ holds for any $\lambda \in \mathbb{C}$ ([76, Ch. 1.2]). Analogously, we say that a realization (4.53) is observable (or the pair (C, A) is observable) provided the pair (A^T, C^T) is controllable ([76, Ch. 3.1]), where we adopt the superscript T as the notation for matrix transposition. A realization that is controllable and observable is called *minimal*. For any minimal realization (4.53) of $G(\lambda)$, $\Lambda(A)$ are called the *poles* of G.

Denote by $\operatorname{rank}_n \Theta(\lambda)$ the *normal rank* of the transfer function matrix (TFM) $\Theta(\lambda)$, *i.e.* the rank of $\Theta(\lambda)$ for almost all $\lambda \in \mathbb{C}$ (but a finite number of points). A square TFM, $\Theta(\lambda) \in \mathbb{R}(\lambda)^{n_y \times n_y}$ that has full normal rank ($\operatorname{rank}_n \Theta(\lambda) = n_y$), has an inverse in $\mathbb{R}(\lambda)^{n_y \times n_y}$.

This paper gives a unified treatment for both the continuous and discrete-time cases. Henceforth, we will denote by Ω the open left half complex plane or the open unit disk, according to the type of system: continuous or discrete–time, respectively. The standard interpretation of Ω in systems theory is related to the *stability domain* of linear systems. We qualify the system (4.53) (or equivalently the TFM $G(\lambda)$) as *stable* if all its poles are in Ω.

A realization (4.53) of order n, (or the pair (A, B)) is called *stabilizable* if for any $\lambda \in \mathbb{C} - \Omega$ we have that rank $\begin{bmatrix} A - \lambda I & B \end{bmatrix} = n$ ([76, Ch. 2.4]). Analogously, we say that a realization (4.53) is *detectable* (or the pair (C, A)) is detectable provided the pair (A^T, C^T) is stabilizable ([76, Ch. 3.4]).

For a given TFM $\Theta(\lambda)$, $\lambda_0 \in \bar{\mathbb{C}}$ is a *zero* of $\Theta(\lambda)$, if the rank of $\Theta(\lambda_0)$ is strictly smaller than the normal rank of $\Theta(\lambda)$. For a square, invertible TFM $\Theta(\lambda)$ it holds true that the *zeroes* of $\Theta(\lambda)$ are the poles (multiplicities counted) of $\Theta^{-1}(\lambda)$. A square TFM $\Theta(\lambda)$ is called *unimodular*, if it is stable, invertible and has a stable inverse, or equivalently if it is invertible and all its poles and all its zeroes are in Ω.

Theorem 4.5.1. *[59, Theorem 1] Let $G(\lambda)$ be some proper $n_y \times n_u$ TFM. The class of* all *DCFs (4.1) of $G(\lambda)$ over Ω is given by*

$$\begin{bmatrix} M(\lambda) & -\widetilde{X}(\lambda) \\ N(\lambda) & \widetilde{Y}(\lambda) \end{bmatrix} = \left[\begin{array}{c|cc} A - BL & B & F \\ \hline -L & I & 0 \\ C - DL & D & I \end{array}\right] T \, , \tag{4.55}$$

$$\begin{bmatrix} Y(\lambda) & X(\lambda) \\ -\widetilde{N}(\lambda) & \widetilde{M}(\lambda) \end{bmatrix} = T^{-1} \left[\begin{array}{c|cc} A-FC & B-FD & F \\ \hline L & I & 0 \\ -C & -D & I \end{array}\right], \tag{4.56}$$

where A, B, C, D, F, L *and* T *are real matrices accordingly dimensioned such that*

i) $T = \begin{bmatrix} V & W \\ O & U \end{bmatrix}$ *has its diagonal* $n_u \times n_u$ *block* V *and* $n_y \times n_y$ *block* U *respectively, invertible,*

ii) F *and* L *are feedback–matrices such that* $\Lambda(A - BL) \cup \Lambda(A - FC) \subset \Omega$,

iii) $G(\lambda) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *is a stabilizable and detectable realization.*

Proposition 4.5.2. *Let* $G(\lambda)$ *be an arbitrary* $n_y \times n_u$ *TFM and* Ω *a domain in* $\mathbb{C}$.

A) *The class of* all *left coprime factorizations of* $G(\lambda)$ *over* Ω, $G = \widetilde{M}^{-1}\widetilde{N}$, *is given by*

$$\begin{bmatrix} \widetilde{N}(\lambda) & \widetilde{M}(\lambda) \end{bmatrix} = U^{-1} \left[\begin{array}{c|cc} A-FC & B-FD & -F \\ \hline C & D & I \end{array}\right], \tag{4.57}$$

where A, B, C, D, F *and* U *are real matrices accordingly dimensioned such that*

i) U *is any* $n_y \times n_y$ *invertible matrix,*

ii) F *is any feedback matrix that allocates the observable modes of the* (C, A) *pair to* Ω,

iii) $G(\lambda) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *is a stabilizable realization.*

B) *The class of* all *right coprime factorizations of* $G(\lambda)$ *over* Ω, $G = NM^{-1}$ *is given* *by*

$$\begin{bmatrix} M(\lambda) \\ N(\lambda) \end{bmatrix} = \left[\begin{array}{c|c} A - BL & B \\ \hline -L & I \\ C - DL & D \end{array}\right] V \tag{4.58}$$

where A, B, C, D, L *and* V *are real matrices accordingly dimensioned such that*

i) V *is any* $n_u \times n_u$ *invertible matrix,*

ii) L *is any feedback matrix that allocates the controllable modes of the* (A, B) *pair to* Ω,

iii) $G(\lambda) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *is a detectable realization.*

Proof. We will only prove point **A)** since the proof for point **B)** follows by duality. The fact that (4.57) is a left coprime factorization of G follows directly from [59, Theorem 1]. One can also note that since for any feedback matrix F the pairs (C, A) and $(C, A - FC)$ have the same observability subspaces ([76]), it follows that the poles of both $\widetilde{M}$ and $\widetilde{N}$ are among the observable modes of $(C, A - FC)$ which are all in Ω due to **ii)**.

Conversely, let $\widetilde{M}$ and $\widetilde{N}$ be such that $G = \widetilde{M}^{-1}\widetilde{N}$ is a left coprime factorization of G over Ω. Then according to [82, Ch.4, Theorem 60] it can always be completed to a doubly coprime factorization (4.1) of G. The respective doubly coprime factorization (4.1) of G, must be of the form (4.55), (4.56) because of [59, Theorem 1]. Furthermore, from (4.56) we get that in fact $\begin{bmatrix} \widetilde{N}(\lambda) & \widetilde{M}(\lambda) \end{bmatrix}$ is given by (4.57) where (A, B) is stabilizable, (C, A) is detectable and the feedback matrix F is such that $\Lambda(A - FC) \subset \Omega$. Finally, we remark that the detectability of (C, A)

is not needed because even if (C, A) is not detectable we take the feedback F to allocate only the observable part of (C, A), while the unobservable part of (C, A) which is invariant under the ...to be completed.

$\square$

State–Space Algorithm for the I/O Decoupled DCF. We provide here a constructive algorithm that produces an Input/Output Decoupled DCF. Due to the many degrees of freedom one has at disposal at certain steps within the algorithm, we point out that in fact, we are able to produce a very broad class of Input/Output Decoupled DCF. For example, one nice feature that is preserved from the classical DCF is that we can place the poles of all factors at any desired locations in Ω.

Given the plant G, we compute using the procedure from Subsection 4.4.1, an Output Decoupled Left Coprime Factorization (4.35) of $G(\lambda)$ (with $G(\lambda) = \widetilde{M}^{*-1}(\lambda)\widetilde{N}^{*}(\lambda)$ and $\widetilde{M}^{*}(\lambda)$ in block–diagonal form). We stop here to remark that there is full leverage in placing the poles of $\widetilde{M}^{*} \stackrel{def}{=} \mathrm{diag}_{i\in\overline{1,p}}\left\{\widetilde{M}_i^{*}\right\}$ from (4.35). Since any invertible factor $\widetilde{M}_i^{*}$ on the block–diagonal is computed by performing a standard left coprime factorization (of the i-th block–row of G), its poles are freely assigned and consequently so are the poles of $\widetilde{M}^{*} = \mathrm{diag}_{i\in\overline{1,p}}\{\widetilde{M}_i^{*}\}$. The state–space representation of this Output Decoupled Left Coprime Factorization can be obtained according to Proposition 4.5.2 **A)** starting from a certain stabilizable state–space realization of $G(\lambda)$ (which we take without loss of generality to be in the Kalman Structural Decomposition, [89]) and which we consider fixed:

$$G(\lambda) = \left[\begin{array}{cccc|c} \star & \star & \star & \star & \star \\ O & A_{22} & O & A_{24} & B_2 \\ O & O & \star & \star & O \\ O & O & O & A_{44} & O \\ \hline O & C_2 & O & C_4 & D \end{array}\right] \tag{4.59}$$

with the $\star$ denoting parts of the realizations that are of no importance in the proof. Continuing with Proposition 4.5.2 **A)**, there also exist an invertible matrix U^* and a feedback matrix F^* (both fixed) such that (with F^* partitioned in accordance with (4.59)) we get

$$\left[\begin{array}{cc} -\widetilde{N}^*(\lambda) & \widetilde{M}^*(\lambda) \end{array}\right] = U^{*-1} \left[\begin{array}{cccc|cc} \star & \star & \star & \star & \star & \star \\ O & A_{22} - F_2^* C_2 & O & A_{24} - F_2^* C_4 & B_2 - F_2^* D & F_2^* \\ O & \star & \star & \star & \star & \star \\ O & -F_4^* C_2 & O & A_{44} - F_4^* C_4 & -F_4^* D & F_4^* \\ \hline O & -C_2 & O & -C_4 & -D & I \end{array}\right] \tag{4.60}$$

with

$$\Lambda\left(\left[\begin{array}{cc} A_{22} - F_2^* C_2 & A_{24} - F_2^* C_4 \\ -F_4^* C_2 & A_{44} - F_4^* C_4 \end{array}\right]\right) \subset \Omega. \tag{4.61}$$

Note that since (4.59) is stabilizable it follows that $\Lambda(A_{44}) \subset \Omega$. After removing the unobservable part from (4.60) (using the same procedure as from (**??**) to (**??**)

in the proof of Proposition 4.5.2 **A)**), we get that

$$\begin{bmatrix} -\widetilde{N}^*(\lambda) & \widetilde{M}^*(\lambda) \end{bmatrix} = U^{*-1} \left[\begin{array}{cc|cc} A_{22} - F_2^* C_2 & A_{24} - F_2^* C_4 & B_2 - F_2^* D & F_2^* \\ -F_4^* C_2 & A_{44} - F_4^* C_4 & -F_4^* D & F_4^* \\ \hline -C_2 & -C_4 & -D & I \end{array}\right] \tag{4.62}$$

Analogously, we compute an Input Decoupled Right Coprime Factorization (4.37) of $G(\lambda)$, (with $G(\lambda) = N^*(\lambda)M^*(\lambda)^{-1}$ and $M^*(\lambda)$ in block–diagonal form). On the same line of reasoning on the poles placement of $\widetilde{M}^*(\lambda)$, notice that the poles of $M^*(\lambda)$ as well, can be placed at will in Ω. According to Proposition 4.5.2 **B)**, there exists a certain detectable state–space realization of $G(\lambda)$ (which we take without loss of generality to be in the Kalman Structural Decomposition and) which we also consider fixed:

$$G(\lambda) = \left[\begin{array}{cccc|c} A_{11} & A_{12} & \star & \star & B_1 \\ O & A_{22} & O & \star & B_2 \\ O & O & \star & \star & O \\ O & O & O & \star & O \\ \hline O & C_2 & O & \star & D \end{array}\right] \tag{4.63}$$

with the $\star$ denoting parts of the realization that are of no importance.

Any two realizations of G will always have the same the controlable and observable part, up to a similarity transformation (4.54). That is to say that if the controlable and stabilizable part of (4.59) is (A_{22}, B_2, C_2, D) then the controlable and stabilizable part of (4.63) must be $(Z^{-1}A_{22}Z, Z^{-1}B_2, C_2 Z, D)$, for some invertible, real matrix Z . We can apply this similarity transformation adequately on (4.63), such that the the controlable and stabilizable part (A_{22}, B_2, C_2, D), appears identical on both realizations (4.59) and (4.63), respectively. This will simplify some of the future computations.

We continue with Proposition 4.5.2 **B)**: along with realization (4.63), there also exist an invertible matrix V^* and a feedback matrix L^* (both fixed) such that (with L^* partitioned in accordance with (4.63))

$$\begin{bmatrix} M^*(\lambda) \\ N^*(\lambda) \end{bmatrix} = \left[\begin{array}{cccc|c} A_{11} - B_1L_1^* & A_{12} - B_1L_2^* & \star & \star & B_1 \\ -B_2L_1^* & A_{22} - B_2L_2^* & \star & \star & B_2 \\ O & O & \star & \star & O \\ O & O & O & \star & O \\ \hline -L_1^* & -L_2^* & \star & \star & I \\ -DL_1^* & C_2 - DL_2^* & \star & \star & D \end{array}\right] V^* \tag{4.64}$$

with

$$\Lambda\left(\begin{bmatrix} A_{11} - B_1L_1^* & A_{12} - B_1L_2^* \\ -B_2L_1^* & A_{22} - B_2L_2^* \end{bmatrix}\right) \subset \Omega, \tag{4.65}$$

Note that since (4.63) is detectable it follows that $\Lambda(A_{11}) \subset \Omega$. After removing the uncontrolable part from (4.64) we get that

$$\begin{bmatrix} M^*(\lambda) \\ N^*(\lambda) \end{bmatrix} = \left[\begin{array}{cc|c} A_{11} - B_1 L_1^* & A_{12} - B_1 L_2^* & B_1 \\ -B_2 L_1^* & A_{22} - B_2 L_2^* & B_2 \\ \hline -L_1^* & -L_2^* & I \\ -DL_1^* & C_2 - DL_2^* & D \end{array}\right] V^* \tag{4.66}$$

We fix now the following stabilizable and detectable state–space realization of $G(\lambda)$:

$$G(\lambda) = \left[\begin{array}{ccc|c} A_{11} & A_{12} & \star & B_1 \\ O & A_{22} & A_{24} & B_2 \\ O & O & A_{44} & O \\ \hline O & C_2 & C_4 & D \end{array}\right] \tag{4.67}$$

Since $\Lambda(A_{11}) \subset \Omega$ we get that (4.67) is detectable and since $\Lambda(A_{44}) \subset \Omega$ we get that (4.67) is stabilizable, hence (4.67) statisfies the hypothesis from Theorem 4.5.1 **iii)**. Starting from realization (4.67) (which is fixed), (4.55) and (4.56) yield a valid DCF of G for any feedback matrices F and L (partitioned in accordance with (4.67) and satisfying Theorem 4.5.1 **ii)**), and any invertible matrix T satisfying Theorem 4.5.1 **i)**. We will denote the factors of this particular DCF with $\big(M(\lambda), N(\lambda), \widetilde{M}(\lambda), \widetilde{N}(\lambda),\ X(\lambda), Y(\lambda),\ \widetilde{X}(\lambda), \widetilde{Y}(\lambda)\big)$. After removing the unobservable part, the $\widetilde{M}$ factor will be (the computation are similar with those for getting from (4.60) to (4.62))

$$\widetilde{M}(\lambda) = U^{-1} \left[\begin{array}{cc|c} A_{22} - F_2C_2 & A_{24} - F_2C_4 & F_2 \\ -F_4C_2 & A_{44} - F_4C_4 & F_4 \\ \hline -C_2 & -C_4 & I \end{array}\right] \tag{4.68}$$

where

$$\Lambda \left(\begin{bmatrix} A_{22} - F_2C_2 & A_{24} - F_2C_4 \\ -F_4C_2 & A_{44} - F_4C_4 \end{bmatrix} \right) \subset \Omega. \tag{4.69}$$

and U is a real, invertible matrix. We compute the factor $\widetilde{\Theta} \overset{def}{=} \widetilde{M}^* \widetilde{M}^{-1}$ using the state–space realizations from (4.60) and (4.68) respectively and we get

$$\widetilde{\Theta}(\lambda) = U^{*-1} \left[\begin{array}{cccc|c} A_{22} - F_2^*C_2 & A_{24} - F_2^*C_4 & F_2^*C_2 & F_2^*C_4 & F_2^* \\ -F_4^*C_2 & A_{44} - F_4^*C_4 & F_4^*C_2 & F_4^*C_4 & F_4^* \\ O & O & A_{22} & A_{24} & F_2 \\ O & O & O & A_{44} & F_4 \\ \hline -C_2 & -C_4 & C_2 & C_4 & I \end{array}\right] U. \tag{4.70}$$

After removing the unobservable part from (4.70) we get that

$$\widetilde{\Theta}(\lambda) = U^{*-1} \left[\begin{array}{cc|c} A_{22} - F_2^*C_2 & A_{24} - F_2^*C_4 & F_2^* - F_2 \\ -F_4^*C_2 & A_{44} - F_4^*C_4 & F_4^* - F_4 \\ \hline -C_2 & -C_4 & I \end{array}\right] U \tag{4.71}$$

and consequently

$$\widetilde{\Theta}^{-1}(\lambda) = U^{-1} \left[\begin{array}{cc|c} A_{22} - F_2C_2 & A_{24} - F_2C_4 & F_2^* - F_2 \\ -F_4C_2 & A_{44} - F_4C_4 & F_4^* - F_4 \\ \hline C_2 & C_4 & I \end{array}\right] U^*, \tag{4.72}$$

which combined with (4.61) and (4.69) shows that $\widetilde{\Theta}(\lambda)$ is unimodular. A similar line of reasoning can be used to prove that $\Theta(\lambda) \overset{def}{=} M(\lambda)^{-1}M^*(\lambda)$ is unimodular.

Finally, compute

$$\left(\begin{bmatrix} \Theta^{-1} & O \\ O & \widetilde{\Theta} \end{bmatrix} \begin{bmatrix} \widetilde{Y}(\lambda) & \widetilde{X}(\lambda) \\ -\widetilde{N}(\lambda) & \widetilde{M}(\lambda) \end{bmatrix} \right) \left(\begin{bmatrix} M(\lambda) & -X(\lambda) \\ N(\lambda) & Y(\lambda) \end{bmatrix} \begin{bmatrix} \Theta & O \\ O & \widetilde{\Theta}^{-1} \end{bmatrix} \right) = I_{n_y+n_u} \tag{4.73}$$

which is still a DCF of G in its own right, due to the unimodularity of Θ and $\widetilde{\Theta}$. Plugging in the definitions of $\widetilde{\Theta}$ and Θ into (4.73) yields

$$\begin{bmatrix} \Theta^{-1}(\lambda)\widetilde{Y}(\lambda) & \Theta^{-1}(\lambda)\widetilde{X}(\lambda) \\ -\widetilde{N}^*(\lambda) & \widetilde{M}^*(\lambda) \end{bmatrix} \left(\begin{bmatrix} M^*(\lambda) & -X(\lambda)\widetilde{\Theta}^{-1}(\lambda) \\ N^*(\lambda) & Y(\lambda)\widetilde{\Theta}^{-1}(\lambda) \end{bmatrix} \right) = I_{n_y+n_u} \tag{4.74}$$

which is an Input/Output Decoupled DCF of G and the proof ends.

Appendix

Proof of Proposition 4.3.1. For this proof we will make extensive references to [82]. The DCF (4.1) of G guarantees that the hypothesis of [82, Ch. 5.2, Theorem 1] and consequently of [82, Ch. 5.2, Corollary 7] are met.

Consider the expression of $H(G, K)$ defined in Subsection 4.1.1 (with K being the central controller from Definition 4.1.3) obtained by taking the expression in

[82, Ch. 5.2, Corollary 7, (8)] with the Youla–parameter equal to zero (according also to Definition 4.1.3) and equate it with the first expression for $H(G,K)$ from [82, Ch. 5.1, pp. 101, (7)] in order to obtain an identity. The bottom right entry for $H(G,K)$ yields the identity $MY = (I+KG)^{-1}$, which is exactly the first relation in (4.13). The bottom left entry of $H(G,K)$ yields the identity $MX = (I+KG)^{-1}K$, which is exactly the second relation in (4.13).

Consider now the expression of $H(G,K)$ obtained by taking the expression in [82, Ch. 5.2, Corollary 7, (9)] with the Youla–parameter equal to zero and equate it with the second expression for $H(G,K)$ from [82, Ch. 5.1, pp. 101, (7)] in order to obtain another identity. The top left entry for $H(G,K)$ yields the identity $\widetilde{Y}\widetilde{M} = (I+GK)^{-1}$, which is exactly the third relation in (4.13). The bottom left entry of $H(G,K)$ yields the identity $\widetilde{X}\widetilde{M} = K(I+GK)^{-1}$, which is exactly the last relation in (4.13).

Chapter 5

Future Research Ideas

5.1 2–Inverses for Binary Matrices

As seen in the second Chapter, for linear, time–invariant systems with sparsity constraints, the quadratic invariance property does not depend on the actual dynamics of the plant or controller. It is exclusively a property of the sparsity patterns of the plant and controller respectively. Specifically, any controller K with $K^{\text{bin}} = \text{Pattern}(K)$ is quadratic invariant with respect to the plant G with $G^{\text{bin}} = \text{Pattern}(G)$if and only if

$$K^{\text{bin}} G^{\text{bin}} K^{\text{bin}} = K^{\text{bin}} \tag{5.1}$$

holds, for the binary matrices K^{bin} and K^{bin}.

If we fix the matrix K^{bin} that all the matrices G^{bin} satisfying (5.1) are called generalized inverses (or 1–inverses) of K^{bin}. Almost 35 years ago by Rao and Rao in their excellent reference [101] have completely characterized and provided computational algorithms for the 1–inverses of binary matrices. Further results have been developed in their second paper [102].

The surprising thing is that apparently people in the control community are not yet aware of this previous work since it contains remarkable results that we have not seen cited anywhere yet. For instance in [101] is proved that if K^{bin} has full–rank as a binary matrix, and we only consider its 1–inverses G^{bin} that are also full–rank, then both K^{bin} and G^{bin} are triangular, modulo some row permutation operation. This result shows that quadratic invariance doesn't actually go to far beyond the so called "nested structured" systems which have been present in the control literature for quite some time.

Of course a separate investigation in needed for the case when the binary matrices involved are not full-rank. Some preliminary results in this respect are already available in the work of Rao and Rao.

For our decentralized control problems, the interesting case is when the matrix G^{bin} is fixed and we want to find all matrices K^{bin} that satisfy (5.1). All K^{bin} matrices satisfying (5.1) are called 2–inverses of G^{bin}. Therefore a systematic study of 2–inverses of binary matrices would be deeply beneficial for understanding the nature of quadratic invariant sparsity structures.

Furthermore, it would be nice if we could parametrize all the 2–inverses of minimum Frobenius norm and then characterize all the 2–inverses around the minimum Frobenius ones. That would reveal which links are actually superfluous in keeping the configuration quadratically invariant. This might prove to be a step forward towards attempting the problem described next.

5.1.1 Reliable Decentralized Stabilization

The problem of reliable decentralized stabilization consists of computing decentralized controllers that are robust to deviations of the closed loop parameters, such as unreliable links between the controller and the plant. Interesting results have been published in [30] and [37].

For the problem of optimal control in a quadratic invariant decentralized configuration, we are interested in investigating how to design controllers that are robustly stabilizing when certain links fail. One possible scenario could be the following. Suppose that we have a feedback system of decentralized, linear time invariant plant as in Figure 2.1 and the sparsity pattern of the controller is quadratically invariant with respect to the plant. This implies that (5.1) holds, where we have used the previously defined notations G^{bin} and K^{bin} to denote the sparsity patterns of the plant and controller respectively. Suppose now that K^{bin} is not a minimum Frobenius norm 2–inverse for G^{bin}. Then, if any link in the controller that represents an entry that does not belong to the minimum Frobenius norm 2–inverses of G^{bin} fails, then the newly obtain sparsity pattern of the controller, call it K_*^{bin} would still be a 2–inverse of G^{bin}. Furthermore, the feedback configuration would remain quadratically invariant and we know that the sparsity pattern of the feedback-loop would be exactly K_*^{bin}. Of course, if the respective link fails, the optimality of the controller is lost, but perhaps we can ensure in the initial design procedure, that at least the stability of the closed–loop is preserved.

Preliminary research shows that the problem of attaining such a design specification is tractable.

5.2 Strong Stabilization and Controller Order Reduction via Fundamental Spaces Analysis

Strong stabilization designates the output feedback stabilization of a given plant with a stable controller. The benefits of strong stabilization are well established in control engineering practice. Equally important, the problem is intimately related with one of the fundamental limitations of feedback control, namely the fact that stabilization via a nonstable controller introduces with necessity additional, undesirable non–minimum phase zeroes in the feedback loop transfer function, beyond those of the original plant. Moreover, the problem of "simultaneous stabilization" is known to be always reducible to a strong stabilization problem of a certain equivalent system [63].

In spite of the considerable research effort that has been and is still being spent in this direction, a general, tractable scheme for the synthesis of a stable controller (when one does exist) is still not available. A key result from [63] states that a certain plant is strongly stabilizable if and only if it satisfies the so–called parity interlacing property . For SIMO plants (and SISO as a special case) several synthesis procedures exist and are anchored on interpolation methods that construct a certain unimodular factor [63, 108]. For the case of MIMO plants, all available techniques are ultimately based on the heavy theoretical machinery from $\mathcal{H}^2$ and $\mathcal{H}^\infty$ optimal synthesis [105, 106, 107, 109, 110, 113, 115, 116]. From the perspective of previous results, our approach is entirely unconventional as it is built solely on the analysis of the fundamental spaces of certain factors of the doubly coprime factorizations of the plant. In fact, the entire research plan proposed in this subsection is envisaged as an important application to the powerful tools developed in [117], that give a complete computational, state–space characterization of the fundamental spaces (range and null space) of an LTI operator. As it will be seen, a cardinal aspect is the fact that in [117] we were able to characterize all vector bases that span the fundamental spaces of a given TFM, and we allow for supplemental restrictions on the allocated poles of these bases.

We remind here that a collection of eight stable TFMs $\big(M(\lambda), N(\lambda), \widetilde{M}(\lambda), \widetilde{N}(\lambda), X(\lambda), Y(\lambda), \widetilde{X}(\lambda), \widetilde{Y}(\lambda)\big)$ is called a *doubly coprime factorization* (DCF) of the plant $G(\lambda)$ if the "denominator" TFMs $\widetilde{M}(\lambda)$ and $M(\lambda)$ are invertible and satisfy $G(\lambda) = \widetilde{M}(\lambda)^{-1}\widetilde{N}(\lambda) = N(\lambda)M(\lambda)^{-1}$ and

$$\begin{bmatrix} Y(\lambda) & X(\lambda) \\ -\widetilde{N}(\lambda) & \widetilde{M}(\lambda) \end{bmatrix} \begin{bmatrix} M(\lambda) & -\widetilde{X}(\lambda) \\ N(\lambda) & \widetilde{Y}(\lambda) \end{bmatrix} = I_{n_y+n_u}. \tag{5.2}$$

To the synthesis problem of a stable controller, it corresponds the problem of existence of a particular DCF (5.2) of the plant, where the denominator of the central controller is unimodular or (without any loss of generality) is has degree McMillan zero (is a constant, invertible matrix). This in its own turn, can be broken down to solving a linear matrix equation $(\widetilde{Y}(\lambda) - N(\lambda)Q(\lambda)) = \Delta(\lambda)$ for the Youla–parameter $Q(\lambda)$, where the parameter $\Delta(\lambda)$ must be unimodular. We remark here that the solution $Q(\lambda)$ (if it exists) must also satisfy the necessary stability constraints, associated with the celebrated Youla parametrization. Noting that the $\widetilde{Y}(\lambda)$ factor is invertible, hence its *Range* is the ambient space, we aim at conveying the existence of such a solution $Q(\lambda)$ in terms of the *Range* of the plant $G(\lambda)$ (as a rational matrix) (which is the same as the range of the factor $N(\lambda)$ from(5.2)).

The most exciting thing about the aforementioned approach is that it suggests a novel way to tackle the problem of computing stabilizing controllers of reduced degree. One can show that if we can solve $(\widetilde{Y}(\lambda) - N(\lambda)Q(\lambda)) = \Delta(\lambda)$ for a stable $Q(\lambda)$, with the parameter $\Delta(\lambda)$ unimodular, then we can immediately build another DCF of $G(\lambda)$, such that $\widetilde{Y}(\lambda)$ has McMillan degree zero (is a constant, invertible matrix). Then one stable controller would be given by $K(\lambda) = \widetilde{X}(\lambda)\widetilde{Y}(\lambda)^{-1}$ and it has the same number of poles as $\widetilde{X}(\lambda)$. We can now try to somehow reduce the McMillan degree of the controller, by looking at all the factors of the form $\widetilde{X}(\lambda) + M(\lambda)Q(\lambda)$, for $Q(\lambda)$ stable TFM, in the *Null* space of $N(\lambda)$. We will do that, by "subtracting" from the *Range* of $\widetilde{X}(\lambda)$, whatever is possible given the constraints on $Q(\lambda)$ and the fact that the factor $M(\lambda)$ is invertible, hence its columns span the ambient space. The outcome would be a strongly stabilizable controller, that at the same time would also have minimal McMillan degree.

For the general (not necessarily strong stabilizable) case, we can easily make use of the ideas above, for outlining what we hope to be a method for computing the minimum order controller. We envision this as a sequential method with two steps, both of them inspired from the strongly stabilizable synthesis above. First, we reduce the McMillan degree of the controller denominator's $\widetilde{Y}(\lambda)$. We do that by "subtracting" the *Range* of $N(\lambda)$ from the *Range* of $\widetilde{Y}(\lambda)$. As a second step, we minimize the degree of the controller's numerator $\widetilde{X}(\lambda)$ after all the Youla parameters $Q(\lambda)$ stable TFM, in the *Null* space of $N(\lambda)$. This leaves the numerator $\widetilde{Y}(\lambda)$ unchanged and in doing so it guarantees that what one does obtain is a very special

DCF of the plant where both the numerator and the denominator of the central controller have reduced their McMillan degrees. Whatismore, it holds true that for any Youla parameter, one would only obtain another controller for which with necessity, either one (or both) of controller's numerator or denominator have larger McMillan degrees. This nice property insinuates that (unless very special poles/zeroes cancelations occur) our specially computed DCF might yield the minimum order stabilizing controller.

Bibliography

[1] F. M. Brasch, Jr., and J. B. Pearson, ”Pole placement using dynamic compensators”, *IEEE Transactions Automatic Control*, vol. AC-15, pp. 34-43, Feb. 1970.

[2] Y. C. Ho and K. C. Chu, “Team decision theory and information structures in optimal control problems - Part I, *IEEE Transactions on Automatic Control* , Volume 17, No 1, 1972, Page(s): 15-22.

[3] Wang, S.H., Davison, E.J., A minimization algorithm for the design of linear multivari- able systems, *IEEE Trans on Automatic Control*, vol AC-18, no 3, June 1973, pp 220-225.

[4] Wang, S.H., Davison, E.J., On the stabilization of decentralized control systems, *IEEE Trans on Automatic Control*, vol AC-18, no 5, October 1973, pp 473-478

[5] J. P. Corfmat and A. S. Morse, Control of linear system through specified input channels, *SIAM J. Contr.*, vol. 14, Jan. 1976.

[6] J. P. Corfmat and A. S. Morse, “Decentralized control of linear multivariable systems”,*Automatica*, Vol. 12, Sept. 1976.

[7] Davison, E.J.,Wang, S.H., New results on the controllability and observability of general composite systems, *IEEE Trans on Automatic Control*, vol AC-20, no 1, 1975, pp 123-128.

[8] Wang, S.H., Davison, E.J.Minimization of transmission cost in decentralized control systems, *International J of Control*, vol 28, no 6, 1978, pp 889-896.

[9] Davison, E.J., The robust decentralized control of a servomechanism problem for com- posite systems with input-output interconnections, *IEEE Trans on Automatic Control*, vol AC-24, no 2, April 1979, pp 325-327.

[10] Davison, E.J., Gesing, W., Sequential stability and optimization of large scale decen- tralized systems, *Automatica*, vol 15, no 3, May 1979, pp 307-324.

[11] Davison, E.J., Ozguner, U., Synthesis of the decentralized robust servomechanism problem using local models, *IEEE Trans on Automatic Control*, vol AC-27, no 3, June 1982, pp 583-600

[12] Davison, E.J., Ozguner, U., Characterizations of decentralized fixed modes for inter- connected systems,*Automatica*, vol 19, no 2, 1983, pp 169-182

[13] Davison, E.J., Wang, S.H., A characterization of decentralized fixed modes in terms of transmission zeros, *IEEE Trans on Automatic Control*, vol AC-30, no 1, January 1985, pp 81-82.

[14] Vaz, A.F., Davison, E.J., A measure for the decentralized assignability of eigenvalues, *Systems & Control Letters*, vol 10, 1988, pp 191-199.

[15] Vaz, A.F., Davison, E.J., The structured robust decentralized servomechanism problem for interconnected systems, *Automatica*, vol 25, no 2, 1989, pp 267-272.

[16] Davison, E.J., Chang, T.N., Decentralized stabilization and pole assignment for general proper systems, *IEEE Trans on Automatic Control*, vol 35, no 6, June 1990, pp 652-664.

[17] Davison, E.J., Ozguner, U., Decentralized control of traffic networks, Joint Special Issue on Large Scale Systems: *IEEE Trans on Circuits and Systems*, vol CAS-30, no 6, June 1983, pp 364-375; *IEEE Trans on Automatic Control*, vol AC-28, no 6, June 1983, pp 677-688; *IEEE Trans on Systems, Man and Cybernetics*, vol SMC-13, no 4, July 1983, pp 476-487

[18] Iftar, A., Davison, E.J., Decentralized control strategies for dynamic routing, *Inter- national J Control*, 1998, vol 69, no 5, 1998, pp 599-632.

[19] Aghdam A., Davison, E.J., Pseudo-decentralized switching Control, *Automatica*, vol 39, No 2, Feb 2003, pp 317 - 324.

[20] Aghdam A.G., Davison E.J., Decentralized Switching Control for Hierarchical Systems, *Automatica*, vol 43, no. 6, 2007, pp 1092 - 1100

[21] M. Athans (Guest editor), *"Special issue on Large-Scale Systems and Decentralized Control,"* IEEE Transactions on Automatic Control, vol 23, no 2, April 1978.

[22] M. Vidyasagar, "Decomposition techniques for large-scale systems with nonadditive interactions: Stability and stabilizability," *IEEE Transactions on Automatic Control* , Volume 25, Issue 4, Page(s):773 - 779, 1980.

[23] B.D.O. Anderson, J.B.Moore "Time-Varying Feedback Laws for Decentralized Control", *IEEE Transactions on Automatic Control*, Vol. AC-26, No. 5, 1981.

[24] B.D.O. Anderson "Transfer Function Matrix Description of Decentralized Fixed Modes", *IEEE Transactions on Automatic Control*, Vol. AC-27, No. 6, 1982.

[25] A. Linnemann, "Decentralized control of dynamically interconnected systems", *IEEE Transactions on Automatic Control,* Volume 29, Issue 11, Page(s):1052 - 1054, 1984.

[26] C. H. Papadimitriou and J. N. Tsitsiklis, " Intractable problems in control theory,"*SIAM J. Control Optim.*, vol 24, 1986, Page(s) 639-654, 1986.

[27] V. D. Blondel and J. N. Tsitsiklis. "A survey of computational complexity results in systems and control". *Automatica*, 36(9):1249–1274, 2000.

[28] A. B. Ozguler "Decentralized control: A stable proper fractional approach", *IEEE Transactions on Automatic Control*, Vol. 35, No. 10, pp. 1109-1117, October 1990

[29] K. A. Unyelioglu and A. B. Ozguler "Decentralized stabilization: Characterization of all solutions and genericity aspects", *International Journal of Control*, Vol. 55, No. 6, pp. 1381-1403, 1992

[30] K. A. Unyelioglu, A. B. Ozguler, and P. P. Khargonekar, "Decentralized simultaneous stabilization and reliable control using periodic feedback", *Systems and Control Letters*, vol. 18, pp. 23-31, 1992.

[31] K. A. Unyelioglu and A. B. Ozguler "Reliable decentralized stabilization of feedforward and feedback interconnected systems", *IEEE Transactions on Automatic Control*, vol. AC-37, no. 8, pp. 1119-1132, 1992.

[32] P. P. Khargonekar and A. B. Ozguler "Decentralized control and periodic feedback", *IEEE Transactions on Automatic Control*, vol. AC-39, No. 4, pp. 877-881, April 1994.

[33] K. A. Unyelioglu, A. B. Ozguler, and U. Ozguner, "Decentralized blocking zeros and the decentralized strong stabilization problem", *IEEE Transactions on Automatic Control*, vol. AC-40, No. 11, pp. 1905-1918, November 1995.

[34] K. A. Unyelioglu, Umit Ozguner, and A. B. Ozguler, "Decentralized fixed zeros of decentralized control systems", *IEEE Transactions on Automatic Control*, Vol. 45, No. 1, pp. 146-151, January 2000.

[35] A. B. Ozguler, "Global stabilization via local stabilizing actions," *IEEE Transactions on Automatic Control*, Vol. 51, No. 12, pp. 530-533, 2006.

[36] Gundes, A.N.; "Coprime factorizations of parallel, cascade, feedback interconnections" *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications,* Volume 43, Issue 9, Sept. 1996 Page(s):813 - 814

[37] Gundes, A.N.; "Reliable decentralized stabilization of linear systems" *IEEE Transactions on Automatic Control,* Volume 43, Issue 12, Dec. 1998 Page(s):1733 - 1739

[38] Javad Lavaei and Amir G. Aghdam, A Graph Theoretic Method to Find Decentralized Fixed Modes of LTI Systems, *Automatica*, vol. 43, no. 2, pp. 2129-2133, Dec. 2007.

[39] Javad Lavaei and Amir G. Aghdam, Control of Continuous-Time LTI Systems by Means of Structurally Constrained Controllers, *Automatica*, vol. 44, no. 1, pp. 141-148, Jan. 2008.

[40] Javad Lavaei and Amir G. Aghdam, Stabilization of Decentralized Control Systems by Means of Periodic Feedback, *Automatica*, vol. 44, no. 4, pp. 1120-1126, Apr. 2008.

[41] C. Fan, J. L. Speyer, and C. R. Jaensch. "Centralized and decentralized solutions of the linear-exponential-gaussian problem", *IEEE Transactions on Automatic Control*, 39(10):1986–2003, 1994.

[42] P. G. Voulgaris. Control under structural constraints: An input-output approach. In *Lecture notes in control and information sciences*, pages 287–305, 1999.

[43] B. Bamieh and P. G. Voulgaris. "Optimal distributed control with distributed delayed measurements". *In Proceedings of the IFAC World Congress*, 2002.

[44] B. Bamieh, F. Paganini and M.A. Dahleh, *"Distributed control of spatially invariant systems,"* Automatic Control, IEEE Transactions on, Volume 47, Issue 7, Page(s): 1091-1107, 2002

[45] B. Bamieh and P.G. Voulgaris, *Optimal Distributed Control with Distributed Delayed Measurements,* in Proc. IFAC World Congress, 2002.

[46] G. A. de Castro and F. Paganini, *"Convex synthesis of localized controllers for spatially invariant systems,"* Automatica, vol. 38, pp. 445 - 456, 2002.

[47] R. Cogill, S. Lall, and P. A. Parrilo, *"On structured semidefinite programs for the control of symmetric systems,"* in Proc. Allerton Conf. Communication, Control, and Computing, 2003.

[48] R. D'Andrea and G.E. Dullerud, *"Distributed control design for spatially interconnected systems,"* Automatic Control, IEEE Transactions on, Volume 48, Issue 9, 2003, Page(s): 1478-1495.

[49] E. Klavins, R. M. Murray; *"Distributed algorithms for cooperative control,"* IEEE Pervasive Computing, Volume 3, Issue 1, Jan.-March 2004 Page(s):56 - 65

[50] J.A. Fax and R.M. Murray, *"Information flow and cooperative control of vehicle formations,"* Automatic Control, IEEE Transactions on, Volume 49, pp. 1465-1476, 2004.

[51] C. Langbort, R.S. Chandra and R. D'Andrea, *"Distributed control design for systems interconnected over an arbitrary graph,"* Automatic Control, IEEE Transactions on, Volume 49, Issue 9, Page(s) 1502-1519, 2004.

[52] X. Qi, M. Salapaka, P. Voulgaris and M. Khammash, *Structured Optimal and Robust Control with Multiple Criteria: A Convex Solution*, IEEE Trans. Aut. Control, Vol.49, No.10, pp 1623–1640, 2004.

[53] B. Recht and R. D'Andrea, *"Distributed Control of Systems over Discrete Groups"*, IEEE Transactions on Automatic Control, Vol 49, No 9, pp. 1446-52, 2004.

[54] J. S. Shamma and G. Arslan, *"A decomposition approach to distributed control of spatially invariant systems,"* IEEE Transactions on Automatic Control, Volume 51, Issue 4, Page(s):701 - 707, 2006 .

[55] M. Arcak. "Passivity as a design tool for group coordination", *IEEE Transactions on Automatic Control*, vol.52, no.8, pages 1380-1390, 2007.

[56] B. Francis " A Course in $\mathcal{H}_\infty$ Control Theory", *Series Lecture Notes in Control and Information Sciences*, New York: Springer-Verlag, 1987, vol. 88.

[57] K. Zhou, J.C. Doyle and K. Glover, " Robust and Optimal Control", *Upper Saddle River, NJ: Prentice Hall*, 1996.

[58] Doyle J., Glover K., Khargonekar P., and Francis B., "State–space solutions to standard $\mathcal{H}^2$ and $\mathcal{H}^\infty$ control problems", *IEEE Trans. Aut. Control*, Vol. 34, 1989. (pp. 831–847)

[59] V. M. Lucic " Doubly Coprime Factorization Revisited", *IEEE Trans. Aut. Control*, Vol.46, No.3, 2001. (pp. 457-459)

[60] C. Oară, A. Varga "Minimal Degree Factorization of Rational Matrices", *SIAM J. Matrix Anal Appl.*, Vol. 21, No.1, 1999. (pp. 245-277)

[61] G. Zames, "Feedback and Optimal Sensitivity: Model reference transformations, multiplicative seminorms and appropriate inverses", *IEEE Trans. Aut. Control*, Vol. 26, 1981. (pp. 301–320)

[62] V. Anantharam, C. Desoer, "On the Stabilization of Nonlinear Systems", *IEEE Trans. Aut. Control*, Vol. 29, 1984. (pp. 569–572)

[63] M. Vidyasagar "Control System Synthesis: A Factorization Approach", *MIT Press*, Signal Processing, Optimization, and Control Series, 1985.

[64] M. Jacobu, M. Jamshidi, C. Abdallah, P. Dorato, D. Bernstein "Suboptimal strong stabilization using fixed order dynamic compensator", American Control Conference, 1974. (pp 2659–2660)

[65] Y. Halevi, "Stable LQG Controllers", *IEEE Trans. Aut. Control*, Vol. 39, 1994. (pp. 2104–2106)

[66] Y. William Wang, D.S. Bernstein, "$\mathcal{H}^2$–suboptimal stable stabilization ", Proceedings. 32nd Conference on Decision and Control, 1993

[67] K. Mori and K. Abe, " Feedback Stabilization over Commutative Rings: Further Study of Coordinate–free Approach", *SIAM J. Control Optim.*, Vol. 39, No. 6, pp 1952-1973, 2001.

[68] K. Mori "Paramaetrization of Stabilizing Controllers over Commutative Rings with Applications to Multidimensional Systems", *IEEE Trans. Circuits Syst. I*, Vol. 49, 2002. (pp. 743-752)

[69] K. Mori "Relationship Between Standard Control Problem and Model–Matching Problem Without Coprime Factorizability", *IEEE Trans. Aut. Control*, Vol. 49, No.2, 2004. (pp. 230-233)

[70] K. Mori "Elementary Proof of Controller Parametrization Wihtout Coprime Factorizability", *IEEE Trans. Aut. Control*, Vol. 49, No. 4, 2004. (pp 589-592)

[71] S. Shankar, V.R. Sule, " Algebraic, Geometric Aspects of Feedback Stabilization", *SIAM J. Control, Optim.*, Vol. 30, 1992. (pp. 11–30)

[72] V.R. Sule, " Feedback Stabilization over Commutative Rings: The Matrix Case", *SIAM J. Control, Optim.*, Vol. 32, 1994. (pp. 1675–1695)

[73] K. Unyelioglu, B. Ozguler, U. Ozguner, "Decentralized Blocking Zeros and the Decentralized Strong Stabilization Problem", *IEEE Trans. Aut. Control*, Vol. 40, No. 11, 1995. (pp. 1905–1918)

[74] V.R. Sule, " Corrigendum: Feedback Stabilization over Commutative Rings: The Matrix Case", *SIAM J. Control, Optim.*, Vol. 36, 1998. (pp. 2194–2195)

[75] C.A. Desoer, R.W. Liu, J. Murray, R. Saeks, "Feedback Systems Design: The Fractional Representation Approach to Analysis and Synthesis", *IEEE Trans. Aut. Control*, Vol. 25, 1980. (pp. 399–412)

[76] Murray W. Wonham, "Linear Multivariable Control", *Springer-Verlag*, 1985.

[77] V.Ionescu, C.Oară, M. Weiss, "Generalized Riccati Theory and Robust Control: A Popov Function Approach", *John Wiley & Sons*, 2000.

[78] M. Rotkowitz, "Tractable Problems in Optimal Decentralized Control", *Ph.D. disertation, Stanford Univ., Stanford CA*, Jun 2005.

[79] M. Rotkowitz, S. Lall "A Characterization of Convex Problems in Decentralized Control", *IEEE Trans. Aut. Control*, Vol.51, No.2, 2006. (pp. 274-286)

[80] M. Rotkowitz, S. Lall "Affine Controller Parametrization for Decentralized Control Over Banach Spaces", *IEEE Trans. Aut. Control*, Vol.51, No.9, 2006. (pp. 1497-1500)

[81] M. Vidyasagar, H. Schneider, B. Francis "Algebraic and Topological Aspects of Feedback Stabilization", *IEEE Trans. Aut. Control*, Vol. 27, No. 4, 1982. (pp. 880-894)

[82] M. Vidyasagar, "Decomposition techniques for large-scale systems with nonadditive interactions: Stability and stabilizability," *IEEE Transactions on Automatic Control* , Volume 25, Issue 4, Page(s):773 - 779, 1980.

[83] "`Matlab` User Guide", The Mathworks Inc., 2009

[84] Ş. Sabău, N.C. Martins, M.C. Rotkowitz, "A convex parameterization of all stabilizing controllers for non-strongly stabilizable plants, under quadratically invariant sparsity constraints", *I.S.R. Technichal Report*, available at

`http://www.lib.umd.edu/drum/bitstream/`

`/1903/8688/3/NonStrStabEdited.pdf`

[85] C. N. Nett, C. A. Jacobson, and M. J. Balas, “A connection between state-space and doubly coprime fractional representations” *IEEE Trans. Automat. Control*, Vol. AC-29, 1984. (pp. 831-832)

[86] G. Verghese, P. Van Dooren, T. Kailath, “Properties of the System Matrix of a Generalized State–Space System”, *Int. Journal Control*, Vol. 30, 1979. (pp 235-243)

[87] H.H. Rosenbrock, “ State-space and Multivariable Theory: Studies in Dynamical Systems” (ISBN: 0177810025 / 0-17-781002-5) Editors: Brockett, Roger and Rosenbrock, H.H., 1970.

[88] Alan J. Laub ”Matrix Analysis for Scientists and Engineers” SIAM

`www.ec-securehost.com/SIAM/ot91.html`

[89] R. E. Kalman, “Canonical Structure of Linear Dynamical Systems”, *Research Institute for Advanced Studies, Baltimore, Maryland*. Communicated by S. Lefschetz, January 23, 1961.

[90] C.A. Desoer, M.J. Chen “Design of Multivariable Feedback Systems with Stable Plant”, *IEEE Trans. Aut. Control*, Vol. 26, No. 2, 1981. (pp 408 – 415)

[91] Weiss. G. “Transfer functions of regular linear systems, Part I; characterizations of regularity.” *Trans. Am. Math. Sot.* 342, 827-854. 1994

[92] Weiss, G. “Regular linear systems with feedback.” *Math. Control, Sig. Syst.*, 7, 23-37., 1994

[93] R. F. Curtain, G. Weiss, and M. Weiss, Coprime factorization of regular linear systems, *Automatica*, Vol.32, No. 11, 1996.

[94] G. Weiss, R. Curtain, “Dynamic Stabilization of Regular Linear Systems”, *IEEE Trans. Aut. Control*, Vol. 42, No.1 1997.

[95] O.J. Staffans “Qudratic, Optimal Control of well–posed linear Systems”, SIAM J. Contr, Optim. 1999.

[96] N. Sebe “Unification of Independent and Sequential Procedures for Decentralized Controller Design”, *Automatica*, Vol. 43, 2007 (pp. 707-713)

[97] R. Date, J. Chow “Decentralized, Stable factors and a parametrization of Decentralized Controllers”, IEEE Trans. Aut. Control, Vol. 39, No. 2, 1994.

[98] K. Braman, R. Byers, R. Mathias "The multishift QR algorithm. Part I: maintaining well-focused shifts and level 3 performance." *SIAM J. Matrix Anal. Appl.*, 23, 929-947, 2002

[99] K. Braman, R. Byers, R. Mathias "The multishift QR algorithm. Part II: aggressive early deflation." *SIAM J. Matrix Anal. Appl.*, 23, 948-973, 2002

[100] G.H. Golub., S. Nash, C.F. Van Loan, "A Hessenberg-Schur method for the problem AX+ XB = C." *IEEE Trans. Aut. Control.* AC-24, 6 1979, (pp 909–913).

[101] P.S.S.N.V.P. Rao, K.P.S.B. Rao "On Generalized Inverses of Boolean Matrices", *Linear Alg. Applications* 11, No. 2, 1975.

[102] K.P.S.B. Rao, P.S.S.N.V.P. Rao "On Generalized Inverses of Boolean Matrices - II", *Linear Alg. Applications* 42, 1982.

[103] A.N. Gundes and H. Ozbay, "Strong Stabilization of MIMO Systems with Restricted Zeros in the Unstable Region", *Proc. 47th IEEE Conference on Decision and Control*, pp. 2220-2225, 2008.

[104] A. E. Barabanov, Design of H optimal stable controller, *Proc. IEEE Conference on Decision and Control*, pp. 734738, 1996.

[105] D. U. Campos-Delgado, K. Zhou, A parametric optimization ap- proach to $\mathcal{H}^2$ and $\mathcal{H}^\infty$ strong stabilization, *Automatica*, vol. 39, no. 7, pp. 1205-1211, 2003.

[106] P. Cheng, Y.-Y. Cao, and Y. Sun, On strong $\gamma_k?\gamma_{cl}\mathcal{H}^\infty$ stabilization and simultaneous $\gamma_k\gamma_{cl}\mathcal{H}^\infty$ control, *Proc. 46th IEEE Conference on Decision and Control*, pp. 5417-5422, 2007.

[107] Y. S. Chou, J. L. Leu, and Y. C. Chu, Stable controller design for MIMO systems: An LMI approach, *IET Control Theory Appl.*, Vol. 1, pp. 817-829, 2007.

[108] P. Dorato, H. Park, and Y. Li, An algorithm with interpolation in units in $\mathcal{H}^\infty$, with applications to feedback stabilization, *Automatica*, 25, pp. 427-430, 1989.

[109] S. Gumussoy, H. Ozbay, Remarks on strong stabilization and stable $\mathcal{H}^\infty$ controller design, *IEEE Transactions on Automatic Control*, vol. 50, pp. 2083-2087, 2005.

[110] Y. Halevi, Stable LQG controllers, *IEEE Transactions on Automatic Control*, vol. 39, pp. 2104-2106, 1994.

[111] A. N. Gundes , C. A. Desoer, Algebraic Theory of Linear Feedback Systems with Full and Decentralized Compensators, Lect. Notes in Contr. and Inform. Sciences, 142, Springer, 1990.

[112] H. Ozbay, A. N. Gundes , Strongly stabilizing controller synthesis for a class of MIMO plants, *17th IFAC World Congress*, pp. 359-363, Seoul, Korea, 2008.

[113] I. Petersen, Robust $\mathcal{H}^\infty$ control of an uncertain system via a strict bounded real output feedback controller, *Proc. 45th IEEE Conference on Decision and Control*, pp. 571-577, 2006.

[114] M. C. Smith, K. P. Sondergeld, On the order of stable compensators, *Automatica*, vol. 22, pp. 127-129, 1986.

[115] M. Zeren, H. Ozbay, On the synthesis of stable H controllers, *IEEE Transactions on Automatic Control*, vol. 44, pp. 431-435, 1999.

[116] M. Zeren, H.Ozbay, On the strong stabilization and stable H- controller design problems for MIMO systems, *Automatica*, vol. 36, pp. 1675-1684, 2000.

[117] C. Oară, Ş. Sabău, "Minimal Indices Cancellation and Rank Revealing Factorizations for Rational Matrix Functions",*Linear Algebra and Its Applications*, vol. 431(10), October 2009, pp. 1785-1814