TECHNICAL RESEARCH REPORT

Modulation and Sampling Techniques for Multichannel Deconvolution

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T.R. 96-54

Sponsored by
the National Science Foundation
Engineering Research Center Program,
the University of Maryland,
Harvard University,
and Industry

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June 3, 1996

[1]Research partially supported by AFOSR Grant F49620-94-1-0196.

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Abstract

The problem of recovering information from single component linear translation invariant systems is inherently ill-posed. However, ill-posedness may be circumvented in a multichannel system if the components of the system satisfy the strongly coprime condition. The signal may be completely recovered from a strongly coprime system by filtering the output of each channel with a deconvolver, and adding. This approach has been labeled *multichannel deconvolution.*

We explore the state of the art for the multichannel theory. We then use sampling theory to develop an alternative method for creating these systems. Modulation techniques are then used to create a strongly coprime system, for which the corresponding deconvolvers are developed. We close by discussing several applications of the theory.

1 Introduction

Linear, translation invariant systems (e.g., sensors, linear filters) are modeled by the convolution equation $s = f * \mu$, where f is the input signal, μ is the system impulse response function (or, more generally, impulse response distribution), and s is the output signal. We refer to μ as a convolver. In many applications, the output s is an inadequate approximation of f, which motivates solving the convolution equation for f, i.e., deconvolving f from μ. If the function μ is time-limited (compactly supported) and non-singular, we have shown that this deconvolution problem is ill-posed in the sense of Hadamard (see [16]).

A theory of solving such equations has been developed. It circumvents ill-posedness by using a multichannel system. If we overdetermine the signal f by using a *system* of convolution equations,

$$s_i = f * \mu_i \ , \ i = 1, \ldots , n \, , \tag{1}$$

the problem of solving for f is well-posed if the set of convolvers $\{\mu_i\}$ satisfies the condition of being what we call *strongly coprime.* In this case, there exist compactly supported distributions (deconvolvers)

$$\nu_i \ , \ i = 1, \ldots , n$$

such that

$$\widehat{\mu_1} \cdot \widehat{\nu_1} + \ldots + \widehat{\mu_n} \cdot \widehat{\nu_n} = 1 \, . \tag{2}$$

Transforming, we get

$$\mu_1 * \nu_1 + \ldots + \mu_n * \nu_n = \delta \, , \tag{3}$$

which in turn gives

$$s_1 * \nu_1 + \ldots + s_n * \nu_n = f \, . \tag{4}$$

The theory of deconvolution presented in this paper is contained in a larger group of results in the theory of residues of analytic functions and their generalities, for example, intersection varieties. These results have appeared in a series of papers by Berenstein, Gay, Taylor, Yger et al. (see [2] – [17]), and can be interpreted as results in division problems, interpolation of analytic functions, analytic continuation, digital to analog conversion, and complexity theory. For deconvolution and other applications to signal and image processing, the theory focuses on solving the *analytic Bezout equation*, i.e., for given holomorphic f_i and ϕ satisfying certain growth conditions, solving for holomorphic g_i satisfying the same growth conditions such that

$$f_1 \cdot g_1 + \ldots + f_n \cdot g_n = \phi \, . \tag{5}$$

For our purposes, we want growth conditions given by the Paley-Wiener-Schwartz Theorem and $\phi = \phi_\lambda$, with $\phi_\lambda \longrightarrow 1$ as $\lambda \longrightarrow \infty$ (ϕ_λ is the transform of an approximate identity).

Solutions to Bezout equations have yielded results in deconvolution, complexity theory, solutions to systems of PDE's, theorems about interpolation and continuation of analytic functions, and results in number theory (see [2] – [17]).

We describe in section 2.1 the strongly coprime condition, and we give examples of sets of strongly coprime system response functions and their deconvolutions for functions in one and several variables. In the language of applications, the set of convolvers $\{\mu_i\}$ models a linear translation invariant multichannel system consisting of an array of sensors or filters. The system is created so that *no* information contained in the input signal f is lost. The signal f is gathered by this system as $\{s_i = f * \mu_i\}$. The signals s_i are then filtered by the ν_i (which have been created digitally, optically, etc., in coordination with the creation of the system and possibly tailored to be optimized under some constraint) and added, resulting in the reconstruction of f. We discuss the various classes of impulse response functions modeled by the theory.

Section 2.2 gives the development of a real-variable method of solving Bezout in a specific case. Using Shannon sampling in the frequency domain, this development utilizes the zero sets of the $\{\widehat{\mu_i}\}$ as two different sampling rates. Section 3 contains a new method for creating strongly coprime systems. In particular, a given fixed convolver is modulated a certain amount, resulting in a strongly coprime pair. The deconvolvers associated with this system are then created. This approach gives a new technique for constructing multichannel systems. Section 3.2 gives a development of these deconvolvers using sampling.

We close in section 4 with a discussion of applications of the multichannel theory to signal and image processing. In a system designed to do signal or image processing, the deconvolvers could act as preliminary image or signal enhancement operators, restoring high frequency information. These operators can work simultaneously with other operators which analyze the image or signal, for example, wavelet operators (see [16]). The work with wavelet analysis in image processing looks particularly promising. Deconvolution will restore the high frequency, fine-detail of the image. Wavelet methods can then be used for image compression and analysis, e.g., for edge detection, motion detection, feature extraction, etc.

We need some background information. For an integrable function f, we define the Fourier transform $\widehat{f}(\omega)$ as

$$\widehat{f}(\omega) = \int_{\mathbb{R}^n} f(t)\, e^{-2\pi i \langle t,\omega\rangle}\, dt\,,$$

for $t \in \mathbb{R}^n$ (time), $\omega \in \widehat{\mathbb{R}^n}$ (frequency)[1]. Given an integrable function g on $\widehat{\mathbb{R}^n}$, the inverse transform is

$$\check{g}(t) = \int_{\widehat{\mathbb{R}^n}} g(\omega)\, e^{2\pi i \langle \omega,t\rangle}\, d\omega\,.$$

If $f(t)$ is a compactly supported function, $\widehat{f}(\omega)$ is an analytic function. Moreover, $\widehat{f}(\omega)$ analytically continues to all of n-dimensional complex Euclidean space $\mathbb{C}^n$. This continuation gives the Fourier-Laplace transform of f, which is defined by

$$\widehat{f}(\zeta) = \int_{\mathbb{R}^n} f(t)\, e^{-2\pi i \langle t,\zeta\rangle}\, dt\,,\ \zeta \in \mathbb{C}^n.$$

[1]We define the Fourier transform in this way so that $\|f\|_2 = \|\widehat{f}\|_2$. Some "standard formulae", e.g., Shannon sampling as presented in [28], have been appropriately adjusted.

(See [19], [24], [28], [29] for further reference.)

The process of deconvolution uses some of the basic theory of distributions. Let $\mathbb{N}$ denote the natural numbers, and let $\mathbb{Z}$ denote the integers. We define three sets of test functions,

$$\mathcal{D}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n) = \{\phi \in C^\infty(\mathbb{R}^n) \text{ with compact support}\},$$

$$\mathcal{S}(\mathbb{R}^n) = \{\phi \in C^\infty(\mathbb{R}^n) \text{ with } \lim_{|t|\to\infty} \left| t^n \frac{d^k}{dt^k}\phi(t) \right| = 0\} \text{ , for all } n, k \in \mathbb{N}, \text{ and}$$

$$\mathcal{E}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n).$$

Each of these has an associated space of distributions, denoted $\mathcal{E}'(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$, and $\mathcal{D}'(\mathbb{R}^n)$. The action of a distribution T on a test function ϕ is denoted by $\langle T; \phi \rangle$. The more restrictive the notion of convergence on the space of test functions, the broader the class of distributions. We have that

$$\mathcal{E}' \subset \mathcal{S}' \subset \mathcal{D}'.$$

The Fourier-Laplace transform is defined on $\mathcal{S}'$ and $\mathcal{E}'$, the classes of tempered and compactly supported distributions, respectively. The sets of transforms are denoted by $\widehat{\mathcal{S}'}$ and $\widehat{\mathcal{E}'}$. For $\lambda \in \mathcal{E}'$, the Fourier transform continues as an analytic function to all of $\mathbb{C}^n$. This continuation defines the Fourier-Laplace transform on $\mathcal{E}'$, which is given by

$$\widehat{\lambda} = \left\langle \lambda; e^{-2\pi i \langle t, \zeta \rangle} \right\rangle.$$

Properties of the function $\widehat{\lambda}$ are given in the following.

Theorem 1.1 (Paley-Wiener-Schwartz (*PWS*)[1]) a.) *The Fourier-Laplace transform of an infinitely differentiable function f with compact support $\subseteq \{|t| \leq A\} \subset \mathbb{R}^n$ is an entire function $\widehat{f}(\zeta)$ in $\mathbb{C}^n$ which satisfies the following property:*

(P_1) *For every integer $N \geq 0$ we can find a positive constant $C = C(N)$ such that*

$$\left|\widehat{f}(\zeta)\right| \leq C(1 + |\zeta|)^{-N} e^{2\pi A |\Im \zeta|} \text{ for all } \zeta \in \mathbb{C}^n. \tag{6}$$

Conversely, every entire function in $\mathbb{C}^n$ satisfying property (P_1) is the Fourier-Laplace transform of a C^∞ function with compact support $\subseteq \{|t| \leq A\}$ in $\mathbb{R}^n$.

b.) *The Fourier-Laplace transform of a distribution λ with compact support $\subseteq \{|t| \leq A\} \subset \mathbb{R}^n$ is an entire function $\widehat{\lambda}(\zeta)$ in $\mathbb{C}^n$ which satisfies the following property:*

(P_2) *There is a positive constant C and an integer $N \geq 0$ such that*

$$\left|\widehat{\lambda}(\zeta)\right| \leq C(1 + |\zeta|)^{N} e^{2\pi A |\Im \zeta|} \text{ for all } \zeta \in \mathbb{C}^n. \tag{7}$$

Conversely, every entire function in $\mathbb{C}^n$ satisfying property (P_2) is the Fourier-Laplace transform of a distribution λ with compact support $\subseteq \{|t| \leq A\}$ in $\mathbb{R}^n$.

(See [1], [24] for further reference.)

2 Deconvolution from a System of Convolution Equations

2.1 An Overview of the Theory

We begin by giving a framework of realistic deconvolution problems. For these problems, our input data is μ and $f * \mu$, and our proposed solution is f. For a fixed μ, we have an associated convolution operator $\mathcal{C}_\mu(f) = f * \mu$. If $\mathcal{C}_\mu$ is injective, then the inverse, or deconvolution operator, is $\mathcal{D}_\mu(f * \mu) = f$. The deconvolution problems we consider are for convolvers μ which are realistic mathematical models of the impulse response functions of linear translation invariant systems. Therefore, we exclude distributions which are not compactly supported (since one would have to integrate for all time to get any information from such a system), distributions of order $k \geq 1$ (since one would have to impose smoothness conditions on any input functions), and any measures that are singular with respect to Lebesgue measure (since such a system is impossible to build). If μ is realizable, then the mapping $\mathcal{C}_\mu \colon L^p(\mathbb{R}^n) \longrightarrow \mathcal{R}_\mu \subset L^p(\mathbb{R}^n)$, where $\mathcal{R}_\mu$ is the range of $\mathcal{C}_\mu$, is continuous and linear. The most general convolver we will consider is a "well-behaved" measure as defined below.

Definition 2.1 *The distribution μ is a* **realizable convolver** *if μ is a compactly supported finite Borel measure which is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^n$.*

The Radon-Nikodym Theorem gives us that all realizable deconvolvers act as compactly supported L^1 functions. The set of realizable convolvers also includes the set of non-singular probability measures of compact support π. Given such a measure, the Radon-Nikodym derivative is the probability density function of π.

We have shown the following.

Theorem 2.1 ([16]) *Let $\mu(t)$ be a realizable convolver. Then for $f \in C(\mathbb{R}^n)$, the convolution operator $\mathcal{C}_\mu(f) = f * \mu$ is not injective. Therefore, the deconvolution problem of recovering f from $f * \mu$ is ill-posed in the sense of Hadamard.*

Remark : The method of proof for the theorem above gives an indication of the "degree of ill-posedness" for inverting a given convolution operator. The procedure involves examining the set of all linear combinations of functions g such that $\text{supp}(\widehat{g}) \subseteq \mathcal{Z}_\mu$, where $\mathcal{Z}_\mu$ is the variety defined by the zero set of $\widehat{\mu}$, the Fourier-Laplace transform of μ. If μ is realizable, $\mathcal{Z}_\mu$ is not empty. Functions g as above will be in the kernel of the operator. The signals modeled by these functions will all be undetectable by the sensor modeled by μ. For example, in one variable, given any $\zeta_0 = \omega_0 + i\eta_0 \in \mathcal{Z}_\mu$, $-\overline{\zeta}_0 \in \mathcal{Z}_\mu$, and so $g(t) = e^{-2\pi\eta_0 t}\cos(2\pi\omega_0 t)$, $h(t) = e^{-2\pi\eta_0 t}\sin(2\pi\omega_0 t)$ are functions such that $\text{supp}(\widehat{g}(\zeta))$ and $\text{supp}(\widehat{h}(\zeta))$ are contained in $\mathcal{Z}_\mu$. Moreover, as there are infinitely many pairs $\{\zeta, -\overline{\zeta}\}$ of zeros in $\mathcal{Z}_\mu$, there are infinitely many such functions (see [16]).

If $f \in L^2(\mathbb{R}^n)$, we have that $\mathcal{C}_\mu \colon L^2(\mathbb{R}^n) \longrightarrow \mathcal{R}_\mu \subset L^2(\mathbb{R}^n)$ is injective with $\|f * \mu\|_2 \leq \|f\|_2\|\mu\|$. We have shown that the deconvolution operator $\mathcal{D}_\mu \colon \mathcal{R}_\mu \longrightarrow L^2(\mathbb{R}^n)$, defined by

$\mathcal{D}_\mu(f * \mu) = f$, is discontinuous. We do this by building a sequence $\{f_n\}$ of $L^2(\mathbb{R}^n)$ functions such that $\|f_n\|_2 \longrightarrow \infty$ as $n \longrightarrow \infty$, but $\|f_n * \mu\|_2 \longrightarrow 0$ as $n \longrightarrow \infty$, for any fixed realizable convolver μ. Thus, this deconvolution problem violates the third condition of well-posedness.

Theorem 2.2 ([16]) *Let μ be a realizable convolver. Then for $f \in L^2(\mathbb{R}^n)$, the deconvolution operator $\mathcal{D}_\mu(f * \mu) = f$ is an unbounded and therefore discontinuous linear operator. Thus, the deconvolution problem of recovering f from $f * \mu$ is ill-posed in the sense of Hadamard.*

These theorems generalize in the following way. If μ is realizable, and $f \in L^p(\mathbb{R}^n)$, then the deconvolution operator $\mathcal{D}_\mu(f * \mu) = f$ is an unbounded and therefore discontinuous linear operator.

We have constructed solutions to the deconvolution problem for various classes of compactly supported convolvers. We have assumed only that our input or initial functions f are of finite energy. Ill-posedness is circumvented by creating a multichannel system. Each channel in the system is a convolver, and the system overdetermines the input function with all of the convolvers in an array chosen so that any information lost by one convolver is retained by another. The theory of deconvolution discussed in the following paragraphs has its roots in the work of Wiener [35] and Hörmander [25], and has been developed into a working theory by Berenstein, Taylor, Gay, Yger, et al. [2] – [17]. These methods are both linear (convolution with deconvolvers) and realizable (the support of the deconvolvers being contained in the bounded support of the kernels of the convolution equations). Thus, deconvolution at a point $t \in \mathbb{R}^n$ depends only on data near t. The theory assumes no *a priori* information about the input signals. Moreover, the theory can be used to develop a stable system for complete signal recovery.

The theory starts with the following existence theorem of Hörmander.

Theorem 2.3 (Hörmander [25]) *For the compactly supported distributions $\{\mu_i\}_{i=1}^n$ on $\mathbb{R}^n$, there exist compactly supported distributions $\{\nu_i\}_{i=1}^n$ such that*

$$\mu_1 * \nu_1 + \ldots + \mu_n * \nu_n = \delta$$

if and only if there exist positive constants A and B and a positive integer N such that

$$\left(\sum_{i=1}^{n} |\widehat{\mu_i}(\zeta)|^2\right)^{\frac{1}{2}} \geq Ae^{-2\pi B|\Im\zeta|}(1 + |\zeta|)^{-N}\,,\ \zeta \in \mathbb{C}^n\,. \tag{8}$$

Definition 2.2 *A set of convolvers $\{\mu_i\}_{i=1}^n$ that satisfy the inequality in the theorem is said to be* **strongly coprime**.

We note that the only way a single compactly supported convolver can be strongly coprime is for it to be a translate of the identity convolver; that is, for it to be the Dirac delta or a translation thereof. This result is one way to state the general ill-posedness of single convolution equations under the constraints imposed by the conditions of the theorem.

Definition 2.3 *Let $\{F_i\}_{i=1}^n$ be a given set of functions in $\widehat{\mathcal{E}'}(\mathbb{R}^n)$. A solution to the* **analytic Bezout equation**

$$\sum_i F_i(\zeta)G_i(\zeta) = 1 \tag{9}$$

is a set $\{G_i\}_{i=1}^n$ in $\widehat{\mathcal{E}'}(\mathbb{R}^n)$ that satisfies the equation.

By PWS and basic properties of the Fourier-Laplace transform, a solution to the analytic Bezout equation is equivalent to solving for a set $\{\nu_i\}$ in $\mathcal{E}'$ such that $\sum_i \mu_i * \nu_i = \delta$, for a given set $\{\mu_i\}$ in $\mathcal{E}'$. By Hörmander's theorem, a strongly coprime set $\{\mu_i\}$ is precisely a set for which the analytic Bezout equation has a solution. The strongly coprime condition guarantees not only that the transforms of the convolvers have no common zeros, but also that these zeros do not cluster too quickly as $|\zeta| \longrightarrow \infty$. Thus, if a given signal f is overdetermined by a strongly coprime system of convolution equations,

$$s_i = f * \mu_i \ , \ i = 1, \ldots, n \, ,$$

then the problem of solving for f is well-posed. We solve for a set $\{\widehat{\nu_i}(\zeta)\}$ of deconvolvers which satisfy the analytic Bezout equation

$$\sum_i \widehat{\mu_i}\widehat{\nu_i} = 1.$$

Taking inverse transforms of both sides of Bezout gives

$$\sum_i \mu_i * \nu_i = \delta \, .$$

The fact that the strongly coprime condition is the inversion of the PWS growth bound allows us to solve for deconvolvers that are compactly supported. This in turn yields

$$\begin{aligned} & s_1 * \nu_1 + \ldots + s_n * \nu_n \\ = \ & (f * \mu_1) * \nu_1 + \ldots + (f * \mu_n) * \nu_n \\ = \ & f * (\mu_1 * \nu_1) + \ldots + f * (\mu_n * \nu_n) \\ = \ & f * (\mu_1 * \nu_1 + \ldots + \mu_n * \nu_n) \\ = \ & f * \delta \\ = \ & f. \end{aligned}$$

Thus, the deconvolution problem can be solved by constructing the Dirac δ for a given class of convolvers. This construction begins with a solution to the analytic Bezout equation. The inverse transforms of these solutions are compactly supported distributions.

To construct compactly supported deconvolving *functions*, we begin by solving a more general analytic Bezout equation, i.e., for given analytic $\widehat{\mu_i}$ and $\widehat{\psi}$ satisfying PWS growth conditions, solving for analytic $\widehat{\nu_i}$ satisfying PWS growth conditions such that

$$\widehat{\mu_1} \cdot \widehat{\nu_1} + \ldots + \widehat{\mu_n} \cdot \widehat{\nu_n} = \widehat{\psi} \, .$$

Moreover, we want $\widehat{\psi} = \widehat{\psi_\lambda}$, with $\widehat{\psi_\lambda} \longrightarrow 1$ as $\lambda \longrightarrow \infty$ ($\widehat{\psi_\lambda}$ is the transform of an approximate identity). This gives us deconvolving functions, i.e., deconvolvers $\{\nu_{i,\psi}\}$ such that $\mu_1 * \nu_{1,\psi} + \ldots + \mu_n * \nu_{n,\psi} = \psi$, which in turn give

$$(f * \mu_1) * \nu_{1,\psi} + \ldots + (f * \mu_n) * \nu_{n,\psi} = f * \psi = f_\psi \, .$$

Then, as $\psi \longrightarrow \delta$, $f_\psi \longrightarrow f$ in the sense of distributions. The deconvolvers in these implementable formulae are periodic functions expressed in their Fourier series expansions.

We give an example. Let $t \in \mathbb{R}$, p be a prime number, and let

$$\mu_1(t) = \chi_{[-1,1]}(t) \ , \ \mu_2(t) = \chi_{[-\sqrt{p},\sqrt{p}]}(t)$$

model the impulse response of the channels of a two-channel system. Thus

$$\widehat{\mu_1}(\zeta) = \frac{\sin(2\pi\zeta)}{\pi\zeta} \ , \ \widehat{\mu_2}(\zeta) = \frac{\sin(2\pi\sqrt{p}\zeta)}{\pi\zeta} \, .$$

Let

$$\mathcal{Z}_1 = \left\{\frac{\pm k}{2}\right\} \ , \ \mathcal{Z}_2 = \left\{\frac{\pm k}{2\sqrt{p}}\right\} \ , \ k \in \mathbb{N}$$

denote the zero sets of $\widehat{\mu_1}(\zeta)$, $\widehat{\mu_2}(\zeta)$, respectively. An examination of the Fourier-Laplace transforms $\widehat{\mu_i}(\zeta)$, $i = 1, 2$, gives that $\{\mu_i\}$ is strongly coprime (see [6]). We choose an arbitrarily close approximation ψ of the Dirac δ based on certain criteria, i.e., ψ in C^4 with support in $(-(1+\sqrt{p}), (1+\sqrt{p}))$. Then $|\widehat{\psi}(z)| \leq \frac{d}{(1+|z|)^4}$ for $z \in \mathcal{Z}_1 \cup \mathcal{Z}_2$. The smoothness and the size of the support of ψ guarantee that the deconvolvers are compactly supported.

Theorem 2.4 ([16]) *The set*

$$\mu_1(t) = \chi_{[-1,1]}(t) \ , \ \mu_2(t) = \chi_{[-\sqrt{p},\sqrt{p}]}(t)$$

is a strongly coprime pair of convolvers. Let $f \in L^2(\mathbb{R})$ and let ψ be a C^4 function with support in $(-(1+\sqrt{p}), (1+\sqrt{p}))$ such that $\psi \geq 0$ and $\int_{-\infty}^{\infty} \psi(t)\, dt = 1$. The deconvolvers $\nu_{i,\psi}$ such that

$$f * \psi = (f * \mu_1) * \nu_{1,\psi} + (f * \mu_2) * \nu_{2,\psi}$$

are given by the formulae

$$\nu_{1,\psi}(t) = \sum_{z \in \mathcal{Z}_2} \frac{\widehat{\psi}(z)}{\widehat{\mu_1}(z) \frac{d}{d\zeta} \widehat{\mu_2}(z)} \left(\frac{1}{z} \left(e^{2\pi i z(t+\sqrt{p})} - 1 \right) \chi_{[-\sqrt{p},\sqrt{p}]}(t) \right) , \tag{10}$$

$$\nu_{2,\psi}(t) = \sum_{z \in \mathcal{Z}_1} \frac{\widehat{\psi}(z)}{\widehat{\mu_2}(z) \frac{d}{d\zeta} \widehat{\mu_1}(z)} \left(\frac{1}{z} \left(e^{2\pi i z(t+1)} - 1 \right) \chi_{[-1,1]}(t) \right) . \tag{11}$$

The function $f * \psi$ is an arbitrarily close approximation of f which converges to f in the sense of distributions as $\mathrm{supp}(\psi) \longrightarrow \{0\}$.

The deconvolvers in Theorem 2.4 were developed by using the Jacobi interpolation formula and the Cauchy residue calculus in the complex plane (see [16]). It is also possible to use real-variable methods, in particular Shannon sampling, to create the $\nu_{i,\psi}(t)$. This is done in section 2.2.

This development works for other classes of convolvers and filters. The current stock of convolvers and their associated deconvolvers includes characteristic functions of squares and (hyper)cubes (see [3], [13], [16]), and characteristic functions of disks and n-dimensional balls (see [4], [5]). The theory has been expanded in one variable to more general convolvers, including convolvers modeled by linear combinations of characteristic functions, linear combinations of n-fold convolutions of characteristic functions with equally space knots (cardinal splines), and truncated sinc, cosine, and Gaussian functions (see [17]). We have shown the conditions for a strongly coprime set of convolvers $\{\mu_i\}$ for each of these types of functions, which, as in the example above, is a condition on the zero sets of the Fourier-Laplace transforms $\{\widehat{\mu_i}\}$. We then have solved for deconvolvers $\{\nu_{i,\psi}\}$ such that $\mu_1 * \nu_{1,\psi} + \ldots + \mu_n * \nu_{n,\psi} = \psi$, where ψ is an approximate convolution identity, by solving the modified Bezout equation $\widehat{\mu_1} \cdot \widehat{\nu_{1,\psi}} + \ldots + \widehat{\mu_n} \cdot \widehat{\nu_{n,\psi}} = \widehat{\psi}$.

In several variables, the formulae for the deconvolvers is simplified by not only solving for an approximation to the δ, but also by strengthening the strongly coprime condition. Let

$$\mathcal{Z} = \{z \in \mathbb{C}^n : \widehat{\mu_1}(z) = \ldots = \widehat{\mu_m}(z) = 0\} .$$

We say that $\mathcal{Z}$ is *almost real* if there exits a constant $\alpha > 0$ such that

$$\mathcal{Z} \subseteq \{z \in \mathbb{C}^n : |\Im z| \leq \alpha \log(2 + |z|)\} .$$

An almost real set is discrete (see [3]). Let $B_r = \{|z| \leq r\}$. Define the counting function

$$n(\mathcal{Z}, r) = \operatorname{card}\{\mathcal{Z} \cap B_r\} .$$

Also define the distance function

$$d(z, \mathcal{Z}) = \min(1, \min\{|z - \zeta| : \zeta \in \mathcal{Z}\}) .$$

Let $H_1(\theta)$ denote the supporting function of the convex hull of $\cup \operatorname{supp} \mu_i$, i.e., for $\theta \in \mathbb{R}^n$,

$$H_1 = \max_j \max\{\langle x, \theta \rangle : x \in \operatorname{supp} \mu_j\} .$$

Definition 2.4 ([3]) *A family of m distributions of compact support in $\mathbb{R}^n$ is* **well behaved** *if there exist positive constants A, B, N, κ and a supporting function H_0 with $0 \leq H_0 \leq H_1$ such that $\mathcal{Z}$ is almost real,*

$$n(\mathcal{Z}, r) = O(r^A) ,$$

and

$$\left(\sum_{i=1}^{m} |\widehat{\mu_i}(\zeta)|^2 \right)^{\frac{1}{2}} \geq B d(z, \mathcal{Z})^{\kappa} e^{H_0(\Im z)} (1 + |\zeta|)^{-N} . \tag{12}$$

Definition 2.5 ([3]) *A well behaved system* $\{\mu_i\}_{i=1}^m$ *is called* **very well behaved** *if there exist positive constants* β *and* M *such that for every* $z \in \mathcal{Z}$, *we have*

$$|J(z)| \geq \beta(1+|z|)^{-M}, \tag{13}$$

where $|J(z)| = \left|\det\left[\frac{\partial\widehat{\mu_i}}{\partial z_j}\right]\right|$.

For a very well behaved system, the zeros of the $\widehat{\mu_i}$ are simple. For convolvers in n-dimensional Euclidean space $\mathbb{R}^n$, we have that a very well behaved system needs only $n+1$ elements.

Theorem 2.5 ([3]) *Given a very well behaved system* $\{\mu_i\}_{i=1}^{n+1}$ *and* ψ *a* C^∞ *function with compact support in* $\mathbb{R}^n$, *there exist compactly supported functions* $\{\nu_i\}_{i=1}^{n+1}$ *such that for all* $\zeta \in \mathbb{C}^n$

$$\widehat{\psi}(\zeta) = \sum_{z\in\mathcal{Z}} \frac{\widehat{\psi}(z)}{J(z)\widehat{\mu_{n+1}}(z)} \begin{vmatrix} \widehat{\nu}_1^1(\zeta,z) & \dots & \widehat{\nu}_1^{n+1}(\zeta,z) \\ \vdots & \ddots & \vdots \\ \widehat{\nu}_n^1(\zeta,z) & \dots & \widehat{\nu}_n^{n+1}(\zeta,z) \\ \widehat{\mu_1}(\zeta) & \dots & \widehat{\mu_{n+1}}(\zeta) \end{vmatrix}, \tag{14}$$

where

$$\widehat{\nu}_k^j(\zeta,z) = \frac{\widehat{\mu_j}\,(\zeta_1,\dots,\zeta_k,z_{k+1},\dots,z_n) - \widehat{\mu_j}(\zeta_1,\dots,\zeta_{k-1},z_k,\dots,z_n)}{(\zeta_k - z_k)}.$$

The function ψ *can be an arbitrarily close approximation of the* δ *which converges to the* δ *in the sense of distributions as* $\operatorname{supp}(\psi) \longrightarrow \{0\}$.

The formula given in the theorem above gives a solution to a generalization of the Bezout equation for a very well behaved family. The function ψ can be chosen arbitrarily close to the δ. Again, the formulae for the deconvolvers are functions expanded in Fourier series, not distributions.

This formula gives the solution to the deconvolution problem when the convolvers are (hyper)rectangular regions in $\mathbb{R}^n$. It has a variety of other applications, including producing a more explicit version of the Ehrenpreis Fundamental Principle, i.e., the representation of all the solutions of a system of linear partial differential equations with constant coefficients [12], characterizations of interpolation varieties, and new versions of gap theorems for Dirichlet series [12].

Certain problems on deconvolution can be phrased as *Pompeiu* problems. We give the following example. Let χ_1, χ_2 be the characteristic functions of the disks $B(0,r_1)$, $B(0,r_2)$, and let E be the collection of positive quotients of zeros of the Bessel function J_1.

Theorem 2.6 ([4]) *Let* f *be a continuous function in the plane. If there exists an* $\alpha > 0$ *such that* $|\frac{r_1}{r_2} - \frac{\xi}{\eta}| \geq \frac{1}{\alpha}\eta^{-\alpha}$ *for all* $\xi,\eta \in E$ *with* $\xi,\eta > 0$, *then the mapping* $P\colon f \longrightarrow (\chi_1 * f, \chi_2 * f)$ *is injective. Moreover, there exist* ν_1,ν_2 *such that*

$$\nu_1 * (\chi_1 * f) + \nu_2 * (\chi_2 * f) = f. \tag{15}$$

In [4], this theorem was also extended to the local problem of reconstructing f in some disk $B(0,R)$, $R > r_1 + r_2$, from its averages on $B(0,r_1)$, $B(0,r_2)$.

The result fits in the larger context of Pompeiu problems. Let $E_1, \ldots, E_m$ be compact sets of positive measure in $\mathbb{R}^n$, let $C(\mathbb{R}^n)$ denote the space of continuous functions, and let $M(n)$ be the group of Euclidean motions in $\mathbb{R}^n$. Then, the (global) Pompeiu transform associated to the sets $E_1, \ldots, E_n$ is the mapping $P\colon C(\mathbb{R}^n) \longrightarrow C(M(n))^m$ given by $(Pf)(g) = \left(\int_{gE_1} f\,dx, \ldots, \int_{gE_m} f\,dx\right)$. The problem is then to give conditions on the sets $E_1, \ldots, E_m$ to guarantee that the mapping P is injective. The problem also has a local formulation (see [5]). We can construct the inverse mapping by constructing deconvolvers which recover the function f, as in the theorem above.

The application of the theory from Pompeiu problems to deconvolution is powerful. These techniques will allow us to create strongly coprime systems by varying a single element. We explore this idea further in Section 3.

2.2 Multichannel Deconvolution and Sampling

We now give a construction of the deconvolvers in Theorem 2.4 using the Shannon Sampling Theorem in the frequency domain. First recall the Shannon Sampling Theorem.

Theorem 2.7 (Shannon [31]) *Let f be a function of finite energy on $\mathbb{R}$ ($f \in L^2(\mathbb{R})$) with Fourier transform $\widehat{f}(\omega) = 0$ for all $|\omega| \geq \Omega$, i.e., $f(t)$ is Ω-band-limited.*

a.) *If $T \leq 1/2\Omega$, then for all $t \in \mathbb{R}$,*

$$f(t) = T \sum_{n=-\infty}^{\infty} f(nT) \frac{\sin(\frac{\pi}{T}(t - nT))}{\pi(t - nT)}. \tag{16}$$

b.) *If $T \leq 1/2\Omega$ and $f(nT) = 0$ for all $n \in \mathbb{Z}$, then $f \equiv 0$.*

Functions in $L^2(\widehat{\mathbb{R}})$ that are Ω-band-limited are the transforms of analytic functions of finite energy which satisfy Paley-Wiener growth estimates. Such functions are said to belong to the set $PW_\Omega(\mathbb{R})$. The Shannon Sampling Theorem has a dual, in that a given Ω-time-limited function can be sampled in frequency and reconstructed (see [28]).

To give our alternate formulae for the approximate deconvolvers $\nu_{1,\psi}$, $\nu_{2,\psi}$, we use the Jacobi interpolation formula to get a solution to the modified Bezout equation. We then show that these deconvolvers satisfy certain PW growth estimates. This in turn allows us to apply sampling. We use the zero sets of the transforms of the convolvers to give us the sampling rates. More general modifications of this construction do not use Jacobi interpolation (see [32, 33]).

Theorem 2.8 *Let $\mu_1(t) = \chi_{[-1,1]}(t)$, $\mu_2(t) = \chi_{[-\sqrt{p},\sqrt{p}]}(t)$, for p prime. Let $f \in L^2(\mathbb{R})$ and let ψ be an even C^4 function with support in $(-(1+\sqrt{p}), (1+\sqrt{p}))$ such that $\psi \geq 0$ and $\int_{-\infty}^{\infty} \psi(t)\,dt = 1$. The deconvolvers $\nu_{i,\psi}$ such that*

$$f * \psi = (f * \mu_1) * \nu_{1,\psi} + (f * \mu_2) * \nu_{2,\psi}$$

are also given by the formulae

$$\nu_{1,\psi}(t) = \left(\frac{1}{2\sqrt{p}} \sum_{j\neq 0} (-1)^{j+1} \frac{\widehat{\psi}\left(j/2\sqrt{p}\right)}{\widehat{\mu_1}(j/2\sqrt{p})} + \frac{1}{2\sqrt{p}} \sum_{n\neq 0} \frac{\widehat{\psi}\left(n/2\sqrt{p}\right)}{\widehat{\mu_1}(n/2\sqrt{p})} e^{\pi i (n/\sqrt{p})t} \right) \chi_{[-\sqrt{p},\sqrt{p}]}\,, \tag{17}$$

$$\nu_{2,\psi}(t) = \left(\frac{1}{2} \sum_{j\neq 0} (-1)^{j+1} \frac{\widehat{\psi}(j/2)}{\widehat{\mu_2}(j/2)} + \frac{1}{2} \sum_{n\neq 0} \frac{\widehat{\psi}(n/2)}{\widehat{\mu_2}(n/2)} e^{\pi i n t} \right) \chi_{[-1,1]}\,. \tag{18}$$

Proof : Let $\omega \in \mathbb{R}$. Since ψ is even, $\widehat{\psi}(\omega)$ is real. Consider first $\nu_{1,\psi}$. The formula for $\nu_{2,\psi}$ will follow similarly. Recall that

$$\mathcal{Z}_1 = \left\{ \frac{\pm k}{2} \right\}\,,\ \mathcal{Z}_2 = \left\{ \frac{\pm k}{2\sqrt{p}} \right\}\,,\ k \in \mathbf{N}$$

denote the zero sets of $\widehat{\mu_1}(\zeta)$, $\widehat{\mu_2}(\zeta)$, respectively. By equation (1) from Theorem 3.3 in [16], we have

$$\nu_{1,\psi} = \sum_{z\in\mathcal{Z}_2} \frac{\widehat{\psi}(z)}{\widehat{\mu_1}(z)\frac{d}{d\zeta}\widehat{\mu_2}(z)} \left(\frac{\widehat{\mu_2}(\zeta)}{\zeta - z} \right)^{\vee}(t)\,.$$

(A similar development is given in Section 3 - see equations (21), (22).) We claim that $\widehat{\nu}_{1,\psi} \in PW_{\sqrt{p}}(\widehat{\mathbb{R}})$. To see this, first note that

$$\sum_{z\in\mathcal{Z}_2} \left| \frac{\widehat{\psi}(z)}{\widehat{\mu_1}(z)\frac{d}{d\zeta}\widehat{\mu_2}(z)} \right| < \infty\,.$$

Now, for each $z \in \mathcal{Z}_2$ and $|\zeta - z| \geq \eta > 0$ with $\eta < 1/2\sqrt{p}$,

$$\left| \frac{\widehat{\mu_2}(\zeta)}{\zeta - z} \right| \leq \frac{1}{\eta} |\widehat{\mu_2}(\zeta)| \leq \frac{C}{\eta} e^{2\pi\sqrt{p}|\Im\zeta|}$$

since $\widehat{\mu_2} \in PW_{\sqrt{p}}(\widehat{\mathbb{R}})$. If $|\zeta - z| \leq \eta$, then

$$\left| \frac{\widehat{\mu_2}(\zeta)}{\zeta - z} \right| \leq \sup_{|\zeta - z|\leq\eta} \left| \frac{d}{d\zeta}\widehat{\mu_2}(\zeta) \right| \leq \| \frac{d}{d\zeta}\widehat{\mu_2} \|_\infty\,.$$

Therefore, there exists $C > 0$ such that

$$\left| \frac{\widehat{\mu_2}(\zeta)}{\zeta - z} \right| \leq C e^{2\pi\sqrt{p}|\Im\zeta|}$$

independently of $z \in \mathcal{Z}_2$. Thus,

$$|\widehat{\nu}_{1,\psi}| \leq \sup_{z\in\mathcal{Z}_2} \left| \frac{\widehat{\mu_2}(\zeta)}{\zeta - z} \right| \sum_{z\in\mathcal{Z}_2} \left| \frac{\widehat{\psi}(z)}{\widehat{\mu_1}(z)\frac{d}{d\zeta}\widehat{\mu_2}(z)} \right| \leq C e^{2\pi\sqrt{p}|\Im\zeta|}\,.$$

Note that for each $z \in \mathcal{Z}_2$,

$$\begin{aligned}\int_{\mathbb{R}} \left|\frac{\widehat{\mu_2}(\zeta)}{\zeta - z}\right|^2 d\zeta &\leq \int_{|\zeta - z| > \eta} \left|\frac{\widehat{\mu_2}(\zeta)}{\zeta - z}\right|^2 d\zeta + \int_{|\zeta - z| \leq \eta} \left|\frac{\widehat{\mu_2}(\zeta)}{\zeta - z}\right|^2 d\zeta \\ &\leq \frac{1}{\eta^2} \int_{\mathbb{R}} |\widehat{\mu_2}(\zeta)|^2 \, d\zeta + \eta \| \frac{d}{d\zeta} \widehat{\mu_2} \|_\infty^2\end{aligned}$$

which is bounded independently of z. Therefore,

$$\int_{\mathbb{R}} |\widehat{\nu}_{1,\psi}(\zeta)|^2 \, d\zeta \leq \sup_{z \in \mathcal{Z}_2} \int_{\mathbb{R}} \left|\frac{\widehat{\mu_2}(\zeta)}{\zeta - z}\right|^2 d\zeta \left(\sum_{z \in \mathcal{Z}_2} \left| \frac{\widehat{\psi}(z)}{\widehat{\mu_1}(z) \frac{d}{d\zeta} \widehat{\mu_2}(z)} \right| \right)^2 < \infty \, .$$

Now, since $\mathcal{Z}_2 = \left\{ \frac{\pm k}{2\sqrt{p}} \right\}$, $k \in \mathbb{N}$, and since for $n, \ m \in \mathbb{Z} \setminus \{0\}$,

$$\frac{\widehat{\mu_2}(m/2\sqrt{p})}{(m/2\sqrt{p}) - (n/2\sqrt{p})} = \delta(n - m) \, ,$$

we have that for $n \in \mathbb{Z} \setminus \{0\}$,

$$\widehat{\nu}_{1,\psi}(\frac{n}{2\sqrt{p}}) = \frac{\widehat{\psi}(n/2\sqrt{p})}{\widehat{\mu_1}(n/2\sqrt{p})}$$

and

$$\widehat{\nu}_{1,\psi}(0) = \sum_{j \neq 0} \frac{\widehat{\psi}(j/2\sqrt{p})}{\widehat{\mu_1}(j/2\sqrt{p}) \frac{d}{d\zeta} \widehat{\mu_2}(j/2\sqrt{p})} \frac{\widehat{\mu}_2(0)}{(-n/2\sqrt{p})} = \sum_{j \neq 0} (-1)^{j+1} \frac{\widehat{\psi}(j/2\sqrt{p})}{\widehat{\mu_1}(j/2\sqrt{p})} \, .$$

Therefore, by Shannon's sampling formula we may write

$$\widehat{\nu}_{1,\psi}(\zeta) = \left(\sum_{j \neq 0} (-1)^{j+1} \frac{\widehat{\psi}(j/2\sqrt{p})}{\widehat{\mu_1}(j/2\sqrt{p})} \right) \frac{\sin(2\pi\sqrt{p}\zeta)}{\pi\sqrt{p}\zeta} + \sum_{n \neq 0} (-1)^{n+1} \frac{\widehat{\psi}(n/2\sqrt{p})}{\widehat{\mu_1}(n/2\sqrt{p})} \frac{\sin \pi(2\sqrt{p}\zeta - n)}{\pi(2\sqrt{p}\zeta - n)} \, .$$

Taking inverse Fourier transforms and simplifying gives

$$\nu_{1,\psi}(t) = \left(\frac{1}{2\sqrt{p}} \sum_{j \neq 0} (-1)^{j+1} \frac{\widehat{\psi}\left(j/2\sqrt{p}\right)}{\widehat{\mu_1}(j/2\sqrt{p})} + \frac{1}{2\sqrt{p}} \sum_{n \neq 0} \frac{\widehat{\psi}\left(n/2\sqrt{p}\right)}{\widehat{\mu_1}(n/2\sqrt{p})} e^{\pi i (n/\sqrt{p}) t} \right) \chi_{[-\sqrt{p}, \sqrt{p}]}(t) \, .$$

Letting $p = 1$ in the above arguments gives

$$\nu_{2,\psi}(t) = \left(\frac{1}{2} \sum_{j \neq 0} (-1)^{j+1} \frac{\widehat{\psi}(j/2)}{\widehat{\mu_2}(j/2)} + \frac{1}{2} \sum_{n \neq 0} \frac{\widehat{\psi}(n/2)}{\widehat{\mu_2}(n/2)} e^{\pi i n t} \right) \chi_{[-1,1]} \, . \quad \square$$

Figures 1 and 2 [20] give simulations of this result.

Figure 1. C^{∞} Function (Ψ) Approximate Deconvolver System.

1.a.) Channel 1 Deconvolver ν_1.

1.b.) Channel 2 Deconvolver ν_2.

1.c.) $\Psi * \mu_1 * \nu_1$.

1.d.) $\Psi * \mu_2 * \nu_2$.

1.e.) $\widetilde{\Psi} = \Psi * \mu_1 * \nu_1 + \Psi * \mu_2 * \nu_2$, the resulting approximate delta.

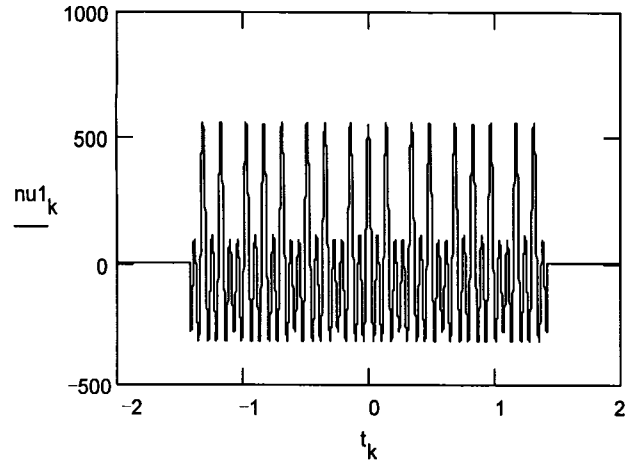
1.f.) The Fourier transform, or frequency response, of this system.

Figure 2. Transfer Response for the System.

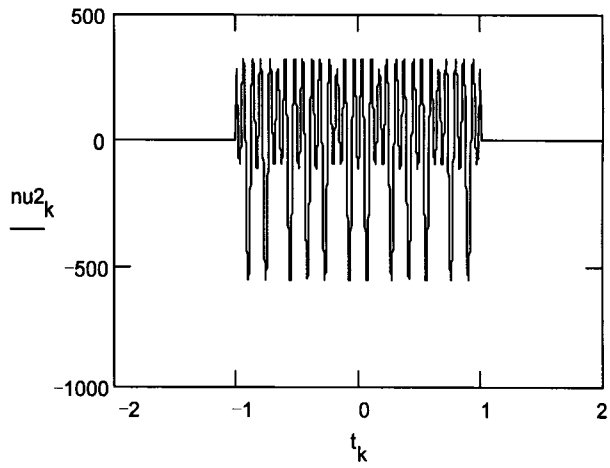
2.a.) The response of each channel, in the two channel system of Figure 1, to two adjacent Gaussian pulses (denoted as input signal f).

$$(\mu_1 * f - \text{solid} \ , \ \mu_2 * f - \text{dotted})$$

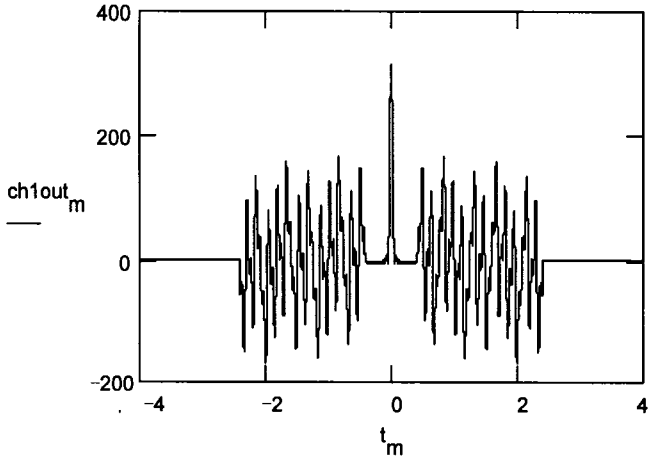
2.b.) Transfer response for the system given in Figure 1. Here the dotted line represents the input function while the solid line shows the system response.



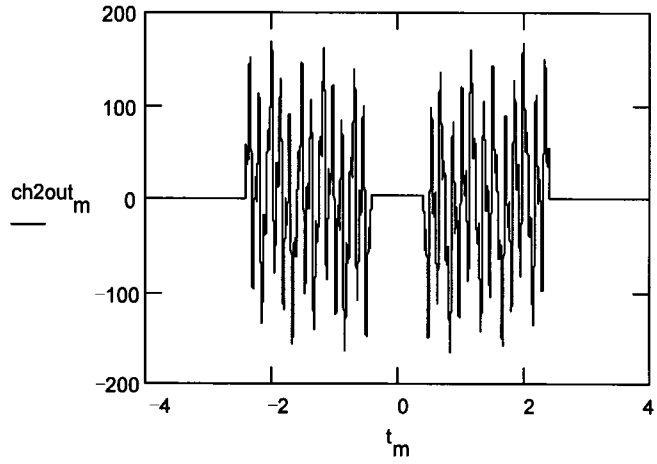
1a. Channel 1 Deconvolver (ν_1).



1a. Channel 2 Deconvolver (ν_2).

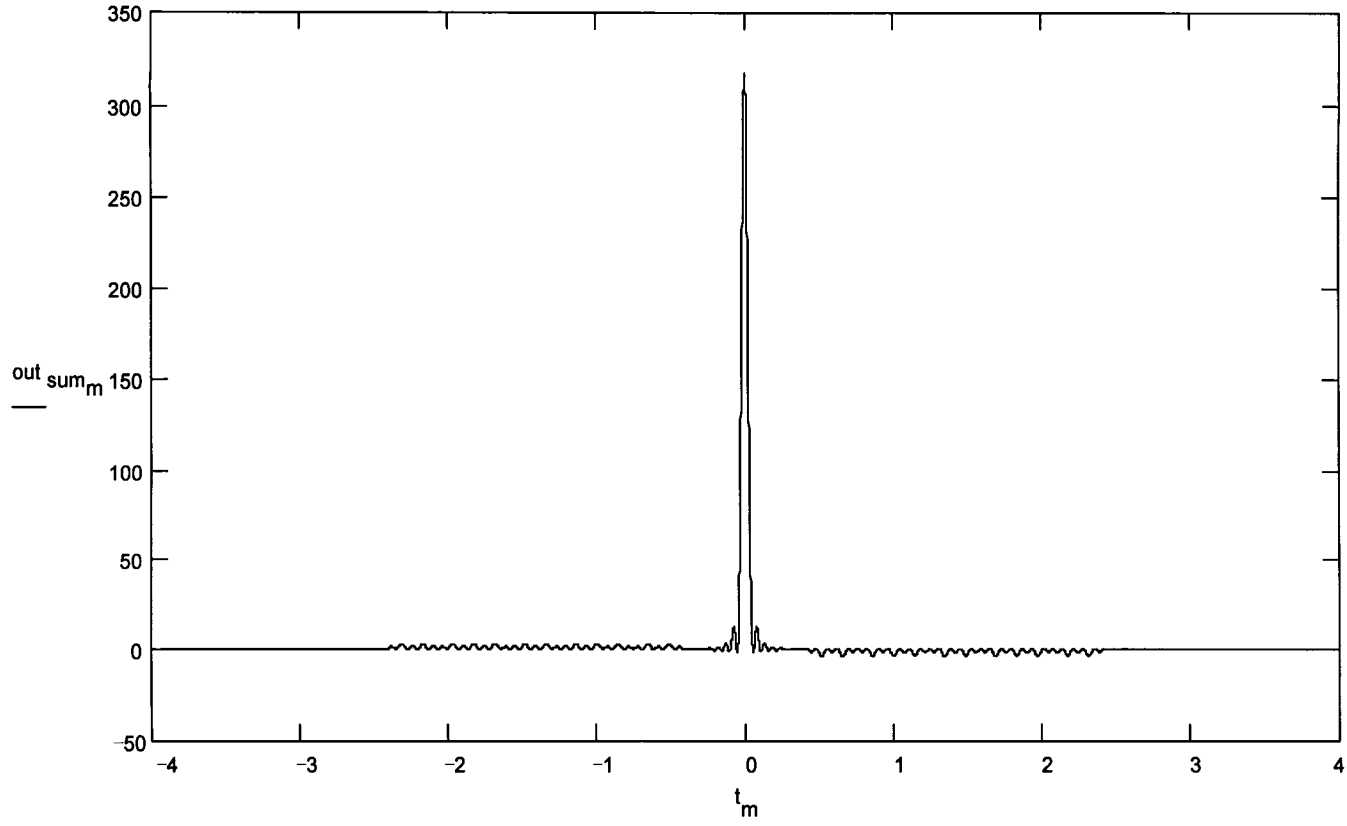


1c. Channel 1 output ($\Psi*\mu_1*\nu_1$)

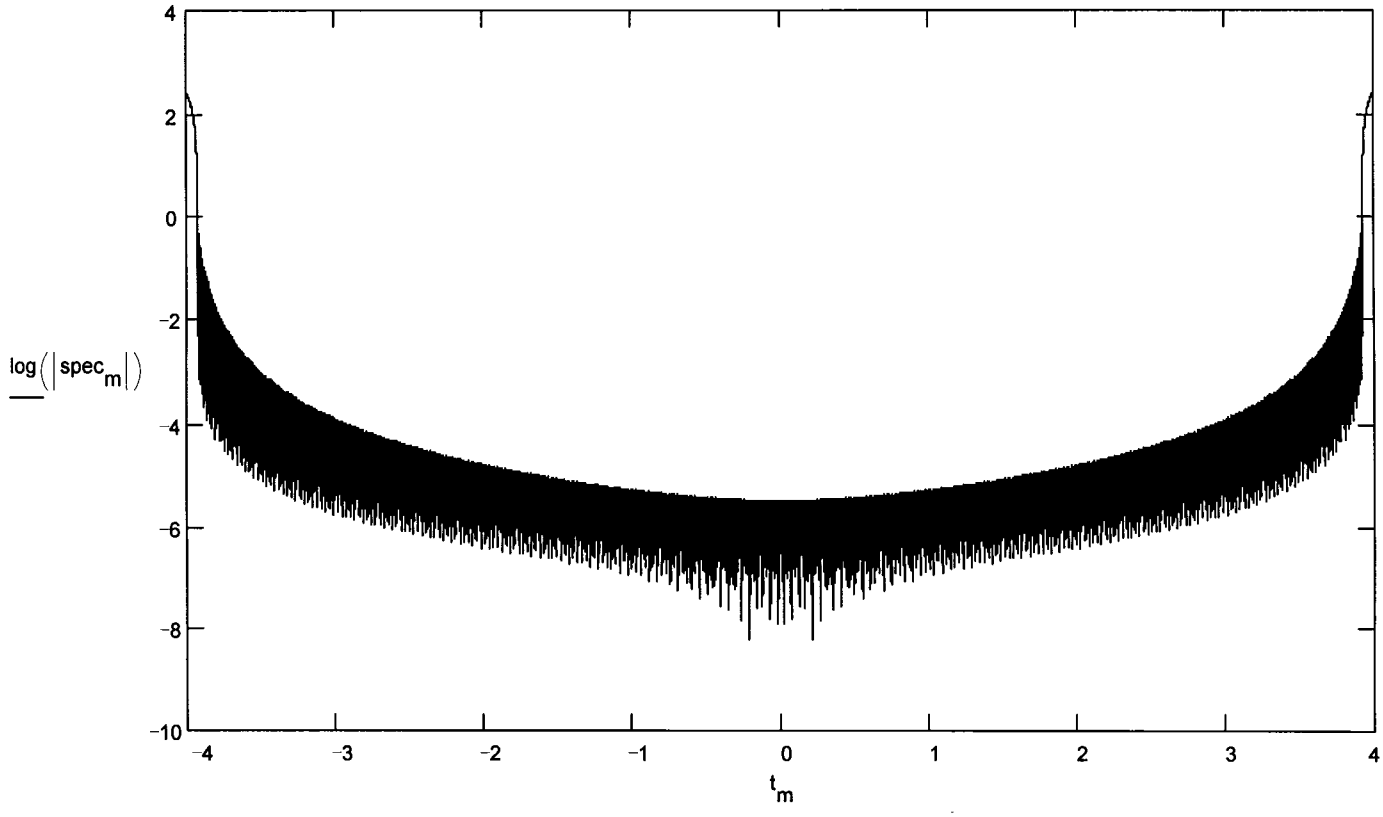


1d. Channel 2 output ($\Psi*\mu_2*\nu_2$)

Fig 1. The C^{∞} (Ψ) Approximate Decomvolver System.

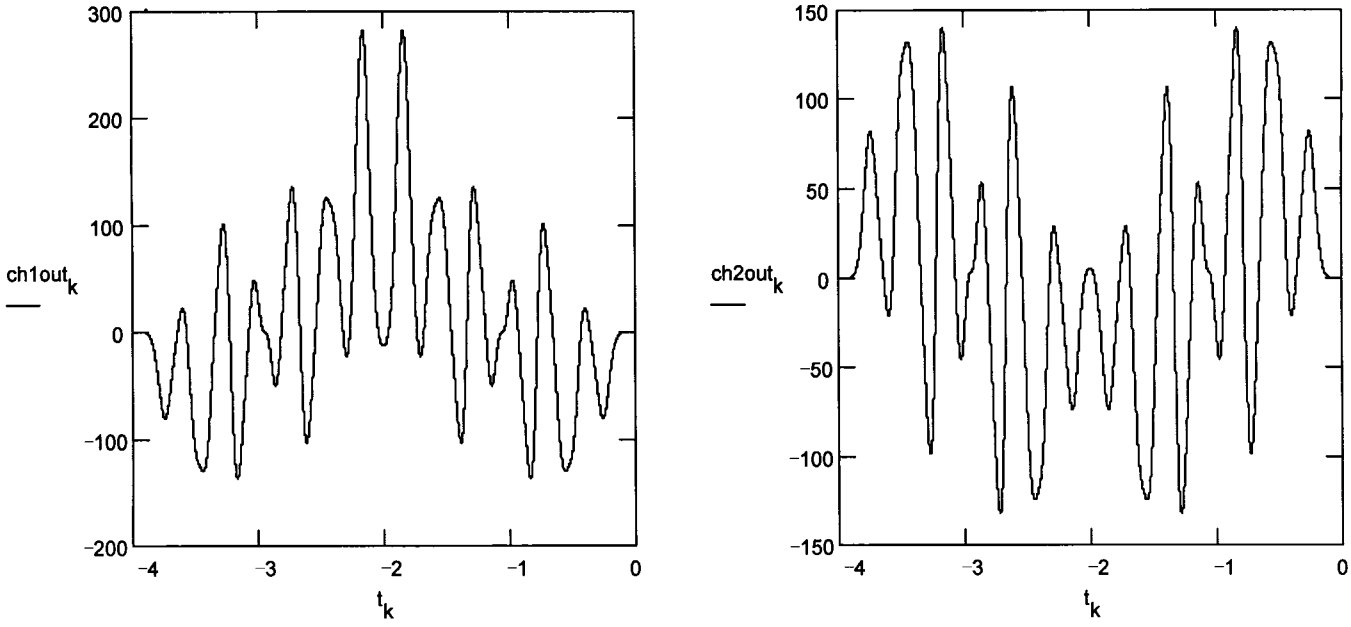


1e. $\widetilde{\Psi} = \Psi * \mu_1 * \nu_1 + \Psi * \mu_2 * \nu_2$, the resulting approximate delta functional (time domain).

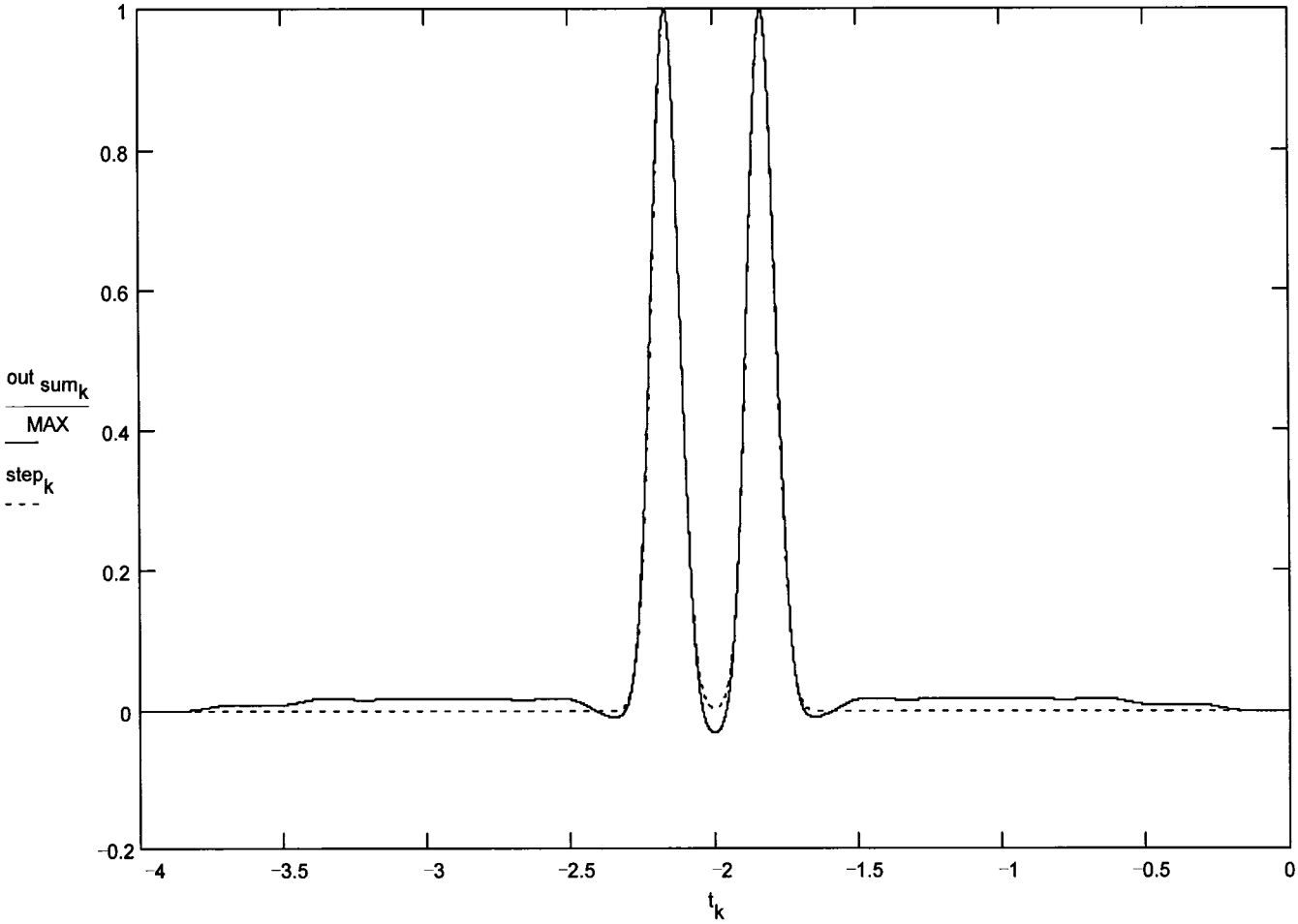


1f. $\widetilde{\Psi} = \Psi * \mu_1 * \nu_1 + \Psi * \mu_2 * \nu_2$, the resulting approximate delta functional (frequency domain).

Fig 1. The C^{∞} (Ψ) Approximate Decomvolver System.



2a. The response of each channel, in the two channel system of Fig. 1, to two Gaussian pulses (denoted as the input signal, f).



2b. System transfer response, for the two channel system of Fig. 1, when the input function is the sum of two time shifted Gaussian pulses. The dotted line is the input function, the solid line is the deconvolved function.

Fig 2. Transfer response for the C^{∞} system shown in Figure 1.

To reconcile the apparent differences between (10), (11) and (17), (18), we compute as follows. Let $r_1 = 1$ and $r_2 = \sqrt{p}$. Then

$$\widehat{\mu_i}(n/2r_j) = \frac{2\sin(n\pi(r_i/r_j))}{(n\pi/r_j)},$$

$$\frac{d}{d\zeta}\widehat{\mu_j}(n/2r_j) = \frac{2r_j\cos(n\pi)}{(n/2r_j)} = \frac{2r_j(-1)^n}{(n/2r_j)}.$$

Thus,

$$\begin{aligned}
&\nu_{i,\psi}(t)\\
&= \sum_{z\in\mathcal{Z}_j} \frac{\widehat{\psi}(z)}{\widehat{\mu_i}(z)\frac{d}{d\zeta}\widehat{\mu_j}(z)} \left(\frac{1}{z}\left(e^{2\pi iz(t+r_j)} - 1\right)\chi_{[-r_j,r_j]}(t)\right)\\
&= \sum_{n\neq 0} \frac{\widehat{\psi}(n/2r_j)}{\widehat{\mu_i}(n/2r_j)} \frac{(n/2r_j)}{2r_j\cos(n\pi)} \left(\frac{1}{(n/2r_j)}\left(e^{2\pi i(n/2r_j)(t+r_j)} - 1\right)\chi_{[-r_j,r_j]}(t)\right)\\
&= \frac{1}{2r_j}\sum_{n\neq 0} \frac{\widehat{\psi}(n/2r_j)}{\widehat{\mu_i}(n/2r_j)}(-1)^n\left((e^{\pi i})^n e^{\pi i(n/r_j)t} - 1\right)\chi_{[-r_j,r_j]}(t)\\
&= \frac{1}{2r_j}\sum_{n\neq 0} \frac{\widehat{\psi}(n/2r_j)}{\widehat{\mu_i}(n/2r_j)}\left((-1)^{n+1} + e^{\pi i(n/r_j)t}\right)\chi_{[-r_j,r_j]}(t).
\end{aligned}$$

The formula for $\nu_{j,\psi}$ is given by a permutation of the indices.

We can simplify these formulae by noting that

$$\widehat{\mu_1}(0)\widehat{\nu_1}(0) + \widehat{\mu_2}(0)\widehat{\nu_2}(0) = \widehat{\psi}(0). \tag{19}$$

Since

$$\lim_{\zeta\to 0}\widehat{\mu_i}(\zeta) = 2r_i,$$

(19) will also hold if

$$\widehat{\nu_i}(0) = \frac{\widehat{\psi}(0)}{4r_i}.$$

This gives

$$\nu_{i,\psi}(t) = \frac{1}{2r_j}\left[\frac{\widehat{\psi}(0)}{4r_i} + \sum_{n\neq 0}\frac{\widehat{\psi}(n/2r_j)}{\widehat{\mu_i}(n/2r_j)}e^{\pi i(n/r_j)t}\right]\chi_{[-r_j,r_j]}(t).$$

Again, the formula for $\nu_{j,\psi}$ is given by a permutation of the indices. However, if $\widehat{\nu_i}(0) = \frac{\widehat{\psi}(0)}{4r_i}$, the deconvolvers are not continuous at $\pm r_i$. There is a subtlety in this development. The two lattices $\mathcal{Z}_1$ and $\mathcal{Z}_2$ overlap at $z = 0$. To uniquely determine the $\nu_{i,\psi}$, we must have either continuity at the endpoints or must specify $\widehat{\nu_{i,\psi}}(0)$ and $\widehat{\nu_{i,\psi}}'(0)$ (see [33]). The deconvolvers given by (17) and (18) are continuous.

3 Creating a System Using Modulation

We can use modulation to create a strongly coprime system. This new technique for creating these systems allows for a greater flexibility in the development of actual systems. The system is created by making two identical copies of a given sensor, splitting the signal into two the two separate channels, and appropriate modulation of both the input and output of one of the channels. These two outputs are then convolved with the appropriate deconvolving filters and added, resulting in the reconstruction of the input signal.

Let μ_1 model the impulse response of a given system. Let $E_{\zeta_0} = e^{2\pi i \zeta_0 t}$. We will refer to E_{ζ_0} as a modulating function in the time domain. In the engineering literature, multiplication by E_{ζ_0} is called "Quadrature Amplitude Shift Keying" or "Quadrature Amplitude Modulation" (see, for example, [26]). "Quadrature" refers to the fact that the real and imaginary parts of the modulating function are $\frac{\pi}{2}$ out of phase with each other.

Lemma 3.1 *If $f \in L^2$ and μ is realizable, then*

$$f * E_{\zeta_0}\mu = E_{\zeta_0}\left[E_{-\zeta_0} f * \mu\right] . \tag{20}$$

Proof : We have that

$$\begin{aligned} f * E_{\zeta_0}\mu &= \int_{-\infty}^{\infty} f(\tau) e^{2\pi i \zeta_0 (t-\tau)} \mu(t-\tau)\, d\tau \\ &= e^{2\pi i \zeta_0 t} \int_{-\infty}^{\infty} e^{-2\pi i \zeta_0 \tau} f(\tau) \mu(t-\tau)\, d\tau \\ &= E_{\zeta_0}\left[E_{-\zeta_0} f * \mu\right] . \ \square \end{aligned}$$

Remarks :

1. Thus, to modulate a channel, we modulate the incoming signal before it goes in and after it comes out of the channel.
2. The lemma holds for more general classes of signals and convolvers.

3.1 Developing a Specific System

We now apply Lemma 3.1 to create a multichannel system for a system with impulse response

$$\mu_1(t) = \chi_{[-1,1]}(t) .$$

The Fourier-Laplace transform of $\mu_1(t)$ is

$$\widehat{\mu_1}(\zeta) = \frac{\sin(2\pi\zeta)}{\pi\zeta} .$$

This has zeros

$$\mathcal{Z}_1 = \left\{ \pm\frac{k}{2} : k \in \mathbb{N} \right\} .$$

Let

$$\mu_2(t) = E_{\frac{1}{4}}\mu_1 = e^{\frac{\pi}{2}it}\mu_1(t)\,.$$

Then

$$\widehat{\mu_2}(\zeta) = \frac{\sin(2\pi(\zeta - \frac{1}{4}))}{\pi(\zeta - \frac{1}{4})}$$

and

$$\mathcal{Z}_2 = \left\{\frac{1}{4} \pm \frac{k}{2}\right\}^2.$$

Theorem 3.1 *The functions $\mu_1(t)$, $\mu_2(t)$ form a strongly coprime pair of convolvers.*

Proof : We want to show that there exist positive constants A and B and a positive integer N such that

$$\left(\sum_{i=1}^{2} |\widehat{\mu}_i(\zeta)|^2\right)^{\frac{1}{2}} \geq Ae^{-2\pi B|\Im\zeta|}(1 + |\zeta|)^{-N}\,, \text{ for all } \zeta \in \mathbb{C}.$$

We need the following.

Proposition 3.1 *If $|\Im(\zeta)| \geq \frac{1}{2}\log(2)$, then $|\sin(\zeta)| \geq \frac{1}{2}e^{-\Im(\zeta)}$.*

Proof : We have that

$$|\sin(\zeta)|^2 = \frac{1}{4}\left[e^{2\Im(\zeta)} + e^{-2\Im(\zeta)} - 2\cos(2\Re(\zeta))\right].$$

If $\Im(\zeta) \geq \frac{1}{2}\log(2) > 0$, then $e^{2\Im(\zeta)} \geq e^{2\frac{1}{2}\log(2)} = 2$, and so

$$e^{2\Im(\zeta)} + e^{-2\Im(\zeta)} - 2\cos(2\Re(\zeta)) \geq 2\left(1 - \cos(2\Re(\zeta))\right) + e^{-2\Im(\zeta)} \geq e^{-2\Im(\zeta)} = e^{-2|\Im(\zeta)|}\,.$$

If $\Im(\zeta) \leq \frac{-1}{2}\log(2) < 0$, then $e^{-2\Im(\zeta)} \geq e^{2\frac{1}{2}\log(2)} = 2$, and so

$$e^{2\Im(\zeta)} + e^{-2\Im(\zeta)} - 2\cos(2\Re(\zeta)) \geq 2\left(1 - \cos(2\Re(\zeta))\right) + e^{2\Im(\zeta)} \geq e^{2\Im(\zeta)} = e^{-2|\Im(\zeta)|}\,.$$

Thus, $|\sin(\zeta)|^2 \geq \frac{1}{4}e^{-2\Im(\zeta)}$. Taking square roots proves the proposition. □

Proposition 3.2 *For any ϵ such that $0 < \epsilon < |\Im(\zeta)|$, there exists $c_\epsilon > 0$ such that $|\sin(\zeta)| \geq c_\epsilon e^{-\Im(\zeta)}$.*

Proof : We have that

$$|\sin(\zeta)|^2 = \frac{1}{4}\left[e^{2\Im(\zeta)} + e^{-2\Im(\zeta)} - 2\cos(2\Re(\zeta))\right] = \frac{1}{2}\left[\cosh(2\Im(\zeta)) - \cos(2\Re(\zeta))\right].$$

[2]Not to be confused with the zero set in Section 2.

Now, $\cosh(t)$ is continuous and $\cosh(t) \geq 1$ for all $t \in \mathbb{R}$ with equality if and only if $t = 0$. Therefore, we have that $|\sin(\zeta)|$ is bounded below by some constant $A_\epsilon > 0$ for all ϵ such that $0 < \epsilon < |\Im(\zeta)|$. Combining this observation with the proposition above, we get

$$|\sin(\zeta)| \geq \begin{cases} \frac{1}{2}e^{-\Im(\zeta)} & \text{if } |\Im(\zeta)| \geq \frac{1}{2}\log(2) \\ A_\epsilon e^{-\Im(\zeta)} & \text{if } 0 < \epsilon < |\Im(\zeta)| < \frac{1}{2}\log(2)\,. \end{cases} \quad \square$$

Therefore, if $|\Im(\zeta)| \geq \frac{1}{2}\log(2)$, $|\sin(2\pi\zeta)| \geq \frac{1}{2}e^{-2\pi\Im(\zeta)}$ and $\left|\sin(2\pi(\zeta - \frac{1}{4}))\right| \geq \frac{1}{2}e^{-2\pi\Im(\zeta)}$. Thus, for $|\Im(\zeta)| \geq \frac{1}{2}\log(2)$,

$$\begin{aligned} \sum_{i=1}^{2} |\widehat{\mu}_i(\zeta)|^2 &= \frac{1}{\pi^2}\left[\frac{|\sin(2\pi\zeta)|^2}{|\zeta|^2} + \frac{\left|\sin(2\pi(\zeta - \frac{1}{4}))\right|^2}{|\zeta - \frac{1}{4}|^2}\right] \\ &\geq \frac{1}{4\pi^2}\left[\frac{|\zeta|^2 + |\zeta - \frac{1}{4}|^2}{|\zeta|^2|\zeta - \frac{1}{4}|^2}\right] e^{-4\pi\Im(\zeta)} \\ &\geq \frac{1}{4\pi^2}\left[\frac{\left(\frac{1}{2}\log(2)\right)^2 + \frac{1}{16}}{\left(1 + |\zeta|\right)^2}\right] e^{-2(2\pi\Im(\zeta))}\,. \end{aligned}$$

So, let $A = \frac{1}{2\pi}\left[\sqrt{(\frac{1}{2}\log(2))^2 + \frac{1}{16}}\right]$, $B = 1$, and $N = 1$. Also, for $0 < \epsilon < |\Im(\zeta)| < \frac{1}{2}\log(2)$,

$$\sum_{i=1}^{2} |\widehat{\mu}_i(\zeta)|^2 \geq \frac{A_\epsilon^2}{\pi^2}\left[\frac{\frac{1}{16}}{\left(1 + |\zeta|\right)^2}\right] e^{-2(2\pi\Im(\zeta))}\,.$$

So, let $A = \frac{A_\epsilon}{4\pi}$, $B = 1$, and $N = 1$. Finally, for $|\Im(\zeta)| \leq \epsilon$, $|\sin(2\pi\zeta)| \geq 2\,|1 - \cos(4\pi\Re(\zeta))|$. Thus,

$$\sum_{i=1}^{2} |\widehat{\mu}_i(\zeta)|^2 \geq \frac{4}{\pi^2}\left[\frac{1}{\left(1 + |\zeta|\right)^2}\left[|1 - \cos(4\pi\Re(\zeta))|^2 + |1 - \cos(4\pi(\Re(\zeta) - \frac{1}{4}))|^2\right]\right]\,.$$

Now, in this last term, the cosines are $\frac{1}{2}$-periodic and are exactly $\frac{1}{4}$ out of phase with each other. Therefore, this term is $\geq \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$. Thus,

$$\sum_{i=1}^{2} |\widehat{\mu}_i(\zeta)|^2 \geq \frac{2}{\pi^2}\left[\frac{1}{\left(1 + |\zeta|\right)^2}\right]\,.$$

So, let $A = \frac{\sqrt{2}}{\pi}$, B be arbitrary, and $N = 1$. Note, all estimates are independent of $\omega = \Re\zeta$.

Combining, we let $B = N = 1$ and

$$A = \begin{cases} \frac{A_\epsilon}{4\pi} & \text{if } 0 < \epsilon < |\Im(\zeta)| \\ \frac{\sqrt{2}}{\pi} & \text{if } 0 < |\Im(\zeta)| \leq \epsilon\,. \end{cases}$$

This proves the theorem. $\square$

The next step is for us to construct deconvolvers $\nu_{i,\psi}$ such that for an approximate identity function ψ, we have

$$\mu_1 * \nu_{1,\psi} + \mu_2 * \nu_{2,\psi} = \psi\,.$$

We will need some computations. We know that

$$\widehat{\mu_1}(\zeta) = \frac{\sin(2\pi\zeta)}{\pi\zeta}\,,\ \widehat{\mu_2}(\zeta) = \frac{\sin(2\pi(\zeta - \frac{1}{4})}{\pi(\zeta - \frac{1}{4})}\,.$$

These have zeros

$$\mathcal{Z}_1 = \left\{\pm\frac{k}{2}\right\}\,,\ \mathcal{Z}_2 = \left\{\frac{1}{4} \pm \frac{k}{2}\right\}\,,\ k \in \mathbb{N}.$$

Now,

$$\frac{d}{d\zeta}\widehat{\mu_1}(\zeta) = \frac{2\cos(2\pi\zeta)}{\zeta} - \frac{1}{\zeta}\widehat{\mu_1}(\zeta)\,.$$

Thus, for $z \in \mathcal{Z}_1$,

$$\left|\frac{d}{d\zeta}\widehat{\mu_1}(z)\right| = \frac{2}{|z|}\,.$$

Also, for $z \in \mathcal{Z}_2$,

$$|\widehat{\mu_1}(z)| = \frac{1}{\pi|z|}\,.$$

Symmetric relationships hold for $\widehat{\mu_2}$ and its derivative.

Let $k \in \mathbb{N}$, and let $\alpha = k + \beta$, where $0 \leq \beta < 1$. We define the *Hölder space* C^α *of exponent* α as

$$\left\{f \in C^k\colon \text{There exists } M > 0 \text{ such that } \left|f^{(k)}(t) - f^{(k)}(\tau)\right| \leq M\,|t-\tau|^\beta\right\}.$$

We also define the related class D^α as

$$\left\{f\colon \widehat{f}(\omega)\,(1+|\omega|)^\alpha \in L^\infty\right\},$$

for ω in frequency space. We have that

$$D^{\alpha+1} \subset C^\alpha.$$

If f has compact support, then

$$f \in C^\alpha \Longrightarrow f \in D^\alpha.$$

Choose ψ to be in the Hölder space $C^{3+\eta}$, $\eta > 0$, with support in $(-2, 2)$ such that $\psi \geq 0$ and $\int_{-\infty}^{\infty} \psi(t)\,dt = 1$. We refer to ψ as the *auxiliary function* of the construction. The conditions that we impose on ψ make it an approximate identity function. Its smoothness and the size of its support guarantee that the deconvolvers are compactly supported. For the compactly supported function ψ, we have that if $\eta > 0$ and $\psi \in C^{3+\eta}$, then

$$\left|\widehat{\psi}(z)\right| \leq \frac{d}{(1+|z|)^{3+\eta}}$$

for $z \in \mathcal{Z}_1 \cup \mathcal{Z}_2$.

We will assume, for technical reasons, that $f \in C^\infty \cap L^2(\mathbb{R})$. We will use a density argument at the end of this section to extend the result to all of L^2. Let

$$\langle f, \psi \rangle = \int_{\mathbb{R}} f(t)\, \overline{\psi(t)}\, dt$$

be the L^2 inner product of f and ψ, and let

$$\langle \nu; \psi \rangle = \int_{\mathbb{R}} \psi(t)\, \nu(t)\, dt$$

denote the dual product between $\psi \in \mathcal{E}$ and $\nu \in \mathcal{E}'$.

We want to show that there exist a sequence of concentric circles Γ_n with centers at the origin and radii ρ_n such that $\lim_{n\to\infty} \rho_n = \infty$ and such that there exists $c > 0$ with

$$|\sin(2\pi\zeta)| \geq c e^{2\pi \Im(\zeta)}$$

and

$$\left|\sin(2\pi(\zeta - \frac{1}{4}))\right| \geq c e^{2\pi \Im(\zeta)}$$

for all $\zeta \in \Gamma_n$. The choice of such Γ_n is obvious. Let

$$\Gamma_n = \left\{ \zeta : |\zeta| = \frac{2n+1}{8}, \, n \in \mathbb{N} \right\} .$$

(This is exactly half-way between the zeros of $\widehat{\mu_1}$ and $\widehat{\mu_2}$.) For $\zeta \in \Gamma_n$, we have that

$$c = \begin{cases} \frac{\sqrt{2}}{4} & \text{if } |\Im(\zeta)| \geq \frac{1}{2}\log(2) \\ \frac{1}{2} & \text{if } |\Im(\zeta)| < \frac{1}{2}\log(2) \end{cases}$$

We choose $c = \frac{\sqrt{2}}{4}$, which is independent of n.

Theorem 3.2 *The set*

$$\mu_1(t) = \chi_{[-1,1]}(t) \, , \; \mu_2(t) = E_{\frac{1}{4}} \chi_{[-1,1]}(t)$$

is a strongly coprime pair of convolvers. The deconvolvers $\nu_{i,\psi}$ such that

$$\langle f, \psi \rangle = \langle \nu_{1,\psi}; f * \mu_1 \rangle + \langle \nu_{2,\psi}; f * \mu_2 \rangle$$

are given by the formulae

$$\nu_{1,\psi}(t) = \sum_{z \in \mathcal{Z}_2} \frac{\overline{\widehat{\psi}(z)}}{\widehat{\mu_1}(z) \frac{d}{d\zeta}\widehat{\mu_2}(z)} \left(\frac{\widehat{\mu_2}(\zeta)}{\zeta - z} \right)^\vee (t), \tag{21}$$

$$\nu_{2,\psi}(t) = \sum_{z \in \mathcal{Z}_1} \frac{\overline{\widehat{\psi}(z)}}{\widehat{\mu_2}(z) \frac{d}{d\zeta}\widehat{\mu_1}(z)} \left(\frac{\widehat{\mu_1}(\zeta)}{\zeta - z} \right)^\vee (t). \tag{22}$$

Also,

$$\psi(t) \longrightarrow \delta \text{ as } \text{supp}(\psi) \longrightarrow \{0\}$$

and

$$\widehat{\psi}(\zeta) \longrightarrow 1 \text{ as } \text{supp}(\psi) \longrightarrow \{0\}$$

in the sense of distributions.

Remark : The formulae for the ν_i are a type of Lagrange interpolation, where the analytic functions $\widehat{\nu_i}(\zeta)$ were constructed from known values on a discrete set of data. The Cauchy residue theory is used to convert these discrete values to analytic functions, in the same fashion as sampling formulae convert discrete functions to analog functions. The formulae are also related to classical Shannon sampling, with the functions

$$\frac{1}{\frac{d}{d\zeta}\widehat{\mu_i}(z)} \left(\frac{\widehat{\mu_i}(\zeta)}{\zeta - z} \right)^{\vee} (t)$$

acting as interpolators.

Proof : We have that

$$\widehat{\mu_1}(\zeta) = \frac{\sin(2\pi\zeta)}{\pi\zeta}, \; \widehat{\mu_2}(\zeta) = \frac{\sin(2\pi(\zeta - \frac{1}{4}))}{\pi(\zeta - \frac{1}{4})},$$

$$\frac{d}{d\zeta}\widehat{\mu_i}(z) \neq 0,$$

and

$$\left| \frac{d}{d\zeta}\widehat{\mu_i}(z) \right| = \frac{2}{|z|}$$

for $z \in \mathcal{Z}_i$.

Since $\widehat{\psi}(\zeta)$ is entire, we may represent $\widehat{\psi}$ by the Cauchy integral formula, i.e.,

$$\widehat{\psi}(\zeta) = \frac{1}{2\pi i} \oint_{\Gamma_n} \frac{\widehat{\psi}(z)}{z - \zeta} \, dz.$$

Lemma 3.2 *For* $\zeta \in \{\zeta : |\zeta| < \rho_n\}$,

$$\begin{aligned} \widehat{\psi}(\zeta) &= \widehat{\mu_1}(\zeta) \sum_{\substack{z \in \mathcal{Z}_2 \\ |z| < \rho_n}} \frac{\widehat{\psi}(z)}{\widehat{\mu_1}(z)\frac{d}{d\zeta}\widehat{\mu_2}(z)} \left(\frac{\widehat{\mu_2}(\zeta)}{\zeta - z} \right) \\ &+ \widehat{\mu_2}(\zeta) \sum_{\substack{z \in \mathcal{Z}_1 \\ |z| < \rho_n}} \frac{\widehat{\psi}(z)}{\widehat{\mu_2}(z)\frac{d}{d\zeta}\widehat{\mu_1}(z)} \left(\frac{\widehat{\mu_1}(\zeta)}{\zeta - z} \right) \\ &+ R_n(\zeta), \end{aligned}$$

where $R_n(\zeta) \longrightarrow 0$ *as* $n \longrightarrow \infty$.

Proof :

$$\begin{aligned}
\widehat{\psi}(\zeta) &= \frac{1}{2\pi i}\oint_{\Gamma_n}\frac{\widehat{\psi}(z)}{z-\zeta}\,dz \\
&= \frac{1}{2\pi i}\oint_{\Gamma_n}\frac{\widehat{\psi}(z)\widehat{\mu_1}(z)\widehat{\mu_2}(z)-\widehat{\psi}(z)\widehat{\mu_1}(\zeta)\widehat{\mu_2}(\zeta)}{(z-\zeta)\left(\widehat{\mu_1}(z)\widehat{\mu_2}(z)\right)}\,dz \\
&\quad + \frac{1}{2\pi i}\widehat{\mu_1}(\zeta)\widehat{\mu_2}(\zeta)\oint_{\Gamma_n}\frac{\widehat{\psi}(z)}{(z-\zeta)\left(\widehat{\mu_1}(z)\widehat{\mu_2}(z)\right)}\,dz \\
&= \frac{1}{2\pi i}\oint_{\Gamma_n}\frac{\widehat{\psi}(z)\left[\widehat{\mu_1}(z)\widehat{\mu_2}(z)-\widehat{\mu_1}(\zeta)\widehat{\mu_2}(\zeta)\right]}{(z-\zeta)\left(\widehat{\mu_1}(z)\widehat{\mu_2}(z)\right)}\,dz \\
&\quad + \frac{1}{2\pi i}\widehat{\mu_1}(\zeta)\widehat{\mu_2}(\zeta)\oint_{\Gamma_n}\frac{\widehat{\psi}(z)}{(z-\zeta)\left(\widehat{\mu_1}(z)\widehat{\mu_2}(z)\right)}\,dz.
\end{aligned}$$

Now, for $z \in \mathcal{Z}_1$ or $z \in \mathcal{Z}_2$, $\left(\widehat{\mu_1}(z)\widehat{\mu_2}(z)\right) = 0$, but $\frac{d}{d\zeta}\left(\widehat{\mu_1}(z)\widehat{\mu_2}(z)\right) \neq 0$. Thus, the function $\left(\frac{1}{\widehat{\mu_1}(z)\widehat{\mu_2}(z)}\right)$ has simple poles at $z \in \mathcal{Z}_1 \cup \mathcal{Z}_2$. Therefore, $\frac{\widehat{\psi}(z)\left[\widehat{\mu_1}(z)\widehat{\mu_2}(z)-\widehat{\mu_1}(\zeta)\widehat{\mu_2}(\zeta)\right]}{(z-\zeta)\left(\widehat{\mu_1}(z)\widehat{\mu_2}(z)\right)}$ has simple poles in $\mathcal{Z}_1 \cup \mathcal{Z}_2$, and so by the Cauchy Residue Theorem,

$$\begin{aligned}
&\oint_{\Gamma_n}\frac{\widehat{\psi}(z)\left[\widehat{\mu_1}(z)\widehat{\mu_2}(z)-\widehat{\mu_1}(\zeta)\widehat{\mu_2}(\zeta)\right]}{(z-\zeta)\left(\widehat{\mu_1}(z)\widehat{\mu_2}(z)\right)}\,dz \\
&= \widehat{\mu_1}(\zeta)\sum_{\substack{z\in\mathcal{Z}_2\\ |z|<\rho_n}}\frac{\widehat{\psi}(z)}{\widehat{\mu_1}(z)\frac{d}{d\zeta}\widehat{\mu_2}(z)}\left(\frac{\widehat{\mu_2}(\zeta)}{\zeta-z}\right) \\
&+ \widehat{\mu_2}(\zeta)\sum_{\substack{z\in\mathcal{Z}_1\\ |z|<\rho_n}}\frac{\widehat{\psi}(z)}{\widehat{\mu_2}(z)\frac{d}{d\zeta}\widehat{\mu_1}(z)}\left(\frac{\widehat{\mu_1}(\zeta)}{\zeta-z}\right).
\end{aligned}$$

Let

$$R_n(\zeta) = \frac{1}{2\pi i}\widehat{\mu_1}(\zeta)\widehat{\mu_2}(\zeta)\oint_{\Gamma_n}\frac{\widehat{\psi}(z)}{(z-\zeta)\left(\widehat{\mu_1}(z)\widehat{\mu_2}(z)\right)}\,dz.$$

To finish, we need to show that $R_n(\zeta) \longrightarrow 0$ as $n \longrightarrow \infty$. We were able to choose Γ_n such that for $z \in \Gamma_n$,

$$|\sin(2\pi z)| \geq \frac{\sqrt{2}}{4}e^{2\pi|\Im z|},$$

$$\left|\sin(2\pi(z-\frac{1}{4}))\right| \geq \frac{\sqrt{2}}{4}e^{2\pi|\Im z|}.$$

By the Paley-Wiener-Schwartz Theorem, there exist positive constants C and A, with $A < 2$ and $C = C(N)$, such that for any positive integer N and any $z \in \mathbb{C}$,

$$\left|\widehat{\psi}(z)\right| \leq C(1+|z|)^{-N}e^{2\pi A|\Im z|}.$$

Therefore, for ζ fixed and any fixed $N > 0$,

$$\begin{aligned}
|R_n(\zeta)| &\leq |\widehat{\mu_1}(\zeta)\widehat{\mu_2}(\zeta)| \left|\frac{1}{2\pi i}\right| \oint_{\Gamma_n} \left|\frac{\widehat{\psi}(z)}{(z-\zeta)\,(\widehat{\mu_1}(z)\widehat{\mu_2}(z))}\right| dz \\
&\leq |\widehat{\mu_1}(\zeta)\widehat{\mu_2}(\zeta)| \sup_{z\in\Gamma_n} \frac{1}{2\pi} \left|\frac{\widehat{\psi}(z)}{(z-\zeta)\,(\widehat{\mu_1}(z)\widehat{\mu_2}(z))}\right| 2\pi\rho_n \\
&\leq |\widehat{\mu_1}(\zeta)\widehat{\mu_2}(\zeta)| \sup_{z\in\Gamma_n} \left[\frac{C(1+|z|)^{-N}\rho_n e^{2\pi A|\Im z|}}{(||z|-|\zeta||)\left(\frac{1}{8}e^{4\pi|\Im z|}\right)}|z|\right] \\
&= |\widehat{\mu_1}(\zeta)\widehat{\mu_2}(\zeta)| \left[\frac{C(1+\rho_n)^{-N}\rho_n^3}{\frac{1}{8}(\rho_n-|\zeta|)} e^{2\pi(A-2)\rho_n}\right].
\end{aligned}$$

Since $A < 2$, this last quantity $\longrightarrow 0$ as $n \longrightarrow \infty$. This completes the proof of the lemma. □

To finish the proof of the theorem, we need to show that the series expressions in the previous lemma converge in the sense of $\widehat{\mathcal{E}'}$, that is, as analytic functions which satisfy the PWS growth estimate.

Lemma 3.3 *The series*

$$\sum_{\substack{z\in\mathcal{Z}_i \\ |z|<\rho_n}} \frac{\widehat{\psi}(z)}{\widehat{\mu_j}(z)\frac{d}{d\zeta}\widehat{\mu_i}(z)} \left(\frac{\widehat{\mu_i}(\zeta)}{\zeta - z}\right), \; i,j = 1,2, \; i \neq j$$

converge in $\widehat{\mathcal{E}'}$.

Proof : If $\widehat{\mu_i}(z) = 0$, then

$$\left|\frac{d}{d\zeta}\widehat{\mu_i}(z)\right| = \frac{2}{|z|}, \; i = 1,2.$$

Also, if $\widehat{\mu_i}(z) = 0$,

$$|\widehat{\mu_j}(z)| = \frac{1}{\pi\,|z|}, \; i \neq j.$$

The functions $\widehat{\mu_i}(\zeta)$ are entire. Therefore, on every compact subset K of $\mathbb{C}$, there exists a constant $c(K)$ such that $|\widehat{\mu_i}(\zeta)| \leq c(K)$ for all $\zeta \in K$. The function $\psi(t)$ is $C^{3+\eta}$ for $\eta > 0$, and therefore its transform will decay on the real axis. In particular, for $z \in \mathcal{Z}_1 \cup \mathcal{Z}_2$, by PWS there exists a constant $d = d(3+\eta)$ such that

$$\left|\widehat{\psi}(z)\right| \leq \frac{d}{(1+|z|)^{3+\eta}}.$$

Therefore,

$$\begin{aligned}
&\frac{\left|\widehat{\psi}(z)\right|}{\left|\widehat{\mu_j}(z)\frac{d}{d\zeta}\widehat{\mu_i}(z)\right|}\frac{|\widehat{\mu_i}(\zeta)|}{|\zeta - z|} \\
\leq\ &\frac{\pi d}{2\,(1+|z|)^{3+\eta}}|z|^2\frac{c(K)}{||\zeta| - |z||} \\
\leq\ &\frac{c(K)\pi d}{2|z|^{\eta}}\frac{1}{||\zeta| - |z||} \quad i,j = 1,2\ ,\ i \neq j\,.
\end{aligned}$$

In particular, for ζ fixed,

$$\frac{\left|\widehat{\psi}(z)\right|}{\left|\widehat{\mu_j}(z)\frac{d}{d\zeta}\widehat{\mu_i}(z)\right|}\frac{|\widehat{\mu_i}(\zeta)|}{|\zeta - z|} \leq \frac{(\text{constant})_i}{|z|^{2+\eta}}\,.$$

Thus, by the Weierstrass M-test, the series

$$\sum_{\substack{z\in\mathcal{Z}_i \\ |z|<\rho_n}} \frac{\left|\widehat{\psi}(z)\right|}{\left|\widehat{\mu_j}(z)\frac{d}{d\zeta}\widehat{\mu_i}(z)\right|}\frac{|\widehat{\mu_i}(\zeta)|}{|\zeta - z|} \quad i,j = 1,2\ ,\ i \neq j$$

both converge uniformly on compact subsets of $\mathbb{C}$, and therefore represent entire functions.

Now, for each $z \in \mathcal{Z}_i$ and $|\zeta - z| \geq \epsilon > 0$ with $\epsilon < 1/2$, there exists a constant a_i

$$\left|\frac{\widehat{\mu_i}(\zeta)}{\zeta - z}\right| \leq \frac{1}{\epsilon}|\widehat{\mu_i}(\zeta)| \leq \frac{a_i}{\epsilon}e^{2\pi|\Im\zeta|}\,.$$

If $|\zeta - z| \leq \epsilon$, then

$$\left|\frac{\widehat{\mu_i}(\zeta)}{\zeta - z}\right| \leq \sup_{|\zeta - z|\leq\epsilon}\left|\frac{d}{d\zeta}\widehat{\mu_i}(\zeta)\right|.$$

Therefore, there exists $\kappa_i > 0$ such that

$$\left|\frac{\widehat{\mu_i}(\zeta)}{\zeta - z}\right| \leq \kappa_i e^{2\pi|\Im\zeta|}$$

independently of $z \in \mathcal{Z}_i$.

From the estimates above, we have that the series

$$\sum_{\substack{z\in\mathcal{Z}_i \\ |z|<\rho_n}} \frac{\left|\widehat{\psi}(z)\right|}{\left|\widehat{\mu_j}(z)\frac{d}{d\zeta}\widehat{\mu_i}(z)\right|}$$

is bounded by some B. Therefore,

$$\begin{aligned}
&\left| \sum_{\substack{z\in\mathcal{Z}_i \\ |z|<\rho_n}} \frac{\widehat{\psi}(z)}{\widehat{\mu_j}(z)\frac{d}{d\zeta}\widehat{\mu_i}(z)} \frac{\widehat{\mu_i}(\zeta)}{(\zeta - z)} \right| \\
\leq & \sum_{\substack{z\in\mathcal{Z}_i \\ |z|<\rho_n}} \frac{\left|\widehat{\psi}(z)\right|}{\left|\widehat{\mu_j}(z)\frac{d}{d\zeta}\widehat{\mu_i}(z)\right|} \frac{|\widehat{\mu_i}(\zeta)|}{|\zeta - z|} \\
\leq & B\kappa_i e^{2\pi|\Im\zeta|},
\end{aligned}$$

Thus, the series converges in $\widehat{\mathcal{E}'}$. This completes the proof of the lemma and the theorem. □

We now give explicit formulae for the deconvolvers without using inverse transforms.

Lemma 3.4 *If* $\widehat{\mu}(\beta) = 0$, *then*

$$\frac{\widehat{\mu}(\zeta)}{(\zeta - \beta)} = \widehat{T}(\zeta),$$

where

$$\langle T(\beta); \psi \rangle = -2\pi i \int_{\mathbb{R}} \left(\int_0^x \psi(t) e^{2\pi i\beta(t-x)} dt \right) \mu(x)\, dx.$$

Proof : Since $T \in \mathcal{E}'$,

$$\begin{aligned}
\widehat{T} &= \left\langle T; e^{-2\pi it\zeta} \right\rangle \\
&= -2\pi i \int_{-\infty}^{\infty} \left(\int_0^x e^{-2\pi it\zeta} e^{2\pi i\beta(t-x)} dt \right) \mu(x)\, dx \\
&= -2\pi i \int_{-\infty}^{\infty} \left(\int_0^x e^{-2\pi it(\zeta-\beta)} dt \right) \mu(x) e^{-2\pi ix\beta}\, dx \\
&= \int_{-\infty}^{\infty} \left(-2\pi i \frac{e^{-2\pi ix\zeta} e^{2\pi ix\beta} - 1}{-2\pi i\,(\zeta - \beta)} \right) \mu(x) e^{-2\pi ix\beta}\, dx \\
&= \int_{-\infty}^{\infty} \frac{\mu(x) e^{-2\pi ix\zeta}}{(\zeta - \beta)}\, dx - \int_{-\infty}^{\infty} \frac{\mu(x) e^{-2\pi ix\beta}}{(\zeta - \beta)}\, dx \\
&= \frac{\widehat{\mu}(\zeta) - \widehat{\mu}(\beta)}{(\zeta - \beta)} \\
&= \frac{\widehat{\mu}(\zeta)}{(\zeta - \beta)},
\end{aligned}$$

since $\widehat{\mu}(\beta) = 0$. □

Lemma 3.5 *For* μ_i *as above and* $z \in \mathcal{Z}_i$, $i, j = 1, 2$,

$$\frac{\widehat{\mu}_i(\zeta)}{(\zeta - z)} = \widehat{T_i}(\zeta),$$

where

$$T_1(t) = \frac{1}{z}\left(e^{2\pi iz(t+1)} - 1\right)\chi_{[-1,1]}(t)\,,$$

$$T_2(t) = \frac{1}{\left(z-\frac{1}{4}\right)}\left(e^{2\pi i\left(z-\frac{1}{4}\right)(t+1)} - 1\right)e^{(\pi/2)it}\chi_{[-1,1]}(t)\,.$$

Proof : For $z \in \mathcal{Z}_1$,

$$\begin{aligned}
\langle T_1;\psi\rangle &= -2\pi i\int_{-\infty}^{\infty}\left(\int_0^x \psi(t)e^{2\pi iz(t-x)}\,dt\right)\chi_{[-1,1]}(x)\,dx \\
&= -2\pi i\int_{-1}^{1}\left(\int_0^x \psi(t)e^{2\pi iz(t-x)}\,dt\right)dx \\
&= 2\pi i\int_{-1}^{0}\left(\psi(t)\int_{-1}^{t} e^{2\pi iz(t-x)}\,dx\right)dt \\
&\quad + -2\pi i\int_0^1\left(\psi(t)\int_t^1 e^{2\pi iz(t-x)}\,dx\right)dt \\
&= \left\langle\frac{1}{z}\left(e^{2\pi iz(t+1)}-1\right)\chi_{[-1,1]}(t);\psi\right\rangle.
\end{aligned}$$

For $z \in \mathcal{Z}_2$,

$$\begin{aligned}
\langle T_2;\psi\rangle &= -2\pi i\int_{-\infty}^{\infty}\left(\int_0^x \psi(t)e^{2\pi iz(t-x)}\,dt\right)e^{(\pi/2)ix}\chi_{[-1,1]}(x)\,dx \\
&= -2\pi i\int_{-1}^{1}\left(\int_0^x \psi(t)e^{2\pi iz(t-x)}\,dt\right)e^{(\pi/2)ix}\,dx \\
&= 2\pi i\int_{-1}^{0}\left(\psi(t)e^{2\pi izt}\int_{-1}^{t} e^{-2\pi i\left((z-\frac{1}{4})x\right)}\,dx\right)dt \\
&\quad + -2\pi i\int_0^1\left(\psi(t)e^{2\pi izt}\int_{-1}^{t} e^{-2\pi i\left((z-\frac{1}{4})x\right)}\,dx\right)dt \\
&= \left\langle\frac{1}{\left(z-\frac{1}{4}\right)}\left(e^{2\pi i\left(z-\frac{1}{4}\right)(t+1)}-1\right)e^{(\pi/2)it}\chi_{[-1,1]}(t);\psi\right\rangle. \quad \square
\end{aligned}$$

These last two lemmas give us explicit formulae for the ν_i in the time domain. We add that the estimates given in the proof of the previous theorem give us that ψ only needs to be a compactly supported $C^{3+\eta}$ function for $\eta > 0$, because this is sufficient to guarantee that the series representations for the ν_i converge to compactly supported functions.

Theorem 3.3 *Let $f \in C^\infty \cap L^2(\mathbb{R})$, and for $\eta > 0$ let ψ be a function with support in $(-2,2)$ in the Hölder space $C^{3+\eta}$ such that $\psi \geq 0$ and $\int_{-\infty}^{\infty}\psi(t)\,dt = 1$. Given*

$$\mu_1(t) = \chi_{[-1,1]}(t)\ ,\ \mu_2(t) = e^{\frac{\pi}{2}it}\chi_{[-1,1]}(t),$$

the deconvolvers $\nu_{i,\psi}$ such that

$$f * \psi = (f * \mu_1) * \nu_{1,\psi} + (f * \mu_2) * \nu_{2,\psi}$$

are given by the formulae

$$\nu_{1,\psi}(t) = \sum_{z\in\mathcal{Z}_2} \frac{\widehat{\psi}(z)}{\widehat{\mu_1}(z)\frac{d}{d\zeta}\widehat{\mu_2}(z)} \left(\frac{1}{\left(z-\frac{1}{4}\right)} \left(e^{2\pi i\left(z-\frac{1}{4}\right)(t+1)} - 1 \right) e^{(\pi/2)it} \chi_{[-1,1]}(t) \right) , \quad (23)$$

$$\nu_{2,\psi}(t) = \sum_{z\in\mathcal{Z}_1} \frac{\widehat{\psi}(z)}{\widehat{\mu_2}(z)\frac{d}{d\zeta}\widehat{\mu_1}(z)} \left(\frac{1}{z} \left(e^{2\pi iz(t+1)} - 1 \right) \chi_{[-1,1]}(t) \right) . \quad (24)$$

The function $f * \psi$ is an arbitrarily close approximation of f which converges to f in the sense of distributions as $\mathrm{supp}(\psi) \longrightarrow \{0\}$.

To finish this section, we need to show that the deconvolution procedure given in Theorem 3.3 works for general L^2 functions. We first note that $C^\infty \cap L^2(\mathbb{R})$ is dense in L^2. We also note that the functions $\nu_{i,\psi}$ are compactly supported and continuous. Therefore, since $f * \mu_i \in L^2$, $f * \mu_i * \nu_{i,\psi} \in L^2$ with

$$\|f * \mu_i * \nu_{i,\psi}\|_2 \leq \|f * \mu_i\|_2 \|\nu_{i,\psi}\|_1.$$

Similarly,

$$\|f * \psi\|_2 \leq \|f\|_2 \|\psi\|_1.$$

Define the operator

$$\mathcal{D}\colon L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$$

by

$$\mathcal{D}(f) = f * \psi - \left(f * \mu_1 * \nu_{1,\psi} + f * \mu_2 * \nu_{2,\psi}\right).$$

The operator $\mathcal{D}$ is clearly linear. The estimates above give us that $\mathcal{D}$ is bounded, and therefore continuous. Since $\mathcal{D} = 0$ on $C^\infty \cap L^2(\mathbb{R})$ and is continuous, $\mathcal{D} = 0$ on $L^2(\mathbb{R})$. Thus, the deconvolution procedure works for general functions of finite energy.

Remark : Thus, given a fixed system, it is possible to modify the system via some easily performed transform to create a strongly coprime system, and consequently recover the complete input function. This modulation technique works for creating strongly coprime systems for B-spline systems. In a similar vein, given $\chi_{[-1,1]\times[-1,1]}$, a strongly coprime system is created by rotating the square by $\frac{\pi}{4}$ and $\frac{\pi}{3}$ (see [13]).

Remark : The deconvolvers are not unique. For example, given two deconvolvers ν_1 and ν_2, $\lambda \in \mathbb{R}$, and a compactly supported function η, the pair

$$\tilde{\nu}_1 = \nu_1 + \lambda\eta * \mu_2 \ , \ \tilde{\nu}_2 = \nu_2 - \lambda\eta * \mu_1$$

is also a set of deconvolvers. The non-uniqueness of the deconvolvers allows one to develop deconvolvers that are optimal with respect to a given condition. In [13], deconvolvers that optimize the signal to noise ratio of the signal relative to white Gaussian noise were constructed. Other types of noise could be dealt with in similar fashion.

Remark : We can also use a family of zero-mean Gaussians for our auxiliary function ψ. If

$$\psi_\lambda(t) = \frac{1}{\sqrt{2\pi\lambda}} e^{-t^2/(2\lambda^2)} ,$$

Figure 3. Gaussian Function G Approximate Deconvolver System.

3.a.) Channel 1 Deconvolver ν_1.

3.b.) Channel 2 Deconvolver ν_2.

3.c.) $G * \mu_1 * \nu_1$.

3.d.) $G * \mu_2 * \nu_2$.

3.e.) $\tilde{G} = G * \mu_1 * \nu_1 + G * \mu_2 * \nu_2$, the resulting approximate delta.

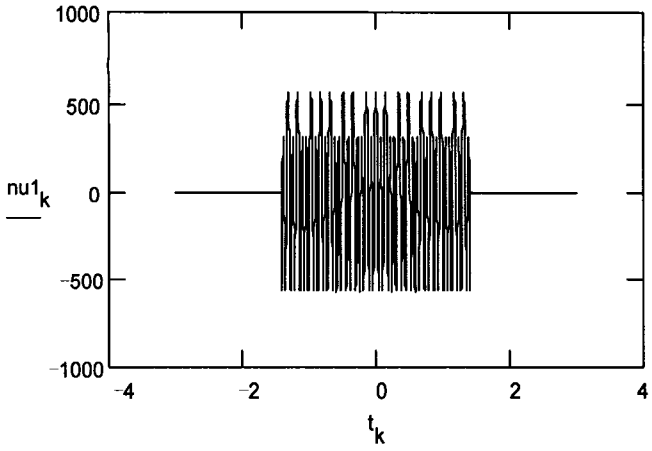
3.f.) The Fourier transform, or frequency response, of this system.

Figure 4. Transfer Response for the System.

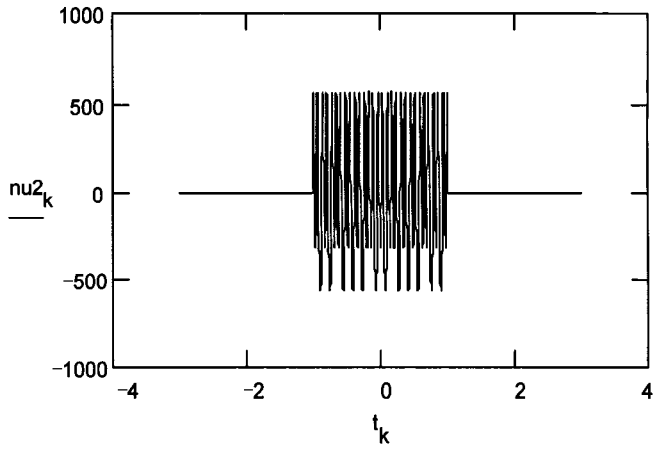
4.a.) The response of each channel, in the two channel system of Figure 3, to two adjacent Gaussian pulses (denoted as input signal f).

$$(\mu_1 * f - \text{solid} \ , \ \mu_2 * f - \text{dotted})$$

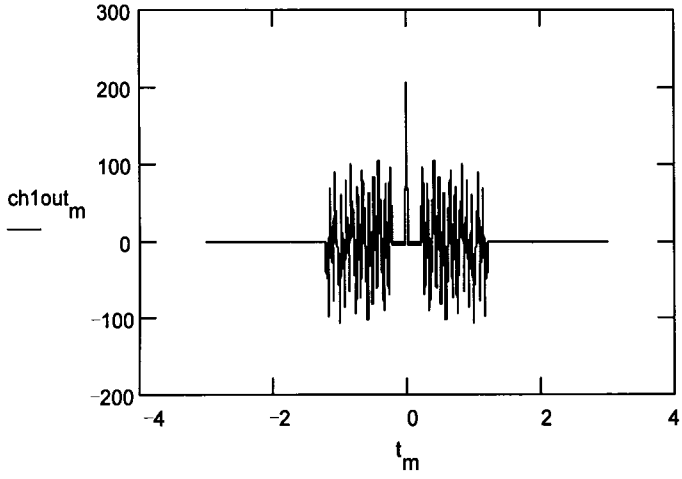
4.b.) Transfer response for the system given in Figure 3. Here the dotted line represents the input function while the solid line shows the system response.



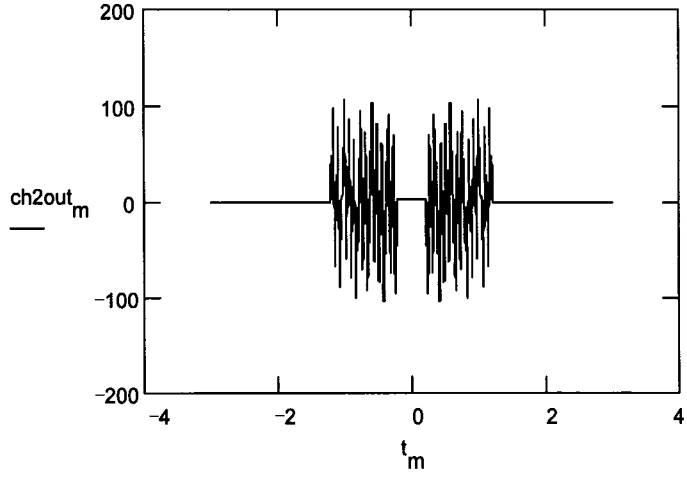
3a. Channel 1 Deconvolver (ν_1)



3b. Channel 2 Deconvolver (ν_2)

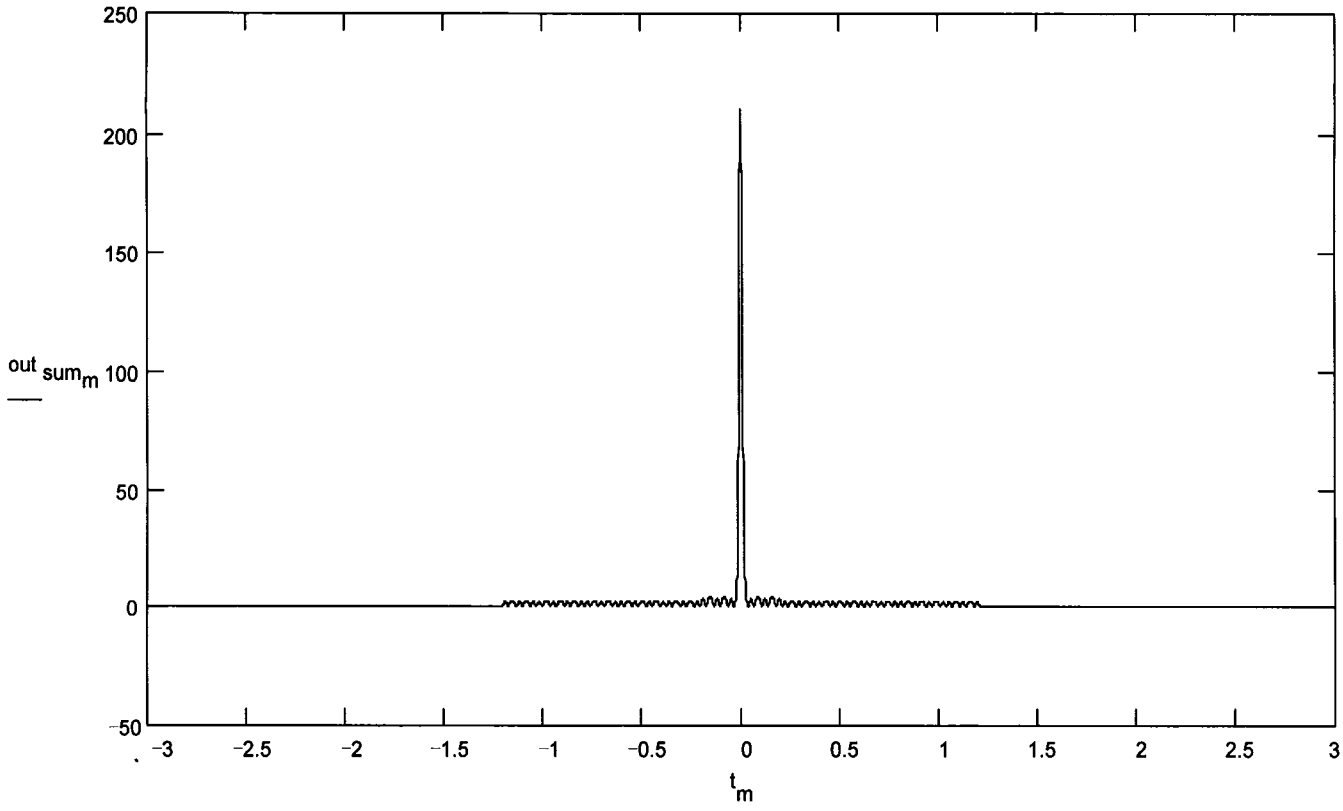


3c. Channel 1 output ($\Psi * \mu_1 * \nu_1$)

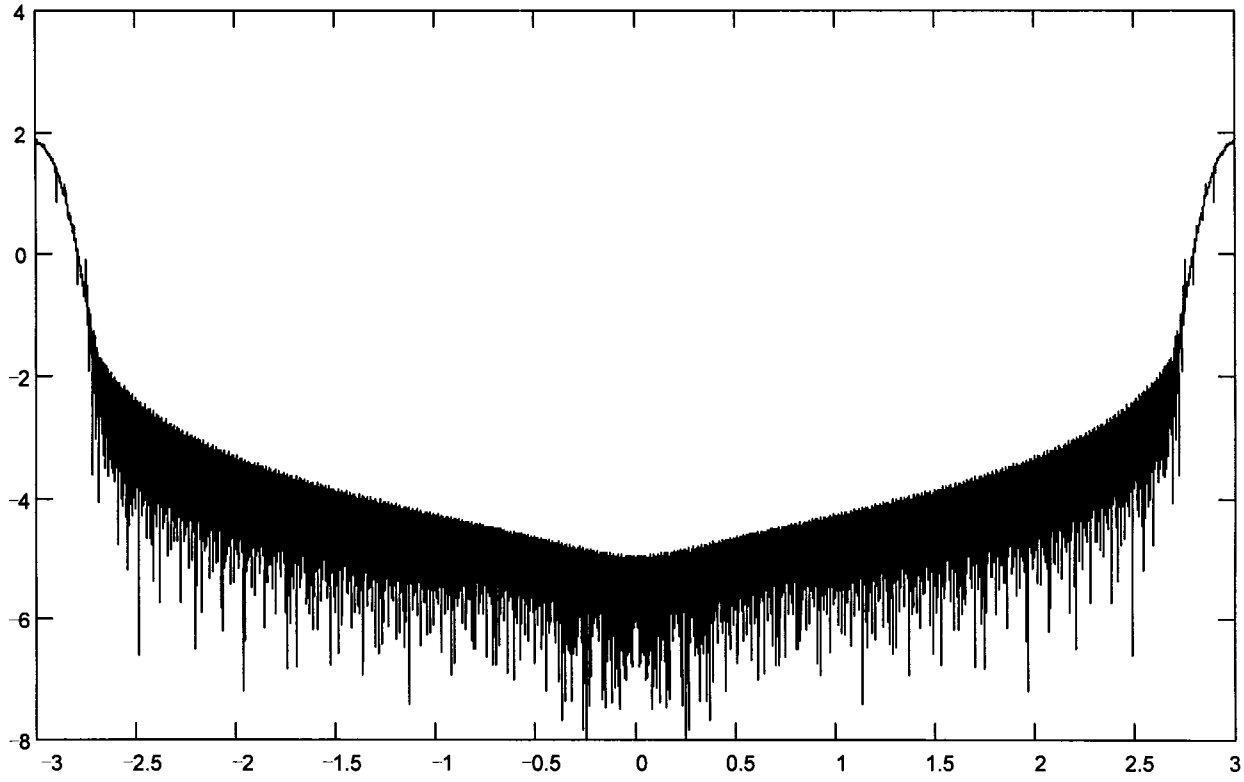


3d. Channel 2 output ($\Psi * \mu_2 * \nu_2$)

Figure 3. The Gaussian (G) approximate deconvolver system.

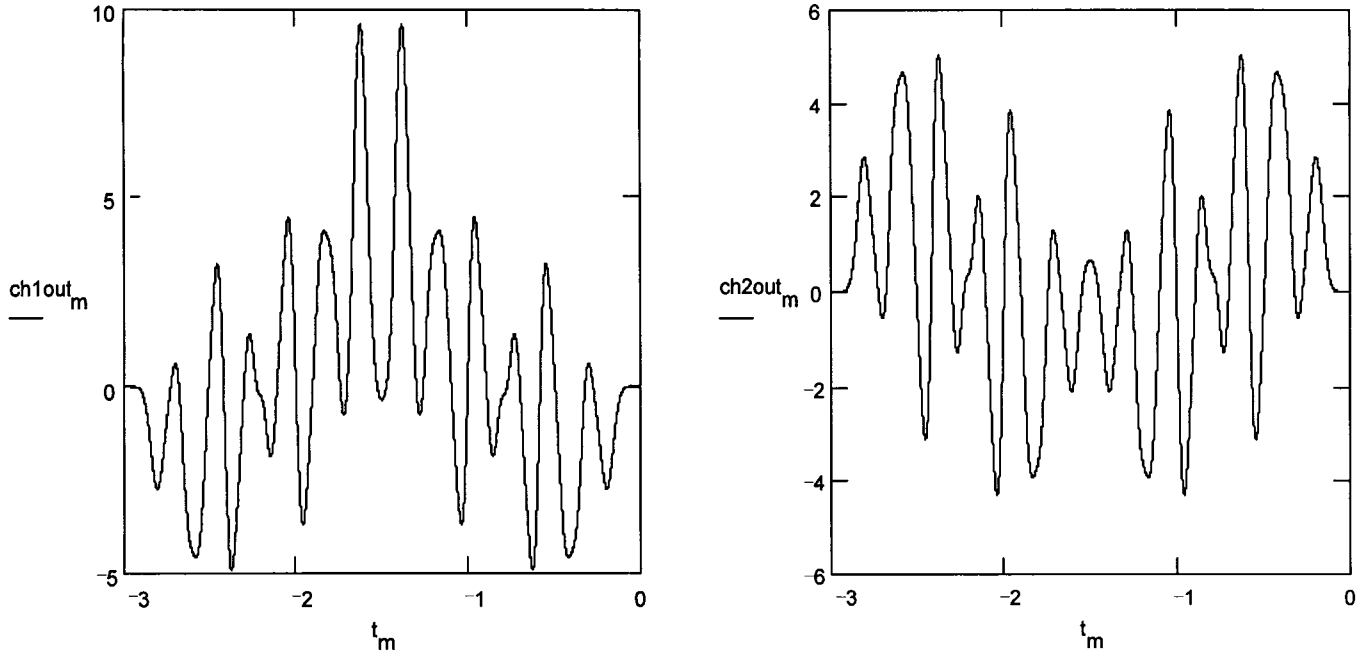


1e. $\widetilde{\Psi} = \Psi * \mu_1 * \nu_1 + \Psi * \mu_2 * \nu_2$, the resulting approximate delta functional (time domain).

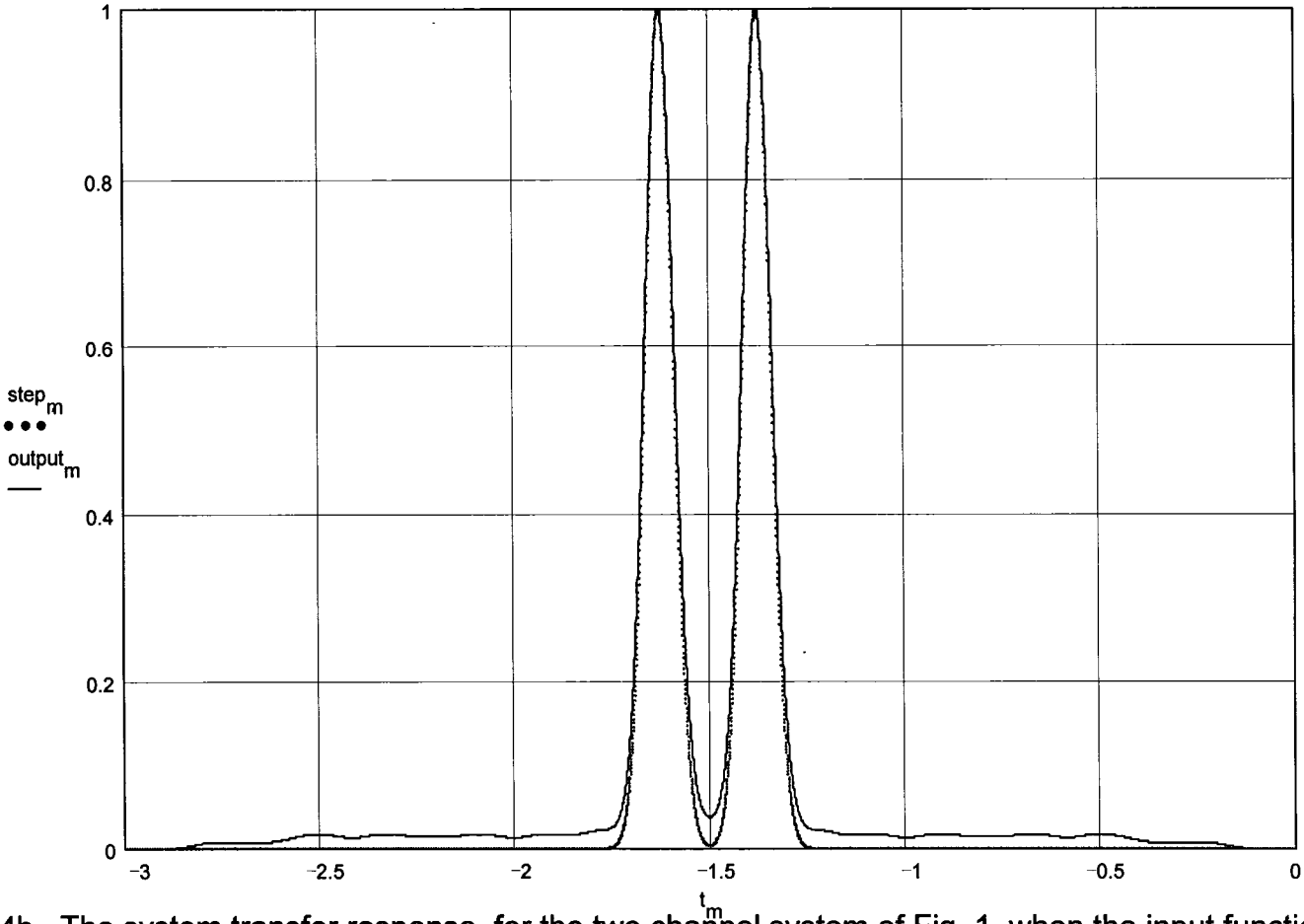


1f. $\widetilde{\Psi} = \Psi * \mu_1 * \nu_1 + \Psi * \mu_2 * \nu_2$, the resulting approximate delta functional (frequency domain).

Figure 3. The Gaussian (G) approximate deconvolver system.



4a. The response of each channel, in the two channel system of Fig. 3, to two Gaussian pulses (denoted as the input signal, f).



4b. The system transfer response, for the two channel system of Fig. 1, when the input function is the sum of two time shifted Gaussian pulses. The dotted line in the input function, the solid line in the deconvolved function.

Figure 4. Transfer response for the system of Figure 3.

then

$$\widehat{\psi_\lambda}(\zeta) = e^{-2\pi^2\lambda^2\zeta^2}.$$

We restrict $\lambda > 4$, and so twice the standard deviation is < 2. As $\lambda \longrightarrow \infty$ $\widehat{\psi_\lambda} \longrightarrow 1$ in the sense of convergence in $\widehat{\mathcal{S}'}$ ($\widehat{\psi_\lambda}$ is the transform of an approximate identity).

Convergence estimates must now be in terms of $\mathcal{S}'$ and $\widehat{\mathcal{S}'}$. The deconvolvers developed by using $\psi_\lambda(t)$ are not compactly supported. However, from a numerical viewpoint, they are "essentially compactly supported," in that for $t > 4\lambda$, the numerical values of the deconvolvers essentially equal zero. Figures 3 and 4 repeat the simulation of Figures 1 and 2 using a Gaussian auxiliary function.

3.2 Sampling Revisited

We first need to recall the following theorem.

Theorem 3.4 ([32]) *Let F be a function of finite energy on $[-T,T]$ ($F \in L^2[-T,T]$). Then*

$$F(t) = \frac{1}{2T}\sum_{n\in\mathbb{Z}} \widehat{F}(n/2T)e^{2\pi i(n/2T)t}\chi_{[-T,T]}(t)\,.$$

Moreover, given any sequence $\{a_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$ with $\sum_n |a_n|^2 < \infty$ (i.e., $\{a_n\} \in \ell^2$), then there exists a function $F \in L^2[-T,T]$ such that

$$\widehat{F}(n/2T) = a_n$$

for all $n \in \mathbb{Z}$. In fact,

$$F(t) = \frac{1}{2T}\sum_{n\in\mathbb{Z}} \widehat{F}(n/2T)e^{2\pi i(n/2T)t}\chi_{[-T,T]}(t)\,.$$

Also,

$$\sum_{n\in\mathbb{Z}} F(t+2nT) = \frac{1}{2T}\sum_{n\in\mathbb{Z}} \widehat{F}(n/2T)e^{2\pi i(n/2T)t}\chi_{[-T,T]}(t)\,,$$

where the sums converge in $L^2[-T,T]$.

Let $k \in \mathbb{N}$. The zeros of $\widehat{\mu_1}(\zeta) = \frac{\sin(2\pi\zeta)}{\pi\zeta}$ are $\mathcal{Z}_1 = \left\{\pm\frac{k}{2}\right\}$, while for $\mu_2(t) = E_{\frac{1}{4}}\mu_1 = e^{\frac{\pi}{2}it}\mu_1(t)$, $\widehat{\mu_2}(\zeta) = \frac{\sin(2\pi(\zeta-\frac{1}{4}))}{\pi(\zeta-\frac{1}{4})}$ and so $\mathcal{Z}_2 = \left\{\frac{1}{4}\pm\frac{k}{2}\right\}$. Thus, for $n \in \mathbb{Z}$,

$$\mathcal{Z}_1 \cup \mathcal{Z}_2 = \left\{\frac{n}{4}\right\} \setminus \left\{0, \frac{1}{4}\right\}.$$

Now, the construction of the deconvolvers in the previous section assumed that the auxiliary function ψ had support $\subset (-2,2)$. A construction of ψ via Shannon sampling in frequency is possible. The Nyquist rate is $\frac{1}{4}$ - exactly the same rate as $\mathcal{Z}_1 \bigcup \mathcal{Z}_2$. These observations yield the following.

Theorem 3.5 *Let $f \in L^2(\mathbb{R})$, and for $\eta > 0$ let ψ be a function with support in $(-2,2)$ in the Hölder space $C^{3+\eta}$ such that $\psi \geq 0$ and $\int_{-\infty}^{\infty} \psi(t)\, dt = 1$. Given*

$$\mu_1(t) = \chi_{[-1,1]}(t)\ ,\ \mu_2(t) = e^{\frac{\pi}{2}it}\chi_{[-1,1]}(t),$$

the deconvolvers $\nu_{i,\psi}$ such that

$$f * \psi = (f * \mu_1) * \nu_{1,\psi} + (f * \mu_2) * \nu_{2,\psi}$$

are given by the formulae

$$\nu_{1,\psi}(t) = \frac{1}{2}\left[K_1 + \sum_{n\neq 0} \frac{\widehat{\psi}((1/4)+(n/2))}{\widehat{\mu_1}((1/4)+(n/2))} e^{\pi i n t}\right] e^{\frac{\pi}{2}it}\chi_{[-1,1]}(t)\,, \tag{25}$$

$$\nu_{2,\psi}(t) = \frac{1}{2}\left[K_2 + \sum_{n\neq 0} \frac{\widehat{\psi}(n/2)}{\widehat{\mu_2}(n/2)} e^{\pi i n t}\right] \chi_{[-1,1]}(t)\,, \tag{26}$$

where

$$K_1 = \sum_{n\neq 0} (-1)^{n+1} \frac{\widehat{\psi}((1/4)+(n/2))}{\widehat{\mu_1}((1/4)+(n/2))}\,,$$

$$K_2 = \sum_{n\neq 0} (-1)^{n+1} \frac{\widehat{\psi}(n/2)}{\widehat{\mu_2}(n/2)}\,.$$

The function $f * \psi$ is an arbitrarily close approximation of f which converges to f in the sense of distributions as $\mathrm{supp}(\psi) \longrightarrow \{0\}$.

Proof : By Theorem 3.1, μ_1 and μ_2 form a strongly coprime pair. Therefore, there exists a positive constant C and a positive integer N such that for all $\omega \in \widehat{\mathbb{R}}$,

$$|\widehat{\mu_1}(\omega)| + |\widehat{\mu_1}(\omega)| \geq C\,(1+|\omega|)^{-N}\ .$$

Thus, since $\widehat{\mu_1}\left(\frac{n}{2}\right) = 0$ and $\widehat{\mu_2}\left(\frac{1}{4}+\frac{n}{2}\right) = 0$ for all $n \in \mathbb{Z}\setminus\{0\}$,

$$\left|\widehat{\mu_1}\left(\frac{1}{4}+\frac{n}{2}\right)\right| \geq C\,(1+|n|)^{-N}\ ,$$

$$\left|\widehat{\mu_2}\left(\frac{n}{2}\right)\right| \geq C\,(1+|n|)^{-N}$$

for all $n \in \mathbb{Z}\setminus\{0\}$. Moreover, estimates developed in the proof of Lemma 3.3 give us that $N = 2$. Now, since ψ is a function in the Hölder space $C^{3+\eta}$,

$$\left|\widehat{\psi}(z)\right| \leq \frac{d}{(1+|z|)^{3+\eta}}$$

for $z \in \mathcal{Z}_1 \bigcup \mathcal{Z}_2$, and so

$$\sum_{n\neq 0}\left|\frac{\widehat{\psi}((1/4)+(n/2))}{\widehat{\mu_1}((1/4)+(n/2))}\right| < \infty, \sum_{n\neq 0}\left|\frac{\widehat{\psi}((1/4)+(n/2))}{\widehat{\mu_1}((1/4)+(n/2))}\right|^2 < \infty,$$

$$\sum_{n\neq 0}\left|\frac{\widehat{\psi}(n/2)}{\widehat{\mu_2}(n/2)}\right| < \infty, \sum_{n\neq 0}\left|\frac{\widehat{\psi}(n/2)}{\widehat{\mu_2}(n/2)}\right|^2 < \infty.$$

Hence, the sums in the definitions of $\nu_{1,\psi}$, $\nu_{2,\psi}$ converge absolutely and uniformly, and the coefficients of the deconvolvers are in ℓ^2.

Now, for $z \in \mathcal{Z}_1 \bigcup \mathcal{Z}_2$,

$$\widehat{\mu_1}(z)\widehat{\nu_{1,\psi}}(z) + \widehat{\mu_2}(z)\widehat{\nu_{2,\psi}}(z) = \widehat{\psi}(z), \tag{27}$$

for if $z \in \mathcal{Z}_i$, $\widehat{\mu_i}(z) = 0$ and $\widehat{\nu_{j,\psi}}(z) = \frac{\widehat{\psi}(z)}{\widehat{\mu_j}(z)}$, for $i \neq j$, $i,j = 1,2$. Also, for $z = 0$ and $z = 1/4$, the formulae agree. Thus

$$\widehat{\mu_1}(n/4)\widehat{\nu_{1,\psi}}(n/4) + \widehat{\mu_2}(n/4)\widehat{\nu_{2,\psi}}(n/4) = \widehat{\psi}(n/4) \tag{28}$$

for all $n \in \mathbb{Z}$. Letting

$$\widehat{F} = \widehat{\mu_1}\widehat{\nu_{1,\psi}} + \widehat{\mu_2}\widehat{\nu_{2,\psi}} - \widehat{\psi},$$

we have that F is continuous and satisfies the hypotheses of Theorem 3.4. Therefore, $F \equiv 0$, which proves the result. $\square$

We can again reconcile the apparent differences between the deconvolver formulae by computation. We have for $z \in \mathcal{Z}_2$

$$\frac{d}{d\zeta}\widehat{\mu_2}(z) = \frac{2\cos\left(2\pi\left(z-\frac{1}{4}\right)\right)}{\left(z-\frac{1}{4}\right)}.$$

Thus,

$$\begin{aligned}
&\nu_{1,\psi}(t)\\
&= \sum_{z\in\mathcal{Z}_2}\frac{\widehat{\psi}(z)}{\widehat{\mu_1}(z)}\frac{1}{\frac{d}{d\zeta}\widehat{\mu_2}(z)}\left(\frac{1}{\left(z-\frac{1}{4}\right)}\left(e^{2\pi i\left(z-\frac{1}{4}\right)(t+1)}-1\right)e^{(\pi/2)it}\chi_{[-1,1]}(t)\right)\\
&= \sum_{z\in\mathcal{Z}_2}\frac{\widehat{\psi}(z)}{\widehat{\mu_1}(z)}\frac{\left(z-\frac{1}{4}\right)}{2\cos\left(2\pi\left(z-\frac{1}{4}\right)\right)}\left(\frac{1}{\left(z-\frac{1}{4}\right)}\left(e^{2\pi i\left(z-\frac{1}{4}\right)(t+1)}-1\right)e^{(\pi/2)it}\chi_{[-1,1]}(t)\right)\\
&= \frac{1}{2}\sum_{n\neq 0}\frac{\widehat{\psi}((1/4)+(n/2))}{\widehat{\mu_1}((1/4)+(n/2))}(-1)^n\left(e^{2\pi i(n/2)(t+1)}-1\right)e^{(\pi/2)it}\chi_{[-1,1]}(t)\\
&= \frac{1}{2}\sum_{n\neq 0}\frac{\widehat{\psi}((1/4)+(n/2))}{\widehat{\mu_1}((1/4)+(n/2))}(-1)^n\left((e^{\pi i})^n e^{\pi int}-1\right)e^{(\pi/2)it}\chi_{[-1,1}(t)\\
&= \frac{1}{2}\sum_{n\neq 0}\frac{\widehat{\psi}((1/4)+(n/2))}{\widehat{\mu_i}((1/4)+(n/2))}\left((-1)^{n+1}+e^{\pi int}\right)e^{(\pi/2)it}\chi_{[-1,1]}(t).
\end{aligned}$$

Similarly, for $z \in \mathcal{Z}_1$

$$\frac{d}{d\zeta}\widehat{\mu_1}(z) = \frac{2\cos(2\pi z)}{z}\,.$$

Thus,

$$\begin{aligned}
&\nu_{2,\psi}(t)\\
&= \sum_{z\in\mathcal{Z}_1} \frac{\widehat{\psi}(z)}{\widehat{\mu_2}(z)} \frac{1}{\frac{d}{d\zeta}\widehat{\mu_1}(z)} \left(\frac{1}{z}\left(e^{2\pi i z(t+1)} - 1\right)\chi_{[-1,1]}(t)\right)\\
&= \sum_{z\in\mathcal{Z}_1} \frac{\widehat{\psi}(z)}{\widehat{\mu_2}(z)} \frac{z}{2\cos(z)} \left(\frac{1}{z}\left(e^{2\pi i z(t+1)} - 1\right)\chi_{[-1,1]}(t)\right)\\
&= \frac{1}{2}\sum_{n\neq 0} \frac{\widehat{\psi}(n/2)}{\widehat{\mu_2}(n/2)}(-1)^n \left(e^{2\pi i (n/2)(t+1)} - 1\right)\chi_{[-1,1]}(t)\\
&= \frac{1}{2}\sum_{n\neq 0} \frac{\widehat{\psi}(n/2)}{\widehat{\mu_i}(n/2)}(-1)^n \left((e^{\pi i})^n e^{\pi i n t} - 1\right)\chi_{[-1,1]}(t)\\
&= \frac{1}{2}\sum_{n\neq 0} \frac{\widehat{\psi}(n/2)}{\widehat{\mu_i}(n/2)} \left((-1)^{n+1} + e^{\pi i n t}\right)\chi_{[-1,1]}(t)\,.
\end{aligned}$$

Remark : The development of the deconvolvers $\nu_{i,\psi}$ via the Jacobi interpolation formula and Cauchy residue theory is now a well-developed tool, given the theoretical base established by Berenstein, Gay, Taylor, Yger, et al. It has a flexibility in the one-variable case that allows for its use in not only general deconvolution problems, but also in the development of filters, etc. The key to this is the flexibility of the Cauchy residue calculus in one-variable. As is well-known, the story in several variables is different. Berenstein, Gay, Taylor, Yger, et al. have given us working formulae for specific situations. There are no general formulae in several variables for the computation of the needed residues.

Sampling lies juxtaposition these methods. It does not have the tremendous flexibility of the complex methods in one variable, but it also does not carry with it comparable computational difficulties in several variables.

Both methods will be used in further developments of the theory.

4 Discussion of Applications

We now give a brief discussion of the applications of these deconvolution techniques, and refer to [13], [14], and [15]. We also mention [17]. The multichannel theory was extended to a much broader class of convolvers in that paper, and was used to design filters in some specific cases.

Deconvolution is applicable in any area of signal and image processing in which it is desirable to know a high degree of detail from the input data. A multichannel system consisting of an array of strongly coprime convolvers $\{\mu_i\}$ guarantees that any information about the input function f lost by one convolver is retained by another in the array as

$s_i = f * \mu_i$ for some i. The signals s_i are then filtered by the ν_i and added, resulting in the reconstruction of f. Again, the methods are both linear (convolution with deconvolvers) and realizable (the support of the deconvolvers being contained in the bounded support of the convolvers), with deconvolution at a point $t \in \mathbb{R}^n$ depending only on data near t. The theory assumes no *a priori* information about the input signals. The non-uniqueness of the deconvolvers allows one to develop deconvolvers that are optimal with respect to a given condition. We also have precise knowledge about digital versions of the deconvolvers, although the sampling rate over those domains still depends on the bandwidth of the received signals, which equal the bandwidth of the original signal. Additionally, we can combine this linear procedure with other linear procedures which extract information about the signal, e.g., discrete wavelet transforms (see [16]).

4.1 Image Processing

Deconvolution will be very useful in image processing. In a system designed to perform image analysis, the deconvolvers would act as an initial enhancing filter for the image. After this process, algorithms such as edge detection, etc. would then be working with data arbitrarily close to the original data, instead of data in which the high frequency information has been lost. Further, these various types of processing could possibly be combined with deconvolution so as to produce the results in a single processing step.

The problem of getting an exact representation of a pixel image is addressed in [13] and [14]. Here, an image is restored from the data gathered by a set of three photo-detectors. Each photo-detector is modeled as the integrator over time of an image f in a compact region of $\mathbb{R}^2$. Thus, the model for the image data is $s = f * \mu$, where μ is the characteristic function χ of some bounded region in $\mathbb{R}^2$. The strongly coprime system is $\mu_1 = \chi_{[-1,1]\times[-1,1]}$, $\mu_2 = \chi_{[-\sqrt{2},\sqrt{2}]\times[-\sqrt{2},\sqrt{2}]}$, $\mu_3 = \chi_{[-\sqrt{3},\sqrt{3}]\times[-\sqrt{3},\sqrt{3}]}$.

To deconvolve, it is necessary to multiplex in time, i.e., the time interval used for integration will be subdivided and used to obtain finer spatial sampling in multiple channels. Although no larger time interval is required, the use of the time interval for multiplexing means that we are not deconvolving the time integration but rather depending upon a relative time invariance of the signal over this interval. Two levels of multiplexing are required for detector arrays: one multiplexing is to approximate a convolution, the second is to obtain multiple convolutions. The deconvolvers used are two dimensional versions of the deconvolvers described in Section 2. These deconvolvers are separable.

For these detector applications an important topic both for hardware development and for models of vision is the design of multiplexed arrays. We want to know how to best arrange in a single array a set of approximately strongly coprime detectors, how to best "scan," "dither," or "sweep" the array in order to obtain the desired oversamples, and how to use these measurements for deconvolution and signal estimation as well as for the determination of information about the scene (source of the signal) from the signal. Associated with these questions are the obvious ones of knowing or estimating temporal and spatial phase of sample points in the array. Because of such considerations, the strongly coprime pair of convolvers given by the characteristic functions of two disks with suitably different radii have nice

properties as well as obvious analogs in biological retinas. The strongly coprime condition is satisfied with only two disks of radii r_1, r_2, with $r_1/r_2 = k \in \mathbb{N}$. Moreover, two such disks can be positioned concentrically, which eliminates one phasing problem. In addition, this configuration may reflect the arrangements in biological retinas that are described by Marr [27] as the "central surround" response used (presumably) for edge detection.

The results on deconvolution are for distributions with compact support. The transforms of these distributions are entire functions and hence have at most isolated zeros. Further, these results are about a set of these distributions such that no isolated zero is common to all in the set. Obviously, these deconvolution results are not applicable to convolvers b that are band-limited. They are, however, applicable to the compactly supported convolvers in a system: if the μ_i, $i = 1, \ldots, n$, are strongly coprime and each μ_i is the composition of all of the compactly supported convolvers of the i^{th} linear system, and if b is the composition of the band-limited convolvers for each system, then, with deconvolvers ν_i, $i = 1, \ldots, n$, $\sum_{i=1}^{n} b * \mu_i * \nu_i = b$. That is, we can reconstruct up to the band-limited response.

A primary class of sensor components that are modeled by band-limited convolvers is the class described by an aperture response function for electromagnetic or acoustic waves, such as the point spread function of an optical lens. (The transform rather than the convolver has compact support.) The band-limit is the diffraction limit. To apply the theory, we would want compactly supported strongly coprime convolvers in a system, and construct up to the band-limit. An example is given by an optical imaging system for incoherent light (e.g., a telescope). The system can be modeled as the convolution $I_i = \widetilde{h} * I_g$ [23] , where I is intensity, and the impulse response is determined by the pupil function of the system (the modulus squared of the Fourier transform of the pupil function). In many cases, the impulse response is radially symmetric and given by Bessel functions. For these systems, the techniques used in solving the Pompeiu problem will have application.

To extend the knowledge of the image past the band-limit requires both an assumption about the image and some other techniques in reconstruction. For systems in which the impulse response is not compactly supported, to get information from the system at any given point, one would have to assume that most of the information contained in the image is contained in a compact region. (Otherwise, to know the system output *anywhere* requires integration over all of $\mathbb{R}^2$.) If we assume that the image is compactly supported (certainly a reasonable assumption for most imaging systems), the transform of the image or image intensity is an entire function in two complex variables. The transform of the system gives us that we know this analytic function inside some ball, determined by the system bandwidth. Moreover, the function is corrupted by noise. To extend our knowledge of the image past the system bandlimit, we could in theory analytically continue to a larger disk, and transform back. We are examining the following technique for reconstructing the function, which involves using the Bergman reproducing kernel. Since the image is L^2 and compactly supported, its transform is L^2 and entire. The transform can therefore be expanded in terms of the kernel - if $t = u + iv$, $f(z) = \int\int f(t)K(z,t)du\,dv$. The Bergman basis functions ω_n for a ball are known explicitly. We can expand the original function in a ball larger than that determined by the bandwidth. Computation of the Bergman representation is approximated by expanding f in terms of these basis functions, i.e., $f \approx \sum_{k=1}^{n} = \langle f, \omega_k \rangle \omega_k$. In this com-

putation, the integration will have the effect of averaging the noise. Exact estimates need to be computed.

4.2 Signal Processing

Multichannel deconvolution theory will be useful in radar and sonar. The sensor components which are modeled by compactly supported convolvers include nearly all of the various pulses used in radar and sonar. The convolvers describe the modulation of the transmitted carrier frequency and the deconvolvers describe the filters used to process the demodulated return signal. The deconvolution theorem says in this context that if a set of strongly coprime pulses are transmitted and if each pulse of the set is separately processed, then the strongly coprime pulses can be completely compressed, without sidelobes, in the case of zero (or known) doppler shift. The types of pulses that can be used to make up a strongly coprime set include any of the pulse types used in radar and sonar. Pairs of strongly coprime chirps is one such example. This feature suggests that it is possible to consider the enhancement of the range resolution performance of pulse types that perform optimally relative to some other performance measure.

The discussion of band-limited convolvers given above is also applicable. The beam pattern for acoustic transducers and for radar antennas, as well as for the real and synthetic arrays of either, is a convolver whose Fourier transform is band-limited. To apply the theory, one would want compactly supported strongly coprime convolvers in a system.

We can apply the deconvolution theory to the acoustic signals $f(x,y,z,t)$. The output signals $s_i(x,y,z,t) = f(x,y,z,t) * \mu_i(x,y,z)$ vary with both the position of the sensors and the time at which the signal is received. The theory has to be suitably modified to take this problem into account. Ideally, we would like to have a Green's function $G(x,y,z,t;x_0,y_0,z_0,t_0)$ for our model of the atmospheric effects. Then, $f(x,y,z,t)$ is represented by an integral equation with terms G, the initial signal information g and $\frac{\partial g}{\partial t} = h$, atmospheric inhomogeneity A, and boundary conditions - namely, $f = \frac{\partial G}{\partial t} * g + G * h + \int^t G * A +$ boundary terms. If the boundary effects can be repressed, then

$$\begin{aligned} f(x,y,z,t) &= \frac{\partial G(x,y,z,t)}{\partial t} * g(x,y,z) \\ &+ G(x,y,z,t) * h(x,y,z) + \int^t G(x,y,z,\tau) * A(x,y,z,\tau). \end{aligned}$$

If f is overdetermined by a strongly coprime set of sensors $\{\mu_i\}$ as $\{s_i = f * \mu_i\}$, convolution with the sensors is over the spatial variables x,y,z. Then for ν_i as above, and as convolution is over x,y,z,

$$\begin{aligned} \sum_{i=1}^n s_i * \nu_i &= \sum_{i=1}^n \left(\frac{\partial G}{\partial t} * g + G * h + \int^t G * A \right) * \mu_i * \nu_i \\ &= \left(\frac{\partial G}{\partial t} * g + G * h + \int^t G * A \right) * \sum_{i=1}^n (\mu_i * \nu_i) \\ &= \frac{\partial G}{\partial t} * f + G * h + \int^t G * A \,. \end{aligned}$$

Finally, in [17], we explored the idea that by using duality and by solving a modified Bezout equation, we may use filters in one class to create a filter in another, for example, to create an ideal low-pass filter out of truncated sinc filters. If $\mu_i = \chi_{[-\alpha_i,\alpha_i]}\,\mathrm{sinc}_{\alpha_i}$, for $\alpha_1 = 1$, $\alpha_2 = \sqrt{p}$, p prime, then $\{\mu_1, \mu_2\}$ form a strongly coprime pair. Therefore, in the time domain, we can solve the modified Bezout equation

$$\underset{\beta}{\mathrm{sinc}}(\zeta) = \widehat{\mu_1} \cdot \widehat{\nu_1} + \widehat{\mu_2} \cdot \widehat{\nu_2}\,.$$

Transforming produces the filter $\chi_{[-\beta,\beta]}$.

Acknowledgments : The author would like to express his sincere thanks to Professor Carlos A. Berenstein of the University of Maryland, Professor Alain Yger of Université de Bordeaux I, and Professor David F. Walnut of George Mason University for many helpful discussions. He would also like to thank his student Matthew Fivash for the preparation of the figures presented in the paper.

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